AI HW #6

Maximilian Schulten (mls384)

2024-10-13

Exercise 1

a)

We check this by verifying the axioms of probability. Namely, every element $0 \leq P(x = i, y = j) \leq 1$, $\forall i, j \in \{1, 2, 3, 4\}$: This clearly holds by inspection. Moreover that $\sum_{i,j} P(x = i, y = j) = 1$. Clearly demonstrated below:

```
0.06+0.08+0.04+0.02+0.12+0.16+0.08+0.04+0.09+0.12+0.06+0.03+0.03+0.04+0.02+0.01
```

```
## [1] 1
```

b)

Define $P_x(x)$ as the marginal distribution of x.

$$P_x(x = i) := \sum_j p(x = i, y = j), \ \forall i \in \{1, 2, 3, 4\}$$

So we fix x and let y vary. Summing the rows we find:

```
joint_prob <- matrix(c(
  0.06, 0.08, 0.04, 0.02,
  0.12, 0.16, 0.08, 0.04,
  0.09, 0.12, 0.06, 0.03,
  0.03, 0.04, 0.02, 0.01
), nrow = 4, byrow = TRUE)

px <- rowSums(joint_prob)
py <- colSums(joint_prob)

px
```

```
## [1] 0.2 0.4 0.3 0.1
```

```
py
```

```
## [1] 0.3 0.4 0.2 0.1
```

So we conclude:

$$P_x(x) = \begin{cases} 0.2\,, & x = 1 \\ 0.4\,, & x = 2 \\ 0.3\,, & x = 3 \\ 0.1\,, & x = 4 \end{cases}$$

c)

X and Y are independent iff their joint probabilities can be factored as a product of their marginal probabilities. Namely we need to check that $P(x=i, y=j) = P_x(x=i) \cdot P_y(y=j)$ for every element of the table. We can easily see that this is the case. Hence X and Y are independent.

Exercise 2

a)

Proposition: $P(a|b,c) = P(b|a,c) \implies P(a|c) = P(b|c)$.

$P(a|b,c) = P(b|a,c) \implies \frac{P(a,b,c)}{P(b,c)} = \frac{P(a,b,c)}{P(a,c)} \implies P(a,c) = P(b,c) \implies P(a|c)P(c) = P(b|c)P(c) \implies P(a|c) = P(b|c)$; **QED.**

b)

Proposition: $P(a|b,c) = P(a) \implies P(b,c) = P(b)$.

$P(a|b,c) = P(a) \iff \frac{P(a,b,c)}{P(b,c)} = P(a) \iff P(b,c) = \frac{P(a,b,c)}{P(a)} = P(b,c|a) \iff P(c|b)P(b) = P(b,c|a) = \frac{P(a|b,c)P(b,c)}{P(a)} \iff P(c|b)P(b) = P(b,c)$

$\therefore P(a|b,c) = P(a) \implies P(b,c) = P(b) \iff P(c|b) = 1$ i.e. c must be guaranteed to occur when b occurs. Hence the proposition is false.

c)

Proposition: $P(a|b) = P(a) \implies P(a|b,c) = P(a|c)$.

$P(a|b,c) = \frac{P(a,b,c)}{P(b,c)} = \frac{P(a,c)P(b|c)}{P(b|c)P(c)} = P(a|c)$; **QED.**

The statement $P(a,b,c) = P(a,c)P(b|c)$ follows from:

$P(a,b,c) = P(a,c)P(b|a,c)$. Since $P(a|b) = P(a)$, a and b are independent. Hence $P(b|a,c) = P(b|c)$. So, $P(a,b,c) = P(a,c)P(b|c)$.

Exercise 3

As a proxy for how indicative the tests are I will say that the test with the higher probability of a patient having the disease given that the test returns positive is the more indicative of the two. To that end I will use Bayesian inference, and define D as the event that a patient is sick.

We are given that, For A:

$$P(T = true|D = true) = 0.95,\ P(T = true|D = false) = 0.1,\ P(D = true) = 0.01$$

Using this information we find:

$$P(D = true|T = true) = \frac{P(T = true|D = true)P(D = true)}{P(T = true)} = \frac{0.95 \cdot 0.01}{\sum_D P(T = true, D)}$$

Notice that:

$$\sum_D P(T = true, D) = \sum_D P(T = true|D)P(D) = 0.95 \cdot 0.01 + 0.1 \cdot 0.99 = 0.1085.$$

Hence:

$$P(D = true|T = true) = \frac{0.95 \cdot 0.01}{0.1085} = 0.08756 = 8.756\%$$

Following a very similar procedure for B, we are given that:

$$P(T = true|D = true) = 0.9,\ P(T = true|D = false) = 0.05,\ P(D = true) = 0.01$$

Using this information we find:

$$P(D = true|T = true) = \frac{P(T = true|D = true)P(D = true)}{P(T = true)} = \frac{0.9 \cdot 0.01}{\sum_D P(T = true, D)}$$

Notice that:

$$\sum_D P(T = true, D) = \sum_D P(T = true|D)P(D) = 0.9 \cdot 0.01 + 0.05 \cdot 0.99 = 0.0585.$$

Hence:

$$P(D = true|T = true) = \frac{0.9 \cdot 0.01}{0.0585} = 0.15385 = 15.385\%$$

We conclude that test B is more indicative of the patient having the virus. This is due to the probabilistic reasoning above yielding almost twice the likelihood of a correct test result given that the disease is being carried when using B versus A.

Exercise 4

a)

Let T represent what color was testified under oath, and let C be the true color of the taxi. We look to find a way to compute the most likely color. We can do this by taking $\max\{P(C = green|T = green), 1 - P(C = green|T = green)\}$. Using Bayes' theorem we can say:

$$P(C = green|T = green) = \frac{P(T = green|C = green)P(C = green)}{P(T = green)}$$

However, we have no information on the distribution of taxis in Athens; namely $P(C)$. So, we cannot find this probability!

b)

Now we have $P(C = green) = 0.9$ and implicitly $P(C = blue) = 0.1$. Returning back to the above we find:

$$P(C = green|T = green) = \frac{P(T = green|C = green)P(C = green)}{P(T = green)} = \frac{0.75 \cdot 0.9}{\sum_C P(T = green, C)}$$

We know:

$$P(T = green) = \sum_C P(T = green, C) = \sum_C P(T = green|C)P(C) = 0.75 \cdot 0.9 + 0.25 \cdot 0.1 = 0.7$$

So:

$$P(C = green|T = green) = \frac{0.75 \cdot 0.9}{0.7} = 0.9643 = 96.43\%$$

It goes without saying then that $\max\{P(C = green|T = green), 1 - P(C = green|T = green)\} = P(C = green|T = green) = 0.9643$. Therefore, given that we testify the taxi is green, there is a 96.43% chance the taxi is in fact green.

Exercise 5

We have:

$p(L,G,V)$	L	G	V
0.87318	F	F	F
0.0072	F	F	T
0.049	F	T	F
0.0004	F	T	T
0.00882	T	F	F
0.0108	T	F	T
0.049	T	T	F
0.0016	T	T	T

a)

$P(L) = \sum_{G,V} P(L,G,V) = 0.00882 + 0.0108 + 0.049 + 0.0016 = 0.07022$

$P(L)$	L
0.0696	T
0.9304	F

b)

$P(G) = \sum_{L,V} P(L,G,V) = 0.049 + 0.0004 + 0.049 + 0.0016 = 0.1$

$P(G)$	G
0.1	T
0.9	F

c)

$P(L = T|G = T) = \frac{P(L=T,G=T)}{P(G=T)} = \frac{0.049+0.0016}{0.1} = 0.5$

$P(L = F|G = T) = \frac{P(L=F,G=T)}{P(G=T)} = \frac{0.049+0.0004}{0.1} = 0.5$

$P(L = T|G = F) = \frac{P(L=T,G=F)}{P(G=F)} = \frac{0.0082+0.0108}{0.9} = 0.0215556$

$P(L = F|G = F) = \frac{P(L=F,G=F)}{P(G=F)} = \frac{0.87318+0.0072}{0.9} = 0.9784444$

$P(L\|G)$	L	G
0.9784	F	F
0.494	F	T
0.0215556	T	F
0.506	T	T

d)

$P(G = T|L = T \vee V = T) = \frac{P(G=T,L=T\vee V=T)}{P(L=T\vee V=T)} = \frac{0.0004+0.049+0.0016}{0.0072+0.0004+0.00882+0.0108+0.049+0.0016} = 0.6554$

e)

Let us translate the problem statement. We do not have any information on V, hence we treat it as we would if it hadn't been mentioned. $P(L = F|G = T) = 0.0215556$

Exercise 6

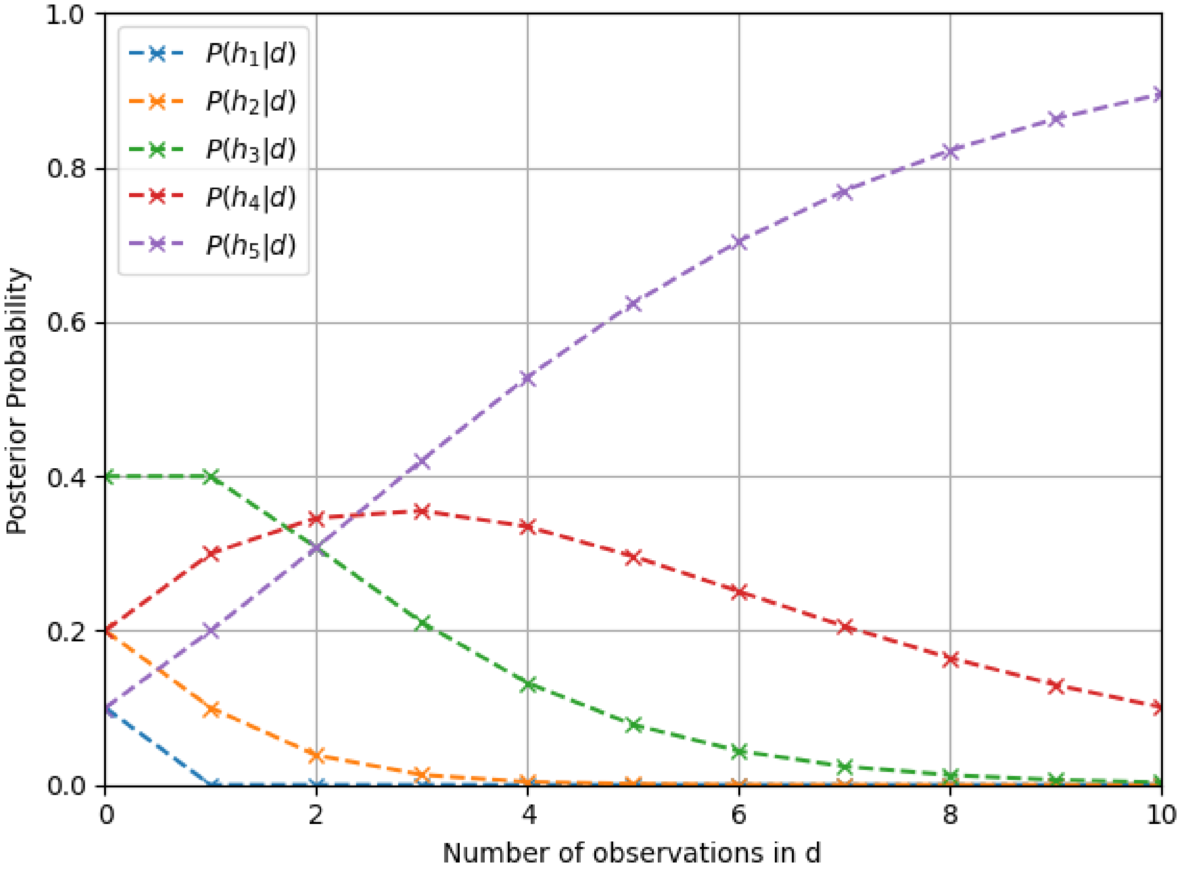
a)

Every hypothesis makes a statement about the probability of drawing a certain flavor. In a sense we can see this as a series of Bernoulli trials modeled via a binomial distribution. As we gather more and more data (i.e. draw more candy from the bag) the probability of success, whichever way we may define it, resembles the actual sample more and more for one of the hypotheses. Hence we become more confident in our guesses. From the text we have the formula:

$$P(\mathbf{d}|h_i) = \prod_j P(d_j|h_i)$$

The reason why certain posterior probabilities begin at 0.2, others at 0.1, and one at 0.4 is defined in the problem. When we haven't observed any candy we have no ability to infer beyond the probabilities of each bag type, which is what is modeled at $n = 0$.

b)



The most relevant code snippets are below:

```
if __name__ == "__main__":
    main(
        bags={
            "P(h_1|d)": {"dist": [1, 0.0], "prob": 0.1},
            "P(h_2|d)": {"dist": [0.75, 0.25], "prob": 0.2},
            "P(h_3|d)": {"dist": [0.50, 0.50], "prob": 0.4},
            "P(h_4|d)": {"dist": [0.25, 0.75], "prob": 0.2},
            "P(h_5|d)": {"dist": [0.0, 1], "prob": 0.1}
        },
        n=15,
        trueBag="P(h_5|d)"
    )

 # Initialize a dictionary to keep track of the respective calculated
    # probabilities at each observation
    probs = {key: [bags.get(key).get("prob")] for key in bags}

    # True bag type
    trueBag = bags.get(trueBag)

    # Keep track of observations
    observations = []

    # Number of observations (x-axis)
    x = np.arange(0, n)

    for i in x:
        # Take a candy from trueBag and add it to the list of observations
        data = take(trueBag)
        observations.append(data)

        for key, hypothesis in bags.items():
            # Calculate the probability of each hypothesis and add it to the
            # dictionary with its corresponding array
            probs.get(key).append(postCalc(prob=hypothesis.get("prob"),
                                           dist=hypothesis.get("dist"),
                                           observation=data,
                                           observations=observations,
                                           bags=bags))

    for key, probabilities in probs.items():
        plt.plot(np.arange(0, n+1), probabilities, label=rf'${key}$',
                 marker='x', linestyle='--')


# Finds p(h_i|d)
def postCalc(prob, dist, observation, observations, bags):
    # Calculating P(d|h_i)
    d_given_h = 1

    # Take product
    for j in observations:
        d_given_h *= dist[j]

    # Multiply by the probability of the hypothesis
    h_given_d = d_given_h * prob

    # Find proportionality constant
    alpha = 0
    for h in bags.values():
        likelihood = 1
        for j in observations:
            likelihood *= h["dist"][j]
        alpha += likelihood * h["prob"]

    return h_given_d / alpha

def take(bag):
    # Select a random index, representing a candy, with given probability
    indices = np.arange(len(bag.get("dist")))

    return np.random.choice(indices, p=bag.get("dist"))
```

Parametrizing the problem As seen in the call to the `main` function I represented the bags as a dictionary of dictionaries in the context of Python. That is each bag is a dictionary which holds: a probability distribution in the form of an array, and a probability of each bag being the bag we are choosing from. Though it has little bearing on the outcome, I parametrized the number of samples taken from the bag as `n` and specified the bag we are truly drawing from as `trueBag`.

Finding probabilities Essentially, the simulation (if you will) works by drawing a candy from the bag. The useful `numpy` library makes quick work of this using their random choice methods which allows me to specify the probability of each index being selected. Here the indices represent the candies, as the candies themselves are really just treated as random selection based on the specified probabilities. Once a selection has been made we estimate the probability $P(h_i|d)$ where each hypothesis corresponds precisely to each respective bag. This is done as given in the textbook, where α is calculated to ensure probabilities sum to 1. The function `postCalc` takes care of this arithmetic.

Plotting Finally, I used `matplotlib` to easily plot the given probabilities of each hypothesis at discrete steps, i.e. every time we pull a candy from the bag.