The University of Warwick: Numerical Integration Approaches to optimal filtering and related problems

Max Richards Adam Johansen

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Introduction

Filtering methods are concerned with the estimation of hidden (latent) states from noisy observations in dynamic systems.

In such models, an unobserved process $\{x_t\}_{t=1}^T$ evolves over time according to a **state transition model**, while an observed process $\{y_t\}_{t=1}^T$ is generated from these latent states through an **observation model**.

Formally, these relationships are often expressed as

$$x_t \sim p(x_t \mid x_{t-1}), \quad y_t \sim p(y_t \mid x_t),$$

where $p(x_t \mid x_{t-1})$ describes the system dynamics, and $p(y_t \mid x_t)$ specifies the likelihood of observations given the state.

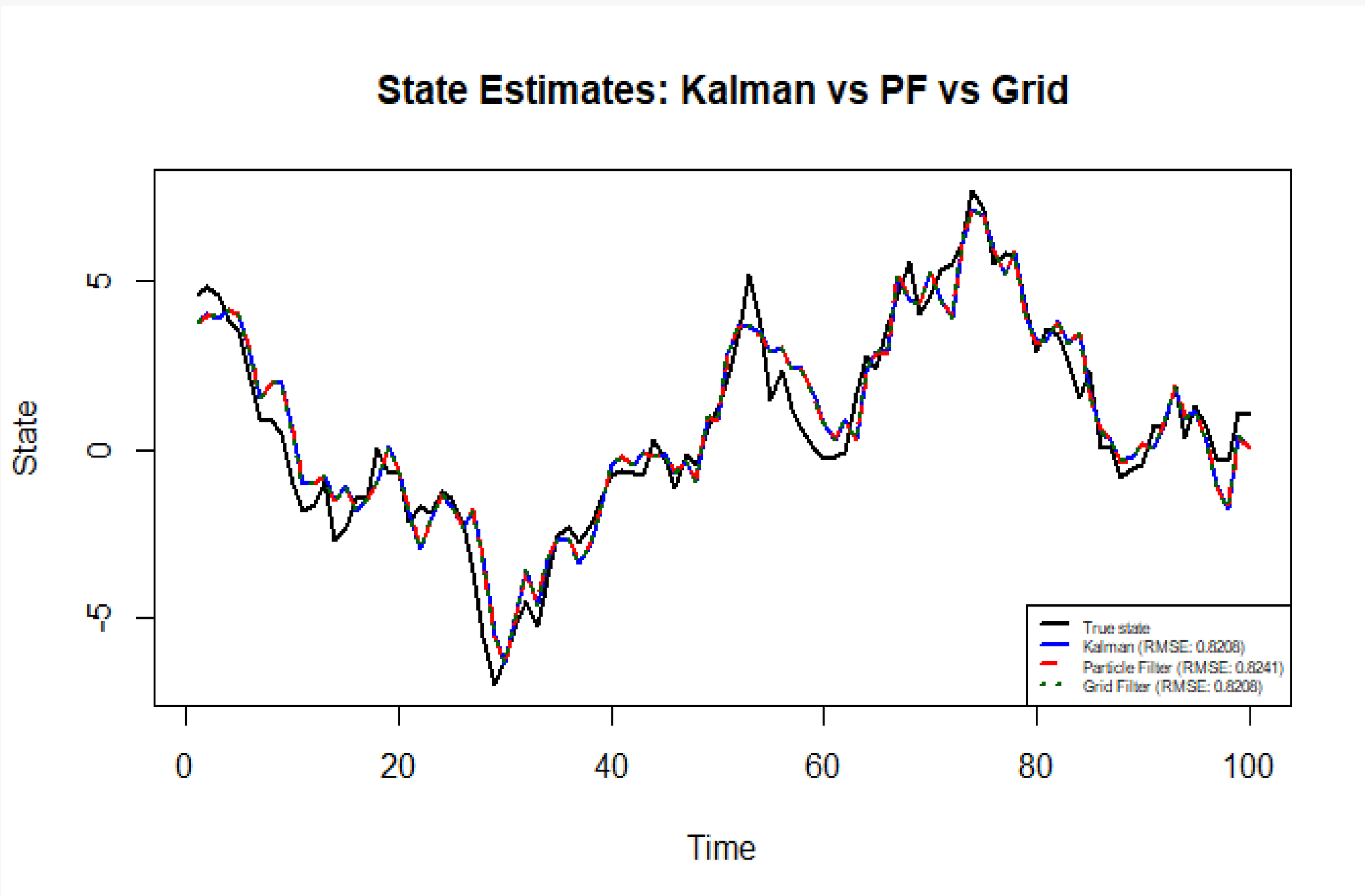
The goal of filtering is to recursively compute the **posterior distribution** over the current state given all available data up to time t:

$$p(x_t \mid y_{1:t}) = \frac{p(y_t \mid x_t) \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}}{p(y_t \mid y_{1:t-1})}.$$

This recursive formulation allows the belief about the hidden state to be updated efficiently as new observations arrive, without reprocessing past data.

Filtering methods form the basis of many algorithms, such as the **Kalman filter** for linear Gaussian systems and **particle filters** for nonlinear or non-Gaussian models. Typical applications include target tracking, signal processing, econometric forecasting, and stochastic volatility modelling in finance.

1D Linear Gaussian State Space Model



1D Linear Gaussian State Space Model

Consider the univariate linear model of the form;

$$y_t = bx_t + \epsilon_t, \ \epsilon_t \sim N(0, r)$$

Let us take the state equation;

$$x_t = ax_{t-1} + \eta_t, \ \eta_t \sim N(0, q)$$

We simulate this model, using $r = 1$, $q = 1$, $a = 0.9$, $b = 1$ and $T = 100$ where T denotes the number of time stamps in our particles, to produce the output seen on the page.

State Estimation Methods

The **Kalman Filter** applies when the model is linear and Gaussian. It combines predictions from the system dynamics with noisy observations to produce an exact estimate of the hidden state. Updates are performed recursively through a prediction and correction step:

$$\hat{x}_{t|t-1} = \hat{x}_{t|t-1} + K_t(y_t - H\hat{x}_{t|t-1})$$

The **Particle (Bootstrap) Filter** represents possible states using many random samples, or 'particles'. Each particle is propogated forward and weighted according to how well it explains the observation:

$$w_t^{(i)} \propto p(y_t|x_t^{(i)})$$

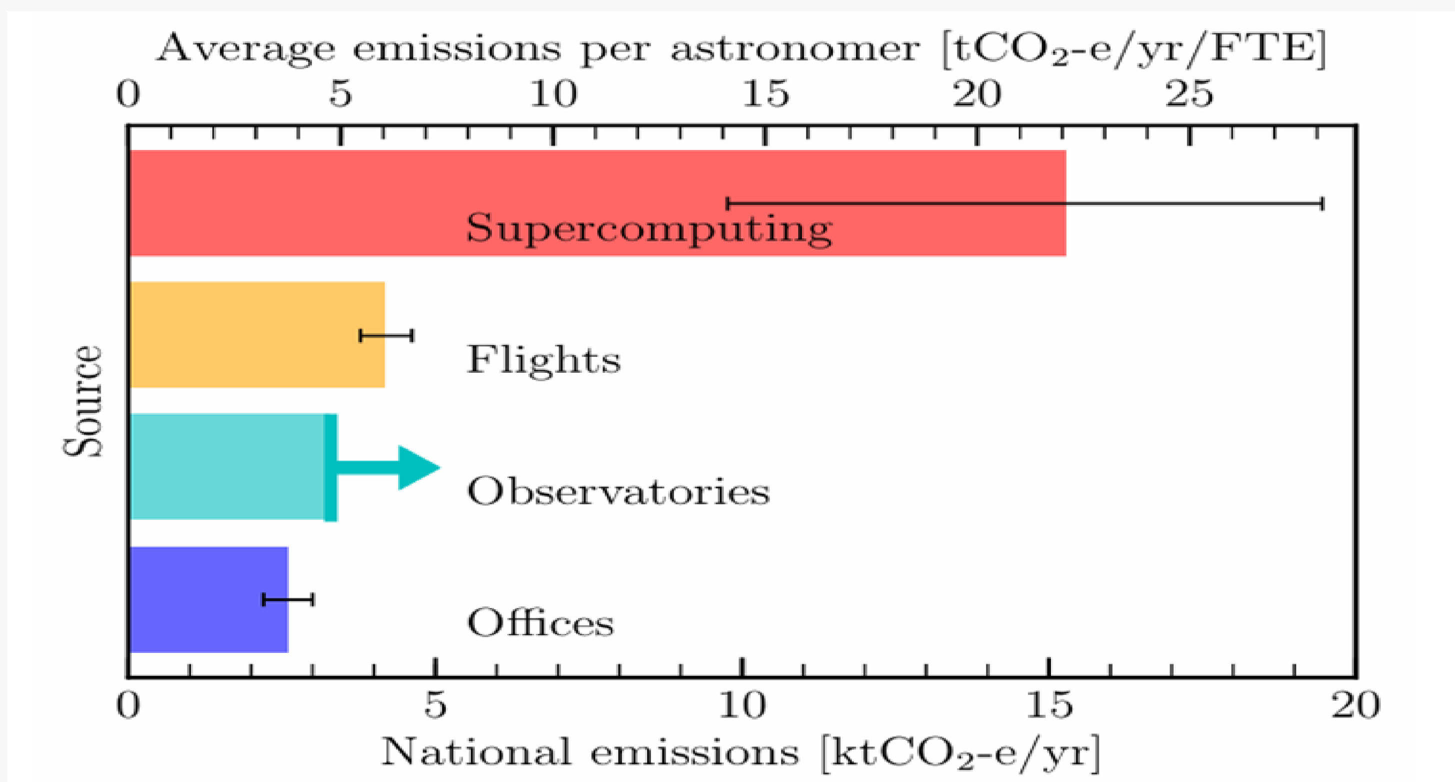
By resampling, the filter concentrates on the most likely states, making it suitable for non-linear or non-gaussian models.

The **Uniform Grid Filter** fivides the state space into fixed grid points and updates the probability attached to each point at every time step:

$$p_t(x^{(j)} \propto p(y_t|x^{(j)}) \times \text{Prediction From } p_{t-1}$$

We have also developed **Adaptive Grid Filters** to improve on these models in different situations/circumstances.

The **Grid Based Methods** provide a full posterior distribution that is often more accurate than other filtering methods, but becomes increasingly more computationally expensive at higher dimensions.



Example: Neuroscience Model

Let $\{x_t\}_{t\geq 0}$ denote the latent state process and $\{y_t\}_{t\geq 1}$ the corresponding observations.

In the univariate setting, each observation $y_t \in \{0, 1, \ldots, M\}$ represents the number of activated neurons observed in M repeated trials at time t, with $M = 50$. The observation model is defined as a Binomial distribution with a logistic link function:

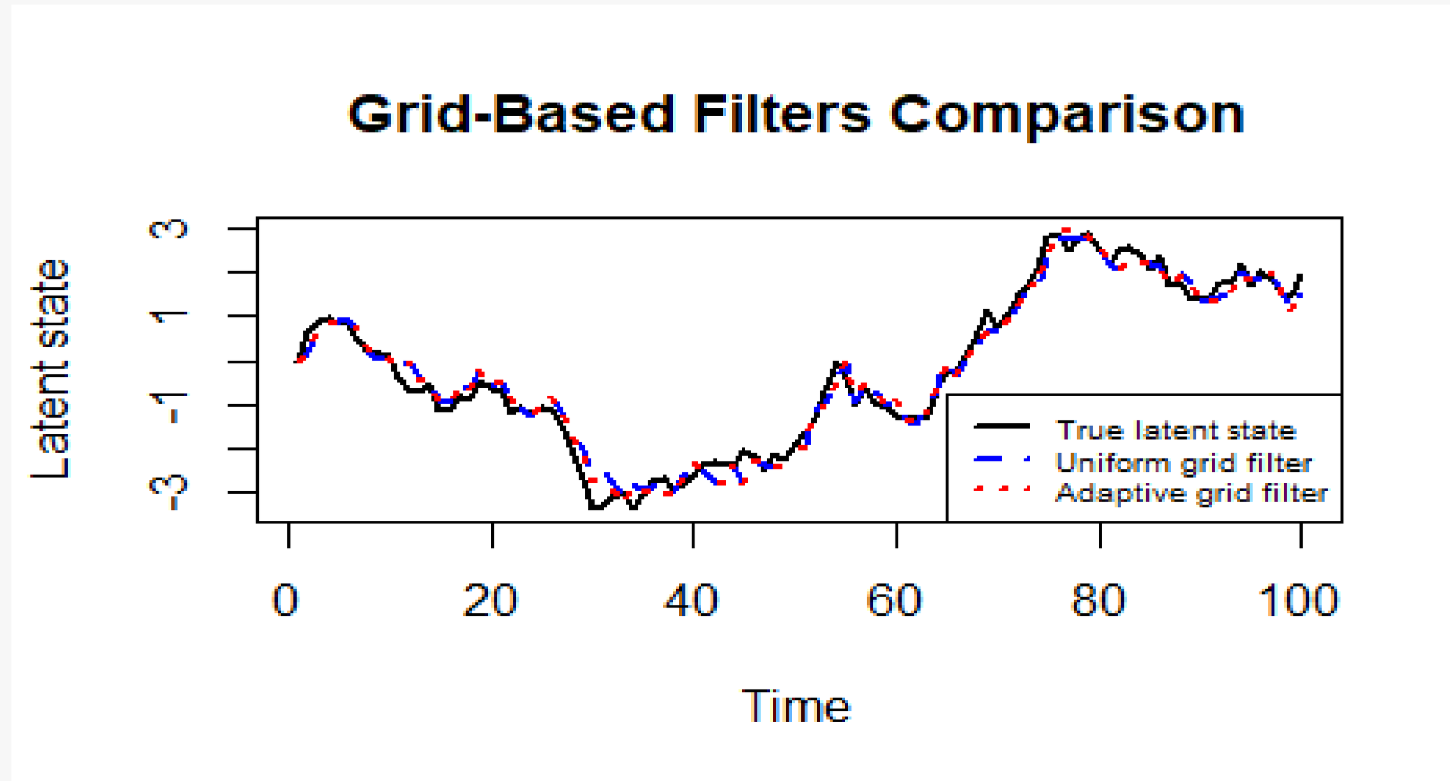
$$y_t \mid x_t \sim \text{Binomial}(M, \kappa(x_t)), \quad \kappa(x_t) = \frac{1}{1 + \exp(-x_t)}.$$

where $\kappa(x_t)$ maps the latent state to the probability of neuronal activation.

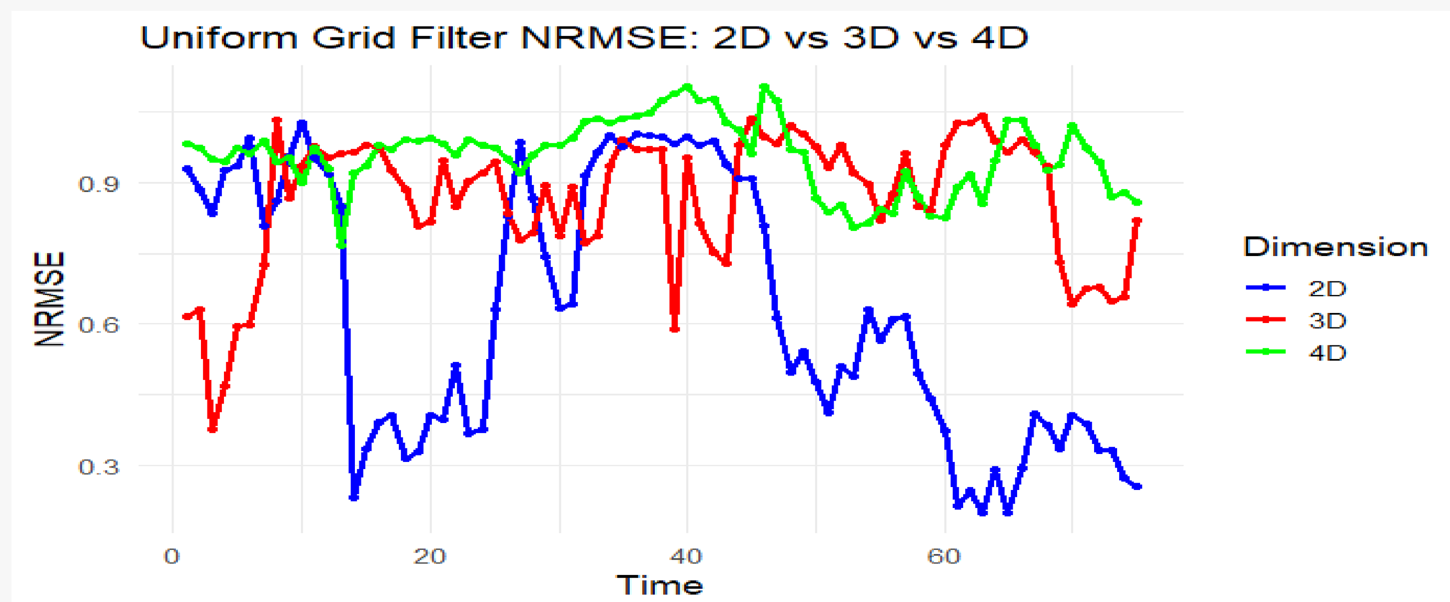
The latent dynamics are specified as a linear Gaussian auto-regressive process:

$$x_0 \sim \mathcal{N}(0, 1), \quad x_t \mid x_{t-1} \sim \mathcal{N}(\alpha x_{t-1}, \sigma^2).$$

where $\alpha = 0.99$ and $\sigma^2 = 0.11$. This formulation captures the temporal dependence of neuronal activity over consecutive trials.



The Curse of Dimensionality



Dimension	Filtering Method	Average NRMSE	Runtime (seconds)
2D	Uniform	0.6325438	2.67
3D	Uniform	0.8624691	17.28
4D	Uniform	0.9569874	77.08

max.richards@warwick.ac.uk