General Information: This assignment contains written and/or programming tasks. Combine all the answers to the written tasks in a single PDF document, named `{lastname}-written.pdf`. You can also scan or take pictures of (readable) handwritten papers. JPEG/PNG image files are accepted in this case and they should be named `{exercisenumber}-{lastname}-written.{jpeg/png}`. Make sure that we can follow the manual calculations. Do not combine too many small steps into one. The programming tasks have to be solved in *Julia* and the source code files have to be submitted using the following naming scheme: `{exercisenumber}-{lastname}.jl`.

(1) (3 points) For $z \in \mathbb{C}$, we have:

$$z = a + \mathrm{i} \cdot b = r(\cos(\varphi) + \mathrm{i} \cdot \sin(\varphi)) = r \cdot e^{\mathrm{i} \cdot \varphi},$$

with $a, b \in \mathbb{R}$ and $|z| = r = \sqrt{a^2 + b^2}$.

a) (1.2 points) Given the complex numbers $x = (\cos(0) + \mathrm{i}\sin(0))$, $y = 5e^{\mathrm{i}0.64}$, and $z = -5 + 2\mathrm{i}$, state the form they are in and determine the missing forms. Think about the meaning of the components to find a formula for φ and make use of sketches to visualize and support your arguments.

b) (0.2 points) Compute the sums $b_1 = x + z$, $b_2 = y + z$.

c) (0.2 points) The products $c_1 = x \cdot z$, $c_2 = y \cdot z$.

d) (0.8 points) Complex division $\frac{a}{b}$ can be achieved by taking the complex conjugate of the denominator and multiplying it with the numerator and denominator $\frac{a\bar{c}}{c\bar{c}}$. The complex conjugate of a complex number $\bar{c}$ is achieved by swapping the sign of the imaginary component. Compute:

$$d_1 = \frac{y}{x},\ d_2 = \frac{x}{y},\ d_3 = \frac{z}{y},\ d_4 = \frac{y}{z}$$

e) (0.2 points) The norm of a complex number $z = a + \mathrm{i}b$ is defined as $|z| = \sqrt{a^2 + b^2}$. Compute the norms for y, and z.

f) (0.4 points) The reciprocal of a complex number is denoted as follows:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Compute the reciprocal for y, and z.

(2) (2 points) Use a Taylor Series to derive Euler's formula. Given the Taylor Series approximation for $x \in \mathbb{R}$:

$$e^x \approx \sum_{k=0}^{N} \frac{x^k}{k!},$$

can be extended as follows:

$$e^{\mathrm{i}x} \approx \sum_{k=0}^{N} \frac{(\mathrm{i}x)^k}{k!}.$$

Show that $e^{\mathrm{i}x} = \cos(x) + \mathrm{i} \cdot \sin(x)$.

Hint: You might want to have a look at the solutions on how to approximate $\sin(x)$ for the exercise sheet about series and sequences. Derive a respective approximation for $\cos(x)$.

(3) (2 points) We are revisiting the Fourier series representation of a square wave function we constructed in the sequences and series exercise. Our goal is to convert the Real series into a Complex series. We had:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) = \sum_{k=-\infty}^{\infty} c_n e^{ikx}$$

with

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \mathrm{dx}, \quad k \geq 0$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \mathrm{dx}, \quad k \geq 1,$$

as coefficients for the real version of the series and similarly we have

$$c_k = \begin{cases} a_0 & , k = 0 \\ \frac{(a_k - \mathrm{i} b_k)}{2} & , k > 0 \\ \frac{(a_{|k|} + \mathrm{i} b_{|k|})}{2} & , k < 0 \end{cases}$$

as coefficient for the complex version of the series.

a) (1 point) Compute c_k by plugging in a_k and b_k.

b) (1 point) Implement the complex version of the square wave Fourier series representation, using `fourier.jl`. You might want to have a look at how to operate complex numbers with Julia[1].

For the sake of completeness, you can check whether your result resembles the solution of the sequences and series exercise sheet just like Figure 1.

[1] https://docs.julialang.org/en/v1/manual/complex-and-rational-numbers/

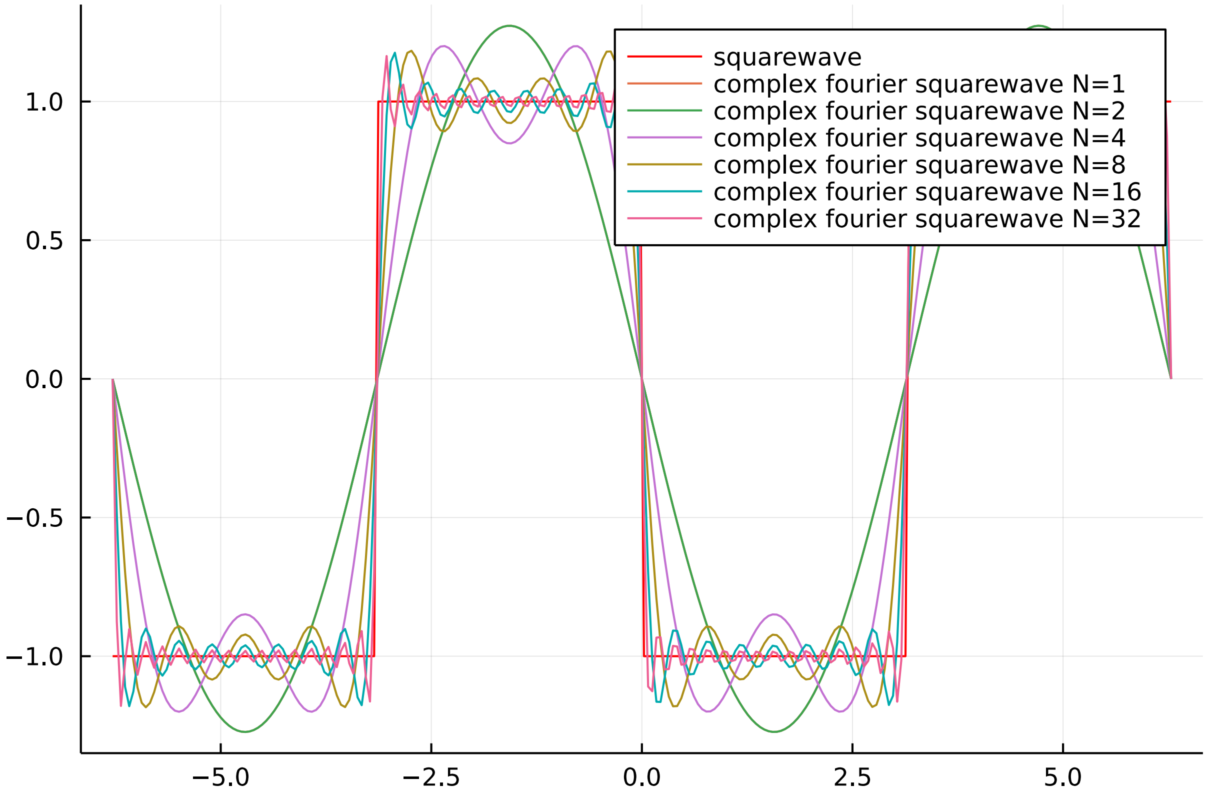

Figure 1: Complex Fourier series representation of a square wave function.