Math 3220-1 HW 1

NAME

Due: DATE

Exercises for Section 1.1: Norm and Inner Product

1. Define the ℓ^1-**norm** on $\mathbb{R}^n$ by
$$\|x\|_1 = \sum_{i=1}^{n} |x^i|,$$
and define the **sup-norm** on $\mathbb{R}^n$ by
$$\|x\|_\infty = \sup\left\{|x^i|\right\}.$$
Show that these satisfy Theorem 1.

   Proof. $\square$

2. Prove that $\|x\| \leq \sum_{i=1}^{n} |x^i|$. In other words, the usual norm is no greater than the ℓ^1-norm.

   Proof. $\square$

3. Prove that $\|x - y\| \leq \|x\| + \|y\|$. (Compare this with part (2) of Theorem 1.) When does equality hold?

4. Prove that $\Big| \|x\| - \|y\| \Big| \leq \|x - y\|$.

5. The quantity $\|y - x\|$ is called the **distance** between x and y. Prove and interpret the "triangle inequality":
$$\|z - x\| \leq \|z - y\| + \|y - x\|.$$

6. Let f and g be integrable on $[a, b]$.
   (a) Prove the integral version of the Cauchy-Schwarz inequality:
$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2}.$$
   Hint: Consider separately the cases $0 = \int_a^b (f - tg)^2$ for some $t \in \mathbb{R}$, and $0 < \int_a^b (f - tg)^2$ for all $t \in \mathbb{R}$.
   (b) If equality holds, must $f = tg$ for some $t \in \mathbb{R}$? What if f and g are continuous?
   (c) Show that the Cauchy-Schwarz inequality is a special case of (a).

7. A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is **norm preserving** if
$$\|T(x)\| = \|x\|,$$
for all $x \in \mathbb{R}^n$, and **inner product preserving** if
$$\langle Tx, Ty \rangle = \langle x, y \rangle,$$
for all $x, y \in \mathbb{R}^n$.
   (a) Prove that T is norm preserving if and only if it is inner product preserving.

(b) Prove that such a linear transformation is 1-1, and T^{-1} is norm preserving (and inner product preserving).

8. If $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $\|T(h)\| \leq M\|h\|$ for all $h \in \mathbb{R}^m$. Hint: Estimate $\|T(h)\|$ in terms of $\|h\|$ and the entries in the matrix for T.

9. If $x, y \in \mathbb{R}^n$, and $z, w \in \mathbb{R}^m$, show that $\langle (x, z), (y, w) \rangle = \langle x, y \rangle + \langle z, w \rangle$, and $\|(x, z)\| = \sqrt{\|x\|^2 + \|z\|^2}$. Note that (x, z) and (y, w) denote points in $\mathbb{R}^{n+m}$.

10. If $x, y \in \mathbb{R}^n$, then x and y are called **perpendicular** (or **orthogonal**), and we write $x \perp y$, if $\langle x, y \rangle = 0$. If $x \perp y$, prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Exercises for Section 1.2: More Topology: Open and Closed Sets in $\mathbb{R}^n$

1. Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence of finitely many) open sets is open. Give a counterexample for the intersection of infinitely many open sets.

2. If $A \subset B \subset \mathbb{R}^n$, prove that
$$\mathrm{cl}A \subset \mathrm{cl}B, \quad \text{and} \quad \mathrm{int}A \subset \mathrm{int}B.$$

3. Prove that if B is an open subset of A, then $B \subset \mathrm{int}(A)$. Note that this says that $\mathrm{int}(A)$ is the largest open subset of A.

4. Prove that the n-dimensional ball centered at a of radius r,
$$B^n(a; r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$
is open.

5. Find the interior, exterior, and boundary of the sets:
$$B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\},$$
$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\},$$
$$\mathbb{Q}^n = \left\{x \in \mathbb{R}^n : \text{ each } x^i \text{ is rational}\right\}.$$

Solution.

6. If $A \subset [0, 1]$ is the union of open intervals (a_i, b_i) such that each rational number in $(0, 1)$ is contained in some (a_i, b_i), show that $\partial A = [0, 1] - A$.

7. If A is a closed set that contains every rational number $r \in [0, 1]$, show that $[0, 1] \subset A$.

8. Graph generic open balls in $\mathbb{R}^2$ with respect to each of the "non-Euclidean" norms, $\|\cdot\|_1$ and $\|\cdot\|_\infty$. What shapes are they?

Solution.