Paper ID: AM-05

Nonlocal Elasticity Theory for Lateral Torsional Buckling of Nanobeam

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Abstract:

In this study, lateral torsional buckling instability of nanobeam is performed in the presence of external bending moment, based on the nonlocal elasticity theory and thin beam theory. At the beginning of the study, total energy (strain energy and potential energy) expressions of nanobeam having doubly cross-sectional symmetry are derived in detail. The variational energy principle is applied to the derived energy expressions to obtain the governing differential equations and boundary conditions. To study the nonlocal nanoscale effect on critical buckling moment, the derived equations of motion are solved for exact solutions and the critical instability buckling moments for various end constrains are presented and discussed in detail. It is observed from the analytical solutions that the critical buckling moment decreases with increasing nonlocal nanoscale and scale free classical model over estimates the critical buckling moment.

Keywords: *Nonlocal elasticity, Nanoscale, Lateral torsional buckling, Critical moment*

1. Introduction

The frequent use of advanced materials and structures in a minute length scale (i.e. micro- or nano-scale) has become the root or progress in nanotechnology. Micro-electromechanical-systems (MEMS) and Nano- electromechanical-systems (NEMS) have great influence in modern world because of its specific and interesting properties and also the reduction of production cost and energy consumption. Due to the reduction of device size into micro- and nano-scale, the scale free classical models and theories are unable to predict the increasingly prominence of size effects [1,2].

There are typically three approaches in the study of size effects in nanomechanics, i.e. experiments [3], numerical atomic-scale simulation [4] and scale dependent continuum mechanics model [5-8]. Among the theories, the scale dependent continuum mechanics models have become frequently used technique not only to its simplicity but also promise to predict accurate analytical results because control experiments in nano-scale are very difficult and numerical atomic-simulations are highly computationally expensive. Based on nonlocal elastic stress approach developed by Eringen and his associate [7,8], a series of research on buckling of nanotubes [9-11] has been conducted recently. In particular, Sudak [9] presented column buckling of MWCNTs. Ru [10] developed multiple shell model to study the buckling of CNTs. This model was further extended to study the thermal effect and van der Waals forces between inner and outer nanotubes by Xiaohu and Qiang [11] for axially compressed MWCNTs. All the above studies, partial nonlocal stress model was conducted to obtain the governing differential equation of equilibrium with nanoscale effect. In this aspect, Lim [12,13] presents a new variational consistent approach for bending of nanobeams based on nonlocal elasticity theory of Eringen [9]. For further study, the author is suggested to see the following articles [14,15].

A plenty of research on bending, buckling, vibration and wave propagation has been found; very limited studies on torsional or lateral torsional behaviors in NEMS are available at present. The aim of this article is to study lateral torsional buckling of nanobeam under external bending moment, based on new nonlocal elasticity theory [12,13] and thin beam theory.To study the effect of nanoscale,a reduced higher-order governing differentialequation is simplified and solved to obtainthe critical buckling loads and deflection of nanobeam for different boundary conditions.

2. Nonlocal modeling and formulation

2.1. Kinematics

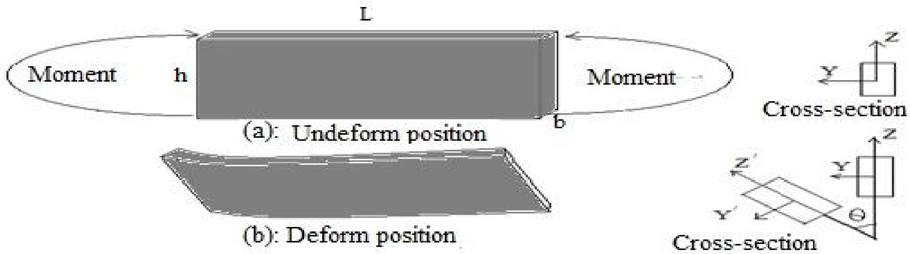

Fig.1. Continuum mode: (a) Undeformed and (b) deformed shape of a doubly symmetric nanobeam loaded to bend about its major axis Y.

Consider a nanobeam of length L, weight b and height h which is shown in **Fig.1**. Let X axis be the beam axis and Y and Z axes are the principle axes of rectangular cross-section. The coordinate of the shear center are generally represented by (y_0, z_0); v and w are the components of displacement of shear center parallel to the rotated axes Y' and Z'; θ is the angle of twist with respect to longitudinal axis X. Let us assume that the bending rigidity of the beam about Y axis is quite large as compared to Z axis. Thus, deflection w in the plane of applied moment is small compared to v and θ. Based on the thin beam theory, the strain displacement relation can be expressed in the following way:

$$\varepsilon_{xx} = y\frac{d^2v}{dx^2} \tag{1}$$

where ε_{xx} is normal strain, y the transverse coordinate measured from the mid-plane, and v the lateral displacement. Considering the relation between the classical shear stress and shear strain, we get

$$\gamma = r\frac{d\theta}{dx} \tag{2}$$

2.1.1. Constitutive relation

For linear, homogeneous and isotropic solids, unlike the classical theory, the nonlocal stress at a point depends not only on the strain at that point but also on the strain at all other points within the continuum. Consequently, in the constitutive relation of nonlocal elasticity theory Hook's law (for local theory) is replaced by an integration which governs the nonlocal material behavior and its relation can be expressed in the following expression [17]

$$t_{i,j}(\mathbf{x}) = \int_V \alpha\left(\left|\mathbf{x}' - \mathbf{x}\right|, \tau\right)\sigma_{i,j}(\mathbf{x}')dV(\mathbf{x}')\left(\tau = \frac{e_0 a}{L} = \frac{\mu}{L}\right) \tag{3}$$

where e_0 is a material constant; a is an internal characteristic length such as lattice parameter or granular distance while L is an external characteristic length; $\mathbf{x}$ is a reference point; $\sigma_{i,j}$ is the local or classical stress tensor; and $\alpha(|\mathbf{x}'-\mathbf{x}|, \tau)$ is a kernel function. Eq. (3) is an integro-partial differential equation and it is extremely difficult mathematically to obtain the solutions in terms of displacement field in nonlocal elasticity due to the presence of spatial derivatives inside the integral. However, by using Green's function with certain approximation error, Eringen [17] transformed the Eq. (3) into second order partial differential equation in the form

$$t_{i,j} - \mu^2\nabla^2 t_{i,j} = \sigma_{i,j} \tag{4}$$

where ∇^2 is a Laplacian operator and for one-dimensional thin structure, Eq. (4) can be written as

$$t_{i,j} - \mu^2\frac{d^2 t_{i,j}}{dx^2} = \sigma_{i,j} \tag{5}$$

For the analysis of nanobeam, the constitutive relation in Eq. (5) for bending and torsion becomes

$$\sigma_{xx} - \mu^2\frac{d^2\sigma_{xx}}{dx^2} = Ey\frac{d^2v}{dx^2};\sigma_{rr} - \mu^2\frac{d^2\sigma_{rr}}{dx^2} = Gr\frac{d\theta}{dx} \tag{6a,b}$$

2.2. Bending strain energy

Reasonable solution of Eq. (6a) can be written as

$$\sigma_{xx} = Ey\sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n}v}{dx^{2n}} \tag{7}$$

As the bending moment is $M_{xx} = \int_A \sigma_{xx} ydA$, the nonlocal moment can be expressed as

$$M_{xx} = EI\sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n}v}{dx^{2n}} \tag{8}$$

where I is the second moment of area. The bending strain energy of the beam is given by

$$U_B = \frac{1}{2}\int_V \sigma_{xx}\varepsilon_{xx} dV = \frac{EI}{2}\sum_{n=1}^{\infty} \mu^{2n-2}\int_0^L \frac{d^{2n}v}{dx^{2n}}\frac{d^2v}{dx^2}dx \tag{9}$$

and the variation of strain energy yields

$$\delta U_B = EI\left[\begin{array}{l} \int_0^L \sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n+2}v}{dx^{2n+2}}\delta v dx - \left[\sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n+1}v}{dx^{2n+1}}\delta v\right]_0^L \\ + \left[\sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n}v}{dx^{2n}}\delta\left(\frac{dv}{dx}\right)\right]_0^L - \frac{1}{2}\left[\sum_{n=1}^{\infty} \mu^{2n} \frac{d^{2n+1}v}{dx^{2n+1}}\delta\left(\frac{d^2v}{dx^2}\right)\right]_0^L + \cdots \end{array}\right] \tag{10}$$

2.3. Torsional strain energy

Similar to Eqs. (7-10), the variation of torsional strain energy is

$$\delta U_T = GJ\left[\begin{array}{l} -\int_0^L \sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n}v}{dx^{2n}}\delta\theta dx + \left[\sum_{n=1}^{\infty} \mu^{2n-2} \frac{d^{2n-1}\theta}{dx^{2n-1}}\delta\theta\right]_0^L \\ -\frac{1}{2}\left[\sum_{n=1}^{\infty} \mu^{2n} \frac{d^{2n}\theta}{dx^{2n}}\delta\left(\frac{d\theta}{dx}\right)\right]_0^L + \frac{1}{2}\left[\sum_{n=1}^{\infty} \mu^{2n} \frac{d^{2n-1}\theta}{dx^{2n-1}}\delta\left(\frac{d^2\theta}{dx^2}\right)\right]_0^L + \cdots \end{array}\right] \tag{11}$$

where $J = \int_A r^2 dA$ is the polar moment of and $T_{rr} = \int_A \sigma_{rr} r dA$ is the torsional moment. Hence, the variation of total strain energy is

$$\delta U = \delta U_B + \delta U_T \tag{12}$$

where δU_B and δU_T are given by Eq. (10) and Eq. (11) respectively. In the presence of external moment M exerted at the end of nanobeam, the variation of work done on the nanobeam is

$$\delta W = M\left\{\left[\frac{dv}{dx}\delta\theta\right]_0^L - \int_0^L \frac{d^2v}{dx^2}\delta\theta dx\right\} + M\left\{\left[\frac{d\theta}{dx}\delta v\right]_0^L - \int_0^L \frac{d^2\theta}{dx^2}\delta v dx\right\} \tag{13}$$

3. Governing equations and boundary conditions

The principle of virtual displacement states that if a body is in equilibrium, we must have

$$\delta(U - W) = 0 \tag{14}$$

Substituting the values of δU and δW into Eq. (14) and since δv, $\delta\left(\frac{dv}{dx}\cdots\right)$, and $\delta\theta$, $\delta\left(\frac{d\theta}{dx}\right)\cdots$ cannot vanish, Eq. (14) yields

$$EI\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n+2}v}{dx^{2n+2}} + M\frac{d^2\theta}{dx^2} = 0 \tag{15}$$

with

$$\begin{aligned} &EI\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n+1}v}{dx^{2n+1}} + M\frac{d\theta}{dx} = 0 \text{ or } v = 0 \ at\ x = 0, L \\ &EI\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n}v}{dx^{2n}} = 0 \text{ or } \frac{dv}{dx} = 0 \ at\ x = 0, L \end{aligned} \tag{16}$$

and

$$-GJ\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n}\theta}{dx^{2n}}+M\frac{d^2v}{dx^2}=0 \tag{17}$$

with

$$\begin{aligned}&GJ\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n-1}\theta}{dx^{2n-1}}-M\frac{dv}{dx}=0 \text{ or } \theta=0 \; at \; x=0,L\\&\frac{GJ}{2}\sum_{n=1}^{\infty}\mu^{2n}\frac{d^{2n}\theta}{dx^{2n}}=0 \text{ or } \frac{d\theta}{dx}=0 \; at \; x=0,L\end{aligned} \tag{18}$$

Let us define new parameters α_1 and α_2 as

$$\alpha_1=(M/EI);\alpha_2=(M/GJ) \tag{19a,b}$$

Therefore, Eq. (16) and Eq. (18) can be written as

$$\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n+2}v}{dx^{2n+2}}+\alpha_1\frac{d^2\theta}{dx^2}=0;-\sum_{n=1}^{\infty}\mu^{2n-2}\frac{d^{2n}\theta}{dx^{2n}}+\alpha_2\frac{d^2v}{dx^2}=0 \tag{20a,b}$$

which represent the higher order equations lateral torsional buckling of nanobeam and the corresponding boundary conditions can be expressed as

$$\left.\begin{aligned}&\frac{dM_{xx}}{dx}+M\frac{d\theta}{dx}=0 \text{ or } v=0 \; at \; x=0,L\\&M_{xx}=0 \text{ or } \frac{dv}{dx}=0 \; at \; x=0,L\\&\frac{dM_{xx}}{dx}=0 \text{ or } \frac{d^2v}{dx^2}=0 at \; x=0,L\end{aligned}\right\} \tag{21}$$

and

$$\left.\begin{aligned}&\frac{dT_{rr}}{dx}-M\frac{dv}{dx}0 \text{ or } \theta=0 \; at \; x=0,L\\&T_{rr}=0 \text{ or } \frac{d\theta}{dx}=0 \; at \; x=0,L\end{aligned}\right\} \tag{22}$$

For simplicity, $n=2$ is chosen in Eqs. (20a,b) and the reduced order governing equations can be written as

$$\mu^2\frac{d^6v}{dx^6}+\frac{d^4v}{dx^4}+\alpha_1\frac{d^2\theta}{dx^2}=0;-\mu^2\frac{d^4\theta}{dx^4}-\frac{d^2\theta}{dx^2}+\alpha_2\frac{d^2v}{dx^2}=0 \tag{23a,b}$$

It is clear that both Eqs. (23a,b) contains nonlocal nanoscale effect. Now substituting $\frac{d^2\theta}{dx^2}=\alpha_2\frac{d^2v}{dx^2}$ (from Eq. (23b)without nonlocal effect) into Eq. (23a) yields

$$\mu^2\frac{d^6v}{dx^6}+\frac{d^4v}{dx^4}+\alpha\frac{d^2\theta}{dx^2}=0 \tag{24}$$

where α is defined as

$$\alpha=\alpha_2\alpha_2=\left(M^2/EIGJ\right) \tag{25}$$

Substituting $v=Ce^{\lambda x}$ into Eq. (24), one gets

$$\mu^2\lambda^6+\lambda^4+\alpha\lambda^2=0 \tag{26}$$

which is six-order polynomial equation and the roots are given by

$$\lambda_1=\lambda_2=0;\lambda_{3,4}=\pm i\sqrt{\left(1-\sqrt{1-4\mu^2\alpha}\right)/\left(2\mu^2\right)};\lambda_{5,6}=\pm i\sqrt{\left(1+\sqrt{1-4\mu^2\alpha}\right)/\left(2\mu^2\right)} \tag{27}$$

Hence, the general solution of Eq. (24) can be written as

$$v=C_1+C_2x+C_3\sin\Lambda_1x+C_4\cos\Lambda_1x+C_5\sin\Lambda_2x+C_6\cos\Lambda_2x \tag{28}$$

where $C_i(i=1,2,\cdots,6)$ are the six unknown constants which are determined from appropriate boundary conditions (Eq. (21) and / Eq. (22)) and Λ_1 and Λ_2 are defined as

$$\Lambda_1=|\lambda_{3.4}|=\sqrt{\left(1-\sqrt{1-4\mu^2\alpha}\right)/\left(2\mu^2\right)};\Lambda_2=|\lambda_{5,6}|=\sqrt{\left(1+\sqrt{1-4\mu^2\alpha}\right)/\left(2\mu^2\right)} \tag{29a,b}$$

4. Examples of nanobeam

4.1 Simply supported nanobeam

For a simply supported nanobeam, six boundary conditions are chosen as

$$v = 0;\ \frac{d^2 v}{dx^2} = 0;\ M_{xx} = 0 \text{ at } x = 0, L \tag{30}$$

Substituting Eq. (28) into Eq. (30)yields

$$C_1 = C_2 = C_3 = C_4 = C_6 = 0 \text{ and } \sin(\Lambda_2 L) = 0 \text{ or } \Lambda_2 = (k\pi/L)(k = 1, 2, \cdots) \tag{31}$$

Hence, the instability deflection can be given by

$$v = C_5 \sin(k\pi/L)x \quad (32$$

Further, substituting $\Lambda_2 = \frac{k\pi}{L}$ into Eq. (29b), the instability lateral torsional buckling moment for simply supported beam is given by

$$R = (M_{nl}/M_l) = \sqrt{\left\{1 - \mu^2 (k\pi/L)^2\right\}} \tag{33}$$

where $M_l = (k\pi/L)\sqrt{EIGJ}$ is the critical buckling moment[17] for a classical column simply supported at both ends.

4.2 Fully Clamped nanobeam

For a fully clamped nanobeam, six boundary conditions are chosen as

$$v = 0;\ \frac{dv}{dx} = 0;\ \frac{dM_{xx}}{dx} = 0 \text{ at } x = 0, L \tag{34}$$

Substituting into Eq. (28) into Eq. (34) yields

$$\begin{aligned} &C_2 = C_3 = C_4 = C_5 = 0;\ C_1 + C_6 = 0;\ \sin(\Lambda_2 L) = 0;\ \cos(\Lambda_2 L) = 1 \\ &\text{or } \Lambda_2 = (2k\pi/L)(k = 1, 2, \cdots) \end{aligned} \tag{35}$$

Hence, the instability deflection can be given by

$$v = C_1\{1 - \cos(2k\pi/L)x\} \tag{36}$$

Further, substituting $\Lambda_2 = (2k\pi/L)$ into Eq. (29b), the instability lateral torsional buckling moment for simply supported beam is given by

$$R = (M_{nl}/M_l) = \sqrt{\left\{1 - \mu^2 (2k\pi/L)^2\right\}} \tag{37}$$

where $M_l = (2k\pi/L)\sqrt{EIGJ}$ is the critical buckling moment for a classical column fully clamped at both ends.The analytical expressions in Eq. (33) for simply supported and in Eq. (37) for fully clamped boundary conditions are presented in **Fig. 2**.

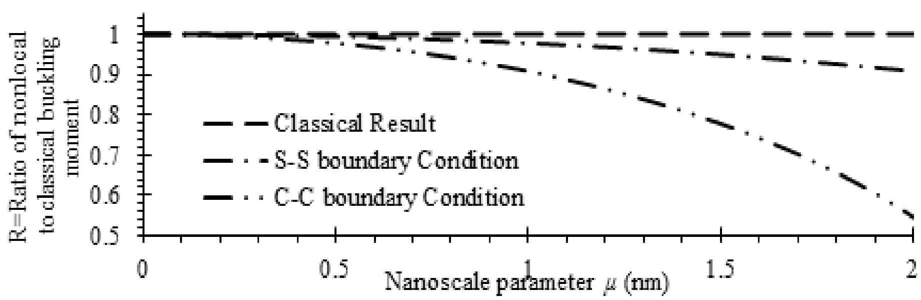

Fig.2: The effect on nanoscale parameter μ on R (for first mode ($k = 1$) in Eq. (34) and Eq. (38)) or the ratio of nonlocal to classical buckling moment for simply supported (S-S) and both side clamped (C-C) boundary conditions for $L = 15\,nm$ with corresponding classical result ($\mu = 0$).

5. General Discussions

The effects of nonlocal nano-scale parameter μ on the ratio of nonlocal to classical buckling moment for s-s and c-c nanobeam for first mode (i.e. $k = 1$) are presented in **Fig. 2**. From the **Fig. 2**, it is demonstrated that the nonlocal nano-scale reduces buckling moment in both cases compared to the classical result as the nonlocal model contains the nanoscale parameter μ in Eq. (33) and Eq. (37).The classical result overestimates the buckling moment presented in **Fig.2**which can be obtained by dropping the nonlocal terms containing μ in Eq. (33) and Eq. (37). It is also noted that buckling moment for c-c nanobeam decreases more rapidly than that of s-s nanobeam. Similar predictions on bending, bucklingand vibration were reported by using nonlocal shell model [10,18,19], nonlocal Timoshenko beam model [20,21],and molecular dynamic simulation [22].The present trend is consistent with the existing results on nanostructures [9-12, 18-22] obtained by different method.

6. Conclusion

Based on the nonlocal constitutive relation developed by Eringen [16], the nonlocal nanoscale size effect for simply supported and fully clamped nanobeam in the presence of external moment is investigated. The nonlocal equations of motion and boundary conditions are developed by means of variational principle. Analytical expressions for s-s and c-c boundary conditions are established and discussed. It is concluded that the analytical nonlocal model captures nanoscale effects and consequently, proposed model predicts that the lateral torsional buckling moment decreases with increasing nonlocal nanoscale parameter. The classical solutions are recovered in the limit of vanishing nonlocal nanoscale and the validity of the present model is validated.

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