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Qualitative behavior of the discrete equilibrium solutions of the Boltzmann equation with respect to the parameters

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Abstract

The nonlinear Boltzmann equationdescribes theevolution of molecules of rarefied gases in which the mean free-path travelled by a molecule between two subsequent collisions is notnegligible compared tothe structure considered.For $f = f\ (t,\ x,\ v)$, a non-negative density function depending on the variables,time $t \in \mathbb{R},\ t \geq 0$, the molecularvelocity $\boldsymbol{v} \in \mathbb{R}^d, d = \{2{,}3\}$,and thespace $x \in \mathbb{R}^m, 1 \leq m \leq d$ the nonlinear Boltzmann equation is given by $(\partial t + \boldsymbol{v}.\nabla x) f(t, x, v) = \boldsymbol{J}[\boldsymbol{f}, \boldsymbol{f}]$. Where$\boldsymbol{J}[\boldsymbol{f}, \boldsymbol{f}]$is a $(2d-1)$-fold integral known as Boltzmann collision operator. The discrete velocity model (DVM) as a deterministic method in which thevelocities of molecules are confined to a finite set vectors has been used for solvingthe Boltzmann equation. The DVM approximates the $(2d-1)$-fold collision integral on a discrete lattice inthe velocity space. In this paper, we describe the qualitative behavior of the discrete equilibrium solutions of the Boltzmann equationwith respect to the parametersfor the generalized N-layer hexagonalgrid $\mathcal{G}_N = \boldsymbol{c} + h \cdot \left(sin\left(\frac{2\pi}{6}(k-0.5)\right),\ cos\left(\frac{2\pi}{6}(k-0.5)\right) \right)_{k=1}^{6}$. The equilibria $f \in \mathcal{E}$ of the discrete Boltzmann equation is described by four parameters characterizing mass, (x,y)-momenta and kinetic energy.

Keywords: Boltzmann equation,Equilibria(equilibrium Solution), Discrete Velocity Model(DVM) and Numerical Simulations.

1. Introduction:

In this paper**, w**e discuss some aspects of the discrete equilibrium solution of the Boltzmann equation of kinetic theory of gases. The discrete equilibrium solutions(eqilibria) of the Boltzmann equation is based on discrete velocity model(DVM) approximation carried on meshes of hexagonal grids.The equilibria of the discrete Boltzmann equation can be expressed in terms of four parameters characterizing mass, (x,y)-momenta and kinetic energy. We construct necessary algorithm for the computation of the equlibria and perform numerical simulation for a 10-layer hexagonal grid in $\mathbb{R}^2$.we derive the discrete equilibrium solutions (equilibria) for the generalized *N*-layer hexagonal grid by induction method as established in [1.5]. The equilibria $f \in \mathcal{E}$ of the discrete Boltzmann equation is described by four parameters characterizing mass, (*x*,*y*)-momenta and kinetic energy. Subsequently, we present some estimations on the discrete equilibria of the Boltzmann equation with respect to the parameters. We also represent subsequent numerical simulations of the equilibrium solution.

2. Boltzmann equation

We present some estimations on the discrete equilibria of the Boltzmann equation with respect to the parameters.For $f = f\ (t,\ x,\ v)$, a non-negative density function depending on the variable,time $t \in \mathbb{R}$, $t \geq 0$, the molecular velocity $\mathbf{v} \in \mathbb{R}^d, d = \{2{,}3\}$,and the space $x \in \mathbb{R}^m, 1 \leq m \leq d$ the nonlinear Boltzmann equation is given by $(\partial t + \mathbf{v}.\nabla x) f(t, x, v) = \mathbf{J}[\mathbf{f}, \mathbf{f}]$ (2.1)

Where $\mathbf{J}[\mathbf{f}, \mathbf{f}] := \int_{\mathbb{R}^\mathbf{d}} \int_{S^{d-1}} k(\mathbf{v} - \mathbf{w}, \eta)[\mathbf{f}(\mathbf{v}')\mathbf{f}(\mathbf{w}') - \mathbf{f}(\mathbf{v})\mathbf{f}(\mathbf{w})] d^2\eta d^3\mathbf{w}$ (2.2)

is a $(2d-1)$-fold integral known as Boltzmann collision operator. Here $k(.,.)$ is the collision kernel in the operator satisfying some symmetry properties, the post collision velocities $\mathbf{v}', \mathbf{w}'$result from the pre-collision velocities $\mathbf{v}, \mathbf{w}$ satisfying the collision relations, conservation of momentum

$$\mathbf{v} + \mathbf{w} = \mathbf{v}' + \mathbf{w}' \tag{2.3}$$

conservation of kinetic energy $|\mathbf{v}|^2 + |\mathbf{w}|^2 = |\mathbf{v}'|^2 + |\mathbf{w}'|^2$ (2.4)

3. AN-layer hexagonal model

Fig. 1 shows a 54-velocity model (as a regular collision model defined in [2]) constructed by adding twolayers of regular basic hexagons centering to a central one and thus called a two-layer model. Similarly by adding one more layer of regular basic hexagons, one can obtain a 3-layer model and so on. In general, we may call such models the *N*-layer model which can be divided into six symmetric partition as shown in the Fig. 1.

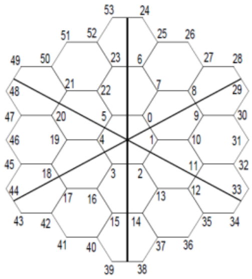

Fig. 1: A 54-velocity model as a two-layer model.

In order to generate a hexagonal mesh for the *N*-layer model, we first collect thecenters of all basic hexagons ordered layer-wise and partition-wise as in algorithm 2.1[3].By algorithm 2.1[3], we obtain the vectors c_x, c_y for the (*x*, *y*)-coordinate of the centers of all regular basic hexagon of the *N*-layer model. The (*x*, *y*)-coordinates of nodes of the model are given by the formula (as stated in (2.1)[3])

$$\mathcal{G}_N = \mathbf{c} + h \cdot \left(sin\left(\frac{2\pi}{6}(k-0.5)\right),\ \ cos\left(\frac{2\pi}{6}(k-0.5)\right)\right)_{k=1}^{6} \tag{3.1}$$

where h is the discretization parameter,

$\mathbf{c} = (c_x, c_y)$ is already obtained by the above algorithm 2.1[8]. By algorithm 2.2[8] we obtain the vectors G_x, G_y for the (*x*, *y*)-coordinate of the grid points of the *N*-layer model and plots the hexagonal mesh as seen in Fig. 2.

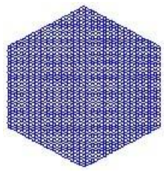

Fig. 2: A 25-layer hexagonal mesh.

Lemma 1: *There are* 6*n number of regular basic hexagons in the n-th*(*n* = 1,…,*N*) *layer of a N-layer grid and the total number of regular basic hexagons of a N-layer grid is given by* 3*N*(*N* + 1) + 1.

Proof: follows lemma 2.11[3].

Lemma 2: *There are* 6(2*n* + 1) *number of nodes in the n-th*(*n* = 0,…,*N*) *layer of a N-layer grid and the total number of nodes in a N-layer grid is given by* $6(N\ +\ 1)^2$.

Proof: follows lemma 2.12[3].

4. Equilibrium solutions

We introduce a function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $J[f,f] \equiv 0$.It will be evident that the solution of the space homogeneous Boltzmann equation converge to the equilibrium solution.

Definition: Equilibrium solution: A function $f \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ with the properties

(a) $f(v) > 0$for all $v \in \mathbb{R}^d$,(b) $\ln(f)J[f,f] \in L^1(\mathbb{R}^d)$

is called equilibrium solution of the Boltzmann equation for which $J[f,f] \equiv 0$.

5. Equilibria for a*N*-layer model

It has been shown in ([2], [3]) that the equilibria$\mathbf{f} \in \mathcal{E}$of the discrete Boltzmann equation can be expressed in terms of four parameters characterizing mass, momenta and energy. In this section we present such equilibrium distribution for a generalized *N*-layer model for any $N \in \mathbb{N}_0$.

Strictly positive density vectors $\mathbf{f} = (f_i)_{\boldsymbol{i=0}}^{\boldsymbol{6(N+1)^2-1}}$ for which $J[f\ ,f\ ] \equiv 0$ is said to be the equilibrium solutions (equilibria) for a*N*-layer hexagonal model.

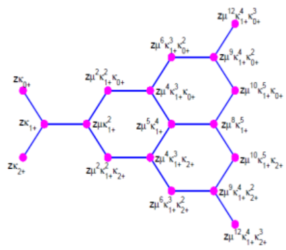

Fig.-3: Equilibria restricted to a partition of a 3-layer model.

The set of equilibria for a *N*-layer hexagonal model is denoted by $\mathcal{E}_N$. Suppose $\mathbf{f} \in \mathcal{E}_N$ be the equilibria of a*N*-layer model and the equilibria at the six nodes of 0-st layer (i.e. at the nodes of the central basic hexagon) is given by$(f_0,\ f_1,\ f_2, f_3,\ f_4, f_5) = z \cdot (k_{0+}, k_{1+}, k_{2+}, k_{0-}, k_{1-}, k_{2-})^{\mathrm{T}}$, where z, $k_{0+}, k_{1+}, k_{2+} > 0$ are arbitrary quantities satisfying $k_{0+}, k_{1-}, k_{2+} = 1$(see prop. 3.3 [2]).For a 3-layer model, the Fig.-3. presents the equilibria for the nodes of the partition corresponding to the triple $z \cdot (k_{0+}, k_{1+}, k_{2+})$.

The values of the equilibria are calculated in a similar way as in Theorem 4.1 in [2], for the layer $n = 1,2,3$respectively as$z(\mu^2 k_{1+} k_{0+}, \mu k_{1+}^2,\ \mu^2 k_{1+}^2 k_{2+}) \in 1\text{st layer}$

$z(\mu^6 k_{1+}^3 k_{0+}^2, \mu^4 k_{1+}^3 k_{0+}, \mu^5 k_{1+}^4, \mu^4 k_{1+}^3 k_{2+}, \mu^6 k_{1+}^3 k_{2+}^2) \in 2\text{nd layer}$

$z(\mu^{12} k_{1+}^4 k_{0+}^3, \mu^9 k_{1+}^4 k_{0+}^2, \mu^{10} k_{1+}^5 k_{0+}, \mu^8 k_{1+}^5, \mu^{10} k_{1+}^5 k_{2+}, \mu^9 k_{1+}^4 k_{2+}^2, \mu^{12} k_{1+}^4 k_{2+}^3) \in 3rd$ layer wherez parameterizes mass, (k_{0+}, k_{1-}, k_{2+}) characterize non-vanishing bulk-velocity, and μ is responsible for kinetic energy. At each n-th layer of a partition we have $(2n + 1)$ nodes and the node numbering is from the top to bottom of at each layer. We generalize these values of equilibria for a partition of aN-layer model as in the proposition below.

Proposition 1:*For a partition (of a N-layer model) corresponding to the triple $z(k_{0+}, k_{1+}, k_{2+})$, the equilibria is described in-terms of the parameters $\mu, k_{0+}, k_{1+}, k_{2+}$as in the following three steps.*

1. Corresponding to the values at the first node (the top one in the figure) of the $(n-1)th$layer (n = 2,…,N), there obtained two values of equilibria with increments $k_{1+}\mu^{n}$ and $k_{0+}k_{1+}\mu^{2n}$ which are assigned respectively to the second and first nodes of the n-st layer. Corresponding to the values at the last node (the bottom one in the figure) of the $(n-1)$-th layer, there obtained also two values of equilibria with increments $k_{1+}\mu^{n}$ and $k_{0+}k_{1+}\mu^{2n}$ which are assigned respectively to the 2n-th and (2n + 1)-th nodes of the nth layer.
2. Corresponding to the values of each 2m-th (even) node ($\mathrm{m} = 1, \dots, \mathrm{n}-1$)of the$(n-1)$-th layer, there obtained values at the*(2m + 1)*-th node of the n-th layer with an increment$k_{1+}^{2}\mu^{2n}$
3. Corresponding to each (2l +1)-*th* (odd) node ($l = 1, \dots, n-2$)of the $(n-1)$-th layer, there obtained values of equilibria with an increment$k_{1+}\mu^{n}$ which is assigned to the (2l+ 2)-th node of the nth layer.

The equilibria for other partitions as well as for the complete N-layer model is determined by symmetry.
Let the equilibria at the ith($i = 1, \dots, 2n + 1$) node of the nth ($n = 0, \dots, N$) layer is given by

$$f(n,i) = z\mu^{m(n,i)}k_{0+}^{k_0(n,i)}k_{1+}^{k_1(n,i)}k_{2+}^{k_2(n,i)};\ \ (m, k_0, k_1, k_2) \in \mathbb{N}_0 \tag{5.1}$$

Then following the statement of the proposition 3.1, one can calculate the exponents $m(n,i)$, $k_0(n,i)$, $k_1(n,i)$, $k_2(n,i)$; $n = 0, \dots, N$; $i = 1, \dots, 2n + 1$ as shown in Algorithm

INITIALIZE $m(0,1) = 0$, $m(1,1) = 2$, $m(1,2) = 1$, $m(1,3) = 2$ FOR n = 2 TO N $m(n,1) = m(n-1,1) + 2n$ FOR i= 2(2)2n $m(n,i) = m(n-1,i-1) + n$ END FOR j = 3(2)(2n−1) $m(n,j) = m(n-1,j-1) + 2n$ END $m(n,2n+1) = m(n-1,2n-1) + 2n$ END	INITIALIZE $k_0(0,1) = 1, k_0(1;\ 1) = 1$ FOR n = 2 TO N $k_0(n,1) = k_0(n-1,1) + 1$ FOR i= 2 TO n $k_0(n,i) = k_0(n-1,i-1)$ END FOR i= n + 1 TO 2n + 1 $k_0(n,i) = 0$ END END
INITIALIZE $k_1(1,1) = 2, k_1(1,2) = 2$, $k_1(1,2) = 2$ FOR n = 2 TO N $k_1(n,1) = k_1(n-1,1) + 1$ FOR i= 2(2)2n $k_1(n,i) = k_1(n-1,i-1) + 1$ END FOR j = 3(2)(2n−1) $k_1(n,j) = k_1(n-1,j-1) + 2$ END $k_1(n,2n+1) = k_1(n-1,2n-1) + 1$ END	INITIALIZE $k_2(0,1) = 1, k_2(1,3) = 1$ FOR n = 2 TO N $k_2(n,2n+1) = k_2(n-1,2n-1) + 1$ FOR i= n + 2 TO 2n $k_2(n,i) = k_2(n-1,i-1)$ END FOR i= 1 TO n + 1 $k_2(n,i) = 0$ END END

Algorithm: To calculate the exponents $m(n,i)$, $k_0(n,i)$, $k_1(n,i)$, $k_2(n,i)$.

Theorem 1. Let $\mathbb{N}_o, \mathbb{N}_e$ denote respectively the set of odd and even natural numbers. The i-thequilibria in the n-th layer of the partition corresponding to the triple $(k_{0+}, k_{1+}, k_{2+})$is given by

$$f(n,i) = z\mu^{m(n,i)}k_{0+}^{\bar{k}_0(n,i)}k_{2+}^{\bar{k}_2(n,i)};\ \ n = 0, \dots, N;\ i = 1, \dots, 2n + 1 \tag{5.2}$$

where for $i = 1, \dots, n + 1$,

$$m(n,i) = n^2 + n - d_i\ \ for\ i \in \mathbb{N}_o d_{i=2k+1} = nk - k^2,$$
$$k = 0, \dots, \frac{n}{2}\ \ if\ \ \mathrm{n} \in \mathbb{N}_e\ and\ k = 0, \dots, \frac{n-1}{2}\ \ if\ \ \mathrm{n} \in \mathbb{N}_o,$$
$$= n^2 - d_i\ \ for\ i \in \mathbb{N}_e d_{i=2k+2} = nk - k(k+1),$$
$$k = 0, \dots, \frac{n-2}{2}\ \ if\ \ \mathrm{n} \in \mathbb{N}_e\ and\ k = 0, \dots, \frac{n-1}{2}\ \ if\ \ \mathrm{n} \in \mathbb{N}_o,$$
$$\bar{k}_0(n,i) = 2n + 1 - d_i\ \ for\ i \in \mathbb{N}_o d_{i=2k+1} = k,$$

$$k = 0,\dots,\frac{n}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_e \ and \ k = 0,\dots,\frac{n-1}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_o,$$
$$= 2n - d_i \ \ for\ i \in \mathbb{N}_e d_{i=2k+2} = k,$$
$$k = 0,\dots,\frac{n-2}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_e \ and \ k = 0,\dots,\frac{n-1}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_o,$$
$$\bar{k}_2(n,i) = \ n + 1 + d_i \ \ for\ i \in \mathbb{N}_o d_{i=2k+1} = k,$$
$$k = 0,\dots,\frac{n}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_e \ and \ k = 0,\dots,\frac{n-1}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_o,$$
$$= n + 1 - d_i \ \ for\ i \in \mathbb{N}_e d_{i=2k+2} = k,$$
$$k = 0,\dots,\frac{n-2}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_e \ and \ k = 0,\dots,\frac{n-1}{2} \quad if \quad \mathrm{n} \in \mathbb{N}_o,$$
$For\ the\ rest\ i = n + 2,\dots,2n + 1,$
$$m\big(n, i_{(=n+2,\dots,2n+1)}\big) = m\big(n, i_{(=n,\dots,1)}\big) \ \ respectively,$$
$$\bar{k}_0\big(n, i_{(=n+2,\dots,2n+1)}\big) = \bar{k}_2\big(n, i_{(=n,\dots,1)}\big) \ \ respectively,$$
$$\bar{k}_2\big(n, i_{(=n+2,\dots,2n+1)}\big) = \bar{k}_0\big(n, i_{(=n,\dots,1)}\big) \ \ respectively,$$

Proof: By lemma-3.2 [2],

Corollary 1. For a regular collision model $(\mathcal{H}_b, \gamma)$, let $f \in \mathcal{E}$ be the equilibria. If we denote the *i*-th component equilibria as $f_i := z\mu^m k$ and the corresponding $r_1 := \sqrt{3n+1}$, where $r_1^2 = v_{x,i}^2 + v_{y,i}^2$, then m = n.

6. Some estimations of discrete equilibria

From the geometrical construction of the discrete equilibria(Fig. 1 and Fig. 3), one can easily read the following properties at a glance.

$$k_{0+} = 1, k_{2+} = 1 \Rightarrow \bar{v}_x = 0, \bar{v}_y = 0$$
$$k_{0+} = k_{2+} > 1 \Rightarrow \bar{v}_x > 0, \bar{v}_y = 0$$
$$k_{0+} = k_{2+} < 1 \Rightarrow \bar{v}_x < 0, \bar{v}_y = 0$$
$$k_{2+} = \frac{1}{k_{0+}}, k_{0+} > 1 \Rightarrow \bar{v}_x = 0, \bar{v}_y > 0$$
$$k_{2+} = \frac{1}{k_{0+}}, k_{0+} < 1 \Rightarrow \bar{v}_x = 0, \bar{v}_y < 0$$
$$k_{2+} < \frac{1}{k_{0+}}, k_{0+} > 1 \Rightarrow \bar{v}_x < 0, \bar{v}_y > 0$$
$$k_{2+} < \frac{1}{k_{0+}}, k_{0+} < 1 \Rightarrow \bar{v}_x < 0, \bar{v}_y < 0$$
$$k_{2+} > \frac{1}{k_{0+}}, k_{0+} > 1 \Rightarrow \bar{v}_x > 0, \bar{v}_y > 0$$
$$k_{2+} > \frac{1}{k_{0+}}, k_{0+} < 1 \Rightarrow \bar{v}_x > 0, \bar{v}_y < 0$$

In the following, we analyze some estimations of the equilibria given by theorem 1 for some special cases of the parameters.

Case 1: $k_{0+} = k_{2+} =: k$, First we consider $k_{0+} = k_{2+} =: k$ then the n*th* layer, the equilibria restricted to the second partition is given by $f(n,i) = z\mu^{m(n,i)} k_{0+}^{\overline{k_0}(n,i) + \overline{k_2}(n,i)} := z\mu^{m(n,i)} k^{k(n,i)}$ (6.1)

Where$k(n,i) = 3n + 2 \ for\ i = 1,3,\dots,2n+1 = 3n + 1 \ for\ i = 2,4,\dots,2n$

Choosing $\mu \in (0,1), k > 1$:For $\mu \in (0,1)$and $k > 1$, $\max_i f(n,i)$ attains for $\mathop{mim}_i m(n,i)$ and $\max_i k(n,i)$. Then it is verified that the $\max_i f(n,i)$ attains for $i = n, n+1, n+2$.

Let $\mathbb{N}_o$ and $\mathbb{N}_e$ denote the set of odd and even integers respectively. Then for $n \in \mathbb{N}_e$

$$m(n, i = n) = \frac{3}{4}n^2 + \frac{n}{2} = m(n, i = n+2)$$

$$m(n, i = n+1) = \frac{3}{4}n^2 + n$$

Then $f(n, i = n) = z\mu^{\frac{3}{4}n^2 + \frac{n}{2}} k^{3n+1} = f(n, i = n+2)$ (6.2)

$f(n, i = n) = z\mu^{\frac{3}{4}n^2 + n} k^{3n+2}$ (6.3)

For $n \in \mathbb{N}_o$, we have $f(n, i = n) = z\mu^{\frac{3}{4}n^2 + n + \frac{1}{4}} k^{3n+2} = f(n, i = n+2)$ (5.4)

$f(n, i = n+1) = z\mu^{\frac{3}{4}n^2 + \frac{n}{2} - \frac{1}{4}} k^{3n+1}$ (6.5)

We choosing now $\mu = 1/k$, then for $n \in \mathbb{N}_e$, equations (5.2) and (5.3) yields

$$f(n, i = n) = zk^{-\frac{3}{4}n^2 + \frac{5}{2}n + 1} =: zk^{k_o(n)} = f(n, i = n+2)$$
$$f(n, i = n+1) = zk^{-\frac{3}{4}n^2 + 2n + 2} =: zk^{k_e(n)}$$

Now for $\max_n f(n,i)$, both $k_o'(n) = 0$, $k_e'(n) = 0$ yields n = 2, and n = 2, $k_o(n) = k_e(n) = 3$

Thus we have three $\max_i f(n,i)$ in the n = 2nd layer.Again for $n \in \mathbb{N}_o$, it follows from equations (5.4) and (5.5) that $f(n, i = n) = zk^{-\frac{3}{4}n^2 + 2n + \frac{7}{4}} =: zk^{k_o(n)} = f(n, i = n+2)$

$$f(n, i = n+1) = zk^{-\frac{3}{4}n^2 + \frac{5}{2}n + \frac{5}{4}} =: zk^{k_e(n)}$$

and both $k_o'(n) = 0$, $k_e'(n) = 0$ yields n = 1, and n = 1, $k_o(n) = k_e(n) = 3$
Thus we have three more $\max_i f(n,i)$ in the n = 1st layer which are equal to those in the n = 2nd layer. Therefore, the six maximum values attains at the nodes $\big(5 + (2 \times n(=1) + 1) + i(=1,2,3)\big) = (9, 10, 11)$ and $\big(5 + (2 \times n(=2) + 1) + i(=1,2,3)\big) = (30, 31, 32)$, where $(30, 31, 32, 11, 10, 9)$ are the six-tupel nodes of the 2nd regular basic hexagon in the 2nd partition of the 2nd layer(see Fig. 1). Then the six maxima of the equilibria $\max_i f(n,i)$ can be obtained from equation (6.1)
Choosing $\mu \in (0,1), k < 1$:
Similarly, if we choose $k < 1$ then for $\mu = k$ we found maximum at six-tupel nodes $(20, 19, 18, 45, 46, 47)$ of the regular basic hexagon in the 5th partition (which is just the opposite of the 2nd) and in the 2nd layer.We collect the results of this subsection as follows.

Corollary 2 For $k := k_{0+} = k_{2+} > 1$ and $\mu = 1/k$ the equilibria $f(n,i) \in \mathcal{E}_N$ given in theorem 1 has six maximum attained at the six-tiple nodes $(30, 31, 32, 11, 10, 9)$ of a regular basic hexagon of the *N*-layer model(Fig. 1).

Corollary 3 For $k := k_{0+} = k_{2+} < 1$ and $\mu = k$ the equilibria $f(n,i) \in \mathcal{E}_N$ given in theorem 1 has six maximum attained at the six-tiple nodes $(20, 19, 18, 45, 46, 47)$ of a regular basic hexagon of the *N*-layer model (Fig. 1).

Case 2: $\boldsymbol{k_{2+} = \frac{1}{k_{0+}}}$ **,**In this case, theorem 1 yields$f(n,i) = z\mu^{m(n,i)} k_{0+}^{-n+i-1}, i = 1, \dots, 2n+1$

$$=: z\mu^{m(n,i)} k_{0+}^{k(n,i)}$$

Then for $\mu \in (0,1)$and $k_{0+} > 1$, $\max_i f(n,i)$ attains for the optimal choice of the pair $\big(m(n,i), k(n,i)\big)$.
For $\mu = 1/k_{0+},\ k_{0+} > 1, f(n,i) = zk_{0+}^{-m(n,i)+n-i+1} =: zk_{0+}^{\tilde{k}(n,i)}, i = 1, \dots, 2n+1$
Then for $n \in N_{0,}$ $\max_i f(n,i)$ attains for $\max_i \tilde{k}(n,i)$ and it is verified that $\max_i \tilde{k}(n,i)$ attain for $i = n-1$.
Therefore $\max_i \tilde{k}(n,i) = -\left(\frac{3}{4}n^2 + \frac{n}{2} + \frac{3}{4}\right) + 2 =: \bar{k}(n)$, $\max_n \tilde{k}(n,i) = zk_{0+}^{\bar{k}(n)}$
exists for $\bar{k}'(n) = 0 \Longrightarrow n = -1$ which is impossible. Thus maxima cannot attain in the 2ndpartition in this case.Now we consider the first partition and For$k_{2+} = 1/k_{0+.}$ Then we have from the theorem 1

$$f(n,i) = z\mu^{m(n,i)} k_{2-}^{\bar{k}_0(n,i)} k_{1+}^{\bar{k}_2(n,i)} = z\mu^{m(n,i)} k_{0+}^{\bar{k}_0(n,i)}$$

Now For $\mu = 1/k_{0+},\ k_{0+} > 1, f(n,i) = zk_{0+}^{-m(n,i)+\bar{k}_0(n,i)} =: zk_{0+}^{\bar{k}_0(n,i)}$
Then $\max_i f(n,i)$ attains for $\max_i \tilde{k}(n,i)$ and it is verified that for $n \in N_{0,}$ $\max_i \tilde{k}(n,i)$ attain for $i = n-1, n+1,$ and for both $i = n-1, n+1,$ we find $\tilde{k}(n, i = n-1) = -\frac{3}{4}n^2 + n + \frac{3}{4} =: \bar{k}(n)$
$\max_n f(n, i = n-1) = zk_{0+}^{\bar{k}(n)}$exists for $\bar{k}'(n) = 0 \Longrightarrow n = 1 \Longrightarrow \bar{k}(n) = 1$. But $i = n-1$ doesn't exist for $n = 1$. However, it can be seen that $\tilde{k}(1,1) = 1$, $\tilde{k}(1,2) = 1$, $\tilde{k}(1,0) = 0$. Thus maxima attained for the first two nodes (6,7) in the n =1-st layer of the first partition is this case.Now for $n \in N_{e,}$ $\max_i \tilde{k}(n,i)$ attains for $i = n$. Then$\tilde{k}(n, i = n) = -\frac{3}{4}n^2 + n + 1 =: \bar{k}(n)$ and $\max_n f(n, i = n) = zk_{0+}^{\bar{k}(n)}$exists for $\bar{k}'(n) = 0 \Longrightarrow n = 0 \Longrightarrow \bar{k}(n) = 1$, and for $i = 0$ we have only one node and it is first node (0) which belongs to the first partition. As in the both $n \in N_{o,e,}\bar{k}(n,i) = 1$, therefore f has maximum at the three nodes (6, 7, 0). Now if we consider the 6th partition with the similar ansatz and arguments it is verified that f has maximum the three nodes (5, 22, 23) for $k_{2+} = \frac{1}{k_{0+}},\ k_{0+} > 1,\ \mu = 1/k_{0+}$. Thus maximum attained at the six-tupel nodes (6,7, 0,5, 22, 23) of the 2nd hexagon of our model. For $\mu = k_{0+},\ k_{0+} < 1$, With $k_{2+} = 1/\ k_{0+}$, it is verified that maximum of f attained at the six-tupel nodes (2, 13, 14, 15, 16, 3)of the 5th hexagon of our model which is just in the opposite side of the previous one.We collect the results of this subsection as follows.

Corollary 4 For $\mu = k_{2+} = 1/k_{0+} < 1$ the equilibria $f(n,i) \in \mathcal{E}_N$ given in theorem 1 has six maximum attained at the six-tiple nodes (6, 7, 0, 5, 22, 23) of a regular basic hexagon of the *N*-layer model(Fig. 1).

Corollary 5 For $k_{2+} = 1/k_{0+} > 1$ and $\mu = k_{0+}$ the equilibria $f(n,i) \in \mathcal{E}_N$ given in theorem 1 has six maximum attained at the six-tiple nodes $(2, 13, 14, 15, 16, 3)$ of a regular basic hexagon of the *N*-layer model(Fig. 1).

Case 3: $\boldsymbol{k_{2+} > \frac{1}{k_{0+}}, k_{0+}} > 1$, We choose the first partition and then $f(n,i) = z\mu^{m(n,i)} k_{2-}^{\bar{k}_0(n,i)} k_{1+}^{\bar{k}_2(n,i)}$
Now for $\mu = \frac{1}{k_{0+}},\ k_{0+} > 1, k_{2+} := \frac{d}{k_{0+}},$ for $d = 1/\mu\ \ k_{2+} = 1$ and we obtain
$f(n,i) = z\mu^{m(n,i) - \bar{k}_2(n,i)} =: z\mu^{\tilde{k}(n,i)}$. Thus $\max_i f(n,i)$ attains for $\operatorname{mim}_i \tilde{k}(n,i)$.For $n \in N_{0,}$ it is seen $\min_i \tilde{k}(n,i)$ attained for $i = n+1, n+3$. For both $i = n+1, n+3, \tilde{k}(n,i) = \frac{3}{4}n^2 - n - \frac{3}{4} =: \bar{k}_0(n)$,

where $\bar{k}_0'(n) = \frac{3}{2}n - 1 = 0 \Longrightarrow n \equiv 1$ and for $\bar{k}_0(1) = -1$and we see that in the n =1-st layer the two max in this partition are $f(1,2) = f(1,3) = z/\mu$. Thus maxima attain at the nodes (7, 8).
For $n \in N_{e,} \underset{i}{min} \tilde{k}(n,i)$ attained for $i = n + 2$and $\underset{i}{min} \tilde{k}(n,i) = \tilde{k}(n, i = n + 2) = \frac{3}{4}n^2 - n - \frac{3}{4} =: \bar{k}_e(n)$, where $\bar{k}_e'(n) = \frac{3}{2}n - 1 = 0 \Longrightarrow n \equiv 0$ and for $\bar{k}_e(1) = -1$ and in the $n = 0$-st layer the only max is $f(0,1) = zk_{0+} = z/\mu$.Thus maxima attained at the three nodes $(0, 7, 8)$. Choosing the 2nd partition, with the similar ansatz it is verified that maxima attained at $(1, 9, 10)$ and the maximum values is equal to the maximum values at the nodes $(0, 7, 8)$. Thus at the nodes $(8, 9, 10, 1, 0, 7)$ (which are the nodes of the third hexagon) maxima attained in this case.We collect the results of this section as follows.
Corollary 6 For $\mu = 1/k_{0+},\ k_{0+} > 1, k_{2+} = 1$ the equilibria $f(n,i) \in \mathcal{E}_N$ given in theorem 1 has six maxima attained at the six-tiple nodes $(8, 9, 10, 1, 0, 7)$ of a regular basic hexagon of the *N*-layer model(Fig. 1).

7. Numerical simulation of discrete equilibria

We presents numerical computation of the discrete equilibria (normalized i.e. $\rho = 1$) based on a 10-layer (725-velocity) model. First of all we compute the equilibria for the trivial case $k_{0+} = k_{2+} = 1$ with $\mu = 0.25$. In this case we computed zero bulk-velocity$\mathbf{v} = \frac{1}{\rho}\sum_{i=0}^{725} f_i v_i = (0,0)$, Which is just the center of the central regular basic hexagon with six-tupel nodes $(0, 1\ 2, 3, 4, 5)$ and the maxima of the equilibria attains at this six nodes as shown in Figure 4

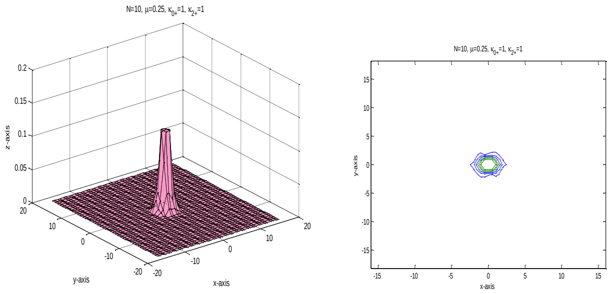

Figure 4: Equilibrium distribution with zero bulk-velocity

We choose case $k_{0+} = k_{2+} = 1$ with $\mu = 0.75$. In this case we computed zero bulk-velocity $\mathbf{v} = \frac{1}{\rho}\sum_{i=0}^{725} f_i v_i = (0,0)$ as shown in the Figure 5.

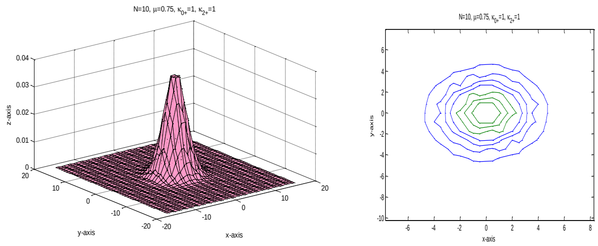

Figure 5: Equilibrium distribution with zero bulk-velocity

8. Conclusion

We have performed numerical simulations of discrete equilibriadistribution of a model Boltzmann Equation based on a hexagonal grid in $\mathbb{R}^2$. The results show the effects of parameters on temperature and bulk-velocity. The analytic results of error estimation and convergence can be investigated as a future work.

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