

Analysis of the Finite Element Solutions of Saint Venant torsion problem for a trapezoidal cross section

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Abstract

In this paper, we consider Finite Element Method (EFM) for solving Saint-Venant Torsion problem for a trapezoidal cross-section. The proposed method is quite efficient and practically well suited for solving this problem. Torsion is an important factor in the design of some load carrying elements such as shafts, curved beams, edge beams in buildings, and eccentrically loaded bridge girders. The finite element formulation of Torsion problem is presented to form the element matrices for higher order elements. In this paper straight quadrilateral elements are divided into two and six linear triangles. This paper is mainly presented global matrices of higher order elements for calculations torsional constants and stress function values for saint venant torsion problem more accurately. The Finite element results are in very good agreement with available analytical solutions for a trapezoidal cross section.

Keywords: Saint-Vennat torsion, stiffness matrices, trapezoidal cross-section and numerical example.

1. Introduction

The finite-element method (FEM) has become a very powerful tool for the numerical solution of a wide range engineering problem, particularly when analytical solutions are not available or very difficult to arriving the results. In FEM, various integrals are to be determined numerically in the evaluation of the stiffness matrix, mass matrix, body force etc. The proposed technique reduces many time consuming steps of FEM solution procedure and that substantially reduces computational effort. For such substantiation, the Saint-Vennat torsion problem studied many authors [1-10] is considered. Saint-Venant's torsion of homogeneous, isotropic, elastic cylindrical body is a classical problem of elasticity [11-13], which was solved using a semi-inverse method by assuming a state of pure shear in the cylindrical body such that it gives rise to a resultant torque over the end cross section [14]. Extension of more complicated cases of anisotropic or non-homogeneous materials has been considered in [15-19]. In this paper, finite element method (FEM) is applied to calculations torsional constants and stress function values of saint venant torsion problem for a trapezoidal cross section. It is well known that the computation of the components of element matrices for linear triangular elements simply can be evaluated analytically. Also, in FEM solution procedure the global (final) matrices are formed by assembling element matrices. Besides, that the properties of the definite integrals help us to split the domain of integrals into a number of sub domains and the sum of integrals over each sub domains produce the result of the original integrals. In this paper a quadrilateral element is splitted by (two, six) linear triangles. Then, over each triangular element the components of element matrices are exactly evaluated. Using such components of (triangular) element matrices in assembly process, the components of (four-sided quadrilateral) element matrices are formed. To maintain the order of the bi-linear and quadratic quadrilateral elements two and six linear triangles are used respectively. As the element matrices of the quadrilateral elements are formed by the assembly of the element matrices of the triangular elements named here as the hybrid type four-sided elements. The element matrices for the four-sided (bilinear and quadratic) elements are presented as the simple expressions of co-ordinates (x_i, y_i) , $i= 1,2,3,4$ and hence we need not to integrate any integrals to form element matrices. The accuracy and efficiency of the presented technique is demonstrated through the calculation of Prandtl stress function values and torsional constant of trapezoidal cross section.

2. Quadrilateral Elements

Finite element modeling means discretizing the big sized element of irregular shape into many number of calculatable regular shapes of small sized elements. In two dimensional objects as shown in fig.- 1 and Fig.-2, it can be discretized into many numbers of triangular elements. Sometimes the discretized element can have other shapes like rectangle, quadrilateral, parallelogram and so on. Among the various shapes, triangle elements are widely used in finite element analysis. Another advantage is to be mentioned that there is no difficulty with triangular elements as the exact shape function are available. We wish to describe two procedures to evaluate element matrices.

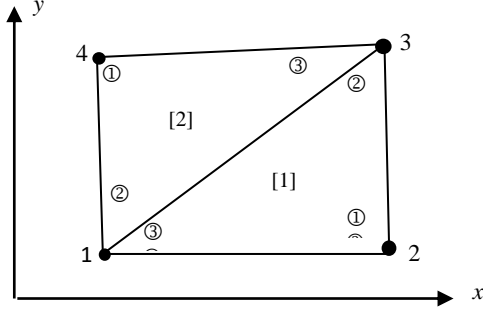


Fig-1: A 4-noded quadrilateral element splitting by two linear triangles.

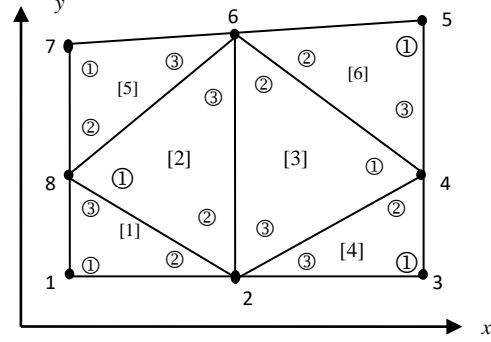
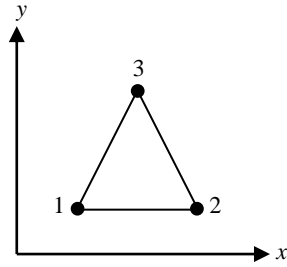
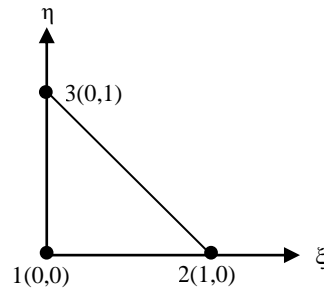


Fig-2: A 4-noded quadrilateral element splitting by six linear triangles.



(a) Unmapped triangle



(b) Mapped triangle

Fig-3: Mapping of a 3-node triangle into right isosceles unit triangle.

The transformation formulae between two coordinates systems are

$$x = \sum_{i=1}^3 x_i N_i(\xi, \eta) \quad (1)$$

$$y = \sum_{i=1}^3 y_i N_i(\xi, \eta) \quad (2)$$

Where (x_i, y_i) are the Cartesian coordinates of the i th node and $N_i(\xi, \eta)$ is the shape function refers to the node i . The shape functions are,

$$N_1 = 1 - \xi - \eta \quad N_2 = \xi \quad N_3 = \eta \quad (3)$$

Form Eqs. (1) & (2) using Eq. (3), we obtain

$$x = (x_2 - x_1)\xi + (x_3 - x_1)\eta + x_1 \quad (4)$$

$$y = (y_2 - y_1)\xi + (y_3 - y_1)\eta + y_1 \quad (5)$$

and the corresponding Jacobian can be expressed as,

$$J^{[e]} = \frac{dx}{d\xi} \frac{dy}{d\eta} - \frac{dx}{d\eta} \frac{dy}{d\xi} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \quad (6)$$

$$\text{and, } \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

$$\text{Hence, } J \frac{\partial N_i}{\partial x} = \frac{\partial y}{\partial \eta} \frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_i}{\partial \eta} \quad (7)$$

$$J \frac{\partial N_i}{\partial y} = -\frac{\partial x}{\partial \eta} \frac{\partial N_i}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial N_i}{\partial \eta} \quad (8)$$

3. Stiffness Matrix Evaluation Procedure

In this section, we consider the integral equations of the elements matrices for torsion problem. In order to find the finite element matrix using quadrilateral elements due to second order linear partial differential equation (Torsion problem) via Galerkin weighted residual formulation the integrals of the product of global derivatives are of the form:

$$k_{ij}^{[e]} = \iint_{\Omega} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dx dy = \int_0^1 \int_0^{1-\xi} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) J^{[e]} d\eta d\xi \quad (9)$$

$$F_i^{[e]} = \iint_{\Omega} N_i(x, y) dx dy = \int_0^1 \int_0^{1-\xi} N_i(\xi, \eta) J^{[e]} d\eta d\xi \quad (10)$$

Matrix [K] is usually known as the stiffness matrix is symmetric and {F} is called the load vector. It is clear that the evaluation of the stiffness matrix [K] requires integrating the product of the global derivatives of shape functions.

Using Eqs. (6) – (8) in Eq. (9) we get,

$$\begin{aligned} k_{11}^{[e]} &= \int_0^1 \int_0^{1-\xi} \left(\frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} + \frac{\partial N_1}{\partial y} \frac{\partial N_1}{\partial y} \right) J^{[e]} d\eta d\xi \\ &= \frac{1}{(J^{[e]})^2} \int_0^1 \int_0^{1-\xi} \{ (y_2 - y_3)^2 + (x_3 - x_2)^2 \} J^{[e]} d\eta d\xi = \frac{1}{J^{[e]}} \{ (y_2 - y_3)^2 + (x_3 - x_2)^2 \} \int_0^1 (1 - \xi) d\xi \\ &= \frac{1}{2J^{[e]}} \{ (y_2 - y_3)^2 + (x_3 - x_2)^2 \} \end{aligned}$$

Similarly, $k_{12}^{[e]} = \frac{1}{2J^{[e]}} \{ (y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3) \}$

$$k_{13}^{[e]} = \frac{1}{2J^{[e]}} \{ (y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1) \}$$

$$k_{22}^{[e]} = \frac{1}{2J^{[e]}} \{ (y_3 - y_1)^2 + (x_1 - x_3)^2 \}$$

$$k_{23}^{[e]} = \frac{1}{2J^{[e]}} \{ (y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1) \}$$

$$k_{33}^{[e]} = \frac{1}{2J^{[e]}} \{ (y_1 - y_2)^2 + (x_2 - x_1)^2 \}$$

Global stiffness matrix of four sided element and load vector for the fig-1 is,

$$K = \begin{bmatrix} K_{33}^{[1]} + K_{22}^{[2]} & K_{31}^{[1]} & K_{32}^{[1]} + K_{23}^{[2]} & K_{21}^{[2]} \\ K_{13}^{[1]} & K_{11}^{[1]} & K_{12}^{[1]} & 0 \\ K_{32}^{[1]} + K_{23}^{[2]} & K_{12}^{[1]} & K_{22}^{[1]} + K_{33}^{[2]} & K_{31}^{[2]} \\ K_{21}^{[2]} & 0 & K_{31}^{[2]} & K_{11}^{[2]} \end{bmatrix}, \quad F = \begin{bmatrix} F_3^{[1]} + F_2^{[2]} \\ F_1^{[1]} \\ F_2^{[1]} + F_3^{[2]} \\ F_1^{[2]} \end{bmatrix}$$

Hence the Global Stiffness Matrix of four-sided element for fig- 2 is,

$$K = \begin{bmatrix} k_{11}^{[1]} & k_{12}^{[1]} & 0 & 0 & 0 & k_{23}^{[2]} + k_{32}^{[3]} & 0 & k_{13}^{[1]} \\ k_{12}^{[1]} & k_{22}^{[1]} + k_{33}^{[2]} + k_{33}^{[4]} & k_{31}^{[4]} & k_{31}^{[3]} + k_{32}^{[4]} & 0 & 0 & 0 & k_{23}^{[1]} + k_{21}^{[2]} \\ 0 & k_{31}^{[4]} & k_{11}^{[4]} & k_{12}^{[4]} & 0 & k_{12}^{[3]} + k_{32}^{[6]} & 0 & 0 \\ 0 & k_{31}^{[3]} + k_{32}^{[4]} & k_{12}^{[4]} & k_{11}^{[3]} + k_{22}^{[4]} + k_{33}^{[6]} & k_{31}^{[6]} & k_{12}^{[6]} & 0 & 0 \\ 0 & 0 & 0 & k_{31}^{[6]} & k_{11}^{[6]} & k_{12}^{[6]} & 0 & 0 \\ 0 & k_{23}^{[2]} + k_{32}^{[3]} & 0 & k_{12}^{[3]} + k_{32}^{[6]} & k_{12}^{[6]} & k_{33}^{[2]} + k_{22}^{[3]} + k_{33}^{[5]} + k_{22}^{[6]} & k_{31}^{[5]} & k_{32}^{[5]} + k_{31}^{[2]} \\ 0 & 0 & 0 & 0 & 0 & k_{31}^{[5]} & k_{11}^{[5]} & k_{12}^{[5]} \\ k_{13}^{[1]} & k_{23}^{[1]} + k_{21}^{[2]} & 0 & 0 & 0 & k_{32}^{[5]} + k_{31}^{[2]} & k_{12}^{[5]} & k_{33}^{[1]} + k_{11}^{[2]} + k_{22}^{[5]} \end{bmatrix},$$

$$F = \begin{bmatrix} F_1^{[1]} \\ F_2^{[1]} + F_2^{[2]} + F_3^{[3]} + F_3^{[4]} \\ F_1^{[4]} \\ F_1^{[3]} + F_2^{[4]} + F_3^{[6]} \\ F_1^{[6]} \\ F_3^{[2]} + F_2^{[3]} + F_3^{[5]} + F_2^{[6]} \\ F_1^{[5]} \\ F_3^{[1]} + F_1^{[2]} + F_2^{[5]} \end{bmatrix}$$

4. Finite Element Equations for Torsion Problem

From Equation (1), the Lagrange interpolant for the Prandtl stress function $\phi(x,y)$ is given by

$$\phi(x,y) = \sum_{i=1}^{NP} \phi_i N_i(\xi,\eta)$$

Where $N_i(\xi,\eta)$ are shape functions (as given) and

$$NP = \begin{cases} 4 & \text{for the 4-noded quadrilateral} \\ 8 & \text{for the 8-noded quadrilateral} \end{cases}$$

By using the Galerkin weighted residual finite element (FE) procedure, we obtain the following FE equations

$$[K]\{\phi\} = \{F\}$$

where the components of matrix $[K]$ and $\{F\}$ are given by

$$K_{ij} = \iint_{\Omega} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dx dy$$

$$F_i = 2 \iint_{\Omega} N_i(x,y) dx dy$$

5. Finite Element Procedure

The calculation process consists of the following steps

- (i) For each element obtain the components of elements matrices i.e., K_{ij} and F_i .
- (ii) Obtain the global FE equations for whole system by assembling element equation.
- (iii) Impose boundary conditions and solve for the generalized stress vectors of the whole system.
- (iv) Calculate the torsional constant k for which $k = 2 \iint_{\Omega} \phi_i(x,y) dx dy$

6. Test Problem

An example of solid cross-section studied by Nguen (1992) for which either exact or approximate and also FE solutions exists are presented. A measure of error, E_k is provided when an exact solution of the torsional constant k is available. Where

$$E_k = 100 \left| 1 - \frac{k}{k_{exact}} \right|$$

7. Application example

To show the application of the derived formula of this paper, we consider the following two-dimensional boundary value (Torsion) problem.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2 \quad \text{with } \Omega$$

$$\phi = 0 \text{ on } \Gamma_1, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma_2$$

Where Γ_1 and Γ_2 constitute the cross-section boundaries.

7.1. Finite Element Solution of Torsion Problem for a Trapezoidal Cross-Section

The cross-section is modeled by 4 four-noded, 9 four-noded, 4 eight-noded and 9 eight-noded elements as shown in the figures: 4.2(a), 4.2(b), 4.2(c), 4.2(d), 4.2(e). Computed stress function values and the torsional constant k are tabulated in table-1. For this problem the approximate solution $k=0.1359$ according to Young (1989). It is reported by Nguen(1992) that the result is erroneous. Calculated value of k in this paper is in agreement with the result of Nguen(1992)

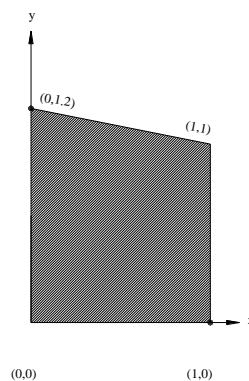


Figure 4.2(a): Trapezoidal cross-section

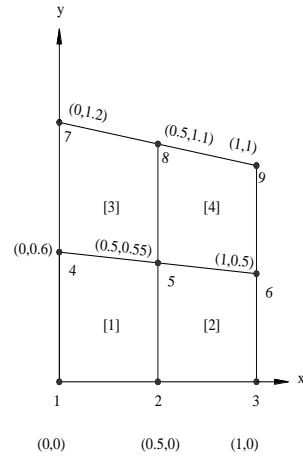


Figure 4.2(b): The FE model-1 with 4 four-noded elements

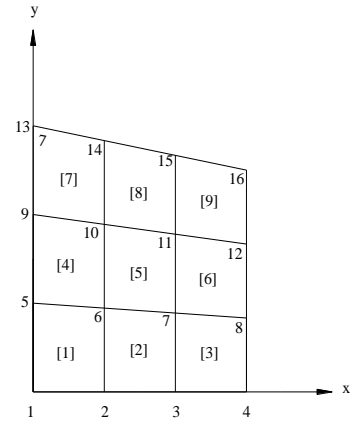


Figure 4.2(c): The FE model-2 with 9 four-noded elements

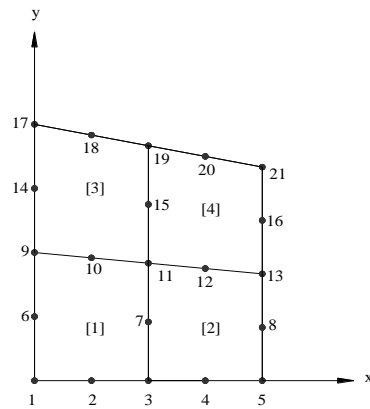


Figure 4.2(d): The FE model-3 with 4 eight-noded element

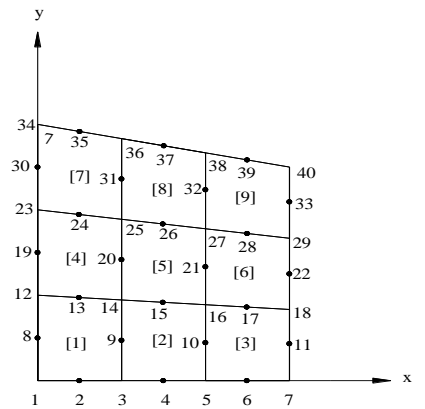


Figure 4.2 (e): The FE model-4 with 9 eight-noded element

TABLE-1

Computed stress function values, torsional constant k for a trapezoidal cross-section.

FE Model No	Φ_i values						Computed Torsional constant k
	i	Φ_i values	i	Φ_i values	i	Φ_i values	
1	5	0.1430260114					0.07866430799
2	6	0.1240339282	7	0.1219963026	10	0.1232362886	0.1205155818
	11	0.1262553158					
3	7	0.1187250833	10	0.1391192828	11	0.1573329713	0.1370739179
	12	0.1346417306	15	0.1178830216			
4	9	0.08918357894	10	0.08435365619	13	0.09850970917	0.1656315513
	14	0.1440438802	15	0.1608443767	16	0.1346706986	
	17	0.09101153550	20	0.1570554096	21	0.1505023114	
	24	0.09140276465	25	0.1354264750	26	0.1581042458	
	27	0.1353916881	28	0.09498622651	31	0.08199244999	
	32	0.08729561247					

8. Conclusions

In Finite Element Method, the Gaussian quadrature rules are being used unanimously for solving boundary value problems. In this paper a quadrilateral element is splitted by (two, six) linear triangles. Then, over each triangular element the components of element matrices are exactly evaluated. Using such components of (triangular) element matrices in assembly process, the components of (four-sided quadrilateral) element matrices are formed. Algebraic expressions in nodal co-ordinates for the components of element matrices are given explicitly. Hence, computing time for the essential integrations are not required. This is the substantial reduction in computing time and also the element matrices are now evaluated exactly. Therefore the demand of the exact evaluation of the element matrices with minimum computing time is now possible. Presented element matrices are tested and found better accuracy. Thus, it is expected that the presented expressions four-sided quadrilateral elements will find better application in Finite Element Method in solving problems of continuum mechanics. Finally through the numerical example, it is clearly shown that this proposed method is very effective for the Saint-Venant torsion of a trapezoidal cross-section.

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