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Analysis of approximate solutions of initial value problems (IVP) for ordinary differential equations (ODE)

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Abstract

In this paper, We have used Euler method and Runge-kutta method for finding approximate solutions of ordinary differential equations(ODE) in initial value problems(IVP). Numerical examples are considered to illustrate the efficiency and convergence ofthe two methods. Numerical results show that the proposed two method are very effective and efficient.We have investigated and computed the error of the proposed two methods. The approximated solutions with different step-size of the methods and analytical solutions are computed in Mathematica software. Approximation accuracy comparison between Euler method and Runge-Kutta methods for ordinary differential equations are done by finding the absolute error. The absolute errors produced by the proposed methods can then be analyzed to determine which one provides more accurate results.

Keywords: Euler method, Runge-kutta method, Ordinary differential equation (ODE).

1. Introduction

A number of problems in science and engineering can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more derivatives. Equations involving derivatives of only one independent variable are called ordinary differential equations and can be classified as either initial value problems (IVP) or boundary value problem (BVP). In most real life situations, the differential equation that models the problem is too complicated to solve exactly. Only a limited number of differential equations can be solved analytically. There are many analytical methods for finding the solution of ordinary differential equations.Even then, there exists a large number of ordinary differential equations whose solutions cannot be obtained in closed form by using well known analytical methods. There, we have to use the numerical methods to get the approximate solution of a differential equation under the prescribed initial condition or conditions. There are many types of practical numerical methods for solving initial value problems for ordinary differential equations. In this paper we present two standard numerical methods, Euler and Runge-Kutta method for solving initial value problems of ordinary differential equations.

We may realize that several works in numerical solutions of initial value problems using Euler method and Runge-Kutta method have been carried out. Many authors have attempted to solve initial value problems (IVP) to obtain high accuracy rapidly by using a numerous methods, including the Euler method and Runge-Kutta method.In [1] the author discussed accuracy analysis of Numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE). In [2] the author discussed accurate solutions of initial value problems for ordinary differential equations with fourth order Runge-kutta method. In [3] studied the Comparative study of the accuracy of an implicit linear multistep method of order six and classical Runge-Kutta method for the solution of initial value problems in ordinary differential equations.In [4,5,6,7,8,9,10,11,12,13,14,15] also numerical solutions of initial value problems for ordinary differential equations are solved using various numerical methods. In this paper Euler method and Runge-kutta method are applied without any discretization, transformation or restrictive assumptions for solving ordinary differential equations in initial value problems. The Euler Method is traditionally the first numerical technique. It is very simple to understand and geometrically easy to articulate but not very practical, the method has limited accuracy for more complicated functions.A more robust and intricate numerical technique is the Runge–Kuttamethod. This method is the most widely used one since it gives reliable starting values and is particularly suitable when the computation of higher derivatives is complicated. The numerical results are very encouraging. Finally, two examples of different kinds of ordinary differential equations are given to verify the proposed formulae. The results of each numerical example indicateconvergence and error analysis are discussed to illustrate the efficiency of the methods. The use of Euler method to solve the differential equation numerically is less efficient since it requires h to be small for obtaining reasonable accuracy.But in Runge-Kutta method, the derivatives of higher order are not required and they are designed to give greater accuracy with the advantage of requiring only the functional values at some selected points on the sub-interval. Runge-Kutta method is a more general and improvised method as compared to that of the Euler method. Euler method uses excessively small step size to converge to analytical solution. So, large number of computation is needed. Runge-kutta method gives more accurate solutions because it requires four evaluations per step and the approximated solutions converge faster to exact solution and involves less iteration to obtain the accurate solution.

2. Initial Value problem (IVP)

In this paper we consider a simple first order differential equation in an initial value problemthat can be defined as $y' = f(x, y(x))$ with initial condition $y(x_0) = y_0$ where $x_0 \le x \le x_n$ (1)

Analytical solution is defined by $y(x)$ and approximate solution is defined by y_n. For solving (1) we divide the interval $[x_0, x_n]$ into n equally spaced subintervals such that $x_n = x_0 + n\alpha$ for each $n = 0,1,2,..., n$ .The parameter α is called the step size. Numerical solutions of the initial value problem are defined to be a set of points $\{(x_n, y_n) : n = 0,1,2,..., n\}$ and each point (x_n, y_n) is an approximation to the corresponding point $(x_n, y(x_n))$ on the solution curve.

3. Numerical Evaluation Procedures

In this section we use two different procedures to solve initial value problems (IVP) for ordinary differential equations (ODE).

3.1. Procedure-1

Euler method is the simplest one-step method. It is a basic explicit method for numerical integration of ordinary differential equations. Euler proposed his method for initial value problems (IVP) in 1768. It is the first numerical method for solving IVP and serves to illustrate the concepts involved in the advanced methods. It is important to study it because it makes the error analysis much easier to understand. The general formula for Euler approximation is $y_{n+1}(x) = y_n(x) + \alpha f(x_n, y_n),\ n = 0,1,2,3...$

3.2. Procedure-2

In Numerical Analysis, the Runge-Kutta method is an important family of implicit and explicit iterative methods, which are used in temporal discretization for the approximation of solutions of ordinary differential equations. These techniques were developed around 1900 by the German mathematicians C. Runge and M. W. Kutta.The fourth order RungeKuttamethod (RK4) is most popular for solving initial value problems (IVP) for ordinary differential equation (ODE).The general formula for Runge-Kutta approximation is

$$y_{n+1}(x) = y_n(x) + \frac{1}{6}(t_1 + 2t_2 + 2t_3 + t_4),\ n = 0,1,2,3...$$

Where $t_1 = \alpha f(x, y),\ k_2 = \alpha f(x + \frac{\alpha}{2}, y + \frac{t_1}{2}),\ t_3 = \alpha f(x + \frac{\alpha}{2}, y + \frac{t_2}{2}),\ t_4 = \alpha f(x + \alpha, y + t_3)$

4. Error Analysis

There are two types of errors in numerical solutions. Round-off errors andTruncation errors occur when ordinary differential equations are solved numerically.Rounding errors originate from the fact that computers can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers or cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is call Round-off error.Truncation errors in numerical analysisarise when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we make the step size, α . A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size α goes to 0. In this paper we consider two IVP problems to verify accuracy of the proposed methods. Then using these methods we find numerical approximations for desired IVP. All the computations are performed by Mathematicasoftware. The convergence of initial value problem (IVP) is calculated by $|y(x_n) - y_n| < \delta$ where $y(x_n)$ denotes the approximate solution and y_n denotes the exact solution and δ depends on the problem which varies from 10^{-7} .The absolute error for this formula is defined by $|y(x_n) - y_n|$.

5. Numerical Examples

In this section,we perform numerical experiments using two proposed methods on illustrative examples of the initial value problems for ordinary differential equations to verify accuracy.

Example 1: we consider the initial value problem $y'(x) = xy - e^{-x}$, $y(0) = 1$ on the interval $0 \le x \le 1$.

The exact solution of the given problem is given $y(x) = \frac{1}{2} e^{\frac{x^2}{2}} (2 + \sqrt{2e\pi}\ erf(\frac{1}{\sqrt{2}}) - \sqrt{2e\pi}\ erf(\frac{x+1}{\sqrt{2}}))$

.The numerical approximations and Absolute errors using two proposed method are shown in Tables5.1-5.4

Table 5.1: Numerical Approximations for different step size using Euler method

x_n	Approximate Solutions				Exact Solutions
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$	
0.1	0.9000000000000000	0.9048135287749642	0.9071825281966918	0.9083581709844192	0.9095281318269378
0.2	0.8185162581964041	0.8275410446371659	0.8320031870900734	0.8342227388971348	0.8364350145966888
0.3	0.753013508052534	0.7658758772439387	0.7722657265605345	0.7754518320655703	0.7786326527713157
0.4	0.7015220912259382	0.7180443542090251	0.7262931266506466	0.730416546408438	0.7345401670168167
0.5	0.6625509702714119	0.682732839287642	0.692861373058415	0.6979380435871402	0.7030241932058348
0.6	0.6350254528137191	0.6590376890154925	0.6711555310054594	0.6772467069626616	0.6833611456192851
0.7	0.6182458163731397	0.646435416774346	0.6607459484358202	0.6679614594504808	0.6752197247016554
0.8	0.6118644931401185	0.6447717407037723	0.6615838022541796	0.6700887376162563	0.6786634544456545
0.9	0.6158807561796058	0.6542695024189834	0.6740165662901743	0.6840420202068181	0.6941744857956834
1.0	0.6306530582617104	0.675556795352521	0.6988254527170644	0.7106843582710795	0.7227015389486589

Table 5.2: Numerical Approximations for different step size using Runge-Kutta method

x_n	Approximate Solutions				Exact Solutions
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$	
0.1	0.9095281891559677	0.9095281354086854	0.9095281320508142	0.9095281318409314	0.9095281318269378
0.2	0.8364351249694311	0.8364350215043004	0.8364350150287971	0.8364350146237085	0.8364350145966888
0.3	0.7786328110327301	0.7786326627020703	0.7786326533933213	0.7786326528102341	0.7786326527713157
0.4	0.7345403673890025	0.7345401796338673	0.73454016780841	0.7345401670663868	0.7345401670168167
0.5	0.7030244293805993	0.7030242081423681	0.7030241941449501	0.7030241932647049	0.7030241932058348
0.6	0.6833614105929027	0.6833611624671793	0.6833611466813637	0.6833611456859505	0.6833611456192851
0.7	0.675220010215858	0.6752197429755409	0.6752197258573754	0.675219724774316	0.6752197247016554
0.8	0.6786637498272825	0.678663473508537	0.6786634556562307	0.6786634545219193	0.6786634544456545
0.9	0.694174775862907	0.6941745047240107	0.6941744870043147	0.6941744858720337	0.6941744857956834
1.0	0.7227018004483059	0.7227015563011094	0.722701540065817	0.72270153901952	0.7227015389486589

Table 5.3: Observed Absolute Error for different step size using Euler method

x_n	Absolute errors			
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$
0.1	9.52813E-03	4.71460E-03	2.34560E-03	1.16996E-03
0.2	1.79188E-02	8.89397E-03	4.43183E-03	2.21228E-03
0.3	2.56191E-02	1.27568E-02	6.36693E-03	3.18082E-03
0.4	3.30181E-02	1.64958E-02	8.24704E-03	4.12362E-03
0.5	4.04732E-02	2.02914E-02	1.01628E-02	5.08615E-03
0.6	4.83357E-02	2.43235E-02	1.22056E-02	6.11444E-03
0.7	5.69739E-02	2.87843E-02	1.44738E-02	7.25827E-03
0.8	6.67990E-02	3.38917E-02	1.70797E-02	8.57472E-03
0.9	7.82937E-02	3.99050E-02	2.01579E-02	1.01325E-02
1.0	9.20485E-02	4.71447E-02	2.38761E-02	1.20172E-02

Table 5.4: Observed Absolute Error for different step size using Runge-kuttamethod

x_n	Absolute errors			
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$
0.1	5.73290E-08	3.58175E-09	2.23877E-10	1.39939E-11
0.2	1.10373E-07	6.90761E-09	4.32109E-10	2.70199E-11
0.3	1.58261E-07	9.93075E-09	6.22006E-10	3.89190E-11
0.4	2.00372E-07	1.26171E-08	7.91594E-10	4.95700E-11
0.5	2.36175E-07	1.49365E-08	9.39116E-10	5.88700E-11
0.6	2.64974E-07	1.68479E-08	1.06208E-09	6.66650E-11
0.7	2.85514E-07	1.82739E-08	1.15572E-09	7.26610E-11
0.8	2.95382E-07	1.90629E-08	1.21058E-09	7.62650E-11
0.9	2.90067E-07	1.89283E-08	1.20863E-09	7.63500E-11
1.0	2.61500E-07	1.73525E-08	1.11716E-09	7.08620E-11

Example 2: we consider the initial value problem $y'(x) = xy + e^{x}$, $y(0) = 1$ on the interval $0 \le x \le 1$. The exact solution of the given problem is given by $y(x) = \frac{1}{2} e^{\frac{x^2}{2}} (\sqrt{2e\pi}\ erf(\frac{x-1}{\sqrt{2}}) + 2 + \sqrt{2e\pi}\ erf(\frac{1}{\sqrt{2}}))$. The numerical approximations and Absolute errors using two proposed method are shown in Tables5.5-5.8

Table 5.5: Numerical Approximations for different step size using Euler method

x_n	Approximate Solutions				Exact Solutions
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$	
0.1	1.100000000000000	1.1051885548188012	1.1078393561464583	1.1091796928070503	1.110530301632017
0.2	1.2215170918075648	1.2328095534593166	1.2386044746837532	1.2415411421210325	1.2445047509926233
0.3	1.3680877094597332	1.3867366550725795	1.3963492404510334	1.4012315340835104	1.4061660780777825
0.4	1.5441162215011255	1.571797058999487	1.5861293379826054	1.5934254824102	1.600811124250039
0.5	1.7550633401252975	1.7939908896249332	1.8142391785103749	1.8245713153222383	1.8350469141081591
0.6	2.007688634201575	2.060774486242448	2.08851857525606	2.102710305819003	2.117123064869392
0.7	2.3103618322927204	2.3814260712775424	2.418749937102355	2.437890885365414	2.457363933871341
0.8	2.6734624313002584	2.767518505694381	2.8171738817509677	2.842707519203811	2.8687320948873327
0.9	3.109893518653526	3.2335322273678173	3.2991608375712325	3.333004318701614	3.3675659276217007
1.0	3.6357442464480383	3.797652829447975	3.884089401273545	3.928797715378466	3.974549507166877

Table 5.6: Numerical Approximations for different step size using Runge-Kutta method

x_n	Approximate Solutions				Exact Solution
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$	
0.1	1.1105302410717133	1.110530297851038	1.1105303013958143	1.1105303016172572	1.110530301632017
0.2	1.2445046270887068	1.2445047432709144	1.2445047505107112	1.2445047509625244	1.2445047509926233
0.3	1.4061658878477006	1.4061660662618751	1.4061660773416873	1.4061660780318517	1.4061660780777825
0.4	1.6008108631429778	1.6008111081125787	1.6008111232475102	1.6008111241875747	1.600811124250039
0.5	1.8350465731825811	1.8350468931762551	1.8350469128126372	1.8350469140276	1.8350469141081591
0.6	2.117122625482692	2.1171230380943378	2.117123063219478	2.117123064767038	2.117123064869392
0.7	2.4573633582017687	2.4573638990299127	2.457363931733263	2.457363933739007	2.457363933871341
0.8	2.8687313095751765	2.8687320475362137	2.8687320919889703	2.8687320947082036	2.8687320948873327
0.9	3.3675647961671404	3.367565859316478	3.3675659234396402	3.367565927363227	3.3675659276217007
1.0	3.974547784090327	3.974549402480526	3.9745495007365212	3.974549506768798	3.974549507166877

Table 5.7: Observed Absolute error for different step size using Euler method

x_n	Absolute errors			
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$
0.1	1.05303E-02	5.34175E-03	2.69095E-03	1.35061E-03
0.2	2.29877E-02	1.16952E-02	5.90028E-03	2.96361E-03
0.3	3.80784E-02	1.94294E-02	9.81684E-03	4.93454E-03
0.4	5.66949E-02	2.90141E-02	1.46818E-02	7.38564E-03
0.5	7.99836E-02	4.10560E-02	2.08077E-02	1.04756E-02
0.6	1.09434E-01	5.63486E-02	2.86045E-02	1.44128E-02
0.7	1.47002E-01	7.59379E-02	3.86140E-02	1.94730E-02
0.8	1.95270E-01	1.01214E-01	5.15582E-02	2.60246E-02
0.9	2.57672E-01	1.34034E-01	6.84051E-02	3.45616E-02
1.0	3.38805E-01	1.76897E-01	9.04601E-02	4.57518E-02

Table 5.8: Observed Absolute errorfor different step size using Runge-Kutta method

x_n	Absolute errors			
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.0125$
0.1	6.05603E-08	3.78098E-09	2.36200E-10	1.47600E-11
0.2	1.23904E-07	7.72171E-09	4.81910E-10	3.00999E-11
0.3	1.90230E-07	1.18159E-08	7.36100E-10	4.59301E-11
0.4	2.61107E-07	1.61375E-08	1.00252E-09	6.24600E-11
0.5	3.40926E-07	2.09319E-08	1.29552E-09	8.05500E-11
0.6	4.39387E-07	2.67751E-08	1.64992E-09	1.02360E-10
0.7	5.75670E-07	3.48414E-08	2.13808E-09	1.32340E-10
0.8	7.85312E-07	4.73511E-08	2.89836E-09	1.79130E-10
0.9	1.13145E-06	6.83052E-08	4.18206E-09	2.58480E-10
1.0	1.72308E-06	1.04686E-07	6.43035E-09	3.98080E-10

6. Discussion of Results

Tables (5.1) and (5.5) show a comparison between the numerical approximation results and the exact solution using Euler method for the different step size with various selected values of x and Tables (5.3) and (5.7) are the computed absolute errors .we can see that the approximate solution when the step sizes $\alpha = 0.1$ and $\alpha = 0.05$ does not converge to exact solution but the step sizes reduces gradually to $\alpha = 0.025$ and $\alpha = 0.0125$ the approximate solution tends to converge slowly to the exact solution with the reduction of the step sizes α , Also the accuracy of the method is not impressive. Hence the Euler method is less accurate. Also Tables (5.2) and (5.6) shows that the approximate solution using Runge-Kutta method at $\alpha = 0.1$, $\alpha = 0.05$, $\alpha = 0.025$ and $\alpha = 0.0125$ is more accurate, efficient and sufficient than the Euler method and Tables (5.4) and (5.8) show the computed absolute errors. We comparing this method to the Euler method, this method give more efficient and accurate results as the approximate solution tends to converge faster to the exact solution. .

7. Conclusion

In this paper, we present Euler method and Runge-Kutta method for solving ordinary differential equation in initial value problems (IVP). To achieve the desired accuracy of numerical solution it is necessary to take step size very, very small. The application of two proposed methods were illustrated by solving an initial value problems (IVP) which is a first order differential equation and using uniform step sizes to obtain numerical results of the approximate solution and compare it with the exact solution. The simple Euler method was found to be less accurate due to the inaccurate numerical results that were obtained from the approximate solution in comparison to the exact solution. In terms of convergence, the approximate solution of the Runge-kutta method was found to converge faster to the exact solution compared to the Euler method. Finally we observe that the Runge-kutta method is more powerful and more efficient in finding numerical solutions of initial value problems compared to the Euler method.

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