Maximum Smoothness Algorithm for Building Commodity Forward Curves

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1 Introduction

This document describes the maximum-smoothness algorithm used to increase the granularity of constructed forward curves. The rest of this document will refer to forward prices and curves, even though the same methodology can be applied to futures and swap instruments as well.

The methodology is an adaption of that described in (Author). This makes reference to Benth et al. (2007) and Lim and Xiao (2002), the later reference which uses maximum smoothness in the context of building interest rate forward curves.

A tension parameter is introduced in this paper, which allows the interpolation to not be based on only maximising the smoothness, but also to penalise the length of the curve. This can be used to reduce oscillations in the interpolated curve.

2 Deriving the Algorithm

2.1 Functional Form

The base of the algorithm is a spline, which by definition is made up of piecewise polynomial functions.

$$p(t) = \begin{cases} p_1(t) & \text{for} \quad t \in [t_0, t_1) \\ p_2(t) & \text{for} \quad t \in [t_1, t_2) \\ \vdots & \\ p_{n-1}(t) & \text{for} \quad t \in [t_{n-2}, t_{n-1}) \\ p_n(t) & \text{for} \quad t \in [t_{n-1}, t_n] \end{cases} \tag{1}$$

Where $t_0 < t_1 < \ldots < t_{n-1} < t_n$ are the boundary points between the polynomials which make up the spline. In the context of building a forward curve, the variable t is defined as the time until start of delivery of a forward contract.

The boundary points are chosen to be start of the input forward prices. It is also assumed that the input forward prices are not for delivery periods which overlap with any other input. Gaps between input forward contracts are permitted, in which case a boundary point will exit for the start of the gap.

The individual polynomial functions p_n themselves are of order 4, hence take the form:

$$p_i(t) = a_i + b_i t + c_i t^2 + d_i t^3 + e_i t^4 \tag{2}$$

The curve fitting algorithm essentially involves solving for the polynomial parameters a_i, b_i, c_i, d_i, and e_i for $i = i \ldots n$.

In many cases the spline described above is not sufficient to derive a forward curve which shows strong price seasonality, especially when this seasonality cannot be directly observed in the traded forward prices. An example of this is the day-of-week seasonality for gas and power prices, which generally are lower at the weekend when demand is lower. As such the function form is as follows:

$$f(t) = (p(t) + S_{add}(t))S_{mult}(t) \tag{3}$$

Where the forward price for the period starting delivery at time t is given by $f(t)$, which consists of $p(t)$ adjusted by two arbitrary seasonal adjustment functions $S_{add}(t)$ an additive adjustment, and $S_{mult}(t)$ a multiplicative adjustment.

2.2 Constraints

2.2.1 Polynomial Boundary Point Constraints

As usual with splines, constraints are put in place that adjascent polynomials have equal value, first derivative, and second derivatives at the boundary points. These three constraints can be respectively expressed as:

$$a_i + b_i t + c_i t^2 + d_i t^3 + e_i t^4 - a_{i+1} - b_{i+1} t - c_{i+1} t^2 - d_{i+1} t^3 - e_{i+1} t^4 = 0 \tag{4}$$

$$b_i + 2c_i t + 3d_i t^2 + 4e_i t^3 - b_{i+1} - 2c_{i+1} t - 3d_{i+1} t^2 - 4e_{i+1} t^3 = 0 \tag{5}$$

$$2c_i + 6d_i t + 12e_i t^2 - 2c_{i+1} - 6d_{i+1} t - 12e_{i+1} t^2 = 0 \tag{6}$$

The above three equations should hold for the boundary points
$t \in \{t_1, t_2, \ldots, t_{n-2}, t_{n-1}\}$.

2.2.2 Forward Price Constraint

The most important constraints is that the derived forward curve averages back to the input traded forward prices. The market inputs to the forward curve model are traded forward prices F_i. Setting this equal to the average of the derived smooth curve:

$$F_i = \frac{\sum_{t \in T_i} (p(t) + S_{add}(t)) S_{mult}(t) w(t)}{\sum_{t \in T_i} w(t)} \tag{7}$$

Where $w(t)$ is a weighting function and T_i is the set of all delivery start times for the delivery periods at the granularity of the curve being built. The weighting function has two meanings from a busines perspective.

- The discount factor, because the no-arbitrage price of a forward contract is equal to the discount factor weighted average of its components. However, in a low interest rate environment, the discount factor can be approximated as 1.0 for all maturities, in which case the discounting component can be ignored.
- The volume of commodity delivered in each period. For example, an off-peak power forward contract in the UK delivers over 12 hours in on weekdays, and 24 hours on weekends, hence $w(t)$ would equal double for t representing weekends compared to $w(t)$ when t represents a weekday delivery. Clock changes can also cause the total volume delivered over a day in a fixed time zone to vary due to hours lost or gained. Hence $w(t)$ can be used to account for this.
- For swaps which only fix on certain days (usually business days) $w(t)$ can be used to account for this by returning the number of fixing days in the period starting at t. For example if deriving a a monthly curve $w(t)$ would evaluate to the number of fixing days in the month starting at t.

Equation 7 can be transformed into an equation linear on the parameters of the piecewise polynomial by substituting in the polynomial representation of $p(t)$:

$$\sum_{t \in T_i} (p_i(t) + S_{add}(t)) S_{mult}(t) w(t) = F_i \sum_{t \in T_i} w(t) \tag{8}$$

Rearranging:

$$\sum_{t \in T_i} (a_i + b_i t + c_i t^2 + d_i t^3 + e_i t^4 + S_{add}(t)) S_{mult}(t) w(t) = F_i \sum_{t \in T_i} w(t) \tag{9}$$

Rearranging again gives a form linear with respect to the unknown polynomial coefficients a_i, b_i, c_i and d_i.

$$\begin{aligned} a_i \sum_{t \in T_i} S_{mult}(t)w(t) + b_i \sum_{t \in T_i} S_{mult}(t)w(t)t + c_i \sum_{t \in T_i} S_{mult}(t)w(t)t^2 + \\ d_i \sum_{t \in T_i} S_{mult}(t)w(t)t^3 + e_i \sum_{t \in T_i} S_{mult}(t)w(t)t^4 = \\ F_i \sum_{t \in T_i} w(t) - \sum_{t \in T_i} S_{add}(t)S_{mult}(t)w(t) \end{aligned} \tag{10}$$

In the case where there are gaps between the input forward contract delivery periods, the forward price constraint is simply omitted.

2.2.3 Matrix Form of Constraints

Equations 4, 5, 6 and 10 can be expressed as the linear system $\mathbf{Ax} = \mathbf{b}$ where:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{n-1} \\ \mathbf{x}_n \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{n-1} \\ \mathbf{b}_n \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & & \mathbf{A}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_n \end{bmatrix}$$

And:

$$\mathbf{x_i} = \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \\ e_i \end{bmatrix}$$

$$\mathbf{b_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ F_i \sum_{t \in T_i} w(t) - \sum_{t \in T_i} S_{add}(t)S_{mult}(t)w(t) \end{bmatrix}$$

$$\mathbf{A}_i = \begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 & t_i^4 & -1 & -t_i & -t_i^2 & -t_i^3 & -t_i^4 \\ 0 & 1 & 2t_i & 3t_i^2 & 4t_i^3 & 0 & -1 & -2t_i & -3t_i^2 & -4t_i^3 \\ 0 & 0 & 2 & 6t & 12t^2 & 0 & 0 & -2 & -6t & -12t^2 \\ f_i^1 & f_i^2 & f_i^3 & f_i^4 & f_i^5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where the components in the last row are defined as:

$$f_i^1 = \sum_{t \in T_i} S_{mult}(t)w(t)$$
$$f_i^2 = \sum_{t \in T_i} S_{mult}(t)w(t)t$$
$$f_i^3 = \sum_{t \in T_i} S_{mult}(t)w(t)t^2$$
$$f_i^4 = \sum_{t \in T_i} S_{mult}(t)w(t)t^3$$
$$f_i^5 = \sum_{t \in T_i} S_{mult}(t)w(t)t^4$$

2.3 Smoothness Criteria

Maximum smoothness is typically obtained by by finding the spline parameters which minimise the integral of the second derivative squared. This paper deviates from this methodology by also including the squared first derivative in the penalty function being minimised. Both the first and second derivative terms penalise oscillations, but in different ways. The first derivative penalises the increased total curve length of oscillations, whereas the second derivative term penalises changes in curve direction. The non-negative tension parameter τ is used to control the contribution of curve length to the penalty function being minimised. Note that although this spline includes a tension parameter, it's behaviour with respect to this parameter is very different to *usual* tension splines which will tend towards linear splines as the tension parameter increases. The effect of the tension parameter τ in this algorithm is more subtle, with the curve alway remaining smooth, but with oscillations generally becoming smaller in amplitude, but with sharper changes in direction when τ is increased.

Writing the penalty function and integrating:

$$
\begin{aligned}
min \int_{t_0}^{t_n} \big(p''(t)^2 + \tau p'(t)^2\big)dt &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} p_i''(t)^2 + \tau p'(t)^2 dt \\
= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \big(2c_i + 6d_it + 12e_it^2\big)^2 &+ \tau\big(b_i + 2c_it + 3d_it^2 + 4e_it^3\big)^2 dt \\
= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big(b_i^2\tau + 4b_ic_i\tau t + c_i^2(4 + 4\tau t^2) &+ 6b_id_i\tau t^2 + c_id_i(24t + 12\tau t^3) + \\
c_ie_i(48t^2 + 16\tau t^4) &+ d_i^2(36t^2 + 9\tau t^4) + \\
8b_ie_i\tau t^3 + d_ie_i(144t^3 + 24\tau t^5) &+ e_i^2(144t^4 + 16\tau t^6)\Big)dt \\
= \sum_{i=1}^{n} b_i^2\tau\Delta_i^1 + 2b_ic_i\tau\Delta_i^2 + 4c_i^2(\Delta_i^1 + \frac{1}{3}\tau\Delta_i^3) &+ 2b_id_i\tau\Delta_i^3 + c_id_i(12\Delta_i^2 + 3\tau\Delta_i^4) \\
&+ c_ie_i(16\Delta_i^3 + \frac{16}{5}\tau\Delta_i^5) + d_i^2(12\Delta_i^3 + \frac{9}{5}\tau\Delta_i^5) \\
&+ 2b_ie_i\tau\Delta_i^4 + d_ie_i(36\Delta_i^4 + 4\tau\Delta_i^6) + e_i^2(\frac{144}{5}\Delta_i^5 + \frac{16}{7}\tau\Delta_i^7)
\end{aligned} \tag{11}
$$

Where Δ_i^j is defined as the difference between t^j at the polynomial boundary points, i.e. $\Delta_i^j = t_i^j - t_{i-1}^j$. Recognising 11 as a quadratic form it can be reformulating in the following matrix form:

$$
\sum_{i=1}^{n} \mathbf{x_i^T H_i x_i} \tag{12}
$$

Where:

$$
\mathbf{H_i} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \tau\Delta_i^1 & \tau\Delta_i^2 & \tau\Delta_i^3 & \tau\Delta_i^4 \\
0 & \tau\Delta_i^2 & 4\Delta_i^1 + \frac{4}{3}\tau\Delta_i^3 & 6\Delta_i^2 + \frac{3}{2}\tau\Delta_i^4 & 8\Delta_i^3 + \frac{8}{5}\tau\Delta_i^5 \\
0 & \tau\Delta_i^3 & 6\Delta_i^2 + \frac{3}{2}\tau\Delta_i^4 & 12\Delta_i^3 + \frac{9}{5}\tau\Delta_i^5 & 18\Delta_i^4 + 2\tau\Delta_i^6 \\
0 & \tau\Delta_i^4 & 8\Delta_i^3 + \frac{8}{5}\tau\Delta_i^5 & 18\Delta_i^4 + 2\tau\Delta_i^6 & \frac{144}{5}\Delta_i^5 + \frac{16}{7}\tau\Delta_i^7
\end{bmatrix}
$$

The objective function 12 can be arranged into a single matrix quadratic form without the summation as:

$$
\mathbf{x^T H x} \tag{13}
$$

Where:

$$
\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{n-1} \\ \mathbf{x}_n \end{bmatrix}
$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & & \mathbf{H}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{H}_n \end{bmatrix}$$

2.4 Minimisation Problem

The sections above show that finding the maximum smoothness curve comes down to finding the polynomial coefficients, vector $\mathbf{x}$, which minimises $\mathbf{x^T H x}$, subject to the linear constraints $\mathbf{Ax} = \mathbf{b}$. This problem is well suited to the method of Lagrange multipliers for which we first define the vector λ.

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{4n-2} \\ \lambda_{4n-1} \end{bmatrix} \tag{14}$$

$$\mathcal{L}(\mathbf{x}, \lambda) = \mathbf{x^T H x} + \lambda^{\mathbf{T}}(\mathbf{Ax} - \mathbf{b}) \tag{15}$$

The minima $\min\limits_{x,\lambda} \mathcal{L}(\mathbf{x}, \lambda)$ is found as the solution where the partial derivatives of $\mathcal{L}(\mathbf{x}, \lambda)$ with respect to $\mathbf{x}$ and λ are zero.

$$\frac{\mathcal{L}(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = \mathbf{2Hx} + \mathbf{A^T}\lambda = 0 \tag{16}$$

$$\frac{\mathcal{L}(\mathbf{x}, \lambda)}{\partial \lambda} = \mathbf{Ax} - \mathbf{b} = 0 \tag{17}$$

These can be arranged into a single linear system:

$$\begin{bmatrix} \mathbf{2H} & \mathbf{A^T} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x_{min}} \\ \lambda_{\mathbf{min}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} \tag{18}$$

Hence the vector of spline polynomial coefficients for the maximum smoothness curve, $\mathbf{x_{min}}$, can be found by solving this system.

$$\begin{bmatrix} \mathbf{x_{min}} \\ \lambda_{\mathbf{min}} \end{bmatrix} = \begin{bmatrix} \mathbf{2H} & \mathbf{A^T} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} \tag{19}$$

Once 19 is solved, the spline parameters are taken from $\mathbf{x_{min}}$ and the derived forward prices are calculated by evaluating 3.

References

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