Tension Spline Algorithm for Building Forward Curves

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1 Introduction

2 Deriving the Algorithm

2.1 Functional Form

The base of the algorithm is a spline, which by definition is made up of piecewise polynomial functions.

$$p(t) = \begin{cases} p_1(t) & \text{for} \quad t \in [t_0, t_1) \\ p_2(t) & \text{for} \quad t \in [t_1, t_2) \\ \vdots & \\ p_{n-1}(t) & \text{for} \quad t \in [t_{n-2}, t_{n-1}) \\ p_n(t) & \text{for} \quad t \in [t_{n-1}, t_n] \end{cases} \tag{1}$$

Where $t_0 < t_1 < \ldots < t_{n-1} < t_n$ are the boundary points between the polynomials which make up the spline. In the context of building a forward curve, the variable t is defined as the time until start of delivery of a forward contract.

The boundary points are chosen to be start of the input forward prices. It is also assumed that the input forward prices are not for delivery periods which overlap with any other input. Gaps between input forward contracts are permitted, in which case a boundary point will exit for the start of the gap.

$$\begin{aligned} p_i(t) = {} & \frac{z_{i-1}\sinh(\tau_i(t_i - t)) + z_i \sinh(\tau_i(t - t_{i-1}))}{\tau_i^2 \sinh(\tau h_i)} \\ & + \frac{(y_{i-1} - z_{i-1}/\tau_i^2)(t_i - t) + (y_i - z_i/\tau_i^2)(t - t_{i-1})}{h_i} \end{aligned} \tag{2}$$

Where $h_i = t_i - t_{i-1}$. $z_i = p''(t_i)$ and $y_i = p(t_i)$, i.e. the (as yet unknown) value of the function at the boundary points.

The curve fitting algorithm essentially involves solving for the parameters z_i, and y_i for $i = 0 \ldots n$.

In many cases the spline described above is not sufficient to derive a forward curve which shows strong price seasonality, especially when this seasonality cannot be directly observed in the traded forward prices. An example of this is the day-of-week seasonality for gas and power prices, which generally are lower at the weekend when demand is lower. As such the function form is as follows:

$$f(t) = (p(t) + S_{add}(t))S_{mult}(t) \tag{3}$$

Where the forward price for the period starting delivery at time t is given by $f(t)$, which consists of $p(t)$ adjusted by two arbitrary seasonal adjustment functions $S_{add}(t)$ an additive adjustment, and $S_{mult}(t)$ a multiplicative adjustment.

2.2 Constraints

2.2.1 Polynomial Boundary Point Constraints

As usual with splines, constraints are put in place that adjascent polynomials have equal value, first derivative, and second derivatives at the boundary points.

2.2.2 Polynomial Value Boundary Point Equality

To make $p(t)$ continuous we need to constrain $p_i(t_{i-1}) = p_{i-1}(t_{i-1})$. Evaluating both of these:

$$\begin{aligned} p_i(t_{i-1}) = \frac{z_{i-1}\sinh(\tau_i h_i) + z_i \sinh(0)}{\tau_i^2 \sinh(\tau_i h_i)} &+ \frac{(y_{i-1} - z_{i-1}/\tau_i^2)h_i}{h_i} \\ &= \frac{z_{i-1}}{\tau_i^2} + y_{i-1} - \frac{z_{i-1}}{\tau_i^2} \\ &= y_{i-1} \end{aligned} \tag{4}$$

$$\begin{aligned} p_{i-1}(t_{i-1}) = \frac{z_{i-2}\sinh(0) + z_{i-1}\sinh(\tau_{i-1}h_{i-1})}{\tau_{i-1}^2 \sinh(\tau_{i-1}h_{i-1})} &+ \frac{(y_{i-1} - z_{i-1}/\tau_{i-1}^2)h_{i-1}}{h_{i-1}} \\ &= \frac{z_{i-1}}{\tau_{i-1}^2} + y_{i-1} - \frac{z_{i-1}}{\tau_{i-1}^2} \\ &= y_{i-1} \end{aligned}$$

Hence, by construction, $p(t)$ is always continuous with value y_{i-1} at the boundary between p_i and p_{i-1}

2.2.3 Polynomial First Derivative Boundary Point Equality

This is to constrain $p_i'(t_{i-1}) = p_{i-1}'(t_{i-1})$. First finding the expression for $p_i'(t)$:

$$p_i(t) = \frac{z_{i-1}\sinh(\tau_i(t_i - t)) + z_i \sinh(\tau_i(t - t_{i-1}))}{\tau_i^2 \sinh(\tau h_i)} + \frac{(y_{i-1} - z_{i-1}/\tau_i^2)(t_i - t) + (y_i - z_i/\tau_i^2)(t - t_{i-1})}{h_i} \quad (5)$$

$$p_i'(t) = \frac{-z_{i-1}\cosh(\tau_i(t_i - t)) + z_i \cosh(\tau_i(t - t_{i-1}))}{\tau_i \sinh(\tau_i h_i)} + \frac{y_i - y_{i-1} + (z_{i-1} - z_i)/\tau_i^2}{h_i} \quad (6)$$

For clarity, rearranging this to highlight the linearity with respect to the parameters:

$$p_i'(t) = z_i\left(\frac{\cosh(\tau_i(t - t_{i-1}))}{\tau_i \sinh(\tau_i h_i)} - \frac{1}{h_i \tau_i^2}\right) + z_{i-1}\left(\frac{1}{h_i \tau_i^2} - \frac{\cosh(\tau_i(t_i - t))}{\tau_i \sinh(\tau_i h_i)}\right) + y_i \frac{1}{h_i} - y_{i-1}\frac{1}{h_i} \quad (7)$$

Evaluating this about the boundary points:

$$p_i'(t_{i-1}) = z_i\left(\frac{1}{\tau_i \sinh(\tau_i h_i)} - \frac{1}{h_i \tau_i^2}\right) + z_{i-1}\left(\frac{1}{h_i \tau_i^2} - \frac{\cosh(\tau_i h_i)}{\tau_i \sinh(\tau_i h_i)}\right) + y_i \frac{1}{h_i} - y_{i-1}\frac{1}{h_i} \quad (8)$$

$$p_{i-1}'(t_{i-1}) = z_{i-1}\left(\frac{\cosh(\tau_{i-1} h_{i-1})}{\tau_{i-1}\sinh(\tau_{i-1} h_{i-1})} - \frac{1}{h_{i-1}\tau_{i-1}^2}\right) + z_{i-2}\left(\frac{1}{h_{i-1}\tau_{i-1}^2} - \frac{1}{\tau_{i-1}\sinh(\tau_{i-1} h_{i-1})}\right) + y_{i-1}\frac{1}{h_{i-1}} - y_{i-2}\frac{1}{h_{i-1}} \quad (9)$$

Setting these equal:

$$
\begin{aligned}
0 = z_i\left(\frac{1}{\tau_i\sinh(\tau_i h_i)} - \frac{1}{h_i\tau_i^2}\right) \\
+ z_{i-1}\left(\frac{1}{h_i\tau_i^2} - \frac{\cosh(\tau_i h_i)}{\tau_i\sinh(\tau_i h_i)} - \frac{\cosh(\tau_{i-1}h_{i-1})}{\tau_{i-1}\sinh(\tau_{i-1}h_{i-1})} + \frac{1}{h_{i-1}\tau_{i-1}^2}\right) \\
- z_{i-2}\left(\frac{1}{h_{i-1}\tau_{i-1}^2} - \frac{1}{\tau_{i-1}\sinh(\tau_{i-1}h_{i-1})}\right) \\
+ y_i\frac{1}{h_i} - y_{i-1}\left(\frac{1}{h_i} + \frac{1}{h_{i-1}}\right) + y_{i-2}\frac{1}{h_{i-1}}
\end{aligned}
\tag{10}
$$

This constraint should be held for $i = 2 \ldots n$.

The above three equation should hold for the boundary points $t \in \{t_1, t_2, \ldots, t_{n-2}, t_{n-1}\}$.

2.2.4 Forward Price Constraint

The most important constraints is that the derived forward curve averages back to the input traded forward prices. The market inputs to the forward curve model are traded forward prices F_i. Setting this equal to the average of the derived smooth curve:

$$
F_j = \frac{\sum_{t\in T_j}(p(t) + S_{add}(t))S_{mult}(t)w(t)D(t)}{\sum_{t\in T_j} w(t)D(t)} \tag{11}
$$

Where $D(t)$ is the discount factor from the settlement date of delivery period t. $w(t)$ is a weighting function and T_i is the set of all delivery start times for the delivery periods at the granularity of the curve being built. The weighting function has two meanings from a busines perspective.

- The volume of commodity delivered in each period. For example, an off-peak power forward contract in the UK delivers over 12 hours in on weekdays, and 24 hours on weekends, hence $w(t)$ would equal double for t representing weekends compared to $w(t)$ when t represents a weekday delivery. Clock changes can also cause the total volume delivered over a day in a fixed time zone to vary due to hours lost or gained. Hence $w(t)$ can be used to account for this.

- For swaps which only fix on certain days (usually business days) $w(t)$ can be used to account for this by returning the number of fixing days in the period starting at t. For example if deriving a a monthly curve $w(t)$ would evaluate to the number of fixing days in the month starting at t.

Equation 11 can be transformed into an equation linear on the parameters of the piecewise polynomial by substituting in the polynomial representation of $p(t)$:

$$\sum_i \sum_{t \in T_j \cap [t_{i-1}, t_i)} p_i(t) S_{mult}(t) w(t) D(t) = F_j \sum_{t \in T_j} w(t) D(t) - \sum_{t \in T_i} S_{add}(t) S_{mult}(t) w(t) D(t) \tag{12}$$

Substituting in for $p_i(t)$:

$$\sum_i \sum_{t \in T_i \cap [t_{i-1}, t_i)} \Bigg(\frac{z_{i-1} \sinh(\tau_i (t_i - t)) + z_i \sinh(\tau_i (t - t_{i-1}))}{\tau_i^2 \sinh(\tau_i h_i)}$$
$$+ \frac{(y_{i-1} - z_{i-1}/\tau_i^2)(t_i - t) + (y_i - z_i/\tau_i^2)(t - t_{i-1})}{h_i} + S_{add}(t) \Bigg) S_{mult}(t) w(t) D(t)$$
$$= F_i \sum_{t \in T_i} w(t) D(t) - \sum_{t \in T_i} S_{add}(t) S_{mult}(t) w(t) D(t) \tag{13}$$

Rearranging again gives a form linear with respect to the unknown polynomial coefficients z_i, z_{-1}, y_i and y_{i-1}.

$$\sum_i \Bigg(z_i \sum_{t \in T_j \cap [t_{i-1}, t_i)} \left(\frac{\sinh(\tau_i (t - t_{i-1}))}{\tau_i^2 \sinh(\tau_i h_i)} - \frac{t - t_{i-1}}{\tau_i^2 h_i} \right) S_{mult}(t) w(t) D(t)$$
$$+ z_{i-1} \sum_{t \in T_j \cap [t_{i-1}, t_i)} \left(\frac{\sinh(\tau_i (t_i - t))}{\tau_i^2 \sinh(\tau_i h_i)} - \frac{t_i - t}{\tau_i^2 h_i} \right) S_{mult}(t) w(t) D(t)$$
$$+ y_i \sum_{t \in T_j \cap [t_{i-1}, t_i)} \frac{(t - t_{i-1})}{h_i} S_{mult}(t) w(t) D(t)$$
$$+ y_{i-1} \sum_{t \in T_j \cap [t_{i-1}, t_i)} \frac{(t_i - t)}{h_i} S_{mult}(t) w(t) D(t) \Bigg)$$
$$= F_j \sum_{t \in T_i} w(t) D(t) - \sum_{t \in T_i} S_{add}(t) S_{mult}(t) w(t) D(t) \tag{14}$$

This constraint should be held for $j = 1 \ldots n$.

2.2.5 Adding in an Extra Constraint

In case we need to add another contraint to ensure are linear system has a unique solution. Constraint the first derivative of one of the knots to be equal to the slope calculated from the two surrounding knots.

$$p_i'(t_{i-1}) = \frac{y_i - y_{i-2}}{h_i + h_{i-1}} \tag{15}$$

Substituting this in:

$$p_i'(t_{i-1}) = \frac{y_i - y_{i-2}}{h_i + h_{i-1}} = z_i\left(\frac{1}{\tau_i \sinh(\tau_i h_i)} - \frac{1}{h_i \tau_i^2}\right) + z_{i-1}\left(\frac{1}{h_i \tau_i^2} - \frac{\cosh(\tau_i h_i)}{\tau_i \sinh(\tau_i h_i)}\right) + y_i \frac{1}{h_i} - y_{i-1}\frac{1}{h_i} \quad (16)$$

Rearranging:

$$0 = z_i\left(\frac{1}{\tau_i \sinh(\tau_i h_i)} - \frac{1}{h_i \tau_i^2}\right) + z_{i-1}\left(\frac{1}{h_i \tau_i^2} - \frac{\cosh(\tau_i h_i)}{\tau_i \sinh(\tau_i h_i)}\right) + y_i\left(\frac{1}{h_i} - \frac{1}{h_i + h_{i-1}}\right) - y_{i-1}\frac{1}{h_i} + y_{i-2}\frac{1}{h_i + h_{i-1}} \quad (17)$$

2.2.6 Matrix Form of Constraints

Start with forward price constraint as less lags (probably)

$$\alpha_i^j = \quad (18)$$

Superscript is for contract, supersci

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_0^1 & \beta_0^1 & \gamma_0^1 & \delta_0^1 & \alpha_1^1 & \beta_1^1 & \dots & 0 & 0 & \alpha_n^1 & \beta_n^1 & \gamma_n^1 & \delta_n^1 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_0^2 & \beta_0^2 & \gamma_0^2 & \delta_0^2 & \alpha_1^2 & \beta_1^2 & \dots & 0 & 0 & \alpha_n^2 & \beta_n^2 & \gamma_n^2 & \delta_n^2 \\
\vdots & & & & & & \ddots & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_0^n & \beta_0^n & \gamma_0^n & \delta_0^n & \alpha_1^n & \beta_1^n & \dots & 0 & 0 & \alpha_n^n & \beta_n^n & \gamma_n^n & \delta_n^n \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} z_0 \\ y_0 \\ z_1 \\ y_1 \\ \vdots \\ z_{n-1} \\ y_{n-1} \\ z_n \\ y_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ f_i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2.3 Smoothness Criteria

$$\begin{aligned}
&min \int_{t_0}^{t_n} \left(p''(t)^2 + \tau_i^2 p'(t)^2\right) dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} p_i''(t)^2 + \tau_i^2 p'(t)^2 dt \\
&= \sum_{i=1}^{n} \frac{(z_{i-1} - z_i)^2 - \tau_i^4 (y_{i-1} - y_i)^2 + \tau_i h_i \left((z_{i-1}^2 + z_i^2)\coth(-\tau_i h_i) - 2 z_{i-1} z_i \operatorname{csch}(-\tau_i h_i)\right)}{-\tau_i^2 h_i} \\
&= \sum_{i=1}^{n} \frac{\tau_i^4 (y_{i-1} - y_i)^2 - (z_{i-1} - z_i)^2 - \tau_i h_i \left((z_{i-1}^2 + z_i^2)\coth(-\tau_i h_i) - 2 z_{i-1} z_i \operatorname{csch}(-\tau_i h_i)\right)}{\tau_i^2 h_i}
\end{aligned} \quad (19)$$

The term inside the summation is a quadratic form, which can be seen from rearranging:

$$
\begin{aligned}
= \sum_{i=1}^{n} \Bigg(& -y_{i-1}^2 \frac{\tau_i^2}{t_{i-1} - t_i} + y_{i-1} y_i \frac{2\tau_i^2}{t_{i-1} - t_i} - y_i^2 \frac{\tau_i^2}{t_{i-1} - t_i} \\
& + z_{i-1}^2 \left(\frac{1}{\tau_i^2 (t_{i-1} - t_i)} - \frac{t_{i-1} \coth(\tau_i (t_{i-1} - t_i))}{\tau_i (t_{i-1} - t_i)} + \frac{t_i \coth(\tau_i (t_{i-1} - t_i))}{\tau_i (t_{i-1} - t_i)} \right) \\
& + z_{i-1} z_i \left(-\frac{2}{\tau_i^2 (t_{i-1} - t_i)} + \frac{2t_{i-1} \operatorname{csch}(\tau_i (t_{i-1} - t_i))}{\tau_i (t_{i-1} - t_i)} - \frac{2t_i \operatorname{csch}(\tau_i (t_{i-1} - t_i))}{\tau_i (t_{i-1} - t_i)} \right) \\
& + z_i^2 \left(\frac{1}{\tau_i^2 (t_{i-1} - t_i)} - \frac{t_{i-1} \coth(\tau_i (t_{i-1} - t_i))}{\tau_i (t_{i-1} - t_i)} + \frac{t_i \coth(\tau_i (t_{i-1} - t_i))}{\tau_i (t_{i-1} - t_i)} \right) \Bigg)
\end{aligned} \tag{20}
$$

Tidying TODO, rearrange for coth and csch of negative number

$$
\begin{aligned}
= \sum_{i=1}^{n} \Bigg(& y_{i-1}^2 \frac{\tau_i^2}{h_i} - y_{i-1} y_i \frac{2\tau_i^2}{h_i} + y_i^2 \frac{\tau_i^2}{h_i} \\
& + z_{i-1}^2 \left(-\frac{1}{\tau_i^2 h_i} - \frac{\coth(-\tau_i h_i)}{\tau_i} \right) \\
& + z_{i-1} z_i 2 \left(\frac{1}{\tau_i^2 h_i} + \frac{\operatorname{csch}(-\tau_i h_i)}{\tau_i} \right) \\
& + z_i^2 \left(-\frac{1}{\tau_i^2 h_i} - \frac{\coth(-\tau_i h_i)}{\tau_i} \right) \Bigg)
\end{aligned} \tag{21}
$$

Writing this in matrix form, as upper triangular, rather than the more typically seen symmetric to aid more efficient inversion later.

$$
= \sum_{i=1}^{n} \begin{bmatrix} z_{i-1} \\ y_{i-1} \\ z_i \\ y_i \end{bmatrix}^T \begin{bmatrix} \left(-\frac{1}{\tau_i^2 h_i} - \frac{\coth(-\tau_i h_i)}{\tau_i} \right) & 0 & 2 \left(\frac{1}{\tau_i^2 h_i} + \frac{\operatorname{csch}(-\tau_i h_i)}{\tau_i} \right) & 0 \\ 0 & \frac{\tau_i^2}{h_i} & 0 & -\frac{2\tau_i^2}{h_i} \\ 0 & 0 & \left(-\frac{1}{\tau_i^2 h_i} - \frac{\coth(-\tau_i h_i)}{\tau_i} \right) & 0 \\ 0 & 0 & 0 & \frac{\tau_i^2}{h_i} \end{bmatrix} \begin{bmatrix} z_{i-1} \\ y_{i-1} \\ z_i \\ y_i \end{bmatrix} \tag{22}
$$

Appendices

A Maximum Smoothness Integral

This section evaluated the integral used in the maximum smoothness criteria. First writing the squares as multiples, and splitting the integral into two:

$$\int_{t_{i-1}}^{t_i} p_i''(t)^2 + \tau_i^2 p_i'(t)^2 dt = \int_{t_{i-1}}^{t_i} p_i''(t)p_i''(t)dt + \int_{t_{i-1}}^{t_i} \tau_i^2 p_i'(t)p_i'(t)dt \tag{23}$$

The first integral on the right-hand-side can be evaluated using integration by parts.

$$\int_{t_{i-1}}^{t_i} p_i''(t)p_i''(t)dt = p_i''(t_i)p_i'(t_i) - p_i''(t_{i-1})p_i'(t_{i-1}) - \int_{t_{i-1}}^{t_i} p_i'''(t)p_i'(t_{i-1})dt \tag{24}$$

Substituing this into the first equation in this appendix and combining the last two integrals into a single integral.

$$\int_{t_{i-1}}^{t_i} p_i''(t)^2 + \tau_i^2 p'(t)^2 dt = p_i''(t_i)p_i'(t_i) - p_i''(t_{i-1})p_i'(t_{i-1}) - \int_{t_{i-1}}^{t_i} \left(p_i'''(t) - \tau_i^2 p_i'(t)\right)p_i'(t)dt \tag{25}$$

Next, focussing on the $p_i'''(t) - \tau_i^2 p_i'(t)$ term. It has previously been shown that the first derivative is as follows:

$$p_i'(t) = \frac{-z_{i-1}\cosh(\tau_i(t_i - t)) + z_i \cosh(\tau_i(t - t_{i-1}))}{\tau_i \sinh(\tau_i h_i)} + \frac{y_i - y_{i-1} + (z_{i-1} - z_i)/\tau_i^2}{h_i} \tag{26}$$

Differentiating:

$$p_i''(t) = \frac{z_{i-1}\sinh(\tau_i(t_i - t)) + z_i \sinh(\tau_i(t - t_{i-1}))}{\sinh(\tau_i h_i)} \tag{27}$$

Differentiating again:

$$p_i'''(t) = \frac{-z_{i-1}\tau_i \cosh(\tau_i(t_i - t)) + z_i \tau_i \cosh(\tau_i(t - t_{i-1}))}{\sinh(\tau_i h_i)} \tag{28}$$

Using the above it can be seen that:

$$p_i'''(t) - \tau_i^2 p_i'(t) = -\frac{(y_i - y_{i-1})\tau_i^2 + z_{i-1} - z_i}{h_i} \tag{29}$$

Crucially this is independent of t, hence can be used to simply the equation.

$$\begin{aligned}\int_{t_{i-1}}^{t_i} \big(p_i'''(t) - \tau_i^2 p_i'(t)\big) p_i'(t) dt &= -\frac{(y_i - y_{i-1})\tau_i^2 + z_{i-1} - z_i}{h_i} \int_{t_{i-1}}^{t_i} p_i'(t) dt \\ &= -\frac{(y_i - y_{i-1})\tau_i^2 + z_{i-1} - z_i}{h_i} \big(p_i(t_i) - p_i(t_{i-1})\big) \quad (30)\end{aligned}$$