An Analysis of Evolutionary Methodologies in the Minority Game

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1 Introduction

Many existing economic, legal and social theories are based on a substrate of rational actors with perfect information, commonly know as Rational Choice Theory (RCT) with its associated theory of rational expectations. [2] [16] The basis for the effective allocation of resources and property rights within the free-flowing dynamic of idealized pure capitalism is individual agents with perfect knowledge of their market environment who are therefore able to perfectly evaluate all potential market transactions, and select those maximizing their individual utility. [16] The latter assumption has been under scrutiny since inception, but Milton Friedman's 1953 interpretation,[1] that takes any agent's actions axiomatically to be in their interest, and thus to the extent one is able to exercise their will, their utility function is fulfilled. Recent advances in econophysics[2] have cast considerable skepticism upon the efficacy of internally consistent neoclassical RCT based economic theory [18], including a fairly thorough refutation of RCT when considered against the random null hypothesis. [2]
This concession to other facets of reality forces economics to deal with the limits of individual agent rationality (as arises from lack of information, inability to understand the complexity of the situation, and just lack of base rationality [1] [2]) and inherent flaws in the signals of aggregate individual preferences[3]. A model recognizing this issue was initially conceived of as the El Farol Bar Problem [1], where the concept of a limited memory length acting as the bound on agent rationality was implemented. In the same paper, William Brian Arthur introduced the minority condition, wherein all agents in a system take a binary vote, and the minority group is declared the winner, implying that there is no deductive[4], a priori solution for an individual agent, i.e. no agent can, through rational recourse, discover a best, or even an a priori *better* solution that will keep them above the group. Instead, inductive reasoning[5] is necessary, using information from the system of agents' collective action to implicitly coordinate and achieve a dynamic equilibrium with a higher overall success rate. The resultant Minority Game attempts to quantify the inevitable irrationality of agents with limited information via finite sum win state[6] scenarios with no steady state solutions.
This paper's primary contribution to the literature on the Minority Game[7] is a more nuanced analysis of the evolutionary methodology that allows for the variation in overall distribution of agent memory lengths and population than originally seen in the original 1997 paper[7]. We employ a range of evolutionary methods, incorporating different reward/penalty regimes, initial distributions, and evolutionary updates. All results are thereafter compared to their static (without evolutionary methods) counterparts, and analyzed with respect to the existing literature.

[1]As synthesized by [14] thusly "According to this argument, it is not the case that actual decision makers consciously maximize their utility function when choosing. Instead, it is the economist who rationalizes the decision makers choices as if they were generated by utility maximization"

[2]A term coined in 1995 by Gene Stanley describing the growing interest and work resulting from the application of statistical physics principles to the field of economics, though the field arguably began with the 1987 Santa Fe conference.

[3]Impossibility of aggregating rational preferences without breaking one of Arrows axioms (those of rationality, non-dictatorship, unrestricted domain, independence of irrelevant alternatives and Pareto efficiency) is summed in Arrows impossibility theorem, and helps desmonstrate that collectively rational action can be irrational without any individual irrationality.

[4]Wikipedia offers the following verbose explanation "in deductive reasoning, a conclusion is reached reductively by applying general rules which hold over the entirety of a closed domain of discourse, narrowing the range under consideration until only the conclusion(s) is left".

[5]As described in [13] summary paper, "Inductive reasoning assumes that by feeding back the information about the game outcome, agents could eventually reach perfect knowledge about the game and arrive to a steady state."

[6]Finite win state solutions are those wherein there is some finite amount of value to be distributed, but competition does not necessarily deny other participants winnings.

[7]Aside from the associated C++ implementation and documentation, made accessible for anybody so inclined to create their own MG variations

2 Minority Game Mechanics

The minority game achieves its dynamic equilibrium through the minority condition, and bounds to rationality via agent memory length. While the original idea for a model of this form was initially conceived as the El Farol Bar Problem by William Author Brian in his 1994 Paper [1], and formalized into its first computationally testable form in 1997 by Challet and Zhang [7], the model has since been continually varied in order to investigate different aspects of the model's emergent behavior, most notably by Zhang and Challet [17] (strategy dependence, 1998, and market prediction game, 2009) and by Andrea Cavagna et al. into the thermal minority game (Thermal Minority Game, 1999).[5] While the implications and variations from these works are discussed in section 6, the following should serve as a comprehensive introduction to the Minority Game's mechanics, as background to this paper or any other based on the original 1997 paper by Zhang and Challet. Analysis of the minority game's results, along with implications of other sections, is provided in section 3.
The Minority Game consists of a set of agents and strategies, wherein agents, given a subset of the overall strategy set, make a binary determination using their highest performing strategy, as determined by the strategy's ability to predict the resultant minority each round. This dynamic can be understood through the initial analogy of going to a bar; if too many people go, the bar is too crowded, and no one enjoys their time there, but the uncrowded bar is enjoyable. If we assume *too many people* is half of the population, and by considering not being in a crowded bar to be a positive gain, we see that at every step, the minority group "wins". Agents then choose, based on their memory (i.e. m number of the last nights) of which side won, whether or not to "**go to the bar**" or not. Adopting the notation assumed in Esteban Morro's summary paper: [13] For a given population of agents, N $\in$ odd $\mathbb{N}$, each agent has access to s $\in \mathbb{N}$, strategies, which are mappings from m $\in \mathbb{N}$ elements of the binary history to a binary response. i.e. $\forall i \in N$, the set of agents, $\exists s \in \mathbb{N}$ strategies, which are mappings $(0,1)^m \rightarrow (0,1)$, with $m \in \mathbb{N}$. The implementation of this processes may be described as follows: (complete picture given in figure 4)

N agents are given a random selection of s strategies, where each strategy is a complete set of permutations of m binary elements, —encompassing all possible binary histories of length m—. The value m is construed to be the memory of the agent, as it reflects how far back into the binary history agents can see, and thus base their strategies on. For example, if an agent has a memory length of 3, we might naively consider a set of strategies, along the lines of arithmetic operations, or simple assembly instructions, [1] i.e. for a given history, one might;

- Choose 1 if the last minority group was one
  This might correspond to a "use the last winning response", or alternatively "because others suspect that the population will choose another anticipatory response, I will stay with this one" this corresponds to the "steady" mindset, as a defining element game in similar congestion experiments.[15]
- Choose 0 if the last minority group was one
  This could correspond to the "I'll shift, because I think the majority will continue to use the last winning answer" mentality, corresponding to the "contrarian" mindset detailed in William Kets's analysis. [9]
- Choose 1/0 if the average of the history of minority groups is less than 0.5
  Or other arithmetic methods
- Choose 1/0 if the last 3 minority groups were all one
  Or other examples of simple, random strategies that make their determination based on an arbitrary configuration of history.

One might initially suspect that via some mental machination, one could conceive of an a priori set of strategies that would allow for better performance than random selection. However, Challet and Zhang

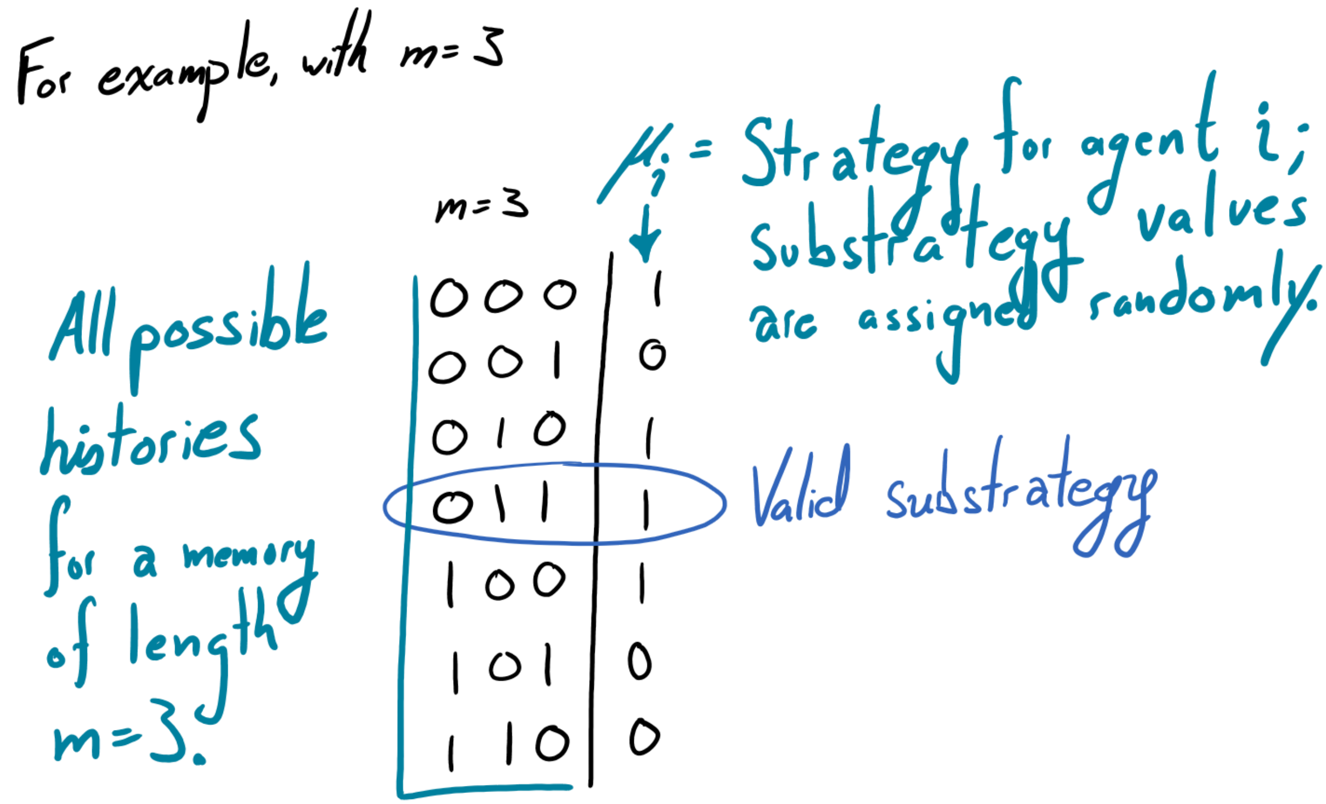


Figure 1: Strategy sub-strategy Diagram

[7] show, there is no strategy that can ensure a better performance than random choice, as the result is entirely dependent upon the minority condition, which simultaneously rules that any successful strategy will be used by a greater number of agents, thus invalidating its initial efficacy [18]. This principle is more simply expressed in that a strategy's success dooms it to failure by its subsequent popularity. Due to the inherent anti-correlation between usage and success of a strategy, this cycle of Success → Popularity → Failure → Rejection → Success → etc. is a defining feature of the minority condition, which governs that behavior of the minority game. This result is only valid if there are enough agents to overlap in strategies, such that all strategies will see an increase in usage in response to their success, undermining their previous efficacy in predicting the minority. It therefore follows, as subsequent research has shown [4] (as well as results in section 5), that the performance (as measured via overall success rate, and variance over population, $\frac{\sigma^2}{N}$ of the unaltered Minority Game) is purely dependent on the critical ratio of agents to unique strategies,

$$\alpha = \frac{2^m}{N} \tag{1}$$

Given that the minority condition ensures the equality of all possible strategies a priori, agents are assigned strategies at random from the pool of all possible strategies. Agents then all vote[8] based on their best strategy's prediction, and then the ratings of the strategies are updated according to a function g(A(t)) ∝ sign(A(t)), where A(t) is the sum of all votes, wherein each binary vote is defined as a ±1, rather than 0 or 1. A(t) is therefore given as

$$A(t) = \sum_{i=0}^{N} a_i(t) \tag{2}$$

[8]or go to a bar, buy a stock, etc. there are as many interpretations of the binary choice as implications of the model, but our operative verb shall remain vote throughout this paper

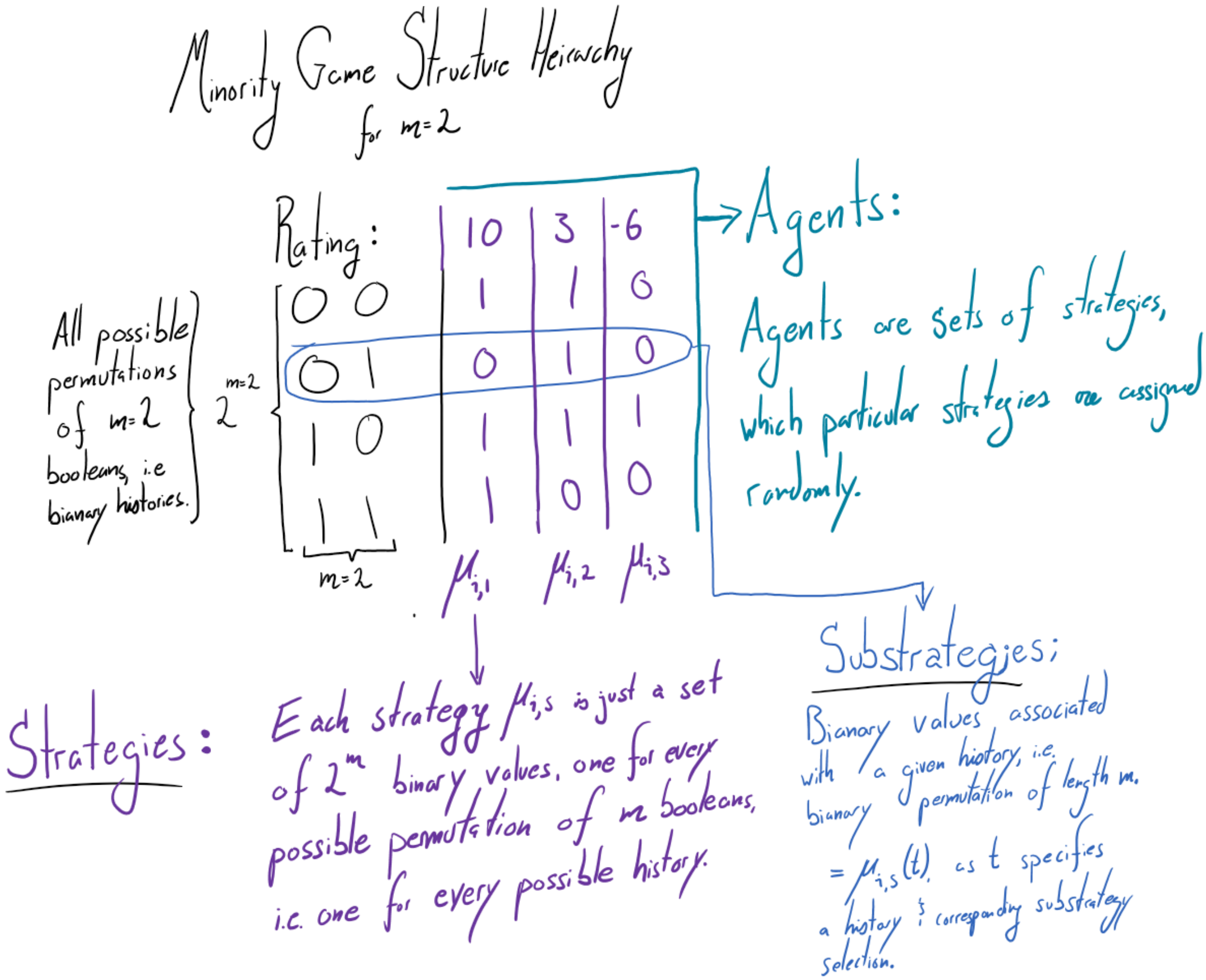


Figure 2: Hierarchy of Agents, Strategies and Sub-strategies

the resulting sign therefore indicates which binary voter block won, and as N is always odd, there are no possible 0 cases. There are a variety of commonly used rating functions, including g(A(t)) = A(t) and the more interpretable g(A(t)) = $\frac{A(t)}{N}$, [13] though these arrangements do not perform qualitatively differently from g(A(t)) = sign(A(t)). All strategies' associated scores are updated (by simply adding the value of $g(A(t_{current}))$ to it's running total) after every round regardless of whether or not they were employed by an agent in that round. [9] The qualitative macro-characteristics are independent of s for $S \geq 2$ [13], though the exact variation in results for varying s are given in section 5. Even when the minority threshold (the fraction of agents that can be in the minority round) is shifted from $\frac{N}{2}$ to $\frac{N}{x}, x \in \{0, N-1\} \in \mathbb{R}$, the Minority Game converges about this new fraction considered the minority.[1] The binary winning group is then recorded by all agents as an update in their running knowledge of the

[9] This basic mechanism is well described in A Cavagna: "A strategy is a choosing device, that is an object that processes the outcomes of the winning room in the last m time steps (each outcome being 0 or 1) and accordingly to this information prescribes in what room to go the next step. The so-called memory m defines 2^m potential past histories (for instance, with $m = 2$ there are four possible pasts, 11, 10, 01 and 00). A strategy is thus formally a vector R_μ, with $\mu = 1, ..., 2m$, whose elements can be 0 or 1. The space Γ of the strategies is an hypercube of dimension $D = 2^m$ and the total number of strategies is 2D"[4]

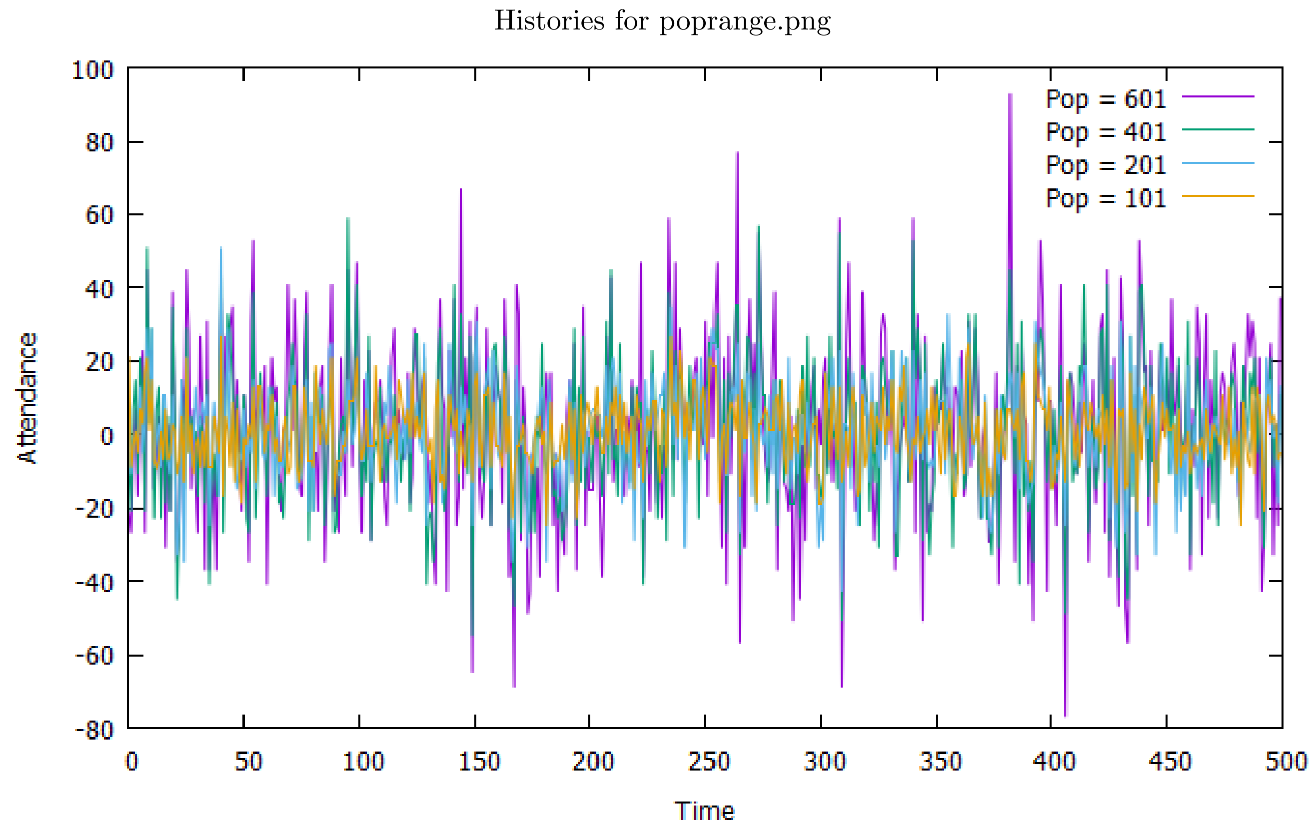


Figure 3: Time evolution of the attendance for the original MG with g(x) = g and N = 101 to 701 and s = 2. Periodic patterns can be observed for lower m values, and the overall variance is proportional to the agent population

last m binary results. In the next round, agents will now make use of sub-strategies consistent with the new history, where each sub-strategy is a unique permutation of m boolean elements (a binary history of length m) and a corresponding binary output. All strategies have exactly 2^m sub-strategies, so as to ensure a response for any given binary sequence given by the Minority Game's resultant history. That is, $\forall s \in S, \exists 2^m$ sub-strategies, which are binary responses associated with a unique permutation of m binary history elements. As each sub-strategy gives a binary response, this results in a factor of 2 in $2 \cdot 2^m$ unique sub-strategies, and another exponential factor in the resultant 2^{2^m} number of unique strategies, i.e. permutations of 2^m sub-strategies. [10]

This hierarchy between agents, strategies and sub-strategies is depicted in figure 2. As in [13], a given strategy (μ_i) may be conceived of as a set of 2^m booleans (± 1 when under computational consideration), where each boolean is, by its position in the set, associated with a given permutation of the binary history. The prediction of that strategy is therefore given as $\mu_i(t) \in \{1, ..., 2^m\}$. It may be noted that for any given strategy, the responses for different values of the binary history are not correlated, as the strategies themselves are generated randomly as the minority condition ensures that there is no possible "optimum" relation, or arrangement of sub-strategies in a given strategy.

[10]This scheme of maintaining a running rating of strategies creates a bias for unused strategies, as in the margin case, i.e. if we imagine all other agents have already voted, any used strategy will put the remaining agent in the majority, and thus leading to a resultant devaluation in the used strategy, and appreciation of any strategy that would have predicted differently, even though were it used, it too would tip the scales and lead to a majority. However, as these only occur on the margin (i.e. when the majority has but one more member than the minority) we estimate the effect of this bias to be proportional to the number of margin cases to the total, which we've found to be —insert empirical result average here— of the total) [10]

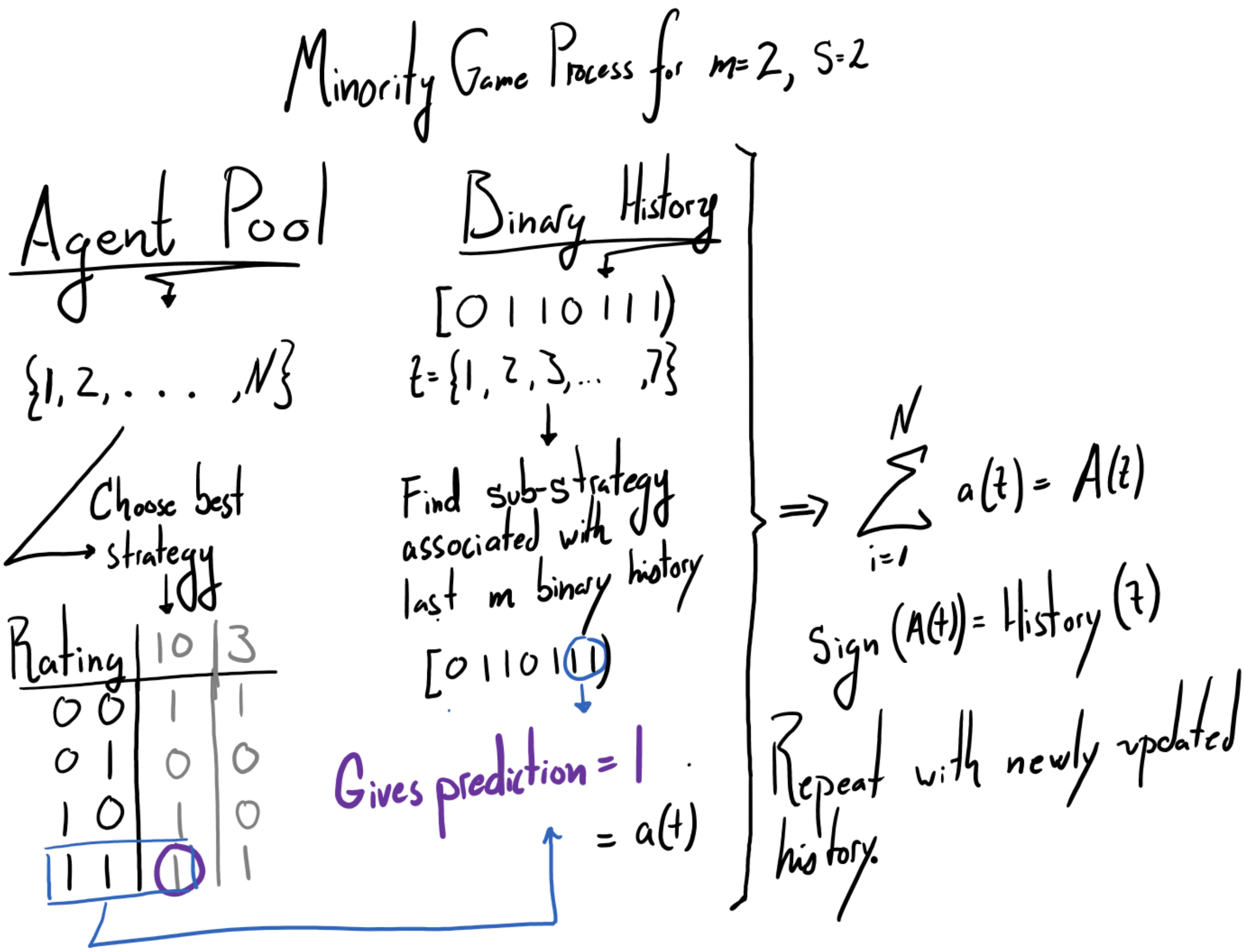


Figure 4: Running Dynamic

A single iteration of the Minority Game therefore proceeds thusly;
All agents vote by finding the binary response in their best performing strategy that corresponds to the last m results from voting, which is seen by all agents and considered a public binary history of length m. The sum of every agent's best strategy's response (given the present history and corresponding sub-strategy) is then summed, and that determines which group (the +1 or -1 group)[11] was in the minority. (i.e. if $A(t) > 0$, then the -1 group is the resultant minority). The history is then updated, with the winning value added as the new beginning of the history sequence, and the process is repeated. This whole cycle is illustrated in figure 4. For longer run time averages, characteristic features of success rates, variance, entropy and information purely dependent on $\alpha^{12} = \frac{2^m}{N}$ are observed.

[11] ± 1 is used as the boolean choice, as it is both compatible with the computational implementation and the analysis above, though in theory any boolean could be configured to work

[12] α could be alternatively defined as $\alpha = \frac{2 \cdot 2^m}{N}$, as $2 \cdot 2^m$ is the number of unique strategy combinations, and revealed under scrutiny to be a more relevant relation. However, as convention among the literature is to use the more simple $\frac{2 \cdot 2^m}{N}$ we shall likewise adopt this notation.

3 Lessons from Minority Game literature

3.1 Minority Game Analysis

The Minority Game's behavior can be generally understood via the afore mentioned ratio of possible strategies (a strategy space) to agent population. While a more analytically conducive construction of strategy space is considered in [5], we will here consider the default construction, as most of the existing literature, as well as this paper's subsequent data, uses the original formulation proposed in [7]. As shown in figure 5, the data generally falls into three separate efficiency regimes that can be analyzed via variance over agent population in relation to the number of unique strategies per agent, as greater variance implies that there was greater fluctuation about the minority comfort level[13] and consequently a lower overall success rate. The society is therefore attempting to minimize variance to get as close to the simple rotating exchange of $\frac{N-1}{2}$ being in the minority each time, and therefore only have an attendance variance of 1 over the course of the long run time dynamic equilibrium.

The three regimes can be seen in figure 5 below, wherein before $\alpha \to \alpha_{critical}$ there are far too few strategies for the number of agents, and therefore agents are left with no choice but to overlap in their decisions, thus are unable to form an effective minority. For $\alpha \approx \alpha_{critical}$ the system enters its most efficient stage, as there is a large enough strategy space to allow the agents to use different strategies (which would lead to a more random response rate), while also pairing off as anti-correlated strategies which perform as perfect opposites[14] and therefore perform better than randomness in attempting to *do the opposite*, as is the optimal behavior in the Minority Game.

The combination of variances from randomness and better than randomness (from anti-correlated strategy pairing) averages to a better than random variance regime, and a corresponding optimized success rate, as seen in the inset of figure 5. The third regime, where $\alpha \to \infty$, performs as random, given that agents strategies are no longer remotely correlated, and all agents have their own unique set of strategies. This randomness does perform much better than overlapping strategies for reasons that follow, but is outperformed by the subtle agent coordination with anti-correlated agents around the $\alpha_{critical}$ regime.

Considering that for every agent, there exists a $(1-\frac{1}{2^s})$ chance of having at least one strategy whose response is opposite the others in their agent strategy set[15], the proportion of sub-strategy responses in a strategy with an alternate binary response in a given agent's strategy set, will converge to

$$\lim_{m\to\infty} \frac{2^m(1-\frac{1}{2^s})}{2^m} = 1 - \frac{1}{2^s}$$

where the increasing sampling in 2^m increases the accuracy of the convergence.

This leads to greater differentiation between the strategies in each agent's strategy set, as well as for all agents universally, which allows agents to make use of their different strategies. As every strategy combination has a $\frac{1}{2\cdot 2^m}$ chance of existence, there is a corresponding $\frac{N\cdot S}{2^{m+1}}$ chance that each strategy will have an entirely anti-correlated pair in the pool of used strategies. Finally subtracting the chance that a given strategy and it's anti-correlated pair will be in the same agent strategy set, and thus inaccessible, we get $\frac{N\cdot S}{2^{m+1}} - \frac{N\cdot S}{N^2} = \frac{N\cdot S}{2^{m+1}} - \frac{S}{N}$. Ergo as m increases, the number of different strategies increases along with their sub-strategy responses, allowing agents to both effectively distinguish between their strategies,and spread out along strategy space, so that fewer agents are forced to perform similar actions for a greater percentage of possible histories.

This process can be visualized initially via a geometric means, though the initial space of all possible strategy configurations in 4D does make it difficult. A visualization of strategy space as points in a

[13] This notion of variance implies that all fluctuations on one side of the minority comfort level have roughly equivalent fluctuations on the other side, which in turn validates variance as a measure for inefficiency. This idea is validated analytically by the principle of the minority condition, and computationally in **??**

[14] Anti-correlated strategies are those that have the exact opposite responses for every sub strategy

[15] Further consider that the number of sub-strategies per strategy is increased exponentially as 2^m

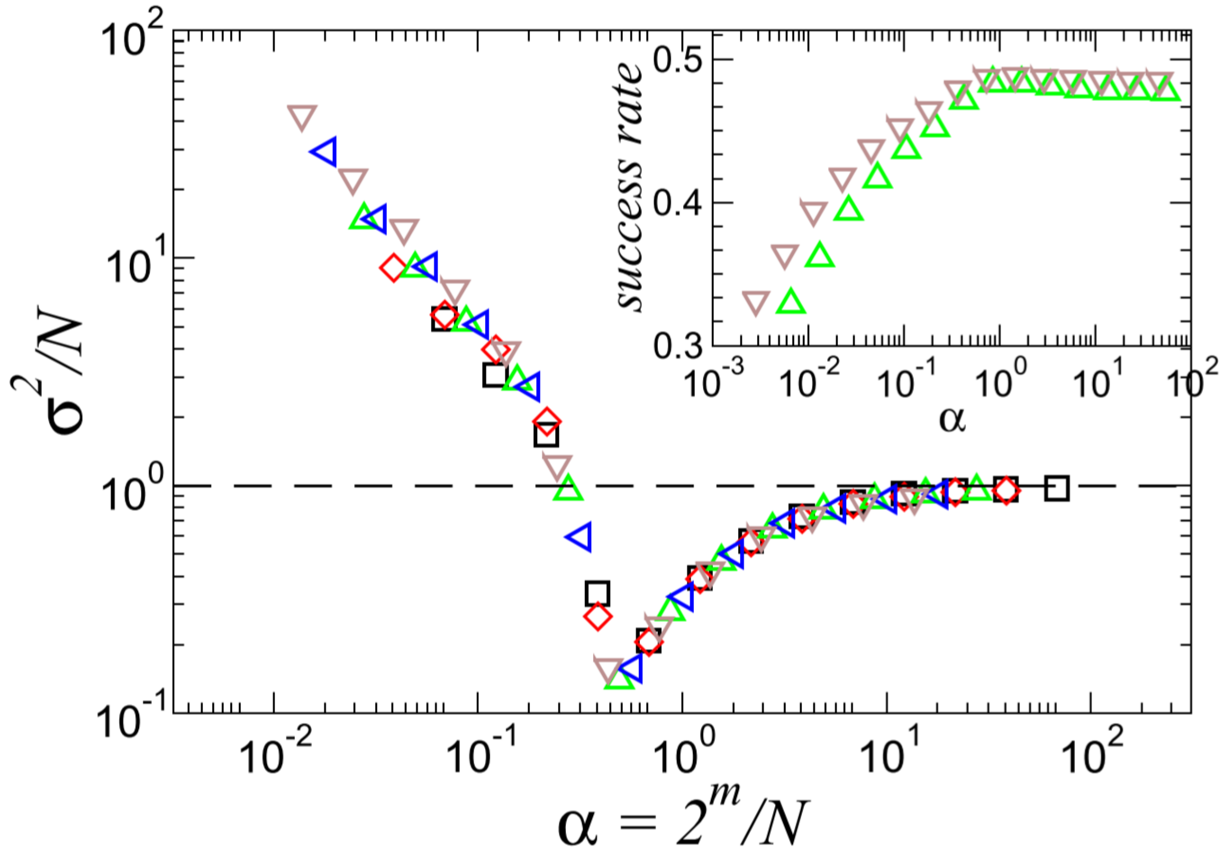


Figure 5: Volatility as a function of the control parameter $\alpha = \frac{2^m}{N}$ for s = 2 and different number of agents N = 101, 201, 301, 501, 701 ($\square, \ldots\diamond$ respectively). Inset: Agents mean success rate as function of α, taken from [13]

higher-dimensional space is attempted in figure 9. The resultant behavior following from this relation of increasing strategy space is rich, and corresponding analogies stretch into many academic realms.
The Minority game has a few notable results fundamentally distinguishing it from other dynamical systems:

- The *minority rule* means that there is never any steady state solution (lest, locally) and thus the entire system is continually adapting and optimizing.
- The heterogeneity of the agents, i.e. that they can use other strategies.
- Dependence on rational capacity as defined by m.
- The independence of agent coordination from actual history, and subsequent exclusive dependence on shared information, regardless of it's veracity.

This helps in dealing with the most conventional problem that rational actor models face: that agents have at best rather limited rationality. The limits of rationality arise in several different ways:

- A lack of information: about any aspect of the given model (how many members there are competing, what other agent's natural logic is, what the existing results were, the limits of agent memory)
- A base lack of rational action, even supplied with the relevant information (as results in the real world from simple stupidity, or, much more commonly, other factors external to the system overall)

- An inability to understand the complexity of the situation (even if given complete information)

Further, Moro [13] elaborates that inductive reasoning implies that by continually remembering the results of the cycle of expectation $\rightarrow$ action $\rightarrow$ precedent $\rightarrow$ expectation, agents could perfectly deduce their optimal state set, and act accordingly, leading to a steady state. However, deductive reasoning, that which is much more familiar to students of economics and other similar disciplines, assumes that the entirety of the relevant knowledge is contained in the immediate precedent to the action, and thus the full optimized action is immediately resolved and performed. However, given the minority condition, it can easily be surmised that there is no optimal deductive solution to this problem, as given only the most recent days attendance as your memory, any number of possible strategies might work.

3.1.1 Irrelevance of the veracity of memory

A. Cavanga et al. point out in [4] that by ignoring the actual history, and instead generating a new, random history every time for the agents to select a different sub-strategy with, the same overall characteristic behavior is obtained. This reduces the complexity of the minority game considerably, from a complex deterministic system to a simple stochastic one. This also reduces the technical complexity of the model, and allows for longer run times on common consumer grade computational hardware. This is especially relevant for evolutionary models, when longer run times are necessary to determine more illusory long-term behavior and equilibrium states. [7]
All following results in section 5 are nonetheless simulated via the actual history for conceptual compatibility with the existing literature. Comparisons with fictional histories will be shown where relevant, and evolutionary behavior will make use of the simplified stochastic model to determine longer term evolutionary behavior. Note that [4] does not refute the importance of memory length, only that its importance lies in determining the dimension of strategy space, 2^m.

3.2 Evolutionary Minority Games

There has been a number of attempts to introduce an evolutionary component to the Minority Game, each revealing their own dynamics and alternate formulations of the Minority Game and its evolutionary nature.

3.2.1 Darwinistic Evolution

In the original 1997 paper, D. Challet and Y.-C. Zhang [7] introduce an evolutionary methodology that deletes poorly performing agents, and spawns new agents with similar (with some sub-strategies altered, as per slight analogous genetic modification) strategy sets with blank strategy scores[16]. This model allows for continual competition between the better performing agents, and thus the characteristic "rich get richer, poor get poorer" behavior seen in the basic model is undermined. Over long run times, this evolutionary adaptation allows for agents to come to a better overall performance, with tighter competition between the best agents leading the overall agent population to come closer to the $\lim t \rightarrow \infty A(t) \rightarrow \frac{N-1}{2}$ limit. While the methodology is as described above, the precise details of this Darwinistic update algorithm are not in the original paper, but characteristics such as optimal rate and method of adaptation will be discussed via results in section 6.
D. Challet et al. also investigate the influence of "inbreeding", or direct cloning of the agents, which naturally lead to far worse outcomes than the slight modifications to each generation.[17] This effect is especially prevalent in the realistic regimes of our population based evolutionary model, as discussed in

[16] This important feature that the children do not inherit their parent's esteem for certain strategy scores allows for the adaption and coordination of the parents and their children, rather than forcing agents to congest with similar esteem, retaining their parents biases.
[17] As is intended to mimic genetic variation via vertical generational shifts

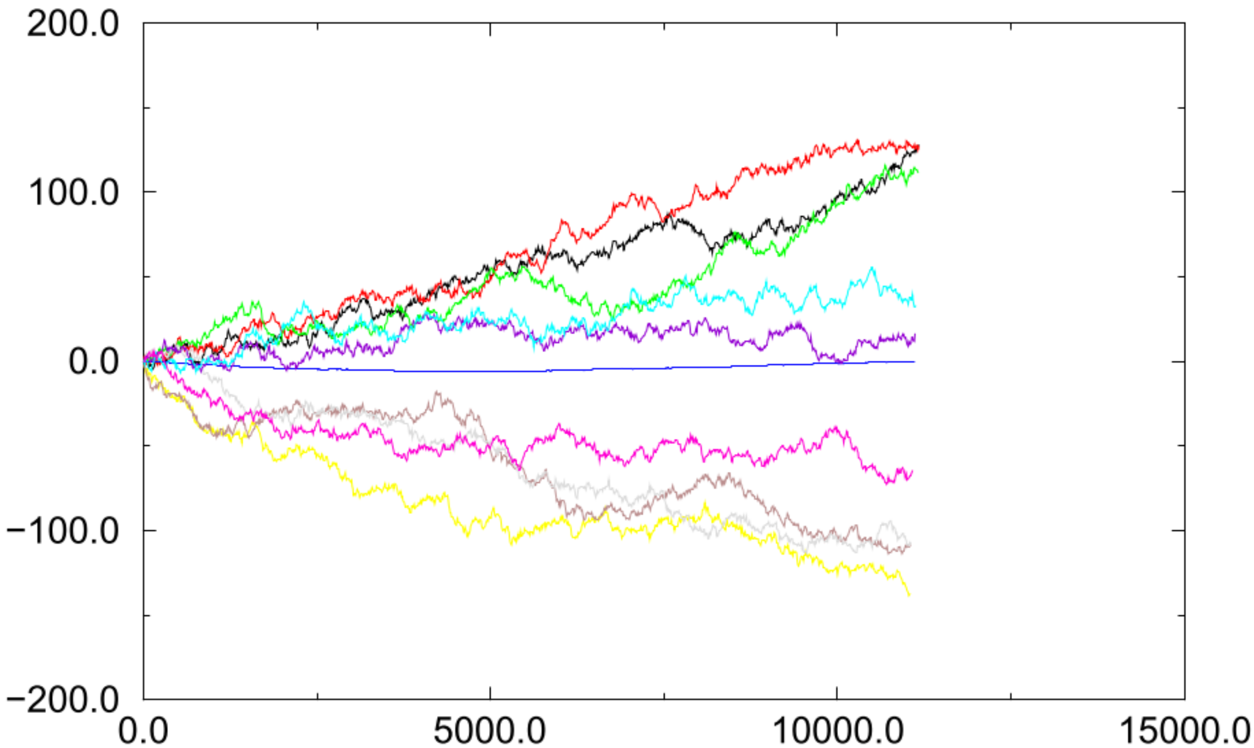


Figure 6: Performance record of the 3 best, the 3 worse and 3 randomly chosen players (N = 1001, M = 10, S = 5), taken from [7]

section 5.2.3. Adding to the existing Darwinian agent evolution, D. Challet et al. add a memory modification methodology that allows for cloned strategies to gain or lose memory (and thus, presumably also lose all connection to their previous strategies), with some small probability. Quoting their corresponding conclusion:

> The temporal record shows that there is an arm race among the players. We know by now that the more brain power leads to advantage, so in the evolution of survival-of-the-fittest the players develop bigger brains to cope with ever aggressive fellow-players. However such an evolution appear to saturate and the arm race to settle at a given level. The saturation values are not universal, having to do with the time intervals of reproduction. In general the larger brains need longer time to learn. Larger population (N = 1001) needs more powerful brains to sustain the apparent equilibrium than the smaller population (N = 101), also the learning rate is smaller.

[7] This naturally leads to qualitative questions which we strive to answer in the following discussion.

3.2.2 Thermal Minority Game

Andrea Cavagna et al. [5] made a minor modification to the Minority Game to allow for a variable probability that the response of an agent will actually be opposite its best strategy prediction. For every strategy an agent possesses, it therefore also possesses its anti-correlated strategy, and as the probability of rejecting the *best choice* increases the number of alternative strategies available to each agent becomes increasingly irrelevant. [11] The probability $\Pi_i^a(t)$ that any given agent's strategy will be used for a given

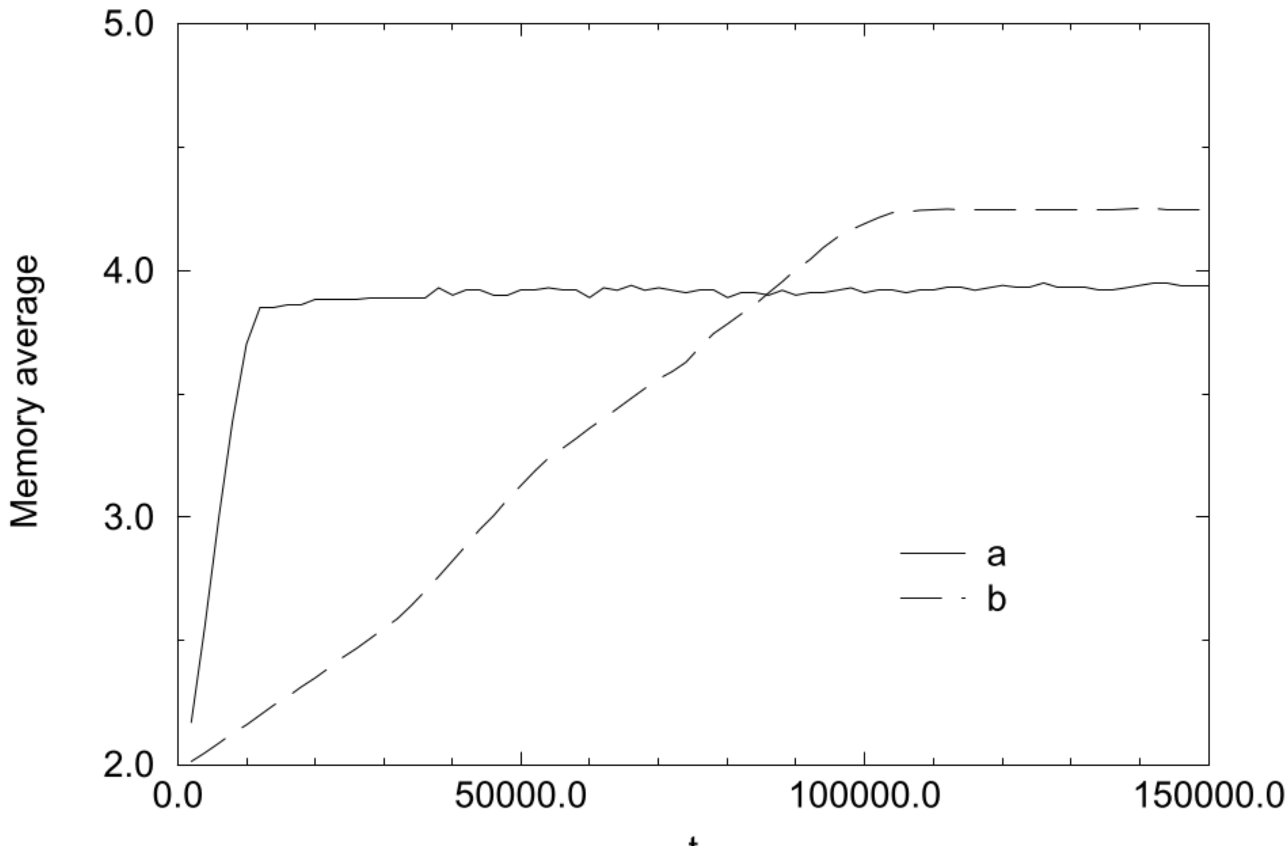


Figure 7: Temporal record of the memory average of a starting from M = 2 population for N = 101 (a) and N = 1001 (b) (S = 5), taken from [7]

moment in time is given by:

$$\Pi_i^s(t) \equiv \frac{e^{\beta P(R_i^s,t)}}{\sum\limits_{s=1}^{S} e^{\beta P(R_i^s,t)}} \tag{3}$$

Where $P(R_i^s(t),t)$[18] is the score (or number of points as initially conceived) of the i^{th} agent's $s(t) \in S$ strategy set, and $\beta \equiv \frac{1}{T}$, where $T = 0$ means agents play pure strategies (always choosing their *best*), and for $T > 0$ is proportional to the chance that the strategy will be *overridden* and another, random strategy will be used.
The results investigate the behavior of variance[19]

$$\sigma^2 = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t_{max}} dt' A(t')^2 \tag{4}$$

and success rates, for intermediate variables that are defined with reference to the evolutionary methodology itself. The behavior here is critically dependent on T; as T increases, the difference between strategy scores for each agent is likely to shrink, and consequently agents will more readily fluctuate between a conglomerate of best strategies and anti-strategies. This process of equalization is induced by the probability that any agent selecting their best strategy will instead lead the use of another strategy. The rating

[18] we deviate slightly from the original notation found in [5] to retain consistency with our standing notation, notably $a(t) \to s(t)$

[19] Though equivalent to the above definition, the above integral in computed explicitly over the discrete set of results via equation 6

methodology is adapted to a continuous bidding, allowing strategies to wager based on their confidence. However, all strategies are still rated on what they would have predicted were they not overridden by chance, and so the general mechanism for the original minority game is only refined.[5] After agent strategies converge, the agent will "do the best it can" by finding the best equilibrium point between using the strategies' predictions and that of the random chance to override it. This communal depression in the predictive capacity of the agents lead to eventual irrelevance of memory lengths, as coordination between each strategy and anti strategy yields identical results in the limiting case $\lim T \to \infty$.

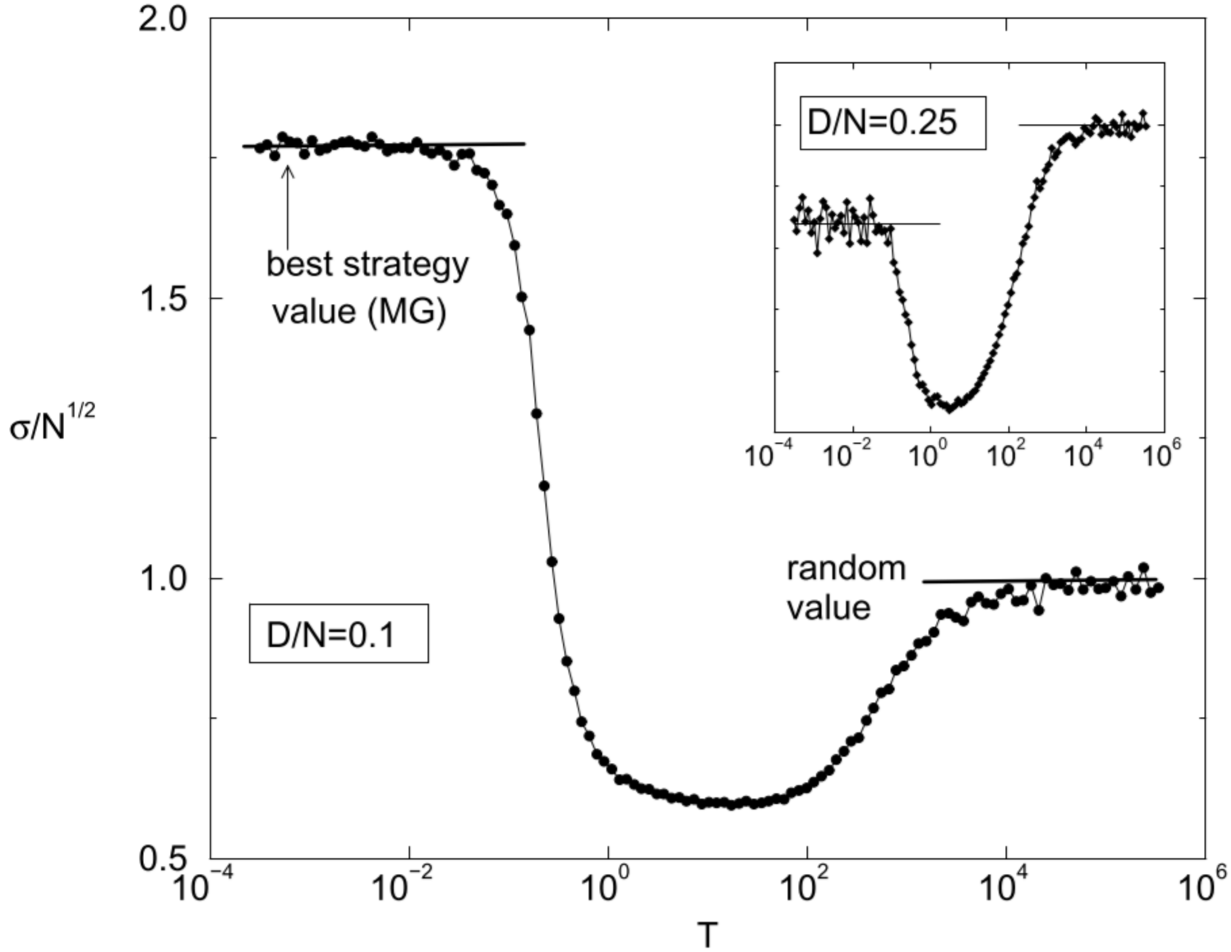


Figure 8: TMG: The scaled variance $\frac{\sigma}{N}$ as a function of the temperature T, at $\frac{2^m}{N} = 0.1$, for s = 2. In the inset we show $\frac{\sigma(T)}{N}$ for $\frac{D}{N} = 0.25$, taken from [5]

3.3 Economic Implications

As mentioned in the introduction, the last thirty years has seen increased scrutiny of internally consistent deductively based economic models, largely through the use of stochastic, highly populated interactive agent models. [13] [9] [14] [2] [17] The Minority Game, though initially conceived of as a deterministic model [1] whose central condition forced non-steady state solutions was itself eventually proved to show statistical parity with a stochastic model when disregarding the actual history that agents produced, and instead utilizing a randomly generated basis of information. [4] Regardless of the precise mechanism behind the agent ensemble, these kind of collective agent models in combination with statistical mechanics analogies and other methodologies familiar to physicists (e.g. kinetic theory of gases, information theory,

percolation models, phase shifts, etc.) have cast particular doubt on rational choice theory as a pillar for neoclassical economics.
A tangential, though particularly well illustrated example of this kind of analysis is highlighted by Moscati et al. in [14], wherein RTC is tested against the most basic economic axioms, through both iterative agent models (including the minority game), randomness, and empirical results, with disappointing results. Specifically, RCT performs worse or at parity with the null hypothesis of randomness for all but one case, leading to two suggested considerations:

- It is possible that equilibrium states take far too long to manifest [20]
- Complex systems (as our economy is proving to show increasing similarity to) may show periodic sensitivity to outside information, with relatively low sensitivity for longer periods where endogenous information determines market confidence, until efficient market correction thresholds are greatly exceeded, after which mild exogenous information leads to market over correction.[21]

As William Brian Author [1] stated

> ...beyond a certain complicatedness [sic], our logical apparatus ceases to cope our rationality is bounded. The other [reason reason is bounded] is that ...agents can not rely upon the other agents they are dealing with to behave under perfect rationality, and so they are forced to guess their behavior. This lands them in a world of subjective beliefs, and subjective beliefs about subjective beliefs. Objective, well-defined, shared assumptions then cease to apply. In turn, rational, deductive reasoningderiving a conclusion by perfect logical processes from well-defined premises itself cannot apply.

This limited irrational behavior is characterized by the Minority Game insofar as the dynamic equilibrium requires constant fluctuations in attendance, and the "rationality" is quantifiable and adjustable. [6] Viewed from this initial perspective as an economic model, several important economic insights from the Minority Game can be gained;

1. The overall behavior of the model is dependent only on the critical ratio $\alpha = \frac{2^m}{N}$ of possible independent agent actions and agent population[4].[22]
2. Despite all actions being of a priori equivalent value at the beginning of the simulation, agents will still sort into systematic winners or losers. [7]
3. The veracity of shared information is irrelevant to the performance of the model, so long as it's the same information that is shared by all agents, the same macrolevel behaviors are observed. [4]
4. Emergent cooperative behavior allows for coordination in the environment where the *society* of agents collectively have sufficiently diverse set of strategies so as to be able to spread out along the strategy space, 2^m which allows for better than random performance.

[20] Quoting Bouchard [2] directly on the real world relevance of this idea, "The convergence to the Eden Garden of economic systems might not be hobbled by regulations but by their tug-induced complexity. One can in fact imagine situations where regulation could nudge free, competitive markets closer to an efficient state, which they would never reach otherwise."

[21] This conclusion is drawn via a particular emphasis on a Random Field Ising Model (RFIM) game, wherein agents have a "herding tendency" where their decision to buy or sell is otherwise determined by exogenous information. However, this herding tendency can lead investors in to invest more than they would were they using only the exogenous information, and thus a market momentum that leads to "hysteria" and thus corresponding over correction, as "In order to trigger the crash, global factors have to degrade far beyond the point where pessimism should prevail." [2]

[22] Note that the same paper, [4] notes that the relevant critical α is not $\alpha = \frac{2 \cdot 2^m}{N \cdot S}$ as once suspected by Challet et al. in [6].

5. As a corollary to the above, the brain size, as measured via the rough analogy of a discrete memory[23] allows for greater coordination up to the saturation point (where there are nearly as many independent strategies as agents) using different strategies to coordinate their actions. [7]

6. So long as agents have two possible choices, dynamics are roughly similar.[24]

While these results alone are promising, by adding evolutionary methods, the literature has thus far been able to further conclude the following:

1. When agents are allowed to reject their best strategy (even at random) the value of greater brain/memory size decreases rapidly, as agents will coordinate by using their strategy and anti-strategies. [5] [11]

2. Evolution with direct cloning leads to overcrowding of existing strategies, and subsequent loss of efficiency. [7] This might be considered in the context of overcrowded investment strategies losing out on dividends from real valued capital returns.[25]

3. A Darwinian evolution methodology that allows for slight genetic variation brings out greater market efficacy than default randomness, as here analogous investors are considering similar evaluations, i.e. have market consensus on value but are still diversifying their investment across the board. [7]

4. By further allowing development of rational capacity (or lest information taken in) on top of a Darwinian evolutionary model, the system finds the optimum solution, being both able to compete with a similar market outlook, diversify and have a sufficient distribution of memory to allow the "low hanging fruit" to be taken up by agents with more limited rational capacity.[26] [7]

4 Adaptive Minority Game

Our adaptation of the Minority Game (dubbed Adaptive Minority Game, or AMG) is largely an expansion and detailed analysis of the evolutionary original 1997 work by D. Challet and Y.-C. Zhang, [7] wherein in the latter quarter of the article, Challet et al. discuss a basic Darwinian adaptive evolutionary methodology and its corresponding results. Their methodology, as described in 3.2.1, allows for cloning, strategy mutation, and finally memory mutation over time, and produces consistently improving results. (Though direct cloning leads to worse strategy outcomes)

4.1 Methodology of Adaptive Memory Minority Games

The first modification to the Minority Game as outlined above is simply reviewing a range of results via the stochastic adaptation, i.e. ensuring that the Adaptive Minority Game could test all its evolutionary methodologies given a falsified history[27]. The subsequent evolutionary methodologies attempt to explain the existing characteristic phenomena by providing observable differences under various adaptive regimes over time, and subsequently examining the systematic changes that arise in known phenomena. An explicitly detailed outline of the methodology in addition to the code itself can be found in the associated repository, [3].0

[23] Though a method using continuous memory by exploiting the stochastic nature of the model was used in [12] to evaluate the minimum with greater accuracy.

[24] The results for an adaptive number of strategies per agent is developed and discussed in 5, but generally the greater number of strategies per agent, the worse each agent performs.

[25] Though this brings into questions from the Random Field Ising Model (RFIM) game outlined in [2]

[26] While there is no explicit description bearing this explanation in the associated initial Minority Game paper, this result is found nonetheless through similar methodology later in this paper

[27] Or in practice a random choice of sub-strategy to universally apply for each time step, as described in the Thermal Minority Game. [5]

Given the great multiplicity of different evolutionary combinations between altering rewards, punishments, evolutionary triggers (time and thresholds before changes), evolutionary methodologies (as briefly enumerated above), and shifting how rewards and punishments pair, there is some necessary discretion exercised in the implementation and investigation of these different regimes. Finding a sub selection of all possible combinations that yield worthwhile results, either as an investigation into new behavior or as a null hypothesis against it, is no trivial or deductively reducible task, but here we attempt an evaluation of a subsection of those regimes we deem worthy. It may be noted that combinations of agent types and evolutionary methodologies may be mixed given the general code provided with the repository [3], and these mixed regimes are recommended for further study, though unfortunately such combinations are outside the purview of the present paper.
The evolutionary methodologies are different ”survival regimes” (i.e. triggers for the death or creation of an agent) and reward regimes (i.e. memory/strategy boost) which can alternately be applied to effective or ineffective or neutral performance. these are further enumerated under Darwinian Evolutionary Mechanisms

4.1.1 Creationism

This is the null case, wherein no agent strategies, memories, or population change. It is therefore used as a test against other evolutionary models, and for consideration of static memory distributions, as examined in section **??**.

4.1.2 Darwinism

Our implementation of Darwinism is intended to mimic that of [7] in broad strokes. Operating on the base Minority Game (as described in section 2), our Darwinistic methodologies have three different evolutionary mechanisms; create/destroy agents (population regimes), add or subtract strategies from agents (strategy modification regime), and modify the memory of strategies (memory modification regime). The population regime destroys an agent if the overall system performs poorly, and adds one if the overall system performs well (as determined by the percentage of agents who ”won”, optimally $\frac{N-1}{2N}$). The strategy configuration of this new agent is the same as the agent whose performance lead to its birth, save that it loses its biases for strategy preference. The strategy and memory regimes add or remove strategies or memory, respectively, based on each agent's given success, as determined by a variable percentage (ϕ) of losses in the last $h \in \mathbb{N}$ elements of history.
There are different methodologies for addressing this last h evolutionary period parameter, and its corresponding ϕ evolutionary threshold. An overall analysis varying them over relevant ranges to reveal the general behavior is detailed in section 5. This static iteration over known parameters however discounts the possibility for h and ϕ to be evolution dependent, which would allow for the evolutionary mechanism itself to find the globally best equilibrium. To this end we have experimented with a variety of evolutionary mechanisms in the hopes of discovering an evolutionary mechanism that would self-discover and equilibrate at the best overall parameters (not merely the dilute $\alpha >> 1$ phase). By having the evolutionary threshold values evolve with the agents individually, the best regimes were passed up in favour of the dilute strategy phase, but when the evolutionary threshold evolves as the best overall population performance through the entire history, optimal regimes are maintained for given evolutionary periods. This means that evolutionary thresholds evolve to the dilute strategy phase, then due to random chance, evolve to be better, and then cannot retreat back due to the shifted evolutionary threshold ϕ. Thus any shift to a better evolutionary condition will not be reversed, and through random shifts it will be incremented to the best possible solution.[28]

[28]This methodology is used for each tested evolutionary mean, but the results are only given as the dynamic ϕ evolution when relevant, as often random noise has a greater effect than the incremental performance improvements expected. This could potentially be partially remedied through longer evolutionary periods, with consequently less noise.

- increase memory length for all strategies (and resetting scores)
- increase memory length for the best performing strategy
- increase number of available strategies
- create another similar agent
- a combination of the above

All relevant results are graphed alongside their thermal counterpart, wherein a new, invented history is shared with all agents each round. This modification could be classified as another evolutionary method, but as the evolutionary mechanism is identical it may be classified as the same method.

4.1.3 Alternative Evolutionary Methodologies

Though not investigated in this paper, the following evolutionary methodologies are nonetheless implemented in [3], and should serve to broaden our understanding of the Darwinian methodology investigated here.

- Counter-Darwinism: wherein agents evolve as in Darwinism, but with the incentives reversed, i.e. it is the poorer players that "reproduce" or "learn", and the richer players that will die our if they win a certain percentage of the last streak.
- Inherited Bias Darwinism: A modification to allow for the continuation of the biases (strategy scores) of the original parents. This may be investigated with varying bias strength $\beta \in \mathbb{R} \cap (0, 1)$ is used as the level of bias received from the parent generation.[29]
- Cyclic Evolution: Here players cycle out, with a life cycle equal to the number of players, as for every day played, the oldest agent is deleted and the newest is created, forcing a maximum age of each agent.

4.2 Evolutionary Triggers

The enumerated methods of reward and breeding (as we presently dub the generation of new agents) are also to be paired with their respective methods of evolutionary trigger. As a default we can reward based on performance through a set period as a percentage of wins above random performance (i.e. if an agent wins 80% of the last 50 moves, it has its memory increased). This will undoubtedly also have to be paired with a punishment as well, i.e. for poor performance, agents loose memory. This introduces several new variables;

- Max Memory Length; maximum value of an evolutionary agent's memory length. (environmental constraint on intelligence)
- Min Memory Length; minimum value of an evolutionary agent's memory length. (environmental constraint on stupidity)
- Evolutionary Period; length of every evolutionary agent's performance history examined to determine whether or not it passes the evolutionary threshold
- Evolutionary Reward threshold; win percentage of evolutionary period necessary for agent to get the evolutionary reward

[29]For example, if a parent (the agent we're drawing the base strategies from, that are then optionally modified) has a rating $\mu_{rating} = 100$ for their strategy, and the universally set $\psi = 0.4$, then the following spawn would give the corresponding child's strategy $\mu_{rating} \cdot \psi = 40$.

- Evolutionary Punishment threshold; lose percentage of evolutionary period necessary for agent to get the evolutionary punishment

In all evolutionary regimes considered, the reward/punishment thresholds were identical (in terms of win/lose percentage of the last evolutionary period outcomes needed to trigger the change) and will be considered through relevant ranges as ϕ. [30]

4.3 Adaptive Evolutionary Mechanics

Via an analysis of the existing literature, as well as the direct results found in [7] [18] [4], we can predict that agents will continually evolve to have better and better memory, given the positive feed back loop of *increased memory → better predictive capacity → better agent performance → increased memory.* However, the boost to "predictive capacity" is in reality a simple expansion of the agent's accessible strategy space, incrementally[31] from $2^{2^m} \to 2^{2^{m+1}}$, or of the more relevant unique response modes of $2^m \to 2^{m+1}$. This leads to an increase in possible strategies from $\sum_{i=0}^{s} 2^{m_i} \to \sum_{i=1}^{s} 2^{m_i} + 2^{m_\alpha+1}$, where i is the different strategies for a given agent, m_i is the memory length for that strategy, and α represents the best performing strategy[32].

This increase in memory is greater, the greater the average memory (or lest the greater the memory of the strategy updated), but does not necessarily lead to a greater expansion of available strategy modes than the strategy update because both reduce to the same total per evolution shift. A strategy shift leads to an expansion (or in the case of evolutionary punishment, retraction) of available strategies from $\sum_{i=0}^{s} 2^{m_i}$ to $\sum_{i=0}^{s+1} 2^{m_i}$ (s = total number of strategies for a given agent), or a 2^m total increase, in comparison to $2^{m+1} - 2^m = 2^m$ for the memory evolution. This matching increase in the sub-strategies available to the agent per evolution is however deceiving, as each sub-strategy available upon a strategy increase has a $\frac{2^m}{s \cdot 2^{2^m}}$ chance of uniqueness[33], whereas each memory update leads to a

$$\frac{2^m}{\sum_{k}^{K} \sum_{i=0}^{s_k} 2^{2^{m_i}}}$$

chance[34] of non-uniqueness, which is identical save the improbability that the strategies now share memory length. K is all memory lengths of a given agent.

Strategy updates therefore continually crowd the existing available strategy space for a given memory, giving agents greater choices within a set strategy space, whereas memory based evolution methods adds exponential multiples to the total strategy space available to agents. If each complete strategy space for a given memory length is conceived of as a 2^m sphere, this extra dimension allows for the folding of this

[30] By default the evolutionary reward and punishment regimes are reflections of each other in success rates (i.e. for reward values r $\in (0,1) \in \mathbb{R}$, the corresponding evolutionary punishment values would be given as $1 - r$) to create a symmetrical evolutionary environment. When conceived thusly, both variables (evolutionary reward and punishment thresholds) may be given simply as their difference, (i.e. $\delta = |2r - 1|$) and this leads to regimes wherein the smaller the delta, the greater the ease of agents to both lose and gain their evolutionary reward. In practice, this should lead to divergence highly dependent on the initial condition, wherein a little success early on leads to an advantage, that due to the low evolutionary reward threshold, leads to ever greater reward rates as their performance improves, and vis-versa.

[31] For a simple single value boot to memory per evolutionary reward, as has been implemented in this program

[32] a variety of memory "boost" choices were investigated, however given the similarity of their results, for simplicity we have chosen to always update the best performing strategy of an agent, boosting its memory. This naturally has the greatest multiplier effect and subsequent positive feedback, however the effect was not so great so as to a permanent division in the agent strategies which got more memory (and thus performed better) and those that did not.

[33] Assuming there is no variation of memory lengths, an assumption which is does not hold for mixed memory and strategy evolutionary methods

[34] Agent memory is all the different memory values an agent has across all its strategies

2^m box into higher dimensions (i.e. $2^{m+1}, 2^{m+2}, 2^{m+3} \dots$), greatly expanding the strategy space available to agents.

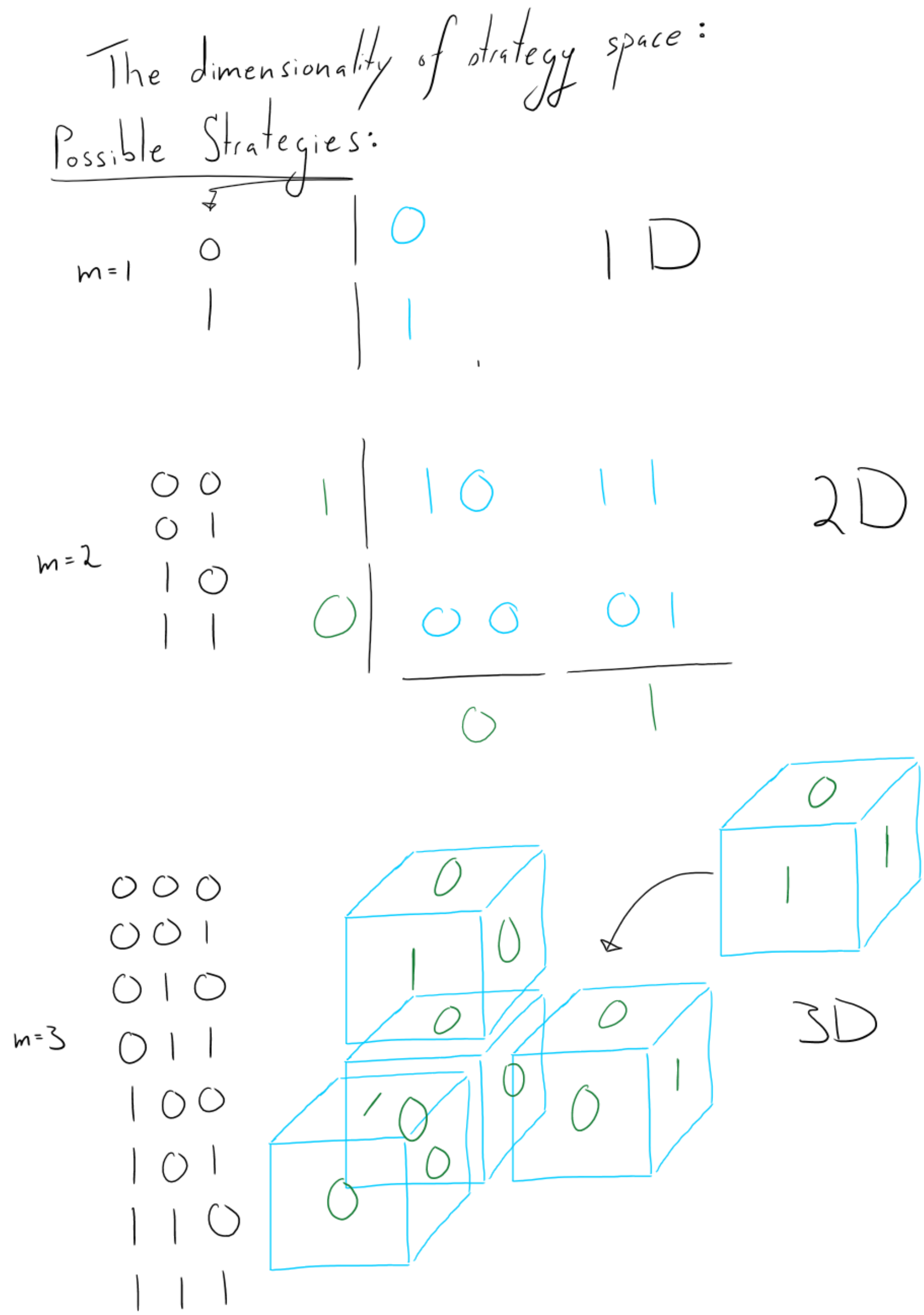


Figure 9: A depiction of the expansion strategy space as folding into higher dimensions

We initially conceived of the above dynamics of evolutionary development leading to an expansion of the available strategy space and the possible strategies implies that for any given agent-evolution configuration, there will be a careful balance between having too many strategies to coordinate anti-correlated cancellation (as seen in the higher α values in figure 5) and having too few strategies, leading to pile-up.

However, this kind of trade off is revealed to be false in our investigation into strategy evolution, as the pile-up effect of agents on shared strategies overpowers all others.
Alternative minima (as explored in the various static memory distributions) can be conceived of as well, as when there is a sufficient distribution of strategy memory lengths to allow for the "low hanging fruit" to be seized by those with low memory strategies, and more complicated, longer memory strategies are able to coordinate with the lower memory, more predictable strategies. A larger number of strategies or high memory strategy options might be necessary for these more "intelligent" agents to effectively coordinate with the "dumber" agents, though our results suggest that any expansion of the number of strategies of an agent comes at the cost of the overall system due to an exaggeration of the effects of the minority condition. Ideally, the optimal distributions may be found through simple evolutionary methods that adjust the relevant variables, however, these results unfortunately do not reveal themselves thusly, for reasons that follow.
Results and initial analytic consideration reveals that all positive feedback evolutionary methods will rapidly[35] evolve into more disparate strategy space regimes. The question, therefore is whether or not each evolutionary regime[36] has an effective mechanism to limit the overall strategy space to achieve the absolute maxima, rather than merely that found from randomness. In order for this to be the case, the difference between the random success rates and those in the ideal local minima would have to be sufficient so as to induce evolutionary retribution and corresponding conformation to the best possible solution, given the evolutionary method being used.
By adjusting how sensitive the evolution is to success/failure, i.e. the percentage win/fails necessary to trigger the evolutionary response (as given by delta as discussed in section 4.2), we can observe for which values of ϕ the ideal minimal variance solutions (and highest overall success rates) are achieved, and whether adjusting evolutionary sensitivity is sufficient to constrain the system to better-than-random solutions. A similar method can be used with different evolutionary periods, as an extension of their evolutionary period leads to a lower standard deviation from random behavior, which agent history resembles [13] [4] and a consequent increase in evolutionary trigger threshold. All evolutionary methodologies are therefore graphed for different evolutionary periods, with key observables vs percentage win/lose threshold for evolution, as well as for values with dynamic evolutionary thresholds. This dynamic evolutionary method, which changes its own parameters to converge on an optimal solution (detailed in section 4.3.1) is then compared to the static distributions, to determine whether or not the dynamic evolutionary method did in fact achieve the absolute globally best solution, or merely the local dilute strategy phase solution.

4.3.1 Dynamic Evolution

Due to the evolutionary dependency on ϕ, for an evolutionary methodology to achieve peak performance, it must have control over this variable, and a method of continually fine-tuning it. To this end, we initially tested a method whereby agents would set their personal phi values to their best ever phi values, enabling agents to rapidly evolve into the random regime, and ideally beyond as the threshold gradually increases, and increases the threshold for random evolution to bring it from its previous equilibrium. However, because agents individually experience much more random variation than the system as a whole, this method was adapted, wherein instead of an individual agent having and adapting their own personal ϕ value, the ϕ value would evolve as the best of the entire system. Results from these simulations are shown when relevant.

[35] The speed at which such evolutionary methods change is largely based on their trigger thresholds (their deltas, in our simulations) and evolutionary periods, or agent's history length considered for the trigger for each evolution

[36] considered to be a given agent population, evolutionary method, evolutionary trigger sensitivity, evolutionary period and memory length combination

4.4 Remarks on Programming

While detailed descriptions of the corresponding program may be found with the linked repository, [3] this briefly details the critical elements for analytic consideration:

- No continuous memory methodology was adopted, even for analytic purity, as in [12]
- For evolutionary methods which adjust the agent population, even agent populations might have indecisive (0) value history elements. In these cases, that history value will be replaced with a value generated within the predicted range (i.e. uniformly as a value $\leq 20\%$ of agent population)
- The simulation relies on a data to initialize, which is dubbed the "prehistory". This data is likewise generated within the $\leq 20\%$ of agent population range[37], and is not considered in any of the following results.
- All data is created with both stochastic and deterministic methods (with real and fake histories, where fake histories are considered as in [5]). Error bars are provided for stochastic methods.
- Deterministic solutions are based off of the procedural generation of a bit by taking the 10^{th} bit of a double sine expansion. This yields sufficiently random data, as can be attested to with the following error margins and visualizations.[38] This was completed for conceptual simplicity and computational power, and a comparison of this sine based bit generator (for counting integer seed) against the c++ random library uniform int dist (with constant seed) is provided below in figure 10.
- Plots are with Gnuplot, though all statistical measures are calculated independently.

5 Results

5.1 Static Memory Distributions

In order to compare with the eventual memory distributions found from evolutionary behavior (section 5.2.1) a variety of memory distributions were tested without evolutionary behavior[39]. For each form of memory distribution either the agents are assigned memories from an overall distribution at random (e.g. for a Poisson distribution, this would lead to higher probabilities that an agent would receive memories more grouped to the center of the distribution) or agents are given consecutive elements of the distribution, such that each agent has either the same memory value or a memory value adjacent in the overall distribution. For example, with the former method (which we shall dub the pachinko machine method), an agent in a system with a negative exponential memory distribution[40], which has three strategies may have one strategy of memory length 3, one of memory length two, and one strategy of memory length nine, where as in the latter method, the strategies for a given agent would all have to have sequential terms, e.g. 2,2,2 or possibly 2,2,3 in this case.

[37] Though only the binary elements are actually necessary
[38] As follows from the periodicity of sin and irrationality of π
[39] i.e. the creationist evolutionary methodology, 4.1.1
[40] i.e. and exponential memory distribution weighted towards the lower strategies

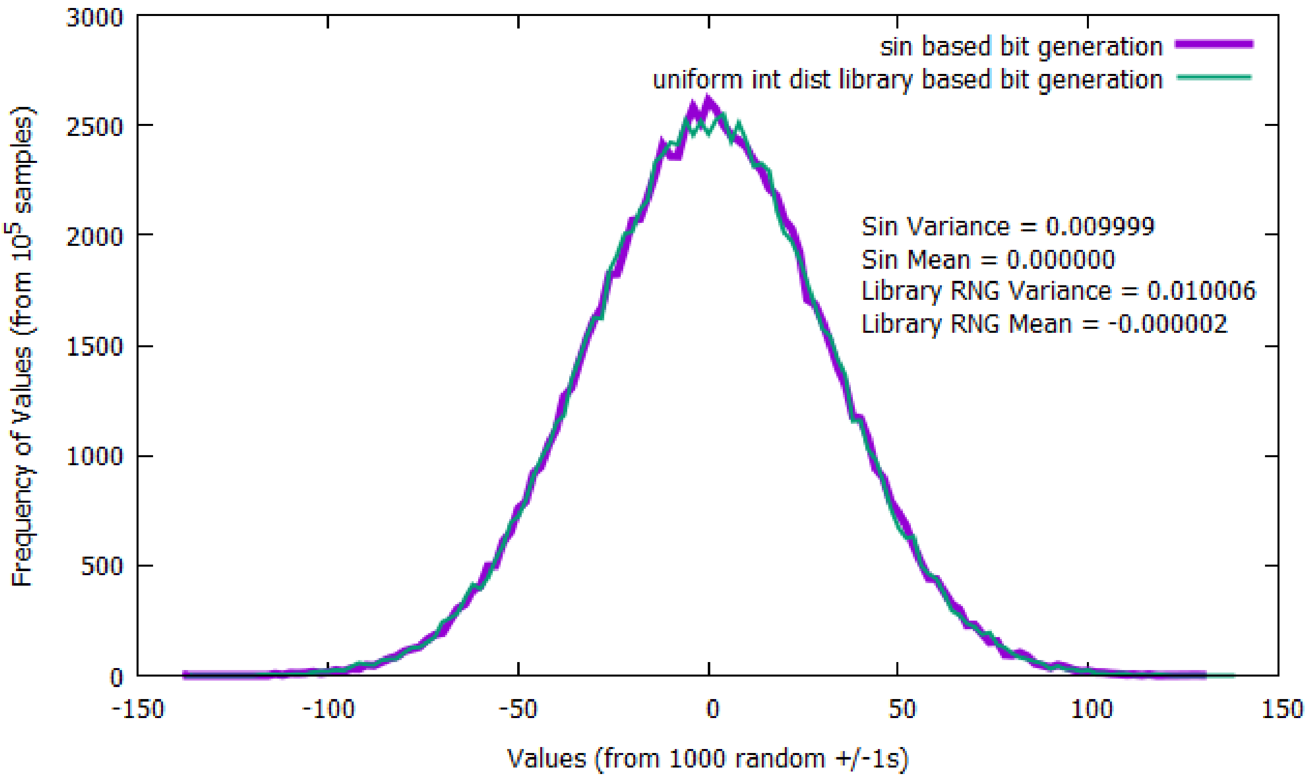
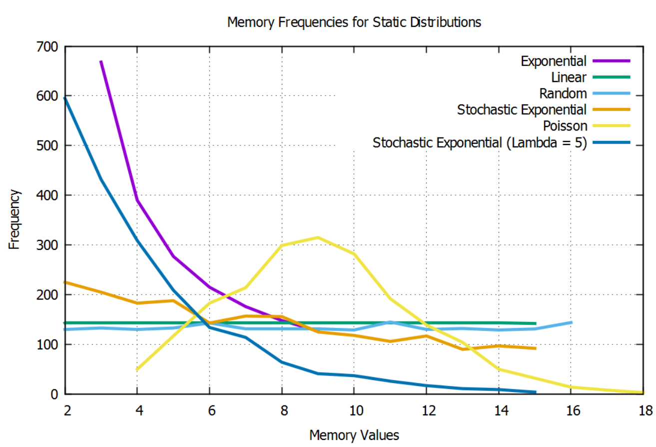
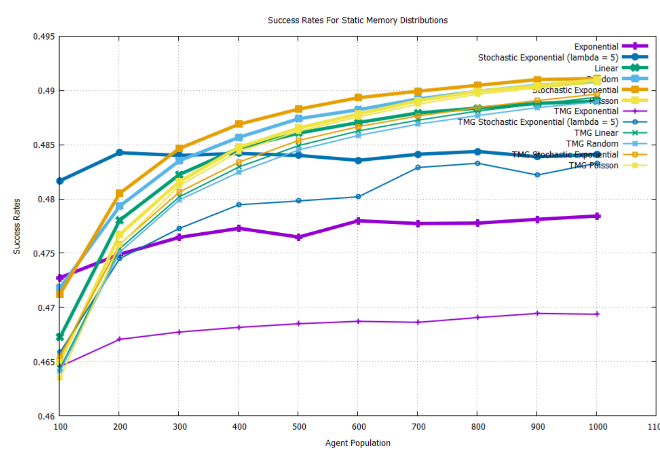


Figure 10: Sine based random bit generator vs c++ random library uniform int dist generator for bits



(a) Memory histograms for static memory distributions



(b) Success Rates for different memory distributions

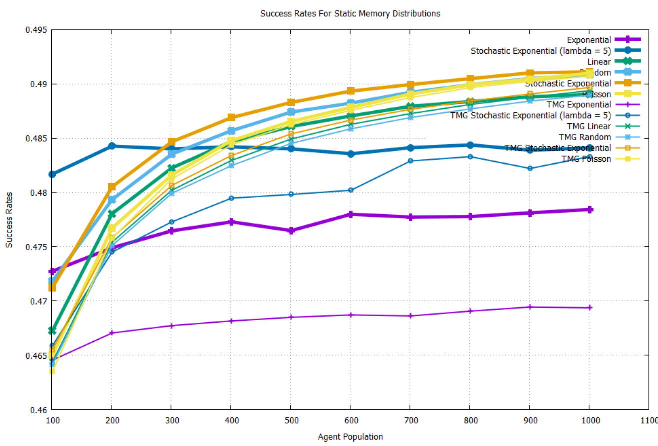
Figure 11: Memory Histogram and success rates for different memory distributions, with both real and fake histories (TMG denotes invented histories)

We consider constant, poisson, exponential, random, and linear distributions, with the poisson, exponential and random distributions also considered with the pachinko method[41]. All measurements are taken using a run-time of $2 \cdot 10^5$ days, memory length of 2, a range of 101 to 1001 agents, and averaged over 10 different possible arrangements of strategy sets. As we are generally interested in the long run-time equilibrium position for evolutionary[42] methods, differences in performance due to an evolutionary

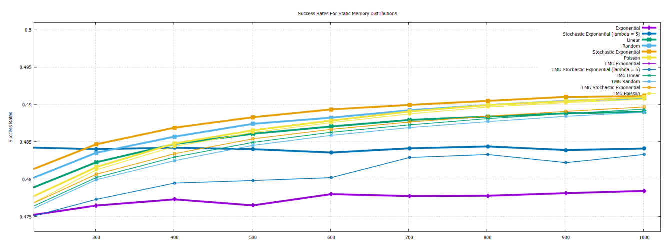
[41] The random and constant memory distributions are a priori the same for both methods

[42] Both with dynamic thresholds and static thresholds over examined over a set range, as explained in 4.3.1

method taking time to settle into its eventual distribution may be determined by examination of the eventual memory histograms from evolutionary methods and those of these set distributions. If they are otherwise the same but there is considerable shift in meta-statistics, then this difference determines the pre-equilibrium evolutionary behavior. The difference in performance for those with the pachinko machine method vs sequential distribution reveals whether or not agents must evolve all their strategies together, or can best optimize when determining between "smart" and "dumb" choices.
After consideration of the stochastic (i.e. with pachinko method) exponential distribution and that with sequential memory values (blue and purple respectively in the below graphs) we determine that sequential memory distributions perform slightly worse than those using the pachinko method. Agents show coordination between both themselves and other agents regardless of their individual memory allocation however, so long as the memory distribution fits a given form, the corresponding performance of that form will be seen in long run behavior, though the sequential distribution performs worse.



(a) Success Rates for different memory distributions



(b) Success Rates for different memory distributions, population 201 to 1001

Figure 12: Success Rates for different memory distributions

These results alone perform admirably (save the outstanding exponential distributions) in comparison to random play[43], with the depressed stochastic exponential distribution performing marginally better than the other distributions, all of which asymptotically approached a success rate of 0.5. Recall that, as seen in figure 5 the success rate of agents decline from 0.5 at $\alpha_{critical}$ due to agent's inability to coordinate and use anti-correlated strategies. [10] Notably, though the distribution of variance over population values were considerably lower than that of randomness, as may be seen in figure 14a, though this did not make a marked change in the eventual success rates. A relevant range of constant values is considered in 14b. When running the minority game with entirely fictitious histories, the overall behavior remains the same, as proven in [4], but the performance is negatively affected, if only marginally[44]. While random, linear, poisson and stochastic exponential distributions all display similar behavior, the poisson distribution and superlinear low lambda, pachikno distributed exponential (the orange line in figures **??** and **??**) distributions have markedly lower variance, and the stochastic, superlinear exponential distribution performed marginally better than all others. The comparatively poor performance of low mean memory distributions suggests that the alpha of the overall data is the most relevant factor in determining its overall performance. The super linear stochastic exponential and poisson distributions best cover the range of critical alpha as the population shifts from 101 to 1001. This is most readily explained exactly, as for an agent population of the afore mentioned range, the memory range to achieve critical alpha

[43] Whose default $\frac{\sigma^2}{N} = 1$

[44] Although it is possible this behavior is a fluke, given that these simulations were of generally exhaustive run-time (20,000 days) and 10 different strategy response modes, it is highly unlikely.

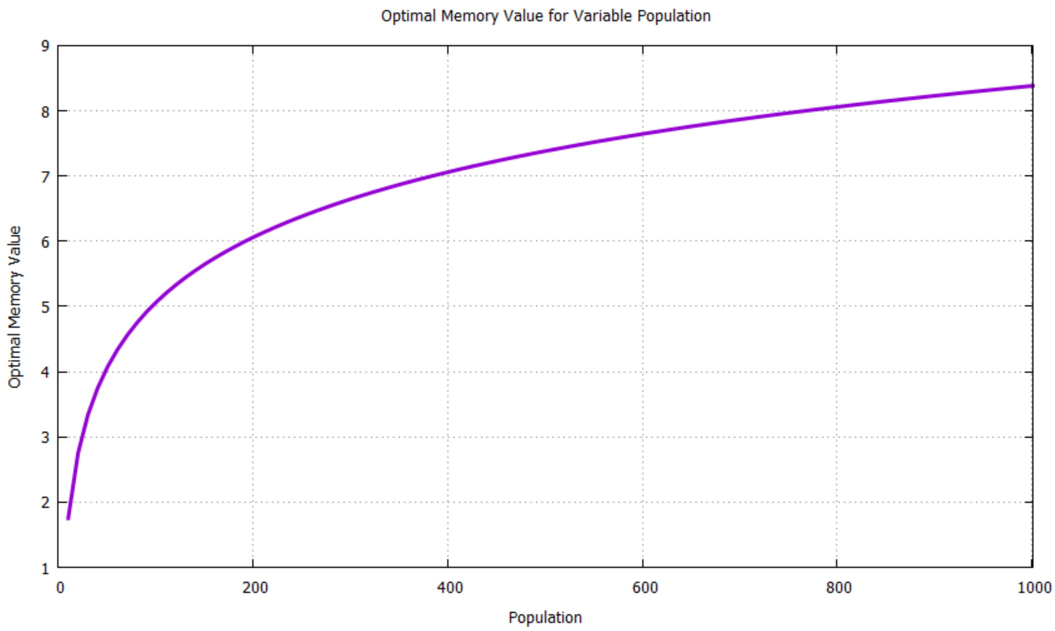
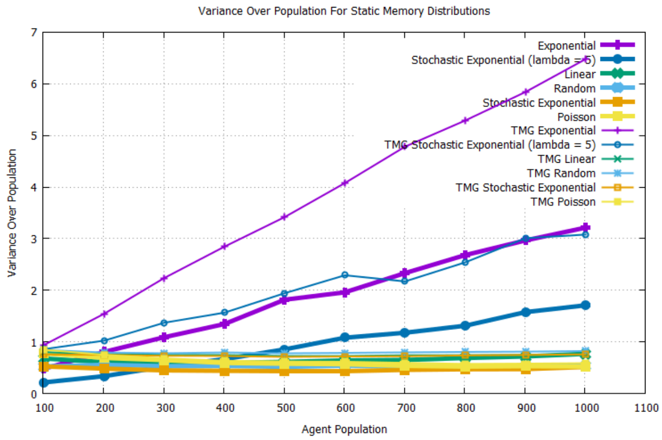


Figure 13: Optimal memory values for population range 0 to 1001, with $\alpha_{critical} = 0.\overline{3}$

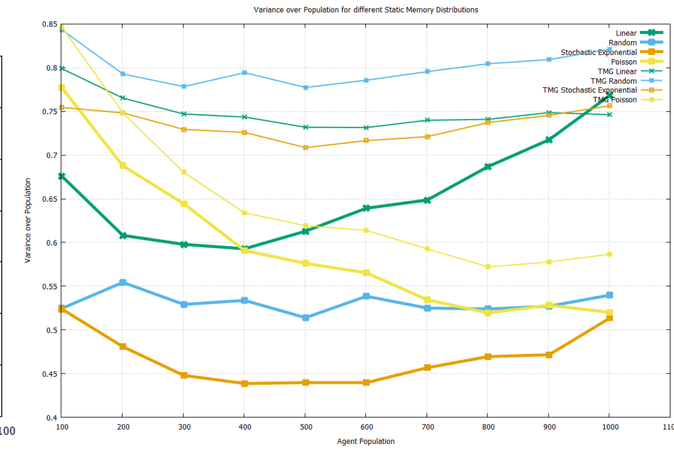
(assumed to be approximately $\frac{1}{3}$ as deduced in figure 15 and [13]) will be $\log_2(\frac{N}{3})$, or logarithmically varying from $5to \sim 8$, which the highlighted ranges best capture with their distributions. The exactly optimal population rates for corresponding memory values is pictured in 13.

5.1.1 Constant Memory Distributions

For clarity and conformation of the theory proposed above and in [18] [8] [13], we have provided figures 15-16, wherein the second compilation of figures (figure 16) shows the performance of constant memory value distributions by individual memory lengths, confirming the above theory. When looking at the success rates of single memory valued distributions, one sees that their performance is directly related to where the memory value's corresponding critical alpha is, e.g. as $2^8 = 256$ and $\frac{256}{800} = 0.32 \approx \frac{1}{3}$, which is as close to $\alpha_{critical}$ as was graphed, it is natural that the performance of the memory $= 8$ line declines thereafter.



(a) $\frac{\sigma^2}{N}$, for populations 101 to 1001



(b) $\frac{\sigma^2}{N}$, for populations 101 to 1001 excluding highly exponential memory distributions

Figure 14: Variance over population for vs Population for different memory distributions, with both real and fake histories (TMG denotes invented histories)

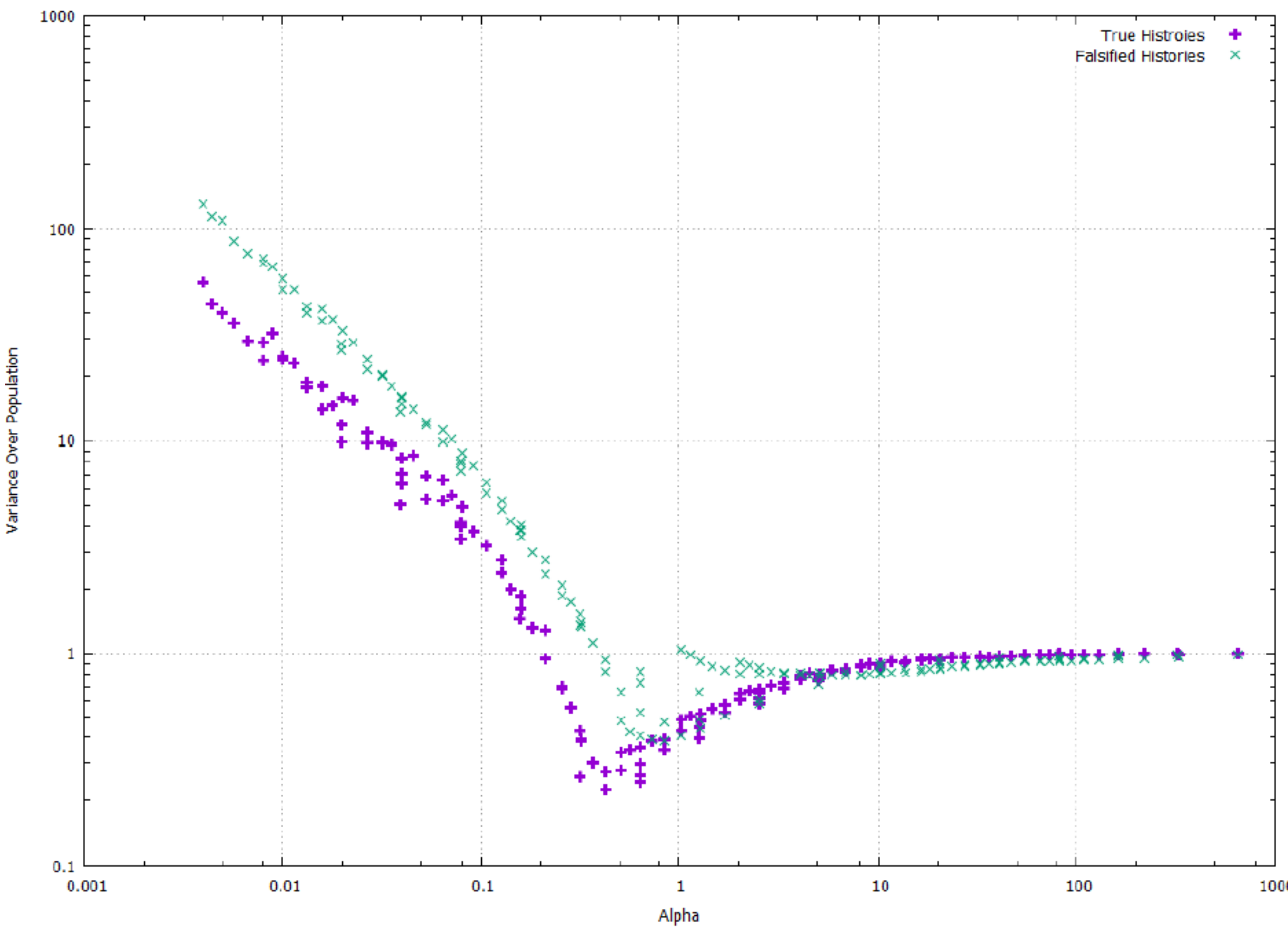


Figure 15: Variance over population vs alpha

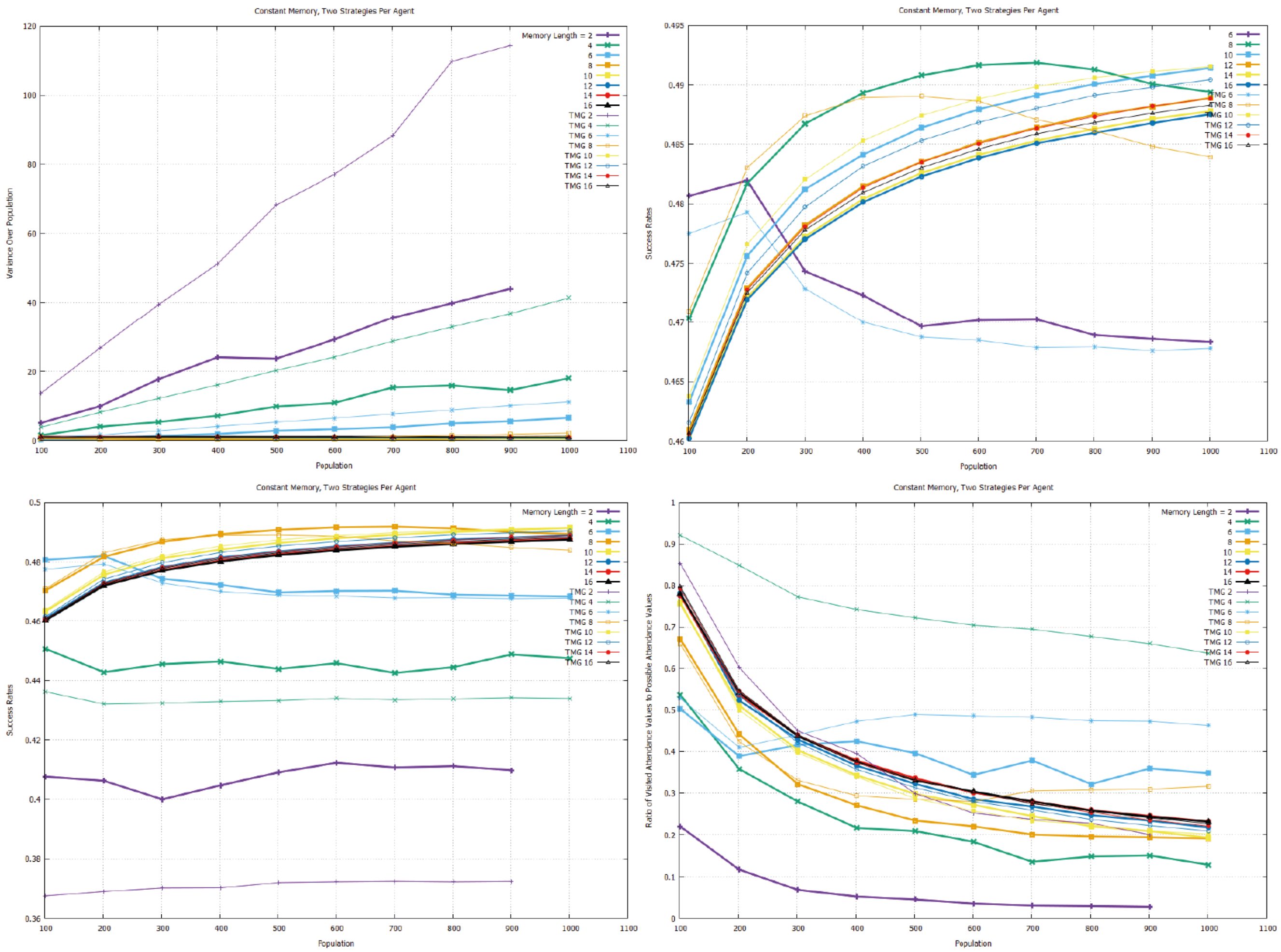


Figure 16: Observable vs population for constant memory distributions

5.2 Darwinian Methodologies

Given the above analysis (section 4.3), we expect all evolutionary methods when under the right meta-conditions to compensate for both the overcrowding and dilution of strategy space to achieve the best result. In order to achieve the best possible regime, better than the relatively low-variance solutions seen with the random play of $\alpha \gg 1$, the environment will have to be sufficiently "harsh" so as not to allow agents to settle for this local minima. Therefore we should expect a two fold shift in agent play as the evolutionary threshold increases, first a shift into a more efficient regime as the increase in evolutionary threshold leads out of constant random fluctuation about the local dilute-strategy phase minima[45] and again when transitioning into the more efficient regime of coordinated agent action. Here the distribution of memory values or strategies reveals the character of the evolution, and how the agents have chosen to spread through strategy space.[46]

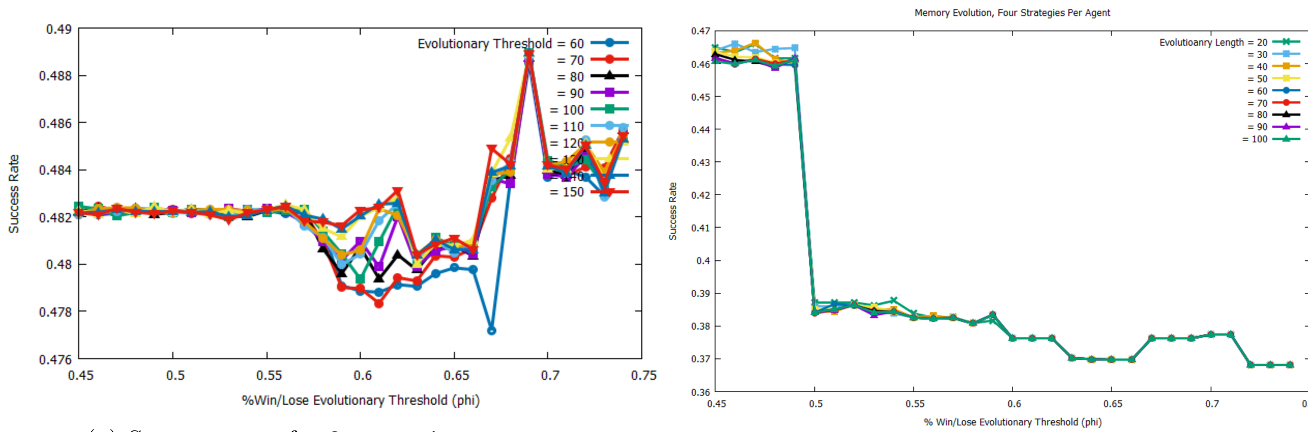
Apart from the afore mentioned variables, there are several different evolution methodologies as introduced in section 4.1.2 which are examined below. As for all data simulation results, we will consider the conventional set of observables established in [13] [8], variance over population ($\frac{\sigma^2}{N}$ formally defined in

[45] For low evolutionary thresholds ϕ, the agents will evolve into to the most dilute strategy phase, and then constantly be evolving back and forth within that regime, leading to no gain above the local minima of dilute strategy induced randomness (dilute strategy phase, meaning many more strategies than agent, or $\alpha >> 1$)

[46] i.e. have all agents acquired a high and low memory pair, and use them alternately, or does each agent settle into a special memory regime, and specialize, with the eventual overall memory distribution following another form?

section 6.4) and success rates (A(t)), mean memory or strategy value at the end of the simulation[47]. This data is paired with agent population, number of elements of the possible full attendance range visited, and evolutionary threshold values when appropriate. When not otherwise stated, all graphs are made with a run-time of $2 \cdot 10^5$ days.

5.2.1 Memory Evolution



(a) Success rates for 2 strategies per agent

(b) Success rates for 4 strategies per agent

Figure 17: Memory Means vs percentage win/lose to cause evolutionary trigger

The overall memory evolution, when directly graphed for varying evolutionary periods h against evolutionary thresholds ϕ, displays consistently $\alpha \gg \alpha_c$ behavior, until the evolutionary threshold ϕ increases and the traditional behavior seen in the above static distributions are seen. The eventual memory distribution are, as revealed by the following memory histogram (figure **??**) exponential in distribution, with more memory values weighted around where the origin. This leads to the characteristic behavior above, with increasingly sub-optimal play for exponential distributions that are more weighted to the minimal side, and thus $\alpha \ll \alpha_c$ behavior. These results can largely be explained through the analysis above, though the eventual minimum is dependent on the initial memory, and how well it suites the existing population[48]. It may be conjectured that for longer run times, agents may eventually evolve to the more optimum memory levels regardless, but this requires an evolutionary mechanic that dynamically shifts ϕ so as to prevent reversion to a less optimal state by chance.

[47] Average α was also considered, but due to the exponential nature of α for varying memory, α serves as a poor proxy for mean strategy/memory lengths

[48] As high ϕ values lead to little evolutionary change, and consequently if the initial regime is $\alpha \gg 1$ or $\alpha \ll 1$ depends on the initial memory values

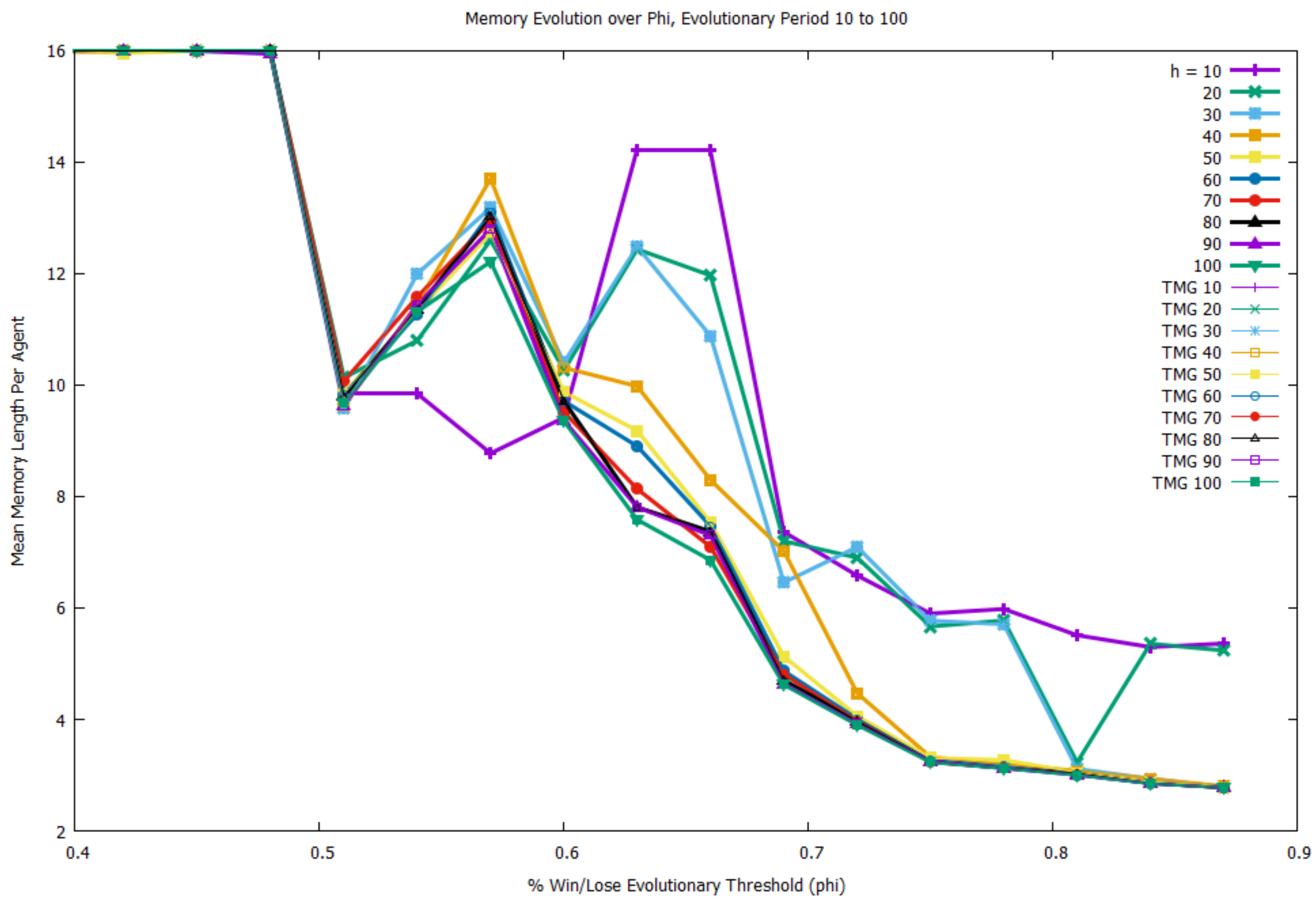
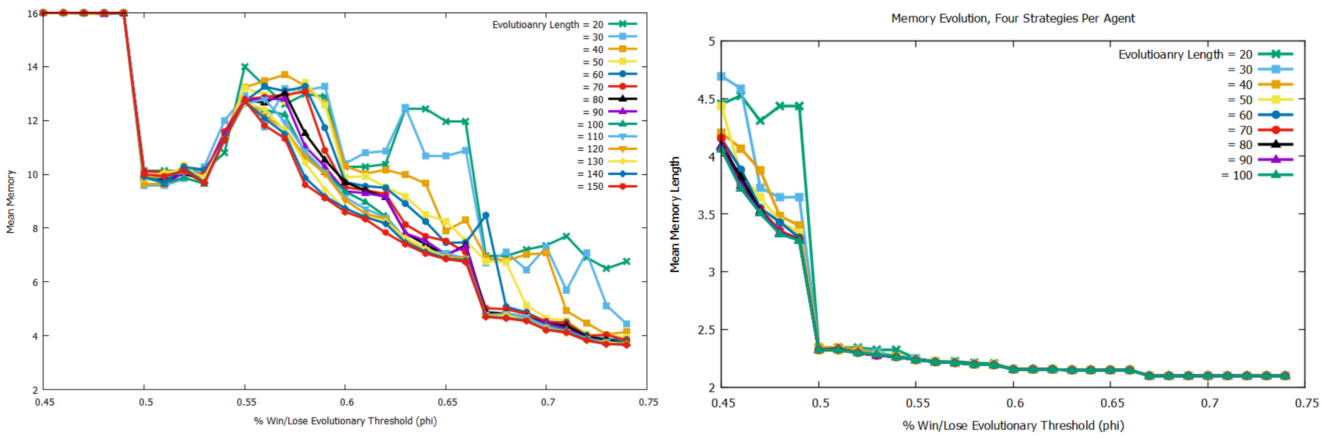


Figure 18: Mean Memory vs phi 0.4 to 0.9, graphed alongside falsified histories (TMG)

All results are given for 1001 agents, initial memory values of 3, maximum memory value at 20, and minimum memory value at 2.[49] This default initial memory value of three then makes $\alpha_{initial} = \frac{2^3}{1001} \approx 0.008$, which would lead to far worse system behavior than the actual data eventually shows, after evolution. (as may be deduced from comparison to constant distributions)
All results were also graphed for longer threshold ranges (ϕ from 0 to 0.9) but as the results before 0.5 are all identical, (as follows from the fact that increasing memory generally improves the system's results) only the shortened graphs are shown, and the behavior after $\phi = 0.6$ is likewise asymptotic. A graph of mean memory with TMG data through 0.9 is show in figure 18.

[49]The minimum memory value is set at 2, rather than 1, so that all agents are still forced to be "Active participants", with the behavior seen in [13] [10] [18]. This does not of course preclude the possibility that by allowing memory (or strategy) count to be 1 other agent might be able to better coordinate, and even these entirely deterministic agents could benefit.

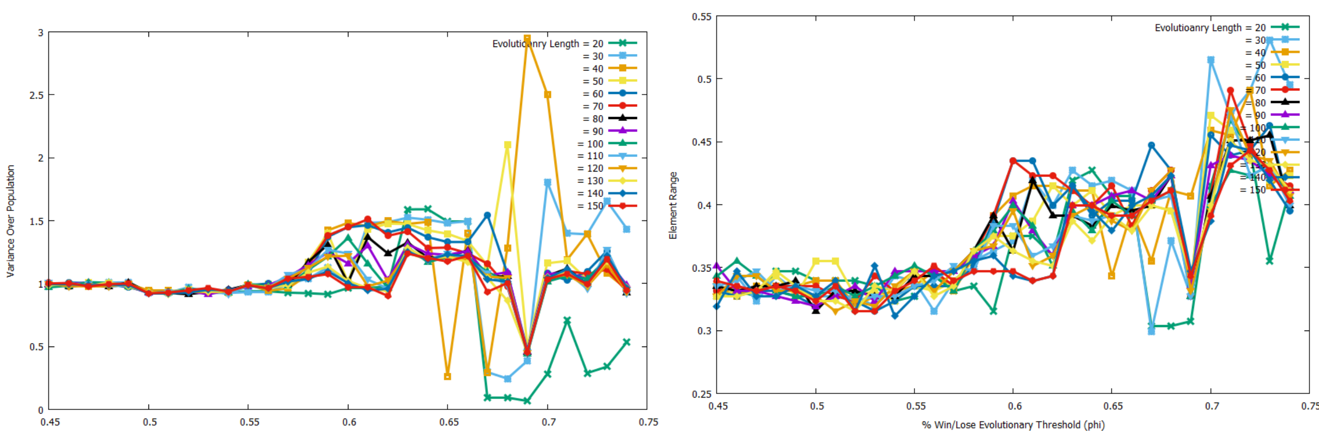


(a) Mean memory for 2 strategies per agent

(b) Mean memory for 4 strategies per agent

Figure 19: Success rates for vs percentage win/lose to cause evolutionary trigger

Though the differences are clearly statistically significant, the behavior in figure 17 should not obscure the scale of the absolute difference in success rates between optimal performance, seen at around $\phi \approx 0.60$ and the worst, $\phi \approx 0.69$ (for two strategies per agent), which is around one percent. The scale of this difference is perhaps the most insightful conclusion from the Minority Game models; not that agents coordinate, but that their coordination provides only very marginal gain so long as the environment remains either too restrictive or too rich. It is therefore understandable if an agent should not evolve into the perfect minimal regime when the local minima dilute phase performs worse only every 100$\cdot$ times, given that this is longer than any run-time used here. Given that the difference in performance for an individual agent is virtually undetectable in the given run time, it is remarkable that the overall system evolves to find the globally optimal solution for any evolutionary system at all. These results are however further confirmed by longer run times and shorter evolutionary lengths, as seen in the following look at the dynamic evolutionary models over a range of evolutionary periods.

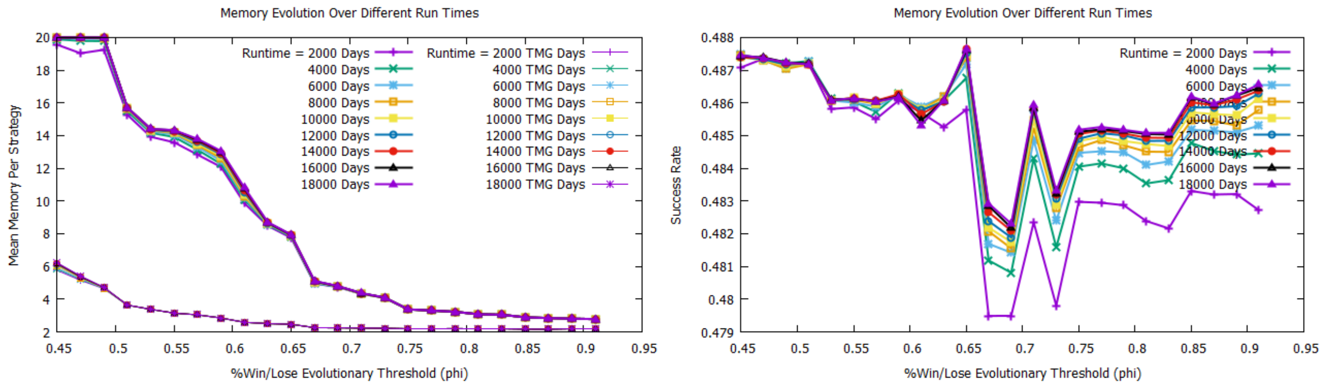


(a) Variance over population for h from 20 to 150

(b) Number of visited market history element normalized by total possible

Figure 20: Variance over population and attendance ratios vs percentage win/lose to cause evolutionary trigger; the attendance ratio is higher the more random the system, such that increased attendance ratio either indicates highly periodic ($\alpha \ll \alpha_c$) behavior or optimal $\alpha = \alpha_c$ coordination behavior

Here the asymptotic behavior for larger ϕ coincides with the previous analytic conclusion that the greater the difference the more binary shifts necessary to reach that level of variance, and lead to a memory shift, leaving the data in the least efficient $\alpha \ll \alpha_c$ regime.



(a) Mean Memory vs ϕ

(b) Success rates vs ϕ

Figure 21: Mean Memory values and Success rates vs ϕ for 2000 to 20000 iterated simulations (market days), h = 100. Note the convergence to specific points despite the jagged nature of the data

There is a subtle fluctuation within the overall trends leading to more jagged peaks than conventionally seen. Though these results seem to lend themselves more to highly localized minima than stochastic noise from initial seeding, given the overlap in behavior for different run-time simulations, as seen in figure 21. As seen in all of the above, there is a correlation between larger evolutionary periods, with a superlinear relation between the value of h and its corresponding convergence in variance, success rates, and other observables. (thus the more detailed graphs examining the higher memory lines alone in figure 19) The poor behavior seen for higher numbers of strategies is understandable in light of the greater number of overall strategies provided to the system, and thus a crowding of the overall strategy space, that consequently leads to the $\alpha \ll \alpha_c$ regime. The fact that the expansion of available strategies leads to overcrowding of the strategy space is better seen by allowing agents to expand their available set of strategies, as in the following section.

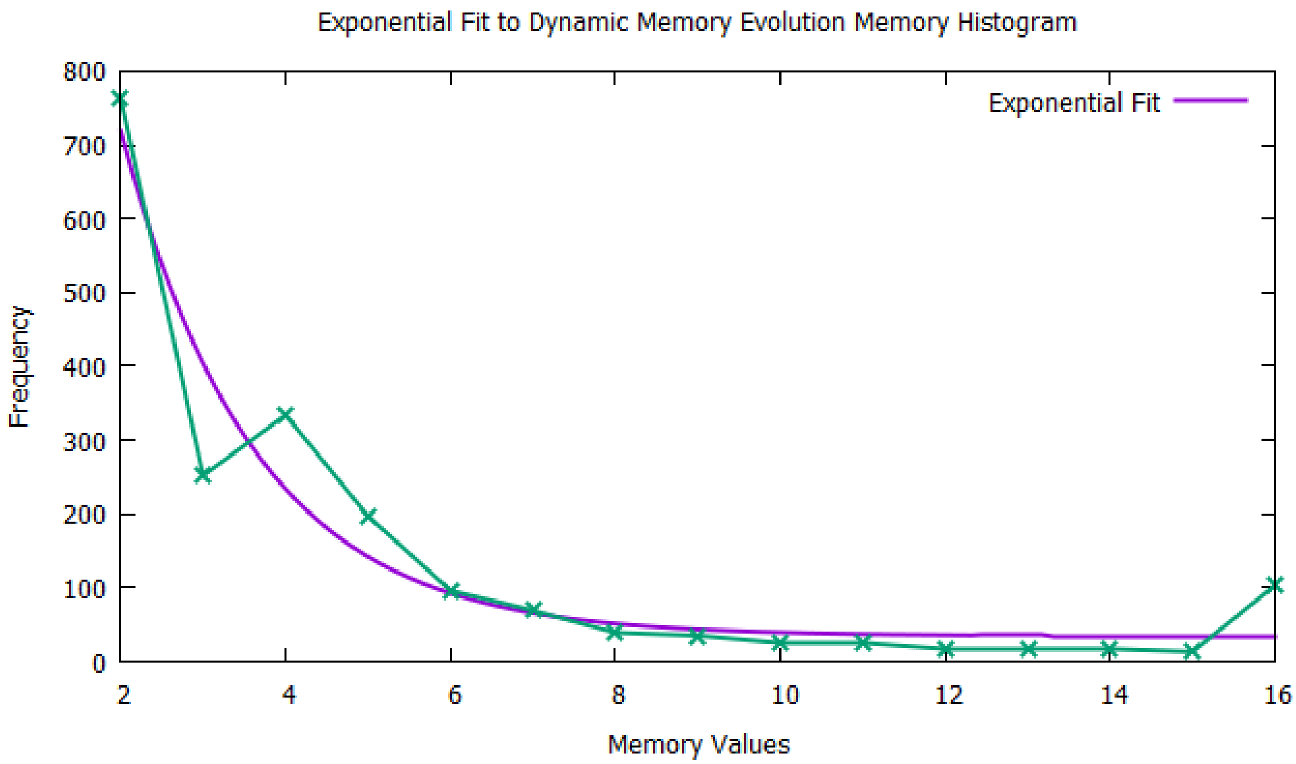


Figure 22: Exponential fit to Memory Histogram

By shifting ϕ to equal the system's best overall success rate (at any point in history) we hoped to allow the system to naturally discover the best ϕ value and corresponding memory arrangement. Though tested for a variety of h values, as all results were identical, only the eventual memory histogram after 20,000 days of simulation and $h = 1,000$ is shown in figure 22. The eventual ϕ value was 0.701299, which lead to a variance over population value of 1.19067, mean memory value rate of 4.65335 and success rate of 0.487177, which is not as impressive a performance as seen in some of the above evolution methods, but this is likely due to the strategy threshold being set higher than absolutely optimal via random variation, with subsequent evolution evolving outside the bounds of the optimal solution. The distribution is plotted against an exponential, of the form $f(x) = b + n \cdot e^{\frac{-x}{u}}$, with subsequent values of b, n and u as 34.3239, 2348.79, and 1.62197, respectively. This distribution of memory values reflects on the evolutionary method more than the optimal solution, as we can expect the dynamic ϕ evolution to proceed in three steps:

1. After several steps, wherein agents choose randomly, their collective performance is as according to their memory and population accords. For low populations, it would start off more or less randomly, for higher populations, the performance would be worse, but still sufficient to increase the evolutionary threshold ϕ.

2. As ϕ increases to 0.5 (faster or slower according to the initial population or memory) fewer agents are able to make the cut, effectively excluding most agents from the evolutionary boon. Those agents that have an evolutionary advantage then go on to use it, and subsequently perform better than those with lower memory. (as similar results in [5] [4] and [18] show)

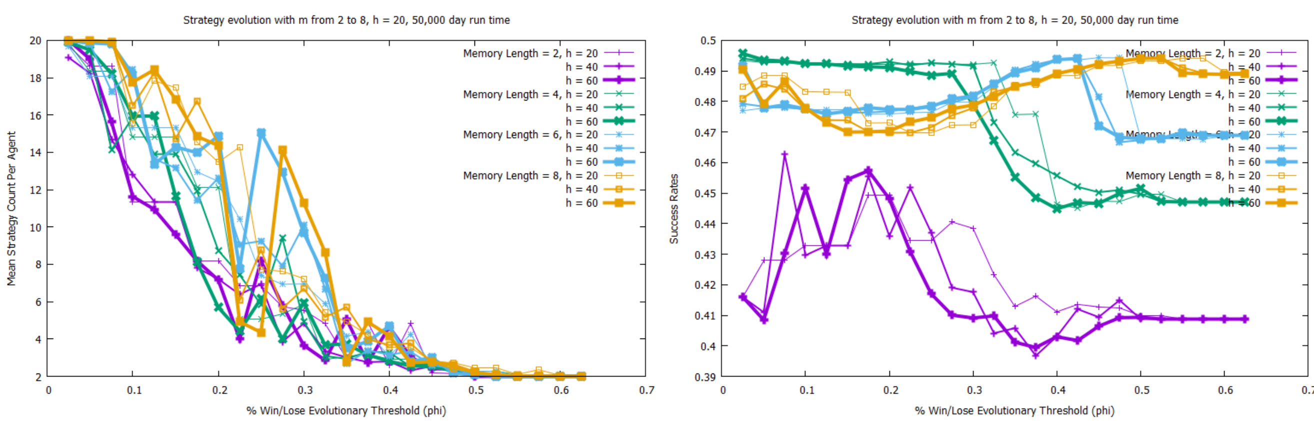
3. For $\phi = 0.5$, agents will play more or less randomly, with the few that break the evolutionary threshold having a slight advantage, and thus more likely to advance in memory. When born out over long time frames, this leads to the exponential behavior seen, as for each increase in memory,

the marginal increase in performance is smaller than the marginal collective probability that they advanced that far in evolution.

Ergo the predicted exponential distribution of agent memory through memory evolution was born out, though not with the predicted effect on performance. Notably, however this arrangement only performs worse than randomness due to the population; for different populations, this memory distribution might be closer to ideal, thus the combination of population and memory evolution is a subject worthy of further study.

5.2.2 Strategy Evolution

As seen previously, by expanding the strategy space available to each agent, the consequence is an overcrowded strategy space that performs worse than previously. Thus, while it was initially suspected that the higher number of strategies per agent would allow agents to coordinate between more strategies, and subsequently achieve a better overall performance (as was seen with the increase in memory), the reality is that agents are unable to distinguish between strategy performance, and thus unable to coordinate and revert to the overcrowded $\alpha \ll \alpha_c$ phase. Agents only have their scoring mechanism to determine whether or not to use a strategy, so when everyone has access to all strategies (to imagine the extreme case) the result is that everyone will always choose the best strategy, and consequently the result will be terrible performance for the overall system. This terrible performance would generally be self-correcting, as evolutionary methods ought to be, but if we observe the curiously *overly bountiful* regime wherein agents are given more strategies even if they perform poorly (i.e. low ϕ regimes), then the agents are fixed in this artificially low alpha regime until ϕ increases, where-after there is a decline to the lowest possible strategy values.



(a) Mean strategy count declines as agents come to perform closer to random

(b) Success rates correspond to their optimal alpha, given the population of 1001 agents

Figure 23: Evolutionary periods (h) from 20 to 60, memory values 2-8, ϕ to 0.65, step length 0.025, 1001 agents and 50,000 day run time

This result is especially remarkable in two ways; in one, the case that agents are unable to coordinate (even when given longer run times[50]) when given a larger fraction of the agent pool, and two, that even when it might be considered beneficial for agents to have more strategies (i.e. for the higher memory values, specifically in ascending order of effect from 8, when given 1001 agents) so that they might coordinate and find anti-correlated pairs, there is no noticeable use of strategy evolution to positive effect. In order to optimize for the marginal difference in success rates between α_c and $\alpha \gg \alpha_c$, agents

[50] Longer run times than otherwise tested in this paper, though as [7] uses run times of over 100,000 runs, there is the possibility 50,000 runs is simply not enough time

must be able to distinguish between their strategy scores, which is only made more difficult the more strategies each agent uses. While this is partly due to marginal difficulties in coordination[51], the more serious problem is that agents will naturally crowd strategies that perform better, as per the minority rule.

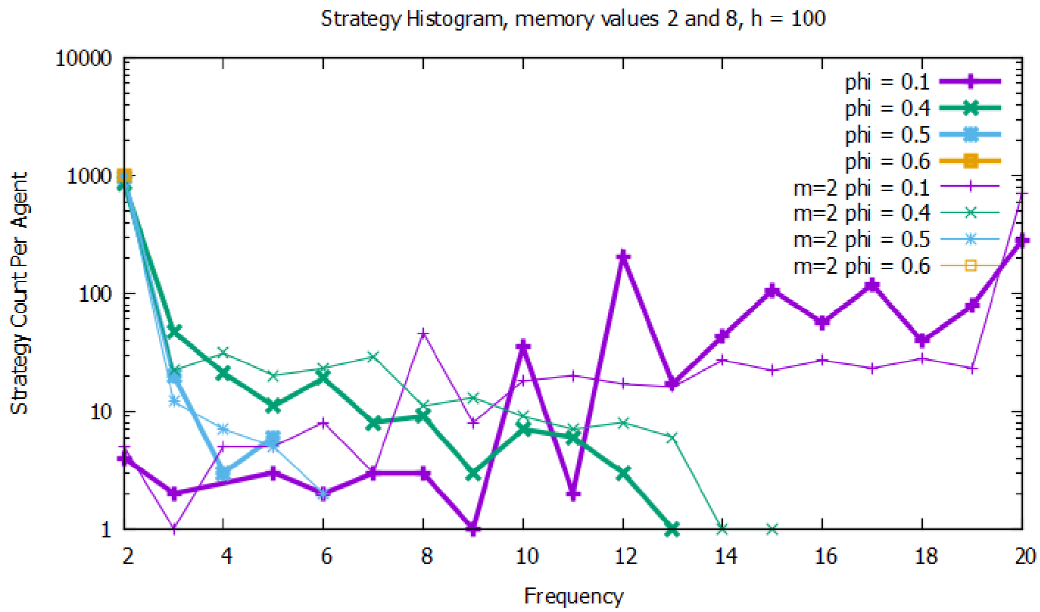


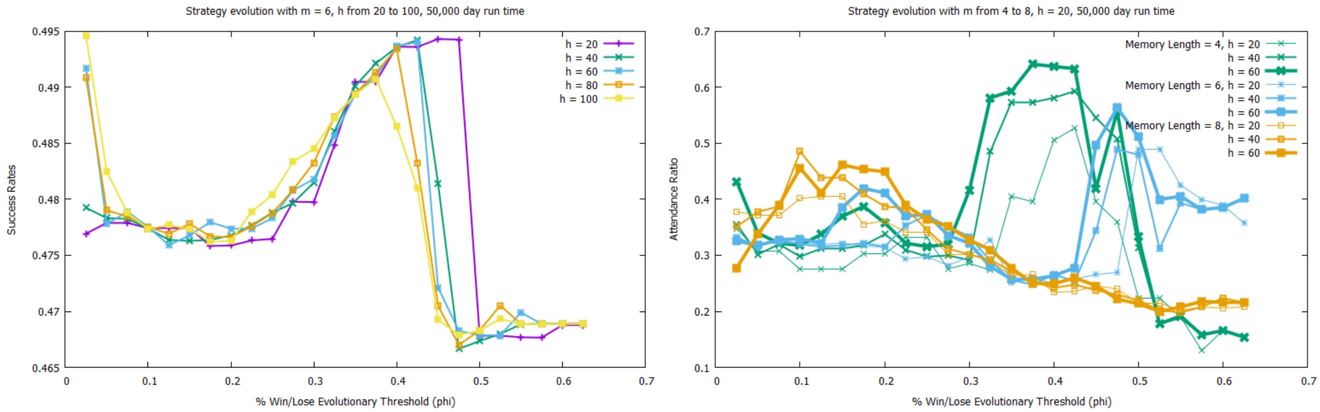
Figure 24: Strategy histogram; number of agents with a given number of strategies

Strategy ratings might not necessarily coincide between agents, even though all strategy ratings through all of history will be the same, as all new strategies created are created without inherited bias. Likewise, strategies that are deleted from an agent's register will not be resurrected with the same biases, even if the same strategy is by chance given back to an agent.[52] This inhomogeneity of scores helps agents coordinate between them, but cannot overcome the "pile-up" effect of agents on well performing strategies, however, and is only as effective as the preexisting biases of a given agent are strong. As explained below, the high system success rates and low strategy count implies a high rate of strategy turnover, and consequently not much time for biases to build up, even if from random play.

By looking at the success rates in figure 23b, we observe the overall behavior responsive to the memory length, as may be expected from the system's overall attempt to match the agent population and memory length with critical alpha. The introduction of variation in the number of strategies modifies the ability of agents to coordinate such that when there are too many strategies, agents will be unable to coordinate (as for the low ϕ regimes, but this may confer slight advantage when the overall strategy space is so overcrowded in the first place, that an increase in agent choice helps agents spread out along what limited strategy space there is, as can be seen in the memory length 2 lines of figure 23. As the number of strategies declines when ϕ drops to levels ($\phi \in (0.2, 0.45)$), The relatively high values of overall success rates when viewed in light of the low strategy count implies that the strategy count per agent values must fluctuate to remain consistent with the high strategy value. This intuitive explanation is

[51] Although agents keep track of their strategies independently, as more agents gain a larger percentage of the overall strategy pool, the chance they will have their own anti-correlated pair also increases, which means as they play against the market, each agent will have to choose which element of the pair they will take. This is further exacerbated by the fact that there is a bias for unused strategies, as noted in [13]

[52] The probability of a given strategy being given back to an agent after it has been lost is directly proportional to the memory length of the system, and the same as the probability that the agent would get any random strategy from the set, i.e. $\frac{1}{2^{2^m}}$

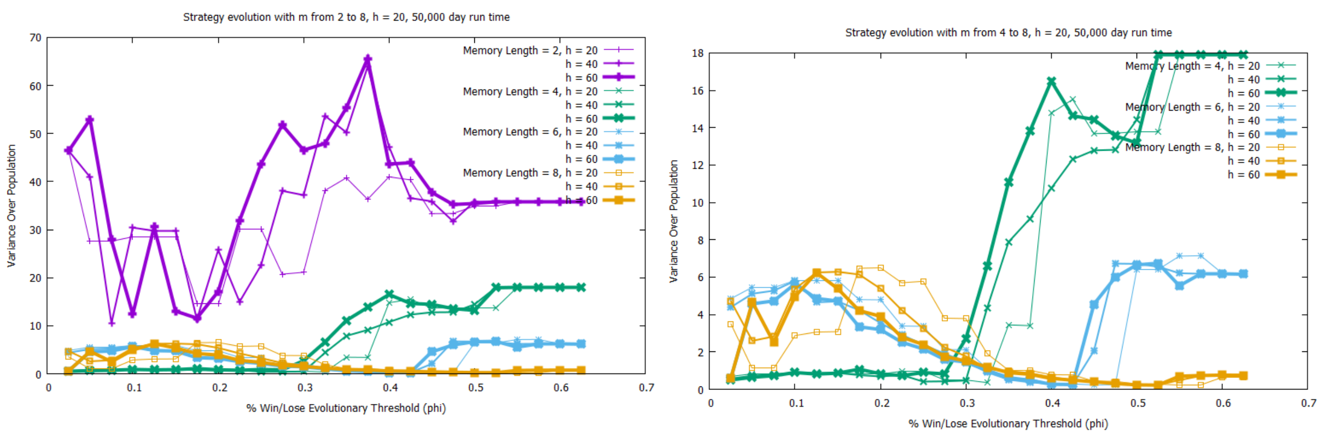


(a) Success rate memory of 6, h from 20 to 100

(b) Attendance ratio vs phi memory 4-8,

Figure 26: Attendance ratio for strategy evolution overall, and success rates for memory = 6, h from 20 to 100

born out by examination of figure 24, as there is a fairly distributed number of strategies per agent, as is consistent with agents being repeatedly rewarded and punished, alternately.



(a) Mean strategy count declines as agents come to perform closer to random

(b) Variance vs phi memory 4-8,

Figure 25: Variance for strategy evolution with evolutionary periods (h) from 20 to 60, memory values 2-8, ϕ to 0.65, step length 0.025, 1001 agents and 50,000 day run time

As observed in [13] [8], the number of strategies has little effect on the overall system behavior, merely the quantitative points, and this still appears to be true given an evolutionary methodology that allows for the expansion of the number of strategies each agent is using. In no ϕ or memory regime did the system perform better than it did with random variation, as increasing the number of strategies available to each agent only allows for agents to better spread out in strategy space, and as highlighted by prior research does not change the overall behavior.

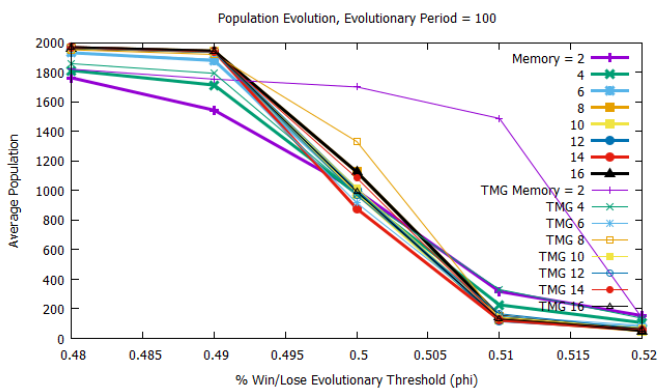
Figure 26a shows how increasing h leads to better resolution at the cost of evolutionary sensitivity, wherein agents are less likely to evolve (given that random data is more likely to be closer to the mean the more samples taken) but this means that each evolution is reflective of a longer consideration, or lifespan, of associated agents. An evolutionary methodology with varying ϕ as the overall best success rate was tested, but lead to no evolutions, and is therefore not shown.[53] This conclusion is not surprising,

[53]Though the corresponding graphs and raw data, as for all simulations and code, may be found at [3]

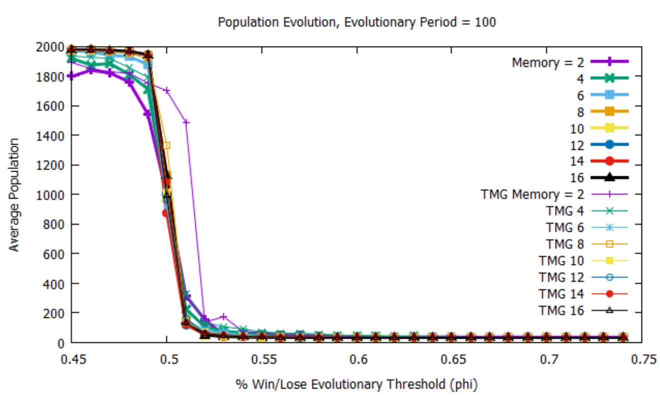
given the overall behavior of the systems when varied over ϕ ranges, and observing the immediate decline after the threshold exceeded that which was expected from randomness.

5.2.3 Population Evolution

To simulate population based evolution, wherein the total population attempts to evolve to achieve their optimal performance, our population model works akin to the memory and strategy evolutionary mechanisms, in that each agent individually has their own success or failure rates considered over given periods and compared against ϕ to determine whether or not they are destroyed, or another agent with the same memory and strategy count, but no biases (strategy score) is created. However, as all the strategy and memory counts are identical when not mixing evolutionary methodologies, this former detail does not come into play. Data was also collected against varying evolutionary period h, but as there was no difference in results, we have omitted the comparisons against varying evolutionary data. The population at the end of the simulation over the ϕ range 0.45 to 0.75 (figure 27b)reveals that the only transition of importance comes around the ϕ value of 0.5 to 0.6, wherein all agents immediately drop in population to the lowest possible, and as this simulation was to range from 2001 and 11, such are the defining bounds on their behavior.



(a) Agent population vs ϕ, 0.48 to 0.52



(b) Agent population vs ϕ, memory lengths 8-16,

Figure 27: Population at end of simulation time vs phi. There is a clear step transition from the over abundant phase $\phi < 0.5$ to the realalistic darwinian evolution, $\phi > 0.5$

When there is no variation in spawned strategy sets as in [7], one should expect a tension between two behaviors. The more similar the strategy pool (as comes from inbreeding, or the effective replication of strategy sets in a limited pool while others theoretically die off) the worse the overall performance, as we learned from the strategy evolution. However, as we saw when dealing with static distributions of

agents, there is an optimum ratio of agents to memory length, and we should expect to see new agents formed in an effort for the system to reach the optimal ratio, or get as close as possible. Given figure 13, and that $log_2(\frac{2001}{3}) = 9.381$, we might therefore expect all memory values greater than 9.381 to have the largest possible population, and those less to have approximately $3 \cdot 2^m$ agents. A glace at figure 27b suffices to show that this is, however not the case. The sharp decline in agent population after leaving the *over-bountiful* regime implies that one the whole, adding agents has a negative effect, and thus cannot be sustained when necessarily performing better than randomness. This is doubtless due to the dominance of high-turnover agent population, which leads to a heavily inbred population, and subsequently very poor performance.
Though a simulation of this population evolution methodology was completed with randomized memory lengths in a range from 2 to 16 (and thus no biases and little overlap) the only consequence was after the step transition at $\phi = 0.5$, the population remained steadily at its initial value, and did not decline as in this simulation. Thus, it appears that that the transition point is strategy space invariant, and turn over rate remains high (as it would have to for an equilibrium rate at non-minimal agent population) but the overcrowding of a narrowed, inbred strategy space is no longer a problem.

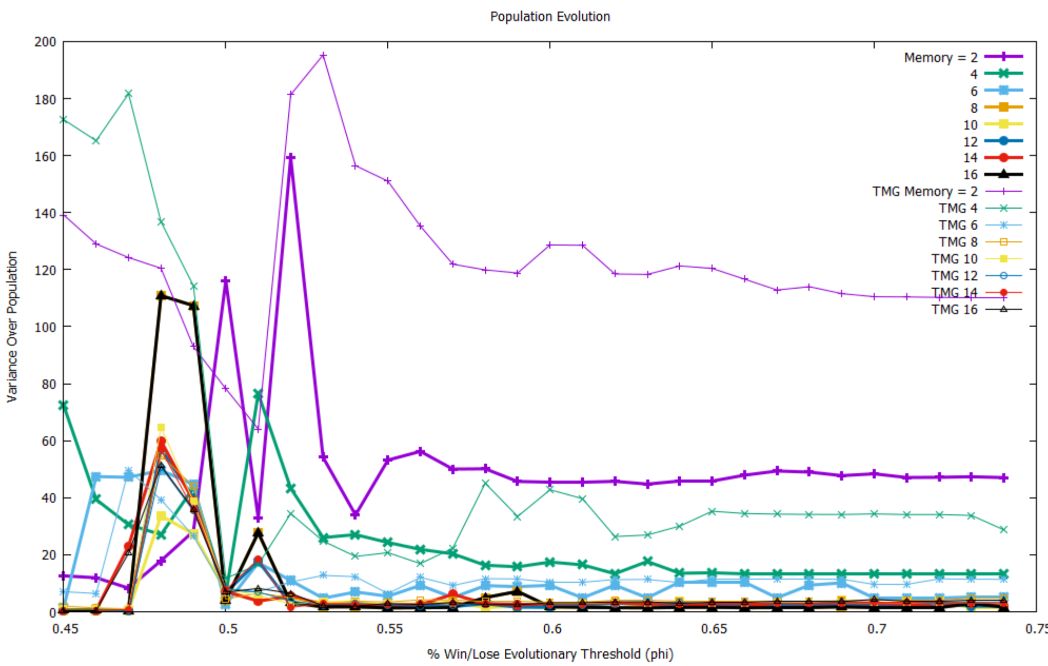


Figure 28: Variance over Population for Population evolution

In figure 28, we see the spike in variance corresponding to the above shift in population, as for low populations (around the minimum population value of of 11) any change in the attendance will lead to a large fluctuation. An implementation with evolutionary dynamic ϕ was implemented with both real and falsified histories, with a full range of memories, however the agent population never exceeded the minimum, and as all results were identical, their corresponding graphs are not shown. This behavior however only further confirms the above result, that inbred populations can only maintain their population if forced to against natural evolutionary regimes, as all $\phi > \frac{1}{2}$ represents, otherwise the eventual overlap in strategy space (the group think) leads to poor performance and crowded strategies.

6 Discussion

6.1 Overview of Results

As evolutionary methods use some parameter ϕ defining the ease by which the evolution is triggered, the overall behavior of the system will be highly dependent on ϕ, even for longer run times. As ϕ increases, random shifts in both directions due to evolutionary rewards and punishments reduce, and random shifts that are beneficial have an opportunity to lead to greater evolutionary reward before being again shifted down due to random fluctuation. For lower ϕ, there is an immediate race to the top (i.e. all agents evolve to the highest possible level) as random fluctuations trigger a positive feedback loop[54], but this race to the top leads to the $\alpha \gg \alpha_c$ regime, which is not the globally best solution. For higher values of ϕ evolution will be rare, as the barrier (number of random runs necessary before there is one above the threshold) to initial evolution is comparable to the overall run-time, meaning that there will be too few evolutions to lead to a positive feedback loop (especially when a single evolution no longer gives the agent sufficient advantage so as to be able to achieve another evolutionary step) and subsequent shift into the strategy dilute phase. Exact probabilities for random binary addition of a given number of times resulting in some value is given by equation 7. While the market histories are not random, their approximation as random is the basis for an analytic approach as given in [13], and can be taken to be true without compromising empirical accuracy[55].

Naturally, there is some happy in-between solution, that allows for enough agents to achieve their best possible equilibrium. As there is no direct cost function associated with constant evolution (as might be considered in real-world biological analogies) agents which perform well will benefit from the positive feedback loop, eventually leading to the longest memory regimes. Considering the multitude of possible strategy responses offered with higher memory values, it seems as though many agents would not be able to effectively distinguish between them, and therefore they perform effectively at random. This random behavior, while clearly better than overcrowding the limited strategy regimes of agents with low memory strategies, is still not optimal, thus one might suspect that this would indirectly lead to reformation, wherein agents then "retreat" in memory space (i.e. are punished by evolution), until they find a memory length that both allows them the flexibility to respond outside crowded memory regimes and coordinate. In reality, agents seem to crowd the edges of the available memory distribution, and those most "intelligent" agents remain so, or use a mixed strategy. This could be simply an issue of run-time, as being "punished" for random behavior might require both the resolution of longer evolutionary periods and very long run times at critical ϕ, where agents would be most sensitive to this larger difference. If this marginal optimization within the globally best equilibrium is a question of longer run times, the critical value of ϕ will have to be found with considerable precision to ensure a sufficient difference within optimal regimes to allow the evolutionary mechanism to hone in on the best memory distribution. Longer run times would also allow for the evolved agents to evolve from their performance (even if it is consistently worse than random due to coordination of the smart strategies) where the same positive feedback loop would then help equalize the memory distribution. Regardless of the possible propensity for agents to regress into their optimal memory distributions, the general behavior for mid ϕ ranges ($\sim 0.6 - 0.72$, for both strategy and memory methods) perform comparatively well, and comparable than random performance.

6.2 Comparison of results to other Minority Games

6.2.1 Comparison to the Original Minority Game

We tested Poisson, exponential, random, linear and constant distributions of memory with and without mandating subsequent elements of memory distribution go to sequential agents, and found that those distributions performed best in regimes where agent populations matched with their mean memory such

[54] random fluctuations $\rightarrow$ positive evolution $\rightarrow$ higher success rates $\rightarrow$ positive evolution etc...

[55] For further verification of the randomness of market history in evolutionary, see [3]

that $\alpha \approx \alpha_{critical}$, without much regard to the shape of the distribution itself. However, when considered over lager population ranges, certain distributions perform better in ensuring emphasis on performance at different population ranges, es defined by their relation to the critical memory to population ratio. All considered distributions perform within a 46% to 50% range, a claim that holds for the evolutionary dynamics as well. This marginal difference is expected given the Minority Game's original phase shift to the random, strategy dilute $\alpha \gg \alpha_{critical}$, and the corresponding distribution of memory and agent populations sampled.

All evolutionary methods were measured over a range of evolutionary thresholds ϕ and evolutionary periods, h, which together determined their sensitivity to random fluctuations. Both the strategy and population evolutionary methodologies lead to worse outcomes for advances in either (i.e. the greater the strategy count per agent, the worse the system as a whole does, and likewise for greater populations), leaving their only use in $\phi < 0.5$ regimes, wherein agents and systems performing worse than random are still forced to acquire new strategies or greater population, respectively. This leads to a paradoxical regime, as may be seen in figure 23b, wherein as $\phi \to 0.5$, agents are less 'forced' to take extra strategies, and thus the system performs better, which in turn leads to it needing to take more strategies. Overall, this leads to a high strategy turnover, with consequent minimally accrued bias, and thus little coordination and sub-random performance in susceptible memory regimes[56].

The Memory evolution methodology however lead to an immediate race to the highest possible memory value, bringing the system to the $\alpha \gg \alpha_c$ phase, which could not be distinguished from the α_c phase because the difference in success rates (which was the deciding factor in the evolutionary mechanism) was too small. Even when provided with an evolutionary mechanism to actively adjust ϕ to the best system performance in an effort to solidify what evolutionary gains were made, the subsequent exponential distribution of memory lengths bore the mark not of optimization to success rates but of the consequent evolutionary mechanism. The evolution showed considerable improvement with increased simulation times (as seen in figure 21), however overall greater simulation times only allowed for asymptotically closer approximations, as it is unlikely a more efficient α_c phase would be found via longer runtimes.

Overall the static memory distributions and the evolutionary mechanisms lead only to distributions whose defining attribute was how well their average memory value approximated m_c, or $log_2(\frac{N}{3})$. All evolutionary methods, while giving interesting behavior, were unable to adapt into the most efficient α_c regime, and instead only achieved the $\alpha \gg \alpha_c$ regime after effective adaption, with memory adaptation leading to these results immediately, and continuing until increases in ϕ lead to a gradual reduction in mean memory, while the strategy and population regimes only maintained their evolutionary behavior in the superabundant $\phi < \frac{1}{2}$ phase.

6.2.2 Comparison to the Thermal Minority Game (TMG)

All results were graphed alongside their Thermal Minority Game counterparts, which revealed a tendency for the fictitious history to perform slightly worse than that of the real history, though for longer run times this difference might disappear. Though our implementation of the thermal minority game did not allow for a continuous "betting" methodology, as in [5] and thus cannot said to be a true Thermal Minority game, as this would be incompatible with the existing implementation of the Adaptive Minority Game and its various evolutionary methodologies. Nonetheless, just by inventing a history at random that is then shared between all the agents, we have effectively created a Thermal Minority Game compatible with discrete Darwinistic evolutionary methods implemented through this paper.

Considering the behavior of the thermal minority game consistently mimicked that of the traditional minority game, we may readily conclude that the conclusion of [4], that the veracity of the memory considered is not important, so long as every shares their memory, holds true through evolutionary means. This is of particular interest when considering the implications of memory distributions, as agents using strategies of different memory length will be given different (invented) histories, and yet their behavior

[56] Those regimes wherein $\alpha \ll 1$

remains similar, albeit requiring a longer time to coordinate. The systematic under-performance (i.e. below that of the traditional minority game) of the thermal data suggests that the genuine history does still have a role to play, albeit not one that describes the overall behavior of the model. This is especially clear when considering variance over population values for different evolutionary models and periods, and the rate of evolutionary advance (and subsequent success rates) for memory evolution, **??**

6.3 Implications

Given that agents are only able to optimize for a given population count and memory size combination, without evolutionary methods having control of both, agents will evolve to suit their surrounding environment. Given that only learning[57] takes place outside the timescale of natural population dynamics, we might imagine our memory methodology analogous to systems wherein learning is available, and population dynamics are not; it is therefore potentially non-coincidental that the development of learning and population shifts occurred simultaneously, from the prospective of a competitive market. Imagining humanity[58] to be an implementation of a market whose overarching goal is to achieve an equilibrium with the highest success rate, this will require the implementation and control over both population and learning. Of course, as with any model, we should be skeptical of any argument which substitutes a binary choice for individual action, but the basic premise of market optimization based on its population is perfectly fundamental, and as agent coordination relies on their accessible choices [5] [16] [2], it follows that any market will be dependant on them both. All these analogies will only hold so long as they are representing a natural evolution, as the game's behavior provides. However, as soon as a force (which for example regulates the market, enforcing optimal results through its omniscient, omnibenevolent ways)) is applied seemingly endogenously, we need ask whether or not such a development is inherent in the system.[59]

The adaptive methodologies, when their ϕ regimes are put into the context of a biological or economic system yield results as tenuous as their analogy. For $\phi < \frac{1}{2}$ we are considering a regime wherein regardless of the agent behavior (unless they perform worse than randomly) they will receive an evolutionary reward. Which in the case of memory, or depth of intelligence (agents now have more ways to solve the same problem) is beneficial, though both over population and an over-abundance of overlapping ideas (strategy evolution in the $\phi < \frac{1}{2}$ regime) leads to the system performing worse than random, which leads the system to auto-correct before it reaches $\phi = \frac{1}{2}$.

6.4 Remarks on Analysis

The variance equation used for all measurements was defined thusly, and is in agreement with earlier definitions by [5] and [13].

$$\sigma^2 = \frac{1}{n}\sum_{i=0}^{n}(x_i - \mu)^2, \mu = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{5}$$

Success rates were calculated as

$$\sum_{i=0}^{N} \frac{N - |a(i)|}{2N} \tag{6}$$

where $a(i)$ is a vote from a given agent i, where each vote $= \pm 1$.

When considering evolutionary triggers, we must consider the probability of their trigger against pure randomness. For an evolutionary period (i.e. how far back evolution considers the agent's actions) N,

[57] For out purposes, learning is non-evolutionary intellectual adaptation

[58] or any other species and their corresponding market, though as the audience is assumed to be human, and we are most familiar with our own markets, we will continue with them in example

[59] i.e. For some level of self-aware systems, we'd call the behavior a form of management, and it is unjust to exclude this binary model from such a decision when it matches some strict definitions therefore.

and a requirement that a given evolutionary agent has over x elements of the last N as successes, we get that the probability of an agent achieving this via random chance is given by

$$\frac{\binom{N-x}{x}}{\binom{N}{k=2}} = \frac{(N-x)!(N-k)!k!}{(N-k-x)!N!x!} \tag{7}$$

7 Conclusion

A formulation of the variable sum Minority Game with adaptive limits on agent rationality and system size was tested against fixed intelligence distributions, and yielded results that compared poorly against random behavior. Through the introduction of the evolutionary threshold ϕ, evolutionary methodologies were tested against variable sensitivity, leading to the prevalence of a phase transition for all methodologies around $\phi \approx 0.5$, where-after agent performance must perform better than random to evolve, and consequently evolution either immediately dropped off akin to a step function behavior, as in the case of population evolution, or else declined slowly due to counteracting pressures. In the case of memory, as ϕ increases beyond 0.5 the positive force of greater memory leading to better performance is counteracted by the heightened threshold, or as in the case of strategy evolution, the negative consequences of overcrowding strategy space is reduced as ϕ approaches 0.5, and agents no longer need to use so many strategies. The under-performance of all evolutionary methodologies is understood in the context of the extremely marginal difference between the optimal α_c regime, and $\alpha \gg \alpha_c$, as random fluctuations from the evolutionary methods themselves and subsequent loss of agent bias, or learning collectively deny agents the ability to coordinate, even when in optimal ϕ regimes. This was more conclusively investigated via an evolutionary method which adapted the system's ϕ according to the system's best success rate, but came to similar conclusions.

An initial investigation into the performance of static memory distributions revealed that performance is based simply on mean memory value and its proximity to the optimal value such that $\alpha_c \approx \frac{1}{3}$, as in [13] [17]. The character of the distribution only changed the performance as it suited varying populations, with each density of memory values corresponding to an optimal population that would allow for effective coordination, and consequently marginally better than random behavior. The evolution of memory, strategy and population regimes was initially suspected to produce optimal solutions corresponding to whatever adjustments were necessary to ensure $\alpha = \alpha_c$, but instead because the difference between performance in $\alpha = \alpha_c$ and $\alpha \gg \alpha_c$ was too small, evolution simply lead to the $\alpha \gg \alpha_c$ dilute strategy space regime.

Given the results exposed above, namely that strategy count has an adverse effect on performance, all evolutionary methodologies perform only as well as their population-memory distribution matches that for α_c, and that the difference in success rates for α_c and $\alpha \gg \alpha_c$ are too small for the adaptive mechanism to detect, we conclude that further research with mixed agent populations, evolutionary regimes, and a cost function tied to evolutionary development could lead to results more compatible with evolutionary biology, and its corresponding analogous systems shown to have impressive parity to both economic and biological systems.

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