1st order PT $\quad \Psi(x,t) = \int_{\mathcal{E}_I}^{\infty} d\omega\, c(\omega)\, |x,\omega\rangle\, e^{-i\omega t} \quad \left\{ c(\omega) = d(\omega)\, cet(\omega,t) \right\} =$

NOTES FOR FOCUSSING OF FREE (PLANE WAVE) IONIZED WAVE PACKETS

$$= \int_{\mathcal{E}_I}^{\infty} d\omega\, d(\omega)\, cet(\omega,t)\, e^{-i\omega t}\, |x,\omega\rangle$$

$$t > t_p = \int_{\mathcal{E}_I}^{\infty} d\omega\, d(\omega)\, 2\pi\, \mathcal{E}(\omega)\, e^{-i\omega t}\, |x,\omega\rangle$$

Gaussian pulse with shaped phase. $\phi(\omega)$

$$\mathcal{E}(\omega) = e^{-\tau(\omega-\omega_0)^2 + i\phi(\omega)}$$

Express $\phi(\omega)$ as polynomial : $\phi(\omega) = \phi(\omega_0) + \dot\phi(\omega-\omega_0) + \frac{\ddot\phi}{2}(\omega-\omega_0)^2$
(Taylor expansion)

In time domain this corresponds to

$$\mathcal{E}(t) = \underbrace{\mathcal{E}_0}_{\text{amplitude}}\, \underbrace{e^{i\phi(\omega_0)}}_{\text{phase shift}}\, \underbrace{e^{-(t-\dot\phi)^2/\tau^2}}_{\text{temporal envelope centered at } t_0=\dot\phi}\, \underbrace{e^{i\delta(t-\ddot\phi)^2}}_{\text{chirp}}\, \underbrace{e^{i\omega_0 t}}_{\text{base freq (oscillation)}}$$

the oscillatory bits are $\alpha = \phi(\omega_0) + \delta(t-\ddot\phi)^2 + \omega_0 t$

$$\frac{d\alpha}{dt} = \omega_0 + 2\delta(t-\ddot\phi) = \omega_0 - 2\delta\ddot\phi + 2\delta t$$

$$\delta = \frac{2\ddot\phi}{\tau_0^4\left(1 + \frac{4\ddot\phi_0^2}{\tau_L^4}\right)} \quad \text{chirp factor}$$

rewrite $\Psi(x,t)$ integral in terms of k. $\quad \omega = \frac{E}{\hbar} = \frac{p^2/2m}{\hbar} = \frac{\hbar^2k^2/2m}{\hbar} = \frac{\hbar k^2}{2m}$

$$\left[\frac{d\omega}{dk} = \frac{2\hbar k}{2m} = \frac{\hbar k}{m} \iff d\omega = \frac{\hbar k}{m}\, dk \right]$$

$$\Psi(x,t) = \int_{\mathcal{E}_I}^{\infty} dk \frac{\hbar k}{m}\, d(\omega(k))\, 2\pi\, \mathcal{E}(\omega(k))\, e^{-i\omega(k)t}\, |x,k\rangle = \quad \left\{ |x,k\rangle = e^{ikx} \right\}$$

$$= \frac{2\pi\hbar}{m} \int_{\mathcal{E}_I}^{\infty} k\, d(\omega)\, \mathcal{E}(\omega)\, e^{ikx - i\omega t}\, dk = \frac{2\pi\hbar}{m} \int_{\mathcal{E}_I}^{\infty} a(k)\, e^{ikx - i\omega t}\, dk$$

$$a(k) = k\, d(\omega)\, \mathcal{E}(\omega), \quad \omega = \frac{\hbar k^2}{2m} = c_0 k^2$$

Mathematica – FT of exp function

$$e^{-m(\omega-\omega_0)^2} \longleftrightarrow \frac{1}{\sqrt{2m}} e^{-t^2/4m - it\omega_0}$$

$$e^{-m(\omega-\omega_0)^2 + ia} \longleftrightarrow \frac{e^{ia}}{\sqrt{2m}} e^{-t^2/4m - i\omega_0 t}$$

$$e^{-m(\omega-\omega_0)^2 + i[a+b(\omega-\omega_0)]} \longleftrightarrow \frac{e^{ia}}{\sqrt{2m}} e^{-(t-b)^2/4m - i\omega_0 t}$$

$\Delta\omega = \omega - \omega_0$

$$e^{-m\Delta\omega^2 + i(a+b\Delta\omega + c\Delta\omega^2)} \longleftrightarrow \frac{e^{ia}}{\sqrt{2}\sqrt{m-ic}} e^{-i\omega_0 t + \frac{m(t-b)^2}{4(c^2+m^2)} + \frac{ic(t-b)^2}{4(c^2+m^2)}}$$

$$\frac{1}{\sqrt{m-ic}} = \frac{\sqrt{m+ic}}{\sqrt{m^2+c^2}} = \frac{(m^2+c^2)^{1/4} e^{i\phi/2}}{(m^2+c^2)^{1/2}} = \frac{e^{i\phi/2}}{(m^2+c^2)^{1/4}}$$

$$\left(m+ic = \sqrt{m^2+c^2}\, e^{i\phi}, \quad \tan\phi = \frac{c}{m}\right)$$

$$\boxed{\frac{e^{i(a+\frac{\phi}{2})}}{\sqrt{2}\,(m^2+c^2)^{1/4}} e^{-i\omega_0 t + \frac{m(t-b)^2}{4(c^2+m^2)} + i\frac{c(t-b)^2}{4(c^2+m^2)}}}$$

Duration (not intensity, but amplitude → not strictly accord. to convention but gives ballpark)

$$e^{-\Delta t^2/4m} = \frac{1}{2} = e^{-\ln 2} \Rightarrow \Delta t^2 = 4m\ln 2 \Rightarrow \Delta t = 2\sqrt{m\ln 2}$$

with chirp

$$e^{-\frac{m}{4(c^2+m^2)}\Delta t^2} = \frac{1}{2} = e^{-\ln 2} \rightarrow \Delta t^2 = \frac{\ln 2 \; 4(c^2+m^2)}{m} \Rightarrow$$

$$\Rightarrow \Delta t = \frac{2\sqrt{\ln 2}\sqrt{c^2+m^2}}{\sqrt{m}}$$

General wave packets

1. $\langle x\rangle_0$ $\quad$ ~~$\langle x(t)\rangle$~~ $\langle x\rangle_t$ $\quad$ $\langle p\rangle$ + dispersion

Free-particle ~~SE~~ TDSE

$$H = \frac{p^2}{2m} = \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

Spectral equation

$$H\psi(x) = E\psi(x) \quad \Rightarrow \quad \psi(x) = e^{\pm ikx}$$

$$E = -\frac{\hbar^2}{2m}(\pm ik)^2 = \frac{\hbar^2k^2}{2m} = \frac{p^2}{2m} \qquad (p = \hbar k)$$

Time dependent

$$(k\in[-\infty,\infty]) \qquad e^{ikx - iEt/\hbar} = \psi(x,t) = e^{ikx-i\omega t}$$

$$\omega = \frac{E}{\hbar} = \frac{(p^2/2m)}{\hbar} = \frac{\hbar k^2}{2m}$$

Fix x, vary t: period $T = \frac{2\pi}{\omega} = \frac{1}{\nu}$, $\omega = 2\pi\nu$ $\qquad E = \hbar\omega = \frac{h}{2\pi}2\pi\nu = h\nu$

Fix t, vary x: period $\lambda = \frac{2\pi}{k}$

Phase velocity

$\psi(x,t) = e^{ikx - iEt/\hbar}$. Find (x,t) trajectory with constant phase.
$x(t)$

$$\phi(x,t) = kx - \frac{Et}{\hbar} = kx - \omega t = c$$

$$k\,x(t) - \omega t = c$$

$$x(t) = \frac{c}{k} + \frac{\omega t}{k} = x_0 + \frac{\omega}{k}t, \quad x_0 = \frac{c}{k}$$

$$\dot{x}(t) = \frac{\omega}{k} = \text{phase velocity} = \frac{\hbar k^2}{2mk} = \frac{\hbar k}{2m} = \frac{p}{2m} = \frac{1}{2}v_{cl}$$

wavepackets

$$\Psi(x,t) = \int a(k,t)\, e^{ikx}\, dk = \int a(k)\, e^{-i\omega t}\, e^{ikx}\, dk \quad , \text{ at } t=0 \quad \Psi(x,0) = \int a(k)\, e^{ikx}\, dk$$

$$a(k,t) = \int \Psi(x,t)\, e^{-ikx}\, dx \quad , \text{ using } \int e^{i(k-k')x}\, dx = 2\pi\, \delta(k-k')$$

This immediately gives us the normalization

$$N^2(t) = \int \Psi^*(x,t)\, \Psi(x,t)\, dx = \frac{1}{4\pi^2} \int dx \left[\int dk\, a(k,t)\, e^{-ikx}\right]\left[\int dk'\, a(k',t)\, e^{-ik'x}\right] =$$

$$= \frac{1}{2\pi} \int dk\, a^*(k,t)\, a(k,t) = \frac{1}{2\pi} \int dk\, a^*(k)\, a(k) = \frac{1}{2\pi} \int dk\, |a(k)|^2$$

$\Rightarrow N^2$ time independent. W-p preserves probability.

$$\int dx\, x\, e^{i(k-k')x} = \int dx \left(i\frac{d}{dk'}\right) e^{i(k-k')x} = i\frac{d}{dk'}\, 2\pi\, \delta(k-k')$$

center of w-p

$$\langle x\rangle_t = \int dx\, \Psi^*(x,t)\, x\, \Psi(x,t) = 2\pi \int dk\, a^*(k,t)\, i\frac{d}{dk}\, a(k,t) =$$

$$= 2\pi i \int dk\, a^*(k)\, e^{i\omega t} \frac{d}{dk}\, a(k)\, e^{-i\omega t} = 2\pi i \int dk\, a^*(k)\, e^{+i\omega t}\left[a'(k)\, e^{-i\omega t} + a(k)\left(-it\frac{d\omega}{dk}\right)e^{-i\omega t}\right]$$

$$= \underbrace{2\pi i \int dk\, a^*(k)\, \frac{d\,a(k)}{dk}}_{I_1} + \underbrace{2\pi t \int dk\, \frac{d\omega}{dk}\, a^*(k)\, a(k)}_{I_2} =$$

$$I_2 = 2\pi t \int dk\, \frac{d\omega}{dk}\, |a(k)|^2 = \frac{2\pi t}{m} \int dk\, (\hbar k)\, |a(k)|^2 = \frac{\langle p\rangle t}{m}$$

$$I_1 = \langle x\rangle_0$$

$$a(k) = |a(k)|\, e^{i\phi(k)} \quad , \quad \frac{d\,a(k)}{dk} = \frac{d\,|a(k)|}{dk}\, e^{i\phi(k)} + |a(k)|\, i\phi'(k)\, e^{i\phi(k)}$$

$$I_1 = 2\pi i \int dk\, |a(k)|\, \frac{d\,|a(k)|}{dk} + |a(k)|^2\, i\frac{d\,\phi(k)}{dk} =$$

$$= \underbrace{2\pi i\left[\tfrac{1}{2}|a(k)|^2\right]}_{=0 \text{ for square integrable}} - 2\pi \int dk\, \frac{d\phi(k)}{dk}\, |a(k)|^2 = -2\pi \int dk\, \frac{d\phi(k)}{dk}\, |a(k)|^2$$

$$\Rightarrow \quad \langle x \rangle_t = \frac{\langle p \rangle t}{m} + \langle x_0 \rangle$$

$$\langle x_0 \rangle = -2\pi \int dk \, \frac{d\phi(k)}{dk} \, |a(k)|^2$$

$$\phi(k) = c \Rightarrow \dot{\phi}(k) = 0 \Rightarrow \langle x_0 \rangle = 0$$

$$\phi(k) = \alpha k \Rightarrow \dot{\phi}(k) = \alpha \rightarrow \langle x_0 \rangle = -2\pi \alpha \int dk \, |a(k)|^2 = -\alpha$$

Group velocity $\frac{d}{dt} \langle x \rangle_t = \frac{\langle p \rangle}{m} = v_g$

$$\omega = \frac{E}{\hbar} = \frac{1}{\hbar}\left(\frac{p^2}{2m}\right) = \frac{\hbar k^2}{2m}$$

$$\frac{d\omega}{dk} = \frac{\hbar k}{m}$$

$$v_g = \frac{1}{m} \int dx \, \psi^*(x,t) \left(-i\hbar \frac{d}{dx}\right) \psi(x,t) = \frac{2\pi}{m} \int a^*(k,t) \, \hbar k \, a(k,t) \, dk =$$

$$= 2\pi \int dk \, \frac{d\omega}{dk} \, |a(k)|^2 = \left\langle \frac{d\omega}{dk} \right\rangle$$

Stationary phase perspective

$a(k)$ peaked at k_0

~~$a(k,t) = a(k)$.~~

$$\psi(x,t) = \int dk \, a(k) \, e^{ikx - i\omega t}$$

$$\phi = kx - \omega t$$

$$\left.\frac{d\phi}{dk}\right|_{k=k_0} = 0 = x - \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} t = x - \frac{\hbar k_0}{m} t = 0 \qquad x = \frac{\hbar k_0}{m} t$$

$$\dot{x} = \frac{\hbar k_0}{m}$$

Dispersion

$$\langle x\rangle_t = \frac{\langle p\rangle}{m} t + \langle x\rangle_0 = v_g t + \langle x\rangle_0$$

$$\boxed{\omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar} = \frac{\hbar k^2}{2m} \qquad \frac{\partial\omega}{\partial k} = \frac{\hbar k}{m} \qquad \frac{\partial^2\omega}{\partial k^2} = \frac{\hbar}{m}}$$

$$\langle x_0\rangle = -2\pi\int dk \frac{d\phi}{dk} |a(k)|^2$$

$$\langle x\rangle_t^2 = (v_g t + \langle x\rangle_0)^2 = v_g^2 t^2 + \langle x\rangle_0^2 + 2v_g t\langle x\rangle_0 = \frac{\hbar^2 t^2\langle k\rangle^2}{m^2} + \langle x\rangle_0^2 + \frac{2\hbar\langle k\rangle t\langle x\rangle_0}{m}$$

$$\boxed{\Delta x_t^2 = \langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle_t - \langle x\rangle_t^2}$$

Need to calculate $\langle x^2\rangle_t$

$$\langle x^2\rangle_t = 2\pi\int dk\, a^* e^{i\omega t} \left(-\frac{d^2}{dk^2}\right) a e^{-i\omega t} = -2\pi\int dk\, a^* e^{i\omega t}\frac{d^2}{dk^2} a e^{-i\omega t}$$

$$\frac{d^2}{dk^2} a e^{-i\omega t} = \ddot{a} e^{-i\omega t} - 2i\dot{a} t\frac{d\omega}{dk} e^{-i\omega t} - i a t\frac{d^2\omega}{dk^2} e^{-i\omega t} - a t^2 e^{-i\omega t}\left(\frac{d\omega}{dk}\right)^2$$

$$\langle x^2\rangle_t = -2\pi\int dk\left[a^*\ddot{a} - 2it\frac{d\omega}{dk}a^*\dot{a} - it\frac{d^2\omega}{dk^2}|a|^2 - t^2\left(\frac{d\omega}{dk}\right)^2|a|^2\right] =$$

$$= \underbrace{-2\pi\int dk\, a^*\ddot{a}}_{I_1} + 4\pi i t\underbrace{\int dk\frac{d\omega}{dk}a^*\dot{a}}_{I_2} + \frac{it\hbar}{m} + \underbrace{\frac{t^2\hbar^2\langle k^2\rangle}{m^2}}_{t^2\langle v_g^2\rangle}$$

(We set $a(k) = \bar{a}(k)\, e^{i\phi(k)}$ — $\bar{a}$ ↑ real, $e^{i\phi}$ ↑ complex)

$$I_1 = \int dk\, a^*\ddot{a} = \int dk\left[\bar{a}\ddot{\bar{a}} + i\ddot{\phi}\bar{a}^2 + 2i\dot{\phi}\bar{a}\dot{\bar{a}} - \dot{\phi}^2\bar{a}^2\right]$$

$$\frac{d^2}{dk^2}\bar{a}e^{i\phi} = \frac{d}{dk}\left[\dot{\bar{a}}e^{i\phi} + i\dot{\phi}\bar{a}e^{i\phi}\right] = e^{i\phi}\left[\ddot{\bar{a}} + i\ddot{\phi}\bar{a} + 2i\dot{\bar{a}}\dot{\phi} - \bar{a}\dot{\phi}^2\right]$$

$$I_2 = \int dk\, a^*\dot{a}\frac{d\omega}{dk} = \underbrace{\int dk\,\bar{a}\dot{\bar{a}}\frac{d\omega}{dk}}_{\text{integrate by parts}} + \int dk\, i\dot{\phi}\bar{a}^2\frac{d\omega}{dk} = -\frac{\hbar}{4\pi m} + i\int dk\,\dot{\phi}\bar{a}^2\frac{d\omega}{dk}$$

$a = \bar{a}e^{i\phi}$

$$\frac{da}{dk} = \dot{\bar{a}}e^{i\phi} + i\dot{\phi}\bar{a}e^{i\phi}$$

$$I = \underbrace{\left[\frac{\bar{a}^2}{2}\frac{d\omega}{dk}\right]_{-\infty}^{\infty}}_{=0} - \frac{1}{2}\int dk\,|\bar{a}|^2\frac{d^2\omega}{dk^2} = -\frac{\hbar}{4\pi m}$$

Dispersion

$$\langle x^2\rangle_t = -2\pi\int dk\left[\bar a\ddot a + i\ddot\phi\,\bar a^2 + 2i\dot\phi\,\bar a\dot a - \dot\phi^2\bar a^2\right] + 4\pi i t\left(-\frac{\hbar}{4\pi m} + i\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk}\right) + \frac{it\hbar}{m} + \frac{t^2\hbar^2\langle k^2\rangle}{m^2} =$$

$$= -2\pi\int dk[\;] - 4\pi t\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk} - \frac{4\pi i t\hbar}{4\pi m} + \frac{it\hbar}{m} + \frac{t^2\hbar^2\langle k^2\rangle}{m^2} =$$

$$\left\{-2\pi\int dk\,\bar a\ddot a = \langle x^2\rangle_0\right.$$

$$= -2\pi\int dk[\;] - 4\pi t\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk} + \frac{t^2\hbar^2\langle k^2\rangle}{m^2} =$$

$$= \langle x^2\rangle_0 + \frac{t^2\hbar^2\langle k^2\rangle}{m^2} \underbrace{-2\pi\int dk\left[i\ddot\phi\,\bar a^2 + 2i\dot\phi\,\bar a\dot a - \dot\phi^2\bar a^2\right] - 4\pi t\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk}}_{Q} \qquad (*)$$

Ensures $\langle x^2\rangle$ is real

Note added 8/3 2009:
$-2\pi i\int dk\,\ddot\phi\,\bar a^2 + 2\dot\phi\,\bar a\dot a = -2\pi i\int dk\left(\frac{d}{dk}\dot\phi\,\bar a^2\right) = \left[\dot\phi\,\bar a^2\right]_{-\infty}^{\infty} = 0$
(square integrable function)

$$\boxed{\Delta x_t^2 = \langle x^2\rangle_t - \langle x\rangle_t^2}$$

$$\frac{d\Delta x_t^2}{dt} = \frac{d\langle x^2\rangle_t}{dt} - \frac{d\langle x\rangle_t^2}{dt} = -4\pi\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk} + \frac{2t\hbar^2\langle k^2\rangle}{m^2} - \frac{2t\hbar^2\langle k\rangle^2}{m^2} - \frac{2\hbar\langle k\rangle\langle x\rangle_0}{m} = 0$$

$$\frac{2t\hbar^2}{m^2}\Delta k^2 = \frac{2\hbar\langle k\rangle\langle x\rangle_0}{m} + 4\pi\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk}$$

$$(YY) \qquad t_0 = \frac{m^2}{2\hbar^2\Delta k^2}\left[\frac{2\hbar\langle k\rangle\langle x\rangle_0}{m} + 4\pi\int dk\,\dot\phi\,\bar a^2\frac{d\omega}{dk}\right]$$

This the time at which we have a minimum width wavepacket in the general case.

$$\Rightarrow (*) \qquad \langle x^2\rangle_t = \langle x^2\rangle_0 + \frac{t^2\hbar^2\langle k^2\rangle}{m^2} + Q$$

$$\langle x\rangle_t^2 = \left(\frac{\hbar t\langle k\rangle}{m} + \langle x\rangle_0\right)^2 = \frac{\hbar^2t^2\langle k\rangle^2}{m^2} + \langle x\rangle_0^2 + \frac{2\hbar t\langle k\rangle}{m}\langle x\rangle_0$$

$$\Delta x_t^2 = \langle x^2\rangle_t - \langle x\rangle_t^2 = \frac{\hbar t^2}{m^2}\Delta k^2 + \Delta x_0^2 + Q - \frac{2\hbar t\langle x\rangle_0\langle k\rangle}{m} \qquad (XX)$$

$\hookrightarrow \Delta k^2 = \langle k^2\rangle - \langle k\rangle^2$

for listening $\frac{d\omega}{dk} = \frac{\hbar k}{m}$

$$(YX) \qquad t_0 = \frac{m^2}{2\hbar^2\Delta k^2}\frac{2\hbar}{m}\left[\langle k\rangle\langle x\rangle_0 + 2\pi\int dk\,\dot\phi\,\bar a^2 k\right] = \frac{m}{\hbar\Delta k^2}\left[\langle k\rangle\left\{-2\pi\int dk\,\dot\phi\,\bar a^2\right\} + 2\pi\int dk\,\dot\phi\,\bar a^2 k\right] =$$

$$= \frac{m}{\hbar\Delta k^2}\left[2\pi\int dk\,(k - \langle k\rangle)\dot\phi\,\bar a^2\right]$$

It follows immediately that for

const $\dot\phi = 0 \Rightarrow t_0 = 0$

linear $\dot\phi = c \Rightarrow t_0 = 0$

square $\dot\phi = ck \Rightarrow t_0 = \cancel{\Delta k}\frac{mc}{\hbar\Delta k^2}\cdot\Delta k^2 = \frac{mc}{\hbar}$

COOL!

~~Dispersion~~

~~special case $\phi = c$, constant, $\dot\phi = 0$~~

~~The expression reduces to~~

$$\langle x^2\rangle_t = -2\pi\int dk\, \bar{a}\ddot{a} + \frac{t^2\hbar^2\langle k^2\rangle}{m^2} = \langle x^2\rangle_0 + \frac{t^2\hbar^2\langle k^2\rangle}{m^2}$$

$$\langle x\rangle_t = \frac{\hbar t\langle k\rangle}{m} + \langle x\rangle_0 \quad (\text{from before and } \langle x\rangle_0 = 0)$$

$$\Delta x_t^2 = \langle x^2\rangle_t - \langle x\rangle_t^2 =$$

Dispersion

special case $\phi = c$, $\dot\phi = 0$

$$\Delta x_t^2 = \frac{\hbar^2 t^2 \Delta k^2}{m^2} + \Delta x_0^2 - \frac{2\hbar t\langle x\rangle_0\langle k\rangle}{m} + Q \qquad (\text{eq. XX})$$

$\dot\phi = 0 \Rightarrow Q = 0$ and $\langle x\rangle_0 = 0$

$$\Rightarrow \Delta x_t^2 = \frac{\hbar^2 t^2 \Delta k^2}{m^2} + \Delta x_0^2$$

Dispersion : $\dfrac{d}{dt}\sqrt{\Delta x_t^2 - \Delta x_0^2} = \dfrac{\hbar\Delta k}{m}$

minimum width at $t = 0$ (eq. YY)

Dispersion

special case $\dot\phi = c \Rightarrow \langle x\rangle_0 = -c$
(linear phase in $a(k)$)

time at which w-p has minimum width?

$$t_0 = \frac{m^2}{2\hbar^2 \Delta k^2}\left[\frac{2\hbar\langle k\rangle\langle x\rangle_0}{m} + 4\pi\int dk\, \dot\phi\, \bar a^2 \frac{d\omega}{dk}\right]$$

$$= \frac{m^2}{2\hbar^2\Delta k^2}\left[\frac{-2\hbar\langle k\rangle c}{m} + 4\pi c\frac{\hbar\langle k\rangle}{2\pi m}\right] = \frac{m^2}{2\hbar^2\Delta k^2}\left[-\frac{2\hbar c\langle k\rangle}{m} + \frac{2\hbar c\langle k\rangle}{m}\right] = 0$$

$\Rightarrow$ even with linear $\phi = kc + \gamma$ the $t=0$ wave packet is the minimum uncertainty one – except it is formed at $\langle x\rangle_0 = -c$

Strategi för att fokusera vågpaket

$$\langle x\rangle_t = \frac{t\hbar\langle k\rangle}{m} + \langle x\rangle_0$$

$$t_0 = \frac{m}{\hbar\Delta k^2}\left[2\pi\int dk(k-\langle k\rangle)\,\dot\phi\,\bar a^2\right]$$

if excitation pulse adds phase that cancels excitation ~~pulse~~ phase so that $a(k) = \bar a(k)$, i.e. real.
then ~~to~~ add a square phase $\phi = \frac{1}{2}ck^2$, $\dot\phi = ck$

this gives

$$\langle x\rangle_0 = -2\pi\int dk\,\dot\phi\,\bar a^2 = -2\pi c\int dk\, k\,\bar a^2 = -c\langle k\rangle$$

$$t_0 = c\frac{m}{\hbar}$$

choose a target distance ~~x~~ x_p at which we want to have a focussed w.p.

$$x_p = \langle x\rangle_{t=t_0} = \frac{cm}{\hbar}\cdot\frac{\hbar\langle k\rangle}{m} - c\langle k\rangle = 0$$

Fokusera vågpaket

$$\begin{cases} \langle x \rangle_t = \dfrac{t\hbar \langle k \rangle}{m} + \langle x \rangle_0 \\ \langle x \rangle_0 = -2\pi \int dk\, \dot{\phi}\, \bar{a}^2 \\ t_0 = \dfrac{m}{\hbar \Delta k^2}\left[2\pi \int dk\, (k - \langle k \rangle\, \dot{\phi}\, \bar{a}^2\right] \end{cases}$$

$$\langle x \rangle_{t=t_0} = \frac{\langle k \rangle}{\Delta k^2}\left[2\pi \int dk\, (k-\langle k \rangle)\,\dot{\phi}\,\bar{a}^2\right] - 2\pi \int dk\, \dot{\phi}\, \bar{a}^2 =$$

$$= \frac{\langle k \rangle}{\Delta k^2}\, 2\pi \int dk \left[k - \langle k \rangle - \frac{\Delta k^2}{\langle k \rangle}\right] \dot{\phi}\, \bar{a}^2$$

$\dot{\phi} = 0 \qquad \langle x \rangle_{t=t_0} = 0$

$\dot{\phi} = c \qquad \langle x \rangle_{t=t_0} = -2\pi \int dk\, \dot{\phi}\, \bar{a}^2 = \langle x \rangle_0$

$\dot{\phi} = ck \qquad \langle x \rangle_{t=t_0} = \dfrac{\langle k \rangle c}{\Delta k^2}\left(\langle k^2 \rangle - \langle k \rangle^2 - \Delta k^2\right) = 0$

$\dot{\phi} = ck^2 \qquad \langle x \rangle_{t=t_0} = \dfrac{c\langle k \rangle}{\Delta k^2}\, 2\pi \int dk \left[k^3 - \langle k \rangle k^2 - \dfrac{\Delta k^2 k^2}{\langle k \rangle}\right] \bar{a}^2 =$

$$= \frac{c\langle k \rangle}{\Delta k^2}\left(\langle k^3 \rangle - \langle k \rangle \langle k^2 \rangle - \frac{\Delta k^2 \langle k^2 \rangle}{\langle k \rangle}\right) = \frac{c\langle k \rangle}{\Delta k^2}\left(\langle k^3 \rangle - \frac{\langle k^2 \rangle^2}{\langle k \rangle}\right) =$$

$$= c\left[\frac{\langle k \rangle \langle k^3 \rangle}{\Delta k^2} - \frac{\langle k^2 \rangle^2}{\Delta k^2}\right]$$

$$\begin{cases} \langle x \rangle = \frac{t\hbar\langle k\rangle}{m} + \langle x\rangle_0 \\ \langle x\rangle_0 = -2\pi\int dk\, \dot{\phi}\, \bar{a}^2 \\ t_0 = \frac{m}{\hbar\Delta k^2}\left[2\pi\int dk\,(k-\langle k\rangle)\,\dot{\phi}\,\bar{a}^2\right] \\ \langle x\rangle_{t=t_0} = \frac{\langle k\rangle}{\Delta k^2}\, 2\pi \int dk\left[k-\langle k\rangle - \frac{\Delta k^2}{\langle k\rangle}\right]\dot{\phi}\,\bar{a}^2 \end{cases}$$

$\phi = c \Rightarrow \dot{\phi} = 0$

$$\langle x\rangle = \frac{\hbar\langle k\rangle}{m}t + \langle x\rangle_0$$

$$\langle x\rangle_0 = 0$$

$$t_0 = 0$$

$$\langle x\rangle_{t=t_0} = 0$$

$\phi = ck \Rightarrow \dot{\phi} = c$

$$\langle x\rangle = \frac{\hbar\langle k\rangle}{m}t + \langle x\rangle_0$$

$$\langle x\rangle_0 = -c$$

$$t_0 = 0$$

$$\langle x\rangle_{t=t_0} = -c$$

$\omega = \frac{\hbar k^2}{2m}$, $\sqrt{\omega} = \sqrt{\frac{\hbar}{2m}}\,k$

$e^{-m(\omega-\omega_0)^2 + ib\sqrt{\omega}} \Rightarrow c = \sqrt{\frac{\hbar}{2m}}\,b$

$\phi = ck^2 \Rightarrow \dot{\phi} = 2ck$

$$\langle x\rangle = \frac{\hbar\langle k\rangle}{m}t + \langle x\rangle_0$$

$$\langle x\rangle_0 = -2c\langle k\rangle$$

$$t_0 = \frac{2cm}{\hbar}$$

$$\langle x\rangle_{t=t_0} = 0$$

$$\boxed{e^{-m(\omega-\omega_0)^2 + ib(\omega-\omega_0)} \longleftrightarrow \frac{1}{\sqrt{2m}}e^{-i\omega_0 t - \frac{(t-b)^2}{4m}}}$$

$\omega = \frac{\hbar k^2}{2m}$ $\left(E = \frac{p^2}{2m}, \omega = \frac{E}{\hbar}, p = \hbar k\right)$

$$\phi = b(\omega-\omega_0) = \frac{b\hbar}{2m}(k^2-k_0^2) = c(k^2-k_0^2)$$

$$\frac{d\phi}{dk} = 2ck, \quad c = \frac{b\hbar}{2m}$$

$$t_0 = \frac{2m}{\hbar}c = \frac{2m}{\hbar}\frac{\hbar}{2m}b = b$$

<u>Hence</u> pulse centered at $t = b$ and minimum width w.p. emerges at $t_0 = b$ with $\langle x\rangle_{t=t_0=b} = 0$

$\phi = c k^4 \Rightarrow \dot{\phi} = 4ck^3$ $\quad \left(\propto e^{-ib(\omega-\omega_0)^2}, \ \omega^2 \propto k^4 \right)$

$$\langle x \rangle = \frac{\hbar \langle k \rangle}{m} t + \langle x \rangle_0$$

$$\langle x \rangle_0 = -4c \langle k^3 \rangle$$

$$t_0 = \frac{4cm}{\hbar \Delta k^2} \left[\langle k^4 \rangle - \langle k \rangle \langle k^3 \rangle \right]$$

$$\langle x \rangle_{t=t_0} = \frac{4c}{\Delta k^2} \left[\langle k^4 \rangle \langle k \rangle - \langle k^3 \rangle \langle k^2 \rangle \right]$$

Duration of pulse

$$e^{-m(\omega-\omega_0)^2 - ib(\omega-\omega_0)^2}$$

$$\Longrightarrow \Delta t = \frac{2\sqrt{\ln 2}\sqrt{b^2+m^2}}{\sqrt{m}}$$

$\langle x \rangle = 0$ $\quad \frac{0 - (-4c\langle k^3 \rangle)}{\cancel{t}} =$

$$t' = \frac{4c\langle k^3 \rangle}{\hbar \langle k \rangle / m} = \frac{4c\langle k^3 \rangle m}{\hbar \langle k \rangle}$$

t'

$\boxed{\text{General}}$

$\cancel{\dot{\phi} = ck^n \quad \left(\phi = \frac{c}{n+1} k^{n+1}\right)}$ $\quad \phi = ck^n \quad \dot{\phi} = nck^{n-1}$

$$\cancel{\langle x \rangle = \frac{\hbar \langle k \rangle t}{m} + \langle x \rangle_0}$$

$$\cancel{\langle x \rangle_0 = -c \langle k^n \rangle}$$

$$\cancel{t_0 = \frac{cm}{\hbar \Delta k^2} \left[\langle k^{n+1} \rangle - \langle k \rangle \langle k^n \rangle \right]}$$

$$\cancel{\langle x \rangle_{t=t_0} = \frac{c}{\Delta k^2} \left[\langle k^{n+1} \rangle \langle k \rangle - \langle k^n \rangle \langle k^2 \rangle \right]}$$

$$\langle x \rangle = \frac{\hbar \langle k \rangle t}{m} + \langle x \rangle_0$$

$$\langle x \rangle_0 = -nc \langle k^{n-1} \rangle$$

$$t_0 = \frac{ncm}{\hbar \Delta k^2} \left[\langle k^n \rangle - \langle k \rangle \langle k^{n-1} \rangle \right]$$

$$\langle x \rangle_{t=t_0} = \frac{c}{\Delta k^2} \left[\langle k^n \rangle \langle k \rangle - \langle k^{n-1} \rangle \langle \right.$$