ALADS - Assignment - 2

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1 Part A

1.1 Question 1

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 16 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

1.1.1 a

We swap row 1 and row 3, so we can know row change matrix P_1:

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\implies P_1 A = \begin{pmatrix} 3 & 10 & 16 \\ 2 & 5 & 10 \\ 1 & 2 & 3 \end{pmatrix}$$

$$row\ 2 \leftarrow row\ 2 - l_{21} \cdot row\ 1$$
$$row\ 2 \leftarrow \begin{pmatrix} 2 & 5 & 10 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 3 & 10 & 16 \end{pmatrix}$$
$$row\ 2 = \begin{pmatrix} 0 & -\frac{5}{3} & -\frac{2}{3} \end{pmatrix}$$

$$row\ 3 \leftarrow row\ 3 - l_{31} \cdot row\ 1$$
$$row\ 3 \leftarrow \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 3 & 10 & 16 \end{pmatrix}$$
$$row\ 3 = \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{7}{3} \end{pmatrix}$$

we can know L_1

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & 10 & 16 \\ 0 & -\frac{5}{3} & -\frac{2}{3} \\ 0 & -\frac{4}{3} & -\frac{7}{3} \end{pmatrix}$$

as $L_1 \cdot (P_1 \cdot A)$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\implies P_2 \cdot L_1(P_1 \cdot A) = \begin{pmatrix} 3 & 10 & 16 \\ 0 & -\frac{5}{3} & -\frac{2}{3} \\ 0 & -\frac{4}{3} & -\frac{7}{3} \end{pmatrix}$$

$$\begin{aligned} &row\ 3 \leftarrow row\ 3 - l_{32} \cdot row\ 2 \\ &row\ 3 \leftarrow \begin{pmatrix} 0 & -\frac{5}{3} & -\frac{2}{3} \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{7}{3} \end{pmatrix} \\ &row\ 3 = \begin{pmatrix} 0 & 0 & -\frac{9}{5} \end{pmatrix} \end{aligned}$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{4}{5} & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3 & 10 & 16 \\ 0 & -\frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{9}{5} \end{pmatrix} = U$$

as $L_2 \cdot P_2(L_1(P_1 A))$

$$\begin{aligned} L_2 \cdot P_2 \cdot L_1 \cdot P_1 \cdot A &= U \\ \implies L_2 P_2 L_1 P_2^{-1} P_2 P_1 A &= L_2 \hat{L_1} P_2 P_1 A \\ \implies \hat{L_1} &= P_2 L_1 P_2^{-1} \end{aligned}$$

$$\begin{aligned} P_2 P_1 &= P \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= P \end{aligned}$$

$$\begin{aligned} L_2 \hat{L_1} P A &= U \\ \implies PA &= LU \end{aligned}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{pmatrix}$$
$$U = \begin{pmatrix} 3 & 10 & 16 \\ 0 & -\frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{9}{5} \end{pmatrix}$$
$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

1.1.2 b

we can know $A = LU$

$$\begin{aligned} Ax &= b \\ \Longrightarrow LUx &= b \end{aligned}$$

we set $Ux = C$ we can get

$$\begin{aligned} Ax &= b \\ \Longrightarrow LUx &= b \\ \Longrightarrow LC &= b \end{aligned}$$

we set $C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, we can know

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

so we can know

$$\begin{aligned} c_1 &= 3 \\ c_2 &= 1 \\ c_3 &= 0 \\ C &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

according $Ux = C$, we can know

$$
\begin{aligned}
Ux &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

we can get

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 1 \\
x_3 &= 0 \\
x &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

1.1.3 c

$$
A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 16 \end{pmatrix}
$$

$$
A = \begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{pmatrix}
$$

Find column 1 in G:

$$
\begin{aligned}
g_{11} : a_{11} = g_{11}^2 &\Longrightarrow g_{11} = \sqrt{a_11} = \sqrt{1} = 1 \\
g_{21} : a_{21} = g_{21}g_{11} &\Longrightarrow g_{21} = \frac{a_{21}}{g_{11}} = 2 \\
g_{31} : a_{31} = g_{31}g_{11} &\Longrightarrow g_{31} = \frac{a_{31}}{g_{11}} = 3
\end{aligned}
$$

For column 2 in G:

$$
\begin{aligned}
g_{22} : a_{22} = g_{21}^2 + g_{22}^2 &\Longrightarrow g_{22} = \sqrt{a_{22} - g_{21}^2} = 1 \\
g_{32} : a_{32} = g_{31}g_{21} + g_{32}g_{22} &\Longrightarrow g_{32} = \frac{a_{32} - g_{31}g_{21}}{g_{22}} = 4
\end{aligned}
$$

For column 3 in G:

$$
g_{33} : a_{33} = g_{31}^2 + g_{22}^2 + g_{33}^2 \Longrightarrow g_{33} = \sqrt{a_{33} - g_{31}^2 - g_{22}^2} = 6
$$

$$\implies G = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 6 \end{pmatrix}$$

$$G^T = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{pmatrix}$$

$$A \neq GG^T$$

1.2 Question 2

Condition Number is used to measure the sensitivity of a function, matrix, or problem, i.e., the degree to which small changes in the input affect the output. In numerical computation, condition numbers help us understand the magnitude of errors that may occur during computation. In particular, when solving systems of linear equations or performing matrix operations, high condition numbers can lead to numerical instability and significant errors, while low condition numbers usually indicate that the computation will be more stable and reliable.

```python
import numpy as np
import pandas as pd
A = np.array([
    [0,6, -2, 0, 0, 0],
    [0,4, 1, 2, 2, 0],
    [0, 0, 0, 0, 2, 4],
    [1, -1, 0, 1, 0, 0],
    [0, 0, 1, 0, -1, 1],
    [0, 0, 0, 1, -1, 1]
])
A_condition_number = np.linalg.cond(A)
A_condition_number
```

Condition Number A = 12.28.
If we use

$$\epsilon_{output} \approx K \times \epsilon_{input}$$

Therefore, the expected relative error in the solution can be estimated to be 12.28%, assuming that the errors are propagated in the worst-case manner as indicated by the condition number.

2 Part B

2.1 Question 3

2.1.1 a

TimeA = 112.87475100000003.
Output look at Figure 1.

Figure 1: Method A

2.1.2 b

TimeB = 5.748755000000017.
Output look at Figure 2.

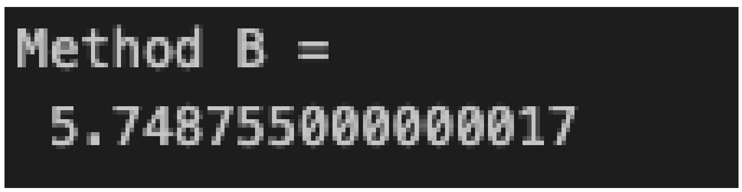

Figure 2: Method B

2.1.3 c

TimeC = 4.8586950000000115.
Output look at Figure 3.

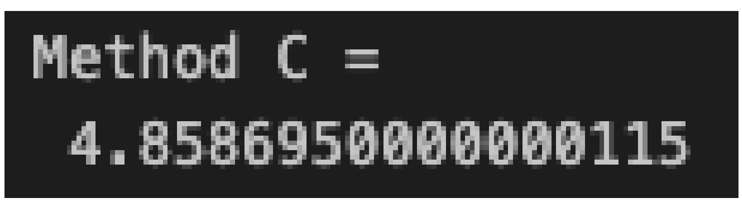

Figure 3: Method C

2.1.4 d

TimeD = 0.0222169999999835.
Output look at Figure 4.

2.2 Question 4

Matrix A is sparse.

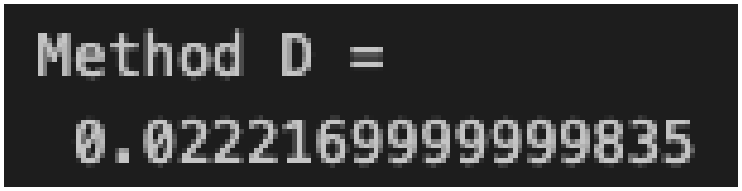

Figure 4: Method D

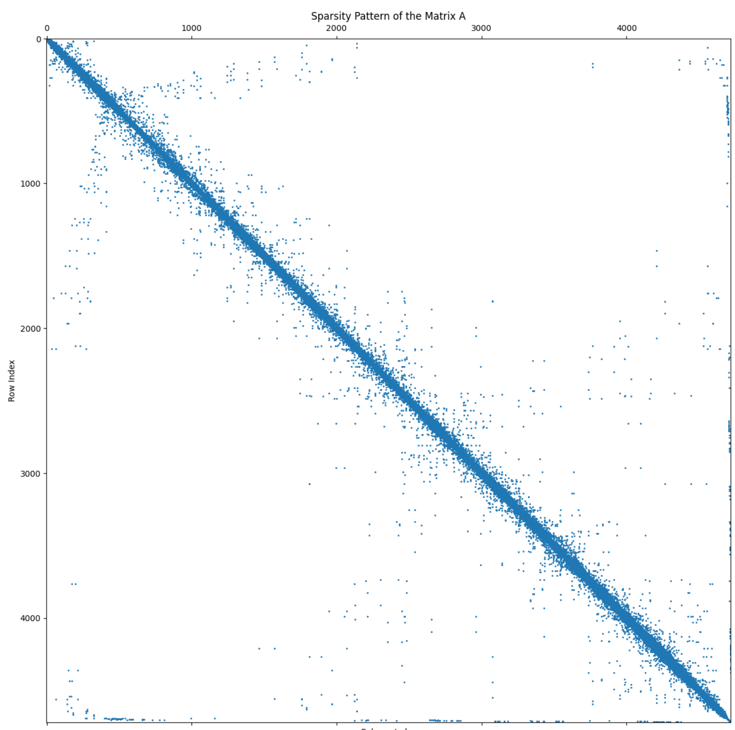

Figure 5: Sparsity Pattern of the Matrix A

2.3 Question 5

The sparse matrix format is a storage scheme that significantly reduces memory usage when dealing with matrices that have a large number of zero entries. Instead of storing all entries including the zeros, the sparse format only stores the non-zero elements and their locations.

- **Compressed Sparse Row (CSR)**: It stores the matrix by three one-dimensional arrays that represent non-zero values, the extents of rows, and column indices.
- **Compressed Sparse Column (CSC)**: It is similar to CSR but focuses on column operations. It uses three arrays to represent column indices, row extents, and non-zero values.
- **Coordinate List (COO)**: It stores a list of (row, column, value) tuples.

We use CSR to store the little sparse matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

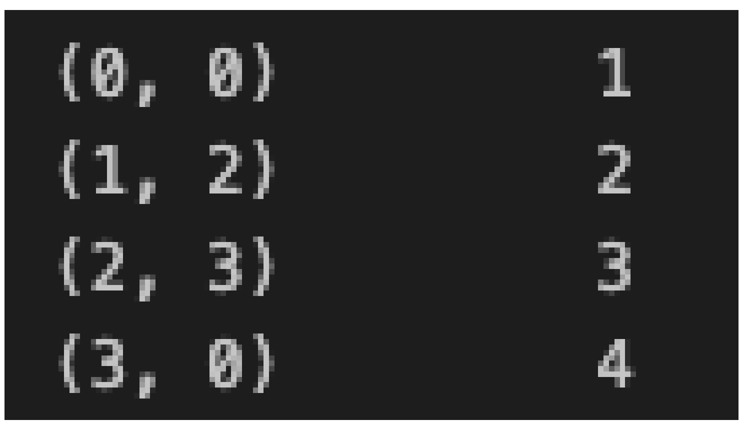

Figure 6: Storing sparse matrix using CSR

2.4 Appendix

Code for Assignment 2 (Github repository)