ALADS - Assignment - 3

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Part A

Question 1

a.

The normal equations for linear regression are given by:

$$\mathbf{A}^T\mathbf{A}\beta = \mathbf{A}^T b$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, b = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

According the table, we can get

$$x = \begin{pmatrix} 8.1 & 10.0 & 11.2 & 12.7 & 13.0 \end{pmatrix}$$
$$b = \begin{pmatrix} 5.0 & 5.0 & 4.0 & 3.0 & 1.0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 8.1 \\ 1 & 10.0 \\ 1 & 11.2 \\ 1 & 12.7 \\ 1 & 13.0 \end{pmatrix}, b = \begin{pmatrix} 5.0 \\ 5.0 \\ 4.0 \\ 3.0 \\ 1.0 \end{pmatrix}, \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

We use Normal Equation to get

$$A^T A\beta = A^T b$$

If we want to solve this equation, we can through QR-decomposition $A = QR$,

$$
\begin{aligned}
A\beta = b &\Longrightarrow A^T A\beta = A^T b \\
&\Longrightarrow (QR)^T QR\beta = (QR)^T b \\
&\Longrightarrow R^T Q^T QR\beta = R^T Q^T b \\
&\Longrightarrow R^T R\beta = R^T Q^T b \\
&\Longrightarrow R\beta = Q^T b
\end{aligned}
$$

Next, we perform QR-decomposition of the Matrix A.

$$
A = \begin{pmatrix} 1 & 8.1 \\ 1 & 10.0 \\ 1 & 11.2 \\ 1 & 12.7 \\ 1 & 13.0 \end{pmatrix}, a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 8.1 \\ 10.0 \\ 11.2 \\ 12.7 \\ 13.0 \end{pmatrix}
$$

$$
e_1 = \frac{a_1}{||a_1||} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{pmatrix}
$$

$$
\begin{aligned}
\mathrm{proj}_{e_1} a_2 &= \left(\frac{a_2 \cdot e_1}{e_1 \cdot e_1}\right) e_1 \\
v_2 &= a_2 - \mathrm{proj}_{e_1} a_2 \\
e_2 &= \frac{v_2}{||v_2||}
\end{aligned}
$$

we can know

$$
\begin{aligned}
Q &= \begin{pmatrix} e_1 & e_2 \end{pmatrix} \\
R &= \begin{pmatrix} e_1 \cdot a_1 & e_1 \cdot a_2 \\ 0 & e_2 \cdot a_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Q &= \begin{pmatrix} 0.4472 & -0.7174 \\ 0.4472 & -0.2474 \\ 0.4472 & 0.0495 \\ 0.4472 & 0.4206 \\ 0.4472 & 0.4948 \end{pmatrix} \\
R &= \begin{pmatrix} 2.2361 & 24.5967 \\ 0 & 4.0423 \end{pmatrix}
\end{aligned}
$$

we use $R\beta = Q^T b$

$$\begin{pmatrix} 2.2361 & 24.5967 \\ 0 & 4.0423 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0.4472 & 0.4472 & 0.4472 & 0.4472 & 0.4472 \\ -0.7174 & -0.2474 & 0.0495 & 0.4206 & 0.4948 \end{pmatrix} \begin{pmatrix} 5.0 \\ 5.0 \\ 4.0 \\ 3.0 \\ 1.0 \end{pmatrix}$$

we can get

$$\beta_0 = 11.4091$$
$$\beta_1 = -0.7099$$
$$\beta = \begin{pmatrix} 11.4091 \\ -0.7099 \end{pmatrix}$$

So this linear regression model is $y = 11.4091 - 0.7099x$.

$$Cond_2(A) = 69.28012237225525$$
$$Cond_2(A^T A) = 4799.735355915109$$

b.

This model

$$y = \beta_0 + \beta_1 \frac{x - \bar{x}}{\sigma(x)}$$

$$\begin{aligned} \bar{x} &= \frac{1}{n} \sum_{i=1}^{n} x_i \\ &= 11.0 \\ \sigma(x) &= 1.8077610461562665 \end{aligned}$$

$$\frac{x - \bar{x}}{\sigma(x)} = \begin{pmatrix} -1.60419432 \\ -0.55317045 \\ 0.11063409 \\ 0.94038977 \\ 1.10634091 \end{pmatrix}$$

so we can know Matrix A

$$A = \begin{pmatrix} 1 & -1.60419432 \\ 1 & -0.55317045 \\ 1 & 0.11063409 \\ 1 & 0.94038977 \\ 1 & 1.10634091 \end{pmatrix}$$
$$A = QR$$
$$R\beta = Q^T b$$

we can get

$$\beta_0 = 3.60$$
$$\beta_1 = -1.43$$
$$\beta = \begin{pmatrix} 3.60 \\ -1.43 \end{pmatrix}$$

So this model is $y = 3.60 - 1.43\frac{x-\bar{x}}{\sigma(x)}$.

$$Cond_2(A) = 1$$
$$Cond_2(A^TA) = 1$$

Question 2

The condition number of a matrix Q with respect to Euclidean norm is defined as:

$$Cond_2(Q) = ||Q||_2 \cdot ||Q^{-1}||_2$$

Proof.

$$||Q||_2 = \sup_{||x||_2 \neq 0} \frac{||Qx||_2}{||x||_2}$$

Since Q is orthogonal, $||Qx||_2 = ||x||_2$ for any vector x:

$$||Q||_2 = 1$$

Since $Q^{-1} = Q^T$ and Q^T is also orthogonal, we have:

$$||Q^{-1}||_2 = ||Q^T||_2 = 1$$

So, we can know

$$Cond_2(Q) = ||Q||_2 \cdot ||Q^{-1}||_2 = 1$$

Listing 1: Orthogonal Matrix Condition Number

```
import numpy as np
np.random.seed(42)
Random_matrix = np.random.rand(5, 5)
#print(Random_matrix)
Q, R = np.linalg.qr(Random_matrix)
Q_cond_number = np.linalg.cond(Q,2)
print(Q_cond_number)
```

$$Cond_2(Q) = 1$$

Part B

3

a.

We exclude data rows that contain missing values. The result is shown in see Figure 1.

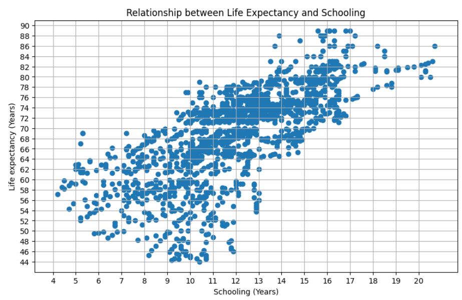

Figure 1: Relationship between Life Expectancy and Schooling

b.

See Figure 2.

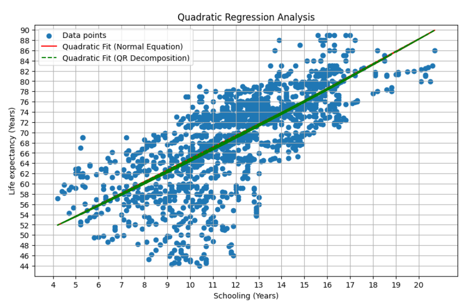

Figure 2: Quadratic Regression Analysis

c.

$$Cond_2(A) = 2216.152618371592$$

$$\frac{||\hat{x}_{exact} - \hat{x}_{NE}||}{||\hat{x}_{exact}||} = 1.4283097275361544 \times 10^{-13}$$
$$\frac{||\hat{x}_{exact} - \hat{x}_{QR}||}{||\hat{x}_{exact}||} = 3.521603460086744 \times 10^{-15}$$

Based on the above calculations, we can see that the normal equation is worse.

d.

Normal Equation, See Figure 3.

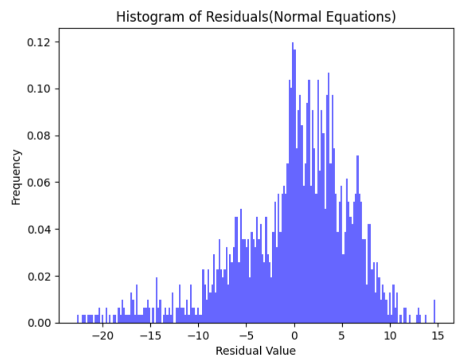

Figure 3: Histogram of Residuals(Normal Equations)

QR-decomposition, See Figure 4.

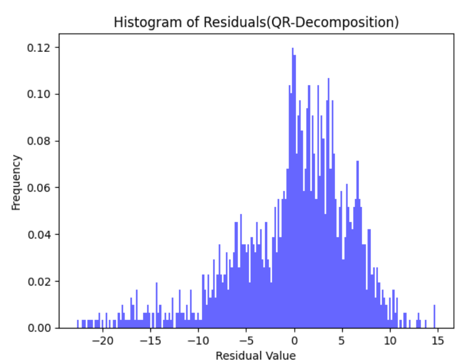

Figure 4: Histogram of Residuals(QR-Decomposition)

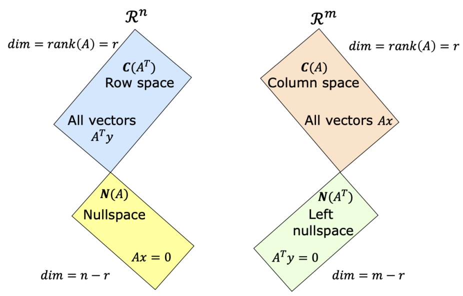

Figure 5: The four fundamental subspaces

4

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ \cdots & \cdots & \cdots \\ 1 & x_m & x_m^2 \end{pmatrix}_{m\times 3}$$

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}_{3\times 1}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}_{m\times 1}$$

$$A\beta = y$$

According the Figure5, the dimension of the column space of A, denoted $C(A)$, is equal to the rank of A, denoted $rank(A)$, the dimension of $C(A) = 3$. $C(A^T)$ and $N(A)$ are subspaces in R^n, and $n = 3$ here so $N(A)$ must be empty. But $C(A)$ and $N(A^T)$ are subspaces in R^m (and m is very large here). So the answer should be $N(A^T) = m - 3$.

5

See appendix.

6

See appendix.

Appendix

Assignment 3