

ALADS - Assignment - 3

Chenglong Li

April 2024

1 Part A

1.1 Question 1

1.1.1 a.

According to the table, we can get

$$A = \begin{pmatrix} 1 & 8.1 \\ 1 & 10.0 \\ 1 & 11.2 \\ 1 & 12.7 \\ 1 & 13.0 \end{pmatrix}, b = \begin{pmatrix} 5.0 \\ 5.0 \\ 4.0 \\ 3.0 \\ 1.0 \end{pmatrix}, \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

We use Normal Equation to get

$$A^T A \beta = A^T b$$

If we want to solve this equation, we can through QR-decomposition $A = QR$,

$$\begin{aligned} A\beta = b &\implies A^T A \beta = A^T b \\ &\implies (QR)^T QR \beta = (QR)^T b \\ &\implies R^T Q^T QR \beta = R^T Q^T b \\ &\implies R^T R \beta = R^T Q^T b \\ &\implies R \beta = Q^T b \end{aligned}$$

Next, we perform QR-decomposition of the Matrix A.

$$A = \begin{pmatrix} 1 & 8.1 \\ 1 & 10.0 \\ 1 & 11.2 \\ 1 & 12.7 \\ 1 & 13.0 \end{pmatrix}, a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 8.1 \\ 10.0 \\ 11.2 \\ 12.7 \\ 13.0 \end{pmatrix}$$

$$e_1 = \frac{a_1}{||a_1||} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{pmatrix}$$

$$\begin{aligned} \text{proj}_{e_1} a_2 &= \left(\frac{a_2 \cdot e_1}{e_1 \cdot e_1} \right) e_1 \\ v_2 &= a_2 - \text{proj}_{e_1} a_2 \\ e_2 &= \frac{v_2}{||v_2||} \end{aligned}$$

we can know

$$\begin{aligned} Q &= (e_1 \ e_2) \\ R &= \begin{pmatrix} e_1 \cdot a_1 & e_1 \cdot a_2 \\ 0 & e_2 \cdot a_2 \end{pmatrix} \\ Q &= \begin{pmatrix} 0.4472 & -0.7174 \\ 0.4472 & -0.2474 \\ 0.4472 & 0.0495 \\ 0.4472 & 0.4206 \\ 0.4472 & 0.4948 \end{pmatrix} \\ R &= \begin{pmatrix} 2.2361 & 24.5967 \\ 0 & 4.0423 \end{pmatrix} \end{aligned}$$

we use $R\beta = Q^T b$

$$\begin{pmatrix} 2.2361 & 24.5967 \\ 0 & 4.0423 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0.4472 & 0.4472 & 0.4472 & 0.4472 & 0.4472 \\ -0.7174 & -0.2474 & 0.0495 & 0.4206 & 0.4948 \end{pmatrix} \begin{pmatrix} 5.0 \\ 5.0 \\ 4.0 \\ 3.0 \\ 1.0 \end{pmatrix}$$

we can get

$$\begin{aligned} \beta_0 &= 11.4091 \\ \beta_1 &= -0.7099 \\ \beta &= \begin{pmatrix} 11.4091 \\ -0.7099 \end{pmatrix} \end{aligned}$$

So this linear regression model is $y = 11.4091 - 0.7099x$.

$$\begin{aligned} \text{Cond}_2(A) &= 69.28012237225525 \\ \text{Cond}_2(A^T A) &= 4799.735355915109 \end{aligned}$$

1.1.2 b.

This model

$$y = \beta_0 + \beta_1 \frac{x - \bar{x}}{\sigma(x)}$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$= 11.0$$

$$\sigma(x) = 1.8077610461562665$$

$$\frac{x - \bar{x}}{\sigma(x)} = \begin{pmatrix} -1.60419432 \\ -0.55317045 \\ 0.11063409 \\ 0.94038977 \\ 1.10634091 \end{pmatrix}$$

so we can know Matrix A

$$A = \begin{pmatrix} 1 & -1.60419432 \\ 1 & -0.55317045 \\ 1 & 0.11063409 \\ 1 & 0.94038977 \\ 1 & 1.10634091 \end{pmatrix}$$

$$A = QR$$

$$R\beta = Q^T b$$

we can get

$$\beta_0 = 3.60$$

$$\beta_1 = -1.43$$

$$\beta = \begin{pmatrix} 3.60 \\ -1.43 \end{pmatrix}$$

So this model is $y = 3.60 - 1.43 \frac{x - \bar{x}}{\sigma(x)}$.

$$\text{Cond}_2(A) = 1$$

$$\text{Cond}_2(A^T A) = 1$$

1.2 Question 2

Listing 1: Orthogonal Matrix Condition Number

```
1 import numpy as np
2 np.random.seed(42)
3 Random_matrix = np.random.rand(5, 5)
4 #print(Random_matrix)
5 Q, R = np.linalg.qr(Random_matrix)
6 Q_cond_number = np.linalg.cond(Q,2)
7 print(Q_cond_number)
```

$$\text{Cond}_2(Q) = 1$$

2 Part B

2.1 3

2.1.1 a.

See Figure 1.

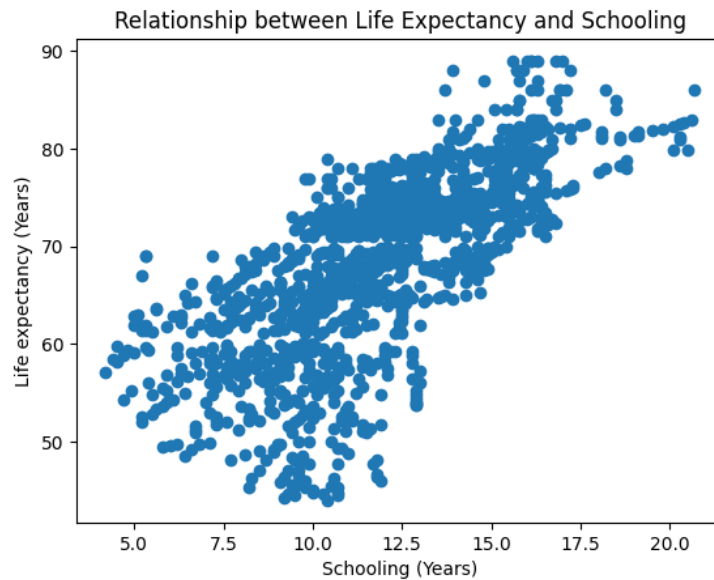


Figure 1: Relationship between Life Expectancy and Schooling

2.1.2 b.

See Figure 2.

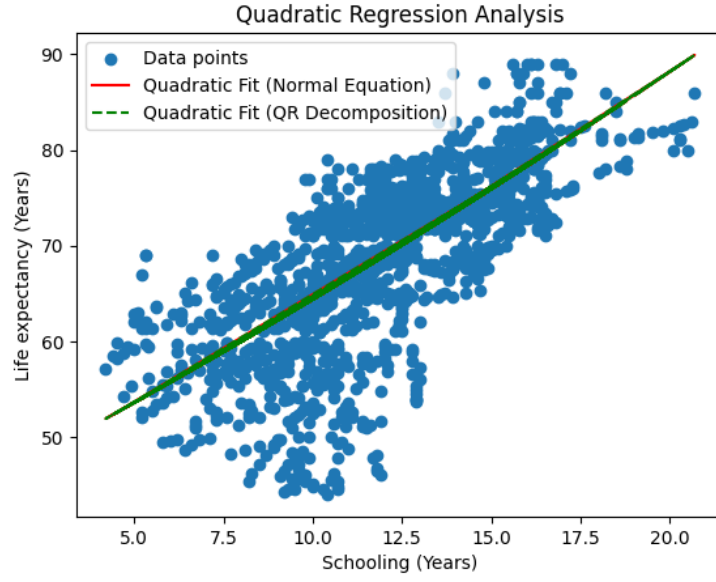


Figure 2: Quadratic Regression Analysis

2.1.3 c.

$$\frac{||\hat{x}_{exact} - \hat{x}_{NE}||}{||\hat{x}_{exact}||} = 1.4283097275361544 \times 10^{-13}$$

$$\frac{||\hat{x}_{exact} - \hat{x}_{QR}||}{||\hat{x}_{exact}||} = 3.521603460086744 \times 10^{-15}$$

2.1.4 d.

Normal Equation, See Figure 3.

QR-decomposition, See Figure 4.

2.2 4

$$\mathbf{C}(A) = 3$$

$$\mathbf{N}(A^T) = 0$$

2.3 5

See appendix.

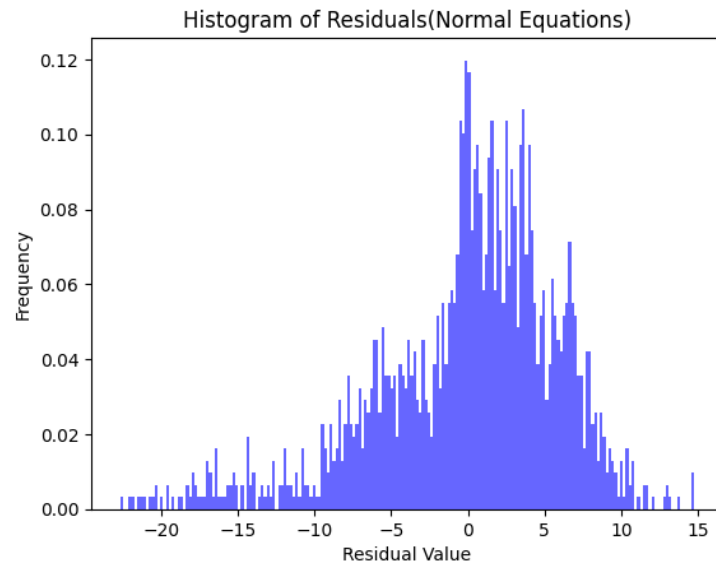


Figure 3: Histogram of Residuals(Normal Equations)

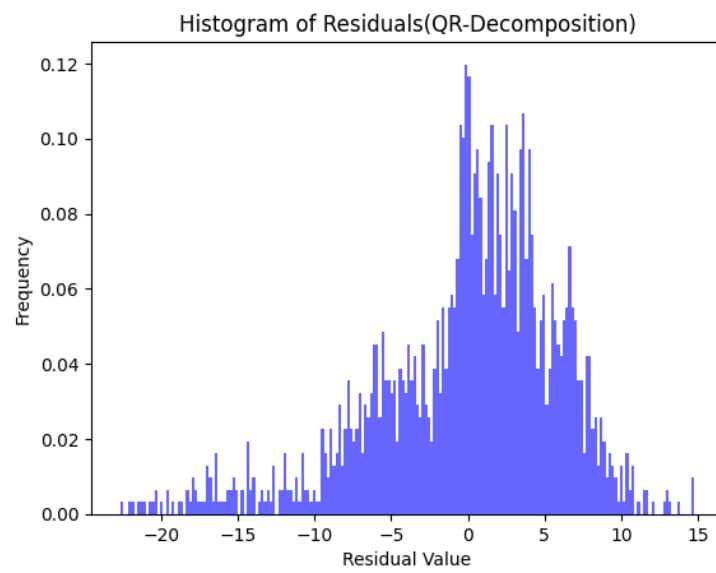


Figure 4: Histogram of Residuals(QR-Decomposition)

2.4 6

See appendix.

3 Appendix

Assignment 3