Computer Intensive Statistics
Group HWA1: Random Number and Monte Carlo

1. (2p) Consider the density

$$p(x) = \frac{1}{4}|\sin(x)|, \quad -\pi < x < \pi.$$

   (a) Use the quantile method to generate random numbers from this distribution.

   (b) Write the code of your transformation method and use the code to generate $N = 10,000$ observations. Attach the code and the histogram of the generated random numbers.

   (c) Propose a rejection sampling algorithm to generate random numbers from this distribution such that the acceptance probability is at least 0.5. What is the acceptance probability (or empirical acceptance percentage) of your rejection algorithm?

2. (1p) In this task, we will complement the ARS algorithm by studying how we can draw X from density $g_n(x)$. By the construction of $g_n(x)$, the upper bound $\bar{h}_n(x) = \log\{M_n g_n(x)\}$ is formed by piecewise linear functions. Suppose that it is formed by r_n linear functions. Then,

$$g_n(x) = M_n^{-1}\Bigg\{\exp(a_{-1} + b_{-1}x)\, 1_{[-\infty,x_0]}(x) + \sum_{j=0}^{r_n} \exp(a_j + b_j x)\, 1_{[x_j,x_{j+1}]}(x) + \exp(a_{r_n+1} + b_{r_n+1}x)\, 1_{[x_{r_n+1},\infty]}(x)\Bigg\},$$

where M_n is the normalizing constant that makes $g_n(x)$ a density function. Propose a transformation method to generate random numbers from the density $g_n(x)$.

3. (1p) Consider again the distribution in Task 1. Approximate its mean by importance sampling. Construct also a 95% confidence interval for the approximated mean as well. Discuss also whether your importance sampling estimator has a bounded variance.

4. (4p) Suppose that we want to approximate $\mathrm{E}[X]$, where the density of X is

$$p(x) = \frac{1}{c}\phi(x)\Phi(x), \quad -\infty < x < \infty,$$

with $\phi(x)$ being the density of $N(0,1)$, $\Phi(x)$ being the cumulative distribution function of $N(0,1)$, and c being an unknown normalizing constant such that

$$c = \int_{-\infty}^{\infty} \phi(x)\Phi(x)\, dx.$$

From now on, you can use the built-in functions to evaluate $\Phi(x)$.

(a) Approximate the value of c using independent Monte Carlo. Construct also a 95% confidence interval for the approximated c.

(b) In the following subtasks, we decide to generate X from the density of X directly. Approximate $\mathrm{E}[X]$ using MCMC, either Metropolis or Metropolis-Hastings. Inspect convergence by plotting the Markov chain.

(c) Approximate $\mathrm{E}[X]$ using HMC. Inspect convergence by plotting the Markov chain.

(d) Approximate $\mathrm{E}[X]$ using quasi-Monte Carlo, where $c = 1/2$.

5. (2p) Consider an Asian option where the payoff depends on the average price over a pre-determined period of time $[0, T]$. Suppose that the price at time t satisfies the stochastic differentiable equation

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t)$$

such that

$$S(t) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right].$$

The Asian option value is

$$\mathrm{E}\left[e^{-rT}\max\left(\frac{1}{M}\sum_{m=1}^{M} S(t_m) - K, 0\right)\right],$$

for the monitoring time $\{t_m = mT/M\}$, $m = 1, ..., M$. We can simulate the process $S(t)$ by

$$S(t_{m+1}) = S(t_m)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_{m+1} - t_m) + \sigma\sqrt{t_{m+1} - t_m}Z_{m+1}\right\},$$

where $\{Z_m\}$ are iid $N(0,1)$. We let $S(0) = 1$,$r = 0.05$, $\sigma = 0.2$, $K = 1$, $M = 200$, and $T = 1$ for simplicity. Approximate such Asian option value. Include also your algorithm. Quantify also the uncertainty, if any.