M1 BBS - EM8BBSEM

Simulation de Systèmes Biologiques

(#6)

Georges Czaplicki, UPS / IPBS-CNRS
Tél. : 05.61.17.54.04, email : cgeorge@ipbs.fr

Projets

Project #2

Suppose that the spruce budworm, in the absence of predation by birds, will grow according to a simple logistic equation of the form

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right)$$

Budworms feed on the foliage of trees. The size of the carrying capacity, ***K***, would depend on the amount of foliage on the trees. We take it to be constant for this model.

a) Draw graphs for how the population might grow if ***r*** were 0.48 and ***K*** were 15. Use several initial values.

b) Introduce predation by birds into this model in the following manner: suppose that for small levels of worm population there is almost no predation, but for larger levels birds are attracted to this food source. Allow for a limit to the number of worms that each bird can eat. A model for predation by birds might have the form

$$P(B) = a\frac{B^2}{b^2 + B^2}$$

where ***a*** and ***b*** are positive. Sketch the graph for the level of predation of the budworms as a function of the size of the population. Take ***a*** and ***b*** to be 2.

c) A model for the budworm population size in the presence of predation could be modeled

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right) - a\frac{B^2}{b^2 + B^2}$$

as

To understand the delicacy of this model and the implications for the care that needs to be taken in modeling, investigate graphs of solutions for this model with parameters ***r*** = 0.48, ***a*** = ***b*** = 2, and ***K*** = 15 or ***K*** = 17.

d) Determine the number and type of solutions for each case. Comment on the significance of the results.

P6. Transport de protéines du réticulum endoplasmique à la bordure de la cellule en passant à travers le Golgi

1. Vous représenterez les données expérimentales suivantes où N représente le nombre total de la protéine considérée dans chacun des compartiments cellulaires:

T (min)	0	10	30	45	60	85
N_{RE}	96	83	60	33	23	16
N_{Golgi}	4	10	25	27	1	8
$N_{bordure}$	0	7	15	40	65	76

2. Vous en déduirez qualitativement des caractéristiques de ce transport.

Il a en fait été proposé que ce transport pouvait être décrit par le système d'équations différentielles suivant :

$$\frac{dN_{RE}}{dt} = -k_{RE}N_{RE} \quad \text{et} \quad \frac{dN_{Golgi}}{dt} = k_{RE}N_{RE} - k_{Golgi}N_{Golgi} \quad \text{et} \quad \frac{dN_{bord}}{dt} = k_{Golgi}N_{Golgi}$$

3. Vous justifierez un tel choix d'équation pour décrire le phénomène de transport (schéma).
4. A partir de la solution analytique trouvée, vous conclurez sur l'adéquation de la description mathématique et les résultats expérimentaux.
5. Vous étudierez ensuite le système suivant d'équations différentielles :

$$\frac{dN_{REint}}{dt} = -k_{REint}N_{REint} \quad \text{et} \quad \frac{dN_{RE}}{dt} = k_{REint}N_{REint} - k_{RE}N_{RE}$$

$$\text{et} \quad \frac{dN_{Golgi}}{dt} = k_{RE}N_{RE} - k_{Golgi}N_{Golgi} \quad \text{et} \quad \frac{dN_{bord}}{dt} = k_{Golgi}N_{Golgi}$$

et déduirez les constante (k_{REint}, k_{RE} et k_{Golgi}) par ajustement non linéaire.

6. Conclusion et lien avec données expérimentalement accessibles.

1.4. Harvesting problem.

Let us consider a population growing according to a logistic dynamics. Let assume that a constant effort E of fishing[1] so that the yield per unit time is qEx, where x is population size q is a coefficient denoting the return to effort.

(a) Write down the differential equation for $x(t)$ which translate these assumptions; find its equilibria and the asymptotic behaviour of solutions, according to parameter values.

(b) Let assume that the unit price at which the fish is sold is p, and that the cost of fishing is proportional (through a coefficient c) to the effort E. Let assume that an enlightened dictator wants to set E at the value that maximizes the gain (= revenue − cost) when the population is at its asymptotically stable equilibrium. Find the value of E and the corresponding equilibrium value for x.

(c) Economic theory predicts that, for an open access fishery, the effort E will in the long run reach the value at which the gain is equal to 0. Find the value of E and the corresponding equilibrium value of x; compare them (i.e, find, if they are greater or smaller) than the previous case.

(d) Let assume that the government taxes at a percentage ρ the gains obtained by fisheries. How does this affect the results obtained with open-access fishery?

[1] or hunting, or harvesting

(e) Let assume that the government taxes according to how much has been fished Y. Let us consisre two separate cases: a constant fraction ρY, or a progressive tax $\tau(Y)$ given by the formula

$$\tau(Y) = \begin{cases} 0 & \text{se } Y \leq Y_0 \\ \rho(Y - Y_0) & \text{se } Y > Y_0 \end{cases}$$

Which are the results of these regulations?

(f) Let us assume that the dynamics of x be described, in absence of fishing, by the generalized logistic equation

$$x'(t) = rx(t)\left(1 - \left(\frac{x(t)}{K}\right)^{\alpha}\right). \qquad \alpha > 0$$

How do previous results change?

OPEN ACCESS Freely available online

PLoS COMPUTATIONAL BIOLOGY

Living on Three Time Scales: The Dynamics of Plasma Cell and Antibody Populations Illustrated for Hepatitis A Virus

Mathieu Andraud[1]*, Olivier Lejeune[1,2], Jammbe Z. Musoro[3,4], Benson Ogunjimi[1], Philippe Beutels[1], Niel Hens[1,3]

1 Centre for Health Economics Research and Modelling of Infectious Diseases (CHERMID), Vaccine & Infectious Disease Institute (VAXINFECTIO), University of Antwerp, Antwerp, Belgium, **2** The SYMBIOS Center, Division of Mathematics, University of Dundee, Dundee, United Kingdom, **3** Interuniversity Institute of Biostatistics and Statistical Bioinformatics, Hasselt University, Diepenbeek, Belgium, **4** Academic Medical Center, University of Amsterdam, Amsterdam, The Netherlands

Abstract

Understanding the mechanisms involved in long-term persistence of humoral immunity after natural infection or vaccination is challenging and crucial for further research in immunology, vaccine development as well as health policy. Long-lived plasma cells, which have recently been shown to reside in survival niches in the bone marrow, are instrumental in the process of immunity induction and persistence. We developed a mathematical model, assuming two antibody-secreting cell subpopulations (short- and long-lived plasma cells), to analyze the antibody kinetics after HAV-vaccination using data from two long-term follow-up studies. Model parameters were estimated through a hierarchical nonlinear mixed-effects model analysis. Long-term individual predictions were derived from the individual empirical parameters and were used to estimate the mean time to immunity waning. We show that three life spans are essential to explain the observed antibody kinetics: that of the antibodies (around one month), the short-lived plasma cells (several months) and the long-lived plasma cells (decades). Although our model is a simplified representation of the actual mechanisms that govern individual immune responses, the level of agreement between long-term individual predictions and observed kinetics is reassuringly close. The quantitative assessment of the time scales over which plasma cells and antibodies live and interact provides a basis for further quantitative research on immunology, with direct consequences for understanding the epidemiology of infectious diseases, and for timing serum sampling in clinical trials of vaccines.

OPEN ACCESS Freely available online PLOS COMPUTATIONAL BIOLOGY

Multi-Scale Modeling of HIV Infection *in vitro* and APOBEC3G-Based Anti-Retroviral Therapy

Iraj Hosseini, Feilim Mac Gabhann*

Institute for Computational Medicine and Department of Biomedical Engineering, Johns Hopkins University, Baltimore, Maryland, United States of America

Abstract

The human APOBEC3G is an innate restriction factor that, in the absence of Vif, restricts HIV-1 replication by inducing excessive deamination of cytidine residues in nascent reverse transcripts and inhibiting reverse transcription and integration. To shed light on impact of A3G-Vif interactions on HIV replication, we developed a multi-scale computational system consisting of intracellular (single-cell), cellular and extracellular (multicellular) events by using ordinary differential equations. The single-cell model describes molecular-level events within individual cells (such as production and degradation of host and viral proteins, and assembly and release of new virions), whereas the multicellular model describes the viral dynamics and multiple cycles of infection within a population of cells. We estimated the model parameters either directly from previously published experimental data or by running simulations to find the optimum values. We validated our integrated model by reproducing the results of *in vitro* T cell culture experiments. Crucially, *both* downstream effects of A3G (hypermutation and reduction of viral burst size) were necessary to replicate the experimental results *in silico*. We also used the model to study anti-HIV capability of several possible therapeutic strategies including: an antibody to Vif; upregulation of A3G; and mutated forms of A3G. According to our simulations, A3G with a mutated Vif binding site is predicted to be significantly more effective than other molecules at the same dose. Ultimately, we performed sensitivity analysis to identify important model parameters. The results showed that the timing of particle formation and virus release had the highest impacts on HIV replication. The model also predicted that the degradation of A3G by Vif is not a crucial step in HIV pathogenesis.

Exercise 6.2:
Change the Lotka-Volterra-Model slightly, in assuming logistic population dynamics for the prey alone,

$$\begin{aligned} \frac{d}{dt}N &= aN(1-N) - bNP \\ \frac{d}{dt}P &= cNP - dP \end{aligned}$$

(a) Compute the stationary points.
(b) Consider parameters, s.t. a stationary solution exists with predator and prey present. Show that this stationary point is locally stable, i.e. the eigenvalues of the linearization have negative real part.

P6. Compétition entre lynxs et lièvres

Les populations de lynx (x) et de lièvres (y) évoluent selon le modèle suivant :

$$\frac{dx}{dt} = rx\left[1 - \frac{x}{K}\right] - b\frac{x}{\alpha + x}y \quad \text{et} \quad \frac{dy}{dt} = -cy + D\frac{x}{\alpha + x}y$$

Où r, K, b, α, c, D sont des constantes positives

1. Vous interpréterez biologiquement les termes apparaissant dans ces équations en donnant les dimensions des paramètres.
2. Par une analyse graphique, vous déterminerez les équilibres rn précisant leur stabilité ainsi que la dynamique de ces deux populations. Vous pourrez choisir comme paramètres :

r	K	b	c	D	α
1,2	10	1	0,1	0,2	0,5 ou 3 ou 4 ou 7

3. Vous en déduirez la dynamique du système lorsque le paramètre a varie au cours du temps selon :

$$\frac{d\alpha}{dt} = -h(\alpha - \alpha_m)$$

4. Conclusion

Project #3

Imagine a *three-species predator-prey* problem, which we identify with grass, sheep and wolves. The grass grows according to a logistic equation in the absence of sheep. The sheep eat the grass and the wolves eat the sheep. Each on its own, the populations of both sheep and wolves decrease with a constant (negative) *per capita* growth rate.

For practical reasons (graphs) assume the following parameter values:

- growth rates: 2 for grass, -1 for sheep and wolves
- carrying capacity for grass: 2
- encounter benefits: -1 for grass because of sheep, +2 for sheep because of grass, -1 for sheep because of wolves, +1 for wolves because of sheep.

a) Write and analyze the system of equations (stationary points, stability, trajectories…).
b) What would be the steady state of grass, with no sheep or wolves present?
c) What would be the steady state of sheep and grass, with no wolves present?
d) What is the revised steady state with wolves present? Does the introduction of wolves benefit the grass?

Project #4

(based on project #3)

The following is a *predator-prey model with child care*. Suppose that the prey is divided into two classes: of young and adults. Suppose that the young are protected from predators. Assume the young increase proportional to the number of adults and decrease due to death or to moving into the adult class. The number of adults is increased by the young growing up and decreased by natural death and predation. Finally, the predators die naturally and benefit from encounters with adults.

Write and analyze the equations (stationary points, stability, trajectories…).

Note:

For practical purposes (graphs) use the following parameter values:

- for the young: birth rate = 2, mortality rate = ½, growing up = ½
- for the adults: natural death rate = ½, predation mortality = 1
- for the predators: death rate = 1, predation benefits = 1.

Exercise 2: (Self intoxicating population) (4)

Some populations (such as algae and bacteria) produce waste products, which in high concentrations are toxic to the population itself. A typical mathematical model for a population $n(t)$ and a toxic waste product $y(t)$ is

$$\begin{aligned} \dot{n} &= (\alpha - \beta - Ky)n \\ \dot{y} &= \gamma n - \delta y \end{aligned}$$

with $\alpha, \beta, \gamma, \delta, K > 0$.

1. Explain each term in the above model and sketch an arrow diagram.
2. Find the nullclines, the steady states and sketch a phase portrait and the vectorfield.
3. Linearize the system at the steady states and determine their stability. Find the regions in parameter space such that the nontrivial (coexistence) equilibrium is a node or a spiral.
4. Sketch some trajectories for the case of $\delta < 4(\alpha - \beta)$ and explain what you see in terms of biology.
5. Consider the case of high dilution $\delta >> 1, \gamma/\delta < \infty$.

Chapter 4

When Zombies Attack!: Mathematical Modelling of an Outbreak of Zombie Infection

Philip Munz[1,*]***, Ioan Hudea***[1,†]***, Joe Imad***[2,‡]***, Robert J. Smith?***[3,§]

[1]School of Mathematics and Statistics, Carleton University,
1125 Colonel By Drive, Ottawa, ON K1S 5B6, Canada

[2]Department of Mathematics, The University of Ottawa,
585 King Edward Ave, Ottawa ON K1N 6N5, Canada

[2]Department of Mathematics and Faculty of Medicine, The University of Ottawa,
585 King Edward Ave, Ottawa ON K1N 6N5, Canada

Abstract

Zombies are a popular figure in pop culture/entertainment and they are usually portrayed as being brought about through an outbreak or epidemic. Consequently, we model a zombie attack, using biological assumptions based on popular zombie movies. We introduce a basic model for zombie infection, determine equilibria and their stability, and illustrate the outcome with numerical solutions. We then refine the model to introduce a latent period of zombification, whereby humans are infected, but not infectious, before becoming undead. We then modify the model to include the effects of possible quarantine or a cure. Finally, we examine the impact of regular, impulsive reductions in the number of zombies and derive conditions under which eradication can occur. We show that only quick, aggressive attacks can stave off the doomsday scenario: the collapse of society as zombies overtake us all.

Population Modeling Using the Leslie Matrix

This lab focuses on the use of the Leslie Matrix to determine the growth of a population, as well as the age distribution within the population over time. The model used here was described by P. H. Leslie in 1945.[1] This model has been used to describe the population dynamics of a wide variety of organisms including: brook trout, rabbits, lice, beetles, pine trees, buttercups, killer whales, and humans. We will apply the Leslie model to find the population and population distribution of a species of salmon that has been recently introduced into the area.

Discussion

The Leslie model uses the following assumptions:

- We consider only the females in the salmon population.
- The maximum age attained by any individual salmon is three years.
- The salmon are grouped into three one-year age classes.
- An individual salmon's chances of surviving from one year to the next is a function of its age.
- The survival rate P_i of each age group is known.
- The reproduction (fecundity) rate F_i for each age group is known.
- The initial age distribution is known.

---

[1] *On the Use of Matices in Certain Population Mathematics*, Leslie, P.H., *Biometrika*, Volume XXXIII, November 1945, pp. 183–212

From these assumptions it is possible to construct a deterministic model by using matrices. Since the maximum age attained by any salmon is three years, the entire population can be broken up into three one-year age classes. Class 1 contains all salmon in their first year, class 2 contains all salmon in their second year, and class 3 contains all salmon in their third and last year of life.

Suppose we know the number of females in each of the three age classes at some time $t = t_0$. Let there be $x_1^{(0)}$ females in the first age class, $x_2^{(0)}$ females in the second age class, and $x_3^{(0)}$ females in the third age class. With these three numbers we form the column vector $\mathbf{x}^{(0)}$.

$$\mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

We call $\mathbf{x}^{(0)}$ the *initial age distribution vector* or the age distribution vector at time $t = t_0$.

As time progresses, the number of females in each of the three age classes changes because of three biological processes: birth, death, and aging. By describing these three processes quantitatively, we shall see how to project the initial age distribution vector into the future.

We will observe the population at discrete one year time intervals defined as $t_0, t_1, t_2, t_3, \ldots$. The birth and death processes between two successive observation times may be described by the means of defining parameters called the *average reproduction rate* and the *net survival rate*.

Let F_1 be the average number of females born to a single female in the first age class, F_2 is the average number of females born to a single female in the second age class, and F_3 is the average number of females born to a single female in the third age class. Each F_i is average reproduction rate of a single female in the ith age group.

Let P_1 be the fraction of females in the first age class that survive the year to live on into the second age class. Let P_2 be the fraction of females in the second age class that survive the year to live on into the third age class. There is no P_3. After the third year, all the salmon die after spawning, so none survive to live on into a fourth age class. In general,

F_i	is the average reproduction rate of a female in the ith age class,
P_i	is the survival rate of females in the ith age class.

By their definitions, $F_i \geq 0$ since the number of offspring produced cannot be negative. In the case of this salmon population, $F_1 = 0$ and $F_2 = 0$ because the salmon only produce offspring in their last year of life. Thus, only F_3 has a positive value. Also, $0 < P_i \leq 1$ for $i = 1, 2$, since we assume that some of the salmon must survive into the next age class. This is true except for the last age class, when all the salmon die after spawning.

We next define the age distribution $\mathbf{x}^{(k)}$ at time t_k by

$$\mathbf{x}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix},$$

where $x_i^{(k)}$ is the number of female salmon in the i-th age class at time t_k. Now, at time t_k, the number of salmon in the first age class $x_1^{(k)}$, are just those salmon born between time t_{k-1} and t_k. The number of offspring produced by each age class can be calculated by multiplying the reproductive rate for the age class times the number of females in the age class. The sum of all these values gives the total number of offspring produced. Thus, we can write

$$x_1^{(k)} = F_1 x_1^{(k-1)} + F_2 x_2^{(k-1)} + F_3 x_3^{(k-1)} \tag{1}$$

which says that the number of females in age class 1 equals the number of daughters born to females in age class 1 between times t_k and t_{k-1}, plus the number of daughters born to females in age class 2 between times t_k and t_{k-1}, plus the number of daughters born to females in age class 3 between times t_k and t_{k-1}. In this example, since salmon only produce offspring in their last year of life, $F_1 = 0$ and $F_2 = 0$, so we get the equation

$$x_1^{(k)} = 0 x_1^{(k-1)} + 0 x_2^{(k-1)} + F_3 x_3^{(k-1)}. \tag{2}$$

The number of females in the second age class at time t_k are those females in the first age class at time t_{k-1} who are still alive at time t_k, or mathematically $x_2^{(k)} = P_1 x_1^{(k-1)}$, the number of females in the third age class at time t_k are those females in the second age class at time t_{k-1} who are still alive at time t_k, or mathematically $x_3^{(k)} = P_2 x_2^{(k-1)}$. We end up with the following system of linear equations.

$$\begin{aligned} x_1^{(k)} &= F_1 x_1^{(k-1)} + F_2 x_2^{(k-1)} + F_3 x_3^{(k-1)} \\ x_2^{(k)} &= P_1 x_1^{(k-1)} \\ x_3^{(k)} &= P_2 x_2^{(k-1)} \end{aligned} \tag{3}$$

We can use matrices to rewrite this system of equations as

$$\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix} = \begin{pmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{pmatrix},$$

and even more compactly as

$$\mathbf{x}^{(k)} = L\mathbf{x}^{(k-1)}, \tag{4}$$

where

$$\mathbf{x}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix}$$

is the age distribution vector at time t_k, and

$$\mathbf{x}^{(k-1)} = \begin{pmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{pmatrix}$$

is the age distribution vector at time t_{k-1}, and

$$L = \begin{pmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix} \tag{5}$$

is called the *Leslie Matrix*.

Finally, because $F_1 = F_2 = 0$, we get

$$L = \begin{pmatrix} 0 & 0 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{pmatrix}. \tag{6}$$

We can now generate a sequence of matrix equations to find the age distribution vector at any time t_k.

$$\begin{aligned} \mathbf{x}^{(1)} &= L\mathbf{x}^{(0)} \\ \mathbf{x}^{(2)} &= L\mathbf{x}^{(1)} = L(L\mathbf{x}^{(0)}) = L^2\mathbf{x}^{(0)} \\ \mathbf{x}^{(3)} &= L\mathbf{x}^{(2)} = L(L^2\mathbf{x}^{(0)}) = L^3\mathbf{x}^{(0)} \\ &\vdots \\ \mathbf{x}^{(k)} &= L\mathbf{x}^{(k-1)} = L(L^{k-1}\mathbf{x}^{(0)}) = L^k\mathbf{x}^{(0)} \end{aligned} \tag{7}$$

Thus, if we know the initial age distribution vector

$$\mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

and the Leslie matrix L, we can determine the female age distribution vector at any later time by multiplying an appropriate power of the Leslie matrix with the initial age distribution vector $\mathbf{x}^{(0)}$.

An Example Using MATLAB

Suppose there are 1,000 females in each of the three age classes, so

$$\mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \begin{pmatrix} 1,000 \\ 1,000 \\ 1,000 \end{pmatrix}.$$

Suppose further that the survival rate for salmon in the first age class is 0.5%, the survival rate for salmon in the second age class is 10%, and that each female in the third age class produces 2,000 female offspring. Then $P_2 = 0.005$, $P_3 = 0.10$, and $F_3 = 2,000$. The corresponding Leslie matrix for this system is

$$L = \begin{pmatrix} 0 & 0 & 2000 \\ .005 & 0 & 0 \\ 0 & .10 & 0 \end{pmatrix}.$$

To find the age distribution vector after one year, we use the equation $\mathbf{x}^{(1)} = L\mathbf{x}^{(0)}$. We can use MATLAB to find $\mathbf{x}^{(1)}$. First, enter the initial age distribution vector and the Leslie matrix.

```
>> x0=[1000;1000;1000];
>> L=[0 0 2000;0.005 0 0;0 .10 0]
L =
   1.0e+003 *
         0         0    2.0000
    0.0000         0         0
         0    0.0001         0
```

Note that Matlab is using scientific notation. The `1.0e+003 *` prior to the display of the matrix indicates that you should multiply each entry of the resulting matrix by 1.0×10^3, effectively moving each decimal point three places to the right. Let's try a different format for our output (Type `help format` to get a complete list of Matlab's formatting possibilities).

```
>> format short g
>> L=[0 0 2000;0.005 0 0;0 0.10 0]
L =

           0           0        2000
       0.005           0           0
           0         0.1           0
```

The command `format short g` prompts Matlab to use the best of fixed or floating point format, making a decision on each entry of the matrix, rather than applying one format to the entire matrix. Now, compute $x^{(1)}$ in the following manner.

```
>> x1=L*x0
x1 =
     2000000
           5
         100
```

The age distribution vector $\mathbf{x}^{(1)}$ shows that after the first year there are 2,000,000 salmon in the first age class, 5 in the second age class, and 100 in the third age class. Use MATLAB to find the age distribution vector $\mathbf{x}^{(2)}$ after two years.

```
>> x2=L*x1
x2 =
      2e+005
       10000
         0.5
```

You can produce exactly the same result with

```
>> x2=L^2*x0
x2 =
      2e+005
       10000
         0.5
```

The age distribution vector $\mathbf{x}^{(2)}$ shows that after two years there are 200,000 salmon in the first age class, 10,000 in the second age class, and 0.5 in the third age class. In real life it is not possible to have 1/2 of a salmon. However, let's postpone this issue for a moment and continue with the computation of the population after three years.

```
>> x3=L*x2
x3 =
        1000
        1000
        1000
```

Again, rather than proceeding with consecutive iterations, one can proceed directly to the answer with

```
>> x3=L^3*x0
x3 =
        1000
        1000
        1000
```

Note that the salmon population has returned to its original configuration, with 1,000 fish in each age category. Use Matlab to perform at least four more iterations; i.e., find $x^{(4)}$, $x^{(5)}$, $x^{(6)}$, and $x^{(7)}$. What pattern do you see?

The Graph of the Age Distribution Vector

One of the best ways to examine trends in population growth is to sketch the graph of the age distribution vector versus time. Also, it's often desirable to track a population for more than three or four years.

Using For Loops

Iterating the equation $x^{(k)} = Lx^{(k-1)}$ in the manner above is inefficient. If you know ahead of time the precise number of times that you wish to perform the iteration, then using a `for` loop in Matlab is the most efficient method.

First, load your Leslie Matrix and the initial age distribution vector.

```
>> L=[0 0 2000;0.005 0 0;0 0.10 0];
>> x0=[1000;1000;1000];
```

Let's iterate the equation $x^{(k)} = Lx^{(k-1)}$ a total of 24 times which will produce 24 generations of the age distribution vector. The makers of Matlab recommend that you plan ahead and reserve space in the memory of your computer to store your results. Let's follow this recommendation and reserve space for the results of 24 iterations by creating a 3×24 matrix of zeros: three rows because each age distribution vector contains three rows, twenty four columns because we will generate 24 age distribution vectors.

```
>> X=zeros(3,24)
```

Next, place the initial age distribution vector in the first column of the matrix X.

```
>> X(:,1)=x0
```

Recall that the Matlab index notation, `X(:,1)`, is read "every row, first column." Consequently, the command `X(:,1)=x0` places the initial conditions, contained in `x0`, into the first column of the matrix X.

Calculate the second through twenty-fourth columns of the matrix X by iterating the equation $x^{(k)} = Lx^{(k-1)}$ for k-values ranging from two through twenty four.

```
>> for k=2:24, X(:,k)=L*X(:,k-1); end
```

When the number of iterations needed are known in advance, Matlab's `for` loop is the ideal construct. Recall that `2:24` produces a row vector, starting at 2 and proceeding in increments of 1 until it reaches the number 24. Therefore, the command `for k=2:24` begins the loop with a k-value equal to 2. The next time through the loop a k-value of 3 is used. Iteration continues and the last time through the loop, a k-value of 24 is used. Note that the `end` command signals the end of the loop.

The command `X(:,k)=L*X(:,k-1)` warrants explanation. Recall that `X(:,k)` is read "matrix X, every row, kth column." Similarly, the command `X(:,k-1)` is read "matrix X, every row, $k-1$st column." Consequently, the command `X(:,k)=L*X(:,k-1)` forms the product of the Leslie matrix L and the $k-1$st column of matrix X and stores the result in the kth column of matrix X, precisely the iteration we need (recall that $\mathbf{x}^{(k)} = L\mathbf{x}^{(k-1)}$).[2]

Once the iteration is complete, you can display the contents of the matrix X by entering X at the Matlab prompt and pressing the Enter key.

```
>> X =
  Columns 1 through 6
       1000       2e+006       2e+005       1000       2e+006       2e+005
       1000            5        10000       1000            5        10000
       1000          100          0.5       1000          100          0.5
  Columns 7 through 12
       1000       2e+006       2e+005       1000       2e+006       2e+005
       1000            5        10000       1000            5        10000
       1000          100          0.5       1000          100          0.5
  Columns 13 through 18
       1000       2e+006       2e+005       1000       2e+006       2e+005
       1000            5        10000       1000            5        10000
       1000          100          0.5       1000          100          0.5
  Columns 19 through 24
       1000       2e+006       2e+005       1000       2e+006       2e+005
       1000            5        10000       1000            5        10000
       1000          100          0.5       1000          100          0.5
```

Type `help plot` at the Matlab prompt and read the resulting helpfile. Pay particular attention to the following lines.

```
>> help plot

 PLOT   Plot vectors or matrices.
   PLOT(Y) plots the columns of Y versus their index.
```

However, the first *row* of the matrix X contains the numbers of female salmon in the first age class (juveniles), the second *row* contains the second age class (subadults), and the third *row* contains the number of female salmon in the third and final age class (adults). We want to plot the *rows* of X versus the index but `plot(X)` plots the *columns* of X versus the index.[3] The solution: plot the *transpose* of X.

The following command will produce an image similar to that in **Figure 1**.

```
>> plot(X')
```

If the command `plot(X')` is supposed to plot each of the three columns of the matrix `X'`, then where are the graphs of the remaining two columns in **Figure 1**? If you look closely you can see a little activity near the x-axis in **Figure 1**. Note that the upper limit on the y -axis in Figure 1 is 2×10^6. When there is such a wide range in the data (values as small as 1/2 and as large as 2,000,000) you can get a better picture by plotting the natural logarithm of the salmon populations versus the time. The following command produces an image similar to that in **Figure 2**.

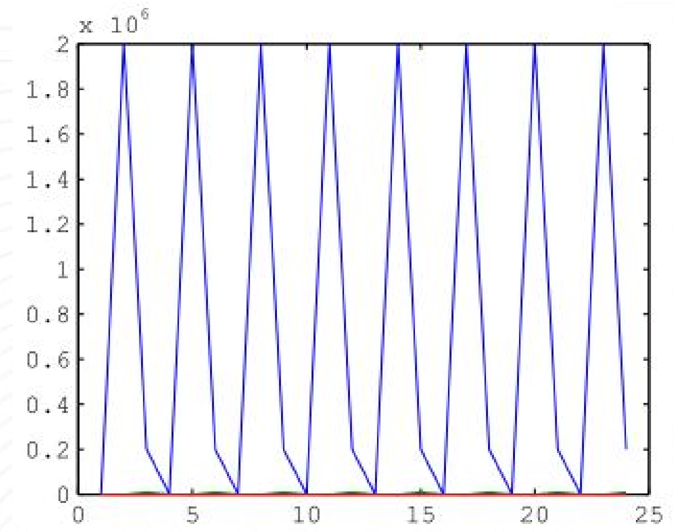

Figure 1 The salmon population over time.

```
>> semilogy(X'
```

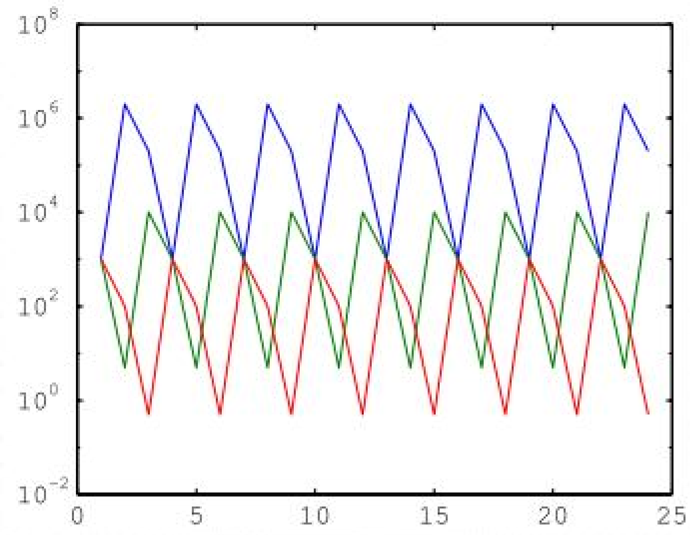

Figure 2 Semilog plot of salmon population over time.

It is helpful to place a legend on your graph. The following command will produce an image similar to that in **Figure 3**.

```
>> legend('Juveniles','Subadults','Adults')
```

It is clear from the graph in **Figure 3** that each age division of the salmon population is oscillating with period 3.

Extra for Experts. The actual graph in **Figure 3** was produced using Matlab's handle graphics capabilities. If you are interested, try the following commands.

```
>> h=semilogy(X')
h =

        9.0011
        12.001
        13.001
>> set(h(1),'LineStyle','--')
>> set(h(2),'LineStyle',':')
>> legend('Juveniles','Subadults','Adults')
>> grid off
```

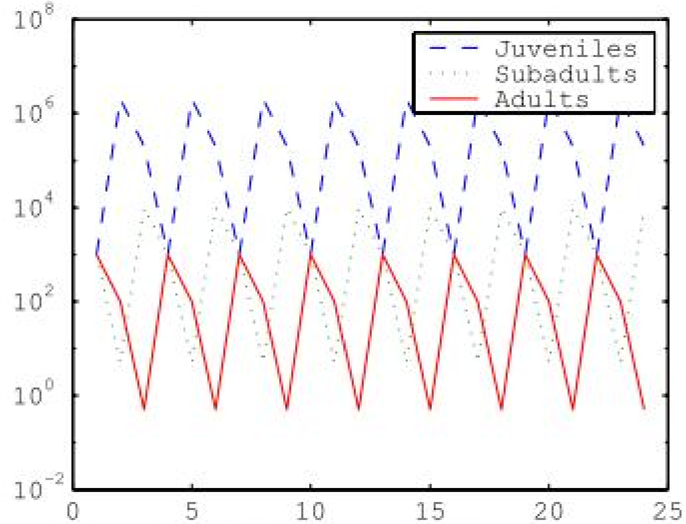

Figure 3 Providing a legend.

In the latest release of Matlab, version 5.3, Release 11, you can change linestyles interactively by enabling plot editing, then right clicking a line with the mouse. A popup menu provides choices for linestyles, colors, and a host of other properties.

Homework

Instructions. For each of the following questions use the printer to produce a hardcopy image of the required graph.

1. Suppose a particular species of salmon lives to *four* years of age. In addition, suppose that the survival rate of salmon in their first, second, and third years is 0.5%, 7%, and 15%, respectively. You also know that each female in the fourth age class produces 5,000 female offspring. The other age classes produce no offspring.

   a. Find the Leslie matrix for this population.

   b. If 1,000 female salmon in each of the four age classes are introduced into the system, find the initial age distribution vector.

   c. Use a `for` loop to iterate the Leslie equation 25 times. Use MATLAB to plot the natural logarithm of each age class of salmon versus time. What is the eventual fate of this salmon population?

   d. Calculate the salmon population on the 50th iteration, without calculating the preceding 49 iterations.

2. Suppose another species of salmon lives to *four* years of age. In addition, the survival rate of salmon in their first, second, and third years is 2%, 15%, and 25%, respectively. You also know that each female in the fourth age class produces 5,000 female offspring. The other age classes produce no offspring.

   a. Find the Leslie matrix for this population.

   b. If 1,000 female salmon in each of the four age classes are introduced into the system, find the initial age distribution vector.

   c. Use a `for` loop to iterate the Leslie equation 25 times. Use MATLAB to plot the natural logarithm of each age class of salmon versus time. What is the eventual fate of this salmon population?

   d. Calculate the salmon population on the 50th iteration, without calculating the preceding 49 iterations.

Discussion

Although **equation 7** gives the age distribution of the population at any time, it does not immediately give a general picture of the dynamics of the growth process. To study the limiting behavior of the population growth we will need to learn about the eigenvalues and eigenvectors of the Leslie matrix. We will return to this problem later in the course.

Properties of Leslie matrices

We list some properties of Leslie matrices (without proof), and we explore the effect of iterating the transition many times, that is, of allowing the population to pass through many reproductive cycles.

Theorems about Leslie Matrices

1. A Leslie matrix **L** has a unique positive eigenvalue λ_1. This eigenvalue has multiplicity 1, and it has an eigenvector $\mathbf{x_1}$ whose entries are all positive.
2. If λ_1 is the unique positive eigenvalue of **L**, and λ_i is any other eigenvalue (real or complex), then $|\lambda_i| \leq \lambda_1$. That is, λ_1 is a *dominant eigenvalue*.
3. If any two successive entries $\mathbf{a_j}$ and $\mathbf{a_{j+1}}$ of the first row of **L** are both positive, then $|\lambda_i| < \lambda_1$ for every other eigenvalue. That is, if the females in two successive age classes are fertile (almost always the case in any realistic population) then λ_1 is a *strictly dominant eigenvalue*.
4. Let $\mathbf{x}^{(k)}$ denote the state vector $\mathbf{L}^k\mathbf{x}^{(0)}$ after **k** growth periods. If λ_1 is a strictly dominant eigenvalue, then for large values of **k**, $\mathbf{x}^{(k+1)}$ is approximately $\lambda_1 \mathbf{x}^{(k)}$, no matter what the starting state $\mathbf{x}^{(0)}$. That is, as **k** becomes large, successive state vectors become more and more like an eigenvector for λ_1.

NOTE: If all the entries of the eigenvector are negative, multiplication by -1 produces an eigenvector with all entries positive, without changing the validity of our reasoning.

Theorem 4 needs careful interpretation. It does not say that the sequence of states converges -- in particular, if the dominant eigenvalue is **> 1**, the sequence does not converge at all. On the other hand, if we "normalize" the state vector at each step -- say, by making its entries sum to 1 -- the sequence of modified state vectors does converge to an eigenvector. Normalized or not, the sequence shows us an *equilibrium age distribution* of the female population, which is approached over time.

Case Study: Dynamical Systems and Spotted Owls

In this case study, we examine how eigenvalues and eigenvectors can be used to study the change in a population over time. We begin by recalling the example of the spotted owl given in the Introduction to Chapter 5.

The population of spotted owls is divided into three age classes: juvenile (up to 1 year old), subadult (1 to 2 years old), and adult (over 2 years old). The population is examined at yearly intervals. Since it is assumed that the number of male and female owls is equal, only female owls are counted in the analysis. If there are j_k juvenile females, s_k subadult females, and a_k adult females at year k, then R. Lamberson et al. (see Reference 3) found that the population of owls could be modelled by the equation

$$\begin{bmatrix} j_{k+1} \\ s_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & .33 \\ .18 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix} \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$$

If we let $\mathbf{x}_k = (j_k, s_k, a_k)$, then we note that the population model has the form $\mathbf{x}_{k+1} = A\mathbf{x}_k$, which is a difference equation. This model is called the **stage-matrix model** for a population. The entries in the matrix A have important meanings. The entries in the first row describe the **fecundity** of the population. Thus in the model above juveniles and subadults do not produce offspring, but each adult female produces (on the average) .33 juvenile females per year. The other entries in the matrix show **survival**. In this model, 18% of the juvenile females survive to become subadults, 71% of the subadults survive to become adults, and 94% of the adults survive each year. Note that the measures of fecundity and survival remain constant through time.

We wish to determine the long-term dynamics of the population: whether the population is becoming extinct or is increasing. To answer these questions we examine the eigenvalues of the matrix A ($\lambda_1 = .98$, $\lambda_2 = -.02 + .21i$, $\lambda_3 = -.02 - .21i$). If we label the corresponding eigenvectors as $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. we may express $\mathbf{x}_k$ as

$$\mathbf{x}_k = c_1(\lambda_1)^k \mathbf{v}_1 + c_2(\lambda_2)^k \mathbf{v}_2 + c_3(\lambda_3)^k \mathbf{v}_3$$

This expression of $\mathbf{x}_k$ is called the **eigenvector decomposition** of $\mathbf{x}_k$ (see page 337 of the text). Since each eigenvalue has magnitude less than 1, we conclude that $\mathbf{x}_k$ is approaching the zero vector as k increases: the population is becoming extinct. Notice that the number of greatest importance to this analysis is $\lambda_1 = .98$, the eigenvalue of greatest magnitude. If λ_1 happened to be greater than 1, the population would instead be increasing steadily.

For example, consider Example 7 on page 345. If the survival rate for juveniles were somehow increased to 30%, the new matrix A would be

$$\begin{bmatrix} 0 & 0 & .33 \\ .30 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda_1 = 1.01$, $\lambda_2 = -.03 + .26i$, $\lambda_3 = -.03 - .26i$. If we let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ now denote eigenvectors of this new matrix, we will again have an eigenvector decomposition for $\mathbf{x}_k$:

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + c_3(\lambda_3)^k\mathbf{v}_3$$

As $k \to \infty$ the second and third vectors tend to the zero vector, but the first does not. Thus $\mathbf{x}_k$ is approaching $c_1(1.01)^k\mathbf{v}_1$ as $k \to \infty$. So the population of owls would be increasing exponentially at a growth rate of 1.01; the population would be increasing by 1% per year. The eigenvector $\mathbf{v}_1$ gives the long-term distribution of the owls by life stages. In this case $\mathbf{v}_1$ is approximately $(10, 3, 31)$, so for every 31 adult females, there will be 10 juvenile females and 3 subadult females. We could further rescale $\mathbf{v}_1$ so that its entries sum to 1, namely: $(.227, .068, .705)$. The entries in this vector show the fraction of the owl population in each class; for example, 22.7% of the owls would be juveniles.

For convenience, here is a summary of our results drawn from Chapter 5.

1. We use the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ to model the population in question; A is an $n \times n$ matrix, where the population has been divided into n classes or stages.

2. We find the eigenvalues of A and list them in descending order of magnitude: $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$.

3. If A is diagonalizable or if A has distinct (possibly complex) eigenvalues, then we may express $\mathbf{x}_k$ in terms of its eigenvalues and corresponding eigenvectors:

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + \ldots + c_3n(\lambda_n)^k\mathbf{v}_n$$

4. If $|\lambda_1| < 1$, then the population is decreasing to extinction.

5. If λ_1 is a real number greater than 1 and all the other eigenvalues are less than 1 in magnitude, then the population is increasing exponentially. As noted on page 338, in this case the eigenvector $\mathbf{v}_1$ gives the stable distribution of the population between classes, and yields the percenatges found in each class if scaled so that its entries sum to 1.

Exercice :

In the 1930's (before its virtual extinction and a great change in its survival rates) a researcher studied the blue whale population (see References 2, 5 and 6 for this data). Due to the long gestation period, mating habits, and migration of the blue whale, a female can produce a calf only once in a two-year period. Thus the age classes for the whale were assumed to be: less than 2 years, 2 or 3 years, 4 or 5 years, 6 or 7 years, 8 or 9 years, 10 or 11 years, and 12 or more years. The matrix for the model is given by

$$\begin{bmatrix} 0 & 0 & .19 & .44 & .50 & .50 & .45 \\ .77 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .77 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .77 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .77 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .77 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .77 & .78 \end{bmatrix}$$

Determine whether the blue whale population is becoming extinct in this model. If the population is not becoming extinct, determine the percentage of each class in the stable population.

Project : Leslie Matrix or Population Projection Matrix

The following table lists reproduction and survivor rates for the female population of a certain species of domestic sheep in New Zealand. (For animal populations, it is conventional to consider only the females, since only they reproduce, and they are usually a fixed percentage of the total population.) Sheep give birth only once a year, which dictates a natural time step of one year. In the species under consideration, sheep seldom if ever live longer than 12 years, which gives a natural stopping point for the age class.

Birth and Survival Rates for Female New Zealand Sheep

age(years)	birth rate	survival rate
0-1	0.000	0.845
1-2	0.045	0.975
2-3	0.391	0.965
3-4	0.472	0.950
4-5	0.484	0.926
5-6	0.546	0.895
6-7	0.543	0.850
7-8	0.502	0.786
8-9	0.468	0.691
9-10	0.459	0.561
10-11	0.433	0.370
11-12	0.421	0.000

(2) Activities:

(a) Enter the Leslie matrix L for the table given above.
(b) Compute the eigenvalues of L. How many real eigenvalues are there?
(c) Find the eigenvectors for the positive eigenvalues. Modify your eigenvectors (if necessary) to make it a state vector, and describe what it tells you about a distribution into age classes. (**Hint**: Any multiple of an eigenvector is also an eigenvector. Your *state* vector should have entries which sum to 1.)
(d) If the state vector you found in the previous step is the age distribution in a given year, what will the distribution be in one year? In ten years?