Max Chung

MATH

lecture 1 (missed)

Defn: a scalar function $f(x_1 \dots x_n)$ of n-variables is a fcn. whose domain & Range. is a subset of $\mathbb{R}^n$

Geometric Intepretations:

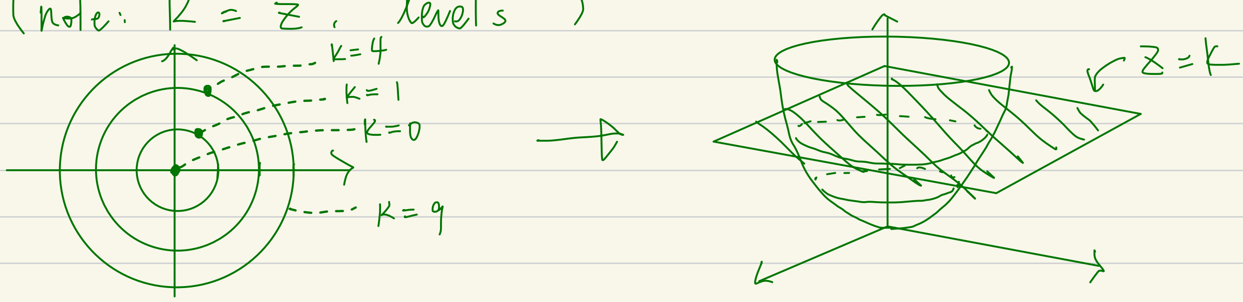
level curves: the curves $f(x,y) = k$ where $k \in \text{Range}(f)$

ex: consider $f(x,y) = x^2+y^2$

$\Rightarrow D(f) = \mathbb{R}^2$ and $R(f) = \{z \in \mathbb{R} : z \geq 0\} \Rightarrow k \geq 0$

inspection: level curves are circles $x^2+y^2=k$, $c=(0,0)$

(note: $k = z$, "levels")



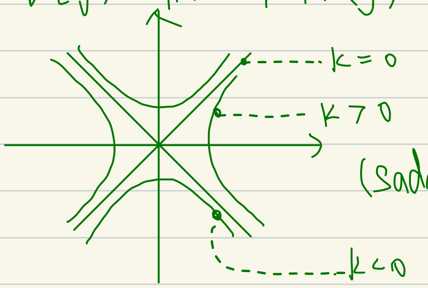
exceptional level curve

paraboloid.

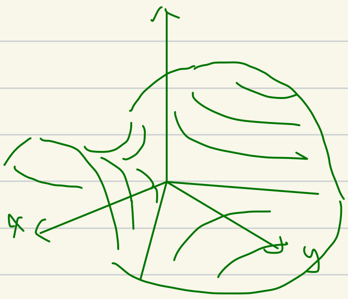
* Review : $(x-a)^2+(y-b)^2 = k$, $c=(a,b)$, $r=\sqrt{k}$

ex: consider $g(x,y) = x^2-y^2$ (hyperbola)

$\Rightarrow D(g) = \mathbb{R}^2$, $R(g) = \mathbb{R} \Rightarrow k \in \mathbb{R}$

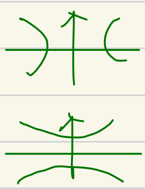


(saddle surface)

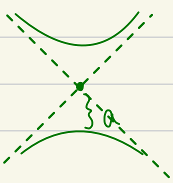


← Pringles

* Review: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-v)^2}{b^2} = 1$, $a > b$, $(x,y) \Rightarrow$ horizontal, $(y,x) \Rightarrow$ vertical

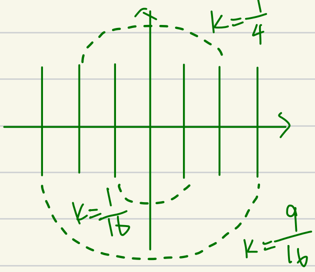


$C = (h,v)$

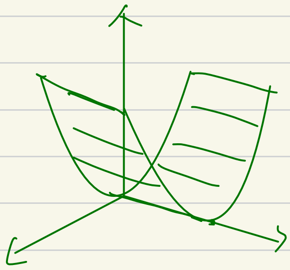


ex: consider $h(x,y) = x^2$

$\Rightarrow D(h) = \mathbb{R}^2,\ R(h) = \{z \in \mathbb{R} : z \geq 0\} \Rightarrow k \geq 0$



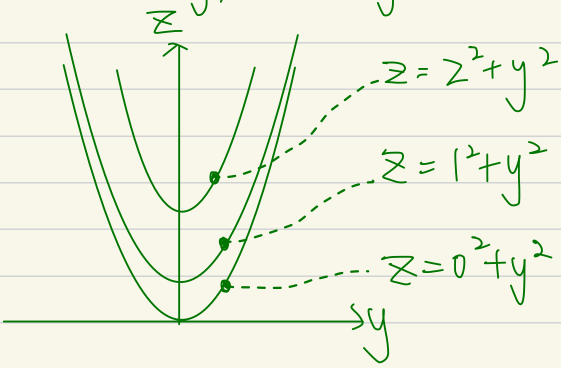
$\Rightarrow$



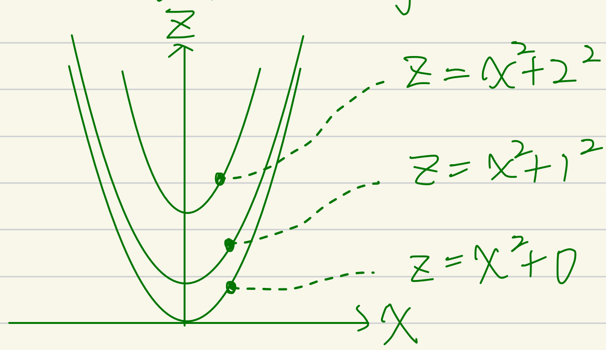
Cross sections of a surface $z = f(x,y)$ is the intersect of the plane ex: vertical planes $x=c$ or $y=d$.

ex: let $f(x,y) = x^2+y^2$ cross section with $x=c$? $y=d$?

x:



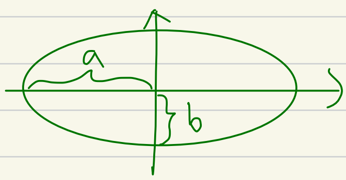
y:



Generalization:

- level surface is defined by $f(x,y,z)=k$, $k \in R(f)$
- level set of $f(x)$, $x \in \mathbb{R}^n$ is defined by $f(x)=k$, $k \in R(f)$

*note: ellipse equ: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



lecture 2.

Defn: an r-neighborhood of a point $(a,b) \in \mathbb{R}^2$ is a set

$$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)-(a,b)\| < r\}$$

Defn: limit (review)

Assume $f(x,y)$ is in neighborhood of (a,b), expect possibly at (a,b)

If for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$0 < \|(x,y)-(a,b)\| < \delta$ implies $|f(x,y) - L| < \varepsilon$

then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

Thm: limit are Unique.!

lecture 3

ex: $\lim_{(x,y)\to(a,b)} \frac{2x^4 y^{\frac{2}{3}}}{x^6+y^2}$, Prove lim DNE

Sol: try $y=mx$, $m\in\mathbb{R}$

$$\lim_{x\to 0} \frac{2x^4(mx)^{\frac{2}{3}}}{x^6+(mx)^2} = \lim_{x\to 0} \frac{2m^{\frac{3}{2}}x^{\frac{8}{3}}}{x^4+m^2} = \frac{0}{m^2} = 0 \quad (m\neq 0)$$

If $m=0 \Rightarrow \lim_{x\to 0} \frac{0}{x^6} = 0 \Rightarrow \lim = 0$ along all $y=mx$

does not imply $\lim_{(x,y)\to(a,b)} = 0$

take $y=x^3$, $\Rightarrow \lim_{x\to 0} \frac{2x^4(x^3)^{\frac{2}{3}}}{x^6+x^6} = \frac{2x^6}{2x^6} = 1 \neq 0$

Thus, lim DNE.

ex: $\lim_{(x,y)\to(1,0)} \frac{x^2-y-1}{x+y-1} \Rightarrow$ Prove DNE

Sol: try lines $y=m(x-1)$ $\to (1,0)$

Shortcut: try $y=0 \Rightarrow \lim_{x\to 1} \frac{x^2-0-1}{x+0-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)} = 2.$

try $x=1 \Rightarrow \lim_{x\to 1} \frac{1^2-y-1}{1+y-1} = -1$ ← different

Thus, lim DNE

Proving that a limit exists.

Squeeze theorem : if $\exists$ a fcn. $B(x,y)$ s.t. ← Bound.

$|f(x,y)-L| \le B(x,y)$ $\forall$ (x,y) in some neighborhood of (a,b), expect possibly at (a,b),

and $\lim_{(x,y)\to(a,b)} B(x,y)=0$ then $\lim_{(x,y)\to(a,b)} f(x,y)=L$

<u>Proof</u>: let $\varepsilon>0$. Since $\lim_{(x,y)\to(a,b)} B(x,y)=0$

we know $\exists$ $\delta>0$ s.t.

$|B(x,y)-0|<\varepsilon$ whenever $0<\|(x,y)-(a,b)\|<\delta$

Now, $|f(x,y)-L| \le B(x,y) < \varepsilon$ whenever $0<\|(x,y)-(a,b)\|<\delta$

↖ hypo ↖ by above.

So, $\lim_{(x,y)\to(a,b)} f(x,y)=L$ by def.

ex.: Show $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}=0$

<u>Sol</u>: consider $\left|\frac{2x^2y}{x^2+y^2}-0\right| = \frac{2x^2|y|}{x^2+y^2} \le \frac{2(x^2+y^2)|y|}{x^2+y^2}$

$=2|y| = B(x,y)$,

Clearly $\lim_{(x,y)\to(0,0)} 2|y|=0$ so by squeez thm. $\lim=0$.

<u>comment</u>: $\frac{2x^2|y|}{x^2+y^2} \le \frac{2x^2|y|}{x^2}$ ↙ nope.

$\to (x,y)\ne(0,0)$ $\to x\ne 0$

$\Downarrow$

ex: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^4+x^2+y^4+y^2}{x^2+y^2}$ or show DNE.

Sol: Along $y=mx$, we have.

$$\Rightarrow \lim_{x\to0} \frac{x^4+x^2+(mx)^4+(mx)^2}{x^2+(mx)^2} = \lim_{x\to0} \frac{\cancel{x^2}(x^2+1+m^4x^2+m^2)}{\cancel{x^2}(1+m^2)}$$

$= \frac{1+m^2}{1+m^2} = 1$, so limit $f(x,y)$ must $=1$ if it exist.

Try squeez $\Rightarrow$ $|f(x,y)-L| = \left| \frac{x^4+x^2+y^4+y^2}{x^2+y^2} - 1 \right|$

$$= \left| \frac{x^4+y^4}{x^2+y^2} \right| \le \frac{(x^2+y^2)^2+(x^2+y^2)^2}{x^2+y^2} = \frac{2(x^2+y^2)^2}{x^2+y^2}$$

$=2(x^2+y^2) = B(x,y)$, $\to 0$ as $(x,y)\to(0,0)$

so $\lim f(x,y)=1$ by sqz thm.

lecture 4

Last time: Sqz thm $|f(x,y)-L| \leq B(x,y) \to 0$

- Common inequality tricks (pg. 19)

$|x+y| \leq |x|+|y|$ ($\triangle$ ineq)

$(x^2+y^2)^2 = x^4+2x^2y^2+y^4 \Rightarrow x^4+y^4 \leq (x^2+y^2)^2$

$|x||y| \leq \frac{x^2+y^2}{2}$ (cosine ineq)

Continuity:

Defn: A fcn. $f(x,y)$ is continuous at (a,b) means.

$$\lim_{(x,y)\to(a,b)} = f(a,b)$$

f is continuous in $D \subseteq \mathbb{R}^2$ if it is cts. at every point in D.

ex: $f(x,y) = \frac{x^4+x^2+y^4+y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$

we showed $\lim_{\to(0,0)} f(x,y) = 1$

if we define $f(0,0)=1$ then f will be cts. at $(0,0)$.

ex: Let $f(x,y) = \begin{cases} \frac{\sin(x^2+2y^2)}{x^2+y^2} & , (x,y) \neq (0,0) \\ k & , (x,y) \neq (0,0) \end{cases}$

Can we find k so that f is cts. at $(0,0)$

Sol: Consider $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{x^2+y^2}$

P.S. try along "simple" line.

$-y=0$: $\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 1$ (prev)

$- x = 0$: $\lim_{y \to 0} \frac{\sin(2y^2)}{y^2} = 2 \lim_{y \to 0} \frac{\sin(2y^2)}{2y^2} = 2$. (different)

Since $1 \neq 2$, $\lim f(x,y)$ DNE.

Continuity Theorem:

let $f(x,y)$, $g(x,y)$ be cts. at (a,b). Then

① $f+g$ and fg are also cts. at (a,b)

② f/g is cts at (a,b), provided $g(a,b) \neq 0$

Continuity composition theorem:

let f be a single var. fcn. and g be a 2-var. fcn.

If g is cts. at (a,b) and f is cts at $g(a,b)$,

then $f(g(x,y))$ is cts. at (a,b)

ex: $\frac{y \sin x - \cos y}{x^2 + y^2}$ is cts. $\forall (x,y)$ except for $(0,0)$ by cty. thm..

ex: $e^{x^3 - \sin(x,y)}$ is cts on $\mathbb{R}$ by cty. thm(s).

ex: $\lim_{(x,y) \to (0,0)} \frac{e^{x^2} + \ln(2 + y^2 + x^4)}{(x-1)^2 + y^4} = ?$

by cty. thm. $f(x,y)$ is cts. at $(0,0)$

thus, plug the fuck in $(0,0)$ $= 1 + \ln(2)$

ex: determine where $f(x,y) = \begin{cases} \frac{x^4 y^6}{x^6 + y^{12}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

is cts.

Sol: for $(x,y) \neq (0,0)$, f is cts. by cty. thm.

for $(x,y) = (0,0)$, use def of continuity.

$$\Rightarrow \lim_{\to(0,0)} f(x,y) \overset{?}{=} f(0,0)$$

$$\lim_{\to(0,0)} \frac{x^4y^6}{x^6+y^{12}}, \text{ along } y=mx: \lim_{x\to 0} \frac{m^6 x^{\cancel{10}\,4}}{\cancel{x^6}(1+m^{12}x^6)} = 0$$

- try a sqz: $\left|\frac{x^4y^6}{x^6+y^{12}} - 0\right| = \frac{x^4y^6}{x^6+y^{12}} = \frac{(x^6+y^{12})^{\frac{4}{6}}(x^6+y^{12})^{\frac{1}{2}}}{x^6+y^{12}}$

$= (x^6+y^{12})^{\frac{1}{6}} \to$ as $(x,y)\to(0,0)$

by sqz thm, $\lim f(x,y) = 0$, which $= f(0,0)$

so f is cts. at $(0,0)$

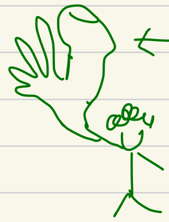
thus f is cts at $\mathbb{R}^2$.

lecture 6.

Two ways to differenciate $f(x,y)$:

① Treat y as fixed x: $\frac{\partial f}{\partial x}$

② Treat x as fixed y: $\frac{\partial f}{\partial y}$



← bro's thumb

ex: $f(x,y) = y^2 \sin(xy)$

$$\frac{\partial f}{\partial x} = y^3 \cos(xy), \quad \frac{\partial f}{\partial y} = 2y \cdot \sin(xy) + y^2 \cos(xy) \cdot x$$

at a point: $\frac{\partial f}{\partial x}(\pi,1) = \frac{\partial f}{\partial x}\Big|_{(\pi,1)} = 1^3 \cdot \cos(\pi) = -1.$

Subscript notation: $\frac{\partial f}{\partial x} = f_x$

$\frac{\partial f}{\partial y} = f_y$

Formal Defn: the partial derivatives of $f(x,y)$ at (a,b) are:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

ex: find "the partial" at $(0,0)$ of

$$f(x,y) = \begin{cases} \frac{x^3+y^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol: use defn.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3+0}{h^2+0} - 0}{h} = 1$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^4}{h^2} - 0}{h} = 0$$

Comment: MUST use defn. when "usual" rules (product, power, quotient, ...) don't apply

ex. page 31~32.

$f(x,y) = (x^3+y^3)^{\frac{1}{3}}$

$$\frac{\partial f}{\partial x} = \frac{1}{3}(x^3+y^3)^{\frac{-2}{3}}(3x^2), \text{ for } x^3+y^3 \neq 0$$

what if $x^3+y^3=0$? i.e. if $y=-x$? i.e. at point $(a,-a)$

above formula not allowed, must use defn.

ex: volume of a ideal gas.

$$V = \frac{82.06T}{P}$$ ← temp. (K) ← pressure (a.t.m)

Find rate of change of volume:

(a) w.r.t T

(b) w.r.t P

when $T = 300k$ and $P = 5$ a.t.m.

Sol: (a) $\frac{\partial V}{\partial T} = \frac{82.06}{P} \Rightarrow \left.\frac{\partial f}{\partial T}\right|_{(300,5)} \approx 16.41 \frac{cm^3}{K}$

(b) $\frac{\partial V}{\partial P} = (82.06T)\cdot(-P^{-2}) \Rightarrow \left.\frac{\partial f}{\partial P}\right|_{(300,5)} \approx -98472 \frac{cm^3}{a.t.m.}$

Second Derivatives: of $f(x,y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1 D_1 f = D_1^2 f$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2 D_1 f$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1 D_2 f$$

$D_1 = \frac{\partial}{\partial x}$

$D_2 = \frac{\partial}{\partial y}$

$\searrow$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = D_2 D_2 f$$

ex: $f(x,y) = xe^{-xy}$

$f_x = 1 \cdot e^{-xy} - yxe^{-xy} = e^{-xy}(1-xy)$

$f_y = -x^2 e^{-xy}$

$f_{xx} = (-y)e^{-xy}(1-xy) + e^{-xy}(-y)$

$f_{xy} = -xe^{-xy}(1-xy) + e^{-xy}(-x) = e^{-xy}(-2x + x^2 y)$

$f_{yx} = -2xe^{-xy} + e^{-xy}(x^2 y) = e^{-xy}(-2x + x^2 y)$ equal!

$f_{yy} = \ldots\ldots$

Clairaut's theorem: If f_x, f_y, f_{xy}, and f_{yx} are all defined in some neighborhood of (a,b), and f_{xy} and f_{yx} are cts. on (a,b) then, $f_{xy} = f_{yx}$.

lecture 7 : Tangent Plane, Linear approximation (43, 44)

But first, more on partial derivatives.

3rd partials of $f(x,y)$, f_{xxx}, f_{xxy}, f_{xyy} ... (8 ways)

For $f(x,y,z)$: $f_x = \frac{\partial f}{\partial x}$

$f \in C^k$: "f is of class C-k"

... means that the k^{th} partial derivatives are continuous.

e.g. $f(x,y) \in C^2$ means that f_{xx}, f_{xy}, f_{yx}, f_{yy} are all cts.

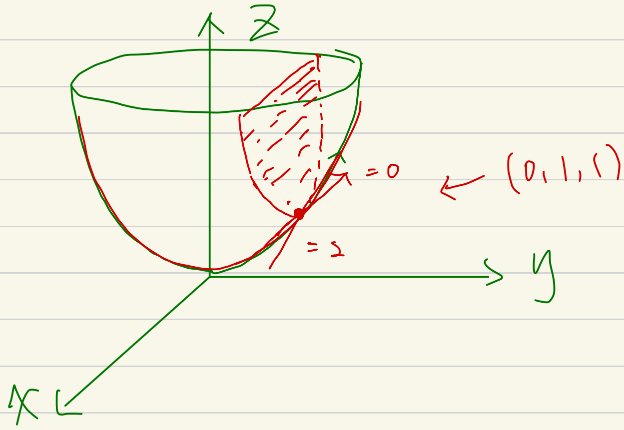
Geometric interp. of f_x and f_y:

ex: $f(x,y) = x^2+y^2$

slope of cross sections.

a) $f_x(0,1) = (2x+0)\big|_{(0,1)} = 0$

$f_y(0,1) = (0+2y)\big|_{(0,1)} = 2.$



Equation of tangent plane to $z=f(x,y)$ at (a,b).

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

ex: find equation of tangent plane to the surface

$z = \frac{xy}{x^2+y^2}$ at $(x,y) = (1,2)$, $f(1,2) = 2/5$

$f_x = \frac{(x^2+y^2)\cdot y - xy(2x)}{x^2+y^2} \Rightarrow \cdots \Rightarrow f_x(1,2) = \frac{6}{25}$

$f_y(1,2) = \frac{-3}{25}$

plugin: $z = \frac{2}{5} + \frac{6}{25}(x-1) - \frac{3}{25}(y-2)$

Defn: Linearization of $f(x,y)$ at (a,b) is

$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Defn: Linear approximation.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

↖ for (x,y) "near" ab

ex: approximate. $\sqrt{3.01^2 + 3.98^2}$

Sol: let $f(x,y) = \sqrt{x^2+y^2}$, and approximate near $(3,4)$

$$f(x,y) \approx f(3,4) + \frac{\partial f}{\partial x}(3,4)(x-3) + \frac{\partial f}{\partial y}(3,4)(y-4)$$

Now, $f(3,4) = 5$,

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)\Big|_{(3,4)} = \frac{x}{\sqrt{x^2+y^2}}\Big|_{(3,4)} = \frac{3}{5}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\Big|_{(3,4)} = \frac{4}{5}$$

So, linear approximation becomes.

$$f(x,y) \approx 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \text{ for } (x,y) \text{ near } (3,4)$$

$$\Rightarrow \sqrt{3.01^2+3.98^2} \approx 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.98-4)$$

$$= 5 + 0.006 - 0.016 = 4.99 \quad (\text{actual value } 4.99004)$$

lecture 8

Last time :

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) \longrightarrow ① \text{ linear approx}$$

$$\Rightarrow \underbrace{f(x,y)-f(a,b)}_{\Delta f} \approx \underbrace{\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)}_{\substack{\nabla f(a,b) \\ \text{"gradient vector"}}} \cdot \underbrace{(x-a, y-b)}_{\substack{=(x,y)-(a,b) \\ = \mathbf{x}-\mathbf{a} \\ = \Delta \mathbf{x}}} \text{ — ②}$$

$\mathbf{x} = (x,y)$

$\mathbf{a} = (a,b)$

$$\Rightarrow \underline{\Delta f \approx \nabla f(x,y) \cdot \Delta \mathbf{x}}$$ "Increment form of linear approx"

Higher dimensions :

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$$

$$\Rightarrow f(x,y,z) - f(a,b,c) \approx \left(f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\right) \cdot (x-a, y-b, z-c)$$

$$\Rightarrow \Delta f \approx \nabla f(\mathbf{a}) \cdot \Delta \mathbf{x}$$ where $\mathbf{a} = (a,b,c)$, $\mathbf{x} = (x,y,z)$, $\Delta \mathbf{x} = \mathbf{x} - \mathbf{a}$.

ex: Let $h(x,y,z) = x,y,z$, and $\mathbf{a} = (1,2,3)$

what is the approx. change in h if x increase by 0.01, y increase by 0.02 and z decreases by 0.01 ?

Sol: $\Delta h \approx \nabla h(1,2,3) \cdot (0.01, 0.02, 0.03)$

$\nabla h = (h_x, h_y, h_z) = (yz, xz, xy)$

$= (6, 3, 2) \cdot (0.01, 0.02, 0.03)$

$= 0.1 \Rightarrow \Delta h \approx 0.1$

ex: try to approx $f(0.1, 0.1)$ where,

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Sol: use linear approx.

$$f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

now $f(0,0) = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

by symmetry, $f_y(0,0) = 0$.

$\Rightarrow f(x,y) \approx 0 + 0 + 0 = 0$

$\Rightarrow f(0.1, 0.1) \approx 0$ (Actual value: 0.5)

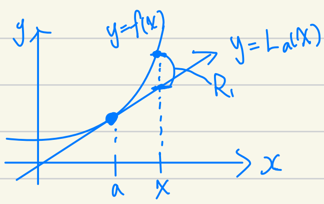
comment: the fcn. is not cts. at $(0,0)$, yet $f_x(0,0)$ and $f_y(0,0)$ exist.

- Need a careful defn. for differentiability.

Flashback: math 137/138.

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{or} \quad f(x) = \underbrace{f(a) + f'(a)(x-a)}_{L_a(x) \text{ 1st degree taylor poly.}} + \overbrace{R_{1,a}(x)}^{\text{Remainder}}$$

Theorem: If $f'(a)$ exists then $\displaystyle\lim_{x \to a} \frac{|R_{1,a}(x)|}{x-a} = 0$



Key idea: Remainder $\to 0$ faster than the distance from x to a

Proof: $\left|\frac{R_{1,a}(x)}{x-a}\right| = \left|\frac{f(x)-(f(a)+f'(a)(x-a))}{x-a}\right|$

$= \left|\frac{f(x)-f(a)}{x-a} - \frac{f'(a)(x-a)}{(x-a)}\right| = |f'(a)-f'(a)| = 0$ as $a \to 0$.

<u>Defn:</u> $f(x,y)$ is differentiable at (a,b) <u>means</u>.

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

where $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

$= f(x,y) - \left(f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)\right)$

and $\|(x,y)-(a,b)\| = \|(x-a, y-b)\| = \sqrt{(x-a)^2+(y-b)^2}$

lecture 9, 10

Theorem: If a fcn satisfies

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y)-\underbrace{f(a,b)-c(x-a)-d(y-b)}_{L(a,b)}|}{\|(x,y)-(a,b)\|} = 0$$

then, $c = f_x(a,b)$ and $d = f_y(a,b)$

Defn: **tangent Plane** of surface $z = f(x,y)$ at $(a,b,f(a,b))$

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

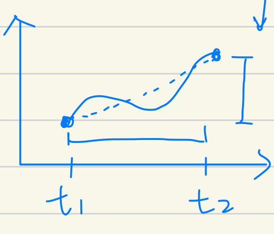
Theorem: if $f(x,y)$ is diff'able at (a,b), then f is cts. at (a,b).

diff'able $\Rightarrow$ cty.

Theorem: **MVT**.

If $f(t)$ is cts on $[t_1, t_2]$ and f is diff'able on (t_1, t_2), then $\exists\ t_0 \in (t_1, t_2)$ s.t.

$$f(t_2) - f(t_1) = f'(t_0)(t_2 - t_1)$$



Theorem: If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are cts at (a,b), then $f(x,y)$ diff'able at (a,b).

partial cts $\Rightarrow$ diff'ability

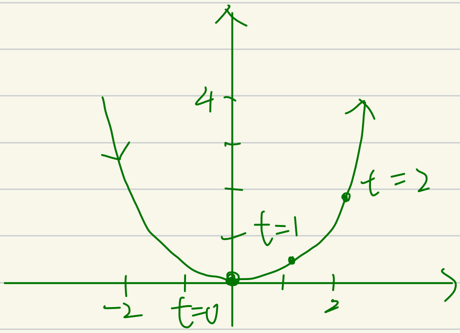
lecture 11

Review of parameter / vector curves

ex: $X(t) = (t, t^2)$

Parametric form:

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$$



eliminate $y = t^2 = (x)^2$ since $x = t \Rightarrow y = x^2$ (lies on $y = x^2$)

If $X(t)$ = position of particle at time t

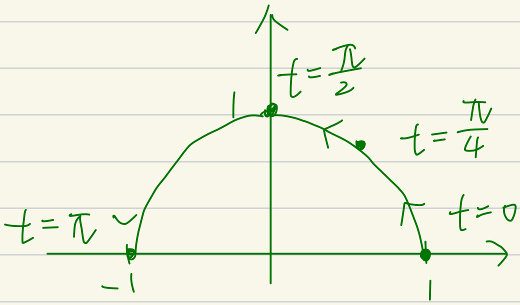
then $X'(t) = (1, 2t)$ is velocity at time t. e.g. $X'(1) = (1,2)$

and $X''(t) = (0,2)$ is acceleration at time t.

ex: $X(t) = (\underbrace{\cos(t)}_{x(t)}, \underbrace{\sin(t)}_{y(t)})$ $0 \leq t \leq \pi$

eliminate t: $x^2 + y^2 = \cos^2 t + \sin^2 t$
$= 1$

so curve lies on circle.



ex: $X(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$, $-\infty < t < \infty$

try to eliminate t:

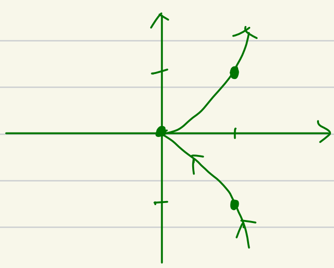
$x^2 + y^2 = e^{-2t}(\sin^2(t) + \cos^2(t))$

$= e^{-2t} \Rightarrow$ "circle" with dec' radius.

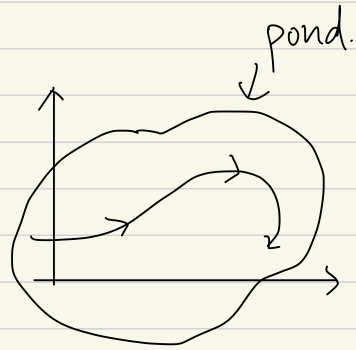
ex: $X(t) = (t^2, t^3)$

$y = t^3 = (\pm\sqrt{x})^3 = \pm x^{\frac{3}{2}}$

or $x = t^2 = (y)^{\frac{2}{3}}$



Chain Rule:



Imagine a duck position: $\mathbf{x}(t) = (x(t), y(t))$

Let $f(x,y)$ = temperature of pond at (x,y)

find r.o.c of temperature experienced by duck.

$\Rightarrow$ in time Δt, duck's position changes by

$$\Delta x = x(t+\Delta t) - x(t)$$

$$\Delta y = y(t+\Delta t) - y(t)$$

by increment form of linear approx.

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \Rightarrow \frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{x(t+\Delta t)-x(t)}{\Delta t} + \frac{\partial f}{\partial y}\frac{y(t+\Delta t)-y(t)}{\Delta t}$$

Let $t \to 0$: $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$ ☆ chain Rule for $f(x(t), y(t))$

Comments:

- $\frac{df}{dt}$ means $\frac{d}{dt} f(x(t), y(t))$
- this is not proof ↗

lecture 12 & 13 , (missed)

Thm: Chain Rule.

let $G(t) = f(x(t), y(t))$ and $a = x(t_0)$, $b = y(t_0)$.

If f is diff'able at (a,b) and $x'(t_0), y'(t_0)$ exist.

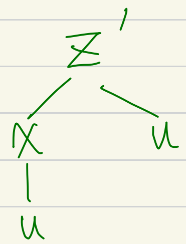
$$G'(t_0) = f_x(a,b)\,x'(t_0) + f_y(a,b)\,y'(t_0)$$

technique: draw a chain of dependence diagram.
and work upward.

now, chain rule of second partials.

ex: let $z = f(x)$, $x = e^u$, verify $z''(u) = x^2 f''(x) + x f'(x)$

$\Rightarrow z'(u) = f'(x)\,x'(u) = \underline{f'(x)}\,\underline{e^u}$, then



$$z''(u) = \frac{\partial z'}{\partial x}\frac{dx}{du} + \frac{\partial z'}{\partial u}$$

$\hookrightarrow \frac{\partial z'}{\partial x} = \frac{\partial}{\partial x}\left(f'(x)e^u\right) = f''(x)e^u$

$\hookrightarrow \frac{\partial z'}{\partial u} = \frac{\partial}{\partial u}\left(f'(x)e^u\right) = f'(x)e^u$

$\Rightarrow z''(u) = f''(x)e^{2u} + f'(x)e^u = x^2 f''(x) + x f'(x)$

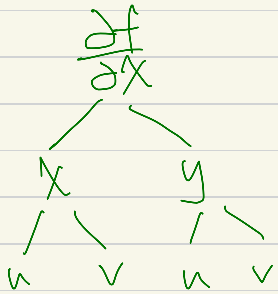
☆ ex: $g(u,v) = f(x,y)$, where $x = u\cos v$, $y = u\sin v$.

Given: $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cos v + \frac{\partial f}{\partial y}\sin v$ & $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}(-u\sin v) + \frac{\partial f}{\partial y}(u\cos v)$

find. $\frac{\partial^2 g}{\partial u^2}$.

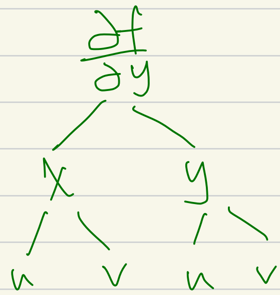
$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\cos v + \frac{\partial f}{\partial y}\sin v\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\cos v + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\sin v.$$



$$\hookrightarrow \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 f}{\partial x^2}\cos v + \frac{\partial^2 f}{\partial y \partial x}\sin v$$



$$\hookrightarrow \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 f}{\partial x \partial y}\cos v + \frac{\partial^2 f}{\partial y^2}\sin v.$$

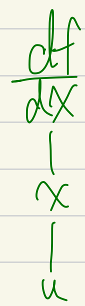
$$\frac{\partial^2 g}{\partial u^2} = (\cos v)(f_{xx}\cos v + f_{xy}\sin v)$$

$$+ (\sin v)(f_{xy}\cos v + f_{yy}\sin v)$$

$$= f_{xx}\cos^2 v + 2 f_{xy}\cos v \sin v + f_{yy}\sin^2 v. \quad \#$$

ex: Method 2 for ex 1:

$$z'(u) = \frac{df}{dx}\frac{dx}{du} = \frac{df}{dx}e^u$$



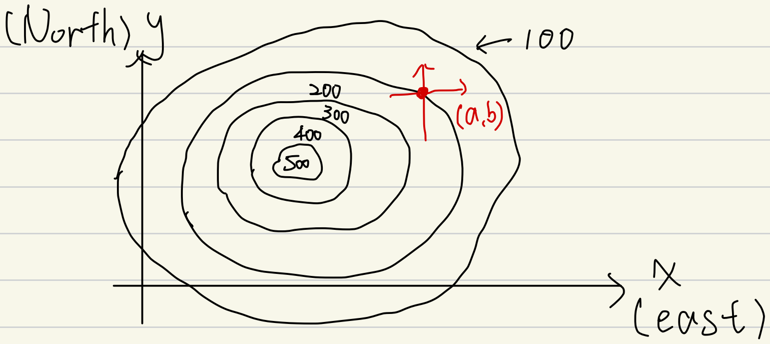
$$z''(u) = \frac{\partial}{\partial u}\left(\frac{df}{dx}e^u\right) = \frac{\partial}{\partial u}\left(\frac{df}{dx}\right)e^u + \frac{df}{dx}\frac{\partial}{\partial u}(e^u)$$

$$= \left(\frac{d^2 f}{dx^2}\frac{dx}{du}\right)e^u + \frac{df}{dx}e^u$$

$$= f''(x)e^{2u} + f'(x)e^u$$

$$= f''(x)x^2 + f'(x)x \quad \Rightarrow \text{same} \ \#$$

lecture 14.



Directional Derivatives:

motivation : mountain climbing

Recall:

$\frac{\partial f}{\partial x}(a,b)$ = slope of cross-sec in x-direc as you walk east.

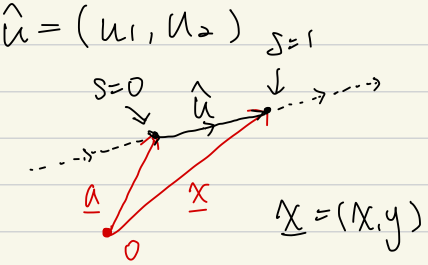
$\frac{\partial f}{\partial y}(a,b)$ = slope of cross-sec in y-direc as you walk north.

Q: What about slope in other direction? i.e. northeast?

given a point $\underline{a} = (a,b)$ and direction vector $\hat{u} = (u_1, u_2)$

a vector equation of a line through $\underline{a}$

in direction $\hat{u}$ is : $\underline{x} = \underline{a} + s\hat{u}$, $s \in \mathbb{R}$



Defn: the directional derivative of $f(x,y)$ at point $\underline{a} = (a,b)$

in the direction of a unit vector $\hat{u} = (u_1, u_2)$ is

$$D_{\hat{u}} f(\underline{a}) = \frac{d}{ds} f(\underline{a} + s\hat{u}) \Big|_{s=0}$$

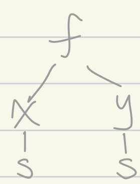
or $\frac{d}{ds} f(a+su_1, b+su_2) \Big|_{s=0}$

Theorem: If $f(x,y)$ is diff'able at $\underline{a}$,

then $D_{\hat{u}} f(\underline{a}) = \nabla f(\underline{a}) \cdot \hat{u}$, $\hat{u}$ is a unit vector

Proof: by defn,

$$D_{\hat{u}} f(\underline{a}) = \frac{d}{ds} f(\underbrace{a+su_1}_{x(s)}, \underbrace{b+su_2}_{y(s)}) \Big|_{s=0} \Rightarrow$$



$$= \frac{\partial f}{\partial x}(a+su_1, b+su_2) \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y}(a+su_1, b+su_2) \cdot \frac{dy}{ds} \Big|_{s=0}$$

with $\frac{dx}{ds} = u_1$, $\frac{dy}{ds} = u_2$

$$= \frac{\partial f}{\partial x}(a,b) \cdot u_1 + \frac{\partial f}{\partial y}(a,b) \cdot u_2$$

$$= \left(\frac{\partial f}{\partial x}(\underline{a}), \frac{\partial f}{\partial y}(\underline{a})\right) \cdot (u_1, u_2) = \nabla f(\underline{a}) \cdot \hat{u}$$

#

Comments:

① if $\hat{u} = (1,0)$, then $D_{\hat{u}} f(\underline{a}) = \nabla f(a,b) \cdot (1,0)$

$$= (f_x(a,b), f_y(a,b)) \cdot (1,0) = f_x(a,b)$$

$\hat{u} = (0,1)$, $D_{\hat{u}} f(\underline{a}) = \dots = f_y(a,b)$

② If direction is given as non-unit vector $\underline{u}$, must normalize: $\frac{\underline{u}}{\|\underline{u}\|}$

③ Defn. and thm. generalize easily for fcn. of n variables.

④ if $f(x,y)$ not diff'able at (a,b), then must use defn.

ex: $f(x,y) = \frac{x}{x^2+y^2}$, calculate directional deriv. at $(2,0)$

in direction vec. $\underline{u} = (1,1)$

Sol: $\frac{\partial f}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = -x(x^2+y^2)^{-2}(2y) = \frac{-2xy}{(x^2+y^2)^2}$

cts. at $(2,0)$ by cty. thm.

$\Rightarrow$ diff'able.

So by theorem,

Aside: $\hat{u} = \frac{(1,1)}{\|(1,1)\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$D_{\hat{u}} f(2,0) = \nabla f(2,0) \cdot \hat{u}$$

$$= \left(\frac{0^2-2^2}{(2^2+0^2)^2}, 0\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{-1}{4\sqrt{2}} + 0 = \frac{-1}{4\sqrt{2}}$$

ex: find directional derivative of $f(x,y,z) = x^2 \cos z + e^y$

in direction $(-1,1,-1)$ at $(1, \ln 2, 0)$

Sol: $f_x = 2x \cos z$

$f_y = e^y$

$f_z = -x^2 \sin z$

clearly all cts. $\Rightarrow$ diff'able by theorem, $D_{\hat{u}} f(1, \ln 2, 0) = \nabla f(1, \ln 2, 0) \cdot \hat{u}$

$$= (2, 2, 0) \cdot \frac{(-1, 1, -1)}{\|(-1, 1, -1)\|} \leftarrow \sqrt{3}$$

$$= 0.$$

Theorem: If f diff'able at (a, b) and $\nabla f(a, b) \neq (0, 0)$, then the largest value of $D_{\hat{u}} f(a, b)$ is $\|\nabla f(a, b)\|$, and occurs when $\hat{u}$ is in direction of $\nabla f(a, b)$

Theorem: If $f(x, y) \in C^1$ in a neighborhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthogonal to level curve $f(x, y) = k$ through (a, b)

same for 3-dim.

p.s. tangent plane $\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$

lecture (didn't go)

Review: one-dimensional taylor-poly.

$f(x)$ degree 2 $\Rightarrow P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2$.

$= L_a(x) + \frac{1}{2!} f''(a)(x-a)^2$

and notice $P''_{2,a}(a) = f''(a)$

now consider 2-D case.

$P_{2,(a,b)}(x,y) = L_{(a,b)}(x,y) + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$

similarly, we can see that, $\frac{\partial^2 P_{2,(a,b)}}{\partial x^2} = 2A$

similarly, $\frac{\partial^2 P(a,b)}{\partial x \partial y} = B$ and $\frac{\partial^2 P(a,b)}{\partial y^2} = 2C$

Defn: second degree taylor $P_{2,(a,b)}$ of $f(x,y)$ at (a,b) is

$P_{2,(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$+ \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$

ex: Use taylor degree 2 to approximate $\sqrt{(0.95)^3 + (1.98)^3}$

$f(x,y) = \sqrt{x^3+y^3} \Rightarrow \nabla f(1,2) = (\frac{1}{2}, 2)$

$Hf(1,2) = \begin{bmatrix} \frac{11}{12} & \frac{-1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ p.s. $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$P_{2,(1,2)}(x,y) = 3 + \frac{1}{2}(x-1) + 2(y-2) + \frac{1}{2}\left[\frac{11}{12}(x-1)^2 - \frac{2}{3}(x-1)(y-2)\right.$

$\left. + \frac{2}{3}(y-2)^2\right]$

thus, $P_{2,(1,2)}(0.95, 1.98) \approx 2.9359$.

lecture (still skipped)

Review 1-D taylor remainder:

Theorem: if $f''(x)$ exist on $[a, x]$ $\exists$ $c \in (a, x)$ s.t.

$$f(x) = f(a) + f'(a)(x-a) + R_{1,a}(x)$$

where $R_{1,a}(x) = \frac{1}{2} f''(c)(x-a)^2$.

For 2-D case:

Theorem:

If $f(x,y) \in C^2$ in some neighborhood $N(a,b)$ of (a,b)

then $\forall (x,y) \in N(a,b)$ $\exists$ a point (c,d) on the line

segment joining (a,b) and (x,y) such that

$$\Rightarrow f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$

where $R_{1,(a,b)}(x,y) = \frac{1}{2}\left[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\right]$

$\hookrightarrow$ Corollary: If $f(x,y) \in C^2$ .... $\exists$ constant M s.t.

$$|R_{1,(a,b)}(x,y)| \le M \|(x,y) - (a,b)\|^2, \quad \forall (x,y) \in N(a,b)$$

Generalization:

Theorem:

If $f(x,y) \in C^{k+1}$ at each point on the line segment joining

(a,b) and (x,y) then there exist a point (c,d) on the line

between (a,b) and (x,y) s.t.

$$f(x,y) = P_{k,(a,b)}(x,y) + R_{k,(a,b)}(x,y), \quad \text{where}$$

$$R_{k,(a,b)}(x,y) = \frac{1}{(k+1)!}\left[(x-a)D_1 + (y-b)D_2\right]^{k+1} f(c,d).$$

corollary 1 : $\exists M > 0$ s.t. $|f(x,y) - P_{k,(a,b)}(x,y)| \le M \|(x,y)-(a,b)\|^{k+1}$

corollary 2 : $\lim\limits_{(x,y)\to(a,b)} \dfrac{|f(x,y) - P_{k,(a,b)}(x,y)|}{\|(x,y) - (a,b)\|^{k}} = 0$

lecture 20.

Defn: Critical Points.

- Local max: $f(x,y) \le f(a,b)$ $\forall (x,y)$ near (a,b)
- Local min: $f(x,y) \ge f(a,b)$ $\forall (x,y)$ near (a,b)

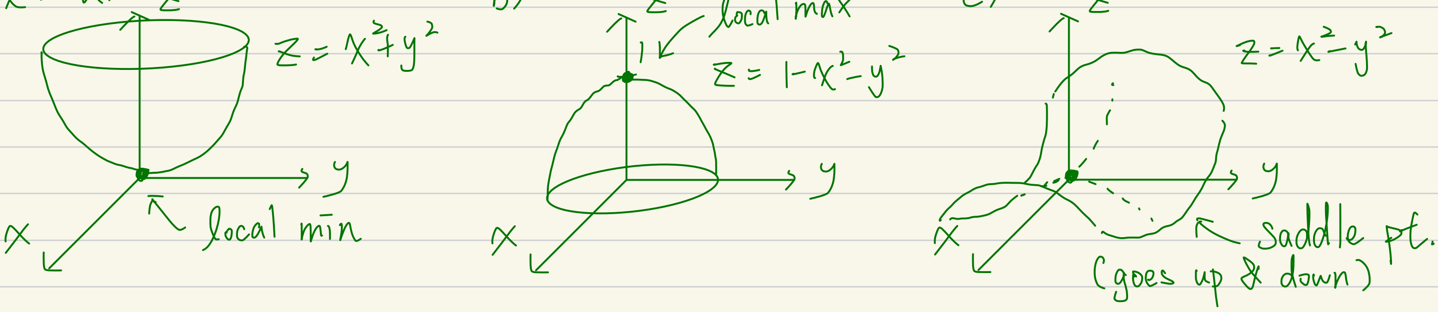
Theorem: If (a,b) is a local max / min, then,

⊛ $f_x(a,b) = 0$ or DNE <u>and</u> $f_y(a,b) = 0$ or DNE

Defn: A point (a,b) satisfying ⊛ is a <u>critical point.</u>

Defn: A critical point which is neither a local max/min is called a <u>saddle point.</u>

ex: a) b) c)



ex: find all c.p. of $f(x,y) = 6xy^2 - 2x^3 - 3y^4$

Sol:
$$\begin{cases} \dfrac{\partial f}{\partial x} = 6y^2 - 6x^2 = 0 & \text{— (1)} \\ \dfrac{\partial f}{\partial y} = 12xy - 12y^3 = 0 & \text{— (2)} \end{cases}$$

eq.(2) $\Rightarrow 12y(x-y^2) = 0 \Rightarrow y = 0$ or $x - y^2 = 0$

If $y=0$, eq.(1) $\Rightarrow 0 - 6x^2 = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

If $x-y^2=0$. eq.(1) $\Rightarrow 6y^2 - 6y^4 = 0 \Rightarrow y = 0, \pm 1 \Rightarrow x = 0, (\pm 1)^2$

Conclusion: c.p. are $(0,0)$, $(1,1)$, $(1,-1)$

Classifying critical points (alt to 9.2)

- If $f \in C^2$ and (a,b) is a critical point, then:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

p.s. $f_x, f_y = 0$ since it's c.p's

$$\Rightarrow \underbrace{f(x,y) - f(a,b)}_{\geq 0?\ <0?} \approx \frac{1}{2}\left[A\underbrace{(x-a)}_{u}^2 + 2B(x-a)\underbrace{(y-b)}_{v} + C(y-b)^2\right]$$ where $f_{xx} = A$, $f_{xy} = B$, $f_{yy} = C$.

$$= \frac{1}{2}\left[Au^2 + 2Buv + Cv^2\right]$$ "quadratic form"

Look at the sign of RHS:

this part not very important

$$A\left(u^2 + 2\frac{B}{A}u\right) + Cu^2 = 0$$
$$= A\left[\left(u^2 + \frac{Bv}{A}\right)^2 - \left(\frac{Bv}{A}\right)^2\right] + Cv^2 = 0$$
$$= A\underbrace{\left(u + \frac{B}{A}v\right)^2}_{\geq 0} + \left(C - \frac{B^2}{A}\right)\underbrace{v^2}_{\geq 0} = 0$$

then - if $A > 0$ and $C - \frac{B^2}{A} > 0$, then RHS ≥ 0 p.s. $(AC - B^2 > 0)$

suggesting $f(x,y) - f(a,b) \geq 0$ (local min)

- if $A < 0$ and $C - \frac{B^2}{A} < 0$, then RHS ≤ 0 p.s $(AC - B^2 > 0)$

suggesting $f(x,y) - f(a,b) \leq 0$ (local max)

- If they have opposite sign, ie. $A \cdot \left(C - \frac{B^2}{A}\right) < 0$, $AC - B^2 < 0$

implies that RHS "sometimes" > 0, "sometimes" < 0

suggesting a saddle point.

"Hessian matrix"

$$\text{Note: } AC - B^2 = \det\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det(Hf(a,b))$$

$\downarrow$

<u>Theorem</u>: Second derivative test.

- If $f \in C^2$ in some neighborhood of critical point (a,b), then.

(i) If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$, then:

(a,b) is a local <u>minimum</u> point.

(ii) If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$, then:

(a,b) is a local <u>maximum</u> point.

(iii) If $\det(Hf(a,b)) < 0$, <u>saddle point</u>.

(iv) If $\det(Hf(a,b)) = 0$, no conclusion.

lecture 20 ~ (missed) 😩

Defn: given a point $f(x,y)$ and set $S \subset \mathbb{R}^2$

1. a point $(a,b) \in S$ is an absolute maximum point of f on S if

$$f(x,y) \le f(a,b) \quad \forall (x,y) \in S$$

2. a point $(a,b) \in S$ is an absolute minimum point of f on S if

$$f(x,y) \ge f(a,b) \quad \forall (x,y) \in S$$

Theorem: EVT (extreme value theorem)

if $f(x)$ is cts. on a finite closed interval I, then $\exists\, c_1, c_2 \in I$ s.t. $f(c_1) \le f(x) \le f(c_2) \quad \forall x \in I$

Defn: A set $S \subset \mathbb{R}^2$ is bounded iff it is contained in some neighborhood.

Defn: Given a set $S \subset \mathbb{R}^2$ is said to be a boundary point of S <u>iff</u> every neighborhood of (a,b) contains at least one point <u>in</u> S and one point <u>not in</u> S.

$\Rightarrow$ the set $B(S)$ of all boundry pts. of S is called boundary of S

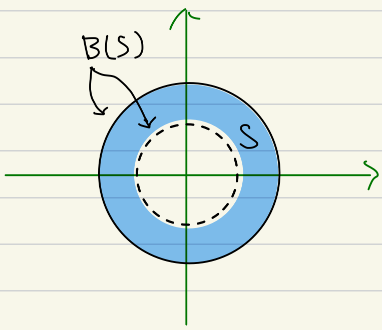
Defn: A set $S \subseteq \mathbb{R}^2$ is said to be closed if S contains all of its boundary pts.

ex: consider $S = \{ (x,y) \in \mathbb{R}^2 \mid 1 < \|(x,y)\| \le 2 \}$

$B(S) = \{ (x,y) \in \mathbb{R}^2 \mid \|(x,y)\| = 1 \text{ or } \|(x,y)\| = 2 \}$

$\Rightarrow$ points s.t. $\|(x,y)\| = 1$ are not on S

$\Rightarrow$ not closed.



Theorem: If $f(x,y)$ is cts. on a closed and bounded set $S \subset \mathbb{R}^2$, then $\exists$ pts. $(a,b), (c,d) \in S$ s.t.

$$f(a,b) \le f(x,y) \le f(c,d) \quad \forall (x,y) \in S.$$

Algorithm: (abs extreme value)

Let $S \subset \mathbb{R}^2$ be closed and bounded. To find max/min of $f(x,y)$ that is cts on S.

① Find all c.f.ps of f contained in S. evaluate each point.

② Find max/min points of f on $B(S)$

③ max & min = largest and smallest found in ① & ②

Algorithm: (Lagrange Multiplier Algorithm) (no ② from ↑)

Assume $f(x,y)$ if a diff'able fcn and $g \in C^1$. To find max/min value of f subject to the constraint $g(x,y) = k$. evaluate $f(x,y)$ at all points (a,b) which satisfies one of following conditions.

① $\nabla f(a,b) = \lambda \nabla g(a,b)$ and $g(a,b) = k$.

② $\nabla g(a,b) = (0,0)$ and $g(a,b) = k$

③ (a,b) is an end point of the curve $g(x,y) = k$.

↳ for ① we want find :

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = k \end{cases}$$

ex: find maximum of $6x+4y-7$ on ellipse $3x^2+y^2=28$

$\Rightarrow$ find max of $f(x,y)=6x+4y-7$ constraint: $g(x,y)=3x^2+y^2=28$

① $\nabla f(x,y) = \lambda \nabla g(x,y)$, $g(x,y)=28$

$\nabla f(x,y) = (6,4)$, $\nabla g(x,y) = (6x, 2y)$

$$\begin{cases} 6 = 6\lambda x & -1. \\ 4 = 2\lambda y & -2. \\ 3x^2+y^2=28 & -3. \end{cases}$$

from 1. , $x \neq 0 \Rightarrow \lambda = \frac{1}{x} \Rightarrow y = 2x$, sub into 3. $\Rightarrow x = \pm 2$

for $x=2$, $y=4$, for $x=-2$, $y=-4$ $\Rightarrow (2,4)$ and $(-2,-4)$

② $\nabla g(x,y) = (0,0)$, $g(x,y)=28$

$\Rightarrow x=y=0$ $g(0,0) \neq 28$ $\Rightarrow$ Nope.

③ check endpoints, $\Rightarrow$ no endpoints since ellipse is a closure.

now evaluate $f(2,4)=21$, $f(-2,-4)=-35$

$\Rightarrow$ thus, max $=(2,4)$

Algorithm: (fun. w/ 3 variables)

... same hypothesis ...

① $\nabla f(a,b,c) = \lambda g(a,b,c)$ and $g(a,b,c)=k$

② $\nabla g(a,b,c) = (0,0,0)$ and $g(a,b,c)=k$

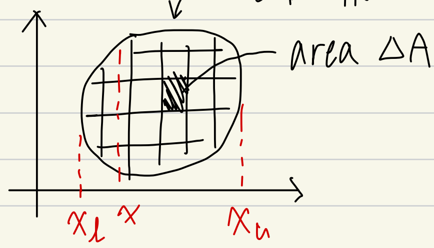
③ (a,b,c) is a edge pt. of surface $g(x,y,z)=k$.

ex: too lazy ... no one's reading this

Defn: let $D \subset \mathbb{R}^2$ be closed and bounded. let P be a partition of D and let $|\Delta P|$ denote the length of the longest side of all rectangles in partition P. A fcn. $f(x,y)$ is Integrable on D if all Riemann sum approach the same value as $|\Delta P| \to 0$

Defn: double integral: $\iint_D f(x,y)\, dA = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$

lecture 26 (I actually went to class)



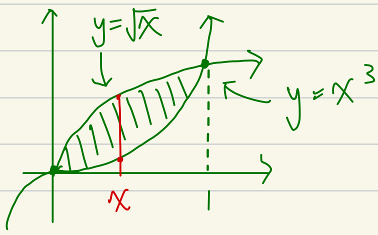
Recap: $\iint_D f(x,y)\, dA = \lim_{\Delta A \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A$

$$= \int_{x_l}^{x_u} \left(\int_{y_l(x)}^{y_u(x)} f(x,y)\, dy \right) dx$$

ex: let D be region bounded by $y = \sqrt{x}$ and $y = x^3$. Eval. $\iint_D x^2 y^2\, dA$.

Sol: Sketch D: $\begin{cases} 0 \le x \le 1 \\ x^3 \le y \le \sqrt{x} \end{cases}$

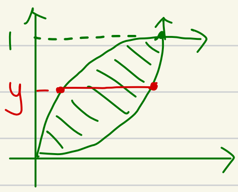
x^3 ↑ lower curve. $\sqrt{x}$ ↑ upper curve.



then. $\iint_D x^2 y^2\, dA = \underbrace{\int_0^1 \Big(\underbrace{\int_{x^3}^{\sqrt{x}} x^2 y^2\, dy}_{\text{fcn.}} \Big) dx}_{\text{constants.}}$

$$= \int_0^1 x^2 \cdot \frac{y^3}{3} \Big|_{x^3}^{\sqrt{x}} dx. = \int_0^1 \frac{1}{3} \left(x^{\frac{7}{2}} - x^{11} \right) dx = \cdots = \frac{5}{108}$$

Another way: $\begin{cases} 0 \le y \le 1 \\ y^2 \le x \le y^{\frac{1}{3}} \end{cases}$

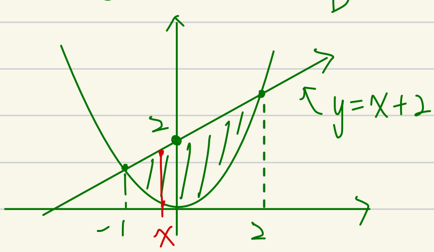


, Get: $\int_0^1 \left(\int_{y^3}^{y^{\frac{1}{3}}} x^2 y^2\, dx \right) dy$

ex: let D be bounded by $y = x^2$ and $y = x+2$. $\iint_D x + 2y\, dA$?

Sol: Sketch D.

$\begin{cases} -1 \le x \le 2 \\ x^2 \le y \le x+2 \end{cases}$



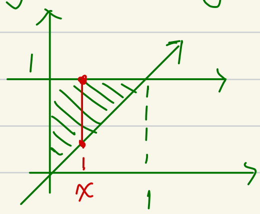
then $\iint_D x + 2y\, dA = \int_{-1}^{2} \left(\int_{x^2}^{x+2} (x+2y)\, dy \right) dx$

$$= \int_{-1}^{2} xy + y^2 \Big|_{x^2}^{x+2} dx = \int_{-1}^{2} \left[x(x+2) + (x+2)^2 - x^3 - x^4 \right] dx.$$

$$= \cdots = \frac{333}{20}.$$

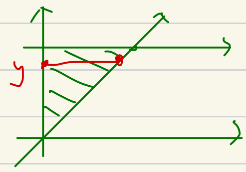
ex: $\iint_D e^{y^2}\,dA = ?$, D bounded by $x=0,\ y=1,\ y=x$.

Sol: Describe D: $\begin{cases} 0\le x\le 1 \\ x\le y\le 1\end{cases}$



$I = \int_0^1 \left(\int_x^1 e^{y^2}\,dy\right) dx \Rightarrow$ undoable (!!)

other approach: $\begin{cases} 0\le y\le 1 \\ 0\le x\le y\end{cases}$



(Integrate w.r.t. x first)

$I = \int_0^1 \left(\int_0^y e^{y^2}\,dx\right) dy = \int_0^1 e^{y^2}x\Big|_0^y\,dy = \int_0^1 (e^{y^2}\cdot y - 0)\,dy.$

$= \frac{1}{2}e^{y^2}\Big|_0^1 = \frac{e-1}{2}$

mf said just guess ??
→ use $u=y^2$. sub.

Properties of $\iint$: $\iint_D f\,dA = \lim \sum f\cdot \Delta A$.

- $\iint_D (f+g)\,dA = \iint_D f\,dA + \iint_D g\,dA$.

- if $f\le g$, then $\iint_D f\,dA \le \iint_D g\,dA$

- $\left|\iint_D f\,dA\right| \le \iint_D |f|\,dA$ → Δ ineq.

- Decomposition property: $\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$ (D_1 | D_2)

ex: eval. $\int_0^1\int_y^{2-y} \frac{1}{1+x^2}\,dx\,dy = I$.

Sol: $I = \int_0^1 \left(\arctan\Big|_y^{2-y}\right) dy = \int_0^1 (\arctan(2-y) - \arctan(y))\,dy$.

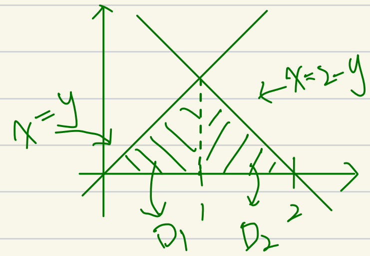
$= \cdots$ (intg by parts) $\cdots =$ we'll see.

Sol②: Reverse the order.

lecture 27

ex: eval. $\int_0^1 \int_y^{2-y} \frac{1}{1+x^2}\, dx\, dy$

Sol: $D \begin{cases} 0 \le y \le 1 \\ y \le x \le 2-y \end{cases}$



$D_1 \begin{cases} 0 \le x \le 1 \\ 0 \le x \le y \end{cases}$

$D_2 \begin{cases} 1 \le x \le 2 \\ 0 \le y \le 2-x \end{cases}$

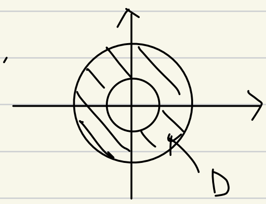
$$I = \iint_D \frac{1}{1+x^2}\, dA = \iint_{D_1} \frac{1}{1+x^2}\, dA + \iint_{D_2} \frac{1}{1+x^2}\, dA$$

$$= \int_0^1 \int_0^x \frac{1}{1+x^2}\, dy\, dx + \int_1^2 \int_0^{2-x} \frac{1}{1+x^2}\, dy\, dx.$$

$$= \cdots = \ln 2 - \frac{1}{2}\ln 5 + 2\arctan 2 - \frac{\pi}{2} \quad \#$$

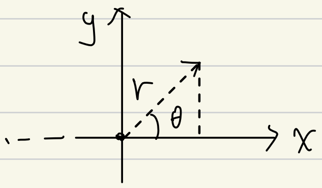
Comment on $\iint$ symmetry.

e.g.



$\iint_D x^2 y\, dA = 0$, by symmetry.

Polar coordinates: (review)



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x}. \end{cases}$

ex: Convert to Cartesians: a) $(r,\theta) = (1, \pi)$ b) $(r,\theta) = (2, \frac{5}{4}\pi)$

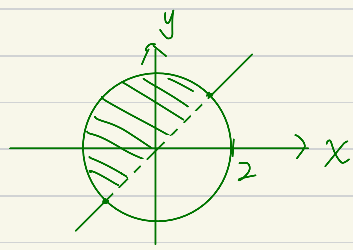
Sol: a) $(x,y) = (1\cos\pi, 1\sin\pi) = (-1, 0)$ b) $(x,y) = (2\cos\frac{5}{4}\pi, 2\sin\frac{5}{4}\pi) = (-\sqrt{2}, -\sqrt{2})$

ex: Convert to Polar: a) $(x,y) = (-1, \sqrt{3})$ b) $(x,y) = (-3,-3)$

Sol: a) $(x,y) = (2, \frac{2}{3}\pi)$ b) $(x,y) = (3\sqrt{2}, \frac{5}{4}\pi)$

ex: Describe in Polar coords.

$D = \{(x,y) \mid x^2+y^2 \le 4 \text{ and } y > x\}$



$\downarrow$

Sol: $\{(r,\theta) \mid 0 < r \le 2,\ \frac{\pi}{4} < \theta < \frac{5}{4}\pi\}$

ex: Describe $\{(x,y) \mid y \le 1\}$

$y=1 = r\sin\theta = 1$

$r = \csc\theta$

Sol: $\{(r,\theta) \mid \pi \le \theta \le 2\pi,\ 0 \le r \le \infty\}$ (bottom)

$\cup \{(r,\theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le \csc\theta\}$

lecture 28 ~

Defn: A function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$ is called a vector-valued function.

Defn: A vector-value function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is a subset of $\mathbb{R}^n$ is called a mapping (or transformation)

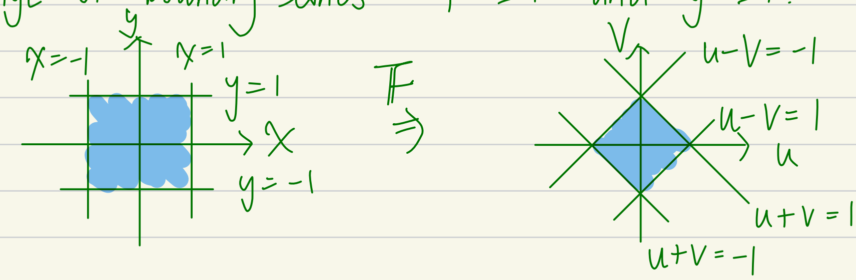
ex: Consider the map $(u,v) = F(x,y) = \left(\frac{1}{2}(x+y), \frac{1}{2}(-x+y)\right)$

find the image of the lines $x=k$ and $y=l$ under F

Sol: observe $x = u-v$, $y = u+v$ $\Rightarrow$ $k = u-v$, $l = u+v$

find image of Square $S = \{(x,y) \mid |x| \le 1, |y| \le 1\}$ under F.

Sol: image of boundry lines: $k = \pm 1$ and $y = \pm 1$.



ex: Find image $D = \{(x,y) \mid -1 \le x \le 3,\ 0 \le y \le 2\}$ under the mapping

$(u,v) = T(x,y) = (x^2 - y^2, xy)$

Sol: find image of boundary lines.

for $x=-1$, $0 \le y \le 2$, $u = 1-y^2$, $v = -y$

eliminate y: $u = 1 - v^2$

since $v = -y$, $0 \le -v \le 2 \Rightarrow -2 \le v \le 0$

for $x = 3$, $0 \le y \le 2$, $u = 9 - y^2$, $v = 3y$

eliminate y: $u = 9 - \frac{1}{9}v^2$

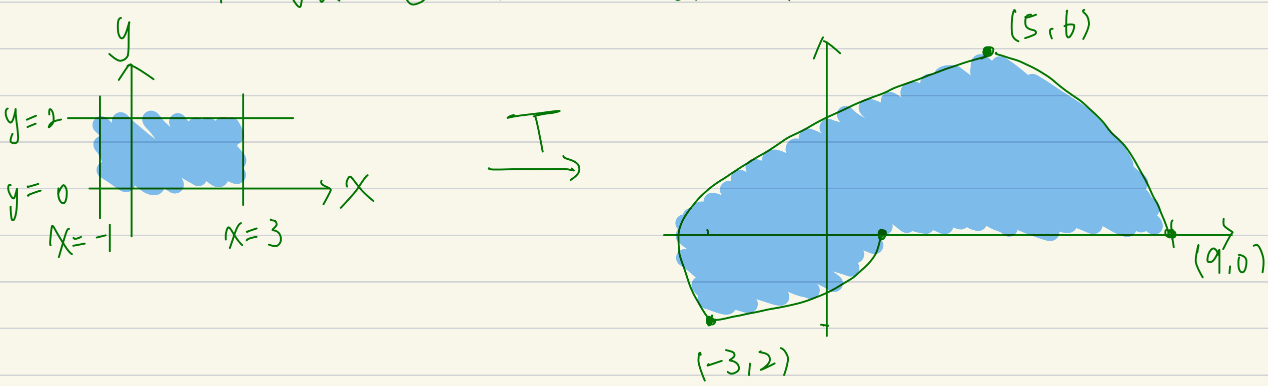
since $v = 3y$, $0 \le \frac{1}{3}v \le 2 \Rightarrow 0 \le v \le 6$.

for $y=2$, $v=2x$, $u=x^2-4=\frac{1}{4}v^2-4$

$-1 \le \frac{1}{2}v \le 3 \Rightarrow -2 \le v \le 6$

for $y=0$, $v=0$, $u=x^2$

$-1 \le \sqrt{u} \le 3 \Rightarrow 1 \le u \le 9$



ex: find image of retangle $R=\{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$

under mapping: $(x,y)=F(r,\theta)=(r\cos\theta,\ r\sin\theta)$

Sol: for $r=1$, $x=\cos\theta$, $y=\sin\theta$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

$\Rightarrow x^2+y^2=1$

for $r=2$, $x=2\cos\theta$, $y=2\sin\theta$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

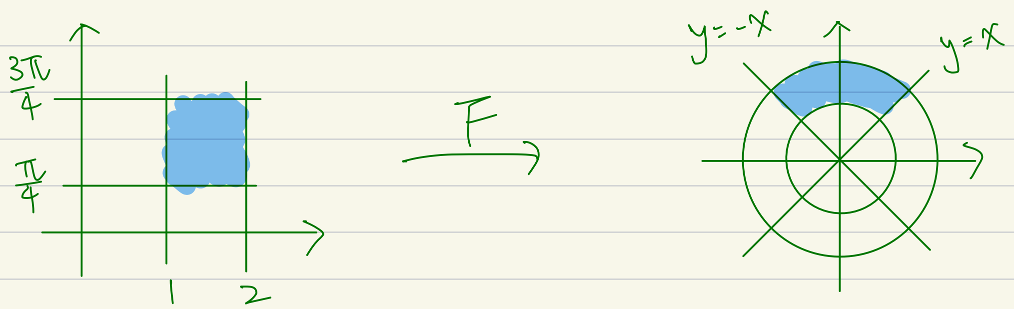
$\Rightarrow x^2+y^2=4$

for $\theta=\frac{\pi}{4}$, $x=\frac{\sqrt{2}}{2}r$, $y=\frac{\sqrt{2}}{2}r$ $\Rightarrow y=x$

$1 \le \frac{2}{\sqrt{2}}x \le 2 \Rightarrow \frac{\sqrt{2}}{2} \le x \le \sqrt{2}$

for $\theta=\frac{3\pi}{4}$, $x=-\frac{\sqrt{2}}{2}r$, $y=\frac{\sqrt{2}}{2}r$ $\Rightarrow y=-x$

$1 \le -\frac{2}{\sqrt{2}}x \le 2 \Rightarrow -\sqrt{2} \le x \le -\frac{\sqrt{2}}{2}$



Defn: let F be a mapping from a set D_{xy} onto a set D_{uv}.

If $\exists$ a mapping F^{-1}, called the inverse of F which maps D_{uv} onto D_{xy} s.t. $(x,y) = F^{-1}(u,v)$ iff $(u,v) = F(x,y)$ then F is said to be invertible on D_{xy}.

Defn: F is one-to-one iff $F(a,b) = F(c,d) \Rightarrow (a,b) = (c,d)$

Theorem: $F: D_{xy} \to D_{uv}$. F one-to-one $\Rightarrow$ F invertible on D_{xy}.

Theorem: $F: D_{xy} \to D_{uv}$. If F has cts partial derivatives at $x \in D_{xy}$ and $\exists$ F^{-1} which has continuous partial derivatives at $u = F(x) \in D_{uv}$,

then $DF^{-1}(u)\, DF(x) = I$

Defn: The Jacobian of mapping $(u,v) = F(x,y) = (u(x,y), v(x,y))$ is denoted $\dfrac{\partial(u,v)}{\partial(x,y)}$ and is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} = \det[DF(x,y)] = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Corollary: consider $(u,v) = F(x,y) = (u(x,y), v(x,y))$

suppose f and g have cts partials on D_{xy}. If F have inverse F^{-1} with cts partials on D_{uv}, Jacobian of $F \neq 0$.

Corollary: $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

chpt. 13.

Inverse Mapping Theorem:

If a mapping $(u,v) = F(x,y)$ has cts partials and $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ then there is a neighborhood of (a,b) in which F has inverse $(x,y) = F^{-1}(u,v)$ which has cts partials.

Geometrical intepretation of Ja caobei:

- given vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, Area $= \left| \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right|$

$$\Rightarrow \Delta A_{uv} \approx \left| \frac{\partial(u,v)}{\partial(x,y)} \right| \Delta A_{xy}$$

ex: area $\Delta x \Delta y$? locate at $(3,4)$ under F:

$$(u,v) = F(x,y) = \left(-x + \sqrt{x^2+y^2},\ x + \sqrt{x^2+y^2}\right)$$

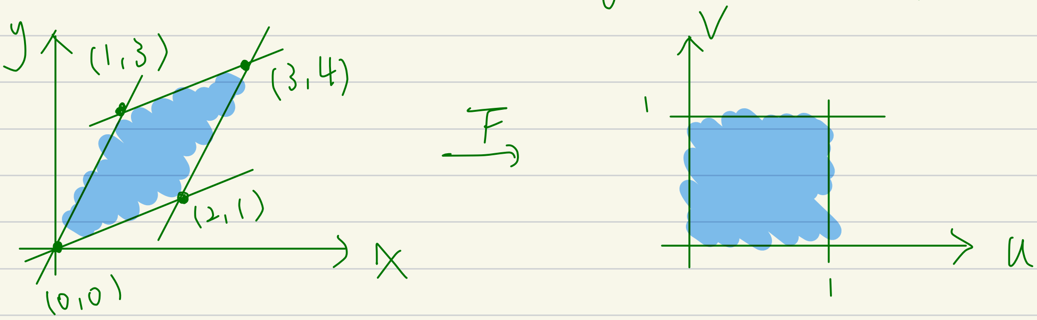
$$\det(DF(3,4)) = \det \begin{bmatrix} \frac{-2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix} = \frac{-8}{5}$$

$$\Rightarrow \Delta A_{uv} \approx \frac{-8}{5} A_{xy}$$

$$\Rightarrow \Delta A_{uv} \approx \left| \frac{\partial(u,v)}{\partial(r,\theta)} \right| \Delta A_{r\theta}$$

Construct Mappings:

ex: Find F that transforms parallelogram with vertices $(0,0)$, $(2,1)$, $(3,4)$, $(1,3)$ into unit square $0 \le u \le 1$, $0 \le v \le 1$



Theorem: Change of variables.

let D_{uv} and D_{xy} be a closed bounded set whose boundry is a piecewise-smooth closed interval. Let

$$(x,y) = G(u,v) = (g(u,v),\ h(u,v))$$

be a one-to-one mapping of D_{uv} onto $D_{x,y}$, with $g, h \in C^1$ and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. If $f(x,y)$ is cts on D_{xy}, then.

$$\iint_{D_{xy}} f(x,y)\,dx\,dy = \iint_{D_{uv}} f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

ex: evaluate $\iint (x+y)\,dA$, D_{xy} bounded by $(0,0), (2,1), (1,3), (3,4)$

we found in construct mapping example that,

$(u,v) = F(x,y) = \left(\frac{1}{5}(2y-x),\ \frac{1}{5}(3x-y)\right)$ maps it into square.

We want $D_{uv} \to D_{xy}$ instead of $D_{xy} \to D_{uv}$, thus

$(x,y) = G(u,v) = F^{-1}(u,v) = (u+2v,\ 3u+v)$

$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-1}{5} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -5$ and $f(g(u,v), h(u,v)) = 4u+3v$

$$\iint_{D_{xy}} (x+y)\,dA = \iint_{D_{uv}} (4u+3v)\,|-5|\,dA = \frac{35}{2}$$

ex: using $(u,v) = F(x,y) = (x+y, -x+y)$ to evaluate

$$\int_0^{\pi}\int_0^{\pi-y} (x+y)\cos(x-y)\,dx\,dy.$$

observe region: $0 \le x \le \pi - y$, $0 \le y \le \pi$

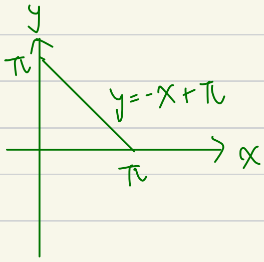
boundries: $x=0$, $y=0$, $x = \pi - y$

for $x=0$, $u = y = v$, $0 \le u \le \pi$

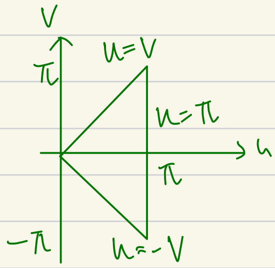
for $y=0$, $u = x = -v$, $0 \le u \le \pi$

for $y = \pi - x$, $u = \pi$ $v = \pi - 2x$ $-\pi \le v \le \pi$

and $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$ then,



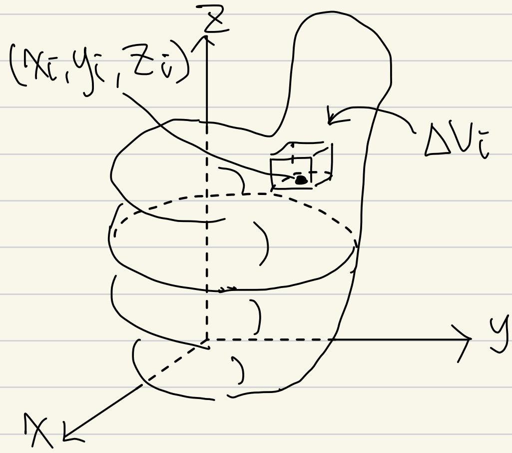
$\Rightarrow$



$$\int_0^{\pi}\int_0^{\pi-y} (x+y)\cos(x-y) = \int_0^{\pi}\int_{-u}^{u} u\cos(-v)\left|\frac{1}{2}\right| dA = \pi.$$

Triple integral:

Label N rectangular blocks that lie completely in D and denote their Volume by ΔV_i, $i=1 \ldots n$, form the Riemann Sum $\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$



Defn: A fcn. $f(x,y,z)$ bounded on a closed bounded set $D \subset \mathbb{R}^3$ is said to be integrable on D iff all Riemann Sum approach the same value as $\Delta P \to 0$

Defn: If $f(x,y,z)$ is integrable, then we define triple integral as.

$$\iiint_D f(x,y,z)\,dV = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

Defn: let $D \subset \mathbb{R}^3$ be c&b with volume $V(D) \neq 0$. and let $f(x,y,z)$ be a bounded and integrable fcn on D. The average value is.

$$f_{avg} = \frac{1}{V(D)} \iiint_D f(x,y,z)\,dV.$$

Theorem : Iterated integral

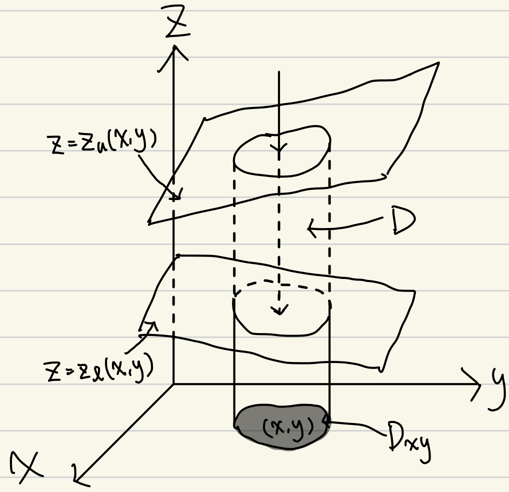
let D be the subset of $\mathbb{R}^3$ defined by

$z_l(x,y) \leq z \leq z_u(x,y)$ and $(x,y) \in D_{xy}$

where z_l and z_u are cts on D_{xy}, D_{xy} is

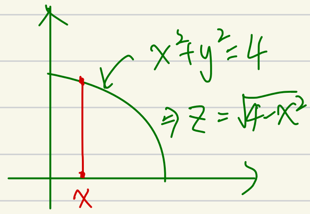
c & b in $\mathbb{R}^2$, If $f(x,y,z)$ is cts then,

$$\iiint_D f(x,y,z)\,dV = \iint_{D_{xy}} \int_{z_l(x,y)}^{z_u(x,y)} f(x,y,z)\,dz\,dA$$



lecture 33.

ex: $\iiint_D \frac{z}{4-x}\, dA = \iint_{D_{xz}} \left(\int_{2-x}^{6-2x} \frac{z}{4-x}\, dy \right) dx\, dz$



then $I = \int_0^2 \int_0^{\sqrt{4-x}} \int_{2-x}^{6-2x} \frac{z}{4-x}\, dy\, dz\, dx$

$= \int_0^2 \int_0^{\sqrt{4-x}} \left(\frac{z}{4-x} \cdot y \right) \Big|_{2-x}^{6-2x} dz\, dx = \int_0^2 \left(\frac{1}{2} z^2 \right) \Big|_0^{\sqrt{4-x^2}} dx$

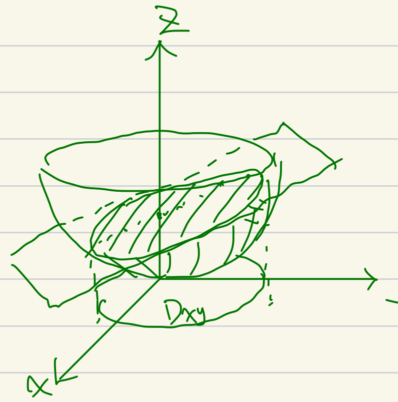
$= \int_0^2 \frac{1}{2}(4-x^2)\, dx = \frac{8}{3}$

ex: Find the volume of region bounded by $z=x^2+y^2$ and $z=y+2$.

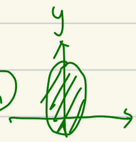
Sol: with $\iint$,

$V = \iint_{D_{xy}} (y+2)-(x^2+y^2)\, dA$

how to do with $\iiint$?



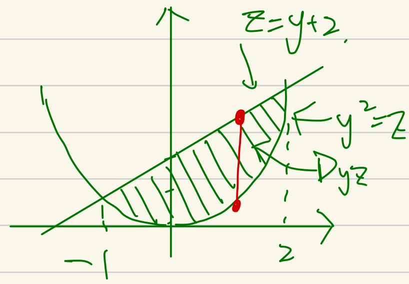
$V = \iiint_D 1\, dv$, how to describe D?



① $(x,y) \in D_{xy}$, $x^2+y^2 \le z \le y+2$ ☹

② $(x,z) \in D_{xz}$, $?? \le y \le$ right side of paraboloid. ☹

√ ③ $(y,z) \in D_{yz}$, $-\sqrt{z-y^2} \le x \le \sqrt{z-y^2}$ ☺



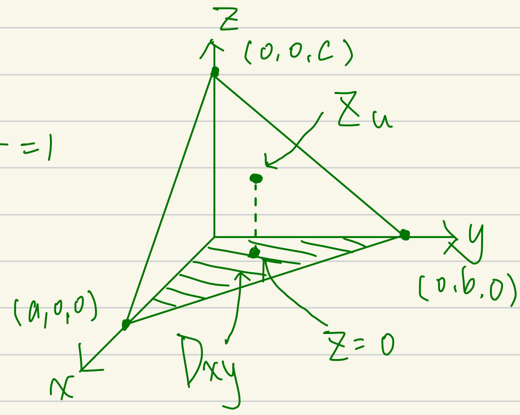
then, $V = \iiint_D 1\, dv = \iint_{D_{yz}} \left(\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1\, dx \right) dA$

$= \int_{-1}^{2} \int_{y^2}^{y+2} \left(x \Big|_{-\sqrt{z+y^2}}^{\sqrt{z+y^2}} \right) dz\, dy = \int_{-1}^{2} \int_{y^2}^{y+2} 2\sqrt{z-y^2}\, dz\, dy$

$= \int_{-1}^{2} \left(\frac{4}{3} (z-y^2)^{\frac{3}{2}} \Big|_{y^2}^{y+2} \right) dy. = \cdots = \frac{81\pi}{32}$

trig sub.
Aside: $(()^2-()^2)^{\frac{3}{2}}$
$y = a\sin\theta$
Math 138.

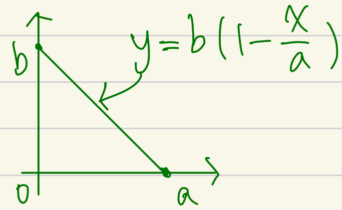
ex: evaluate $\iiint_D z\,dV$, where D is a tetrahedron with verticies $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, $(0,0,0)$



Sol: bounded by $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

(graph) $z_u = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$$\Rightarrow \iiint_D z\,dV = \iint_{D_{xy}} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} z\,dz\,dA$$



$$☆ = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} z\,dz\,dy\,dx$$

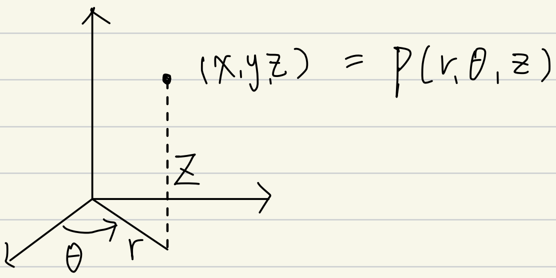
$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \frac{1}{2}\left(c-\frac{c}{a}x-\frac{c}{b}y\right)^2 dy\,dx$$

$$= \int_0^a \frac{-b}{6c}\left(c-\frac{c}{a}x-\frac{c}{b}y\right)^3 \Big|_0^{b(1-\frac{x}{a})} dx = \cdots = \frac{1}{24}abc^2.$$

How to generalize polar coords to $\mathbb{R}^3$?

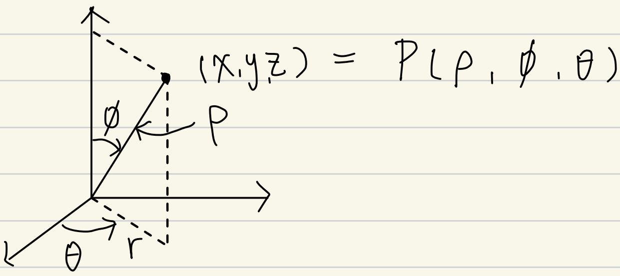
- Cylindrical coordinates :

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \\ z = z \end{cases}$$



- Spherical coordinates :

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \tan\theta = \dfrac{y}{x} \\ \cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$



$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases}$$