
100

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Office : MC 5468
OH : Mon & Wed 1:30 ~ 2:30
WAs due Monday, 5:30 pm.

5 Quizzes → 15% (lowest drop)
4 WA → 10%
Midterm → 25%
Final → 50%

Chapters :

- Chapter 1: Vectos in $\mathbb{R}^n$
- Chapter 2: Span, lines, and Spans.
- Chapter 3: System linear equations.
- Chapter 4: Matrices.
- Chapter 5: linear transformations.
- Chapter 6: The determinant.
- Chapter 7: Eigenvalues and Diagonalization
- Chapter 8: Subspaces and Bases.

Chapter 1

(1.1)

- $\mathbb{R}^n$ is defined as $\left\{ \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1, \ldots, x_n \in \mathbb{R} \right\}$

- A vector is an element $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ of $\mathbb{R}^n$

ex: $\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^3$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{100} \end{bmatrix} \in \mathbb{R}^{100}$

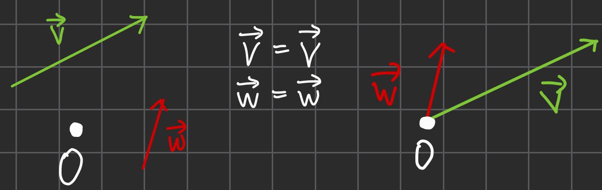
- row notation for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is $\vec{v} = [v_1, v_2, \ldots v_n]^T$. $T \leftarrow$ (transpose)

$\hookrightarrow [v_1, v_2 \ldots v_n] \neq [v_1, v_2 \ldots v_n]^T$

(1.2)

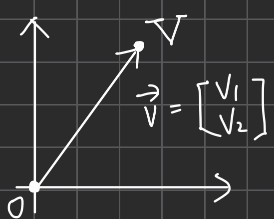
- vector's: algebraic representation : $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

geometric representation :



- Original point 0: the initial point of $\vec{v}$
- Terminal point V: coordinates $(v_1, v_2, \ldots v_n) \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

ex: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$ $\Rightarrow$



Equality

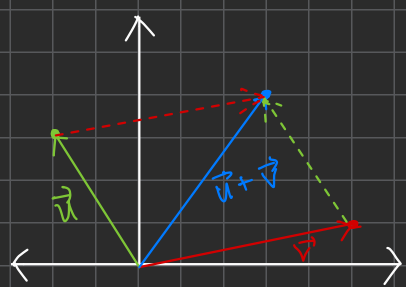
(1.3)

- $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ are equal if $u_i = v_i$, $\forall i = 1, 2, 3 \ldots n$

then, $\vec{u} = \vec{v}$

$$\Rightarrow \vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \quad \vdots \\ u_n \quad v_n \end{bmatrix}$$

ex: (geo) $\vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$ $\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



- Properties of Vector Addition.

這邊純講幹話

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$,

(a) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (symmetry)

(b) $\vec{u}+\vec{v}+\vec{w}=\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ (associativity)

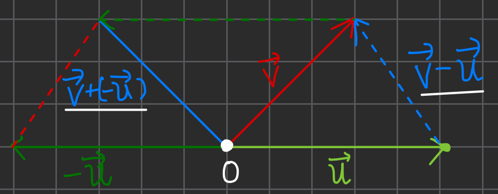
(c) There is zero vector, $\vec{0}=[0,0,\dots 0]^T$ in $\mathbb{R}$

with the property $\vec{v}+\vec{0}=\vec{0}+\vec{v}=\vec{v}$

- $\vec{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ additive inverse $-\vec{u}=\begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$.

p.s. $\vec{u}-\vec{u}=\vec{u}+(-\vec{u})=(-\vec{u})+\vec{u}=\vec{0}$ (負號就是方向相反)

- Subtraction: $\vec{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{bmatrix}$, $\vec{v}-\vec{u}=\vec{v}+(-\vec{u})=\begin{bmatrix} v_1-u_1 \\ v_2-u_2 \\ \vdots \quad \vdots \\ v_n \quad u_n \end{bmatrix}$



- Scalar multiplication: $c \in \mathbb{R}$, $\vec{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \Rightarrow c\vec{v}=c\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$

we say $\vec{v}$ is scaled by c,

幹話

P.S. – when $c>0$, $c\vec{v}$ points in the direction of $\vec{v}$ c times as long.
– when $c<0$, $c\vec{v}$ points in the opposite of $\vec{v}$ $|c|$ times as long.

- Properties of Scalar Multiplication:

Let $c, d \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$.

(a) $(c+d)\vec{v}=c\vec{v}+d\vec{v}$

(b) $(cd)\vec{v}=c(d\vec{v})$

(c) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$

(d) $0\vec{v}=0$

(e) If $c\vec{v}=0$, $c=0$ or $\vec{v}=\vec{0}$ (cancellation law)

- Standard Basis for $\mathbb{R}^n$:

  In $\mathbb{R}^n$, let $\vec{e_i}$ be the vector whose i^{th} component is 1, with all other components 0.

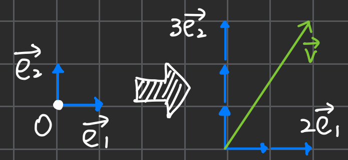
  The set $\mathcal{E} = \{\vec{e_1}, \vec{e_2}, \vec{e_3} \dots \vec{e_n}\}$ is the standard basis.

  ex: standard basis for $\mathbb{R}^3$ is $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

- If $\vec{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} = v_1\vec{e_1} + v_2\vec{e_2} \dots v_n\vec{e_n}$, then we call $v_1, v_2, \dots v_n$

  $\rightarrow$ components of $\vec{v}$

  ex: In $\mathbb{R}^2$, $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix} = 2\vec{e_1} + 3\vec{e_2} = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$



(1.4)

- Vectors in $\mathbb{C}^n$

  Set. $\mathbb{C}^n$ is defined as: $\left\{ \begin{bmatrix}z_1\\\vdots\\z_n\end{bmatrix} : z_1, \dots, z_n \in \mathbb{C} \right\}$

  vector $\vec{z} = \begin{bmatrix}z_1\\\vdots\\z_n\end{bmatrix}$ of $\mathbb{C}^n$. (相加，減都一樣)

  Standard basis is the same as $\mathbb{R}$

(1.5)

- Dot Product in $\mathbb{R}^n$: 表示兩個向量之間的夾角 or b 在 a 上的投影

  Let $\vec{u} = \begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ in $\mathbb{R}^n$

  dot product : $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \dots + u_nv_n$.

  ex: $\begin{bmatrix}3\\-5\\2\end{bmatrix} \cdot \begin{bmatrix}-4\\4\\-2\end{bmatrix} = 3(-4) + (-5)4 + 2\cdot 2 = -36$

- Properties of Dot Product:

  Let $c \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w}$ are vectors in $\mathbb{R}^n$.

  (a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (symmetry)

  $$\left.\begin{array}{l}\text{(b) } (\vec{u}+\vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \\ \text{(c) } (c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})\end{array}\right\} \text{linearity.}$$

(勾股定理)

- Length (norm or magnitude) of $\vec{v} \in \mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

  ex: $\left\|\begin{bmatrix}2\\4\end{bmatrix}\right\| = \sqrt{\begin{bmatrix}2\\4\end{bmatrix} \cdot \begin{bmatrix}2\\4\end{bmatrix}} = \sqrt{2(2)+4(4)} = \sqrt{20} = 2\sqrt{5}$

- If $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$, then $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

  ex: $\left\|\begin{bmatrix}-3\\-6\end{bmatrix}\right\| = \left\|-3\begin{bmatrix}1\\2\end{bmatrix}\right\| = |-3|\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\| = 3 \cdot \sqrt{1+4} = 3\sqrt{5}$

- $\vec{v} \in \mathbb{R}^n$ is a Unit vector if $\|\vec{v}\| = 1$ (單位向量)

  ex: $\vec{v} = \begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{bmatrix}^T = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\2\end{bmatrix}$

  then, $\|\vec{v}\| = \frac{1}{\sqrt{6}}\sqrt{1+1+4} = \frac{1}{\sqrt{6}}\sqrt{6} = 1$ $\longrightarrow$ Yes, unit vector.

- Normalization (produce a unit vector)

  $\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$ (用腦子導)

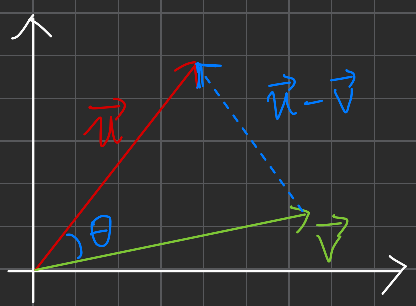
- Angle Between vectors:

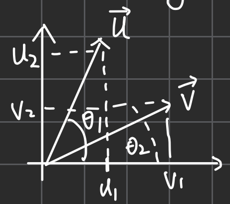
  Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^n$, the angle θ, $(0 \le \theta \le \pi)$

  $$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

  Proof:

  Given the graph,





$\vec{u} \cdot \vec{e}_1 = u_1 = \|\vec{u}\| \cos\theta_1$

$\vec{u} \cdot \vec{e}_2 = u_2 = \|\vec{u}\| \sin\theta_1$

$\vec{v} \cdot \vec{e}_1 = v_1 = \|\vec{v}\| \cos\theta_2$

$\vec{v} \cdot \vec{e}_2 = v_2 = \|\vec{v}\| \sin\theta_2$

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

$= \|\vec{u}\|\|\vec{v}\| \cos\theta_1 \cos\theta_2 + \|\vec{u}\|\|\vec{v}\| \sin\theta_1 \sin\theta_2$

$= \|\vec{u}\|\|\vec{v}\| \cos(\theta_1 - \theta_2)$

$= \|\vec{u}\|\|\vec{v}\| \cos\theta$

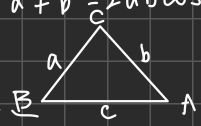
$\theta = \arccos \ldots$

p.s. 老師解

Then, the law of cosine shows that:

law of cos..

$c^2 = a^2 + b^2 - 2ab\cos C$



$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

then, $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

$$= \underline{\vec{u}\cdot\vec{u}} - \underline{\vec{u}\cdot\vec{v}} - \underline{\vec{v}\cdot\vec{u}} + \underline{\vec{v}\cdot\vec{v}}$$

$$= \|\vec{u}\|^2 - 2\vec{v}\cdot\vec{u} + \|\vec{v}\|^2$$

Plug it back in, $\|\vec{u}\|^2 - 2\vec{v}\cdot\vec{u} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

$$\Rightarrow \quad \vec{v}\cdot\vec{u} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\Rightarrow \quad \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|\|\vec{v}\|} = \cos\theta$$

$$\Rightarrow \quad \theta = \arccos\left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

(我真牛逼) 真滴

(不重要吧)

- Cauchy-Schwarz Inequality: $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\| \quad \forall\, \vec{u},\vec{v} \in \mathbb{R}^n$.

prc. find the θ between $\begin{bmatrix}1\\3\\5\end{bmatrix}$ and $\begin{bmatrix}2\\-4\\3\end{bmatrix}$ in $\mathbb{R}^3$

$$\theta = \arccos\left(\frac{2-12+15}{(\sqrt{1+9+25})(\sqrt{4+16+9})}\right) = \arccos\left(\frac{5}{\sqrt{35}\cdot\sqrt{29}}\right) \approx 1.413$$

(垂直)

- If $\vec{u}\cdot\vec{v} = 0$ in $\mathbb{R}^n$ we say it's orthogonal (or perpendicular)

- From the proof up there, we have $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

  p.s. If $\vec{v}$ & $\vec{w}$ are orthogonal, $0 = \cos\theta$, $\theta = \frac{\pi}{2}$ (90°)

  & for every $\vec{v}\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} \in \mathbb{R}^n$ is orthogonal to $\vec{0}$

prc. determine a vector $\vec{u}$ that's orthogonal to $\vec{v} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ $\quad u_1 - u_2 = 0$, $\underline{u_1 = u_2}$

$u_1 + 2u_2 + 3u_3 = 0 \quad 3u_1 + 3u_3 = 0$, $\underline{u_3 = -u_1}$

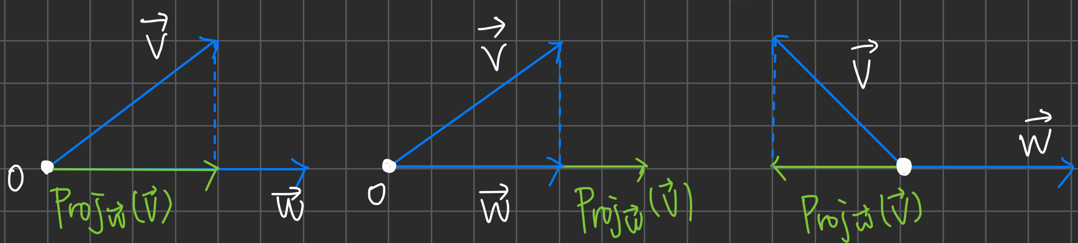
$$\Rightarrow \vec{u} = \begin{bmatrix}u_1\\u_1\\-u_1\end{bmatrix}, \; u \in \mathbb{R} \quad \#$$

(1.6)

- **Projection**: (the part of $\vec{v}$ that lies on $\vec{w}$)

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, with $\vec{w} \neq 0$, Projection of $\vec{v}$ onto $\vec{w}$ is

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v})\cdot(\vec{w})}{\|\vec{w}\|^2}\vec{w} = \frac{(\vec{v})\cdot(\vec{w})}{\vec{w}\cdot\vec{w}}\vec{w}$$ (答案是$\vec{w}$的parts)



pre: find projection of $\vec{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$ onto $\vec{w}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v})\cdot(\vec{w})}{\|\vec{w}\|^2}\vec{w} = \frac{28-8}{(49+1)}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{2}{5}\vec{w}$$

pre: find projection of $\vec{v} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ onto $\vec{w}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{3-8+15}{(1+4+9)}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{5}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ or } \frac{5}{7}\vec{w}$$

Variation:

- $\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v})\cdot(\vec{w})}{\|\vec{w}\|^2}\vec{w} = \left(\vec{v}\cdot\frac{\vec{w}}{\|\vec{w}\|}\right)\frac{\vec{w}}{\|\vec{w}\|} = (\vec{v}\cdot\hat{w})\,\hat{w}$
- $\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v})\cdot(\vec{w})}{\|\vec{w}\|^2}\vec{w} = \frac{\|\vec{v}\|\|\vec{w}\|\cos\theta}{\|\vec{w}\|^2}\vec{w} = (\|\vec{v}\|\cos\theta)\,\hat{w}$

- **Component**:

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{w} \neq 0$ then,

$\|\vec{v}\|\cos\theta = \vec{v}\cdot\hat{w}$ (is the component of $\vec{v}$ along $\vec{w}$)

pre: determine component of $\vec{v} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ along $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

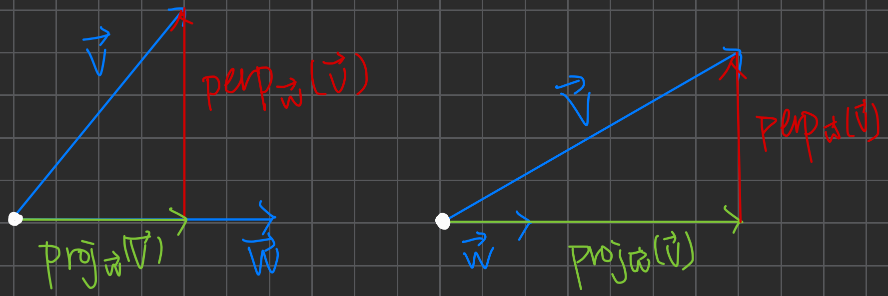
$\|\vec{w}\| = \sqrt{1+4+9} = \sqrt{14}$, $\hat{w} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

component of $\vec{v}$ along $\vec{w} = \frac{1}{\sqrt{14}}(3-8+15) = \frac{10}{\sqrt{14}}$

- **Perpendicular:**

  Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\vec{w} \neq 0$, then perpendicular of $\vec{v}$ onto $\vec{w}$ is

  $$\operatorname{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \operatorname{proj}_{\vec{w}}(\vec{v}).$$



  pr. perpendicular of $\vec{v} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ onto $\vec{w} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

  $$\vec{v} - \operatorname{proj}_{\vec{w}}(\vec{v}) = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} - \left(\frac{3-8+15}{1+4+9}\right)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

  $$= \frac{1}{7}\begin{bmatrix} 21 \\ -28 \\ 35 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16 \\ -38 \\ 25 \end{bmatrix}$$

- $\operatorname{perp}_{\vec{w}}(\vec{v}) \cdot \operatorname{proj}_{\vec{w}}(\vec{v}) = 0$

  proof: $\underline{\underline{\operatorname{perp}_{\vec{w}}(\vec{v}) \cdot \vec{w}}} = (\vec{v} - \operatorname{proj}_{\vec{w}}(\vec{v})) \cdot \vec{w}$

  $$= \vec{v} \cdot \vec{w} - \operatorname{proj}_{\vec{w}}(\vec{v}) \cdot \vec{w}$$

  $$= \vec{v} \cdot \vec{w} - \left(\frac{(\vec{v} \cdot \vec{w})}{\|\vec{w}\|^2}\vec{w}\right) \cdot \vec{w}$$

  $$= \vec{v} \cdot \vec{w} - \left(\frac{(\vec{v} \cdot \vec{w})}{\|\vec{w}\|^2}\right)\|\vec{w}\|^2$$

  $$= \vec{v} \cdot \vec{w} - (\vec{v} \cdot \vec{w}) = 0$$

  thus, $\underline{\operatorname{perp}_{\vec{w}}(\vec{v}) \text{ is orthogonal to } \vec{w}}$

  $\underline{\text{and} \Rightarrow \text{also orthogonal to } \vec{w}}$

- Recall different ways to determine magnitude.

  $x \in \mathbb{R} \Rightarrow$ magnitude (absolute value) $= |x| = \sqrt{x(x)}$

  $z \in \mathbb{C} \Rightarrow$ magnitude (modulus) $= |z| = \sqrt{z\bar{z}}$

  (cuz $z\bar{z} = |z|^2$

ex: Show $\text{perp}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v}) = 0$

$$LS = (\vec{v} - \text{proj}_{\vec{w}}(\vec{v})) \cdot \text{proj}_{\vec{w}}(\vec{v})$$

$$= \vec{v} \cdot \text{proj}_{\vec{w}}(\vec{v}) - \text{proj}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v})$$

$$= \vec{v} \cdot \left[\left(\frac{(\vec{v} \cdot \vec{w})}{\|\vec{w}\|^2}\right)\vec{w}\right] - \|\text{proj}_{\vec{w}}(\vec{v})\|^2$$

$$= \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)(\vec{v} \cdot \vec{w}) - \left\|\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w}\right\|^2$$

$$= \frac{(\vec{v} \cdot \vec{w})^2}{\|\vec{w}\|^2} - \frac{(\vec{v} \cdot \vec{w})^2}{\|\vec{w}\|^{\cancel{4}\,2}} \cancel{\|\vec{w}\|^2} = 0 \#$$

ex: Show $\text{proj}_{\vec{u}}(\text{proj}_{\vec{u}}(\vec{v})) = \text{proj}_{\vec{u}}(\vec{v})$

$$LS = \text{proj}_{\vec{u}}\left\{\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\vec{u}\right\} = \left[\frac{\left[\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\vec{u}\right] \cdot \vec{u}}{\|\vec{u}\|^2}\right]\vec{u}$$

$$= \left[\frac{\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\|\vec{u}\|^2}{\|\vec{u}\|^2}\right]\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\vec{u} = \text{proj}_{\vec{w}}(\vec{v})$$

1.7

- **Standard Inner Product on $\mathbb{C}$** (就是 Complex 的 dot) ⟨ ⟩

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$

the $\langle \vec{v}, \vec{w} \rangle = v_1\overline{w_1} + v_2\overline{w_2} + v_3\overline{w_3} \dots + v_n\overline{w_n}$

pr. $\vec{v} = \begin{bmatrix} 1+i \\ 1+2i \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2+i \\ 1+i \end{bmatrix}$ evaluate $\langle \vec{v}, \vec{w} \rangle$ 這面教授跳過了

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= (1+i)(2-i) + (1+2i)(1-i) \\ &= 2-i+2i+1+1-i+2i+2 \\ &= 6+2i\end{aligned}$$

- **Properties** of Standard inner product.

Let $c \in \mathbb{C}$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$, then.

(a) $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ (conjugate symmetry)

(b) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

(c) $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$

} (linearity in the 1st arguement)

(d) $\langle \vec{v}, \vec{v} \rangle \geq 0$, with $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\Leftrightarrow$ $\vec{v} = \vec{0}$

- **length** (magnitude) of vector $\vec{v} \in \mathbb{C}^n$ is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

pr. determine length of vector $\vec{v} = \begin{bmatrix} 2-i \\ -3+2i \\ -4-5i \end{bmatrix}$

$$l = \sqrt{\langle v, v \rangle} = \sqrt{(2-i)\overline{(2+i)} + (-3+2i)\overline{(-3-2i)} + (-4-5i)\overline{(-4+5i)}}$$

$$= \sqrt{4+1+9+4+16+25} = \sqrt{59} \#$$

Note that $\langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2$.

- **Properties of length** ($\mathbb{C}$)

let $c \in \mathbb{C}$ and $\vec{v} \in \mathbb{C}^n$. Then

(a) $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ ($|c| \to$ modulus

(b) $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| \Leftrightarrow \vec{v} = 0$

Defn: ($\mathbb{C}$) we say two vectors are **orthogonal** if $\langle \vec{u}, \vec{v} \rangle = 0$.

pre. (a) find a vector that is org to $\vec{w} = \begin{bmatrix} 2+i \\ 1+i \end{bmatrix}$

$$x(2-\bar{i}) + y(1-\bar{i}) = 0$$
$$y(1-\bar{i}) = x(\bar{i}-2) \qquad \times\ (\overline{1-\bar{i}})$$
$$2y = x(\bar{i}-2)(\bar{i}+1)$$
$$2y = x(-3-\bar{i})$$
$$y = \frac{-3-\bar{i}}{2} x \qquad (接下來隨便選個數代入)$$

let $y=2$ $\quad \vec{v} = \begin{bmatrix} 2 \\ -3-i \end{bmatrix}$ #

(b) find unit vector org to $\vec{w}$ (normalize $\vec{v}$)

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} \qquad \|\vec{v}\| = \sqrt{2(2) + (-3-i)(-3+i)} = \sqrt{4+10} = \sqrt{14}$$

$$\hat{v} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3-i \end{bmatrix} \quad (\text{org to } \vec{w} \text{ since same direction to } \vec{v})$$

- **Projection ($\mathbb{C}$)**. let $\vec{v}, \vec{w} \in \mathbb{C}^n$ with $\vec{w} \neq \vec{0}$, projection =

$$\mathrm{Proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2} \vec{w} = \langle \vec{v}, \hat{w} \rangle \hat{w}$$

pre. find proj. of $\vec{v} = \begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix}$ onto $\vec{w} = \begin{bmatrix} 1-i \\ 2-i \\ 3+i \end{bmatrix}$

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= 1(1+i) + i(2+i) + (1+i)(3-i) \\ &= 1+i+2i-1+3+1+2i \\ &= 4+5i \end{aligned}$$

$$\|\vec{w}\|^2 = \langle \vec{w}, \vec{w} \rangle = 1+1+4+1+9+1 = 17$$

thus, $\mathrm{proj}_{\vec{w}}(\vec{v}) = \dfrac{4+5i}{7} \begin{bmatrix} 1-i \\ 2-i \\ 3+i \end{bmatrix}$

1.8. **Standard Inner Product** of $\vec{v}, \vec{w} \in \mathbb{F}^n$ is

$$\langle \vec{v}, \vec{w} \rangle = v_1\overline{w_1} + v_2\overline{w_2} + v_3\overline{w_3} \cdots + v_n\overline{w_n}$$

Remark: If $\mathbb{F} \in \mathbb{C}$, this is correct. for $\mathbb{C}^n$

If $\mathbb{F} \in \mathbb{R}$, $w_i \in \mathbb{R}$ and $\overline{w_i} = w_i$, thus, 跟 dot 一樣

一直跳到這邊

1.9 **Cross Product**

let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ v_1u_2 - u_2v_1 \end{bmatrix}$$

note: only define in $\mathbb{R}^3$.

Properties of $\times$:

let $\vec{u}, \vec{v} \in \mathbb{R}^3$ and let $\vec{z} = \vec{u} \times \vec{v}$

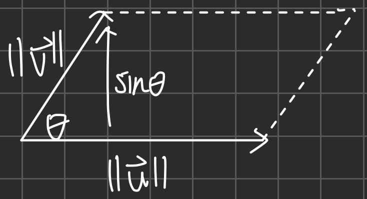
(a) $\vec{z} \cdot \vec{u} = 0$ and $\vec{z} \cdot \vec{v} = 0$ (org to both $\vec{u}$ and $\vec{v}$

(b) $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$

(c) If $\vec{u} \neq 0$ and $\vec{v} \neq 0$ then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$

(where θ is the angle between $\vec{u}$ and $\vec{v}$)

Geometric representation:



$\|\vec{v}\|\|\vec{u}\| \sin\theta$ = Area of parallelogram

• **linearity of $\times$:**

(a) $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

(b) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$

} linearity in first arguement.

(廢話)

(c) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

(d) $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

} linearity in 2nd arguement

Chapter 2.

- **Linear Combination**: Let $C_1, C_2 \dots C_K \in \mathbb{F}$ and let $\vec{v}_1, \vec{v}_2 \dots \vec{v}_K$ be vectors in $\mathbb{F}^n$. We refer to any vector of the form $c\vec{v}_1 + C_2\vec{v}_2 \dots C_K\vec{v}_K$ as linear combination of $\vec{v}_1 \dots \vec{v}_K$

- **Span**: Let $\vec{v}_1, \vec{v}_2 \dots \vec{v}_K$ be vectors in $\mathbb{F}^n$. We define span of $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_K\}$ to be a set ALL linear combo of $\vec{v}_1 \dots \vec{v}_K$.

  That is, $\text{Span}\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_K\} = \{C_1\vec{v}_1, C_2\vec{v}_2, C_3\vec{v}_3 \dots C_K\vec{v}_K\}$

note: - If $\vec{v} = \vec{0}$, $S = \{\vec{0}\}$

- S always pass thru orign.

ex: Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$, what does $S = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ look like geometrically?

- If $\vec{v}_1 = c\vec{v}_2$ then S is a line through 0.
- if $\vec{v}_1 \neq c\vec{v}_2$ then S is a plane through orign.

- **lines in $\mathbb{R}^2$.**

  - $y = mx + b$, $m \Rightarrow$ slope, $b \Rightarrow$ y-int
  - $y - y_1 = m(x - x_1)$, $m \Rightarrow$ slope $(x_1, y_1) \Rightarrow$ a pt. on the line.
  - $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$, $(x_1, y_1), (x_2, y_2) \Rightarrow$ two distinct points.

- **Parametric equation**. Let p, q be fixed real numbers and $q \neq 0$ then parametric equation of a line in $\mathbb{R}^2$ through the point (x_1, y_1) with slope $\frac{p}{q}$ are

$$\begin{cases} x = x_1 + qt \\ y = y_1 + pt \end{cases}$$

  - t is called a parameter, each value of t is a pt. on the line.

ex: Find the parametric eqn of line with eqn

$y = 3x + 2$

Sol: point $(0,2)$, slope $= \frac{3}{1}$

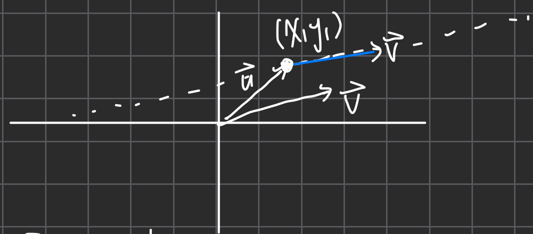
$x = 0 + t$
$y = 2 + 3t$ $\quad t \in \mathbb{R}$

- Vector equation of a line $\mathcal{L}$ in $\mathbb{R}^2$,

let $\begin{bmatrix} q \\ p \end{bmatrix}$ be a non-zero vector in $\mathbb{R}^2$. The expression

$$\vec{l} = \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{(\vec{u})} + t \underbrace{\begin{bmatrix} q \\ p \end{bmatrix}}_{(\vec{v})}, \quad t \in \mathbb{R}$$

note: (x_1, y_1) is a pt. on the line (terminal pt.)



ex: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} t$, $t \in \mathbb{R}$

we know, $(0,2)$ is a pt. on the line.

$(1,3)$ is not on the line but parallel.

Defn: Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ with $\vec{v} \neq \vec{0}$ we refer to the set of vectors.

$$\mathcal{L} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$$

as a line in $\mathbb{R}^2$ through $\vec{u}$ with direction $\vec{v}$.

Defn: lines in $\mathbb{R}^n$: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$, the expression,

$$\vec{l} = \vec{u} + \vec{v}t \quad , \quad t \in \mathbb{R}$$

is the vector equation of the line $\mathcal{L}$ in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$.

Note: if $\vec{l_1} = \vec{l_2}$, with direction $\vec{v_1}$ and $\vec{v_2}$,

we say they have same direction $c\vec{v_1} = \vec{v_2}$, $c \in \mathbb{R}$

ex: Consider Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}$

Line in $\mathbb{R}^5$ through the orign

direction vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Parametric equation.

$$\begin{aligned} x_1 &= 0 + t \\ x_2 &= 0 + 2t \\ x_3 &= 0 + 3t \quad t \in \mathbb{R} \\ x_4 &= 0 + 4t \\ x_5 &= 0 + 5t. \end{aligned}$$

Defn, Vector equation of a Plane.

let $\vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{v} \neq c\vec{w}$ for any $c \in \mathbb{R}$

the expression $\vec{p} = s\vec{v} + t\vec{w}$ (span of 2 vectors)

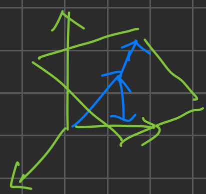
is a vector equation of the plane in $\mathbb{R}^n$ through orign with direction $\vec{v}$ and $\vec{w}$

Defn: Plane in $\mathbb{R}^n$ through the orign.

let $\vec{v}, \vec{w} \in \mathbb{R}^n$, with $\vec{v} \neq c\vec{w}$ for any $c \in \mathbb{R}$

$$P = \text{Span}\{\vec{v}, \vec{w}\} = \{s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$

is a plane through the orign with direction vectors $\vec{v}$ and $\vec{w}$.



Defn: Vector equation of a Plane through $\vec{u}$.

let $\vec{u} \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$, with $\vec{v} \neq c\vec{w}$ then,

$$\vec{p} = \vec{u} + s\vec{v} + t\vec{w}, \quad s, t \in \mathbb{R}$$

is a vector equation of the plane in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$ and $\vec{w}$.

Defn: Plane in $\mathbb{R}^n$:

let $\vec{u} \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$, with $\vec{v} \neq c\vec{w}$ then,

$$P = \{\vec{u} + s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$

is a plane in $\mathbb{R}^n$ through $\vec{u}$ with direction vectors $\vec{v}$ and $\vec{w}$

ex: find a vector equation of the plane through the points $A(1,0,2)$, $B(-3,-2,4)$ and $C(1,8,-5)$

$$\vec{v} = \vec{b} - \vec{a} = \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} \qquad \vec{w} = \vec{c} - \vec{a} = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

ex: find an non-zero vector orthogonal to the plane.

$$P = \left\{ s\underset{(\vec{v})}{\begin{bmatrix}1\\0\\2\end{bmatrix}} + t\underset{(\vec{w})}{\begin{bmatrix}0\\-1\\1\end{bmatrix}} : s, t \in \mathbb{R} \right\}$$

$$\vec{v} \times \vec{w} = \begin{bmatrix}0-(-2)\\-(1-0)\\-1-0\end{bmatrix} = \begin{bmatrix}-2\\-1\\-1\end{bmatrix}$$

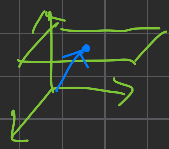
let $\vec{p} = \begin{bmatrix}x\\y\\z\end{bmatrix}$

$\leftarrow$ vector to the plane.

$\vec{p}$ is going to parallel the the plane P goes through orign

ex: $\vec{p} = \begin{bmatrix}1\\0\\2\end{bmatrix} + s\begin{bmatrix}-4\\-2\\2\end{bmatrix} + t\begin{bmatrix}0\\8\\-7\end{bmatrix}$

Does this plane go through the orign?



$1-4s=0 \qquad s=\frac{1}{4}$

$-2s+8t=0 \qquad t=\frac{1}{16}$

$2+2s-7t=0 \qquad 2+\frac{1}{2}-\frac{7}{16} \neq 0$ So, No.

$\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{p}$ (vector to the plane) will not parallel to P

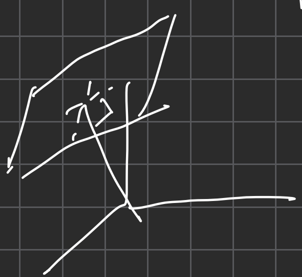
However, $\vec{p} - \begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}x-1\\y\\z-2\end{bmatrix}$ will parallel to the plane.

Defn: Normal vector

let $\vec{u}, \vec{w} \in \mathbb{R}^3$

$\vec{n} = \vec{u} \times \vec{w}$ (orth $\llcorner$)

($\vec{n}$ 垂直 to plane.)

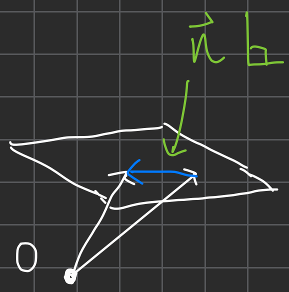


Defn: Normal form, Scalar Equation of plane in $\mathbb{R}^3$

- let P be a plane in $\mathbb{R}^3$ with direction vector $\vec{v}$ & $\vec{w}$
  and a normal vector $n = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$

  let $\vec{u} \in P$ and $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in P$ where $\vec{p} \neq \vec{u}$

  A <u>Normal form</u> of P is given by



$$\vec{n} \cdot (\vec{p} - \vec{u}) = 0$$

Expanding this ⇒ scalar equation.

$$ax + by + cz = d = \vec{n} \cdot \vec{u}$$

Chapter 3

Defn: linear equation

n variables $x_1, x_2, x_3 \dots x_n$ that can be written

$$\underbrace{a_1 x_1 + a_2 x_2 + a_3 x_3}_{\text{coeff}} \dots + a_n x_n = \underbrace{b}_{\text{Constant term}}$$

Defn: System of linear equation

collection of m linear equ in n variables.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots + a_{1n}x_n = b_1$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \dots + a_{mn}x_n = b_m$$

$\Rightarrow$ a_{ij} is the coeff of x_j in the i^{th} equ.

Defn: Solve, solution

we say scalars $y_1, y_2, y_3 \dots y_n$ in $\mathbb{F}$ solve a system of linear equ when we set $x_1 = y_1, x_2 = y_2 \dots x_n = y_n$ each equ is satisfied.

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \dots + a_{1n}y_n = b_1$$
$$a_{21}y_2 + a_{22}y_2 + a_{23}y_3 \dots + a_{2n}y_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}y_1 + a_{m2}y_2 + a_{m3}y_3 \dots + a_{mn}y_n = b_m$$

we say $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is the solution.

Defn: Solution Set

The set of all soln to a linear eqn is a soln set

Solution Set is either:

a) empty (inconsistent)

b) exactly 1 element. } (consitent.

c) infinite element.

p.s. we say solution are equivalent when same soln set.

Elementary Operations:

let equation ordered from e_1 to e_m

type ① Equation Swap $e_i \leftrightarrow e_j$

type ② Equation Scale $e_i \to m e_i$, $m \in \mathbb{F} / \{0\}$

type ③ Equation Addition $e_j \to c e_i + e_j$, $c \in \mathbb{F}$

trival eqn $0 = 0$ ie. non-trival for else.

Defn: Matrix, Entry.

$m \times n$ matrix, a rectangle array of scalars of m rows and n columns. The scalar in i^{th} row, j^{th} column is the $(i,j)^{th}$ entry. aka. a_{ij} or $(A)_{ij}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Defn: Coeff Matrix, Augmented Matrix

Given:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots + a_{1n}x_n = b_1$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \dots + a_{mn}x_n = b_m$$

- Coeff matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the coeff of x_j in i^{th} equ.

- Augmented matrix $[A|\vec{b}]$ of system is.

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

where $\vec{b}$ is a column of constant terms.

Defn: Elementary Row Operations. (EROs)

	equation	Rows.
I Row Swap:	$e_i \leftrightarrow e_j$	$R_i \leftrightarrow R_j$
II Row Scale:	$e_i \to ce_i,\ c \neq 0$	$R_i \to cR_i,\ c \neq 0$
III Row addition:	$e_i \to ce_j + e_i\ \ i \neq j$	$R_i \to cR_j + R_i,\ i \neq j$

Zero Row: a row that has a 0 entries

Defn: (s)

Leading Entry: left-most non-zero entry in any non-zero row.

Leading one: If leading entry is 1, is called leading one.

Row Echelon form (REF): when,

a) all zero rows occurs as the last rows.

b) The leading entry appears in a column to the right of the columns containing the leading entry.

Pivot: In REF, leading entries are called Pivot.

Reduced Row echelon form (RREF): when,

a) It is in REF

b) All pivot are leading ones

c) the only non-zero entry in a pivot column is the pivot.

Defn: Unique RREF

let A be matrix with REFs R_1 and R_2, then R_1, R_2 will have the same set of pivot positions.

And, there's a Unique matrix $R \ni R$ is RREF of A.

we write, $R = RREF(A)$

Defn: Basic, free variable.

- Basic variable: we call x_i basic v. if the i^{th} column contains a pivot.
- Free variable: vice versa. (Basically $\Rightarrow 0$)

pro:

$$\left[\begin{array}{cccc|c} -1 & -2 & 2 & 4 & 3 \\ 2 & 4 & -2 & -2 & 4 \\ 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -6 & 8 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -2 & -2 & 4 \\ -1 & 2 & 2 & 4 & 3 \\ 2 & 4 & -6 & -8 & -4 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -4 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -4 & -4 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 - x_3 - 2x_4 = 0 \\ x_3 + 2x_4 = 3 \\ 2x_4 = 4 \end{cases}$$

$x_4 = 2, \; x_3 = -1$

$\Rightarrow x_1 + 2x_2 = 3 \quad \Rightarrow \quad$ let $x_2 = t$

$x_1 = 3 - 2x_2 \qquad x_1 = 3 - 2t$

then, $S = \left\{ \begin{bmatrix} 3-2t \\ t \\ -1 \\ 2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$ (REF)

to get RREF,

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 + 2x_2 = 3$

$x_3 = -1 \qquad$ let $= x_2$ ... same

$x_4 = 2$

p.s. If we get $[\,0\;0\;0 \ldots 0 \mid b\,]$ $b \neq 0 \Rightarrow$ inconsistent prc:

$$\left[\begin{array}{ccc|c} 3 & 5 & 3 & 9 \\ -2 & -1 & 6 & 10 \\ 4 & 10 & -3 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ -2 & -1 & 6 & 10 \\ 4 & 10 & -3 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ 0 & 7 & 24 & 48 \\ 0 & -6 & -39 & -78 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ 0 & 1 & -15 & -30 \\ 0 & -6 & -39 & -78 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ 0 & 1 & -15 & -30 \\ 0 & 0 & -129 & -258 \end{array}\right] \xRightarrow{(REF)} \left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ 0 & 1 & -15 & -30 \\ 0 & 0 & -1 & -2 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ 0 & 1 & -15 & -30 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 9 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Defn:

- $M_{m\times n}(\mathbb{R})$: set of all $m\times n$ matrices with real entries.
- $M_{m\times n}(\mathbb{C})$: set of all $m\times n$ matrices with complex entries.

p.s. if $m=n$, then write. $M_n(\mathbb{F})$

Defn: Rank:

let $A \in M_{m\times n}(\mathbb{F})$ such that RREF(A) has exactly r pivots.
then, we say the Rank of A is r, $\text{rank}(A)=r$

Thm: Rank Bounds.

If $A \in M_{m\times n}(\mathbb{F})$, then $\text{rank}(A) \leq \min\{m,n\}$

Thm: Consistent System test:

let A be coeff matrix $\Rightarrow [A\,|\,\vec{b}]$ be the augmented matrix of system.
The system is consistent iff $\text{rank}(A) = \text{rank}[A\,|\,\vec{b}]$

Thm System Rank Theorem

let $A \in M_{m \times n}(\mathbb{F})$ with $\operatorname{rank}(A) = r$

a) let $\vec{b} \in \mathbb{F}^m$, if $[A \mid \vec{b}]$ is consistent, then the Solution set contain $n-r$ parameters.

b) The system with augmented matrix $[A \mid \vec{b}]$ is consistent for every $\vec{b} \in \mathbb{F}^m$ iff $n = m$.

pro:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = s$, $x_4 = t$

$x_1 = 2 + 2s - t$

$x_3 = 1 + 2t$

$$= \left\{ \begin{bmatrix} 2+2s-t \\ s \\ 1+2t \\ t \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$ #

pro:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & -1 \\ 2 & -4 & 1 & 0 & 4 \\ 1 & -2 & 2 & -3 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & -1 \\ 0 & 0 & 3 & -6 & 6 \\ 0 & 0 & 3 & -6 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & -1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = s$, $x_4 = t$

$x_1 = 1 + 2s - t$

$x_3 = 2 - s + 2t$

$$= \left\{ \begin{bmatrix} 1+2s-t \\ s \\ 2+2t \\ t \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

Defn:

Homogeneous: all constant terms ($\vec{b}$) are zero. (consistent)

non-homogeneous: $\exists$ non-zero constants.

p.s. for homogeneous system, the trivial solution is $x_1 = x_2 = \dots x_n = 0$.

Null space: solution set of homogeneous system.

Defn: linear equ over $\mathbb{C}$

Recall for $z = a+bi \in \mathbb{C}$, that $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-bi}{a^2+b^2}$

Ex.
$$\left[\begin{array}{cc|c} 1+i & -2-3i & -15i \\ 1+3i & -1-8i & 15-30i \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & \frac{-2-3i}{1+i} & \frac{-15i}{1+i} \\ 1+3i & -1-8i & 15-30i \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & \frac{-2-3i}{1+i} & \frac{-15i}{1+i} \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{5}{2}-\frac{i}{2} & -\frac{15}{2}-\frac{15i}{2} \\ 0 & 0 & 0 \end{array}\right]$$

let $x_2 = t$

$$x_1 = -\frac{15}{2} - \frac{15i}{2} - \left(-\frac{5}{2} - \frac{i}{2}\right)t$$

$$= \left\{ \begin{bmatrix} -\frac{15}{2} - \frac{15i}{2} - \left(-\frac{5}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix} : t \in \mathbb{C} \right\}$$

$$= \left\{ \begin{bmatrix} -\frac{15}{2} - \frac{15i}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{5}{2} + \frac{i}{2} \\ 1 \end{bmatrix} : t \in \mathbb{C} \right\}$$

Defn: Row Vector

For matrix $A \in M_{m\times n}(\mathbb{F})$ we denote ith row

$$\overrightarrow{row}_i(A) = [a_{i1}, a_{i2}, a_{i3} \dots a_{in}]$$

Defn: Product $A\vec{x}$

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \vdots & \vdots & \ddots & \\ a_{m1}x_1 & a_{m2}x_2 & & a_{mn}x_n \end{bmatrix}$$

$$1 \rightarrow 2 \quad \cdots \rightarrow n$$

pro. $\begin{bmatrix} 1 & 6 & 1 \\ 3 & 4 & 5 \\ 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+18+6 \\ 3+12+30 \\ 5+6-18 \end{bmatrix} = \begin{bmatrix} 25 \\ 45 \\ -7 \end{bmatrix}$

Defn: linearity of Matrix-vector multiplication.

let $A \in M_{m\times n}(\mathbb{F})$. Let $\vec{x}, \vec{y} \in \mathbb{F}^n$ and $c \in \mathbb{F}$. then,

a) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

b) $A(c\vec{x}) = cA\vec{x}$

Proposition: Let $A \in M_{m\times n}(\mathbb{F})$, if for every vector $\vec{e_i}$ in the standard basis of $\mathbb{F}^m$ the system of equations $A\vec{x} = \vec{e_i}$ is consistent, than $\text{rank}(A) = m$.

Defn: Associated homogenous system.

Let $A\vec{x} = \vec{b}$ where $\vec{b} \neq 0$, then the associated homogenous system is $A\vec{x} = \vec{0}$

Defn: Particular solution

Let $A\vec{x} = \vec{b}$ be a consistent system of linear equations.

then $\vec{x}_p$ is a particular solution.

Thm: Solutions to $A\vec{x}=\vec{0}$ and $A\vec{x}=\vec{b}$

let solution of $A\vec{x}=\vec{b}$, $\vec{b}\neq 0$ be $\tilde{S}$

let associate homo-soln of $A\vec{x}=\vec{0}$ be S,

then $$\tilde{S}=\{\vec{x}_p+\vec{x} : \vec{x}\in S\}$$

pro: let $A=\begin{bmatrix}1&2\\2&4\end{bmatrix}$, solve

a) $A\vec{x}=0$ S

b) $A\vec{x}=\begin{bmatrix}4\\8\end{bmatrix}$ T

c) $A\vec{x}=\begin{bmatrix}-2\\-4\end{bmatrix}$ U

(super-augmented matrix)

$$\left[\begin{array}{cc|ccc}1&2&0&4&-2\\2&4&0&8&-4\end{array}\right]\Rightarrow\left[\begin{array}{cc|ccc}1&2&0&4&-2\\0&0&0&0&0\end{array}\right]$$

then, $$S=\left\{\begin{bmatrix}-2\\1\end{bmatrix}t : t\in\mathbb{R}\right\}$$

$$T=\left\{\begin{bmatrix}4\\0\end{bmatrix}+\begin{bmatrix}-2\\1\end{bmatrix}t : t\in\mathbb{R}\right\}$$

$$U=\left\{\begin{bmatrix}-2\\0\end{bmatrix}+\begin{bmatrix}-2\\1\end{bmatrix}t : t\in\mathbb{R}\right\}$$

Extension Thm: Solution to $A\vec{x}=\vec{b}$ and $A\vec{x}=\vec{c}$

$\tilde{S}_b$ and $\tilde{S}_c$ are respective solution. then,

$$\tilde{S}_c=\{(\vec{x}_c-\vec{x}_b)+\vec{z} : \vec{z}\in\tilde{S}_b\}$$

Chapter 4

Defn: Column space of a A, denoted by Col(A) is a span of the columns of A

$$\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3 \dots \vec{a}_n\}$$

Prop: Consistent System and column space

the system $A\vec{x} = \vec{b}$ is consistent iff $\vec{b} \in \text{Col}(A)$

Defn: Transpose let $A \in M_{m \times n}(\mathbb{F})$,

We define transpose of A, denoted A^T, by $(A^T)_{ij} = (A)_{ji}$

ex:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix}$$

Defn: Row Space

$$\text{Row}(A) = \text{Span}\{(\overrightarrow{\text{row}_1}(A))^T, (\overrightarrow{\text{row}_2}(A))^T \dots (\overrightarrow{\text{row}_n}(A))^T\}$$

ex:

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & -1 \\ 3 & 2 & 1 \\ 4 & 1 & -1 \end{bmatrix}, \text{Row}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$\Rightarrow$ Then we see $\text{Row}(A) = \text{Col}(A^T)$

Nullspace Null(A): solution set of homogeneous system A

Column space $\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_n\}$

Row Space

(三個都是 Span)

$\vec{x} \in \text{Null}(A)$ 可 $A\vec{x} = \vec{0}$

Defn: Matrix Equality.

Let $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{p\times q}(\mathbb{F})$. A & B are equal iff.

1. A and B have the same size, that is, $m=p$ and $n=q$
2. The corresponding entries of A and B are equal. $a_{ij} = b_{ij}$

then $A=B$

lemma: Column Extraction.

let $A = [\vec{a}_1, \vec{a}_2 \dots \vec{a}_n] \in M_{m\times n}(\mathbb{F})$, Then, $A\vec{e}_i = a_i$

Defn: Matrix Multiplication

let $A = M_{m\times n}(\mathbb{F})$ and $B = M_{n\times p}(\mathbb{F})$. We define matrix product $AB = C$ to be matrix $C \in M_{m\times p}(\mathbb{F})$

$$C = AB = A[\vec{b}_1, \vec{b}_2 \dots \vec{b}_p] = [A\vec{b}_1, A\vec{b}_2 \dots A\vec{b}_p]$$

$$\vec{c}_j = A\vec{b}_j$$

pre

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -1+4 & 3-8 \\ -3+10 & 9-20 \\ -8+14 & 24-28 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 7 & -11 \\ 6 & -4 \end{bmatrix}$$

multiplication 是加

Column of 1st Matrix 必须 = Row of 2nd Matrix

Defn: Matrix Sum

let $A, B \in M_{m\times n}(\mathbb{F})$ $\quad A + B = C$

$$C_{ij} = A_{ij} + B_{ij}$$

Defn: Additive Inverse.

let $A \in M_{m\times n}(\mathbb{F})$, additive inverse of $A = -A$

where $a_{ij} = -a_{ij}$

Defn: zero matrix

Zero matrix $O_{m\times n} \in M_{m\times n}(\mathbb{F})$

↳ all entries are 0.

$$\vec{0} = O_{m\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Properties of Matrix transpose.

a) $(A+B)^T = A^T + B^T$

b) $(sA)^T = sA^T$

c) $(AC)^T = C^T A^T$

d) $(A^T)^T = A$

Defn: Square Matrix: $A \in M_{n\times n}(\mathbb{F})$ where row = column $\Rightarrow M_n(\mathbb{F})$

Defn: Upper triangular: if $a_{ij} = 0$ for $i > j$ with $i,j = 1 \dots n$.

ex: $\begin{bmatrix} 3 & -4 & 6 \\ 0 & 5 & 9 \\ 0 & 0 & 7 \end{bmatrix}$

Defn: Lower triangular: if $a_{ij} = 0$ for $i < j$ with $i,j = 1 \dots n$.

ex: $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 7 & 0 \end{bmatrix}$

p.s. - transpose for Upper / lower trangle is lower / Upper.

- the product of Upper / lower is Upper / lower.

Defn: Diagonal: if $a_{ij} = 0$ for $i \neq j$ with $i,j = 1 \dots n$.

we refer a_{ii} as diagonal entries.

we denote matrix as $A = \text{diag}(a_{11}, a_{22}, \dots a_{nn})$

ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

p.s. all diagonal matrix are also upper / lower triangle

Defn: Identity Matrix: $\text{diag}(1, 1, 1 \dots 1)$

ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $I_m A = A$ (multiplicative identity)

$A I_n = A$

Defn: Elementary Matrix: matrix that can be obtain by performing <u>a single</u> ERO

Thm: let $A \in M_{m\times n}(\mathbb{F})$ and a single ERO is perform to get matrix B

Suppose, we perform the same ERO on I_m to produce elementary matrix E then, $B = EA$

Thm: $A \in M_{m\times n}(\mathbb{F})$ and a finite number of EROs $1 \sim k$ are performed on A to produce matrix B.

Suppose E_i = elementary matrix corresponding to the i^{th} ERO $(1\sim k)$ applied to I_m, then $B = E_k \dots E_2 E_1 A$

$$= (E_k \dots (E_2(E_1 A)))$$

$\boxed{m\times \underline{n \quad n} \times k \Rightarrow m\times k}$

multiplication.

Defn: Invertible:

- We say $n\times n$ matrix is invertable if $\exists$ $n\times n$ matrix B and C such that. $AB = CA = I_n$

→ If its a square $(n\times n)$ $\Rightarrow B = C$

→ $\exists B \Leftrightarrow \exists C$

Defn: If $A_{n\times n}$ is invertable $B = A^{-1}$

$$\Rightarrow AA^{-1} = A^{-1}A = I_n$$

(nxn)

Invertable = Rank(A) = n = RREF(A) = I_n

Prep

1. Construct $[A \mid I_n]$
2. Find RREF $[R \mid B]$ of $[A \mid I_n]$
3. If $R \neq I_n \Rightarrow A$ is not invertible
   If $R = I_n \Rightarrow A$ is invertible. $\Rightarrow A^{-1} = B$

Defn: Inverse of 2x2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc$ → determinant of A

Chapter 5

Defn: function defined by Matrix

- let $A \in M_{m\times n}(\mathbb{F})$, the function defined by matrix A is the function $T_A : \mathbb{F}^n \to \mathbb{F}^m$

defined by $T_A(\vec{x}) = A\vec{x}$

Thm T_A is linear $\Rightarrow$

1. $T_A(\vec{x}+\vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$
2. $T_A(c\vec{x}) = cT_A(\vec{x})$

Defn: linear transformation

We say $T: \mathbb{F}^n \to \mathbb{F}^m$ is linear transformation if two properties hold for any $\vec{x}, \vec{y}$.

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

② $T(c\vec{x}) = cT(\vec{x})$

p.s $\mathbb{F}^n$ is the <u>Domain</u> of T

$\mathbb{F}^m$ is the <u>codomain</u> of T.

or ③ $T(c\vec{x}+\vec{y}) = cT(\vec{x}) + T(\vec{y})$

④ $T(\vec{0}) = \vec{0}$

Defn: Range

let $T: \mathbb{F}^n \to \mathbb{F}^m$ be l tr

we define $\text{Range}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}$

↳ subset of $\mathbb{F}^m$

p.s. $\text{Range}(T_A) = \text{Col}(A)$

$A\vec{x} = \vec{b}$ have a solu <u>iff</u> $\vec{b} \in \text{Range}(T_A)$ or $\text{Col}(A)$

Defn: Onto

We say $T: \mathbb{F}^n \to \mathbb{F}^m$ is Onto if $\text{Range}(T) = \mathbb{F}^m$

same: ① T_A is onto

② $\text{Col}(A) = \mathbb{F}^m$

③ $\text{rank}(A) = m$

Defn: Kernel

set of inputs of T whose output is 0

$$\ker(T) = \{\vec{x} \in \mathbb{F}^n : T(\vec{x}) = \vec{0}_{\mathbb{F}^m}\}$$

p.s. $\ker(T_A) = \text{Null}(A)$

Defn: one-to-one

when $T(\vec{x}) = T(\vec{y})$ then $\vec{x} = \vec{y}$

T is one-to-one iff $\ker(T) = \{\vec{0}_{F^n}\}$

Same: ① T_A is one-to-one

② $\text{Null}(A) = \{\vec{0}\}$

③ $\text{nullify}(A) = 0$

④ $\text{rank}(A) = n$

Equivalent statements:

a) A is invertible

b) A is one-to-one

c) A is Onto

d) $\text{Null}(A) = \{\vec{0}\}$ → only soln to $A\vec{x} = \vec{0}$ is $\vec{0}$

e) $\text{Col}(A) = \mathbb{F}^n$ → for every $\vec{b} \in \mathbb{F}^n$, $A\vec{x} = \vec{b}$ is consistent.

f) $\text{nullify}(A) = 0$

g) $\text{rank}(A) = n$

h) $\text{RREF}(A) = I_n$

Defn: Standard Matrix:

let $T: \mathbb{F}^n \to \mathbb{F}^m$, we define standard Matrix of T, by $[T]_\varepsilon$ to be $m \times n$

$$[T]_\varepsilon = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3) \dots T(\vec{e}_n)]$$

$$= \left[T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) T\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) \dots T\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \right]$$

Thm: every linear T is determine by a matrix.

$$T(\vec{x}) = [T]_\varepsilon \vec{x}$$

$\Rightarrow T = T_{[T]_\varepsilon}$ is the linear transformation determine by matrix $[T]_\varepsilon$

Proposition: let $T: \mathbb{R} \to \mathbb{R}$ then, $\exists\ m \in \mathbb{R}$ s.t. $T(x) = mx\ \forall x \in \mathbb{R}$

Properties: $A \in M_{m \times n}(\mathbb{F})$, $T_A: \mathbb{F}^n \to \mathbb{F}^m$
$T: \mathbb{F}^n \to \mathbb{F}^m$

a) $T_{[T]_\varepsilon} = T$

b) $[T_A]_\varepsilon = A$

c) T is onto iff $\operatorname{rank}([T]_\varepsilon) = m$

d) T is one to one iff $\operatorname{rank}([T]_\varepsilon) = n$

Defn: Compositions of linear T

let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$, $T_2 : \mathbb{F}^m \to \mathbb{F}^p$

then $T_2 \circ T_1 : \mathbb{F}^n \to \mathbb{F}^p$

is : $(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$

Proposition: Standard M of composition.

$$[T_2 \circ T_1]_\mathcal{E} = [T_2]_\mathcal{E} [T_1]_\mathcal{E}$$

Defn: Identity transformation

$id_n : \mathbb{F}^n \to \mathbb{F}^n$ s.t. $id_n(\vec{x}) = \vec{x}$ $\forall \vec{x} \in \mathbb{F}^n$

is called identity transformation.

p.s. $T^p = T \circ T^{p-1}$

and $T^0 = id_n$.

$\hookrightarrow [T^p]_\mathcal{E} = ([T]_\mathcal{E})^p$

Chapter 6.

Defn: Determinant (basic) (invertibality)

If $A \in M_{1\times1}(\mathbb{F})$, $\det(A) = a_{11}$

If $A \in M_{2\times2}(\mathbb{F})$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

p.s. invertible if none zero.

Defn: Submatrix & Minor

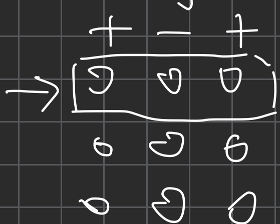
<u>SubM</u>, detoned $M_{ij}(A)$ is obtained by removing the ith row and jth column from A.

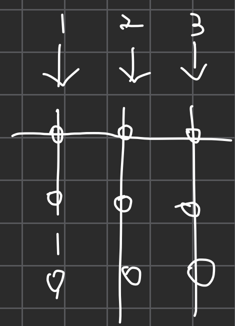
$(i,j)^{th}$ <u>Minor</u> of A, is the determinant of $M_{ij}(A)$.

Defn: Determinat of nxn

let $A \in M_{n\times n}(\mathbb{F})$ and $n \geq 2$

$$\det(A) = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det(M_{1j}(A))$$





ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$

$\det(A) = 1\cdot\det\left(\begin{bmatrix} 5 & 6 \\ 8 & 10 \end{bmatrix}\right) + -2\cdot\det\left(\begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix}\right) + 3\cdot\det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right)$

$= (50-48) + -2(40-42) + 3(32-35)$

$= 2 + 4 + (-9) = -3$

Prep: i^{th} Row Expansion of the determinant.

let $i \in \{1, 2, 3 \ldots n\}$ (把 I 改成 i)

$$\det(A) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij}(A))$$

基數(+)，偶數(-)

ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ 2nd row.

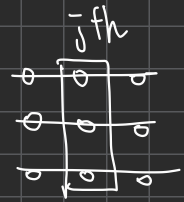
$$\det(A) = (-4)\det\left(\begin{bmatrix} 2 & 3 \\ 8 & 10 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 3 \\ 7 & 10 \end{bmatrix}\right) + (-6) \cdot \det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right)$$

$$= (-4)(20-24) + 5(10-21) + (-6)(8-14)$$

$$= 16 + (-55) + 36 = -3$$

Prep: j^{th} Column Expansion of determinant. (不一樣)

let $j \in \{1, 2, \ldots, n\}$ $(1 \Rightarrow i,\ j \Rightarrow \text{const})$



$$\det(A) = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(M_{ij}(A))$$

基(+)，偶(-)

ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ 3rd column.

$$\det(A) = 3 \cdot \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right) + (-6)\det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) + 10 \cdot \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\right)$$

$$= 3(32-35) + (-6)(8-14) + 10(5-8)$$

$$= (-9) + 36 + (-30) = -3$$

P.S. Determinants (easier to calculate)

(a) If A has a Row / Column only zeros, $\det A = 0$.

(b) If A is uppertriangle, $\det A = a_{11} a_{22} a_{33} a_{44} \ldots a_{nn}$

(c) $\det(I_n) = 1$

(d) $\det(A) = \det(A^T)$

Thm. Effects of ERO on determinant.

(1) If B is obtained from A by Row swap

then, $\det(B) = -\det(A)$

(2) If B is obtained from A by Row scale, m,

then $\det(B) = m\det(A) \longrightarrow \frac{1}{m}\det(B) = \det(A)$

(3) If B is obtained from A by Row addition

the, $\det(B) = \det(A) \Rightarrow$ no change.

P.S. If A same Row / column, $\det(A) = 0$

Determinants of elemantary Matricies.

(a) Row swap, $\det(E) = -1$

(b) Row scale, $\det(E) = M$

(c) Row addition, $\det(E) = 1$

(d) Suppose perform ERO on A to produce B

an the elementary Matix $= E$

$\det(B) = \det(E)\det(A)$

Since $B = E_k \cdots E_3 E_2 E_1 A$

$\Rightarrow \det(B) = \det(E_k) \cdots \det(E_2)\det(E_1)\det(A)$

Preposition:

① let $A, B \in M_{n \times n}(\mathbb{F})$

then, $\det(AB) = \det(A)\det(B)$

p.s. $\det(A+B) \neq \det(A) + \det(B)$

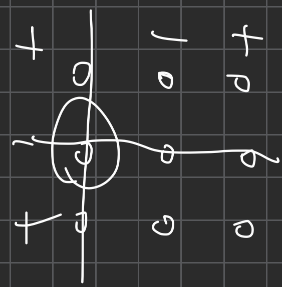
② $\det(AB) = \det(BA)$

③ $\det(A^{-1}) = 1/\det(A)$

Defn: Cofactor

The $(i,j)^{th}$ cofactor is:

$$C_{ij}(A) = (-1)^{i+j} \det(M_{ij}(A))$$



p.s. $\Rightarrow \det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}(A)$

Defn: Adjugate:

$\mathrm{adj}(A)$ is the $n \times n$ matrix whose $(i,j)^{th}$ entry is

$$(\mathrm{adj}(A))_{ij} = C_{ji}(A) = [C_{ij}(A)]^T$$

$\Rightarrow$ adjugate of A is the transpose of the matrix Cofactors of A

ex.: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$C_{11} = 1 \cdot \det(d) = d$ $\quad$ $C_{12} = -1 \cdot \det(c) = -c$

$C_{21} = -1 \cdot \det(b) = -b$ $\quad$ $C_{22} = 1 \cdot \det(a) = a$

$$\mathrm{adj}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thm $A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A) I_n$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Cramer's Rule

Consider $A\vec{x} = \vec{b}$ where $\det(A) \neq 0$

If we construct B_j from A by replacing the j^{th} column of A by $\vec{b}$ then soln to $A\vec{x} = \vec{b}$

is $x_j = \dfrac{\det(B_j)}{\det(A)}$ $\forall j = 1 \ldots n$.

Area of Parallelogram: (平行四边形)

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$A = \left| \det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \right) \right|$$

Chapter 7

Defn: Eigenvector, Eigenvalue, Eigenpair.

$A \in M_{n\times n}(\mathbb{F})$, A non-zero vector $\vec{x}$ is an eigenvector of A over $\mathbb{F}$ iff $\exists$ a scalar λ s.t.

$$A\vec{x} = \lambda\vec{x}$$

The scalar λ is called eigenvalue of A over $\mathbb{F}$ and the pair $(\lambda, \vec{x})$ is an eigenpair of A over $\mathbb{F}$

p.s. $A\vec{x} = \lambda\vec{x}$

$$\Rightarrow A\vec{x} - \lambda\vec{x} = \vec{0} \Rightarrow A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}.$$

Defn: eigenvalue equation / problem

$$\Rightarrow A\vec{x} = \lambda\vec{x} \quad \text{or} \quad (A - I\lambda)\vec{x} = \vec{0}$$

Defn: characteristic polynomial of A

$$C_A(\lambda) = \det(A - \lambda I) \quad (找\ \lambda)$$

charateristic equation:

$$C_A(\lambda) = 0.$$

proposition: A is invertible iff $\lambda = 0$ is Not eigenvalue.

Defn: trace

$$tr(A) = \sum_{i=1}^{n} a_{ii} \quad \text{(sum of diagonal entries)}$$

Features of characteristic polynomial.

let $A \in M_{n\times n}(\mathbb{F})$, $C_A(\lambda) = \det(A - \lambda I)$

$$C_A(\lambda) = c_n \lambda^n + c_{n-1}\lambda^{(n-1)} + \cdots + c_1\lambda + c_0$$

where

$$\begin{cases} a)\ c_n = (-1)^n \\ b)\ c_{n-1} = (-1)^{(n-1)}\, tr(A) \\ c)\ c_0 = \det(A) \end{cases}$$

* (Same but over $\mathbb{C}$):

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

where.

$$\begin{cases} a)\ c_{n-1} = (-1)^{(n-1)} \sum_{i=1}^{n} \lambda_i \\ b)\ c_0 = \prod^{n} \lambda_i. \end{cases}$$

P.S. $(\mathbb{C})$

a) $\sum_{i=1}^{n} \lambda_i = tr(A)$

b) $\prod_{i=1}^{n} \lambda_i = \det(A) \quad (c_0)$

Proposition: Linear combinations of Eigenvectors.

let $c, d \in \mathbb{F}$ and $(\lambda_1, \vec{x})$ and $(\lambda_1, \vec{y})$ are eigenpairs

If $c\vec{x} + d\vec{y} \neq 0$, then $(\lambda_1, c\vec{x} + d\vec{y})$ is ↗

Defn: eigenspace

the eigenspace associate with λ, denoted by.

$E_\lambda(A)$ is the solution set to $(A - \lambda I)\vec{x} = \vec{0}$

$\Rightarrow E_\lambda(A) = \text{Null}(A - \lambda I)$

Defn: Similar

We say that A is similar to B over $\mathbb{F}$ if

$\exists P \in M_{n \times n}(\mathbb{F})$ s.t. $A = PBP^{-1}$.

Defn: diagonalize

We say A is diagonalizable over $\mathbb{F}$ if it is similar to $\mathbb{F}$ over to a diagonal matrix D

if $\exists P$ s.t. $P^{-1}AP = D$ we say P diagonalize A

Chapter 8

Defn: Subspace

a subset V of $\mathbb{F}^n$ is called subspace of $\mathbb{F}^n$ if

$$\begin{cases} 1. & \vec{0} \in V \\ 2. & \forall \vec{x}, \vec{y} \in V, \quad \vec{x}+\vec{y} \in V \\ 3. & \forall \vec{x} \in V \text{ and } c \in \mathbb{F}, \quad c\vec{x} \in V \end{cases}$$

ex: $A \in M_{m \times n}(\mathbb{F})$, then

$\Rightarrow$ Null(A), Col(A), E_λ is subspace of $\mathbb{F}$

If $T: \mathbb{F}^n \to \mathbb{F}^m$, then,

$\Rightarrow$ Range(T), Ker(T) is subspace of $\mathbb{F}$

Defn: Subspace test

If V is a subset of $\mathbb{F}^n$, V is a subspace iff

$$\begin{cases} a) & V \text{ is non-empty} \\ b) & \forall \vec{x}, \vec{y} \in V \text{ and } c \in \mathbb{F}, \ c\vec{x}+\vec{y} \in V. \end{cases}$$

Defn: Linearly dependent

$\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ are linearly dependent if $\exists$ scalars $c_1, c_2 \dots c_k$ s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_k\vec{v}_k = \vec{0}$

If $U = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$ we say U is a linearly depencelt set.

Defn: Linearly independent

<u>$\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ are linearly dependent</u> if $\nexists$ scalars $c_1, c_2 \dots c_k$ s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_k\vec{v}_k = \vec{0}$

$\to$ if the solution to $c_1\vec{v}_1 + c_2\vec{v}_2 \dots c_k\vec{v}_k = \vec{0}$ is the trivial solution $\vec{0}$.

Defn: Basis

let V be a subspace of $\mathbb{F}^n$ and $B = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$

B is a Basis for V if

1. B is linearly independent.
2. $V = \text{span}(B)$

Proposition: Linear Dependence check

1. $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ are <u>linearly dependent</u>

   iff one of the vectors can be written as linear combination of other vectors.

2. $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ are <u>linearly independent</u>

   iff $c_1\vec{v}_1 + \dots c_k\vec{v}_k = \vec{0} \Rightarrow c_1 = \dots = c_k = 0$

P.S. let $S \subseteq \mathbb{F}^n$

1. if $\vec{0} \in S$, S is linearly dependent
2. if $S = \{\vec{x}\}$ contains only 1 vector, S is linearly dependent iff $\underline{\vec{x} = \vec{0}}$
3. if $S = \{\vec{x}, \vec{y}\}$ contains only 2 vectors, S is linearly dependent iff they are <u>scalar multiples</u>.

Proposition

$\in \mathbb{F}^n$

let $S = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$, $A = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_k]$

$\text{Rank}(A) = r$ and A has pivot columns $q_1, q_2, \dots q_r$

let $U = \{\vec{v}_{q_1}, \vec{v}_{q_2} \dots \vec{v}_{q_r}\}$ then,

a) S is linearly indenpendent iff $r = k$

b) U is linearly independent.

c) If $\vec{v}$ is in S but not U, the set $\{\vec{v}_{q_1}, \vec{v}_{q_2} \dots \vec{v}_{q_r}\}$ is linearly independence.

d) $\text{Span}(U) = \text{Span}(S)$

P.S. If $n < k$, S is linearly dependent (長 < 寬)

原來的 vector set

8.4

Thm: Every Subspace has a spanning set.

Let V be subspace of $\mathbb{F}^n$, $\exists\ \vec{v}_1 \dots \vec{v}_k \in V$ s.t. $V = \text{Span}\{\vec{v}\}$

Proposition: Span $\mathbb{F}^n$ iff rank $= n$

Let $S = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$ be a set of K vectors,

Let $A = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_k]$ be the matrix

Then, $\text{Span}(S) = \mathbb{F}^n$ iff $\text{rank}(K) = n$

8.5

Thm: Every Subspace has a basis.

Let V be subspace of $\mathbb{F}^n$, then V has a basis

Defn: Standard Basis

The set $\mathcal{E} = \{\vec{e}_1, \vec{e}_2 \dots \vec{e}_n\}$ is called standard Basis.

Proposition: Size of Basis for $\mathbb{F}^n$

Let $S = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\} \in \mathbb{F}^n$ so if S is a basis of $\mathbb{F}^n$,

then $K = n$.

Proposition: n Vectors Span iff independent

Let $S = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\} \in \mathbb{F}^n$ S independent $\Leftrightarrow$ $\text{Span}(S) = \mathbb{F}^n$

Thm: Let $S = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$ be a subset of $\mathbb{F}^n$

(a) If $\mathrm{Span}(S) = \mathbb{F}^n$ $\exists$ a subset B of S which is basis of $\mathbb{F}^n$

(b) If $\mathrm{Span}(S) \neq \mathbb{F}^n$ and S is linearly independent,
$\exists$ vectors $\vec{v}_{k+1}, \dots, \vec{v}_n$ in $\mathbb{F}^n$ s.t.
$B = \{\vec{v}_1 \dots \vec{v}_k, \vec{v}_{k+1} \dots \vec{v}_n\}$ is ~~Bass~~ Basis for $\mathbb{F}^n$.

8.6.

Proposition: Basis for $\mathrm{Col}(A)$

Let $A = [\vec{a}_1, \vec{a}_2 \dots \vec{a}_n] \in M_{m \times n}(\mathbb{F})$, and Suppose RREF(A) has $q_1, q_2 \dots q_r$ pivot columns, Then,

$$\{\vec{a}_{q_1}, \vec{a}_{q_2} \dots \vec{a}_{q_r}\} \text{ is basis for } \mathrm{Col}(A)$$

8.7.

Thm: Dimensioned. well defined

Let V be a subspace of $\mathbb{F}^n$ If $B = \{\vec{v}_1 \dots \vec{v}_k\}$ and $C = \{\vec{w}_1 \dots \vec{w}_\ell\}$ are both bases of V,
then $k = \ell$.

(vectors)

Defn: Dimensions: # of elements in a basis. $\Rightarrow \dim(V)$

p.s. for $\mathbb{F}^n$, $\dim(V) \leq n$

Propositions: Rank and Nul as Dimensions

(a) $\text{rank}(A) = \dim(\text{Col}(A))$

(b) $\text{nullity}(A) = \dim(\text{Nul}(A))$

Thm: $A \in M_{m \times n}(\mathbb{F})$

$$n = \text{rank}(A) + \text{nullity}(A)$$
$$= \dim(\text{Col}(A)) + \dim(\text{Null}(A))$$

Thm: Unique Representation Theorem. Coordinates

let $B = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$ be basis of $\mathbb{F}^n$, then, for every vector $\vec{v} \in \mathbb{F}^n$ $\exists$ unique scalars $c_1 \dots c_n \in \mathbb{F}^n$

s.t. $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_n\vec{v}_n$

We call $c_1 \dots c_n$ the coordinates of $\vec{v}$ with respect to B, or the B-coordinates of $\vec{v}$.

Defn: Ordered basis

An ordered basis for $\mathbb{F}^n$ is a basis $B = \{\vec{v}_1 \dots \vec{v}_n\}$ with a fix ordering.

P.S. a basis give rise to $n!$ ordered basis.

Defn: coordinate Vector

take the coordinates of a ordered basis $\Rightarrow [\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$

(a) $\forall \vec{u}, \vec{v} \in \mathbb{F}^n$, $[\vec{u}+\vec{v}]_B = [\vec{u}]_B + [\vec{v}]_B$

$\forall \vec{v} \in \mathbb{F}^n$, $[c\vec{v}]_B = c[\vec{v}]_B$

Defn. change of coordinate/basis matrix.

let $B = \{\vec{v}_1, \dots \vec{v}_n\}$, $C = \{\vec{w}_1 \dots \vec{w}_n\}$ be ordered basis of $\mathbb{F}^n$

The change-of-basis / coordinates from B to C

is the $n \times n$ matrix $\ni$ ${}_C[I]_B = [[\vec{v}_1]_C \dots [\vec{v}_n]_C]$

Similarly, from C to B

is the $n \times n$ matrix $\ni$ ${}_B[I]_C = [[\vec{w}_1]_B \dots [\vec{w}_n]_B]$

P.S. $[\vec{x}]_C = \underline{{}_C[I]_B}[\vec{x}]_B$ and $[\vec{x}]_B = {}_B[I]_C[\vec{x}]_C$

and $\ni$ $[\vec{x}]_C = \underline{{}_C[I]_\varepsilon}[\vec{x}]_\varepsilon$

and $\ni$ $\underline{{}_B[I]_C}\ \underline{{}_C[I]_B} = I_n$ and $\underline{{}_C[I]_B}\ \underline{{}_B[I]_C} = I_n$

(互相 inverse)