9/6 week 1 Lec 1

- What mathematics ?

→ a language we can describe the world.

Shortnotation

- $\forall$ → for all , $\exists$ → there exist , $\ni$ → such that
- $\therefore$ such that , wlog → without loss of generality
- $\in$ → is a element of , WSIC → why should I care ?

- In highschool, |abs| means get rid of negative.

However $\forall x \in \mathbb{R}, |-x| = x$ NO!!! $|-(-2)| \neq -2$

* Mathematic def of $|x|$

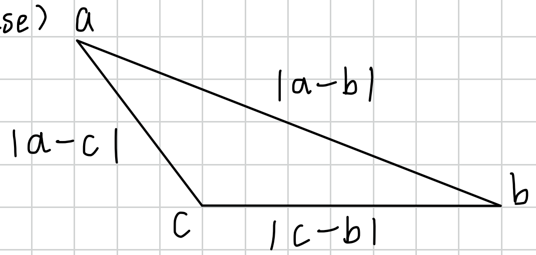
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- In words, the abs val gives the distance from 0

TA → in general, $\forall a, b \in \mathbb{R}$, the distance between a & b is $|b-a| = |a-b|$

* The triangle Inequality

(2D case)

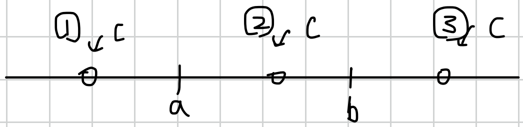


Thm 1 : The Δ Inequality

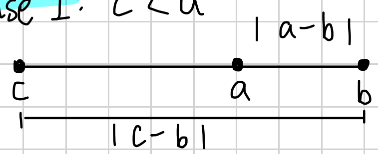
For $a, b, c \in \mathbb{R}$, we have

$$|a-b| \leq |a-c| + |c-b|$$

Pf : (for 1 dimensional case) Since $|a-b| = |b-a|$, assume wlog that $a < b$



Case 1: $c < a$

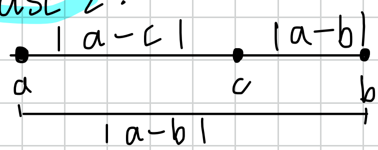


From the pic, $|a-b| < |c-b|$

Then, as $|a-c| > 0$, we have

$|a-b| \leq |c-b| + |a-c|$ ✓

Case 2:

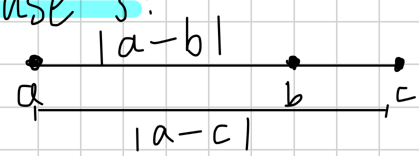


From the pic,

$|a-c| + |c-b| = |a-b|$

Thus, $|a-c| + |c-b| \leq |a-b|$

Case 3:



From the pic $|a-c| < |a-b$

Then as $|b-c| > 0$, we

have ✓

9/8 Week 1 lecture 2

* **Thm 2**: The triangle inequality 2.0

For $a, b \in \mathbb{R}$, we have that $|a+b| \leq |a| + |b|$

Pf: $|a+b| = |a-(-b)| \leq |a-0| + |0-(-b)|$
$\leq |a| + |b|$
Δ Inq 1, $c=0$

* **Interval vs set notation**

The largest set of value using is $\mathbb{R}$

- We also deals with subset of $\mathbb{R}$:

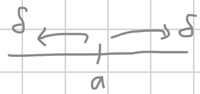
$1 < x < 2 \rightarrow x \in (1,2)$
$1 < x \leq 2 \rightarrow x \in (1,2]$

$-4 \leq x \leq -2$ or $\geq 3 \longrightarrow x \in [-4,-2] \cup [x, \infty)$

- Common Ineq used in 3 forms.

1. variable $x \in \mathbb{R}$
2. fixed point $a \in \mathbb{R}$
3. tolerance $\delta > 0 \in \mathbb{R}$

1) $|x-a| < \delta$ (think, all real num whose distance from a is $< \delta$)



$x \in (a-\delta, a+\delta)$

2) $|x-a| \leq \delta$

$x \in [a-\delta, a+\delta]$

3) $0 < |x-a| < \delta$

difference.

$x \in (a-\delta, a+\delta) / \{a\}$
or $\rightarrow x \in (a-\delta, a) \cup (a, a+\delta)$

Ex 1:

$|-2x+6| < 5$

$\rightarrow -5 < -2x+6 < 5$

$\rightarrow -11 < -2x < -1$

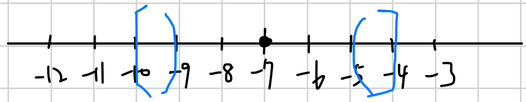
$\rightarrow \frac{11}{2} > x > \frac{1}{2}$

Ex 2:

$2 < |x+7| \leq 3$

① Rewrite it into distance form $2 < |x-(-7)| \leq 3$

So: all points whose distance from -7 is most equal to 3 or at least 2.



$\rightarrow x \in [-10, 9) \cup (-5, -4]$

Or:

②

$2 < |x+7| \leq 3$

$2 < |x+7|$ and $|x+7| \leq 3$

$\begin{cases} 2 < x+7, & x > -5 \\ 2 < -x-7, & x < -9 \end{cases}$ $\longleftrightarrow$ overlap

$\rightarrow -3 \leq x+7 \leq 3$

$\rightarrow -10 \leq x \leq -4$

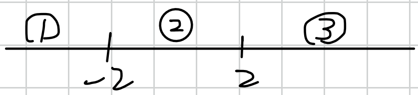
$\rightarrow$ The intersection of $x > -5$, $x < -9$ and $-10 \leq x \leq -4$

is $[-10, -9) \cup (-5, -4]$

Ex 3:

$\frac{|x+2|}{|x-2|} > 5$

$\rightarrow |x+2| > 5|x-2| \quad (x \neq 2)$



① $x < -2$ 不符合

$\rightarrow -(x+2) > 5(-(x-2))$

$\rightarrow x > 3$, no solution for $x < -2$, $x > 3$

② $-2 \leq x < 2$ overlap

$\rightarrow (x+2) > 5(-(x-2))$

$\rightarrow x > 4/3 \longrightarrow \frac{4}{3} < x < 2$

③ $x > 2$

$\rightarrow x+2 > 5(x-2)$

$\rightarrow x < 3 \longrightarrow 2 < x < 3$

$\Rightarrow x \in (\frac{4}{3}, 2) \cup (2, 3)$

9/11 Week lec 3

P.S. Eventually : functions. , Now sequences.

- A sequence is basically an order list.
  - ↳ discrete data (list of data points)

Many continuous process can be modeled with discrete data

Approximations are often done with sequences.

↳ We can approx solus to equs that can't be solved explicitly

The CN highlights how to approx sqrt root using sequences (p.14)

* We can express an infinite seq explicitly as:

$$\{\underbrace{a_1, a_2}_{\text{terms}}, \ldots\ldots, \underset{\uparrow \text{ index}}{a_n} \ldots\} \leftarrow \text{set}$$

OR $\{a_n\}_{n=1}^{\infty}$

OR $\{a_n\}$

Ex: $\{\frac{1}{n}\}_{n=1}^{\infty} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$

$\{n^2\}_{n=1}^{\infty} = \{1, 4, 9 \ldots\ldots\}$

$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1 \ldots\}$

* We can also express seqs recursively:

- $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ $(n \geq 3)$ (Fibonacci)
- $a_1 = 16$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{260}{a_n}\right)$ (Heron's, approx $\sqrt{260}$)

* We revisit $\{\frac{1}{n}\}$ and $\{(-1)^n\}$

What's happening at large n?

→ $\frac{1}{n}$ gets closer and closer to 0

→ $(-1)^n$ oscillates forever.

What's a sequence approaching, if anything?

* Terminology

- We can extract infinitely many terms of a given sequence to build a new sequence.

Ex: $\{\frac{1}{n}\}_{n=1}^{\infty}$ → take every odd term → $\{\frac{1}{1}, \frac{1}{3}, \frac{1}{5} \dots\}$

→ $\{a_{2k+1}\} = \{\frac{1}{2k+1}\}_{k=0}^{\infty}$

- We have extracted a subsequence.

Definition:

Let $\{a_n\}$ be a sequence. Let $\{n_1, n_2, n_3 \dots n_k \dots\}$ be a sequence of natural numbers $n_1 < n_2 < n_3 \dots n_k < \dots$

Then $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, \dots a_{n_k}\}$ is a subsequence of $\{a_n\}$.

* We care about $\{a_n\}$ at large n. So we are interested in the subsequence which is all terms of $\{a_n\}$ beyond some cutoff term a_k

Definition of tail:

Let $\{a_n\}$ be a sequence. Let $k \in \mathbb{N}$. Then, the subsequence $\{a_k, a_{k+1}, a_{k+2} \dots\}$ is called the tail of $\{a_n\}$ with cutoff k.

* If a sequence converges / approaches to some fixed value L as index n becomes large, we say the sequence is convergent with limit L

We have to be careful. It isn't enough to say a limit is a value we approach at large n.

× [Ex. $\{\frac{1}{n}\}$ approaches -1 too, so would also consider a limit?]

* The correct way:

"L is the limit of $\{a_n\}$ if as n gets larger, a_n gets infinitely close to L."

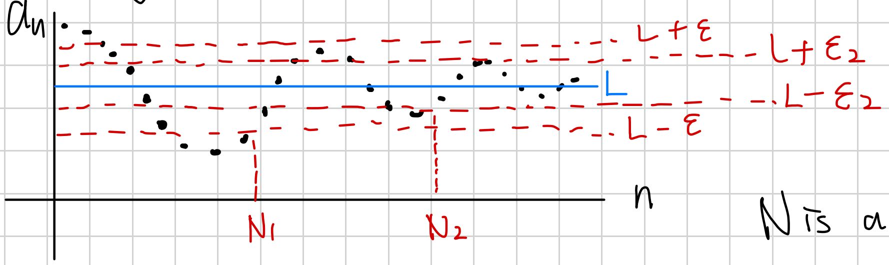
* For any positive tolerance $\varepsilon > 0$ given, we can find an $N \in \mathbb{N}$ such that a_n approximates L with an error less than ε for $n \geq N$.

* Formal def of the limit of a Sequence

- L is the limit of $\{a_n\}$ (as $n \to \infty$)

  if for every $\varepsilon > 0$ $\exists N \in \mathbb{N} \ni$ if $n \geq N$ then $|a_n - L| < \varepsilon$

Pictorially:



N is a cutoff

For smaller ε, we need to move to larger n to be within the error band.

(The Game to think abt it)

Evil Mathematician vs You

$\downarrow$ $\varepsilon > 0$

$\downarrow$ return an $N \in \mathbb{N}$ such that for $n \geq N$ $|a_n - L| < \varepsilon$

This game keeps repeating

To win: Return N that is dependent on ε

Week 2, lecture 4

* Definition of convergent & divergent.

- If an L exist such that the formal def of a limit holds, than $\{a_n\}$ is convergent.

$$\left[\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L\right]$$

Else, sequence is divergent.

Ex: show $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, using the formal definition.

a) $\varepsilon = \frac{1}{1000}$ b) $\varepsilon > 0$

a):

Rough work (as, aside)

We must find an $N \in \mathbb{N} \ni$ for $n \geq N$

$$|a_n - 0| \leq \frac{1}{1000}$$

$$\left|\frac{1}{\sqrt[3]{n}} - 0\right| < \frac{1}{1000}$$

drop |abs| since $n \in \mathbb{N}$

$$\to \frac{1}{\sqrt[3]{n}} - 0 < \frac{1}{1000}$$

$$\to 1000 > \sqrt[3]{n}$$

$$\to n > 1000000000$$

這邊+1是為了讓 $n \geq N$ $(n > 1000000000)$

Pf / solution:

So, given $\varepsilon = \frac{1}{1000}$, Let $N = 1000000001$

Then, for $n \geq N$, we have

$$|a_n - 0| = \left|\frac{1}{\sqrt[3]{n}}\right| = \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \frac{1}{\sqrt[3]{1000000000}} = \frac{1}{1000}$$

$\therefore |a_n - 0| < \varepsilon$ #

Ex:

Show $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, using the formal definition.

a) $\varepsilon = \frac{1}{1000}$ b) $\varepsilon > 0$

b):

We must find an $N \in \mathbb{N} \ni$ for $n \geq N$

Rough work

$|a_n - 0| < \varepsilon$

$\to \left|\frac{1}{\sqrt[3]{n}}\right| < \varepsilon$ since $n \in \mathbb{N}$

$\to \frac{1}{\sqrt[3]{n}} < \varepsilon$

$\to \frac{1}{\varepsilon} < \sqrt[3]{n}$

$\to \frac{1}{\varepsilon^3} < n$ ← 反正把 n 變回原樣

這樣寫是因為 $n > \frac{1}{\varepsilon^3}$

Pf / solution:

Given $\varepsilon > 0$, let $N > \frac{1}{\varepsilon^3}$, $N \in \mathbb{N}$

Then, for $n \geq N$ we have $|a_n - 0| = \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}}$

$= \frac{1}{1/\varepsilon} = \varepsilon$

$\Rightarrow |a_n - 0| < \varepsilon \quad \forall \varepsilon > 0$ #

Pf $\lim_{n\to\infty}$ using ε

we need find $N \in \mathbb{N}$ for which $n \geq N$

$|a_n - 0| < \varepsilon$

$\left|\frac{1}{\sqrt[3]{n}}\right| < \varepsilon \rightarrow n \in \mathbb{N}$

$\frac{1}{\sqrt[3]{n}} < \varepsilon$

$\frac{1}{\varepsilon} < \sqrt[3]{n}$

$\frac{1}{\varepsilon^3} < n$

Another way of saying $n \geq N$

$N = 1000000001$, (From last page)

Given $\varepsilon > 0$, let $N > \frac{1}{\varepsilon^3}$, $N \in \mathbb{N}$ for $n \geq N$ we have $|a_n - 0| = \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}}$

$< \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}} = \varepsilon$

$\Rightarrow |a_n - 0| < \varepsilon \quad \forall \varepsilon > 0.$

Ex 2:

Show $\lim_{n\to\infty} \dfrac{3n^2+2n}{4n^2+n+1} = \dfrac{3}{4}$, using the formal def.

We need $N \in \mathbb{N}$ such $\underbrace{n \geq N}_{\text{beyond that cutoff}}$

Rough work

$$|a_n - \tfrac{3}{4}| < \varepsilon$$

$$\to \left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| < \varepsilon$$

$$\to \left| \frac{4(3n^2+2n) - 3(4n^2+n+1)}{4(4n^2+n+1)} \right| < \varepsilon$$

$$\to \left| \frac{5n-3}{16n^2+4n+4} \right| < \varepsilon$$

$\longrightarrow$ now note:

$$\frac{5n-3}{16n^2+4n+4} < \frac{5n}{16n^2+4n+4} < \frac{5n}{16n^2+4n} < \frac{5n}{16n^2} = \frac{5}{16n}$$

$\longrightarrow$ So, we want $\dfrac{5}{16n} < \varepsilon \Rightarrow \dfrac{5}{16\varepsilon} < n$

$$\frac{3(4n^2+1) - 4(3n^2+2n)}{4(4n^2+n+1)}$$

$$\frac{\cancel{12n^2}+3-\cancel{12n^2}+8n}{16n^2+4n+4} = \frac{8n+3}{16n^2+4n+4}$$

(反正化簡到 n 就對了)

$\langle n \geq N$

Given $\varepsilon > 0$, $N > \dfrac{5}{16\varepsilon}$, $N \in \mathbb{N}$

$|a_n - \tfrac{3}{4}| < \dfrac{5}{16n} <$

Pf/ solution:

Given $\varepsilon > 0$, let $N > \dfrac{5}{16\varepsilon}$, $N \in \mathbb{N}$

Then, for $n \geq N$, we have

$$\left|a_n - \frac{3}{4}\right| = \left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| = \left| \frac{5n-3}{16n^2+4n+14} \right| \leq \left| \frac{5n}{16n^2} \right|$$

$$= \frac{5}{16n} \leq \frac{5}{16N} < \frac{5}{16\left(\frac{5}{16\varepsilon}\right)} = \varepsilon$$

$\Rightarrow |a_n - \tfrac{3}{4}| < \varepsilon$, $\forall \varepsilon > 0$ #

cuz $n \geq N$, 要想辦法變回 ε 才會 $|a_n - L| < \varepsilon$

Thus, $\lim_{n\to\infty} \dfrac{3n^2+2n}{4n^2+n+1} = \dfrac{3}{4}$

Note that as we have seen $|a_n - L| < \varepsilon$ $\longrightarrow -\varepsilon < a_n - L < \varepsilon$

gives $a_n \in (L-\varepsilon, L+\varepsilon)$

Remember that $\{a_n\}$ with index $a_n \geq N$ is a tail

Next page

Formal definiton 2.0

→ $\lim_{n\to\infty} a_n = L$ if $\forall\, \varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$

Further, for any interval (a,b) containing L,
we can find an $\varepsilon > 0$ small enough that

$$L \in (L-\varepsilon, L+\varepsilon) \subseteq (a,b)$$

So, any interval (a,b) containing L has a tail of $\{a_n\}$

Thm 3:

These are equivilant:

1. $\lim_{n\to\infty} a_n = L$
2. Every interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$
3. Every interval $(L-\varepsilon, L+\varepsilon)$ contains all but finetly many terms of $\{a_n\}$
4. Every interval (a,b) containing L contains a tail of $\{a_n\}$
5. Every interval (a,b) containing L contains all but finitely many terms of $\{a_n\}$

* change finitely many terms of a sequence $\{a_n\}$ does not change the convergence.

Now, uniqueness of lim

Ex: $\{(-1)^n\}$

↳ Neither -1 nor 1 can be limits.

↳ We can make a interval around either which excluders the other.

↳ Ex: $(1.5, -0.5)$ → contains -1 (we claim $L=-1$)

↳ every tail includes the term 1 ⟶

→ 1 is not in this interval
↳ this interval does not contain a tail

↳ Can there be another L?
↳ Suppose so, let's say given $\varepsilon = \frac{3}{4}$
→ Then we must have that $(L - \frac{3}{4}, L + \frac{3}{4})$ contains a tail
↳ so 1 must be in this interval
↳ $|1 - L| < \frac{3}{4} \Rightarrow L \in (\frac{1}{4}, \frac{7}{4})$
↳ So -1 must be

Week 2, lecture 5

Thm 4: Uniqueness of limits of sequence → 基本上就是在說一個 seq 只有一個 lim

If the sequence $\{a_n\}$ has limit L, it is unique

Pf: (By contridiction)

Suppose $\{a_n\}$ has 2 different lims, L & M

wlog $L < M$

s L lies in between

Consider that $L \in (L-1, \frac{L+M}{2})$ and $M \in (\frac{M+L}{2}, M+1)$

with these intervals disjoint. ← no overlap

We note that by definition each of these intervals contains a tail of $\{a_n\}$ with infinitly many terms.

- Since there infinity many terms, eventually at some index at least one term lies in both intervals.

→ Which is impossible, (interval disjoint → contradiction ⇒ $L=M$)

只有一個

Proposition 5:

If $a_n \geq 0$ $\forall n \in \mathbb{N}$ in $\{a_n\}$, then $\lim_{n\to\infty} a_n = L \geq 0$

Pf: 基本上就是如果 $a_n \geq 0$, $L \geq 0$ (不會小於 0)

Assume $L<0$. Consider $L \in (L-1, L/2)$

This interval is strictly negative, which means no term of $\{a_n\}$ are included, so no tail.

A contradiction. Thus $L<0$ is not a limit of $\{a_n\}$ #

Now : $\{n^2\} \to$ this grows without bound.

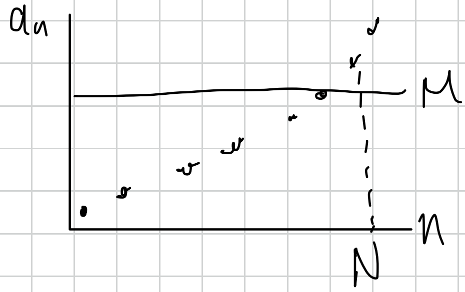
We denote such a situation as $\lim_{n\to\infty} a_n = \infty$ ← not a number → diverges.

Divergence to $+\infty$

$\lim_{n\to\infty} a_n = \infty$

No matter what number evil Mathematician gives

- if $\forall M > 0 \ \exists N \in \mathbb{N} \ni$ for $n \geq N$, $a_n > M$
- every interval (M, ∞) contains a tail of $\{a_n\}$



You can always find a bigger tail.

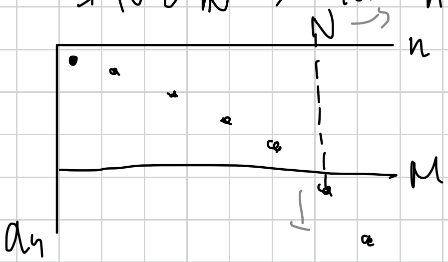
Pri: $\forall M < 0 \ \exists N \in \mathbb{N}$ $\ni$ for $n > N$, $a_n < M$

every interval $(-\infty, M)$ contains a tail of $\{a_n\}$

Divergence to $-\infty$

$\lim_{n\to\infty} a_n = -\infty$

- if $\forall M < 0 \ \exists N \in \mathbb{N} \ni$ for $n \geq N$ $a_n < M$



You can always find a small n

- every interval $(-\infty, M)$ contains a tail of $\{a_n\}$

$\lim_{n\to\infty} n^3 = \infty$

Rough W:

We want that given $\forall M > 0, \exists$

$a_n > M$

$n^3 > M$

$n > \sqrt[3]{M}$

$\forall M > 0$, $N > \sqrt[3]{M}$, $N \in \mathbb{N}$

Then, for $n > N$, we have

$a_n = n^3 > N^3 > (\sqrt[3]{M})^3 = M$

$\therefore a_n > M$, $\forall M > 0$, when $n > N$

So, $\lim_{n\to\infty} n^3 = \infty$

Ex: Show $\lim_{n\to\infty} n^3 = \infty$

這邊M基本上取代了ε的作用

Aside: Rough work

For a given $M > 0$, we want $N \in \mathbb{N} \ni$

$a_n > M$

$n^3 > M$

$n > \sqrt[3]{M}$

一樣先把n化到最簡

Pf.:

Given $M > 0$, let $N > \sqrt[3]{M}$, $N \in \mathbb{N}$

Then, for $n \geq N$, we have

$a_n = n^3 \geq N^3 > (\sqrt[3]{M})^3 = M$

$\therefore a_n > M$, $\forall M > 0$, when $n \geq N$

So $\lim_{n\to\infty} n^3 = \infty$ #

Thm 6:

i) If $\alpha > 0$ then $\lim_{n\to\infty} n^{\alpha} = \infty$ divergent

ii) If $\alpha < 0$ then $\lim_{n\to\infty} n^{\alpha} = 0$ convergent

Thm 7: Arithmetic Rules for sequence limits.

Let $\{a_n\}\{b_n\}$ be sequences with $\lim_{n\to\infty} a_n = L \in \mathbb{R}$

and $\lim_{n\to\infty} b_n = M \in \mathbb{R}$

Then:

* To use these, the lim must exist.

1) If $a_n = c \in \mathbb{R}\ \forall n$, then $L = c$.
2) For $c \in \mathbb{R}$, $\lim_{n\to\infty} c a_n = cL$ Constant 直接乘
3) $\lim_{n\to\infty} (a_n + b_n) = L + M$ Seq 相加 = lim 相加
4) $\lim_{n\to\infty} a_n b_n = LM$ 先後相乘沒差
5) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$ 先後相除沒差
6) If $a_n \geq 0\ \forall n$ and $\alpha > 0$ then $\lim_{n\to\infty} a_n^{\alpha} = L^{\alpha}$ 次方先後都一樣
7) For any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$. yep

We want $|(a_n+b_n) - (L+M)| < \varepsilon$

$|(a_n - L) + (b_n - M)| < \varepsilon$

$|(a_n - L)| < \varepsilon_1 \quad |(b_n - M)| < \varepsilon_2 \quad$ let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$

for cutoffs $N_1 \& N_2 \in \mathbb{N}$, pick $N = \max\{N_1, N_2\}$

$\triangle$ Ineq $|(a_n - L) + (b_n - M)| \leq |(a_n - L)| + |(b_n - M)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$

Thus, for $N = \max\{N_1, N_2\}$, we have for $n \geq N$

$|(a_n + b_n) - (L+M)| < \varepsilon \quad \forall \varepsilon > 0$

Thus, $\lim_{n\to\infty} (a_n + b_n) = L + M$

Pf: of No.3 (previous page) (seq 相加 = lim 相加)

let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$ & $\lim_{n\to\infty} b_n = M$, we know by defn that $\forall \varepsilon_1, \varepsilon_2 > 0$ we have

$$|a_n - L| < \varepsilon_1 \quad \& \quad |b_n - M| < \varepsilon_2$$

Aside) Rough work:

We want $|(a_n+b_n) - (L+M)| < \varepsilon \ \forall \varepsilon > 0$

$|(a_n - L) + (b_n - M)| < \varepsilon$

By Δ Ineq,

$|(a_n-L) + (b_n-M)| \leq |a_n - L| + |b_n - M|$

We have freedom, we choose

$\varepsilon_1 = \varepsilon_2 = \varepsilon/2$

因 → Let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$

for cutoffs N_1 & $N_2 \in \mathbb{N}$.

Take $N = \max\{N_1, N_2\}$

Now, examine:

$|(a_n+b_n) - (L+M)| \leq |a_n - L| + |b_n - M|$

果 → $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus, for $N = \max\{N_1, N_2\}$ we have for $n \geq N$

$|(a_n+b_n) - (L+M)| < \varepsilon, \ \forall \varepsilon > 0$

Thus, $\lim_{n\to\infty} (a_n + b_n) = L + M$ #

Thm 8: 假如分母是的 seq $=0$，分子的 $\lim = 0$

Assume $\{a_n\}$ & $\{b_n\}$ are sequences, and that $\lim_{n\to\infty} b_n = 0$

Assume $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists. Then, $\lim_{n\to\infty} a_n = 0$.

Pf:

let $\lim_{n\to\infty} \frac{a_n}{b_n} = k \in \mathbb{R}$, Since $a_n = \frac{a_n}{b_n} \cdot b_n$

相乘=相乘

arith rule #4 shows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot b_n$$
$$= \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \lim_{n\to\infty} b_n$$
$$= (k \cdot 0) = 0$$

Ex 1 正常解法

highest power

$$\lim_{n\to\infty} \frac{3n^2+2n}{4n^2+2n+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{4+\frac{1}{n}+\frac{1}{n^2}}$$

代入

arith rules

$$= \frac{3+0}{4+0+0} = \frac{3}{4}$$

注意：是要乘分子&分母各自最大次方

* There is a generalization / shortcut for that kind of limit

in Coursenote p36, Ex: 13. , P.S. can't use in quiz

Ex 2:

$$\lim_{n\to\infty} \sqrt{n^2+n} - n \rightarrow \text{looks like "}\infty - \infty\text{"}$$

→ Multiply by conjugate.

$$= \lim_{n\to\infty} \sqrt{n^2+n} - n \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} \begin{matrix}\times\frac{1}{n}\\ \times\frac{1}{n}\end{matrix}$$

plug in

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}+\frac{n}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

Ex: 3

$a_1 = 16$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{260}{a_n}\right)$. Say that we know $\lim\limits_{n\to\infty} a_n = L$

Then, using arith rule 7#, $\lim\limits_{n\to\infty} a_{n+1} = L$

So, $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \frac{1}{2}\left(a_n + \frac{260}{a_n}\right)$

$$L = \frac{1}{2}\left(L + \frac{260}{L}\right)$$

$$L^2 = \frac{1}{2}L^2 + 130$$

$$\frac{1}{2}L^2 = 130$$

$$L^2 = 260$$

$$L = \sqrt{260}$$

$+\sqrt{260}$ or $-\sqrt{260}$?

Prop #5, $\lim\limits_{n\to\infty} a_n > 0$, $L > 0$

Then, we have $L \geq 0$

$\therefore L = \sqrt{260}$

Week 3 lecture 6

Collary to thm 8:

Assume $\{a_n\}$ & $\{b_n\}$ are sequences.

If $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} a_n \neq 0$ then, $\lim_{n\to\infty} \frac{a_n}{b_n}$ DNE

- More tools of looking for convergence of seq

Let's look at $\left\{\frac{\sin n}{n}\right\}$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin(n)}{n} \overset{?}{=} \lim_{n\to\infty} \sin(n) \cdot \cancel{\lim_{n\to\infty}\left(\frac{1}{n}\right)}$ 0

(No, cuz both lim have to exist & lim sin(n) oscillates (diverges))

what else can we do?

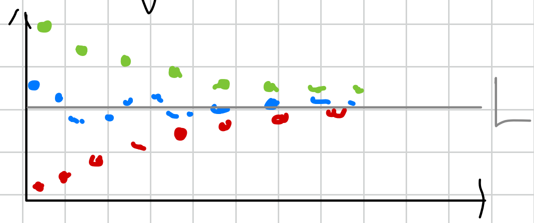
Thm 9: Squeeze thm / Sandwich thm

or eventually.

Assume $a_n \le b_n < c_n \ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \boxed{L} = \lim_{n\to\infty} c_n$

Then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$

Pictionally,



Pf: Since $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then for $\varepsilon > 0$ given, we can find $N \in \mathbb{N} \ni$ for $n \ge N$ we have $a_n \in (L-\varepsilon, L+\varepsilon)$ and $c_n \in (L-\varepsilon, L+\varepsilon)$.

- That is, for $n \ge N$, $L-\varepsilon < a_n \le b_n \le c_n < L+\varepsilon$
- This means $b_n \in (L-\varepsilon, L+\varepsilon)$ for $n \ge N$ $\forall \varepsilon > 0$
- Thus $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ ∎

Ex 1:

$$\lim_{n\to\infty} \frac{\sin(n)}{n}$$

note that $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

and we know $\lim_{n\to\infty} \pm\frac{1}{n} = 0$

then by squeeze thm, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

Ex 2:

$$\lim_{n\to\infty} \frac{4+(-1)^n}{n^3+n^2-1}$$

bounce between (-1, 1)

note: $\frac{3}{n^3+n^2-1} \leq \frac{4+(-1)^n}{n^3+n^2-1} \leq \frac{5}{n^3+n^2-1}$

And $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} C_n$

Thus, by Sqz thm, $\lim_{n\to\infty} b_n = 0$

* For a special class of sequences we can state a nice result.

First terminology:

Defn: we say that a sequence $\{a_n\}$ is $(\forall n \in \mathbb{N})$

- Increasing if $a_n < a_{n+1}$
- non-decreasing if $a_n \leq a_{n+1}$
- decreasing if $a_n > a_{n+1}$
- non-increasing if $a_n \geq a_{n+1}$
- monotonic if non-dec or non-inc

More Def:

- Let $S \subseteq \mathbb{R}$. We say that α is an upper bound of S if $x \leq \alpha \ \forall x \in S$

  If $\exists$ an upper bound, we say S is bounded above.
- We say β is a lower bound of S if $x \geq \beta \ \forall x \in S$.

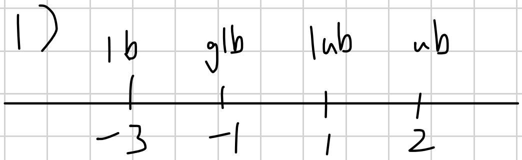
  If $\exists$ a lower bound, we say S is bounded below
- We say S is bounded if it is bounded above and below.

  That is, $\exists M \in \mathbb{R} \ni S \subseteq [-M, M]$

Even more Def:

- Let $S \in \mathbb{R}$. We say α is the least upper bound (lub or supremum) of S if α is the smallest upper bound of S. That is, if $x \leq \gamma \ \forall x \in S$, then $\alpha \leq \gamma$.
- We say β is the greatest lower bound (glb or infimum) of S if β is the largest lower bound of S. That is, if $x \geq \gamma$, $\forall x \in S$, then $\beta \geq \gamma$

Ex: $[-1, 1)$



Now, a key Axiom:

Axiom 10:

Let $S \in \mathbb{R}$ be non-empty & bounded above / below

Then S has a lub / glb.

Week 3 Lecture 7

Thm II Monotone Convergence thm (MCT)

Let $\{a_n\}$ be a ~~non-decreasing~~ sequence.

1) If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = \text{lub}(\{a_n\})$

2) If $\{a_n\}$ is not bounded above, then $\{a_n\}$ diverges to ∞. [$\{a_n\}$ converges $\Leftrightarrow$ bounded above]

Note: similar statement for non-increasing sequences

↳ replace with glb & $-\infty$

Pf:

1) Let $\{a_n\}$ be non-decreasing. Let $\text{lub}\{a_n\} = L$.

Let us be given $\varepsilon > 0$. Then $L - \varepsilon < L$.

Thus, $L - \varepsilon$ cannot be an upper bound for $\{a_n\}$.

Then $\exists N \in \mathbb{N} \ni a_N > L - \varepsilon$. Then, for $n \geq N$

we have $a_n \geq a_N > L - \varepsilon$. But since $L = \text{lub}(\{a_n\})$

we get $L \geq a_n \geq a_N > L - \varepsilon$. Also, note $L + \varepsilon > L$

Thus, $L + \varepsilon > L \geq a_n \geq a_N > L - \varepsilon$.

→ That is, for $n \geq N$, $a_n \in (L - \varepsilon, L + \varepsilon)$ $\forall \varepsilon > 0$

Thus $\lim\limits_{n \to \infty} a_n = L$

$\Rightarrow$) Let a_n be non-decreasing. Let $\{a_n\}$ be not bounded above. Let us be given $M \in \mathbb{R}^+$.
Note that M is not an upper bound for $\{a_n\}$
Thus $\exists\, N \in \mathbb{N} \ni a_N > M$. Thus, for $n \geq N$ we have $a_n \geq a_N > M \quad \forall M > 0$.
$\rightarrow$ Thus, by definition, $\lim_{n\to\infty} = \infty$ ■

MCT:

- To utilize MCT, we need a new proof technique:

<u>Induction</u>: * useful for proofs in recursive situations. (M135)

① Prove a base case true. ($n=1$)
② Make an induction hypothesis (IH) ($n=k$ for some $\boxed{k \geq 1}$)
③ Use the IH to show the next step is true ($n=k+1$)
Then, we have shown the claim to be true $\forall n \in \mathbb{N}$

- To use MCT on recursive sequences, we will..

① Prove the sequence is monotonic. $\boxed{\text{non-dec or non-Inc}}$ therefore monotonic
② Prove the sequence is bounded (above or below)] Induction.
③ Conclude converge by MCT
④ Use lim law #7 $\lim_{n\to\infty} a_{n+1} = L$ to find L

ex: next page $\rightarrow$

Ex 1

Let $a_n = 1$ and $a_{n+1} = \sqrt{3+2a_n}$, Show converge, find L

First, prove monotonic ; the $\boxed{\text{guess}}$ is non-decreasing

Base Case :

$a_1 = 1$, $a_2 = \sqrt{3+2} = \sqrt{5} \geq 1$ (term 2 is > term 1)

$\boxed{\therefore a_2 \geq a_1}$

Inductive Hypothesis :

Assume $a_k \leq a_{k+1}$ for some $k \geq 1$. → next, form the original Seq.

Then, $2a_k \leq 2a_{k+1}$

Then, $3+2a_k \leq 3+2a_{k+1}$

Then, $\sqrt{3+2a_k} \leq \sqrt{3+2a_{k+1}}$ $\boxed{\text{That is, } a_{k+1} \leq a_{k+2}}$

Thus, non, decreasing. → (show upper bound)

Thus, $\boxed{\text{monotonic by induction}}$

Now, lets proof that we are bounded above.

I guess bounded above by 7. → Just a random ass number.

→ next page

Base Case:

$a_1 = 1 \leq 7 \quad \therefore a_1 \leq 7$

Inductive Hypothesis

Assume $a_k \leq 7$ for some $k \geq 1$

Then, $2a_k \leq 14$

Then, $3+2a_k < 17$,

Then, $\sqrt{3+2a_k} \leq \sqrt{17}$

- Thus, $\boxed{a_{k+1} \leq \sqrt{17} \leq 7}$

- Thus, bounded above by 7, $\boxed{\text{by Induction}}$

* Thus, by MCT, the sequences converges

Now, $\lim_{n\to\infty} a_n = L$

Then, by lim laws, $\lim_{n\to\infty} a_{n+1} = L$.

So, we have $\lim_{n\to\infty} \sqrt{3+2a_n} = L$

$\rightarrow \sqrt{3+2L} = L$

$3+2L = L^2$

$L^2 - 2L - 3 = 0$

$(L-3)(L+1) = 0 \Rightarrow L = \cancel{-1}, 3$

Limit uniqueness so only one limit

We know $a_n = 1$ & $\{a_n\}$ is non-decreasing, so $L \neq -1$.

$\boxed{\therefore L = 3}$

Ex 2:

$a_1 = 4$, $a_{n+1} = \frac{7+a_n}{22}$. Show converge & find L.

First, prove monotonicity. Guess, non-increasing.

Base Case:

$a_1 = 4$, $a_2 = \frac{7+4}{22} = \frac{1}{2} \leq a_1$

$\therefore a_2 < a_1$

Induction Hypothesis: → we need to prove $a_{k+1} \geq a_{k+2}$

Assume $a_k \geq a_{k+1}$ for some $k \geq 1$

Then $7 + a_k \geq 7 + a_{k+1}$

Then $\frac{7+a_k}{22} \geq \frac{7+a_{k+1}}{22}$

That is, $a_{k+1} \geq a_{k+2}$

Alert, be careful for sequence manipulations. If you have to invert fraction, you may need to show you're bounded below by zero. (non-negative)

- Thus, non-increasing, Thus monotonic by induction.

Now, proof bounded below, Guess bounded below by -10

Base Case:

$a_1 = 4 \geq -10$.

$\therefore a_1 \geq -10$.

Induction Hypothesis:

Assume $a_k \geq -10$ for some $k \geq 1$

Then, $7 + a_k \geq -3$

Then, $\frac{7+a_k}{22} \geq \frac{-3}{22}$

$a_{k+1} \geq \frac{-3}{22} \geq -10$.

Thus, bounded below by -10, by induction.

Thus, by MCT, this sequence converges

→ next page

Now, $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$

So, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{7+a_n}{22} = L$

$$\frac{7+L}{22} = L$$

$$7+L = 22L$$

$$L = \frac{7}{21} = \frac{1}{3} \quad (\text{glb})$$

Week 3 lecture 8.

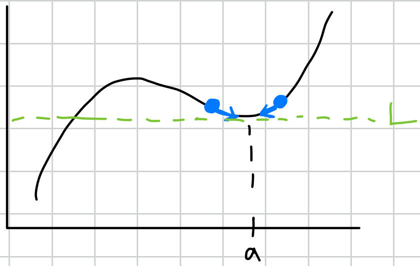
It's finally time for the functions Quiz 4 not 3

We are interested in the limits of functions.

That is, $\lim_{x \to a} f(x) = L$

In words,

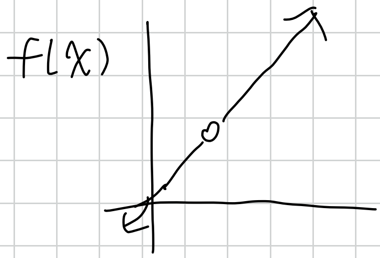
As x get infinitely closer to a without ever reaching a, $f(x)$ gets infinitely closer to L.

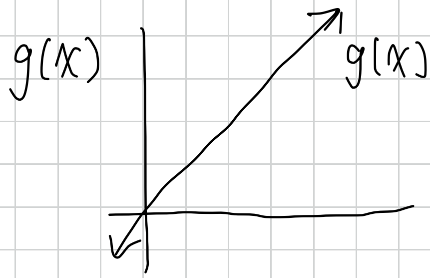


A quick cautionary tale:

Examine $f(x) = \frac{x^2-3x+2}{x-1}$. we would like it if we could say this is the same function as $g(x) = x-2$

But it's not





However $\lim_{x \to 1} f(x) = \lim_{x \to x} g(x)$

↳ note we approach but never reach $x=1$.

Formal Defn of the limit of a function.

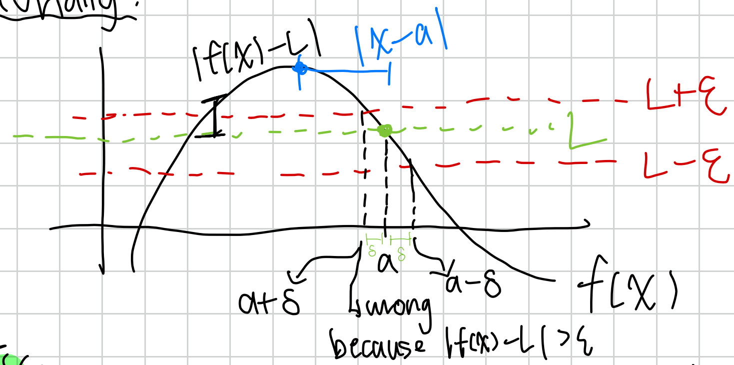
Let $f: \mathbb{R} \to \mathbb{R}$ be a function & $a \in \mathbb{R}$. Then,

$$\lim_{x \to a} f(x) = L$$ approach, never touch a.

if $\forall \varepsilon > 0, \exists \delta > 0 \ni$ if $0 < |x-a| < \delta$ ← delta → cut off difference

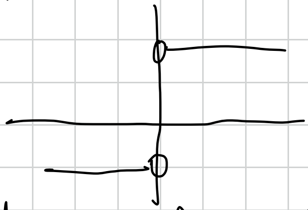
then $|f(x) - L| < \varepsilon$
↖ allowable error / tolerance.

Pictorially:



Ex:

$$f(x) = \begin{cases} 3 & \text{if } x > 0 \\ -2 & \text{if } x < 0 \end{cases}$$



- Examine $\lim_{x \to 0} f(x)$. Approach -2 from the left, 3 from left

Try to show the lim DNE

Pf: Suppose $\lim_{x \to 0} f(x) = L$. Say we are given $\varepsilon = 1$

(assume limit exist) (choose a ε)

From the formal definition we should find $\delta > 0$

$\Rightarrow$ for $0 < |x - 0| < \delta$ we got $|f(x) - L| < 1$
↖ a ↖ the ε that we chose

↓

$x \in (-\delta, 0) \cup (0, \delta)$

First, lets look at $x \in (-\delta, 0)$: think with distance

Here, $f(x) = -2$, Then $|-2 - L| < 1$. That is $L \in (-3, -1)$

Next, lets look at $x \in (0, \delta)$

Here, for $f(x) = 3$. Then $|3 - L| < 1$. That is $L \in (2, 4)$

These interval disjoint. Contradiction

$\therefore \lim_{x \to 0} f(x)$ DNE.

Ex: Show $\lim_{x\to 7} 8x-3=53$

Rough Work:

(If $0<|x-7|<\delta$) — a — we control δ

we want — $f(x)-L$

$|8x-3-53|<\varepsilon$

$8x-56<\varepsilon$

$8|x-7|<\varepsilon$

$|x-7|<\varepsilon/8$

now, backward.

we can choose

Pf: Given $\varepsilon>0$. Let $\delta=\frac{\varepsilon}{8}$

Then, if $0<|x-7|<\delta$,

we have,

$$0<|x-7|<\frac{\varepsilon}{8}$$

$$0<8|x-7|<\varepsilon$$

$$0<|8x-56|<\varepsilon$$

$$0<|(8x-3)-53|<\varepsilon$$

That is, $|(8x-3)-53|<\varepsilon\ \forall\varepsilon>0$

Thus, $\lim_{x\to 7} 8x-3=53$

Ex:

Show $\lim_{x\to 1} x^2+3x+4=8$

a

Rough work:

(If $0<|x-1|<\delta$)

we want $|f(x)-L|<\varepsilon$

we want

$$|x^2+3x+4-8|<\varepsilon$$

$$|x^2+3x-4|<\varepsilon$$

$$|(x+4)(x-1)|<\varepsilon$$

$$|(x+4)||(x-1)|<\varepsilon$$

↳ this depends on x, we can't choose, so ε [illegible] nono

Trick. if we find δ that works for ε, then any smaller δ will also work.

~~work for ε ⇒ 把 ε 變 δ~~

closer than 1 unit away. certainly.

So: Assume $\delta \le 1$

$\Rightarrow \; 0<|x-1|<\delta\le 1$ ← redo

$$|x-1|<1$$

↳ $x\in(0,2)$ ← upper bound

$$|x+4|<|2+4|=6 \Rightarrow |x+4|<6$$

∴ returning to lair

$$|x+4||x-1|<6|x-1|<\varepsilon$$

$$|x-1|<\frac{\varepsilon}{6}$$

Asking $\delta=\min\{1,\varepsilon/6\}$ means:

- $|x+4|<6$, since $\delta<1$
- $|x-1|<\varepsilon/6$, since $|x-1|<\delta$ & $\delta\le\varepsilon/6$

key.

next page →

Week 4, lecture 9

→ From last pg.

Pf:

let $\varepsilon > 0$ be given. Let $\delta = \min\{1, \varepsilon/6\}$

Then, if $0 < |x-1| < \delta$, we have

$$|x^2+3x+4-8| = |x+4||x-1|$$

Since $|x-1| < \delta \le 1 \Rightarrow 0 < x < 2$ ← upper bound.

Thus, $|x+4| < 6$

Thus, $|(x^2+3x+4) - 8| < 6|x-1|$

$$< 6\delta \le 6\left(\varepsilon/6\right) = \varepsilon$$

That is $|(x^2+3x+4) - 8| < \varepsilon$, $\forall \varepsilon > 0$

Thus, $\lim_{x\to 1} x^2+3x+4 = 8$

Ex. 4 pg 64 of CN → read !!!

Notes:

1) For $\lim_{x\to a} f(x)$ to exist f must be defined on an open interval (α, β) combining $x=a$, expect possibly at $x=a$.
2) The value of $f(a)$, if defined at all, does not effect the existence of the limit or its value.
3) If two functions are equal, expect possibly at $x=a$, then their limiting behaviour at a is identical

WSIC?

we must understand limits of functions in order to examine the instantaneous rate of change of functions. That is, we need limit to define derivatives.

How can sequences be useful when examining fun limits?

Thm 1: sequential Characterization of limits.

Let f be defined on an open interval containing $x=a$, except possibly at $x=a$. Then,

$\lim_{x \to a} f(x) = L \Leftrightarrow$ $\{x_n\}$ is a sequence with $x_n \neq a$ and $x_n \to a$ then, $\lim_{x \to \infty} f(x_n) = L$

function limits. ↳ sequence limits

泰酷辣!!

Pf for ($\Rightarrow$) direction

Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, we can find $\delta > 0 \ni$ if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

And since $x_n \to a$, we can find cutoff $N \in \mathbb{N}$ $\ni$ for $n \geq N$, we have $|x_n - a| < \delta$ ($\forall \delta > 0$, including the one we found.)

Thus, we have $|f(x_n) - L| < \varepsilon$ for $n \geq N$

$\therefore \lim_{n \to \infty} f(x_n) = L$. ∎

→ Since we know the sequence limits are unique, functions limits are unique.

Thm 2: Uniqueness of limits of functions.

Assume $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} f(x) = M$, then $L = M$.

That is, function limit are unique.

→ next pg.

Sequential characterization can help lim DNE:

1) Find $\{x_n\}$ with $x_n \to a$, $x_n \neq a$, for which $\lim_{x \to x_n} f(x_n) = \text{DNE}$

2) Find $\{x_n\}$ and $\{y_n\}$ with $x_n, y_n \to a$, $x_n, y_n \neq a$, for which $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} f(y_n) = M$, But $L \neq M$

找兩個 →0 的 seq, 但 L 不一樣.
↳ 代入原式 $(f(x_n), f(y_n))$ 不相等 ⇒ DNE
(uniqueness)

Example of (2)

$\lim_{x\to 0} \cos(\frac{1}{x})$ [look like garbage around $x=0$]

Lets $x_n = \frac{1}{2\pi n}$, $y_n = \frac{1}{\pi + 2\pi n}$ foresight, 為下面 1,-1.

Note: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ and x_n and $y_n \neq 0$

However, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \cos\left(1 / \frac{1}{2\pi n}\right) = \lim_{n\to\infty} \underbrace{\cos(2\pi n)}_{=1\ \forall n}$
$= 1$

And $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \cos\left(1 / \frac{1}{\pi + 2\pi n}\right) = \lim_{n\to\infty} \cos(\pi + 2\pi n)$
$= -1\ \forall n$
$= -1$

Thus, $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, so $\lim_{x\to 0} \cos(\frac{1}{x})$, DNE ∎

→ Just as with Sequences, we don't tend to use formal definiton for practice.

$\lim_{x \to a} f(x)$ $x, y \to a$ but $\lim x \neq \lim y$ → DNE

→ vevl p?

Thm 3 Arithmetic Rules for limits of functions.

Let f & g be functions with $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} g(x) = M$

Let $a \in \mathbb{R}$

1) If $f(x) = c$ $\forall x \in \mathbb{R}$, then $\lim_{x\to a} f(x) = c$ $\qquad c = \lim c$

2) For any $c \in \mathbb{R}$, $\lim_{x\to a} [c f(x)] = cL$ $\qquad$ c相乘 = 相乘c

3) $\lim_{x\to a} f(x) + g(x) = L + M$ $\qquad$ 相加 = 相加

4) $\lim_{x\to a} [f(x) g(x)] = LM$ $\qquad$ 相乘 = 相乘

5) $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$ if $M \neq 0$ $\qquad$ 相除 = 相除

6) $\lim_{x\to a} [f(x)]^{\alpha} = L^{\alpha}$ $\forall \alpha > 0, L > 0$ $\qquad$ 裡外次方 $\boxed{\text{沒差}}$

Another carryover for sequence

Thm 4:

Assume $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]$ $\boxed{\text{exist}}$ and $\lim_{x\to a} g(x) = 0$

Then, $\lim_{x\to a} f(x) = 0$. $\qquad$ 分母 = 0 $\longrightarrow$ lim = 0

Week 4 lecture 10

- A few common function limits (basic)

1) If $P(x)$ is polynomial, then $\lim_{x\to a} P(x) = P(a)$
2) If $r(x) = \frac{p(x)}{q(x)}$ is rational function [p&q are polynomial]
   a) if $q(a) \neq 0$ then $\lim_{x\to a} r(x) = r(a)$
   $\frac{1}{0}$ b) if $q(a)=0$ and $p(a)\neq 0$, then $\lim r(a)$. $\boxed{\text{DNE}}$
   c) if $q(a)=0$ and $p(a)=0$, then $(x-a)$ must be a factor of both p&q. $\Rightarrow$ Factor out, cancel, and look at the new function. (maybe long division)

- Think back to $f(x) = \begin{cases} 3 & \text{if } x>0 \\ -2 & \text{if } x<0 \end{cases}$ (no limit at 0)

  we formally showed that $\lim_{x\to 0} f(x)$ DNE.

- However, we can understand visually & conceptually that the function approaches different value from different sides.

Definitions:

Right side limit: $\lim_{x\to a^+} f(x) = L$

$\to$ f have a limit approaching from right

$\to$ $\forall \varepsilon>0 \ \exists \delta>0 \ni$ if

$0<|x-a|<\delta$ and $x>a$ Show that from right

then $|f(x)-L|<\varepsilon$.

Left side limit: $\lim_{x\to a^-} f(x) = L$

$\to$ f have a limit approaching from the Left.

$\to$ $\forall \varepsilon>0 \ \exists \delta>0 \ni$ if

$0<|x-a|<\delta$ and $x<a$ $\to$ show from left

then $|f(x)-L|<\varepsilon$.

- To relate one-side limit with two-side ones, we have:

Thm 6:

$$\lim_{x\to a} f(x) = L \ \boxed{\overset{\text{Iff}}{\Longleftrightarrow}} \ \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

* All arithmetic rules (limit laws) & sequential characterization still hold for one side limits.

- Another carry-over.

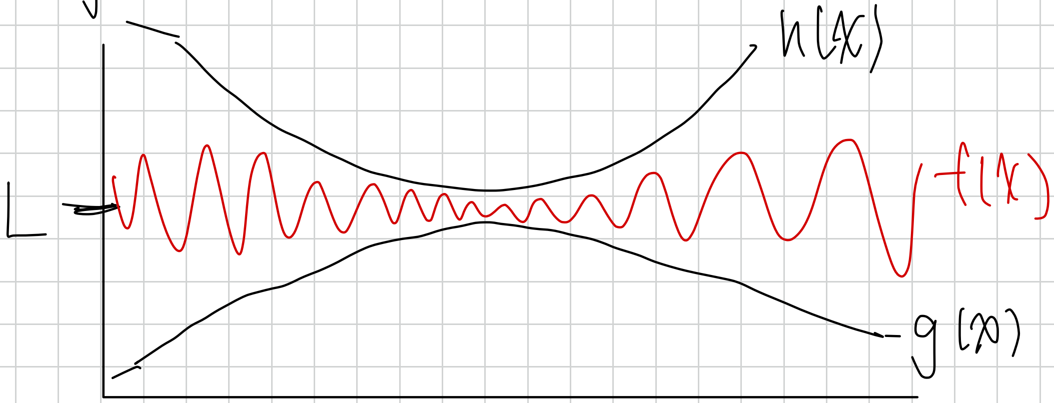
Thm 7: The sqz thm, sandwich thm.

Assume functions. f, g, h are defined on an open interval I, containing $x=a$, expect possibly at $x=a$. Assume that $\forall\, x \in I$, except possibly at $x=a$, that $g(x) \le f(x) \le h(x)$, and that

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$$

Then, $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} f(x) = L$

Pictorially:



Ex: $\lim\limits_{x\to 0} x^8 \cos\left(\frac{1}{x}\right)$

基本款

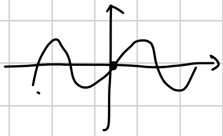
Note that: $-1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad \forall x \in \mathbb{R}\setminus\{0\}$

$\therefore \quad -x^8 \le x^8\cos\left(\frac{1}{x}\right) \le x^8$

note: $\lim\limits_{x\to 0} -x^8 = 0 = \lim\limits_{x\to 0} x^8$

$\therefore$ By sqz thm $\lim\limits_{x\to 0} x^8\cos\left(\frac{1}{x}\right) = 0$ ■

Ex: $\lim\limits_{x\to 0} \sin(x)$

From drawing 

The function looks to 0 as $x \to 0$

We can show $\lim\limits_{x\to 0}\sin(x) = 0$ using sqz thm

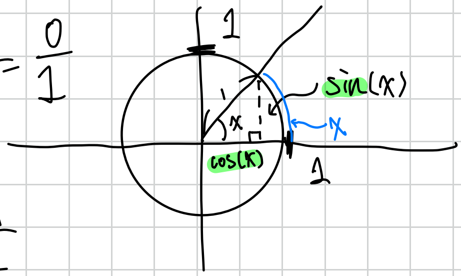
$\Rightarrow$ Examine the unit circle for $0 < x < \frac{\pi}{2}$ (looking $\lim\limits_{x\to 0^+}$)

$\sin(x) = \frac{O\text{pposite}}{H\text{ypotenuse}} = \frac{O}{1}$

$\Rightarrow O = 1\sin(x)$

$\cos(x) = \frac{A}{H} = \frac{A}{1}$

$\Rightarrow A = 1\cos(x)$



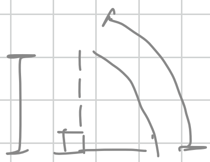
$P = 2\pi r$
$= 2\pi 1$

$P = 2\pi$

arclength $= 2\pi \times \frac{x}{2\pi}$
$= x$

— We see from the diagram that.

on $0 < x < \frac{\pi}{2}$ that $0 < \sin(x) < x$ (弧)

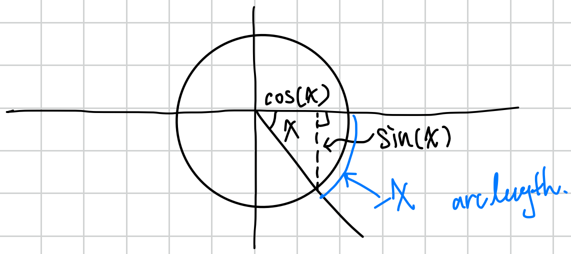


We know that $\lim\limits_{x\to 0^+} 0 = 0$ & $\lim\limits_{x\to 0^+} x = 0$

Thus, $\lim\limits_{x\to 0^+}\sin(x) = 0$ by sqz thm.

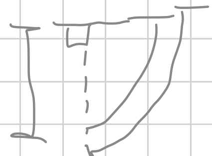
next pg.

Now look at unit circle for $-\frac{\pi}{2} < x < 0$ (looking at $\lim_{x\to 0^-}$)



- We can see that on $-\frac{\pi}{2} < x < 0$ that

$$0 < -\sin(x) < -x$$ (弧)



Then we have $0 > \sin(x) > -x$, we know that $\lim_{x\to 0^-} 0 = 0 = \lim_{x\to 0^-} -x$. Then, by sqz thm, we have $\lim_{x\to 0^-} \sin(x) = 0$

$\therefore \lim_{x\to 0} \sin(x) = 0$ ■

Ex: As a sidebar, we know,

$$\cos(x) = \sqrt{1-\sin^2(x)}$$

Then, by lim laws, $\lim_{x\to 0} \sqrt{1-\sin^2(x)}$

$$= \sqrt{\lim_{x\to 0} 1 - \lim_{x\to 0} \sin^2(x)}$$

$$= \sqrt{1-0} = 1$$

$\therefore$ we recover $\boxed{\lim_{x\to 0} \cos(x) = 1}$ as expected.

And for $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\therefore \lim_{x\to 0} \tan(x) = \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0$$

- We can use sqz thm in a similar manner.

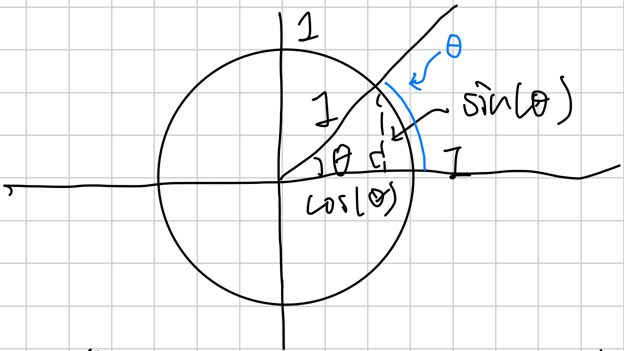
To figure out: $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$ (Fundamental trig limits)

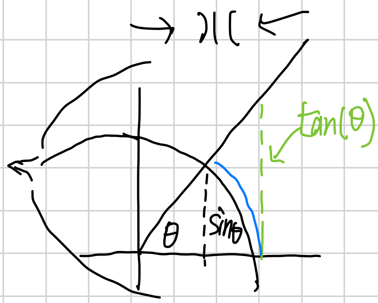
$= 1$ (L'H)

Let's revisit the idea of a unit circle.

we will only argue $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta}$, but $\lim_{\theta \to 0^-}$ is very similar.

Look at $\theta \in (0, \frac{\pi}{2})$:





$\tan = \frac{O}{A} \left(\frac{y}{x}\right)$

$\tan = O$ 砍

Here, there are 3 area of interest.:

1) smaller triangle ($\cos\theta$, $\sin\theta$, 1)
2) sector of triangle (1, 1, θ)
3) large triangle (1, $\tan\theta$, ?)

→ Note that 1) < 2) < 3)

We get formulate for the areas:

1) $\frac{1}{2}\cos\theta\sin\theta$
2) $(\pi(1)^2) \cdot \left(\frac{\theta}{2\pi}\right) = \frac{1}{2}\theta$
3) $\frac{1}{2}\tan\theta$

So we have: $(0 < \theta < \frac{\pi}{2})$ so all positive

$$\frac{1}{2}\cos\theta\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

→ next pg.

Week 4, lecture 11

It is very important to note that for $\theta \in (0, \frac{\pi}{2})$, we have $\sin\theta > 0$, $\cos\theta > 0$, $\tan\theta > 0$, $\theta > 0$

Now multiplying everything by $\frac{2}{\sin\theta}$:

$$\cos\theta < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos\theta}$$

Reciprocate:

$$\frac{1}{\cos\theta} > \frac{\sin(\theta)}{\theta} > \cos\theta$$

Then, since $\lim_{\theta\to 0^+} \cos\theta = 1 = \lim_{\theta\to 0^+} \frac{1}{\cos\theta}$, by squeeze thm

we have $\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$.

The LH limit follow similarly going.

$$\rightarrow \boxed{\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1}$$

$$\delta = \frac{72\cos(72x)}{9\sec^2(9x)}$$

LH

真不行就 LH

Ex 1:

$$\lim_{x\to 0} \frac{\sin(72x)}{\tan(9x)} = \lim_{x\to 0} \sin(72x) \cdot \frac{\cos(9x)}{\sin(9x)}$$

$$= \lim_{x\to 0} \sin(72x) \cdot \frac{72x}{72x} \cdot \frac{\cos(9x)}{\sin(9x)} \cdot \frac{9x}{9x}$$

$$= \lim_{x\to 0} \frac{\sin(72x)}{72x} \cdot \frac{9x}{\sin(9x)} \cdot \cos(9x) \cdot \frac{72x}{9x}$$

sidebar: note that $\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}}$

$$= \frac{\lim 1}{\lim \frac{\sin\theta}{\theta}} = \frac{1}{1} = 1$$

$$= (1)\cdot(1)\cdot(1)\cdot(8) = 8$$

Ex 2:

$$\lim_{x\to 1}\frac{\sin(x^2-1)}{\sin(x-1)} = \lim_{x\to 1}\frac{\sin(x^2-1)}{\sin(x-1)}\cdot\frac{x^2-1}{x^2-1}$$

$$=\lim_{x\to 1}\frac{\sin(x^2-1)}{\sin(x-1)}\cdot\frac{(x-1)(x+1)}{(x^2-1)}$$

$$=\lim_{x\to 1}\frac{\sin(x^2-1)}{x^2-1}\cdot\frac{x-1}{\sin(x-1)}\cdot(x+1)$$

$$=(1)\cdot(1)\cdot(2)=2$$

L'H ($\frac{0}{0}$)

$$\frac{2x\cos(x^2-1)}{\cos(x-1)}$$

$$=\frac{2}{1}=2$$

Ex: ① mathmetize:

$$\lim_{x\to 0}\frac{\tan(x)}{x} = \lim_{x\to 0}\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{x}$$

$$=\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{1}{\cos x}$$

$$=(1)\cdot(1)=1$$

②

$$\lim_{x\to 0}\frac{\cos(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{x}\cdot\frac{\sin(x)}{\sin(x)}$$

$$=\lim_{x\to 0}\frac{\cos(x)}{\sin(x)}\cdot\frac{\sin(x)}{x}$$

$\Rightarrow$ DNE!

$\Rightarrow$ only $\frac{0}{0}$ 需要解

this is $\frac{1}{0}$, 所以 $\to$ DNE, 上面有 thm.

Pf: for $\lim_{x\to 2} 2x^2+x-3=7$

Rough work ✓

$0<|x-2|<\delta$

$|2x^2+x-3-7|<\varepsilon$

$|2x^2+x-10|<\varepsilon$

$(x-2)(2x+5)<\varepsilon$

→ Assume $\delta\le 1$

$|x-2|<\delta\le 1$

$|x-2|\le 1$ ↙ ub

$x\in(1,3)$

$|2x+5|<|6+5|=11$

$\Rightarrow |2x+5|<11$

Back, $|x-2||2x+5|<11|x-2|<\varepsilon$

$11|x-2|<\varepsilon$

$|x-2|<\frac{\varepsilon}{11}$

let $\delta=\min\{1,\varepsilon/11\}$

Pf

Pf ③

$\boxed{0<|x-2|<\delta}$

$\boxed{|(2x^2+x-3)-7|<\varepsilon}$

$2x+5$

$x-2$

$|2x+5||x-2|<\varepsilon$

几嘆: nono

Assume $\delta\le 1$

$|x-2|<\delta\le 1$

$|x-2|\le 1$ ↙ up

② $x\in(1,3)$

$|2x+5|<|6+5|<11$

$\Rightarrow |2x+5|<11$

Back ④ $\begin{cases}11|x-2|<\varepsilon\\ |x-2|<\varepsilon/11\end{cases}$

① let $\delta=\min\{1,\varepsilon/11\}$

最後回到①

→結論

let $\varepsilon>0$, $\delta=\min\{1,\varepsilon/11\}$

Then, since $\delta\le 1$, we have that for

$|x-2|<\delta \Rightarrow x\in(1,3)$.

Then for $0<|x-2|<\delta$, we have that

$|(2x^2+x-3)-7|=|x-2||2x+5|$

$<11|x-2|$

$\le 11(\varepsilon/11)$ $\quad\delta=\min\{1,\varepsilon/11\}\Rightarrow\delta\le\varepsilon/11$

That is, $|(2x^2+x-3)-7| < \varepsilon \quad \forall \varepsilon > 0$

$\therefore \lim_{x \to 2} 2x^2+x-3 = 7$

for $\lim_{x \to 2} 2x^2+x-3 = 7$

RW: $0 < x-2 < \delta$

$|2x^2+x-3-7| < \varepsilon$

$|x-2||2x+5| < \varepsilon$

$x \ -2$
$2x \ 5$

$|x-2| < \delta \leq 1, \ |x-2| \leq 1$

$x: (1,3)$ up

$|2x+5| < 11$

then, $11|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{11}$

Proof:

Given $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{11}\}$

then if $0 < |x-2| < \delta$

we have $|2x^2+x-3-7|$

$= |x-2||2x+5|$

Since $|x-2| < \delta \leq 1 \quad |2x+5| < 11$

then we have $|x-2||2x+5| < 11|x-2| \leq 11(\frac{\varepsilon}{11}) = \varepsilon$,

That is $|(2x^2+x-3)-7| < \varepsilon \quad \forall \varepsilon > 0$,

$\therefore \lim_{x \to 2} 2x^2+x-3 = 7$

Mid 11/23 7~8 50 pm

135 Test I 11/16

7~8:50 pm

Week 5 lecture 12.

We now want to look at limits at and going to ∞.

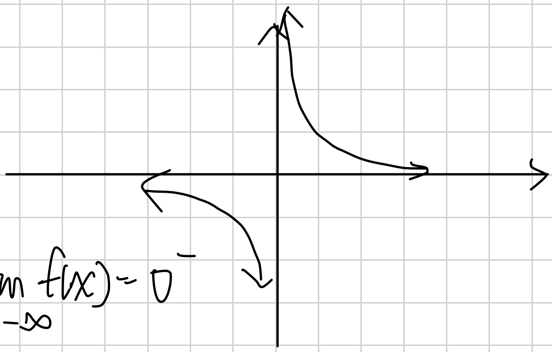
You may have heard of end behaviour.

↳ what's happening to the function at the end of x-axis

↳ $\lim_{x \to +\infty} f(x)$

<u>For ex:</u>

$f(x) = \frac{1}{x}$



Defn of limits at Infinity:

f is a function, $L \in \mathbb{R}$

- $\lim_{x \to +\infty} f(x) = L$: $\forall \varepsilon > 0, \exists N > 0 \in \mathbb{R}$ $\ni$ if $x > N$, then $|f(x) - L| < \varepsilon$
- $\lim_{x \to -\infty} f(x) = L$: $\forall \varepsilon > 0, \exists N < 0 \in \mathbb{R}$ $\ni$ if $x < N$ then $|f(x) - L| < \varepsilon$

We can also define the followings:

Defn: Horizontal asymptote

Assume $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. Then

the line $y = L$ is a horizontal asymptote of $f(x)$.

Ex: for $f(x) = \frac{1}{x}$, $y = 0$ is a horizontal asymptote.

Some functions may shoot off without bounds as $x \to \pm\infty$, rather than approach a value.

Defns: $\pm$ limit at $\pm\infty$

- $\lim_{x \to +\infty} f(x) = +\infty$: $\forall M > 0$, $\exists N > 0$ $\in$ if $x > N$, then $f(x) > M$
- $\lim_{x \to +\infty} f(x) = -\infty$: $\forall M < 0$, $\exists N > 0$ $\in$ if $x > N$, then $f(x) < M$
- $\lim_{x \to -\infty} f(x) = +\infty$
- $\lim_{x \to -\infty} f(x) = -\infty$

The squeeze thm still holds way out at $\pm\infty$

Thm 9: Sqz thm at $\pm\infty$

- Assume $g(x) \le f(x) \le h(x)$, $\forall x \ge N$

  If $\lim_{x \to \infty} g(x) = L = \lim_{x \to \infty} h(x)$,

  then $\lim_{x \to \infty} f(x) = L$

- Assume $g(x) \le f(x) \le h(x)$ $\forall x \le N$

  If $\lim_{x \to -\infty} g(x) = L = \lim_{x \to \infty} h(x)$,

  then $\lim_{x \to -\infty} f(x) = L$

Mathematica Q:

① $\lim_{x \to \infty} \frac{\cos(x)}{x}$

$$\frac{-1}{x} \le \frac{\cos(x)}{x} \le \frac{1}{x}$$

$-1 \le \cos(x) \le 1$

$\Rightarrow \frac{-1}{x} \ge \cos(x) \ge \frac{1}{x}$

因x是負的

一點結

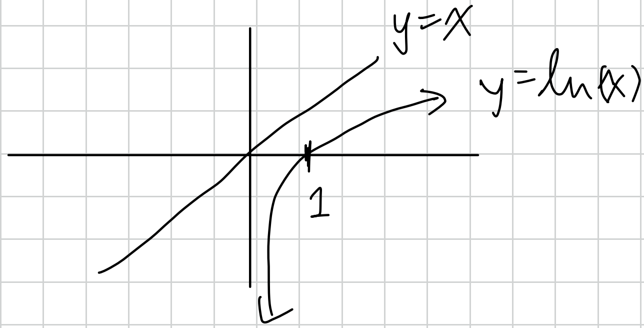
$$\lim_{x \to \infty} \frac{-1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}$$

According to Sqz, $\lim_{x \to \infty} \frac{\cos(x)}{x} = 0$

We will now apply Sqz thm at ∞ to examine

L'H $\frac{\frac{1}{x}}{1} = \frac{1}{x} \Rightarrow 0$ $\quad \lim_{x\to\infty} \frac{\ln(x)}{x}$ = "Fudamental log lim"

Examine



Note that for $x \geq 1$ we have that $x, \ln(x) \geq 0$.

As well, note that for $x \geq 0$, $\ln(x) \leq x$

- We are looking as $x \to +\infty$, so we are written these regimes.

Then, we have for $x \to \infty$ that $0 \leq \frac{\ln(x)}{x}$

By being a lil sneaky, we can note:

$$\frac{\ln(x)}{x} = \frac{\ln\left(x^{\frac{1}{3}}\right)^3}{x^{\frac{2}{3}} x^{\frac{1}{3}}} = \frac{3}{x^{\frac{2}{3}}}\underbrace{\left(\frac{\ln\left(x^{\frac{1}{3}}\right)}{x^{\frac{1}{3}}}\right)}_{\leq 1}$$

That is, $0 \leq \frac{\ln(x)}{x} = \frac{3}{x^{\frac{2}{3}}}\left(\frac{\ln\left(x^{\frac{1}{3}}\right)}{x^{\frac{1}{3}}}\right) \leq \frac{3}{x^{\frac{2}{3}}}$

$$0 \leq \frac{\ln(x)}{x} \leq \frac{3}{x^{\frac{2}{3}}}$$

Since $\lim_{x\to\infty} 0 = 0 = \lim_{x\to\infty} \frac{3}{x^{\frac{2}{3}}}$

By Sqz thm $\boxed{\lim_{x\to\infty} \frac{\ln(x)}{x} = 0}$

Ex 1:

$$\lim_{x\to\infty} \frac{\ln(x)}{x^p} = \lim_{x\to\infty} \frac{\ln(x^p)^{\frac{1}{p}}}{x^p} = \lim_{x\to\infty} \frac{1}{p}\left(\frac{\ln(x^p)}{x^p}\right)$$

$$= \left(\frac{1}{p}\right)\cdot(0) = 0 \quad \#$$

Ex 2:

$$\lim_{x\to\infty} \frac{\ln(x^p)}{x} = \lim_{x\to\infty} \left(\frac{\ln(x)}{x}\right)\cdot p = p\cdot(0) = 0 \quad \#$$

Ex 3:

$$\lim_{x\to\infty} \frac{x^p}{e^x} \quad (p>0)$$

$\Rightarrow$ let $u = e^x$ note: for $x\to\infty$, $u\to\infty$

$\Rightarrow x = \ln(u)$

Then, $$\lim_{x\to\infty} \frac{x^p}{e^x} = \lim_{u\to\infty} \frac{[\ln(u)]^p}{u}$$

$$= \lim_{x\to\infty} \left(\frac{\ln(u)}{u^{\frac{1}{p}}}\right)^p$$

$$\frac{\ln(x^p)^{\frac{1}{p}}}{x^p} \quad \frac{1}{p} = 0 \cdot 0 = 0$$

$u = e^x$

$x = \ln(u)$

$$p\cdot\frac{\ln(u)^x}{u} = p\cdot 0 = 0$$

Week 5, lecture 13

Ex: 4 $\lim_{x \to 0^+} x^p \ln(x) \Rightarrow$ let $u = \frac{1}{x}$ note that as $x \to 0^+$, $u \to \infty$

$\Rightarrow x = \frac{1}{u}$

then, $\lim_{x \to 0^+} x^p \ln(x) = \lim_{u \to \infty} \left(\frac{1}{u}\right)^p \ln\left(\frac{1}{u}\right)$

$= \lim_{u \to \infty} \left(\frac{1}{u^p}\right) \ln(u^{-1})$

$= \lim_{u \to \infty} -\frac{\ln(u)}{u^p}$ ← from previous.

$= -(0) = 0$

$\frac{\ln(x)}{x^p} = \frac{1}{p} \cdot \frac{\ln(x^p)}{x^p}$

$= 0 \cdot 0 = 0.$

* Observation (for $p > 0$)

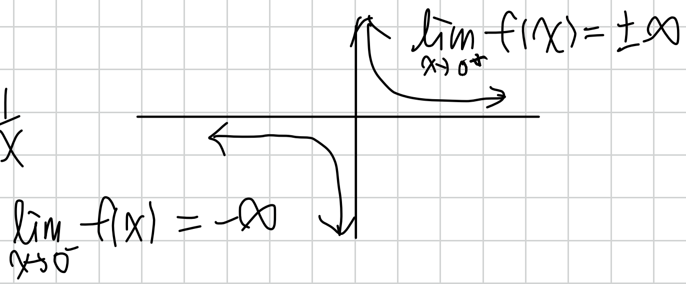
$[\ln(x)]^p \ll x^p \ll p^x \ll x^x$ as $x \to \infty$

We can have functions with limits of $\pm\infty$ for $x \to a^{\pm}$ (in addition to $x \to \pm\infty$ as we saw last class)

That is, $\lim_{x \to a^{\pm}} f(x) = \pm\infty$

Ex:

For $f(x) = \frac{1}{x}$

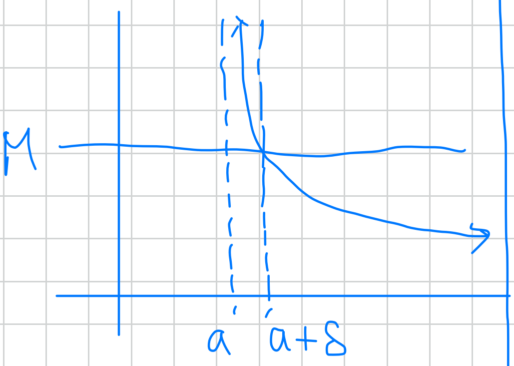


Definitions: Infinite limits:

- Right sided

$\lim_{x \to a^+} f(x) = +\infty$:

$\forall M > 0 \ \exists \delta > 0 \ \ni$ if $|x-a| < \delta$ and $x > a$, that is, $a < x < a+\delta$, then $f(x) > M$



$\lim_{x \to a^-} f(x) = -\infty$:

$\forall M < 0, \exists \delta > 0 \ \ni$ if $|x-a| < \delta$ and $x > a$ that is, $a < x < a+\delta$ then $f(x) < M$.

- Left side

$\lim_{x \to a^-} f(x) = +\infty$

$\lim_{x \to a^-} f(x) = -\infty$

As would follow from our understanding of one vs two sided limits, we say:

$$\lim_{x \to a} f(x) = \pm\infty, \text{ if, } \lim_{x \to a^-} f(x) = \pm\infty = \lim_{x \to a^+} f(x)$$

Remember: $\pm\infty$ are not values, just notation, lim DNE.

Defn: Vertical asymptote.

If any of $\lim_{x \to a^{\pm}} f(x) = \pm\infty$, we say the line $x=a$ is a vertical asymptote of $f(x)$.

Ex: For $f(x) = \frac{1}{x}$, we have $x=0$ is the VA.

HA: check $\lim_{x \to \pm\infty} f(x)$ 如果相等 → HA

VA → $\pm\infty$ 分母 $=0$ or $\to 0$

The next fundamental concept of calc.

Defn: continuity. (in 3 flavours.)

- We say a function f is continuous (cts) at a point $x=a$ if
  a) $\lim_{x \to a} f(x)$ exists, and
  b) $\lim_{x \to a} f(x) = f(a)$

Otherwise the function is discts at $x=a$, which is a point of discontinuity.

- We say a function is cts at $x=a$ if
  $\forall \varepsilon > 0, \exists \delta > 0 \ni$ if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$
- A function is cts at $x=a$
  iff $\iff$
  $\{x_n\}$ is a sequence with $\lim_{n \to \infty} x_n = a$
  then $\lim_{n \to \infty} f(x_n) = f(a)$

* We can rewrite our continuity statement in a different manner:

Notice that for $x \neq a$, $x = a+h$ $(h \neq 0)$

Then, $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

So, a function is cts at $x=a$

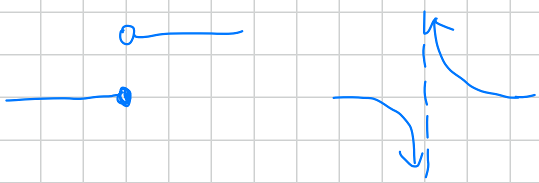
$$\iff$$

$$\lim_{h \to 0} f(a+h) = f(a)$$

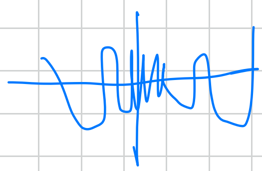
There are a few different kind of discontinuities.

- hole / removeable discty: $\lim_{x \to a}$ exist, but $\neq f(a)$

- jumps: finite / infinite: $\lim_{x \to a^+} \neq \lim_{x \to a^-}$



- Oscillatory: $\lim_{x \to a}$ DNE due to infinite ascillations.
  - $\hookrightarrow$ ex: $\cos(\frac{1}{x})$



Week 5 lecture 14

We will examine continuity of common functions.

Polynomials: we have seen that that $\lim_{x\to a} p(x) = p(a)$
This is literally the definition of cty.
Thus, polynoms are cts $\forall a \in \mathbb{R}$

$\sin(x)\ \cos(x)$: We showed $\lim_{x\to 0} \sin(x) = 0 = \sin(0)$

using the sqz thm, as well as using that result to show $\lim_{x\to 0} \cos(x) = 1 = \cos(0)$

Then $\lim_{x\to a} \sin(x) = \lim_{h\to 0} \sin(a+h)$

$= \lim_{h\to 0} (\sin(a)\cos(h) + \sin(h)\cos(a))$

$= \sin(a)(1) + (0)\cos(a)$

$= \sin(a)$

$\therefore$ $\sin(x)$ is cts $\forall a \in \mathbb{R}$

- Try showing $\cos(x)$ is cts $\forall a \in \mathbb{R}$ yourself.

$e^x / \ln(x)$: These are not simple to show.
You would need 'power series' (M138)
But, if we assume e^x is cts at $x=0$,
we can show it is cts $\forall a \in \mathbb{R}$
So: take as fact that $\lim_{x\to 0} e^x = 1 = e^0$

Then, $\lim_{x\to a} e^x = \lim_{h\to 0} e^{a+h} = \lim_{h\to 0} e^a e^h = e^a(1) = e^a$

$\therefore$ cts $\forall a \in \mathbb{R}$ (on it's domain)

We can make a geometric arguement for $\ln(x)$

Note $\ln(x)$ is the inverse function of e^x. That is, it is the reflection of $y=e^x$ across the line $y=x$.

Then, since there are no breaks in e^x, there will be no breaks in $\ln(x)$.

$\therefore$ $\ln(x)$ is cts on its domain.

Thm 12: Continuity of Inverses.

If $y=f(x)$ is invertible with inverse $f^{-1}(y)=x$ and $f(a)=b$, and $f(x)$ is cts at $x=a$, then $f^{-1}(x)$ is cts at $x=b$

As with limits, we have tools at our disposal to extend, identify, and work with continuity.

Thm 13: Continuity of Sums & Products

Let f & g be cts at $x=a$. Then,

1) $f+g$ is cts at $x=a$

2) fg is cts at $x=a$

如果都 cts,
相加也 cts
相乘也 cts.

Thm 14: Continuity of Quotients.

Let f & g be cts at $x=a$. If $g(x)\neq 0$, then f/g is cts at $x=a$. (分母≠0) 相除也 cts.

(Note, pfs. of Thms 13 & 14 come from limit laws)

Thm 15: continuity of Compositions.

Let f be cts at $x=a$ and let g be cts at $x=f(a)$. Then $h(x) = g\circ f = g(f(x))$ is cts at $x=a$.

Pf:

Let f be cts at $x=a$ and let g be cts at $x=f(a)$. Let $h(x) = g(f(x))$.

Let $\{x_n\}$ be a sequence $\ni x_n \to a$.

Then since f is cts at $x=a$, by sequential characterization defn. we have

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Then, $\{f(x_n)\}$ is a sequence $\ni f(x_n) \to f(a)$.

Then, since g is cts at $x=f(a)$, by sequential characterization defn, we have

$$\lim_{n\to\infty} g(f(x)) = g(f(a))$$

This means that $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} (f(x_n)) = g(f(a)) = h(a)$

This is the sequential characterization defn of cty for $h(x)$. $\therefore h(x)$ is cts $\forall a \in \mathbb{R}$ ■

Ex:

$$f(x) = 2^{2x \sin(e^x)}$$

This fcn is cts $\forall a \in \mathbb{R}$ since e^x is cts $\forall a \in \mathbb{R}$, and since $\sin(x)$ is cts $\forall a \in \mathbb{R}$, and since $2x$ is cts $\forall a \in \mathbb{R}$, and since 2^x is cts $\forall a \in \mathbb{R}$

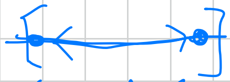
(This is a combination of product & composition rules)

- There are some nuances to cty on intervals.

Defn: Continuity on (a,b) / $\mathbb{R}$

We say f is cts on the open interval (a,b) / $\mathbb{R}$ iff it is cts at each $x \in (a,b)$ / $\mathbb{R}$.

Defn: Continuity on $[a,b]$



We say f is cts on the closed interval $[a,b]$ if

1) it is cts $\forall x \in (a,b)$

2) $\lim_{x \to a^+} f(x) = f(a)$

3) $\lim_{x \to b^-} f(x) = f(b)$

Ex:

We could say that $f(x)\ x^{1/4}$ is cts on $[0,\infty]$ since it is cts $\forall x \in (0,\infty)$ and $\lim_{x \to 0^+} x^{1/4} = 0 = 0^{1/4}$

WSIC?

Continuity of functions is a crucial requirement for many of the major theorems in calc. We will need to be mindful that our functions are cts before applying these theorems.

midterm close Wed 😮

Week 6 (MT) lecture 15

We now state a key theorem without proof

Thm 16: Intermediate Value theorem (IVT)

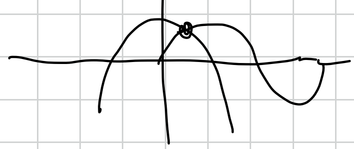
Assume that f is cts on the closed interval $[a,b]$ and either $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$, Then, there exist $c \in (a,b) \Rightarrow f(c) = \alpha$

This make intuitive sense. A cts fcn which start above / below α and ends below / above α at two points must hit α at some point between

Ex1:

Prove that $\sin(x)$ & $1-x^2$ intersect for some $c \in (0,1)$

First for sanity we confirm graphically



$\Rightarrow$ We are trying to show $\sin(x) = 1-x^2$ for some $c \in (0,1)$

Well this the same as showing that

$f(x) = \sin(x) - (1-x^2) = \boxed{0}$

for some $c \in (0,1)$

* First and foremost, note that $f(x)$ is cts on $[0,1]$

  $\hookrightarrow$ sum of cts functions

Now note $f(0) = \sin(0) - (1-0^2) = -1 \quad < 0$

$f(1) = \sin(1) - (1-1^2) = \sin(1) \quad > 0$

* Then by IVT, $f(x) = 0$ for some $c \in (0,1)$

That is, $\sin(x) = 1-x^2$, for some $c \in (0,1)$.

Ex 2.

Prove that x^3+3x^2-x-3 $\boxed{\text{has a root}}^{=0}$ on $[-5,-2]$

→ First note that $f(x)=x^3+3x^2-x-3$ is a polynomial, so it is cts on $[-5,-2]$

→ Now $f(-5)=-48<0$

and $f(-2)=3>0$

Then, by IVT $f(x)=0$ for some $c\in(-5,-2)$ #

Now, we can make quite a approximation for solus to $f(x)=0$ for cts fcns $f(x)$.

We saw for $f(x)=x^3+3x^2-x-3$, that $f(-5)<0$ and $f(-2)>0$, so $f(x)=0$ b/w $x=-5$ & $x=-2$

- lets check the midpoint of the interval $x=-\frac{7}{2}$

We'd find $f(-\frac{7}{2})=\frac{-45}{8}<0$

Then since $f(-2)>0$, we know that by IVT that $f(x)=0$ b/w $x=-1/2$ & $x=-2$

① 代入左右

② 代入 midpoint.

③ since，另一边 <>0

by IVT (左，右)

- Check a new midpoint $x=\frac{-11}{4}$

this gives $f(\frac{-11}{4})=\frac{105}{64}>0$

Then since $f(\frac{-7}{2})<0$, we know by IVT that, $f(x)=0$ b/w $x=\frac{-7}{2}$ and $x=\frac{-11}{4}$

We contine iterating by taking further midpoints of the relevant intervals. we get closer to the exact solution $(x=-3)$ every iteration.

Every iteration cuts your interval in half.

Our newest interval has length $|-3.5-(-2.75)|=0.75$

whereas the original interval length was $|(-5-(-2))|=3$.

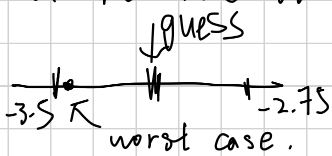
So, our final interval had length 長度

$$\frac{1}{2^{3}}(3)$$

original interval length.

# of iteration

Our final guess for the solution would be the midpoint of our most recent interval.



At worst we could be half length of the current interval away from the true solution

That is, $x=\frac{-25}{8}=-3.125$ is at most $\frac{1}{2^3}(3)$ away from the solution, So, the error for this guess is at most 0.375 (we see the actual error is 0.125)

Notes:

- $\frac{1}{2^4}=\frac{1}{16}<\frac{1}{10}$: so every 4 iterations improves accuracy by at least one decimal place.
- $\frac{1}{2^{10}}=\frac{1}{1024}<\frac{1}{1000}$: so every 10 iterations leads to at least 3 decimal place improvement in accuracy.

We formalize the previous procedure into the

Bisection Method

Suppose want to approximate the solution to $F(x)=0$.

(So for $f(x)=g(x)$, we set $F(x)=f(x)-g(x)$), with F being being cts (on relevant interval) for error less than ε.

Step 1: Find two points $a_0 < b_0 \ni F(a_0)$ and $F(b_0)$ are on opposite sides of 0. Then IVT guarantees, $\exists\ c \in (a_0, b_0) \ni F(c) = 0$

Step 2: Find the midpoint of interval $[a_0, b_0]$, $d = \frac{a_0 + b_0}{2}$ and evaluate $f(d)$.

Step 3: If $f(a_0)$ & $f(d)$ have the same sign, let $a_1 = d$ and $b_1 = b_0$. This gives new interval $[a_1, b_1]$ of length $\frac{1}{2}(a_0 - b_0)$ containing a soln to $F(x) = 0$.

Otherwise, let $a_1 = a_0$ and $b_1 = d$.

Step 4: Repeat step 2&3 to obtain intervals each containing a soln to $F(x) = 0$

The k^{th} interval $[a_k, b_k]$ will have length $\frac{1}{2^k}(b_0 - a_0)$

Step 5: Stop when $\frac{1}{2^{k+1}}(b_0 - a_0) < \varepsilon$, taking the final approx $d = \frac{a_k + b_k}{2}$

Then, we know $\exists\ c \ni F(c) = 0$ where $|d - c| < \varepsilon$

We will later learn Newton's method (faster

Week 6 (MT) lecture 16

Another key thm first requires some defns.

Defn: Global (absolute) Maxima & minima.

Suppose $f: I \to \mathbb{R}$ where I is an interval

- We say c is the global maximum for f on I if $c \in I$ and $f(x) \leq f(c)$ $\forall x \in I$

- We say c is the global minimum for f on I if $c \in I$ and $f(x) \geq f(c)$ $\forall x \in I$

- We say c is the global extremum for f on I if it is either a global min or max

Thinking back to different kind of intervals as well as cty, we can say that if f is defined on a non-empty interval I, then f achieves a global max and min on I. No!! → ex:

Counter example:

Ex 1:

take $f(x) = x^2$ on $[0, 1)$

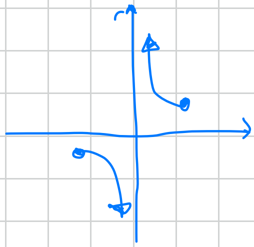
Here, $f(x)$ has a global min of 0 at $x = 0$ but no global max.

Notice that an endpoint on the interval is a key feature of obtaining a global extremum.

Counter example:

Ex 2:

take $f(x) = \frac{1}{x}$ on $[-1, 1]$



there $f(x)$ has neither max nor min.

Thm 17: Extreme value thm (EVT)

Suppose f is cts on $[a, b]$. Then, there exists two number c_1 & $c_2 \in [a, b]$ such that

$$f(c_1) \leq f(x) \leq f(c_2)$$
$$\forall x \in [a, b]$$

That is, there exist a global max or min.

Note, this gives us existence, not value.

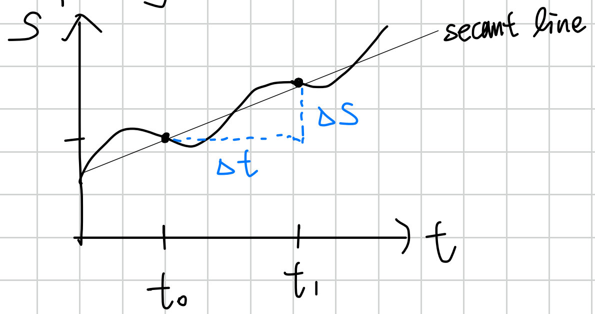
Mid term til here.

We now take our next big step in calculus.
We are interested in rate of change.
- Let's start by displacement & velocity.

The average velocity between t_0 & t_1 is

$$V_{avg} = \frac{\Delta S}{\Delta t} = \frac{S(t_1) - S(t_0)}{t_1 - t_0}$$

Graphically:



So, the slope of the secant line is the avg r.o.c

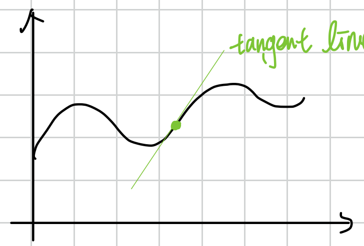
Now, what if we want instantaneous velocity.
↳ the velocity at t_0

This is like taking the average velocity b/w t_0 & points getting closer to t_0

To find the velocity at time t_0 we need limits

$$V_{inst} = \lim_{t \to t_0} \frac{S(t) - S(t_0)}{t - t_0} = V(t_0)$$

Graphically:



Now, as with continuity, we can reframe this limit: we note that for $t \neq t_0$ $t = t_0 + h$

↓

$$V_{inst} = v(t_0) = \lim_{t \to t_0} \frac{S(t) - S(t_0)}{t - t_0} = \lim_{h \to 0} \frac{S(t_0+h) - S(t_0)}{(t_0+h) - t_0}$$

$$= \lim_{h \to 0} \frac{S(t_0+h) - S(t_0)}{h}$$

We can extend this to any given fcn:

Defn: Average R.O.C.

The avg roc of a fcn f b/w $x = a$ & $x = b$

is $f_{avg} = \frac{f(b) - f(a)}{b - a}$

Defn: Instantaneous R.O.C / The derivative of $x = a$

The inst roc of f at $x = a$, or the derivative of at $x = a$, denoted as $f'(a)$, is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Week 6 (144) lecture 17

Slope Intercept form of a line.

① $y = mx + b$

② $y - y_1 = m(x - x_1)$ (point slope formula)

or $y = m(x - x_1) + y_1$

Defn: Tangent line:

If f is differentiable at $x = a$, then the tangent line to f at $x = a$ is the line passing thru $(a, f(a))$ with slope $f'(a)$

This has equ:

$$y - f(a) = f'(a)(x - a)$$

or $y = f'(a)(x - a) + f(a)$

Ex 1: find the instantaneous velocity of

$s(t) = 5t^2 + 6t - 7$

a) at $t = 7$

b) at $t = t_0$

Part a)

$$S'(7) = \lim_{h \to 0} \frac{S(7+h) - S(7)}{h}$$

$$= \lim_{h \to 0} \frac{5(7+h)^2 + 6(7+h) - 7 - (5(7^2) + 6(7) - 7)}{h}$$

$$= \lim_{h\to 0} \frac{76h + 5h^2}{h}$$

$$= \lim_{h\to 0} 76 + 5h = 76 \quad \#$$

Part b)

$$s'(t_0) = \lim_{h\to 0} \frac{s(t_0+h) - s(t_0)}{h}$$

$$= \lim_{h\to 0} \frac{5(t_0+h)^2 + 6(t_0+h) - 7 - (5(t_0^2) + 6(t_0) - 2)}{h}$$

$$= \lim_{h\to 0} \frac{(10t_0 + 6)h + 5h^2}{h}$$

$$= \lim_{h\to 0} 10t_0 + 6 + 5h$$

$$s'(t_0) = 10t_0 + 6 = v(t_0)$$

$$v(t) = x'(t)$$

Ex 2:

Find the equation of the tangent line at $x=3$ for $f(x) = \frac{1}{x+5}$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3+h+5} - \frac{1}{8}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{8-8-h}{64+8h}}{h} = \lim_{h\to 0} \frac{\frac{-h}{64+8h}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{-h}{h(64+8h)} = \lim_{h\to 0} \frac{-1}{64+8h} = \frac{-1}{64}$$

$x=3$, $y=f(x)=\frac{1}{8}$

the equ of tangent line to $f(x)$ at $x=3$

is: $y=\frac{-1}{64}(x-3)+\frac{1}{8}$

lets connect differentiability with continuity.

Thm 1: Differentiability Implies Continuity.

If f is differentiable at $x=a$, then it is cts at $x=a$.

[By contrapositive: if f is discontinuous at $x=a$ then f is not differentiable at $x=a$.]

Pf:

Let f be differentiable at $x=a$. Then

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists.}$$

Then, by a thm from earlier of the course, since $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and since $\lim_{x\to a} x-a=0$ we must have $\lim_{x\to a}[f(x)-f(a)]=0$.

$$\lim_{x\to a}[f(x)]-f(a)=0.$$

$$\lim_{x\to a}[f(x)]=f(a)$$

that is, $f(x)$ is cts at $x=a$.

Now, does cty implies differentiability.

↳ No Counterexample : $f(x) = |x|$ at $x=0$

We know $\lim_{x \to 0} |x| = 0 = |0|$ so, cts.

But $f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$

$= \lim_{h \to 0} \frac{|h|}{h}$

Here, $\lim_{h \to 0^-} \frac{|h|}{h} = \frac{-h}{h} = -1$

$\lim_{h \to 0^+} \frac{|h|}{h} = \frac{h}{h} = 1$

That is, $\lim_{h \to 0} \frac{|0+h| - |0|}{h}$ DNE.

That is, $f'(0)$ DNE $\Rightarrow f$ not diff'able at $x=0$

Week 7 lecture 18. 初始微分

Defn: The derivative Fcn.

The derivative fcn f is defined as

$$f'(x) = \frac{f(x+h)-f(x)}{h}$$

We say f is differentiable on interval I
if $f'(a)$ exists $\forall a \in I$
So the derivative fcn is the derivative of f
at each $x \in I$

This "prime" notation is courtesy of Newton.

Leibniz seperately and simultaneously build up concepts of calculus, with his own notation:

Given $y=f(x)$ the derivative is:

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{df}{dx} = \frac{d}{dx}(f)$$

differential operator.

We write $f'(a)$ as $\left.\frac{dy}{dx}\right|_{x=a}$

Defn: nth derivative

If f is n time diff'ble, we note the nth derivative as: $f^{(n)} = \frac{d^n}{dx^n}(f) = \frac{d}{dx}(f^{(n-1)})$

Ex:
a) If f is 2 times diff'ble, the second derivative is:

$$f'' = \frac{d^2}{dx^2}(f) = \frac{d}{dx}(f')$$

b) If f is 17 times diff'ble, the 17th derivative is

$$f^{(17)} = \frac{d^{17}}{dx^{17}}(f) = \frac{d}{dx}(f^{(16)})$$

Derivative of constant fcn.

Let $c \in \mathbb{R}$, $f(x) = c$, Then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h}$$

$$= \lim_{h \to 0} 0 = 0$$

That is, for a constant fcn, $f(x) = c$, we have $f'(x) = 0$

Derivative of a linear fcn

Let $f(x) = mx + b$, $m, b \in \mathbb{R}$, Then $f'(x) = m$

Derivative of a quadratic fcn

Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$, Then $f'(x) = 2ax + b$

polynomial always diff'able.

Derivative of $\cos(x) = -\sin(x)$

We are trying to find:

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Trig Identity.

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

If both lim exist

$$\stackrel{?}{=} \lim_{h\to 0} \left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \underbrace{\sin(x)\left(\frac{\sin(h)}{h}\right)}_{=\sin(x)\cdot 1}\right]$$

$= \sin(x) \cdot 1 \leftarrow$ fundamental trig lim

$= \sin(x)$

We need to work on:

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} \cdot \frac{\cos(h)+1}{\cos(h)+1}$$

$$= \lim_{h\to 0} \frac{\cos^2(h) - 1}{h(\cos(h)+1)} = \lim_{h\to 0} \frac{(1-\sin^2(h))-1}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2(h)}{h(\cos(h)+1)} = \lim_{h\to 0} \frac{\sin(h)}{h} \cdot \frac{-\sin(h)}{\cos(h)+1}$$

$$= (1) \cdot \left(\frac{-0}{2}\right) = 0$$

Returning to the main problem:

$$f'(x) = \lim_{h\to 0} \left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right]$$

$$= \cos(x)(0) - \sin(x)(1)$$

$$= -\sin(x)$$

Derivative of $\sin(x)$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Derivative of e^x

There are several ways to define e.

Most common : $\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to 0}(1+n)^{\frac{1}{n}} = e$$

For our purpose, we define, we define e to be the unique value for which a fcn form $f(x) = a^x$, $a \in \mathbb{R}^+$ has a tangent line of slope 1 thru $(0,1)$

That is, we defined for $f(x) = e^x$ that $f'(0) = 1$

$$f'(0) = 1 = \lim_{h\to 0}\frac{e^{0+h} + e^0}{h}$$

$$1 = \lim_{h\to 0}\frac{e^h - 1}{h}$$

Then, for $f(x) = e^x$, we have:

$$f'(x) = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} e^x\left(\frac{e^h - 1}{h}\right)$$

$$= e^x(1) = e^x$$

That is, for $f(x) = e^x$, we have $f'(x) = e^x$

Thm 7: Arithemic rules for differentiation.

Assume f & g is differentiable on $x=a$.

① Constant Multiple Rule: Let $h(x) = cf(x)$ then h is differentiable at $x=a$ and $h'(a) = cf'(a)$

② Sum Rule: $h(x) = f(x)+g(x)$. Then h is differentiable at $x=a$ and $h'(a) = f'(a)+g'(a)$

③ Product Rule: Let $h(x) = f(x)g(x)$ then h is differentiable at $x=a$ and $h'(a) = f'(x)g(x) + f(x)g'(x)$

④ Reciprocal Rule: Let $h(x) = 1/g(x)$. If $g(a) \neq 0$, then h is differentiable at $x=a$ and

$$h'(a) = \frac{-g'(a)}{[g(a)]^2}$$

⑤ Quotient Rule: Let $h(x) = f(x)/g(x)$, If $g(a) \neq 0$ then h is differentiable at $x=a$ and

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Week 7 lecture 19

Pfs of Arithamic Diff Rule

3) Product rule:

By def $(fg)'(a) = \lim_{h\to 0} \dfrac{f(a+h)g(a+h) - f(a)g(a)}{h}$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a+h) + f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \left(\underbrace{f(a+h)}_{①} \left[\frac{g(a+h)-g(a)}{h} \right] + g(a) \left[\frac{f(a+h)-f(a)}{h} \right] \right)$$

$$= f(a)\, g'(a) + g(a) f'(a)$$

① because f is diff'able, which implies that f is cts

Being cts mean by def $\lim_{x\to a} f(x) = f(a)$

or $\lim_{h\to 0} f(a+h) = f(a)$

4) Reciprocal Rule

By defn. $\left(\frac{1}{g}\right)'(a) = \lim_{h\to 0} \dfrac{\frac{1}{g(a+h)} + \frac{1}{g(a)}}{h}$

$$= \lim_{h\to 0} \frac{g(a) - g(a+h)}{h(g(a)g(a+h))}$$

$$= \lim_{h\to 0} \left[-\frac{g(a+h)-g(a)}{h} \cdot \frac{1}{g(a)\,g(a+h)} \right]$$

$$= -g'(a) \cdot \frac{1}{g(a)g(a)} = \frac{-g'(a)}{g(a)^2}$$

5) Quotient Rule:

We can combine 3) and 4) here

$$\left(\frac{f}{g}\right)'(a) = \left(f\cdot\frac{1}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{[g(a)]^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{\{g(a)\}^2}$$

A few rules we use without proves.

Thm 8: Power Rule

Assume that $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $f(x) = x^\alpha$

Then, f is diff'able and $f'(x) = \alpha x^{\alpha-1}$ wherever $x^{\alpha-1}$ is defined.

Ex: $f(x) = x^\pi$ $f'(x) = \pi x^{\pi-1}$

Thm 9: Chain Rule

Assume $y=f(x)$ is diff'able at $x=a$ & $z=g(y)$ is diff'able at $y=f(a)$. Then, $h(x) = g\circ f = g(f(x))$ is diff'able at $x=a$ and $h'(a) = f'(x)g'(f(x))$

Note that Leibniz notation pays off here:

For $z = g(y)$ and $y = f(x)$ we get:

$$g'(y) = \frac{dz}{dy} \quad \text{and} \quad f'(x) = \frac{dy}{dx}$$

Then, for $z = g(y) = g(f(x))$, we have that

$$\frac{dz}{dx} = g'(f(x)) \cdot f'(x) = \left.\frac{dz}{dy}\right|_{f(x)} \left.\frac{dy}{dx}\right|_{x}$$

that is: $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

Ex 1:

$$f(x) = \sin(x^2) \implies f'(x) = 2x\cos(x^2)$$

Other key derivatives

1) $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

$$= \frac{\frac{d}{dx}[\sin(x)]\cos(x) + \sin(x)\frac{d}{dx}[\cos(x)]}{[\cos(x)]^2}$$

$$= \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$

2) $\csc(x) = \dfrac{1}{\sin(x)}$

Q' rule $= \dfrac{-\frac{d}{dx}[\sin(x)]}{\sin^2(x)} = \dfrac{-\cos(x)}{\sin^2(x)}$

$= \dfrac{-1}{\sin(x)} \cdot \dfrac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$

3) $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

4) $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$

5) $f(x) = a^x$, $a > 0$

$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$

$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln(a)})$

$= \ln(a)\, e^{x\ln(a)}$

$= \ln(a)\, a^x$

More Practice with the rule.

Ex: $f(x) = 2^x \sin(x)$

$f'(x) = \ln(2) 2^x \sin(x) - 2^x \cos(x)$

Ex: $f(x) = (\sqrt[3]{x^4} + 5x + 7)(\tan x)$

$= (x^{\frac{4}{3}} + 5x + 7)(\tan x)$

$f'(x) = (\frac{4}{3}x^{\frac{1}{3}} + 5)(\tan x) + (x^{\frac{4}{3}} + 5x + 7)(\sec^2 x)$

$= (5 \cdot 0) + (7 \cdot 1)$

$= 7$

Week 7 lecture 20

We can do product rule to more than 3 fcns.

Ex: $(fgh)'$:

Notice $(fgh) = (fg)(h)$

So; $(fgh)' = [(fg)'](h) + (fg)(h')$

$= [(f)'g + f(g)']h + (fg)(h')$

豪豪完 $= f'gh + fg'h + fgh'$

Mathmatize: Q.

1) find $(fgh)'(7)$

- $f(7)=2$, a c function
- $g'(7)=3$, $g(-7)=4$, g is even fcn
- $h'(x)=h(x)$, $h(0)=3$ $\quad (h(x) = 3e^x$

Plug in → $0 + (2\cdot 3\cdot 3e^x) + (2\cdot 4\cdot 3e^x)$

$= 18e^x + 24e^x = 42e^x$

Ex: $f(x)\ \dfrac{2x}{\sin(e^x)}$

$$f'(x) = \frac{2\cdot \sin(e^x) + 2xe^x\cos(e^x)}{\sin^2(e^x)}$$

Ex: (Mathmatize)

$f(x) = \cot(\sin(x^{e^7}))$

$f'(x) = -e^7x^{e^7-1}\cos(x^{e^7})\csc^2(\sin(x^{e^7}))$

$f'(1) = -e^7\cdot(1)\cdot\cos(1)\csc^2(\sin(1))$

$= -e^7\csc^2(\sin(1))\cos(1)$

end of week 6

let's revisit def of $f'(a)$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

Then, for x values very close to $x=a$, we have

$$f'(a) \approx \frac{f(x)-f(a)}{x-a}$$

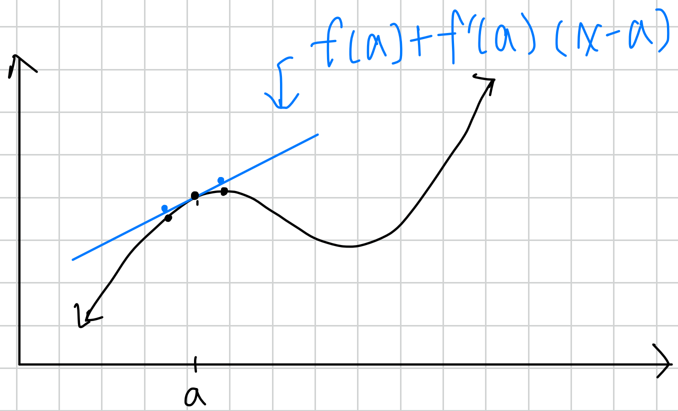
which we can rearrange to:

$$f(x) \approx f(a) + f'(a)(x-a) \quad \circledast$$

for x-value very close to a → tangent line equ

Note: We recognize $\circledast$ as the equation of the tangent line to $f(x)$ at $x=a$

Pictorially:



We can approximate the fcn value close to $x=a$ by taking a linearization or tangent line approx.

Week 8 lecture 21

Defn: linear Approximation *

Let f be diff'able at $x=a$, the linear approx to f at $x=a$ is the fcn:

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

其實就是 tangent line

Note: We just write $L_a(x)$ if f is clear

Ex:

Use the lin approx to estimate $\sin(\sqrt{10})$

take $\sin(\sqrt{10})$ and the approx to be at $a=\pi^2$

$\pi^2 \approx 10$ & $\sin(\sqrt{10})$ is easily found.

$$L_{\pi^2}^f(x) = f(\pi^2) + f'(\pi^2)(x-\pi^2)$$

Now, $f(\pi^2) = \sin(\sqrt{\pi^2}) = \sin(\pi) = 0$

And, $f'(x) = \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$

$$\Rightarrow f'(\pi^2) = \cos\pi\left(\frac{1}{2\pi}\right) = \frac{-1}{2\pi}$$

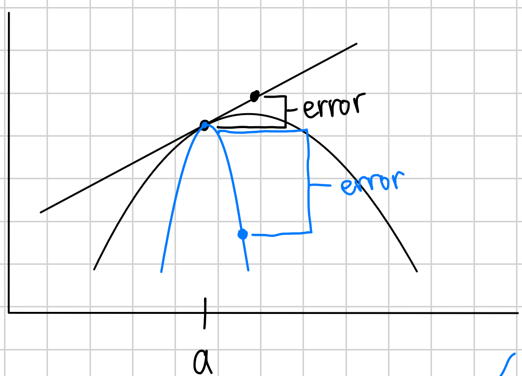
$$\therefore L_{\pi^2}^f(x) = 0 + \frac{-1}{2\pi}(x-\pi^2)$$

Thus, $\sin(\sqrt{10}) \approx L_{\pi^2}^f(10) = \frac{-1}{2\pi}(10-\pi^2) = \frac{-5}{\pi} + \frac{\pi}{2}$

$\doteq -0.020753$

(actual: 0.020683.)

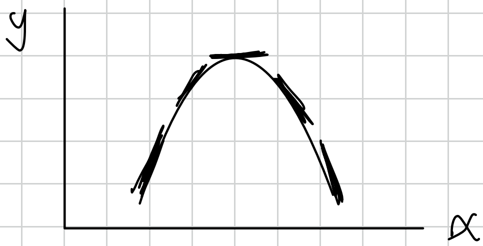
What affects how good our linear approximation is?



- how far we are from a
- how curve the fcn is near $x=a$
  ↳ more curve = more error

(here it is a over approx)

How do I describe the idea of "curvature"?



f 普狼普浪

→ r.o.c of r.o.c → 2nd derivative.

Thm 6. error in a linear approx ★

Assume that f is such that $|f''(x)| \leq M$ for each x in a interval I containing a point a

Then, $\underbrace{|f(x) - L_a^f(x)|}_{\text{error}} \leq \underbrace{\frac{M}{2}(x-a)^2}_{\text{upper bound of the error, the worst case.}}$

for each $x \in I$.

Ex:

Find an upper bound for the error on $L_{27}^{\sqrt[3]{x}}(x)$ on $[25, 30]$

① We have to find $M \Rightarrow (f''(x))$

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$f''(x) = -\frac{2}{9} x^{-\frac{5}{3}} = \frac{-2}{9\sqrt[3]{x^5}}$

Then, $|f''(x)| = \left|\frac{-2}{9\sqrt[3]{x^5}}\right| = \frac{2}{9\sqrt[3]{x^5}}$ on $[25, 30]$

★ This is maximized at $x = 25$ on $[25, 30]$

$\therefore M = \frac{2}{9(25)^{\frac{5}{3}}} \doteq 1.03965 \times 10^{-3}$

So, $\left| \sqrt[3]{x} - L_{27}(x) \right| \leq \frac{1.03965 \times 10^{-3}}{2} \times (x-27)^2$

★ RHS is maximize by 30 on [25, 30]

$$\leq \frac{1.03965 \times 10^{-3}}{2} \times 3^2$$

$$\left| \sqrt[3]{x} - L_{27}(x) \right| \leq 4.67892 \times 10^{-3}$$

<u>WSIC?</u>

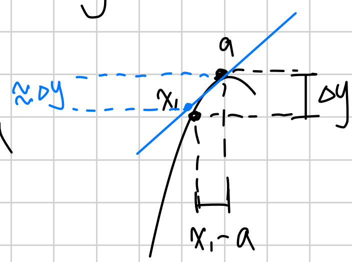
The linear approx is useful for estimating fcn values for nasty fcns. We always care abt errors to ensure our approx are not garbage.

↳ We will see similar ideas when we get to Taylor poly.

<u>Estimating Changes.</u>

Assume we know $f(a)$. How does $f(x)$ change if we move to x_1 near $x = a$.

$$\Delta y = f(x_1) - f(a) \qquad \Delta x = x_1 - a$$



Using the fact that $f(x_1) \approx L_a(x_1)$:

$$\Delta y \approx L_a(x_1) - f(a)$$
$$\approx (\cancel{f(a)} + f'(a)(x_1 - a)) - \cancel{f(a)}$$
$$\approx f'(a)(x_1 - a)$$
$$\Delta y \approx f'(a)\,\Delta x$$

Ex:
An Icecube of side length 3cm shrinks such that its side length reduces by 1mm

Estimate the change in volume of the ice cube.

↗ 3公分時體積變化率 × -1毫米

We have that $\Delta V \approx V'(30)(-1)$ where $V(s) = s^3$

$V'(s)$ = 體積 r.o.c.

$V'(s) = 3s^2 \quad V'(30) = 2700$

$\therefore \Delta V \approx 2700(-1) = \underline{-2700} \text{ mm}^3$

(actual: -2611 mm^3)

Recall: We use IVT and use bisection method for root finding.

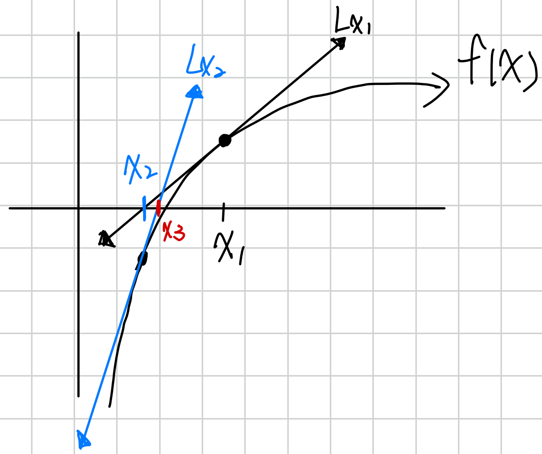
We can use linear approx to motivate another algorithm

$\Rightarrow$ Newton's method.

Week 8 lecture 22

Newton's Method

Pictorially:



The Method:

- Make a guess x_1 of where the root is (IVT useful)
- ⊛ Take the linear approx L_x^F, and find where it intersects the x-axis, call the value x_2
- Repeat at x_2 and find x_3 and so far.

Examining ⊛

We looking for $L_x^F(x_2) = 0$

$$L_x^f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) = 0$$

$$\Rightarrow f'(x_1)x_2 = f'(x_1)x_1 - f(x_1)$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{if } f'(x_1) \neq 0$$

$\therefore$ we iteratly use: $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

Note: this doesn't always converge.

ex. - L_{x_n} is a horizontal tangent line.

- pick a point too far from the root, the seq diverge.

- Converges much faster than bisection method.

Ex. Find the root of $x^3+5x^2-3x-17$ between $[1,3]$, accurate to 7 d.p.

IVT 然後取中間

Note that $f(1)<0$ and $f(3)>0$, and $f(x)$ is cts everywhere. Then, by IVT there is a root on $[1,3]$

Lets take $x_1 = 2$

Now, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Here, $f'(x) = 3x^2+10x-3$

So, $x_{n+1} = x_n \frac{x_n^3+5x_n^2-3x_n-17}{3x_n^2+10x_n-3}$

Then, $x_2 = 2 - \frac{2^3+5(2)^2-3(2)-17}{3(2)^2+10(2)-3}$

$x_2 = 2-\frac{5}{29} \doteq 1.82758621$

$x_3 \doteq 1.81486224$

$x_4 \doteq 1.81479452$

$x_5 \doteq 1.81479452$

$\therefore$ the root is 1.8147945.

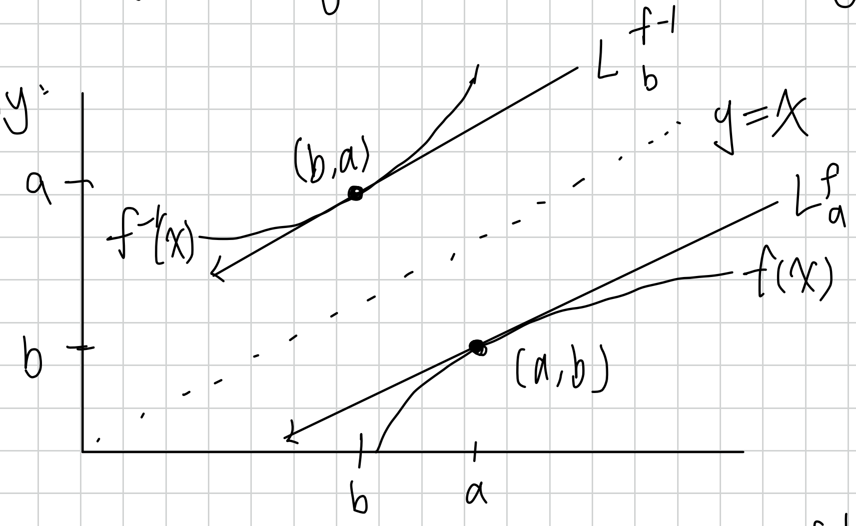
We now return to derivative rules.

The linear approx is useful for derivative of inverse fcns.

Remember, <u>geometrically</u>, inversion is the reflection of a fcn accross the line $y=x$

Algebraically, we find inverse by swapping x & y.

Pictorially:



There are two ways we can formulate $L_b^{f^{-1}}$:

1) Directly, by formula : $L_b^{f^{-1}} = f^{-1}(b) + (f^{-1})'(b)(x-b)$ ⊛

2) Algebraically the inverse of L_a^f

$$L_a^f = y = f(a) + f'(a)(x-a)$$

swap → $\quad = x = f(a) + f'(a)(y-a)$

solve → $\quad y = \frac{1}{f'(a)}(x - f(a)) + a \quad (f'(a) \neq 0)$

Note: $f(a) = b$ and $f^{-1}(b) = a$ by defn.

This gives $L_b^{f^{-1}} = \frac{1}{f'(a)}(x-b) + f^{-1}(b)$ ⊛⊛

Comparing terms of ⊛ and ⊛⊛ we conclude:

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

this leads to:

Thm 10: Inverse fcn. Thm (IFT)

Assume $y = f(x)$ is cts and invertable on $[c, d]$ we inverse $x = f^{-1}(y)$, and f is diff'able at $a \in (c, d)$. If $f'(a) \neq 0$, then f^{-1} is diff'able at $b = f(a)$, and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

Moreover, L_a^f is also invertable and.

$$(L_a^f)^{-1}(x) = L_b^{f^{-1}}(x) = L_{f(a)}^{f^{-1}}(x)$$

(Silly Goofy) Ex:

Let $f(x) = 5\sqrt{x}$. Find $(f^{-1})'(5)$,

By the IFT, $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))}$

$(f^{-1})'(5)$

$= \dfrac{1}{f'(f^{-1}(5))}$

Given $f(x) = 5\sqrt{x}$, we know

$$f'(x) = \frac{5}{2}\left(x^{-\frac{1}{2}}\right) = \frac{5}{2\sqrt{x}}$$

Find f^{-1}: $y = 5\sqrt{x}$

swap: $x = 5\sqrt{y} \xrightarrow{\text{solve}} \frac{x}{5} = \sqrt{y} \rightarrow y = \frac{x^2}{25} = f^{-1}(x)$

then, $f^{-1}(5) = 1$.

then, $f'(f^{-1}(5)) = f'(1) = \frac{5}{2}\left(1^{-\frac{1}{2}}\right) = \frac{5}{2}$

So, by IFT $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{\frac{5}{2}} = \dfrac{2}{5}$

check directly: $f^{-1}(x) = \frac{x^2}{25} \rightarrow (f^{-1})'(x) = \frac{2x}{25}$

$\Rightarrow (f^{-1})'(5) = \frac{2}{5}$ ✓

We can arrive IFT via chain rule:

Assume f has inverse f^{-1}, where both are diff'able.

Then, we know by defn: $f(f^{-1}(x)) = x$

$$\frac{d}{dx}[f(f^{-1}(x))] = \frac{d}{dx}[x]$$

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

e^x 微分 → 带入 $f^{-1}(x)$

练习

$$\frac{1}{f'(f^{-1}(x))} \qquad \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Week 8 lecture 23

We can us IFT to find derivative of $\ln(x)$

Ex:

let $f = e^x$ we know $f^{-1}(x) = \ln(x)$, $x > 0$

By IFT, $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

that is, $(\ln(x))'(x) = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$

$\therefore \dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$

We can use IFT to find inverse trig.

Ex: $\dfrac{d}{dx}(\arccos(x))$

Note: the domain is $[-1,1]$

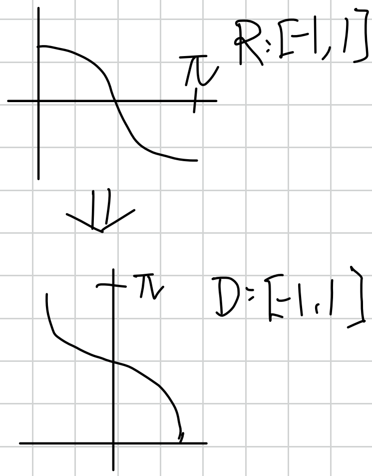
The IFT tells us it's diff'able.

sidebar:

$\cos(x)$

restrict $\to [0,\pi]$

cause normal arccos is not a fcn.



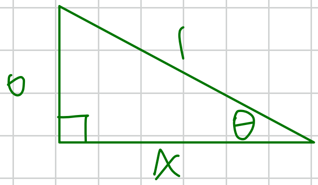
We have $f(x) = \cos(x)$

and $f^{-1}(x) = \arccos(x)$

Plugin IFT: $(f^{-1})'(x) = \dfrac{1}{-\sin(\arccos(x))}$

Let's rewrite: $\sin(\arccos(x))$

Remember $\arccos(x)$ is an angle θ



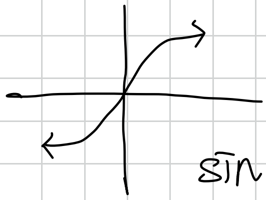
$\arccos(x) = \theta \Rightarrow \cos(\theta) = \dfrac{x}{1}$

By 勾股, o side is $\sqrt{1-x^2}$

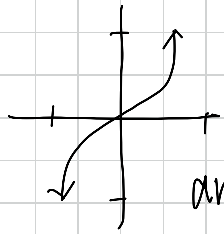
Then, $\sin\theta = \dfrac{\sqrt{1-x^2}}{1}$

Thus, we have that $\dfrac{d}{dx}[\arccos(x)] = \dfrac{-1}{\sqrt{1-x^2}}$

Similarily:

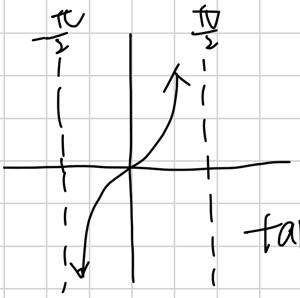


$\sin(x)$: D: $[-\pi/2, \pi/2]$ R: $[-1, 1]$ $\Rightarrow$

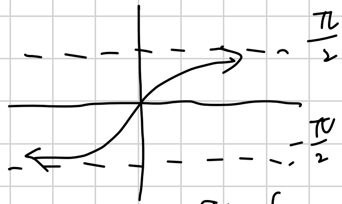


$\arcsin(x)$: D: $[-1, 1]$ R: $[-\pi/2, \pi/2]$

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$



$\tan(x) \to$ D: $(-\pi/2, \pi/2)$ R: $(-\infty, \infty)$ $\Rightarrow$



arctan: D: $(-\infty, \infty)$ R: $(-\pi/2, \pi/2)$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

Ex1:

$$f(x) = \arcsin(2^{5x})$$

$$f'(x) = \frac{1}{\sqrt{1-2^{10x}}} \cdot 5 \cdot \ln(2) \cdot 2^{5x}$$

Ex2:

$$f(x) = \ln(\arctan(e^{\sin(x)}))$$

$$f'(x) = \frac{1}{\arctan(e^{\sin(x)})} \cdot \frac{1}{1+e^{2\sin(x)}} \cdot \cos(x)e^{\sin(x)}$$

$$= \frac{\cos(x)e^{\sin(x)}}{(\arctan(e^{\sin(x)}))(1+e^{2\sin(x)})}$$

Week 9, lecture 24

So far, we've done explicit fcns. , $y=f(x)$

Now, Implicit fcns.

Ex:

Find y' for $x^3y^5+2x = y^3+4$ (y is dependent, x is independent)

$\Rightarrow 3x^2y^5 + x^3 5y^4 y' + 2 = 3y^2 y'$

write y' or $\frac{dy}{dx}$

$\Rightarrow x^3 5y^4 y' - 3y^2 y' = -3x^2y^2 - 2$

$\Rightarrow y'(x^3 5y^4 - 3y^2) = -3x^2y^2 - 2$

$$\Rightarrow y' = \frac{-3x^2y^2-2}{x^3 5y^4 - 3y^2}$$

Ex: (MM)

Find $\frac{dy}{dx}\Big|_{(x,y)=(2,0)}$ $xe^y = x - y$

$\Rightarrow e^y + xe^y y' = 1 - y'$

$\Rightarrow xe^y y' + y' = 1 - e^y$

$\Rightarrow y'(xe^y+1) = 1 - e^y$

$$\Rightarrow y' = \frac{1-e^y}{xe^y+1}\Big|_{(2,0)} = \frac{1-1}{2+1} = \frac{0}{3} = 0$$

Note: Be careful, Given $\overbrace{x^2y^2}^{+} = \overbrace{-72}^{-}$ we could do the process outlined above and we'd find $y' = \frac{-y}{x}$

But there is no such curve $x^2y^2 = -72$, so meaningless.

Extend the idea of implicit and derivative rules to:

"mother of all rules" Logarithmic Differentiation.

allow us to find fcns form:

$$h(x) = g(x)^{f(x)} \qquad (g(x) > 0)$$

Ex: 1:

$y = x^x$ Find y' $(x > 0)$

Start by take "ln" of both side,

$$\ln(y) = \ln(x^x)$$

$$\Rightarrow \ln(y) = x\ln(x)$$

f' $\Rightarrow y'\frac{1}{y} = \ln(x) + 1 \Rightarrow y' = y(\ln(x) + 1)$ plugin.

$$\Rightarrow y' = x^x(\ln(x) + 1) \ *$$

But, look at this,

Ex 2:

$$y = \frac{(x-3)^3(x+4)^2(x-1)}{(x+1)^2(x^2+x+1)^3}$$, Find y', doable, but hella long.

So,

$$\ln(y) = 3\ln(x-3) + 2\ln(x+4) + \ln(x-1) - 2\ln(x+1) - 3\ln(x^2+x+1)$$

$$y'\frac{1}{y} = \frac{3}{x-3} + \frac{2}{x+4} + \frac{1}{x-1} - \frac{2}{x+1} - \frac{3(2x+1)}{x^2+x+1}$$

$$y' = y\left(\frac{3}{x-3} + \frac{2}{x+4} + \frac{1}{x-1} - \frac{2}{x+1} - \frac{6x+3}{x^2+x+1}\right) \text{ (sub } y\text{)}$$

$$y' = \frac{(x-3)^3(x+4)^2(x-1)}{(x+1)^2(x^2+x+1)^3}\left(\frac{3}{x-3} + \frac{2}{x+4} + \frac{1}{x-1} - \frac{2}{x+1} - \frac{6x+3}{x^2+x+1}\right)$$

#

We now revisit the idea of extrema.

Recall EVT, if cts $[a,b]$ $\exists$ global extrema.

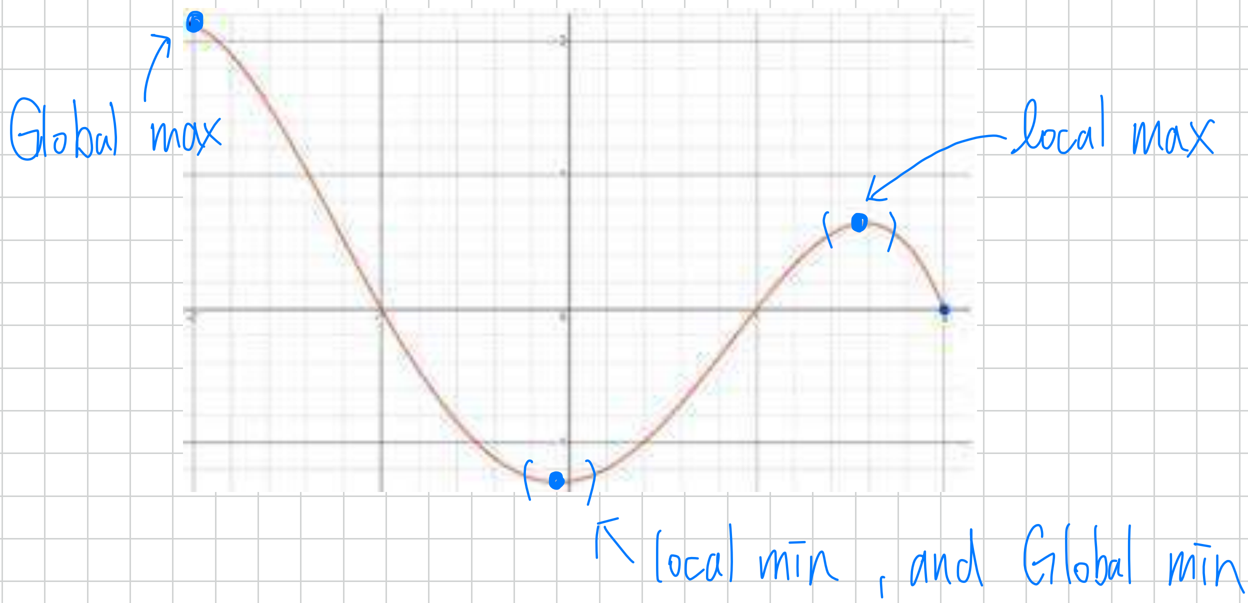
Defn: Local Max / Min.

A point c is called a local max/min for a fcn f if $\exists$ an open interval (a,b) containing c such that

$$f(x) \le f(c) \;/\; f(x) \ge f(c) \quad \forall x \in (a,b)$$

Note, the means Endpoints are NOT local extrema and, Global extrema occur at non-endpoints are also Local extrema.

Ex:



Week 9 lecture 25

Thm 11 local extrema theorm (LET)

If c is a local extremum for f and $f'(c)$ exists, then $f'(c)=0$

Pf:

Assume wlog we have c is a local min, and $f'(c)$ exist.

Since $f'(c)$ exist, we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

$$= \lim_{h\to 0^+} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$$

Since c is a local min $\exists (a,b) \ni c \in (a,b)$ and $f(c) \leq f(x)$ $\forall x \in (a,b)$

Then, for $h>0$ small enough that $c < c+h < b$ we have that $f(c+h) \geq f(c)$

This means that

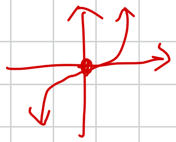
$$f'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \geq 0 \quad \left(\frac{\geq 0}{>0}\right)$$

And, for $h<0$ small enough that $a < c+h < c$, we have $f(c+h) \geq f(c)$ as well. h is negative.

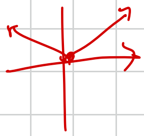
Then, $f'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \leq 0 \quad \left(\frac{\geq 0}{<0}\right)$

(similar for local max) ∎

Note: • $f'(c)=0 \not\Rightarrow$ local extrema.

(for ex: $y=x^3$, $y=0$ )

• local max/min $\not\Rightarrow f'(c)=0$

(for ex: $y=|x|$, $y=0$ )

We are not just focus on $f'(c)=0$ to find local extrema.

Our candidate point of interest:

Defn: critical points (cps),

A point c in the domain of the function f is called a critical point for f if either

$$f'(c)=0 \text{ or } f'(c) = \text{DNE}$$

We note that on a close interval $[a,b]$ (for a cts fcn.) EVT guaranteed a global max/min, either at endpoints or (a,b).

If they occur in (a,b), then, they will be a c.p's.

as we note that global extrema are also local extrema.

Here's how we find global extrema for cts fcn f on $[a,b]$

1. calculate $f(a)$ & $f(b)$ (endpoints)
2. find $f'(x)$ mention it everytime.
3. find all cp's. ($f'=0$ or DNE)
4. Calculate the fcn value at cp's.
5. Global max is the greatest from $1 \sim 4$
   min is the least from $1 \sim 4$

Ex: Find the global extrema of $f(x) = x\sqrt{4-x^2}$ on $[-1,2]$

1. $f(-1) = -\sqrt{3}$ and $f(2) = 0$

2. $f'(x) = \sqrt{4-x^2} + \dfrac{-x^2}{\sqrt{4-x^2}}$

3. cp's $f'(x)$ DNE $\longrightarrow$ when $x = \pm 2$ (-2 out of bound)

$f'(x) = 0 \Rightarrow \quad \sqrt{4-x^2} = \dfrac{x^2}{\sqrt{4-x^2}}$

$$4 - x^2 = x^2$$

$$x = \pm\sqrt{2} \quad (-\sqrt{2} \text{ out of bound})$$

4. Check cp's $^{(1)+(2)+(3)}$ $f(2) = 0$, $f(\sqrt{2}) = 2$

$\therefore$ Global max: $(\sqrt{2}, 2)$

Global min: $(-1, -\sqrt{3})$

Next, we build up to a central calc thm.

First: A scenario

My drive to work is 5km. If I make it in to work in 5 mins, and the speed limit is 50 km/hr can you guarantee I speed at some point?

avg velocity $\dfrac{\Delta s}{\Delta t} = \dfrac{5\text{km}}{1/12\,\text{hr}} = 60\ \text{km/hr}$.

Then either:

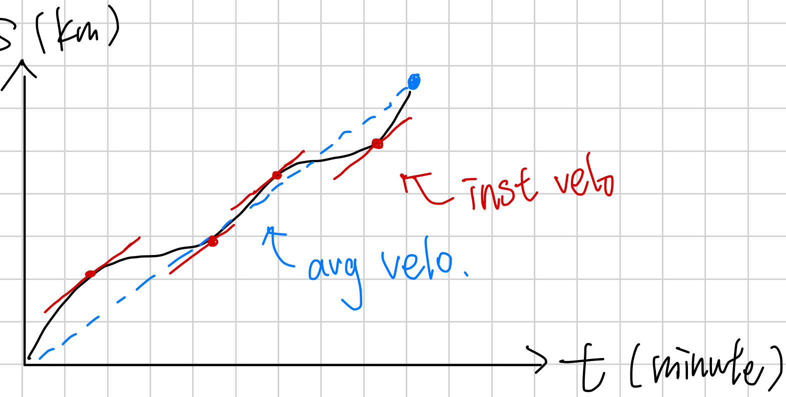
- I'm always driving 60 km/hr
- Driving below & above average.

$\rightarrow$ because the speed is like a cts fcn. I must hit 60.

Takeaway: At some point, the inst roc = avg roc.

$\downarrow$

Pictorially:



Motivating Idea: between the zeros of $f(x)$, there is a zero of $f'(x)$.

"What comes up must come down theorm" 

Thm 2: Rolls thm

Assume $f(x)$ is cts on $[a,b]$ and diff'able on (a,b), and that $f(a)=f(b)=0$.
Then, $\exists\ c \in (a,b) \ni f'(c)=0$

Pf:

Case 1: $f(x)=0$. Then, $f'(x)=0$, $f'(c)=0\ \forall c \in (a,b)$

Case 2: $f(x_0)>0$ for some $x_0 \in (a,b)$

Then, by EVT, since f is cts on $[a,b]$ $\exists$ a global max on $[a,b]$.

Since $f(a)=f(b)=0<f(x_0)$, then the global max must be at a $c\in(a,b)$ thus also a local max.

Since f is diff'able on (a,b), then by LET it must be that $f'(c)=0$

Case 3: $f(x_0) < 0$ for some $x_0 \in (a, d)$

same $\rightarrow$

Week 9, lecture 26

Thm I : Mean Value theorm (MVT)

Assume $f(x)$ is a cts on $[a,b]$ ① and diff'able on (a,b) ②

Then $\exists\ c \in (a,b)$ such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$
↑ inst roc ↖ avg roc

Pf:

we introduce the fcn $h(x) = f(x) - \Big[\underbrace{f(a) + \frac{f(b)-f(a)}{b-a}(x-a)}_{\text{secant line between } a \,\&\, b.}\Big]$

This is essentially the height of or fcn $f(x)$ above the secant line.

Notice : • $h(x)$ is cts on $[a,b]$ & diff'able on (a,b)

• $h(a) = h(b) = 0$ (so we can apply Rolls thm)

That means,

$\exists\ c\ (a,b) \ni h'(c) = 0$

Now, $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

Then $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Note: Always check the hypothesis, ①, ② cts $[a,b]$ diff (a,b)

→ the fcn is cts and diff'able.

WSIC2 : MVT is a central theorm as it is important for future cal/cul concept.

Defn: Antiderivative

Given a fcn f, an antiderivative of a function F such that $F'(x) = f(x)$.

If $F'(x) = f(x)$ $\forall x \in I$, we say F is an antiderivative for f on I

Note: Unlike derivative, antiderivative are not unique.

Ex: $f(x) = 5 \to F(x) = 5x$, $F(x) = 5x+1$, $F(x) = 5x-1$

We now work towards that anti-D differ by constant.

Thm 3: Constant fcn theorm:

If $f'(x) = 0$ $\forall x \in I$ then $\exists \alpha \in \mathbb{R}$ s.t. $f(x) = \alpha$ $\forall x \in I$

Pf:

Choose $x_1, x_2 \in I$, $x_1 \neq x_2$, wlog let $x_2 > x_1$

Let $f(x_1) = \alpha$. Let $f'(x) = 0$, $\forall x \in I$

Since f is diff'able an cts on I,

we can apply MVT on $[x_1, x_2]$

$$\therefore \exists c \in (x_1, x_2) \ni f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

hypothesis.

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) = f(x_1)$$

$$\Rightarrow f(x_2) = \alpha$$

Since x_2, x_1 are arbitrary, $f(x) = \alpha$, $\forall x \in I$ ∎

Thm 4: Antiderivative theorm

If $f'(x) = g'(x)$ $\forall x \in I$ then $\exists\, \alpha \in \mathbb{R}$ s.t.

$$f(x) = g(x) + \alpha \quad \forall x \in I.$$

Pf: Consider $h(x) = f(x) - g(x)$

Note: • $h(x)$ is cts and diff'able on I as f & g are.

• $h'(x) = f'(x) - g'(x) = 0$ $\forall x \in I$

by hyp $f' = g'$

Then, we apply CFT to say:

$$h(x) = \alpha \quad \forall x \in I \quad (\alpha \in \mathbb{R})$$

Thus, $f(x) - g(x) = \alpha$ $\forall \alpha \in I$

$$\Rightarrow f(x) = g(x) + \alpha \quad \forall x \in I. \blacksquare$$

Leibiniz Notation for Antiderivatives.

$\int f(x)\,dx$ denotes the family of antiderivative of $f(x)$

→ indefinite integral of f

↳ $f(x)$ is intergrand.

Ex: $\int 5\,dx = 5x + C$

Power Rule for Anti-D

If $\alpha \neq -1$, then,

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

絕對值

- $\int 0\,dx = C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
- $\int \cos x\,dx = \sin x + C$
- $\int e^x\,dx = e^x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
- $\int (\alpha_1 f(x) + \alpha_2 g(x) + \alpha_3 h(x) + \dots)\,dx$

$$= \alpha_1 \int f(x)\,dx + \alpha_2 \int g(x)\,dx + \alpha_3 \int h(x)\,dx \dots$$

If the anti-D exist.

Week 10 lecture 27

Use MVT to prove thms!

Thm 6: Increasing / Decreasing fcn thm.

Let I be an interval and $x_1, x_2 \in I$ where $x_1 < x_2$

1) If $f'(x) > 0 \ \forall x \in I$, then $f(x_1) < f(x_2)$ 後>前
   → That is, f is increasing on I
2) If $f'(x) \geq 0 \ \forall x \in I$, then $f(x_1) \leq f(x_2)$ 後≥前
   → that is, f is non-decreasing on I
3) If $f'(x) < 0 \ \forall x \in I$, then $f(x_1) > f(x_2)$ 後<前
   → that is, f is decreasing on I
4) If $f'(x) \leq 0 \ \forall x \in I$, then $f(x_1) \geq f(x_2)$ 後≤前
   → that is, f is non-increasing on I.

Pf: (4)

Since $f'(x)$ exist on I, f is diff'able and cts on I

Thus, f is cts on $[x_1, x_2]$ and diff'able on (x_1, x_2)

↳ state hyp for MVT

So we can apply MVT on $[x_1, x_2]$

That is $\exists \ c \in (x_1, x_2) \ni$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(x) \leq 0 \ \forall x \in I$

$$\text{So, } f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$$

Since $x_2 > x_1$, $x_2 - x_1 > 0$

Thus, $f(x_2) - f(x_1) \le 0$

$\Rightarrow f(x_2) \le f(x_1)$, non-inc.

Note:

- f inc $\not\Rightarrow f' > 0$ ex:  $y = x^3$
- f dec $\not\Rightarrow f' < 0$ ex  $y = -x^3$

f' could be 0 or DNE

Thm 7: Bounded Derivative Theorm. (BDT)

same as MVT

Assume f is cts on $[a,b]$ and diff'able (a,b), and $m \le f'(x) \le M \quad \forall x \in (a,b)$

Then, $f(a) + m(x-a) \le f(x) \le f(a) + M(x-a) \quad \forall x \in [a,b]$

① 找接近f的整数 a, b ② 把 a, b 代入 f(x) 找 m, M.
③ 全部代入公式

Pf: By the hypothesis of the theorm, we can apply MVT to f on $[a,b]$

Further, MVT would also apply on $[a,x]$ for some $x \in (a,b)$.

That is, $\exists\, c \in (a, x_1) \ni$

$$f'(c) = \frac{f(x_1) - f(a)}{x_1 - a}$$

From the hypothesis, we have that since $c \in (a,b)$ we know $m \le f'(c) \le M$.

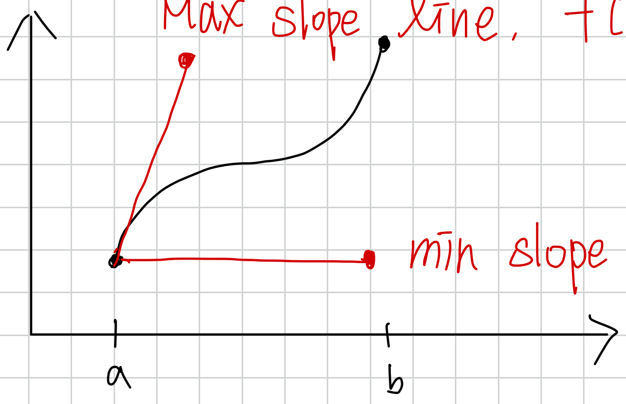
Thus, $m \le \dfrac{f(x_1) - f(a)}{x_1 - a} \le M$, $x_1 > a \Rightarrow x_1 - a > 0$

$$m(x_1-a) \le f(x_1)-f(a) \le M(x_1-a)$$

$$f(a)+m(x_1-a) \le f(x_1) \le f(a)+M(x_1-a)$$

Since x_1 was arbitrary from $[a,b]$, holds $\forall x \in [a,b]$ ∎

Pictorally:



Ex:

$[49, 64]$

Prove $\sqrt{50} \in [7+\frac{1}{16}, 7+\frac{1}{14}]$ using BDT

Pf:

take $f(x)=\sqrt{x}$, then, $f'(x)=\frac{1}{2\sqrt{x}}$

we know that $f(x)$ is cts on $[49, 64]$ and we see $f'(x)$ exists on $(49, 64)$. form 大 and 小.

On $(49, 64)$, we see that

$$\frac{1}{2\sqrt{64}}=\frac{1}{16} \le f'(x) \le \frac{1}{2\sqrt{49}}=\frac{1}{14}$$

Then, by BDT,

$$\underset{f(a)}{\sqrt{49}}+\underset{m}{\frac{1}{16}}(x-\underset{a}{49}) \le f(x) \le \underset{f(a)}{\sqrt{49}}+\underset{M}{\frac{1}{14}}(x-\underset{a}{49})$$

then, $7+\frac{1}{16}(x-49) \le \sqrt{x} \le 7+\frac{1}{14}(x-49)$

$\Rightarrow 50$, $7+\frac{1}{16}(50-49) \le \sqrt{50} \le 7+\frac{1}{14}(50-49)$

$\Rightarrow 7+\frac{1}{16} \le \sqrt{50} \le 7+\frac{1}{14}$

Thm 8:

Assume f & g are cts on $x=a$ with $f(a)=g(a)$

1) If both f & g are diff'able for $x>a$ and if $f'(x) \le g'(x)$ $\forall x>a$, then

$$f(x) \le g(x) \quad \forall x>a$$

2) If both f & g are diff'able for $x<a$ and if $f'(x) \le g'(x)$ $\forall x<a$, then

$$f(x) \ge g(x) \quad \forall x<a$$

Pf of (2):

Take all the hyp. Define $h(x)=f(x)-g(x)$

Then, h is cts at $x=a$ & diff'able at $x<a$

Further, we have $h'(x)=f'(x)-g'(x)$.

and since $f'(x) \le g'(x)$ $\forall x<a$, $h(x) \le 0$ $\forall x<a$.

Note: we can apply MVT on $[x,a]$. That is,

$\exists\, c \in (x,a) \ni$

$$h'(c) = \frac{h(a)-h(x)}{a-x} \quad (\text{since } c<a)$$

Now, since, $x<a$, $a-x>0$, and since $f(a)=g(a)$ we have $h(a)=0$.

Thus we have that $0-h(x) \le 0$

$$h(x) \ge 0$$

$$\Rightarrow f(x)-g(x) \ge 0$$

$$\Rightarrow f(x) \ge g(x) \quad \text{for } x<a \quad \blacksquare$$

Note:

- If $f'(x) < g'(x)$ we get $f(x)<g(x)$ $\forall x>a$

$$f(x)>g(x) \quad \forall x<a$$

Ex: Prove that $x^2 > \ln(1+x^2)$ for $x<0$

Pf: let $f(x) = x^2$ Let $g(x) = \ln(1+x^2)$

Also, note that $f(0) = g(0) = 0$

$f'(x) = 2x$

$g'(x) = \frac{2x}{1+x^2}$

↳ which we note exist everywhere.

for $x<0$ we have that $1+x^2 > 1$

then, for $x<0$, we have $\frac{2x}{1} < \frac{2x}{1+x^2}$ (蛤?!, 因為x是負的)

that is, for $x<0$, we have, $g'(x) > f'(x)$

then, by thm, for $x<0$, $f(x) > g(x)$ ∎

小心看

Week 10 lecture 28

Intermediate form:

- $0/0$ • $\pm\frac{\infty}{\infty}$ • $0\times\infty$
- $\infty-\infty$ • 1^∞ • ∞^0 • 0^0

Thm 14: L'Hôspital's rule (L'HR)

Assume $f'(x)$ & $g'(x)$ exist near $x=a$, $g'(x)\neq 0$ near $x=a$, except possibly at $x=a$, and that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \pm\frac{\infty}{\infty}$$

Then,
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note: L'H applies to $a\in\mathbb{R}$, a "=" $\pm\infty$ and for one side lims
- you may have to apply L'H repeatly.

Type I: Direct

Ex.1:
$$\lim_{x\to 0}\frac{e^x-x-1}{5\cos(x)+x^2-5} \quad \text{"}\frac{0}{0}\text{"} \quad \overset{L'HR}{=} \lim_{x\to 0}\frac{e^x-1}{-5\sin(x)+2x} \quad \text{"}\frac{0}{0}\text{"}$$

$$\overset{L'HR}{=} \lim_{x\to 0}\frac{e^x}{-5\cos(x)+2} = -\frac{1}{3} \ \#$$

Ex 2:
$$\lim_{x\to 0}\frac{\sin(x)}{x} \overset{\text{"}\frac{0}{0}\text{"}}{\underset{L'HR}{=}} \lim_{x\to 0}\cos(x) = 1 \ \#$$

E3:
$$\lim_{x\to 1}\frac{\ln(x)}{1-\cos(x-1)} \overset{L'HR}{=} \lim_{x\to 1}\frac{\frac{1}{x}}{\sin(x-1)} = \frac{1}{0} = \text{DNE} \ \#$$

Type 2: $\frac{\pm\infty}{\infty}$ Direct

Ex1: $\lim\limits_{x\to\infty} \frac{\ln(x)}{x} \overset{L'HR}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$ #

Ex2: $\lim\limits_{x\to\infty} \frac{x^2+4x+1}{3x^2-1} \overset{L'HR}{=} \lim\limits_{x\to\infty} \frac{2x+4}{6x} \overset{L'HR}{=} \lim\limits_{x\to\infty} \frac{2}{6} = \frac{1}{3}$ #

Type 3: $0\cdot\infty$ Indirect

Strategy: make it look like $\frac{\infty}{\infty}$

by dividing the reciprocal of one of the fcns.

Ex1: $\lim\limits_{x\to 0^+} x\ln(x) \; \text{"}=0\cdot-\infty\text{"} = \lim\limits_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \; \text{"}=\frac{\infty}{\infty}\text{"} \overset{L'HR}{=} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{-x^2}}$

$= \lim\limits_{x\to 0^+} \frac{-x^2}{x} = \lim\limits_{x\to 0^+} -x = 0$ #

Ex2: $\lim\limits_{x\to -\infty} e^x x^{5/3} \; \text{"}=0\cdot\infty\text{"} = \lim\limits_{x\to -\infty} \frac{x^{5/3}}{e^{-x}} \; \text{"}=\frac{-\infty}{\infty}\text{"} \overset{L'HR}{=} \lim\limits_{x\to -\infty} \frac{\frac{5}{3}x^{\frac{2}{3}}}{-e^{-x}} \; \text{"}\frac{-\infty}{-\infty}\text{"}$

$\overset{L'HR}{=} \lim\limits_{x\to -\infty} \frac{\frac{10}{9}x^{-\frac{1}{3}}}{e^{-x}} = \frac{0}{\infty} = 0$ #

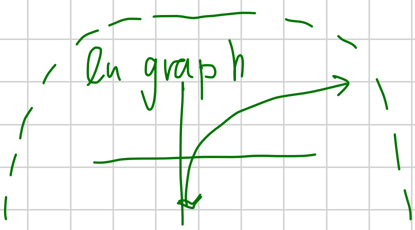
Type 4: $\infty-\infty$ Indirect.

Strategy: combine it into 1 term so it looks like type 3.

Ex1: $\lim\limits_{x\to 0} \left[\cot(x) - \frac{1}{x}\right] \; \text{"}=\infty-\infty\text{"} = \lim\limits_{x\to 0} \left[\frac{1}{\tan(x)} - \frac{1}{x}\right] = \lim\limits_{x\to 0} \left[\frac{x-\tan(x)}{x\tan(x)}\right] \; \text{"}=\frac{0}{0}\text{"}$

$\overset{L'HR}{=} \lim\limits_{x\to 0} \left[\frac{1-\sec^2(x)}{\tan(x)+x\sec^2(x)}\right] \; \text{"}=\frac{0}{0}\text{"} \overset{L'HR}{=} \lim\limits_{x\to 0} \left[\frac{-2\sec^2(x)\tan(x)}{\sec^2(x)+\sec^2(x)+2x\sec^2(x)\tan(x)}\right]$

$= \frac{0}{2} = 0$



EX2: $\overset{\text{"}=\infty+(-\infty)\text{"}}{\lim_{x\to\infty} \ln(3x)+\ln\left(\frac{17}{x+7}\right)} = \lim_{x\to 0} \ln\left(3x\cdot\frac{17}{x+7}\right) = \lim_{x\to 0} \ln\left(\frac{51x}{x+7}\right)$

Since ln is cts: $= \ln\left(\lim_{x\to\infty} \frac{51x}{x+7}\right)^{\text{"}\frac{\infty}{\infty}\text{"}} \overset{L'HR}{=} \ln\left(\lim_{x\to\infty} \frac{51}{1}\right) = \ln(51)$ #

Type 5: 1^{∞}, ∞^{0}, and 0^{0} Indirect.

Strategy: use $e^{\ln[\text{indet}]}$ which means the exponent will be "$0\cdot\infty$" we can use e^{x} to distribute the limit.

$\left[\ \ln[\text{"}1^{\infty}\text{"}] \text{ "="} \infty \ln(1) \text{ "="} \infty\cdot 0\ \right]$

Week 10 lecture 29

Ex1: $\lim_{x\to 0^+} x^x \overset{``0^0"}{=} \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln(x)}$ cause e is constant.

Examine: $\lim_{x\to 0^+} x\ln(x) = 0$ (last lecture)

$= e^0 = 1$ #

Ex2: $\lim_{x\to\infty} \left(1+\frac{P}{x}\right)^x \overset{``1^\infty"}{=} \lim_{x\to\infty} e^{x\ln(1+\frac{P}{x})}$, by continuity, we have,

$= e^{\lim_{x\to\infty} x\ln(1+\frac{P}{x})}$

Examine $\lim_{x\to\infty} x\ln\left(1+\frac{P}{x}\right) \overset{``\infty\cdot 0"}{=} \lim_{x\to\infty} \frac{\ln(1+\frac{P}{x})}{\frac{1}{x}}$

$\overset{L'HR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{P}{x}}\cdot\frac{-P}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{P}{1+P/x} = P$

$\lim_{x\to\infty} \left(1+\frac{P}{x}\right)^x = e^P$

Note: this gives $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = 1$

or $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = 0$

Mathmatize: "$\frac{0}{0}$"

1. $\lim_{x\to -4} \frac{\sin(\pi x)}{x^2-16} \overset{L'HR}{=} \lim_{x\to -4} \frac{\pi\cos(\pi x)}{2x} = \frac{\pi}{-8}$ #

2. $\lim_{x\to 1^+} \left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right] \overset{``0\cdot\infty"}{=} \lim_{x\to 1^+} \frac{(x-1)}{\cot(\frac{\pi}{2}x)}$ "$\frac{0}{0}$"

$\overset{L'HR}{=} \lim_{x\to 1^+} \frac{1}{\frac{\pi}{2}\left(-\csc^2(\frac{\pi}{2}x)\right)} = \frac{-2}{\pi}$ #

3. $\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right) \overset{\text{"}\infty-\infty\text{"}}{=} \lim_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right) \overset{\text{"}\infty\cdot 0\text{"}}{=} \lim_{x\to\infty} \frac{\left(e^{\frac{1}{x}}-1\right)}{\frac{1}{x}}$ $\text{"}\frac{0}{0}\text{"}$

$\overset{L'HR}{=} \lim_{x\to\infty} \frac{\frac{-1}{x^2}e^{\frac{1}{x}}}{\frac{-1}{x^2}} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1$ #

4. $\lim_{x\to 0}(\cos(x))^{\csc(x)} = \lim_{x\to 0} e^{\csc(x)\ln(\cos(x))} = e^{\lim_{x\to 0}\csc(x)\ln\cos(x)}$

Examine: $\lim_{x\to 0}\csc(x)\ln(\cos(x)) \overset{\text{"}\infty\cdot 0\text{"}}{=} \lim_{x\to 0} \frac{\ln(\cos(x))}{\frac{1}{\csc(x)}}$ $\text{"}\frac{0}{0}\text{"}$

$\overset{L'HR}{=} \lim_{x\to 0} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{\cos(x)}{1}} = \lim_{x\to 0}\frac{-\sin(x)}{\cos^2(x)} = \frac{0}{1} = 0$

$= e^0 = 1$ #

HW:

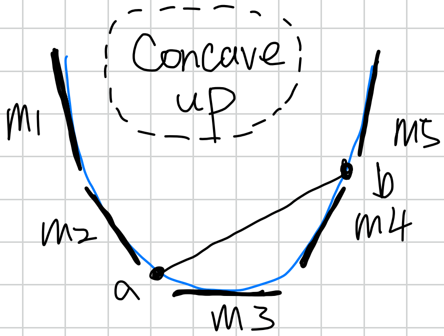
Solve the Mean Girls limit:

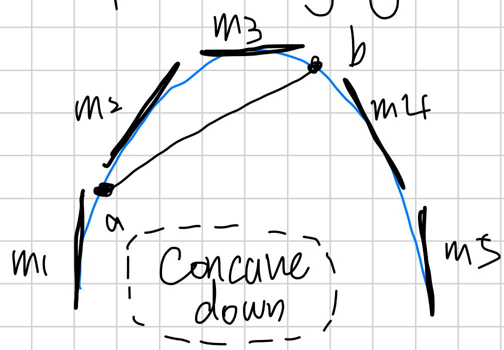
$$\lim_{x\to 0}\frac{\ln(1-x)-\sin(x)}{1-\cos^2(x)}$$

Week 11 lecture 30

We focused on the first derivative → Inst. r.o.c.
Now, lets focus on double derivative.

→ r.o.c of r.o.c, how r slopes changing?





- $m_1 < m_2 < m_3 < m_4 < m_5 \Rightarrow f'' > 0$
- secant line is above

- $m_1 > m_2 > m_3 > m_4 > m_5 \Rightarrow f'' < 0$
- secant line is below

Defn: Concavity

The graph of f is concave upwards (c.u) / downwards (c.d.) on an interval I if $\forall\, a, b \in I$, the secant line joining $(a, f(a))$ & $(b, f(b))$ sits above / below the graph.

1) If $f''(x) > 0$ $\forall x \in I$, then graph of f is concave up on I
2) If $f''(x) < 0$ $\forall x \in I$, then graph of f is concave down on I

Defn: Inflection Point (POI)

A point $(c, f(c))$ is called POI of f is cts at c and the concavity of f changes at $(c, f(c))$.

↳ these occur when f'' changes sign. If f'' is cts at the POI, IVT requires $f''(c) = 0$.

Thm 11: Test for POI

If f'' is cts at $x=c$ & $(c, f(c))$ is an inflection point of f, then, $f''(c)=0$.

Ex: Find the intervals of concavity & any POI of f:

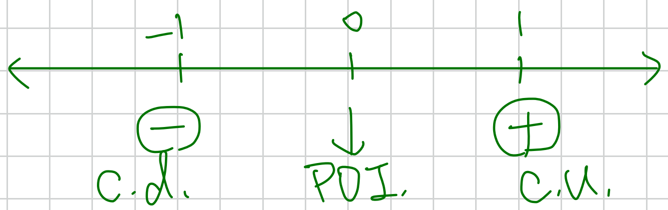
a) $y=x^3$ b) $y=-1/x$

---

a) $y=x^3 \Rightarrow y'=3x^2 \Rightarrow y''=6x$

y'' is a polynomial, so cts $\forall x \in \mathbb{R}$

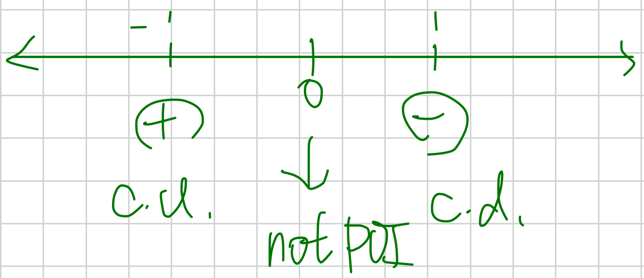
$y''=0$ when $x=0$.



$\therefore$ y is c.d. on $(-\infty, 0)$ and c.u. on $(0,\infty)$ and $(0,0)$ is a POI

b) $y=\frac{-1}{x} \Rightarrow y'=\frac{1}{x^2} \Rightarrow y''=\frac{-2}{x^2}$

note: y'' DNE @ $x=0$



$\therefore$ y has no POI.

We have seen that if $x=c$ is a local, then it is a c.p. ($f'=0$ or DNE)

But once we find a c.p. we must determine if it is local max/min.

Thm 12: First Derivative Test (FDT)

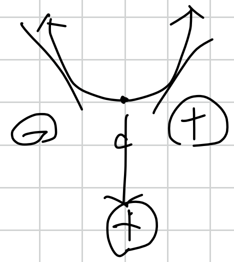
Assume c is a c.p. of f & f is cts at c.
If there is a interval (a,b) containing c s.t.

1) $f'(x) < 0 \ \forall x \in (a,c)$ & $f'(x) > 0 \ \forall x \in (c,b)$ than, c is a local min.
2) Vice versa.
3) If $f'(x)$ doesn't change sign, then c is neither local max/min.

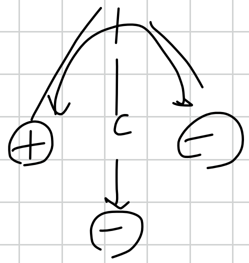
Pictorially:

1) $f' =$

$f'' =$



2) $f' =$



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Thm 13: Second Derivative Test (SDT)

Assume $f'(c) = 0$ and that $f''(c)$ is cts at c
If:

1) $f''(c) > 0$ then c is local min
2) $f''(c) < 0$ then c is local max
3) $f''(c) = 0$ then we have no info.

Ex1: find local max/mins of $y = x^3 - 13x + 12$ using FDT & SDT.

$\Rightarrow \quad y' = 3x^2 - 13 = 0 \quad , \quad y'' = 6x.$

then, c.p.s: $x = \pm\sqrt{\frac{13}{3}}$, (no DNEs since polynomial)

$-3 \quad -\sqrt{\frac{13}{3}} \quad 0 \quad \sqrt{\frac{13}{3}} \quad 3$

(FDT): $f' =$ ⊕ ⊖ ⊕

local max   local min.

(SDT): $f'' =$ ⊖ ⊕

local max   local min.

$f = \nearrow \frown \searrow \smile \nearrow$

Ex:2: Find all max/mins (Global & local.) of $y = x\sqrt[3]{2-x}$, on $[0,3]$

Note: $y' = \dfrac{2 - \frac{4}{3}x}{(2-x)^{\frac{2}{3}}}$, $y'' = \dfrac{\frac{4}{9}x - \frac{4}{3}}{(2-x)^{\frac{5}{3}}}$

Endpoints: $f(0) = 0$ & $f(3) = -3$

C.P.s : $f' = 0$, $x = \frac{3}{2}$, $f' = $ DNE : $x = 2$.

$0 \quad \frac{3}{2} \quad \frac{5}{3} \quad 2 \quad 3$

(FDT) $f' =$ ⊕ ⊖ ⊖

local max

(SDT) $f'' =$ ⊖   DNE, so nope.

$f' : \nearrow \frown \searrow \quad \searrow \qquad f\left(\frac{3}{2}\right) = \dfrac{3}{2^{4/3}}$

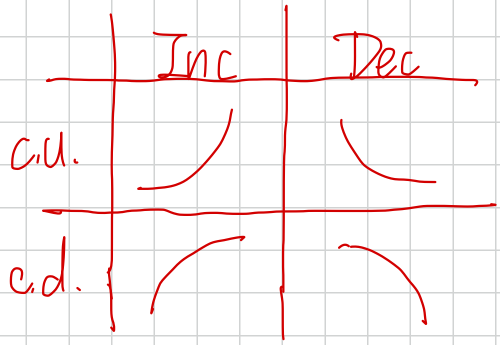
Global & local max : $x = \frac{3}{2}$

Global min : $x = 3$.

Week 11 lecture 31

Curve Sketching process

1) Identify the function Domain & possibly fcn values and endpoints.
2) Identify x & y intercepts.
3) Identify horizontal asymptotes. $\left(\lim\limits_{x\to\pm\infty} f(x)\right)$
4) Identify any holes or vertical asymptotes. $\left(\lim\limits_{x\to a^{\pm}} f(x)\right)$
5) Find all c.p.'s ($f'(x)=0$ or DNE).
6) Find where $f''(x)=0$ or DNE
7) Investigate the interval divided by the points from 5) & 6) for concavity & inc/dec.
8) Identify local extrema & POI's from 7)
9) Plot (and lable)



Ex: I

Sketch $f(x)=\dfrac{x^2-1}{x^2+3x}$ given $f'(x)=\dfrac{3x^2+2x+3}{x^2(x+3)^2}$

$$f''(x)=\frac{-6(x+1)(x^2+3)}{x^3(x+3)^3}$$

Domain

1) $f(x)=\dfrac{(x+1)(x-1)}{x(x+3)}$, $x\neq 0,-3$.

So: D: $(-\infty,-3)\cup(-3,0)\cup(0,\infty)$

x, y Intercept.

2) x-Int ($y=0$)

$$0=\frac{(x+1)(x-1)}{x(x+3)} \Rightarrow x=\pm 1$$

y-int ($x=0$) $\Rightarrow$ none.

H.A. 3) $\lim_{x\to\pm\infty} \frac{x^2-1}{x^2+3x} = 1$

$\therefore$ H.A. at $y=1$ for $x\to\pm\infty$

holes & V.A. 4) We are interested in $x=0$ & $x=-3$

$\lim_{x\to 0^-} \frac{x^2-1}{x(x+3)} = +\infty$ $\qquad$ $\lim_{x\to -3^-} \frac{x^2-1}{x(x+3)} \frac{+}{+} = +\infty$

$\lim_{x\to 0^+} \frac{x^2-1}{x(x+3)} = -\infty$ $\qquad$ $\lim_{x\to -3^+} \frac{x^2-1}{x(x+3)} \frac{+}{-} = -\infty$

C.P's 5) $f'(x) = \frac{3x^2+2x+3}{x^2(x+3)^2}$

C.P.S: $f'(x)=0 \longrightarrow 0 = 3x^2+2x+3$

$x = \frac{-2\pm\sqrt{4-(4\cdot3\cdot3)}}{2(3)}$ $\quad \sqrt{-20}$, no solu for $f'=0$

$f'(x) = \text{DNE} \longrightarrow x^2(x+3)^2 = 0 \Rightarrow x = 0, -3$ ✗ (not in the domain)

$\Rightarrow$ there are no C.p's.

$\Rightarrow$ no local extrema.

Note: we still investigate Inc/dec, concavity.

POIs 6) $f''(x) = \frac{-6(x+1)(x^2+3)}{x^3(x+3)^3}$

$f''(x) = 0 \longrightarrow 0 = -6(x+1)(x^2+3) \Rightarrow x = -1$ (candidate POI)

$f''(x)$ DNE when $x^2(x+3)^3 = 0 \Rightarrow x = 0, -3$ (not in Domain)

Invest. 7) Inc/Dec:

C.P.S

	-5	-3 DNE	-1	0 DNE	1
f':	$+$		$+$		$+$
f:	↗		↗		↗

Concavity:

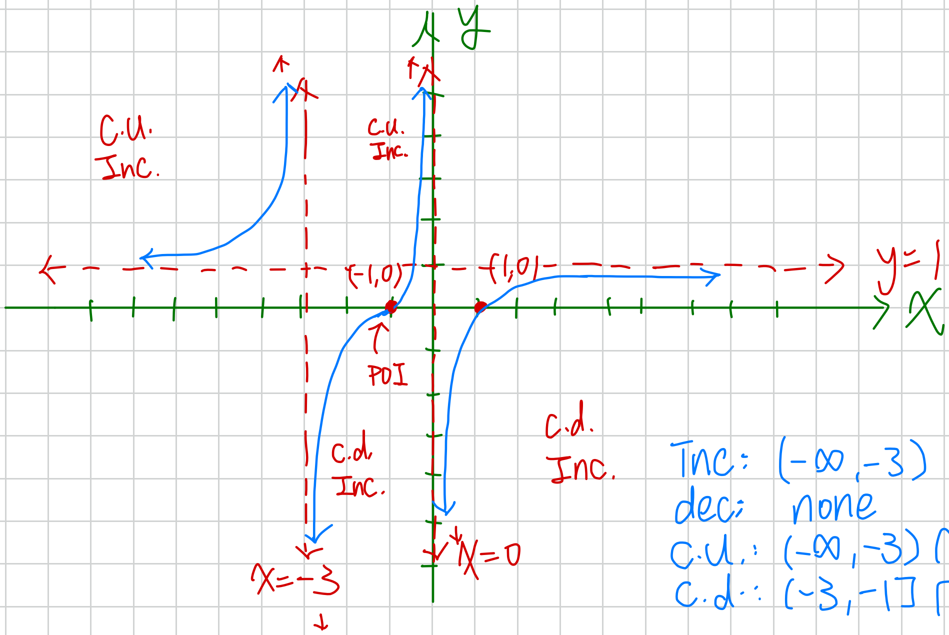
POI candidate,

x		-3 DNE		-1 (0)		0 DNE	
f'':	$+$		$-$		$+$		$-$
f:	∪		∩		∪		∩

Confirmed POI

$f(-1) = 0$

8) As discussed in 5) & 7), no local extrema.
and $(-1, 0)$ is a POI.



Inc: $(-\infty, -3) \cap (-3, 0] \cap (0, \infty)$
dec: none
C.U.: $(-\infty, -3) \cap [-1, 0)$
C.d.: $(-3, -1] \cap (0, \infty)$

Week 11 lecture 32

EX2:

Sketch $f(x) = \frac{e^x(x-2)}{x^2-2x}$ given $f'(x) = \frac{e^x(x-1)(x-2)}{x^3-2x^2}$

$$f''(x) = \frac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3}$$

D 1) $f(x) = \frac{e^x(x-2)}{x^2-2x} = \frac{e^x(x-2)}{x(x-2)}$, $x \neq 0, 2$.

We have $x \in (-\infty, 0) \cup (0, 2) \cup (2, \infty)$

x,y int 2) x-int $(y=0)$

$0 = \frac{e^x(x-2)}{x^2-2x}$ $\quad x = 2$ X (not in domain)

$\Rightarrow$ no x-int.

y-int $(x=0)$ $\Rightarrow$ no y-int (not in domain)

HA 3) $\lim_{x\to\infty} \frac{e^x(x-2)}{x^2-2x} = \lim_{x\to\infty} \frac{e^x}{x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\underset{=}{L'HR}} \lim_{x\to\infty} \frac{e^x}{1} = \infty$

$\lim_{x\to-\infty} \frac{e^x}{x} = \frac{0}{-\infty} = 0$

$\therefore$ HA at $y=0$ for $x \to -\infty$

Hole or V.A 4) $\lim_{x\to0^-} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to0^-} \frac{e^x}{x} \overset{+}{\underset{-}{=}} -\infty$

$\lim_{x\to0^+} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to0^+} \frac{e^x}{x} \overset{+}{\underset{+}{=}} \infty$

$\therefore$ V.A at $x=0$

$\lim_{x\to2} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to2} \frac{e^x}{x} = \frac{e^2}{2}$

$\therefore$ a hole on $(2, \frac{e^2}{2})$

c.p's 5) $f'(x) = \frac{e^x(x-1)(x-2)}{x^3-2x^2} = \frac{e^x(x-1)(x-2)}{x^2(x-2)}$

c.p's when: $f'=0$: $e^x(x-1)(x-2) = 0$

$\Rightarrow x = 1, \cancel{2}$ ← (not in domain)

$f' = DNE \Rightarrow x = \cancel{0, 2}$. (not in Domain)

$\therefore$ we have a c.p at $x = 1$ (candidate of max/min)

6) $f''(x) = \dfrac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3} = \dfrac{e^x(x^2-2x+2)(x-2)}{x^3(x-2)}$

$f''=0$: $\underset{\neq 0}{e^x}\underset{\neq 0}{(x^2-2x+2)}(x-2)=0$

$x = \cancel{2}$ (not in domain)

$f'' = DNE \Rightarrow x = \cancel{0, 2}$ (not in domain)

$\Rightarrow$ No candidate POI (不代表 Inflection 不變)

7) Inc/dec

	-1	0 DNE	$\frac{1}{2}$	1	$\frac{3}{2}$	2 DNE	3
f':	$-$		$-$		$+$		$+$
f:	↘		↘		↗		↗

FDT classify $x=1$ as a local min

(SDT : $f''(1) > 0$ also classifies)

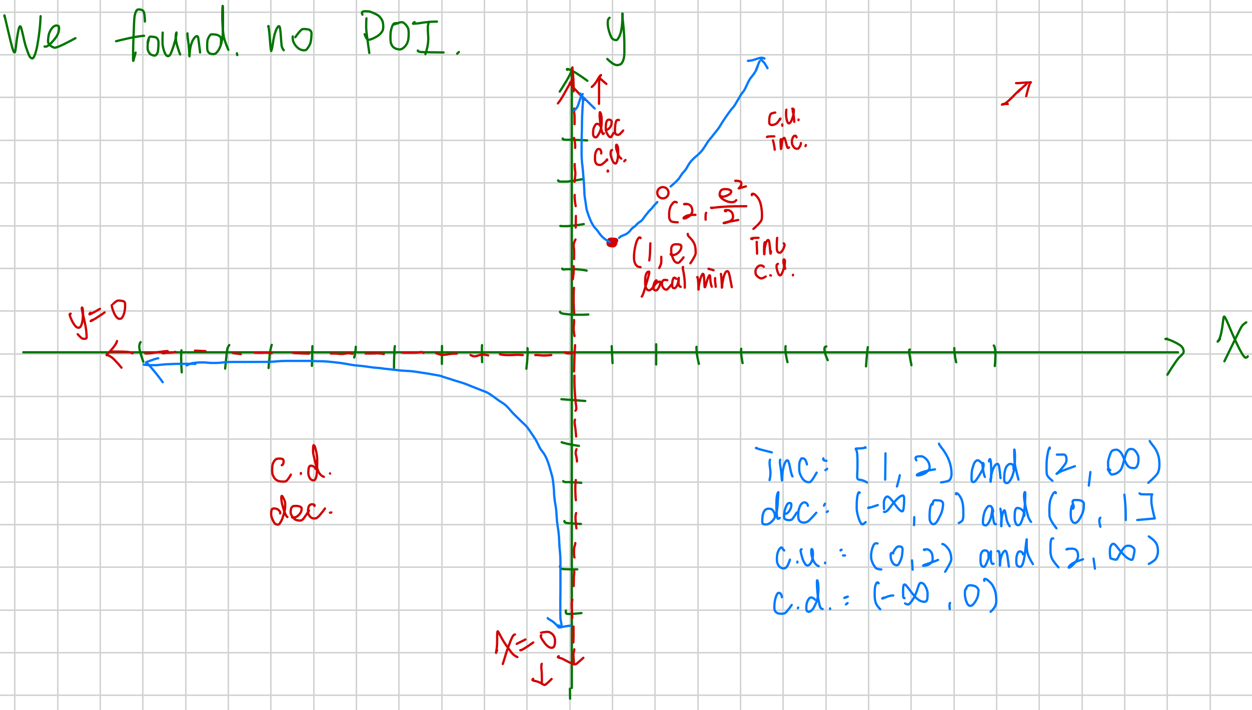
Concavity

	-1	0	$\frac{1}{2}$	2	3
f'':	$-$		$+$		$+$
f:	∩		∪		∪

8) We found local min at $x=1$, $f(1) = e$

so local min at $(1, e)$

We found no POI.



inc: $[1, 2)$ and $(2, \infty)$

dec: $(-\infty, 0)$ and $(0, 1]$

c.u.: $(0, 2)$ and $(2, \infty)$

c.d.: $(-\infty, 0)$

Week 12 lecture 33

Recall linear approx:

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

Basically tagent line.

Some key features:

- $L_a^f(x) = f(a)$
- $L_a^{f\prime}(x) = f'(a)$
- $\underbrace{|f(x) - L_a^f(x)|}_{\hookrightarrow \text{ error}} \leq \frac{M}{2}(x-a)^2$ $\quad |f''(x)| \leq M \;\; \forall x \in I$

We did this to approximate fcns with linear fcns.

Now we want a larger nth degree polynomial which is a better approx. that agrees up to n^{th} derivative of $f'(a)$ at $x=a$ and is equal to $f(x)$ at $x=a$.

centre

$$T_{n,a}(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 \; \cdots \; C_n(x-a)^n$$

taylor degree

First we want: $T_{n,a}(a) = f(a)$

$$T_{n,a}(a) = C_0 + C_1\cancel{(a-a)}^{0} + C_2\cancel{(a-a)}^{0} \cdots\cdots$$

$$\therefore C_0 = f(a)$$

Next we want: $T_{n,a}'(a) = f'(a)$

$$T_{n,a}'(x) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 \cdots\cdots$$

$$T_{n,a}'(a) = 0 + C_1 + 2C_2\cancel{(a-a)}^{0} + 3C_3\cancel{(a-a)^2}^{0} \cdots$$

$$\therefore \; C_1 = f'(a)$$

(notice T degree 1 = linear approx)

Now we want: $T_{n,a}''(a) = f''(a)$

$$T_{n,a}''(x) = 2C_2 + (3)(2)(x-a) + \cdots$$
$$T_{n,a}''(a) = 2C_2 + (6)\cancel{(a-a)}^{0} + \cdots$$

$$\therefore \; 2C_2 = f''(a)$$
$$C_2 = \frac{f''(a)}{1 \cdot 2}$$

We could then continue demanding $T_{n,a}'''(x) = f'''(a)$

↳ This gives, $C_3 = \dfrac{f'''(x)}{6} = \dfrac{f'''(x)}{3 \cdot 2 \cdot 1} = \dfrac{f'''(x)}{3!}$

That is, the k^{th} derivative would demand.

$$C_k = \frac{f^{(k)}(a)}{k!}$$

Defn: Taylor Polynomial

If f is n times diff'able at $x = a$, we say "the n^{th} degree Taylor polynomial for f centered at $x = a$" is the polynomial:

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \cdots$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note, if the center $a=0$, it's "Mclaurin Polynomial"

Ex1:

Find up to $T_{5,0}(x)$ for $f(x) = \sin(x)$

First we note that $f(0)=0$. So $T_{0,0}(x) = 0$

Now, $f'(x) = \cos(x) \Rightarrow f'(0) = 1$

$f''(x) = -\sin(x) \Rightarrow f'(0) = 0$

$f'''(x) = -\cos(x) \Rightarrow f'(0) = -1$

$f^4(x) = \sin(x) \Rightarrow f'(0) = 0$

$\therefore T_{1,0}(x) = 0 + \frac{1}{1!}(x-0)^1 \Rightarrow T_{1,0}(x) = \boxed{x}$

$T_{2,0}(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 \Rightarrow T_{2,0}(x) = \boxed{x}$

$T_{3,0}(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3$

$\Rightarrow T_{3,0}(x) = \boxed{x - \frac{1}{6}x^3}$

$T_{4,0}(x) = 0 + \text{-----} + \frac{0}{4!}(x-0)^4 \Rightarrow T_{4,0}(x) = \boxed{x - \frac{1}{6}x^3}$

$T_{5,0}(x) = 0 + \text{----} + \frac{1}{5!}(x-0)^5$

$\Rightarrow T_{5,0}(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$

Ex2

Find $T_{5,0}(x)$ for $f(x) = \cos(x)$

we have $f(0) = 1$

$f'(x) = -\sin(x) \Rightarrow f'(0) = 0$

$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$

$$f'''(x) = \sin(x) \Rightarrow f'''(0) = 0$$
$$f^{4}(x) = \cos(x) \Rightarrow f^{4}(0) = 1$$

$$1 \to 0 \to -1 \to 0 \dashrightarrow$$

$$T_{5,0}(x) = 1 + \frac{0}{1!}(x)^1 + \frac{-1}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{1}{4!}(x)^4$$
$$+ \frac{1}{5!}(x)^5 =$$
$$\Rightarrow 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

Week 12 lecture 34

Mathmatize:

Ex1: Find $T_{3,0}(1)$ for $f(x) = e^x$

$$f^k(x) = e^x$$

$$T_{3,0}(1) = 1 + \frac{1}{1!}(x)^1 + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6}$$

$$= \frac{8}{3}$$

As this is a approx, we're concern abt error.

Defn: Taylor remainder:

Assume f is n-times diff'able at $x=a$. Then.

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

is called the nth degree Taylor remainder fcn. centered at $x=a$.

Notes:

- Error: $|R_{n,a}(x)|$
- $R>0 \Rightarrow$ under estimate.
- $R<0 \Rightarrow$ over estimate

Thm I: Taylor's thm

Assume f is $n+1$ times diff'able on Interval I containing $x=a$. Let $x \in I$, Then $\exists$ c between x and a such that.

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Notes:

- For $n=1$, we have $T_{1,a}(x) = L_a^f(x)$

  Taylor's thm tells us

  $$R_{1,a}(x) = \frac{f''(c)}{2}(x-a)^2$$

  and if we have $|f''(c)| \leq M$ $\forall x \in I$, then

  $$|R_{1,a}(x)| \leq \frac{M}{2}(x-a)^2$$

  This is what we previously said the approx error is.

- For $n=0$, we have $T_{0,a}(x) = f(a)$

  Taylor's thm tells us:

  $$f(x) - T_{0,a}(x) = R_{0,a}(x) = f'(c)(x-a)$$

  $$\Rightarrow f(x) - f(a) = f'(c)(x-a)$$

  $$\Rightarrow \frac{f(x)-f(a)}{x-a} = f'(c)$$

This is MVT !!! Taylor thm is just a higher order MVT.

- Taylor doesn't say how to find c, just that it exist

As we did in linear approx, we seek upper bound.

Corollary: Taylor's Inequality

If we have $|f^{(n+1)}(c)| \le M$ $\forall c$ between x & a then,

$$|R_{n,a}(x)| \le \underbrace{\frac{M}{(n+1)!}|x-a|^{n+1}}_{\text{up. b}}$$

$\forall c$ between x & a.

Ex 1:

a) Estimate $\cos(0.1)$ with $T_{5,0}(x)$ of $\cos(x)$.

b) What is an upper bound on the error in our approx in a)?

c) Is $T_{5,0}(x)$ an under/over-estimate on $[0,1]$?

d) based on a), b), and c) give an interval for the true value of $\cos(0.1)$

---

a) We found last lecture that $T_{5,0}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Then $\cos(0.1) \approx T_{5,0}(0.1) = 1 - \frac{1}{2}\left(\frac{1}{100}\right) + \frac{1}{24}\left(\frac{1}{1000}\right)$

$$= \frac{238801}{24000} \quad (\doteq 0.9950417)$$

b) by Taylor's thm, $\exists\, c \in (0, 0.1)$ such that

$$R_{5,0}(0.1) = \frac{f^{(6)}(c)}{6!}(0.1-0)^6$$

Now $f^{(6)}(x) = -\cos(x)$

We note that $|f^{(6)}(x)| = |-\cos(x)| \le 1$ $\forall c \in (0, 0.1)$

$\therefore M = 1$ is a valid choice.

$\therefore \; |R_{5,0}(0.1)| \leq \frac{1}{6!}|0.1 - 0|^6 = \frac{1}{720\,000\,000}$

c) $\cos(x) - T_{5,0}(x) = R_{5,0}(x) = \frac{-\cos(c)}{720}x^6$

now, for $x \in [0,1]$, we have $c \in [0,1]$

For $c \in [0,1]$, $-\cos(c) \leq 0$

Also, note that $x^6 \geq 0$ on $[0,1]$

$\therefore \; R_{5,0}(x) \leq 0$ on $[0,1]$

$(f(x) - T_{5,0}(x) \leq 0 \Rightarrow f(x) \leq T_{5,0}(x))$

$\therefore$ overestimate.

d) Our approx in a) $\frac{238801}{240000}$ and

In c) we said this is an over estimate.

In b) we said at worst case error was $\frac{1}{720000000}$

So $\cos(0.1) \in \left[\frac{238801}{240000} - \frac{1}{720000000}, \frac{238801}{240000}\right]$

Week 12 lecture 35

Ex2:

a) Estimate $\sqrt[3]{30}$ using $T_{2,27}(x)$ of $\sqrt[3]{x}$

b) What is an upper bound on the error of our approx in a)?

c) What is the upper bound of the error of $T_{2,27}(x)$ for $x \in [20, 35]$

d) For what value, $x > 0$, is T over / under est

---

a) $T_{2,27}(x) = \sqrt[3]{27} + \frac{1}{27}(x-27) + \frac{-2}{2187(2!)}(x-27)^2$

$\hookrightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \quad f'(27) = \frac{1}{27}$

$f''(x) = \frac{-2}{9}x^{-\frac{5}{3}} \quad f''(a) = \frac{-2}{2187}$

$T_{2,27}(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$

$\therefore \sqrt[3]{30} \approx T_{2,27}(30) = 3 + \frac{1}{27}(30-27) - \frac{1}{2187}(30-27)^2 = \boxed{\frac{755}{243}}$

b) By Taylor Thm,

$\exists\, c \in (27, 30)$ s.t.

$$R_{2,27}(30) = \frac{f'''(c)}{3!}(30-27)^3$$

$$|R_{2,27}(30)| = \frac{|f'''(c)|}{|3!|}|30-27|^3 \leftarrow \text{error}$$

Now, for reference, $f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}$

Now, on $[27, 30]$, we note that

$$|f'''(c)| \le \underbrace{\frac{10}{27}(27)^{-\frac{8}{3}}}_{M} = 10(27)^{-\frac{11}{3}}$$

since $f'''(x)$ is decreasing on the interval

$$\therefore |R_{2,27}(x)| \le \frac{10(27)^{-\frac{11}{3}}}{3!}(30-27)^3 = \frac{5}{19683}$$

c) By Taylor's Ineq

$$|R_{2,27}(x)| \leq \frac{M(x-27)^3}{3!}$$

where $|f'''(c)| \leq M$ for $c \in [20, 35]$

we saw previously $f'''(x) = \frac{10}{27} x^{-\frac{8}{3}}$

Then, $|f'''(c)| \leq \left| \frac{10}{27} (20)^{-\frac{8}{3}} \right|$ on $[20, 35]$

since f''' is decreasing on the interval

So we have:

$$|R_{2,27}(x)| \leq \frac{10}{27(3!)} (20)^{-\frac{8}{3}} |x-27|^3$$

Then we want to maximize RHS to find u.b.

take $x = 35$.

$$\therefore |R_{2,27}(x)| = \frac{2560}{81} (20)^{-\frac{8}{3}} \doteq 0.0107$$

d) we had:

$$R_{2,27}(x) = \frac{f'''(c)}{3!}(x-27)^3 \quad \text{for some } c \in (x, 27)$$

$$= \underbrace{\left(\frac{1}{3!}\right)}_{+} \underbrace{\left(\frac{10}{27} c^{-8/3}\right)}_{+} \underbrace{(x-27)^3}_{\text{(varies)}}$$

Overestimate: $T > f \Rightarrow R < 0$: $\underline{(x-27) < 0}$

$\hookrightarrow$ $\underbrace{0 <}_{\text{given in problem.}} x < 27$

Underestimate: $T < f \Rightarrow R > 0$: $(x-27) > 0$

$\hookrightarrow$ $x > 27$ give us underest.