lecture 1: intro to vector spaces.

Defn: A vector space over $\mathbb{F}$ is a set of V together with operation "vector addition" and "scalar multiplication."

$\forall \vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} \in V$ and. $c\vec{x} \in V$

lecture 2.

Vector space axioms:

① $\forall \vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

② $\forall \vec{x}, \vec{y}, \vec{z} \in V$, $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$

③ $\forall \vec{x}, \vec{y} \in V$, $c \in \mathbb{F}$, $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$

④ $\forall \vec{x} \in V$, $c, d \in \mathbb{F}$, $(c+d)\cdot\vec{x} = c\vec{x}+d\vec{x}$

⑤ $\forall \vec{x} \in V$, $c, d \in \mathbb{F}$, $(cd)\vec{x} = c(d\vec{x})$

⑥ $\forall \vec{x} \in V$, $1\cdot\vec{x} = \vec{x}$

⑦ $\exists\, \vec{0} \in V$ s.t. $\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}$

⑧ $\forall \vec{x} \in V$, $\exists\, -\vec{x}$ s.t. $\vec{x}+(-\vec{x}) = (-\vec{x})+\vec{x} = 0$

Prop: Let V be vector space over $\mathbb{F}$. then

① $\vec{0}$ is unique.

② let $\vec{x} \in V$. then $-\vec{x}$ is unique.

③ $\vec{0}\cdot\vec{x} = \vec{0}$, $\forall \vec{x} \in V$.

④ $(-1)\cdot\vec{x} = -\vec{x}$, $\forall \vec{x} \in V$.

☆ main things to check for vector space ① $+$, ② $\cdot$ and ③ $\vec{0} \in V$?

↳ ④ and additive inverse

lecture 3:

Defn: Let V be vector space over $\mathbb{F}$. let $W \subseteq V$ be a subset of V. If W is itself a vector space, then we say W is a subspace of V.

Theorem: (Subspace test) let $W \subseteq V$. then W is a subspace of V iff.

① $W \neq \emptyset$ or $\vec{0} \in W$ (non-empty)

② $\forall \vec{x}, \vec{y} \in W$, $\vec{x} + \vec{y} \in W$ (closed under addition)

③ $\forall c \in \mathbb{F}$, $\vec{x} \in W$, $c\vec{x} \in W$ (closed under scalar multi)

Defn: let $S = \{\vec{v}_1 \dots \vec{v}_k\}$ be a finite set of vectors in V

the span of S is $\text{Span}(S) = \{c_1\vec{v}_1 + \dots c_k\vec{v}_k : c_i \in \mathbb{F}\}$

Prop: $\text{Span}(S)$ is a subspace of V.

Defn: let W be a subspace of V. If $W = \text{span}(S)$ for some subset $S \subseteq V$ then we say W is spanned by S or. S is a spanning set for W.

↳ that's said, if you want to show for ex W is a subspace of V. you can instead show for some $S \subseteq V$, $\text{span}(S) = W$.

lecture 4

last time: $\vec{v}_1 \dots \vec{v}_k$ LI if no $\vec{v}_i$ is a linear comb of the others:

$\vec{v}_i \neq c_1\vec{v}_1 \dots c_k\vec{v}_k$, if not LI, $\rightarrow$ LD

Prop: $\vec{v}_1 \dots \vec{v}_k$ are LI iff the only soln. to

$c_1\vec{v}_1 + c_2\vec{v}_2 \dots + c_k\vec{v}_k = \vec{0}$ * is the trivial soln.

Proof: (sketch)

($\Rightarrow$) Assume $\vec{v}_1 \dots \vec{v}_k$ are LI, Consider eq * If there is a soln w/ $c_i \neq 0$, then.

$$\frac{c_1\vec{v}_1}{c_i} + \frac{c_2\vec{v}_2}{c_i} + \dots \frac{c_k\vec{v}_k}{c_i} = \cancel{c_i}\vec{v}_i$$

thus $\vec{v}_i$ is a linear comb, contradiction.

ex: $V = P_2(\mathbb{R})$, $S = \{1-x, 1+x, 1-x^2, 1+x^2\}$, is S LI?

$\hookrightarrow$ $\underline{a} + \underline{b}x + \underline{c}x^2 \Rightarrow$ "dim $V = 3$" $0+0x+0x^2$

Sol: consider equation: $c_1(1-x) + c_2(1+x) + c_3(1-x^2) + c_4(1+x^2) = \vec{0}$

Equate coeff of $1, x, x^2$:

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0 \\ -c_1 + c_2 = 0 \\ -c_3 + c_4 = 0 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

Thus, S is LI $\Leftrightarrow$ the only soln is the trivial soln.

$\Leftrightarrow$ nullity $(A) = 0$ (nullity = dim(nullspace))

$\Leftrightarrow$ rank $(A) = 4$ (n)

Since A is 3×4 rank$(A) \leq 3$, thus S is not LI

p.s. nullity = non-pivot columns in RREF (# of cols − rank) ex: $V = P_2(\mathbb{R})$, $S = \{1+x, 1+x^2\}$, does S span V?

$\hookrightarrow \dim(V) = 3$

Sol: to show $V = \text{span}(S)$ we must show $\overset{①}{V \subseteq \text{Span}(S)}$, $\overset{②}{\text{Span}(S) \subseteq V}$.

② is trivial

$\Rightarrow$ to show $V \subseteq \text{span}(S)$, we must show that

$a + bx + cx^2 \in \text{span}(S)$, $\forall a, b, c$

equivalently, we want to find soln. to

$$a + bx + cx^2 = c_1(1+x) + c_2(1+x^2)$$

$$\Rightarrow \quad * \begin{cases} a = c_1 + c_2 \\ b = c_1 \\ c = \quad\quad c_2 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{array}\right] \leftarrow \text{Inconsistent.}$$

thus, $\text{span}(S) = V \Leftrightarrow *$ have a soln $\forall a, b, c$

$\Leftrightarrow [a, b, c]^T \in \text{Col}(A)$ $\forall a, b, c$

$\Leftrightarrow \text{Col}(A) = \mathbb{R}^3$

$\Leftrightarrow \dim(\text{Col}(A)) = \dim(\mathbb{R}^3)$

$\Leftrightarrow \text{rank}(A) = 3$

Since A is 3×2 $\text{rank}(A) \leq 2$, $\text{Span}(S) \neq V$

Generalization:

If $\dim(V) = n$ then,

- a set of $> n$ must be LD
- a set of $< n$ cannot span V

Defn: Let V be a vector space, A basis for V is a set $B \subseteq V$ is LI and spans V.

lecture 6

ex: stadard basis for $\mathbb{F}^n$ is $B = \{\vec{e_1}, \vec{e_2} \dots \vec{e_n}\}$

ex. Standard basis for $P_n(\mathbb{F})$ is $B = \{1, x, x^2 \dots x^n\}$

ex: standard basis for $M_{m\times n}(\mathbb{F})$ is $B = \left\{ \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right\}$

There's no <u>standard</u> basis for arbitrary V.

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ex: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

ex: $\{1+2x, 3+3x\}$ is a basis for $P_1(\mathbb{R})$

Show that every vector space has a basis.

ex: $V = C([0,1]) = \{f : [0,1] \to \mathbb{R}, \text{cts}\}$, basis?

$\Rightarrow$ impossible to write out, the basis is itself.

Defn: We say V is finite dimensional if is has a finite spanning set. Else, it's infinite dimensional.

ex: $\mathbb{F}^n$, $M_{m\times n}(\mathbb{F})$, $P_n(\mathbb{F})$, finite dimensional.

ex: $C([0,1])$ is infinite dimensional.

Theorem: every finite dimensional vector space V has a basis.

Proof: Assume $S = \{\vec{v_1}, \dots \vec{v_k}\}$ spans V.

① If S is LI — it's done! it's a basis.

② If S is LD than some vector $\vec{v_k}$ is linear comb of other.

Let $S' = \{\vec{v_1} \dots \vec{v_{k-1}}\}$, than $\text{Span}(S') = \text{Span}(S)$

If S' is LI, done, other wise repeat.

Connection. : $\emptyset$ (empty) is LI and basis for $\{\vec{0}\}$

Defn: Dimension. If V is a finite-dim vector space then we define $\dim(V)$ to be the size of any basis for V

lecture 7

Prep^n: Suppose $\dim V = n$ let $S \subseteq V$ have size k.

① If $k > n$ then S cannot be LI

② If $k < n$ then S cannot span V

③ If $k = n$ then $\underline{S \text{ is LI}} \Leftrightarrow \underline{\text{Span}(S) = V}$.

Prove either

☆ important general proof.

ex: $V = P_2(\mathbb{R})$, $S = \{1+x, 1-x^2, 2x+x^2\}$

↳ $\dim V = 3$.

Claim: S is a basis for V.

Proof ① S is LI:

Consider $c_1(1+x) + c_2(1-x^2) + c_3(2x+x^2) = \vec{0}$ $(0 + 0x + 0x^2)$

Equate coeff $1, x, x^2$:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 2c_3 &= 0 \\ -c_2 + c_3 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$\therefore$ S is LI $\Leftrightarrow$ only soln to this system is trivial soln.

$\Leftrightarrow$ Nullspace$(A) = \{\vec{0}\}$

$\Leftrightarrow$ Nullity$(A) = 0$

Proof ② S spans V.

Given $\vec{v} \in V$, check if we can express it as linear comb. of vectors in S

$\Rightarrow$ Solve $c_1(1+x) + c_2(1-x^2) + c_3(2x+x^3) = a + bx + cx^2$

↳ same coeff matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 2 & b \\ 0 & -1 & 1 & c \end{array}\right]$$

(the first three columns are marked (A))

$\therefore$ S spans V $\Leftrightarrow$ system has a soln $\forall\, [a, b, c]^T \in \mathbb{R}^3$

↓

$\Leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in Col(A), \forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$

$\Leftrightarrow Col(A) = \mathbb{R}^3$

$\xrightarrow{\text{sketchy. (1)}} \Leftrightarrow \dim(col(A)) = \dim(\mathbb{R}^3)$

$\Leftrightarrow rank(A) = 3$

thus S is a basis. #

We just saw : ① S is LI $\Leftrightarrow$ nullity $(A) = 0$

② S spans V $\Leftrightarrow$ rank $(A) = 3$ } are the same.

rank(A) + nullity (A) = # of columns. (n)

So, rank (A) = 3 iff nullity (A) = 0

S span V iff S is LI

Problem: Suppose V is a n-dim vector space.

Suppose U is a subset of V

Prove : ① $\dim U \leq \dim V$

and ② $\dim U = \dim V \Leftrightarrow U = V$

Proof: ① Let B be a basis for U, then $\dim(U) = |B|$

Since $B \subseteq U$, $B \subseteq V$.

Since B is LI, $|B| \leq \dim(V)$

② Assume now $\dim U = \dim V$, So $\dim(V) = |B|$

Since B is a basis for U, $U = span\ B$

Since $|B| = \dim V$ and B is LI

$\Rightarrow B$ spans V $\Rightarrow V = span\ B = U$

Coordinates: In $\mathbb{R}^3$, we have std basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$

and every $\vec{v} \in \mathbb{R}^3$ looks like $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$

- In a vector space B w/ basis $\mathcal{B} = \{\vec{b_1}, \dots \vec{b_n}\}$

  then every $\vec{v} \in V$ can be written as $\vec{v} = a_1\vec{b_1} + \dots + a_n\vec{b_n}$

* Fact: (*unique representation) these a_n are unique.

We call them the coordinates of $\vec{v}$

We create the $\mathcal{B}$-coord vectors for $\vec{v}$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

lecture. 8

If V is a vector space w/ ordered basis B, then every $\vec{v} \in V$ can be written *uniquely in the form

$$\vec{v} = a_1\vec{b_1} + a_2\vec{b_2} + \cdots a_n\vec{b_n} \text{ for some } a_i \in \mathbb{F}$$

Defn: We record the B-coordinate vector of $\vec{v}$ $[\vec{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

ex: In $\mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$

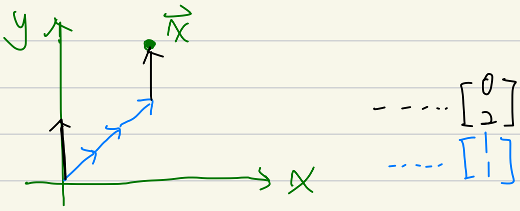
$$[\vec{x}]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \vec{x} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \vec{x} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

ex: now let $\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \in \mathbb{R}^2$. Then $[\vec{x}]_S = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

determine $[\vec{x}]_B$.

$$\vec{x} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad a=3, b=1, \quad [\vec{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



ex: $V = \{ A \in M_{2\times2}(\mathbb{F}) : tr(A) = 0 \}$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

let $A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} \in V$, determine $[A]_B$

Soln: $\begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

by inspection, $b=3$, $a=1$, $c=5$

ex: $V = P_3(\mathbb{R})$, $B = \{-1+2x+2x^2,\ 2+x^2,\ -3+x\}$

Prove that B is a basis and determine $[1+2x+3x^2]_B$

Solu: Since $|B| = 3$ and $\dim V = 3$,

It's sufficient to prove one of ① B is LI, ② B spans V.
↖ pick ②

To show $\mathrm{Span}(B) = V$, take an arbituary $a+bx+cx^2 \in V$

and find c_1, c_2, c_3 such that

$$c_1(-1+2x+2x^2) + c_2(2+x^2) + c_3(-3+x) = a+bx+cx^2$$

coeff's:
$$\begin{aligned} -c_1 + 2c_2 - 3c_3 &= a \\ 2c_1 + 0c_2 + c_3 &= b \\ 2c_1 + c_2 + 0c_3 &= c \end{aligned}$$

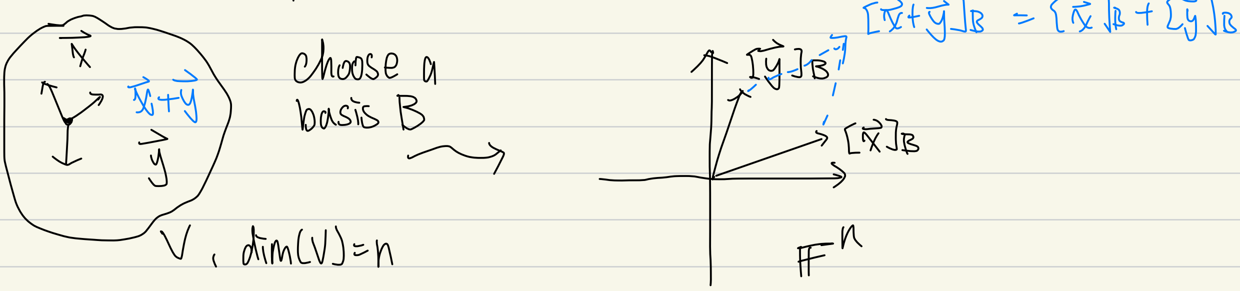
$$\left[\begin{array}{ccc|c} -1 & 2 & -3 & a \\ 2 & 0 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a+3b-2c \\ 0 & 1 & 0 & -2a-6b+5c \\ 0 & 0 & 1 & -2a-5b+4c \end{array}\right] \begin{array}{l} \to c_1 \\ \to c_2 \\ \to c_3 \end{array}$$

So $\mathrm{span}(B) = V$ hence b is a basis for V.

Now, $[1+2x+3x^2]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1+6-6 \\ -2-12+15 \\ -2-10+12 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ #

↳ $a=1$, $b=2$, $c=3$

* Abstract vector space V.S. $\mathbb{F}^n$



Theorem: let V be a n-dim vector space
let B be a basis for V.

Then: (a) $[\vec{x}+\vec{y}]_B = [\vec{x}]_B + [\vec{y}]_B \quad \forall \vec{x}, \vec{y} \in V$

(b) $[c\vec{x}]_B = c[\vec{x}]_B$

Lecture 9

Last time: Linear maps,

function: $L: V \to W$ is linear if

$$\left.\begin{array}{ll} a) & L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y}) \\ b) & L(c\vec{x}) = cL(\vec{x}) \end{array}\right\} \forall\, \vec{x},\vec{y} \in V,\ V \in \mathbb{F}$$

Remarks:

① Properties (a) and (b) is equivalent to

$$L(c\vec{x}+\vec{y}) = cL(\vec{x}) + L(\vec{y}),\ \forall\, \vec{x},\vec{y},c$$

② Linear map respect + and · What else do they respect.

③ Does L (linear map) respect $\vec{0}$?

$\Leftrightarrow L(\vec{0}_V) = \vec{0}_W$, yes.

④ Does L respect $-$?

$\Leftrightarrow L(-\vec{x}) = -L(\vec{x})$, yes.

⑤ What else does linear map respect?

- Bases? If $\{b_1 \dots b_n\}$ is a basis for V,

  is $\{L(b_1) \dots L(b_n)\}$ ________ W?

- LI? if ________ is LI in V ________ LI in V?

ex: of linear maps.

① The zero map $L: V \to W$ define by $L(\vec{v}) = \vec{0}_W,\ \forall\, \vec{v} \in V$

$\vec{v} \mapsto \vec{0}_W$

② The identity map $L: V \to V$ define by $L(\vec{v}) = \vec{v},\ \forall\, \vec{v} \in V$

$\vec{v} \mapsto \vec{v}$

ex: Fundamental examples.

① Let V be n-dimensional, Let B be order basis for V.

The B-coord map $[\ ]_B : V \to \mathbb{F}^n$ $(\vec{v} \longmapsto [\vec{v}]_B)$

Special case: $V = P_3(\mathbb{F})$, $B = \{1, x, x^2, x^3\}$

Then $[\ ]_B = P_3(\mathbb{F}) \to \mathbb{F}^4$ $(a+bx+cx^2+dx^3 \longmapsto [a+bx+cx^2+dx^3]_B)$

☆ ② Let $A \in M_{m\times n}(\mathbb{F})$, we can define linear map.

$L_A : \mathbb{F}^n \to \mathbb{F}^m$ $(\vec{x} \longmapsto A\vec{x})$

ex: Various examples:

① Differentiation: $D : P_n(\mathbb{F}) \longrightarrow P_{n-1}(\mathbb{F})$ $(p(x) \longmapsto p'(x))$

check $D(c\vec{x}+\vec{y}) = D(c\vec{x}) + D(\vec{y}) = cD(\vec{x}) + D(\vec{y})$.

② Integration: $I : P_n(\mathbb{F}) \to P_{n+1}(\mathbb{F})$ $(p(x) \longmapsto \int p(x)\,dx)$

③ Evaluation: Let $\alpha \in \mathbb{F}$

Define $ev_\alpha : P_n(\mathbb{F}) \to \mathbb{F}$ $(p(x) \longmapsto p(\alpha))$

④ Transpose: $L : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ $(A \longmapsto A^T)$

ex: is $L : P_3(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$

$p(x) \longmapsto \begin{bmatrix} p(1) & p'(1) \\ p''(1) & p'''(1) \end{bmatrix}$ linear?

yes.

ex: Non-linear map.

① Determinant $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

if $n=1$, det is linear (identity map)

↓

② $L: M_{2\times 2}(\mathbb{F}) \to P_2(\mathbb{F})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a+b) + (c+d)x + x^2$$

L is non-linear cause $L(\vec{0}) \neq \vec{0}$

The most important feature of linear map are:

① what it destroys.

② what it creates

Defn: Let $L: V \to W$ be linear

① The kernel of L is $\text{Ker}(L) = \{\vec{v} \in V : L(\vec{v}) = \vec{0}\}$

② The range of L is $\text{Range}(L) = \{L(\vec{v}) : \vec{v} \in V\}$

lecture 10

Defn: Let $L: V \to W$ be a LT. The Kernel (nullspace) of L is the set of all vectors in V that are mapped to zero vector of W. That is,

$$\ker(L) = \{\vec{v} \in V : L(\vec{v}) = \vec{0}_W\}$$

and the Range is the set of all outputs.

$$\text{Range}(L) = \{L(\vec{v}) : \vec{v} \in V\}$$

Theorem: Let $L: V \to W$ be a LT, then

① $\ker(L)$ is a subspace of V

② $\text{Range}(L)$ is a subspace of W.

Defn: Let $L: V \to W$ be a LT. then,

① $\text{Rank}(L) = \dim(\text{Range}(L))$

② $\text{nullity}(L) = \dim(\ker(L))$

Theorem: Let $L: V \to W$ be a LT, where $\dim(V) = n$, then $\text{rank}(L) + \text{nullity}(L) = n$

$\Rightarrow$ dim of range + dim of ker = dim of domain.

lecture 11

last time: If $L: V \to W$ is a linear map, then

$\text{rank}(L) = \dim(\text{Range}(L))$

$\text{nullity}(L) = \dim(\ker(L))$ (# non-pivot columns)

ex: warmup. $L: \mathbb{R}^3 \to P_1(\mathbb{R})$ $\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto (a+b+c)x \right)$

what are rank(L) and nullity(L)?

↳ seems like 1 ↳ guess 2.

Sol: proof need $\text{Range}(L) = \left\{ L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \right\}$

$= \left\{ (a+b+c)x : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \right\}$

$\subseteq \text{Span}\{x\}$.

On the other hand, $\text{span}\{x\} \subseteq \text{Range}(L)$:

$\alpha x = L\left(\begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}\right)$ $\forall \alpha \in \mathbb{R}$, So every $\alpha x \in \text{Range}(L)$

thus, $\text{Range}(L) = \text{span}\{x\}$.

$\Rightarrow \text{rank}(L) = \dim(\text{Range}(L)) = 1$ #

Now, Let's determine ker(L)

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker(L) \Leftrightarrow L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = 0 + 0x$

$\Leftrightarrow (a+b+c)x = 0 + 0x$

$\Leftrightarrow a+b+c = 0 \Leftrightarrow a = -b-c$

$\Leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So $\ker(L) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{nullity}(L) = \dim(\ker(L)) = 2$ #

Observation: $L: \mathbb{R}^3 \to P_1(\mathbb{R})$

$\operatorname{rank}(L) + \operatorname{nullity}(L) = 1 + 2 = 3 \quad (= \dim(\mathbb{R}^3))$

Fundemental theorem of linear alg. (Rank-nullity theorem):

If $L: V \to W$ is linear then $\underbrace{\dim(V)}_{\text{domain dim}} = \underbrace{\operatorname{rank}(L)}_{\text{codomain dim}} + \underbrace{\operatorname{nullity}(L)}_{\text{dim lost.}}$

Proof: Let $\{\vec{k}_1, \vec{k}_2 \ldots \vec{k}_n\}$ be a basis for $\ker(L)$

(so $n = \dim(\ker(L)) = \operatorname{nullity}(L)$)

Extend to basis for V — say $B = \{\vec{k}_1, \vec{k}_2 \ldots \vec{k}_n, \vec{v}_1, \vec{v}_2 \ldots \vec{v}_r\}$

(so $\dim(V) = n + r$)

let $C = \{L(v_1), L(v_2) \ldots L(v_r)\} \subseteq W$

Claim, C is a basis for $\operatorname{Range}(L)$.

Proof • spanning Need to show $\operatorname{span}(C) = \operatorname{Range}(L)$

$\subseteq$: let $\vec{w} \in \operatorname{Span}(C)$

then $\vec{w} = a_1 L(v_1) + \cdots + a_r L(v_r)$

$= L(a_1 v_1 + \cdots a_r v_r)$

So $\vec{w} = L(\text{sth in } V)$, thus, $\vec{w} \in \operatorname{Range}(L)$.

$\supseteq$: let $\vec{w} \in \operatorname{Range}(L)$. Then $\vec{w} = L(\vec{v})$ for some $\vec{v} \in V$

Since $\vec{v} \in V$, and B is a basis for V, we have

$V = a_1 \vec{k} + \cdots + a_n \vec{k}_n + b_1 \vec{v}_1 + \cdots b_r \vec{v}_r$

$\Rightarrow L(\vec{v}) = a_1 \cancel{L(\vec{k}_1)}^{0} + \cdots + a_n \cancel{L(\vec{k}_n)}^{0} + b_1 L(\vec{v}_1) + \cdots b_n L(\vec{v}_r)$

$= \sum b L(\vec{v})$, thus $\vec{w} = L(\vec{v}) \in \operatorname{Span}(C)$

ex: $tr : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

Let $W = \ker(tr) = \{A \in M_{n\times n}(\mathbb{F}) : tr(A) = 0\}$, determine $\dim W$.

Sol: note $\dim W = \dim(\ker(tr))$

$= \text{nullity}(tr) = \dim(M_{n\times n}(\mathbb{F})) - \text{rank}(tr)$

$= n^2 - \text{rank}(tr)$.

What is $\text{rank}(tr)$?

$tr : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ $= \cancel{0}$ or 1

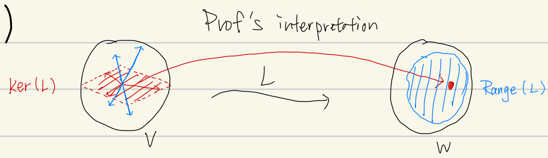
because tr is not zero map.

thus, $\dim W = n^2 - 1$

lecture 12.

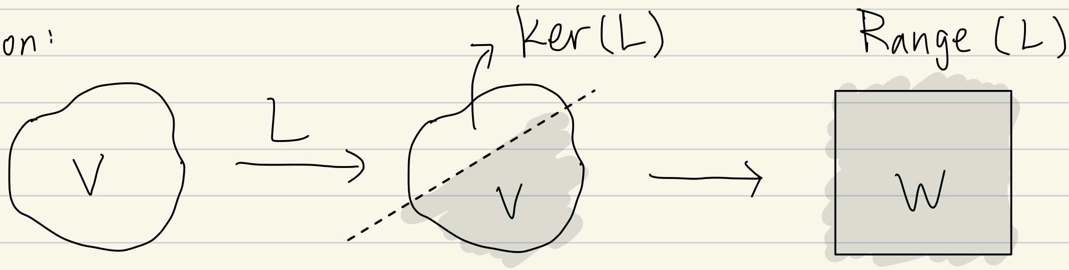
Last time we see that if $L: V \to W$ is linear map,

then $\dim(V) = \text{rank}(L) + \text{nullity}(L)$



This is fairly intuitive:

My version:



In ideal situation, L "preserves" V and just place it in W somehow (maybe stretch or rotate). In this case, we'd expect. $\dim(V) = \dim(\text{Range}(L))$. However, in general L might squash away some dimensions (it destroys $\ker(L)$), so we need to correct this: we lose $\dim(\ker(L))$ degrees of freedom in passage from V to Range(L). So we should have:

$$\dim(V) - \dim(\ker(L)) = \dim(\text{Range}(L))$$

↳ also rank & nullity thm.

this suggest:

Defn: Let $L: V \to W$ be linear

(a) If $\ker(L) = \{\vec{0}\}$ ($\Leftrightarrow$ nullity $= 0$)

then we say L is **injective**. (one-to-one)

(b) If $\text{Range}(L) = W$ ($\Leftrightarrow \text{rank}(L) = \dim(W)$)

then we say L is **surjective** (onto)

↳ "covers" entire codomain.

ex: Let $L: P_1(\mathbb{F}) \to P_2(\mathbb{F})$ be defined by $L(a+bx) = bx^2$

① is L injective?

Let's compute $\ker(L) = \{a+bx \in P_1(\mathbb{F}) : L(a+bx) = \vec{0}\}$

$= \{a+bx \in P_1(\mathbb{F}) : bx^2 = 0 + 0x + 0x^2\}$

$= \{a+bx \in P_1(\mathbb{F}) : b = 0\}$

$= \{a : a \in \mathbb{F}\}$ (constant polynomial)

$\ker(L) \neq \{\vec{0}\}$, thus not injective.

② is L surjective?

Let's compute $\text{Range}(L) = \{L(a+bx) : a+bx \in P_1(\mathbb{F})\}$

$= \{bx^2 : b \in \mathbb{F}\}$

$= \text{span}\{x^2\}$

$\text{Range}(L) \neq P_2(\mathbb{F})$, thus not surjective.

ex: (Exercise)

Determine if each is injective / surjective.

(a) identity map: $id: V \to V$ defined by $id(\vec{v}) = \vec{v}$.

- injective, since $\ker(id) = \{\vec{0}\}$

- surjective, since obviously Range(L)

dimension lost

Prep: ① $L: V \to W$ if $\dim(V) > \dim(W) \Rightarrow L$ can't be injective.

② $L: V \to W$ if $\dim(W) > \dim(V) \Rightarrow L$ can't be injective.

doesn't "cover" whole codomain.

*Prep: if $\dim(V) = \dim(W)$, L injective $\Leftrightarrow$ L surjective.

lecture 13.

ex: Consider $L: P_3(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ defined by

$$L(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p'(0) & p''(0) \end{bmatrix}$$

Sol: Since $\dim(P_3) = 4$ and $\dim(M_{2\times 2}(\mathbb{F})) = 4$

thus injective $\Leftrightarrow$ surjective

Now, injective $\Leftrightarrow$ nullity $(L) = 0$

Or, surjective $\Leftrightarrow$ rank$(L) = 4$

Instead of doing directly, let's observe.

$\rightarrow L: P_3(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ is given by:

$$L(a+bx+cx^2+dx^3) = \begin{bmatrix} a & a+b+c+d \\ b & 2c \end{bmatrix}$$

this looks like:

$T: \mathbb{F}^4 \to \mathbb{F}^4$ given by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a \\ a+b+c+d \\ b \\ 2c \end{bmatrix}$

* The connection is coordinates! we are taking coordinate vector $[p(x)]_B$ and $[L(p(x))]_C$ with standard basis $B = \{1, x, x^2, x^3\}$ and $C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$\Rightarrow$ we'll show how every $L: V \to W$ can look like $T: \mathbb{F}^n \to \mathbb{F}^m$ where $n = \dim(V)$ and $m = \dim(W)$.

why?, easier to study.

ex: let $L: \mathbb{F}^2 \to \mathbb{F}^3$ defined $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x \end{bmatrix}$

then we claim there is a matrix A s.t.

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 2x \end{bmatrix}$,

in this case, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$ works.

In general, given a linear map $L: \mathbb{F}^n \to \mathbb{F}^m$, we define its <u>standard matrix</u> to be $[L] = [L(\vec{e}_1) \mid L(\vec{e}_2) \mid \dots L(\vec{e}_n)]$ where $\{\vec{e}_1, \vec{e}_2 \dots \vec{e}_n\}$ is the standard basis

* key fact: matrix $[L]$ "know everything" abt map L

Here is how to create a matrix $\hat{A}$ that performs $L: V \to W$

- Step 1: choose ordered basis $B = \{\vec{v}_1 \cdots \vec{v}_n\}$ (so $n = \dim(V)$)

  $C = \{\vec{w}_1 \cdots \vec{w}_m\}$ (so $m = \dim(W)$)

- Step 2: describe the effect of L on coordinate vector.

  This gives a linear map $\mathbb{F}^n \to \mathbb{F}^m$

- Step 3: find the standard matrix of the map $\mathbb{F}^n \to \mathbb{F}^m$

  $\Rightarrow$ We can streamline this process.

Defn: let $L: V \to W$ with ordered basis $B = \{\vec{v}_1 \dots \vec{v}_n\}$ $C = \{\vec{w}_1 \dots \vec{w}_n\}$

then the matrix of L wrt B and C is the $m \times n$ matrix.

$${}_C[L]_B = \left[\, [L(\vec{v}_1)]_C \;\; [L(\vec{v}_2)]_C \;\; \cdots \;\; [L(\vec{v}_n)]_C \,\right]$$

* <u>key property</u>: $[L(\vec{x})]_C = {}_C[L]_B [\vec{x}]_B$

ex: Let's look at differentiation map $D: P_3(\mathbb{R}) \to P_2(\mathbb{R})$

Let B std basis $= \{1, x, x^2, x^3\}$

and C std basis $= \{1, x, x^2\}$

$$
{}_C[D]_B = \big[\ [D(1)]_C\ \ [D(x)]_C\ \ [D(x^2)]_C\ \ [D(x^3)]_C\ \big]
$$

$$
= \big[\ [0]_C\ \ [1]_C\ \ [2x]_C\ \ [3x^2]_C\ \big]
$$

$$
= \big[\ [0+0x+0x^2]_C\ \ [1+0x+0x^2]_C\ \ [0+2x+0x^2]_C\ \ [0+0x+3x^2]_C\ \big]
$$

$$
= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}
$$

* The idea here is that the matrix "performs" D 概念

i.e. we can multiply it by a polynomial.

BUT, we can multiply it by the coordinate vect. $[p(x)]_B$ ☆

Here's what we get:

$$
\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}\ \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}} = \underbrace{\begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}}
$$

this basically says that $D(a+bx+cx^2+dx^3) = b+(2c)x+(3d)x^2$

BUT what it actually says is $\underline{{}_C[D]_B}\ \underline{[p(x)]_B} = \underline{[D(p(x))]_C}$

* Takeaway: The key property tells us that we can "perform" D by multiplying coordinate vectors by ${}_C[D]_B$

lecture 14.

<u>Prop</u>: let $L: V \to W$ and let B & C be bases for V & W,

let $A = {}_C[L]_B$, then,

(a) $\vec{v} \in \ker(L) \Leftrightarrow [\vec{v}]_B \in \mathrm{Null}(A)$: $\longrightarrow A\vec{x} = \vec{0}$

(b) $\vec{w} \in \mathrm{Range}(L) \Leftrightarrow [\vec{w}]_C \in \mathrm{Col}(A)$.

Corollary : let $A = {}_C[L]_B$

(a) $\dim(\ker(L)) = \dim(\mathrm{Null}(A))$
(i.e nullity(L) = nullity(A))

(b) $\dim(\mathrm{range}(L)) = \dim(\mathrm{Col}(A))$
(i.e rank(L) = rank(A))

lecture 15: Iso

Defn: An isomorphism is a linear map $L: V \to W$ that is injective and surjective. We say V is isomorphic to W.

Useful comments: ☆

- L injective means it place a copy of V in W without destroying anything.
- L surjective means the copy of V fill up all of W.

So, in essense, L "morphs" V into W perfectly.

Defn: let $L: V \to W$ be an isomorphism. The inverse of L is the unique linear map $T: W \to V$ that satisfies.

$L(T(\vec{w})) = \vec{w}$, $\forall \vec{w} \in W$ and $T(L(\vec{v})) = \vec{v}$, $\forall \vec{v} \in V$.

We describe it by L^{-1}.

lecture 16

Defn: change of coord:

$$_C[I]_B = \begin{bmatrix} | & | & & | \\ [\vec{v}_1]_C & [\vec{v}_2]_C & \cdots & [\vec{v}_n]_C \\ | & | & & | \end{bmatrix}$$ from B to C coord.

where $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

Thm:

(a) $_C[I]_B [\vec{x}]_B = [\vec{x}]_C$ → Basically performs lin map different way...

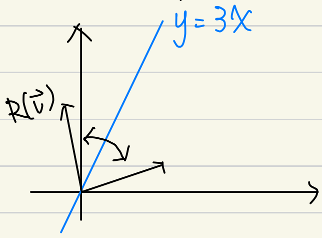
(b) $(_C[I]_B)^{-1} = {_B[I]_C}$

(c) $_D[I]_C \; _C[I]_B = {_D[I]_B}$

lecture 17.

* <u>Motavating problem:</u>

Suppose we want to <u>determine</u> a closed form expression for linear map $R: \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across $y = 3x$



$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow R(\vec{v}) = ??$$

p.s. This is certainly doable with some plane geometry

$\longrightarrow$ but there's a simpler approach.

1st key: R is completely determined by any of its matrix representation.

$\Rightarrow {}_B[R]_B = \left[[R(\vec{v}_1)]_B \; [R(\vec{v}_2)]_B \right]$, B is a basis for R.

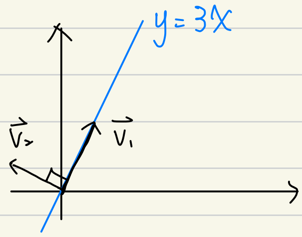
choosing a useful basis, $B = S = \{\vec{e}_1, \vec{e}_2\}$, then

$\Rightarrow R(\vec{x}) = {}_S[R]_S \, [\vec{x}]_S = {}_S[R]_S \, \vec{x}$

p.s. here, the problem is that hard to determine $R(\vec{e}_1)$ and $R(\vec{e}_2)$

2nd key: pick a basis that "respects" action of R

Here, we take $B = \{\vec{v}_1, \vec{v}_2\}$, where



$\vec{v}_1$ = any vector on line $y = 3x$

$\vec{v}_2$ = any vector $\perp$ to line $y = 3x$

Then, we easily have $R(\vec{v}_1) = \vec{v}_1$ and $R(\vec{v}_2) = -\vec{v}_2$

specifically, ${}_B[R]_B = \left[[R(\vec{v}_1)]_B \; [R(\vec{v}_2)]_B \right]$

$$= \left[[\vec{v}_1]_B \; [-\vec{v}_2]_B \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

so if $\vec{x} \in \mathbb{R}^2$ has $[\vec{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$ then,

$$[R(\vec{x})]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix} \longrightarrow \text{nice!}$$

But, it all take place at B-coords. How can we get $R(\vec{x})$ in standard-coords?

Ans: Change coordinates:

$$_S[I]_B = [\,[\vec{v}_1]_S \;\; [\vec{v}_2]_S\,]$$

So,

$$\Rightarrow [R(x)]_B = {}_B[R]_B [\vec{x}]_B \Rightarrow {}_S[I]_B [R(x)]_B = {}_S[I]_B \, {}_B[R]_B [\vec{x}]_B$$

$$\Rightarrow [R(x)]_S = {}_S[R]_B [\vec{x}]_B$$

$$= {}_S[R]_B \, {}_B[I]_S [x]_S$$

$$= {}_S[R]_S [x]_S$$

This shows that ${}_S[R]_S = {}_S[I]_B \, {}_B[R]_B \, {}_B[I]_S$

$$= {}_S[I]_B \, {}_B[R]_B \, ({}_S[I]_B)^{-1}$$

Theorem: Let $L: V \to V$ be a linear map. let B & C be basis for V

then, ${}_B[L]_B = {}_B[I]_C \, {}_C[L]_C \, {}_C[I]_B$

If let $A = {}_B[L]_B$, $B = {}_C[L]_C$, $P = {}_B[I]_C$, then $A = PBP^{-1}$

Defn: Let $A, B \in M_{n\times n}(\mathbb{F})$, we say A is similar to B if $\exists$ an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ s.t. $A = PBP^{-1}$ $\Rightarrow$ $A \sim B$

properties: ① $A \sim A$, ② $A \sim B \Rightarrow B \sim A$, ③ $A \sim B$ and $B \sim C \Rightarrow A \sim C$

Theorem: Let $A, B \in M_{n\times n}(\mathbb{F})$. If $A \sim B$ then $\exists$ a linear map $L: \mathbb{F}^n \to \mathbb{F}^n$ and bases C_1 & C_2 for $\mathbb{F}^n$ s.t. $A = {}_{C_1}[L]_{C_1}$ and B ${}_{C_2}[L]_{C_2}$

* Conceptual tip: $A \sim B$ $\Rightarrow$ they give the "same" tranformation but in "different" coordinates!!!

* So, matrices being similar meaning they come from the same map.

Prop: Let $A, B \in M_{n\times n}(\mathbb{F})$. If $A \sim B$, then (a) $\det(A) = \det(B)$

(b) $tr(A) = tr(B)$

(c) rank, nullity (A) = " " (B)

Defn: A linear map $L: V \to V$ is diagonalizable if there is a basis $\mathcal{B}$ for V s.t. ${}_{\mathcal{B}}[L]_{\mathcal{B}}$ is digonal matrix

Lecture 18.

Prop: let $L: V \to V$ be a linear operator and let $B = \{\vec{v}_1 \dots \vec{v}_n\}$ be a basis for V then:

$$_B[L]_B = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \text{ iff } L(\vec{v}_i) = \lambda \vec{v}_i \quad \forall i = 1 \dots n.$$

Defn: A eigenvector for a linear operator $L: V \to V$ is a vector $\vec{v} \in V$ s.t.: (a) $\vec{v} \neq \vec{0}$.

(b) $L(\vec{v}) = \lambda \vec{v}$ (this λ is the eigenvalue)

Corollary: $L: V \to V$ is diagonalizable iff $\exists$ a basis $B = \{\vec{v}_1 \dots \vec{v}_n\}$ for V s.t. each $\vec{v}_i$ is an eigenvector for L.

(a.k.a eigenbasis or diagonalizing basis.)

Defn: The eigenspace of L corresponding to λ is

$$E_\lambda(L) = \ker(L - \lambda \cdot Id)$$

$$= \{\vec{v} \in V : L(\vec{v}) = \lambda \vec{v}\}$$

note: $E_\lambda(L)$ consist of all eigenvectors and λ.

Prop: let $L: V \to V$ be a linear operator and let $A = {}_B[L]_B$

Then: $\vec{v} \in V$ is an eignvector of L w/ eigenvalue λ $\iff$ $[\vec{v}]_B \in \mathbb{F}^n$ is an eigenvector of A w/ eigenvalue λ

corollary: $E_\lambda(L)$ is isomorphic to $E_\lambda(A)$

Defn: let $A \in M_{n\times n}(\mathbb{F})$. We say A is diagonalizable if A is similar to a digonal matrix $D \in M_{n\times n}(\mathbb{F})$ (meaning: $\exists$ P s.t $A = PDP^{-1}$).

So: $L: V \to V$ is diagonalizable $\Leftrightarrow$ any of reps. ${}_B[L]_B$ are diagonalizable.

Q: how to determine if diagonalizable? (lin I review)

Step 1: compute characteristic polynomial of A:

$$C_A(\lambda) = \det(A - \lambda I)$$

it's roots in $\mathbb{F}$ are eigenvalues of A.

$\to$ if A does not factor to linear terms, $\to$ not diagonalizable.

Step 2: let $\lambda_1 \ldots \lambda_k$ be distinct eigenvalues of A,

For each λ_i:

- compute the algebraic multiplicity a_{λ_i} (# of occurences)
- compute the geometric multiplicity g_{λ_i} ($\dim(E_{\lambda_i}) = \text{nullity}(A - \lambda I)$)

$\to$ if $a_{\lambda_i} \neq g_{\lambda_i} \to$ not diagonalizable.

$\to$ if $a_{\lambda_i} = g_{\lambda_i} \to$ YES

ex: let $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, then $C_B(\lambda) = (\lambda - 3)^2$, so $\lambda_1 = 3$, $\lambda_2 = 3$

and $a_\lambda = 2$,

$g_\lambda = \text{nullity}(B - \lambda I) = \text{nullity}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1. \neq 2.$

thus, NO.

lecture 19.

$A = PDP^{-1}$

note: - a_λ tells us how many times λ appears in D

- g_λ tells us how many columns in P can produce a λ in D

How to find D and P?

let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of A

- Step 1: for each λ_i, find a basis for $E_\lambda(A) = \mathrm{Null}(A - \lambda I)$

then the columns of P are the vectors of these basis.

$P = [\text{Basis for } E_{\lambda_1} \mid \text{basis for } E_{\lambda_2} \mid \cdots \mid \text{basis for } E_{\lambda_k}]$

- Step 2: let $D = \begin{bmatrix} \overbrace{\lambda_1 \ddots \lambda_1}^{a_{\lambda_1} \text{ times}} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\lambda_n \ddots \lambda_n}^{a_{\lambda_n} \text{ times}} \end{bmatrix}$

Then $A = PDP^{-1}$

ex: let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$. Then $C_A(\lambda) = (1-\lambda)(2-\lambda)(4-\lambda)$

$\Rightarrow \lambda_1 = 1,\ a_1 = 1$ & $\lambda_2 = 2,\ a_2 = 1$ & $\lambda_3 = 4,\ a_3 = 1$

to find P we need $E_1(A)$, $E_2(A)$, $E_4(A)$

- $E_1(A) = \mathrm{null}(A - 1I) = \mathrm{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\right) = \mathrm{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

- $E_2(A) = \mathrm{null}(A - 2I) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

- $E_4(A) = \mathrm{null}(A - 4I) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}\right\}$

So, $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

lecture 20.

Defn: Let V be a <u>real</u> vector space. An inner product on V is a fcn. $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies:

(a) linearity in 1st argument. $\langle a\vec{u}+\vec{w}, \vec{v} \rangle = a\langle \vec{u}, \vec{v}\rangle + \langle \vec{w}, \vec{v}\rangle$

(b) symmetry. $\langle \vec{u}, \vec{v}\rangle = \langle \vec{v}, \vec{u}\rangle$

(c) positive definiteness $\langle \vec{u}, \vec{u}\rangle \geq 0$ and $\langle \vec{u}, \vec{u}\rangle = 0$ iff $\vec{u} = 0$

A vector space with an inner product is called inner vector space.

ex:

① the dot product is a inner product (a.k.a. standard inner product)

② On $M_{2\times 2}(\mathbb{F})$, $\langle A, B\rangle = \text{trace}(B^T A)$ (a.k.a. Frobenius inner product)

* Remark: to define an inner product on a arbitrary n-dim vector space, choose a basis B and get isomorphism $V \to \mathbb{R}^2$ then copy over the dot product.

Defn: An inner product on a vector space over $\mathbb{F}$ is a fcn.

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$

that satisfies:

(a) linearity in 1st argument. $\langle a\vec{u}+\vec{w}, \vec{v} \rangle = a\langle \vec{u}, \vec{v}\rangle + \langle \vec{w}, \vec{v}\rangle$

(b) conjugate symmetry. $\langle \vec{u}, \vec{v}\rangle = \overline{\langle \vec{v}, \vec{u}\rangle}$

(c) positive definiteness. $\langle \vec{u}, \vec{v}\rangle \geq 0$ and $\langle \vec{u}, \vec{v}\rangle = 0$ iff $\vec{u} = 0$

ex:

① Std. inner product on $\mathbb{F}^n$ is $\langle \vec{x}, \vec{y}\rangle = x\bar{y} + \ldots + x_n \bar{y}_n$.

② $\langle A, B\rangle = tr(\overline{B^T} A)$ is a inner p.

lecture 21 (skipped)

Defn: let V be an inner space. The norm of a vector $\vec{v} \in V$ is

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

using this we can define the distance between $\vec{u}$ & $\vec{v}$ by

$$\text{dist}(\vec{u}, \vec{v}) \; \|\vec{u} - \vec{v}\|$$

* Prop: (properties of norm)

① $\|\vec{x}\| \geq 0$

② $\|a\vec{x}\| = |a| \, \|\vec{x}\|$

③ $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ $\quad \triangle$ ineq

Cauchy schwarz ineq: $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$

* Prop: (properties of distance)

① $d(\vec{x}, \vec{y}) \geq 0$ and $d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$

② $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$

③ $d(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{y})$

Defn: Two vectors $\vec{x}$ & $\vec{y}$ in an inner product space are said to be orthogonal (or perpendicular) if $\langle \vec{x}, \vec{y} \rangle = 0$. we denote this by $\vec{x} \perp \vec{y}$.

Pythagorean theorem: if $\vec{x} \perp \vec{y}$ then $\|\vec{x} + y\|^2 = \|\vec{x}\|^2 + \|y\|^2$.

lecture 22 (skp)

Defn: A set of vectors $\{\vec{v}_1 \dots \vec{v}_n\}$ in an inner product space is said to be an orthogonal set if $\langle \vec{v}_i, \vec{v}_j \rangle = 0 \ \forall\, i \neq j$.

Theorem I: (generalized Pythagoras)

If $\{\vec{x}_1 \dots \vec{x}_n\}$ is orthogonal then $\|\vec{x}_1 + \dots + \vec{x}_n\|^2 = \|\vec{x}_1\|^2 + \dots + \|\vec{x}_n\|^2$

Theorem 2:

If $S = \{\vec{x}_1 \dots \vec{x}_n\}$ is a orthgonal set and if NO $\vec{x}_i = \vec{0}$ then S is linearly independent.

Theorm 3:

If $B = \{\vec{x}_1 \dots \vec{x}_n\}$ is an orthogonal set AND if B is a basis for V then every $\vec{x} \in V$ can be written as

$$\vec{x} = c_1\vec{x}_1 + \dots + c_n\vec{x}_n \quad \text{w/} \quad c_j = \frac{\langle \vec{x}, \vec{x}_j \rangle}{\langle \vec{x}_j, \vec{x}_j \rangle} \ \star$$

ex: in $\mathbb{R}^3$, given $\vec{x} = [0, -1, 2]^T$, then

$$\frac{\vec{x}\cdot\vec{e}_1}{\vec{e}_1\cdot\vec{e}_1} = 0, \quad \frac{\vec{x}\cdot\vec{e}_2}{\vec{e}_2\cdot\vec{e}_2} = -1, \quad \text{and} \quad \frac{\vec{x}\cdot\vec{e}_3}{\vec{e}_3\cdot\vec{e}_3} = 2$$

lecture 23 (skp)

Theorem (add on last time):

If $B\{\vec{v}_1 \dots \vec{v}_n\}$ is orthogonal basis for V then every $\vec{v} \in V$ can be expressed as $\vec{x} = \sum_{i=1}^{n} \frac{\langle \vec{x}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i$

If B is orthonormal (so $\langle \vec{v}_i, \vec{v}_i \rangle = 1 \ \forall i$) then

$$\vec{x} = \sum_{i=1}^{n} \langle x, \vec{v}_i \rangle \vec{v}_i$$

normal vectors.

p.s. could find coordinates.

ex: $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. (orthogonal basis), $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, find $[\vec{x}]_B$.

$$\frac{\vec{x} \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1 = \frac{a+b}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{\vec{x} \cdot \vec{b}_2}{\vec{b}_2 \cdot \vec{b}_2} \vec{b}_2 = \frac{a-b}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus, $[x]_B = \left[\frac{a+b}{2}, \frac{a-b}{2} \right]^T$

ex: let $V = P_2(\mathbb{R})$ w/ $\langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} pq$, and let $B = \{1-x, 1+3x\}$ orthogonal

find $[p(x)]_B$ where $p(x) = a + bx$.

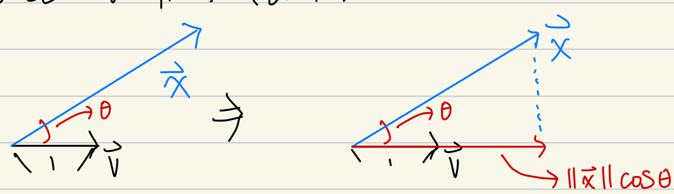
$$C_1 = \frac{\vec{p} \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} = \frac{\int_{-1}^{1} (a+bx)(1-x)\,dx}{\int_{-1}^{1} (1-x)^2\,dx} = \frac{\int_{-1}^{1} a + (b-a)x - bx^2\,dx}{\int_{-1}^{1} 1 - 2x + x^2\,dx} = \frac{3a-b}{4}$$

$$C_2 = \frac{\vec{p} \cdot \vec{b}_2}{\vec{b}_2 \cdot \vec{b}_2} = \frac{\int_{-1}^{1} (a+bx)(1+3x)\,dx}{\int_{-1}^{1} (1+3x)^2\,dx} = \dots = \frac{a+b}{4} \quad \#$$

So what does $\frac{\langle \vec{x}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$ mean? , suppose $V = \mathbb{R}^2$. (dot)

$$= \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = (\|\vec{x}\| \cos\theta) \frac{\vec{v}}{\|\vec{v}\|}$$

direction



Defn: let $\vec{v}, \vec{x} \in V$ and assume $\vec{v} \neq \vec{0}$. then projection of $\vec{x}$ onto $\vec{v}$ is

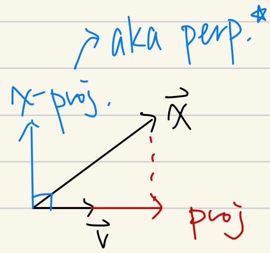
$$\text{proj}_{\vec{v}}(\vec{x}) = \frac{\langle \vec{x}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}.$$

onto

Prop: (projection)

① $\text{Proj}_{\vec{v}}(\vec{v}) = \vec{v}$

② $\vec{x} - \text{proj}_{\vec{v}}(\vec{x}) \perp \text{proj}_{\vec{v}}(\vec{x}) \Rightarrow$



We can now rewrite the formula:

Theorem:

If $B = \{\vec{v}_1 \dots \vec{v}_2\}$ is an orthogonal basis for V, then $\forall \vec{x} \in V$

$$\vec{x} = \text{proj}_{\vec{v}_1}(\vec{x}) + \cdots + \text{proj}_{\vec{v}_n}(\vec{x}).$$

lecture 24 (skp)

Goal: show every finite-dim inner product space has a <u>orthogonal basis</u>.

The Gram-Schmidt Orthogonalize process: (basis $\to$ orth basis)

- let V be an inner product space w/ basis $\{\vec{v}_1 \dots \vec{v}_n\}$,

Define vector $\vec{w}_1 \dots \vec{w}_n$ iteratively as follows:

- $\vec{w}_1 = \vec{v}_1$
- $\vec{w}_2 = \vec{v}_2 - \dfrac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1$ (perp)

$\vdots$

perp$_{W}(\vec{v})$ $\leftarrow$ vector space.

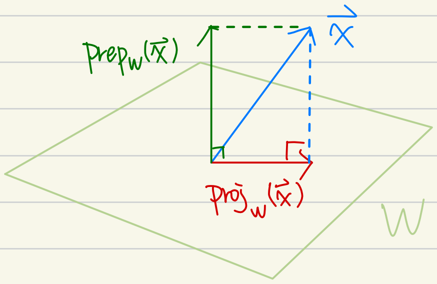
- $\vec{w}_n = \vec{v}_n - \dfrac{\langle \vec{v}_n, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 - \dfrac{\langle \vec{v}_n, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} \vec{w}_2 - \dots \dfrac{\langle \vec{v}_n, \vec{w}_{n-1} \rangle}{\langle \vec{w}_{n-1}, \vec{w}_{n-1} \rangle} w_{n-1}$

Then $\{\vec{w}_1, \vec{w}_2 \dots \vec{w}_n\}$ is a orthogonal basis for V.

Corollary: every finite-dim inner p-space has an orthonormal vector.

lecture 25 (skp)

Goal: Given a subspace W of inner product space V, define $\text{proj}_W(\vec{x})$ & $\text{perp}_W(\vec{x})$



$$\vec{x} = \text{proj}_W(\vec{x}) + \text{perp}_W(\vec{x})$$

this is an example of $W = \text{span}\{\vec{w}\}$ is a one-dim vec. space. how in general?

* two basic defn. of $\text{proj}_W(\vec{x})$

① a vector $\vec{p} \in W$ s.t. $\vec{x} - \vec{p}$ is $\perp$ to every vector in W.

② a vector $\vec{p} \in W$ closest to $\vec{x}$,

Theorem: let W be a subspace of finite-dim inner product space V. Then we can find an orthogonal basis $\{\vec{w}_1, \dots \vec{w}_k, \vec{n}_1 \dots \vec{n}_\ell\}$ for V s.t. $\{\vec{w}_1 \dots \vec{w}_k\}$ is an orthogonal basis for W.

Defn: let W be a subspace of inner product space V. let $\{\vec{w}_1 \dots \vec{w}_k\}$ be an <u>orthogonal</u> basis for W. Let $\vec{x} \in V$. The projection of $\vec{x}$ onto $\vec{w}$ is the vector

$$\text{proj}_W(\vec{x}) = \text{proj}_{\vec{w}_1}(\vec{x}) + \dots + \text{proj}_{\vec{w}_2}(\vec{x})$$

$$= \frac{\langle \vec{x}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 + \dots + \frac{\langle \vec{x}, \vec{w}_k \rangle}{\langle \vec{w}_k, \vec{w}_k \rangle} \vec{w}_k$$

lecture 26

Goal: if given another ort. basis $U = \{u_1 \dots u_k\}$, does

$$\text{proj}_{w_1}(\vec{x}) + \dots + \text{proj}_{w_k}(\vec{x}) = \text{proj}_{u_1}(\vec{x}) + \dots + \text{proj}_{u_k}(\vec{x}) \ ?$$

$\Rightarrow$ does $\text{proj}_W(\vec{x}) = \text{proj}_U(\vec{x})$? , it has to.

Theorem: let W be a subspace of a finite-dim inner product spc. V. then we can find another orthogonal basis $\{\vec{w}_1 \dots \vec{w}_k, \vec{n}_1 \dots n_l\}$ for V s.t. $\{\vec{w}_1 \dots \vec{w}_k\}$ is an orthogonal basis for W.

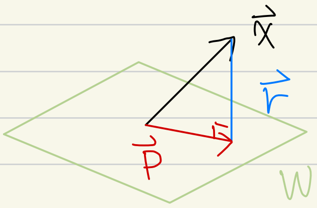
The orthogonal decomposition theorem:

let W be a subspace of a finite-dim inner product space V.

then every $\vec{x} \in V$ can be expressed uniquely as

$$\vec{x} = \vec{p} + \vec{r}.$$

where $\vec{p} \in W$ and $\vec{r}$ is $\perp$ W (i.e. $\vec{r} \perp \vec{w} \ \forall \ \vec{w} \in W$).



Defn: let W be a subspace of a finite-dim inner product space V.

the orthogonal complement of W in V, denoted by $W^\perp$ (read "W perp") is the set:

$$W^\perp = \{\vec{r} \in V : \vec{r} \perp W\}$$

$$= \{\vec{r} \in V : \langle \vec{r}, \vec{w} \rangle = 0, \ \forall \ \vec{w} \in W\}$$

$$\begin{cases} \vec{x} = \vec{p} + \vec{r} \\ \vec{p} \in W \end{cases}$$

Corollary: let W be a .... space V. Then if $\{\vec{w}_1 \dots \vec{w}_k\}$ and $\{\vec{u}_1 \dots \vec{u}_k\}$ are orthogonal bases for W, we have:

$$\text{proj}_{w_1}(\vec{x}) + \dots + \text{proj}_{w_k}(\vec{x}) = \text{proj}_{u_1}(\vec{x}) + \dots + \text{proj}_{u_k}(\vec{x})$$

lecture 27.

Goal: is $\text{proj}_W(\vec{v})$ the closest on to V in all the vectors in W?

or, $\|\vec{x} - \text{proj}_W(\vec{x})\| \leq \|\vec{x} - \vec{w}\| \quad \forall \vec{w} \in W$?

$\hookrightarrow \text{perp}_W(\vec{x})$

Theorem: let W be a ... V. let $\vec{v} \in V$ and $\vec{p} = \text{proj}_W(\vec{v})$.

then, $\forall \vec{w} \in W$, $\underbrace{\|\vec{v} - \vec{p}\|}_{\text{dist}(\vec{v},\vec{p})} \leq \underbrace{\|\vec{v} - \vec{w}\|}_{\text{dist}(\vec{v},\vec{w})}$ w/ equality iff $\vec{w} = \vec{p}$

ex: let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y - z = 0 \right\}$. find vect. closest to $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Sol: basis $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right\}$

$$\text{proj}_W\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 13/6 \\ 5/2 \end{bmatrix}$$

Prop^n: let W be a subspace of an inner product space V. then,

a) $W^\perp$ is a subspace of V.

b) if V is a finite-dim space then $\dim(V) = \dim(W) + \dim(W^\perp)$

c) if V is a finite-dim space then $(W^\perp)^\perp = W$.

Prop^n: If $\{\vec{w}_1 \dots \vec{w}_k\}$ is a spanning set for W then,

$\vec{x} \in W^\perp$ iff $\langle x, \vec{w}_i \rangle = 0 \quad \forall i = 1, \dots k$.

lecture 28.

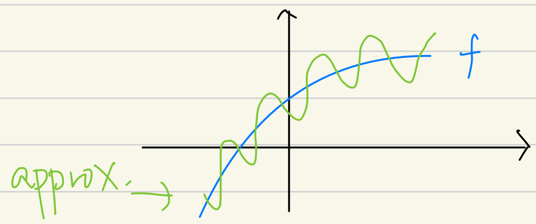
goal: Applications of proj_W

App ①: Fourier Series.

let V be vector space of cts fcns on $[-\pi, \pi]$ (random)

Give it the L^2 inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} fg$

we want to approx. any given $f \in V$ trig fcns. – imagine taking a cts fcn and approx. it with "waves".



let $W_n = \text{Span}\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$.

, this is a space of "waves". We want a fcn in W_n that approx. our $f \in V$. take $P_n = \text{proj}_{W_n}(f) \rightarrow$ the fcn in W_n closest to f

(Fact, $\{1, \cos x, \sin x \ldots \cos nx, \sin nx\}$ is a orthogonal set (basis))

So we can compute: $P_n = \text{Proj}_1(f) + \sum_{k=1}^{n} \text{proj}_{\cos kx}(f) + \sum_{k=1}^{n} \text{proj}_{\sin kx}(f)$

$\Rightarrow \text{proj}_{W_n}(x) = \sum_{l=1}^{n} \frac{2}{l}(-1)^{l+1} \sin(lx)$

approx.: $n=1$ $n=10$ $n=100$

App ②: Polynomial approximation.

We want to approx. a cts fcn by a polynomial.

let V = cts fcns on $[-1, 1]$, $W_n = \text{span}\{1, x, \ldots, x^n\}$

for inner product, let's use $\langle f, g \rangle = \int_{-1}^{1} fg$.

now given a f, we want to compute $\text{proj}_{W_n}(f)$.

Since $\{1, x, \dots x^n\}$ is not orthogonal, we apply Gram-Schmidt to get a orthogonal basis $\{q_1, \dots q_{n+1}\}$ then,

$$\Rightarrow \text{proj}_{W_n}(f) = \sum_{k=1}^{n+1} \text{Proj}_{q_k}(f) = \sum_{k=1}^{n+1} \frac{\langle f, q_k \rangle}{\langle q_k, q_k \rangle} q_k.$$

App③: Least Squares solution. (Important)

Goal: $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$. Solve $A\vec{x} = \vec{b}$ for $\vec{x} \in \mathbb{R}^n$.

we know if $\vec{b} \in \text{Col}(A)$, we can use row reduction to find it.

but what if $\vec{b} \notin \text{Col}(A)$? can we find $A\vec{x} \approx \vec{b}$ (approx.).

$\Rightarrow$ then we want $A\vec{x} - \vec{b} \approx \vec{0}$ ($\|A\vec{x} - \vec{b}\|$ as small as possible)

Defn: A least square solution (LSS) to $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{x}_0$ that minimize

Prop: let $A \in M_{m\times n}(\mathbb{R})$, then,

$\vec{x}_0 \in \mathbb{F}^n$ is an LSS to $A\vec{x} = \vec{b}$ $\Leftrightarrow$ it's a solution to $A\vec{x} = \text{proj}_{\text{Col}(A)} \vec{b}$.

lecture 29. unitary Diagonalization.

Problem: Let $L: V \to V$ be a linear operator on a inner product space V.

can we find an orthonormal basis $\mathcal{B}$ for V. s.t. $[L]_{\mathcal{B}}$ is a diag. matrix?

↳ not always possible to find.

⇒ Problem (matrix ver.): Let $A \in M_{n\times n}(\mathbb{F})$, can we find diag. matrix $D \in M_{n\times n}(\mathbb{F})$

and an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ whose column are orthonormal

s.t. $A = PDP^{-1}$?

Defn:

$U^* = U^{-1}$

(a) If $P \in M_{n\times n}(\mathbb{F})$ has orthonormal columns, we say P is unitary.

(b) If yes, we say A is unitarily diganalizable

Defn: The adjoint of $A \in M_{m\times n}(\mathbb{F})$ is the $n\times m$ matrix $A^* = \overline{A}^T$

Key property: Let $\vec{x}, \vec{y} \in \mathbb{F}^n$. then the standard inner product of $\vec{x}, \vec{y}$

is given by $\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x}$

Prop: let $P \in M_{n\times n}(\mathbb{F})$. P is unitary $\Leftrightarrow P^* = P^{-1}$

so now we can say:

☆ $A \in M_{n\times n}(\mathbb{F})$ is unitarily diag'zible $\Leftrightarrow \exists$ unitary $U \in M_{n\times n}(\mathbb{F})$ and

diagzable $D \in M_{n\times n}(\mathbb{F})$ s.t. $A = UDU^*$.

Prop: Basic Property of Adjoint.

① $(AB)^* = B^*A^*$

③ $(cA)^* = \bar{c}A^*$

② $(A+B)^* = A^* + B^*$

④ $(A^*)^* = A$

③ A invertible ⇒ A^* invertible and $(A^*)^{-1} = (A^{-1})^*$

Prop: ☆ Key property of Adjoint.

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^{*}\vec{y} \rangle$$

lecture 30.

Recap: let $A \in M_{n\times n}(\mathbb{F})$. Can we unitarily diagonalize A?

meaning: Can we find orthonormal basis for $\mathbb{F}^n$ consisting of eigenvalues of A?

goal: geometric definition & test for unitary diagonalizability.

Consider ex:

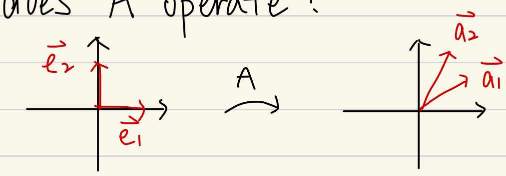
take $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ that has $\lambda = 3$ & $\lambda = 1$ w/ eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^{-1} \end{bmatrix}$, this is diagonalization.

we want to find unitary diagonalization. the vectors are orthogonal but not normal. fix $\Rightarrow \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

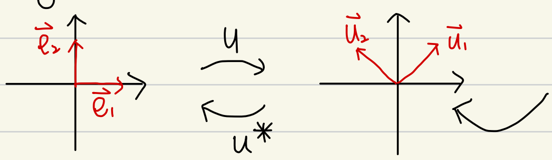
$$\therefore A = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{U^{-1}} \leftarrow \text{easy since } U^{-1} = U^* = U^T$$

how does A operate?



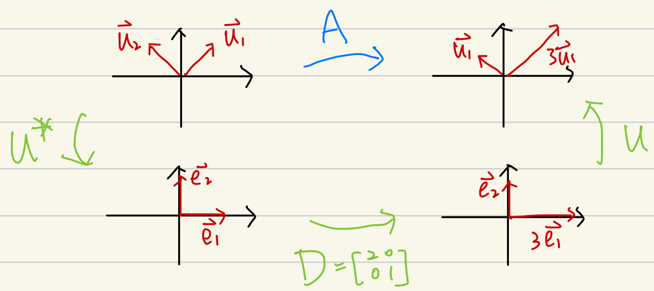
Instead of looking action of A directly, lets look at $A\vec{x} = UDU^*\vec{x}$

let's begin with $U^* = U^{-1}$ & U, they act:



Note: notice $\vec{u}_1 \perp \vec{u}_2$ and $\|\vec{u}_1\| = \|\vec{u}_2\| = 1$ b/c U is unitary. So the effect of U in this case is <u>rotation</u>. This is how unitary matrices act. they are reflection/rotation.

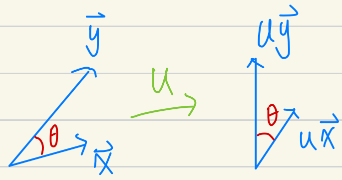
Now let's look at what it does on the basis $\{\vec{u}_1, \vec{u}_2\}$ instead of $\{\vec{e}_1, \vec{e}_2\}$



Note: A stretches in the direction of these axes by a factor of corresponding eigenvalues.

Prop: (Effect of a unitary matrix)

Let $U \in M_{n\times n}(\mathbb{F})$ be unitary, then



(a) $\langle U\vec{x}, U\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$ ← preserve angles.

(b) $\|U\vec{x}\| = \|\vec{x}\|$ ← preserve lengths.

Now, "test" of unitary matrix.

Theorem: (Spectral theorem – over $\mathbb{C}$)

Let $A \in M_{n\times n}(\mathbb{C})$. Then:

A is $\underline{\text{unitarily diagonalizable}}$ over $\mathbb{C}$ $\iff AA^* = A^*A$

← $\exists$ U and D s.t. $A = UDU^*$

to prove $\Leftarrow$ we need a new theorem.

Theorem: (Schur's Trangularization theorem)

Let $A \in M_{n\times n}(\mathbb{C})$, then $\exists$ unitary $U \in M_{n\times n}(\mathbb{C})$ and upper-Δ $T \in M_{n\times n}(\mathbb{C})$ s.t.

$$A = UTU^* = U \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*.$$

furthermore, the diagonal entries of T are complex eigenvalues of A, repeated according to alg. multiplicity.

Also, if $A \in M_{n\times n}(\mathbb{R})$, and if all eigenvalues are real, then we can choose $U \in M_{n\times n}(\mathbb{R})$ and $T \in M_{n\times n}(\mathbb{R})$.

Theorem: let $A \in M_{n\times n}(\mathbb{C})$ be normal. If $\lambda \neq \mu$ are distinct eigenvalues of A then $E_\lambda \perp E_\mu$, that is, if $\vec{v} \in E_\lambda$ and $w \in E_\mu$ then $\vec{v} \perp \vec{w}$

↳ basically, eigenspace of distinct eigenvalues are orthogonal.

Algorithm for Unitary Diagonalization:

Step 0: is A normal ($A^*A = AA^*$)? If not, STOP.

Step 1: Diagonalize as usual.

- find eigenvalues $\lambda_1 \dots \lambda_n$
- find bases $B_1, \dots B_n$. for eigenspace $E_{\lambda_1} \dots E_{\lambda_n}$

* Step 2: Apply Gram-Schmidt to each of $B_1 \dots B_n$ to obtain bases $\tilde{B}_1 \dots \tilde{B}_n$

Step 3: let $U = [$ vectors in $\tilde{B}_1$ ... vectors in $\tilde{B}_n$ $]$.

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_k \end{bmatrix}$$

then $A = UDU^*$

unitary → $AA^* = I$

orthogonal → $AA^T = I$

self-adjoint → $A = A^*$

normal → $AA^* = A^*A$

Done