Chapter I

Defn: Partition, P_n for the interval $[a,b]$ is a finite seq of increasing numbers of the form

$$a = t_0 < t_1 < t_2 \dots < t_{n-1} < t_n = b$$

· This partition subdivides the interval $[a.b]$ into n subintervals:

$[t_0, t_1], [t_1, t_2] \dots [t_{n-2}, t_{n-1}]\ [t_{n-1}, t_n]$.

Defn: Riemann Sum, Given a bounded function f on $[a,b]$, a partition P_n of $[a,b]$ and a set $\{c_1, c_2, \dots c_n\}$, where $c_i \in [t_{i-1}, t_i]$ then a Riemann Sum for f respect to P is,

$$S_n = \sum_{i=1}^{n} f(c_i)\,\Delta t_i$$

Defn: Integrable. We say f is integrable on $[a,b]$ if there exist a unique number $I \in \mathbb{R}$ such that for any sequence of partitions, $\{P_n\}$ with $\lim_{n\to\infty} \|P_n\| = 0$ and any seq of Reimann Sums $\{S_n\}$ associate with P_n's, we have $\lim_{n\to\infty} S_n = I$

In this case we call I the Integral of f over $[a,b]$ and denote it by

$$\underbrace{\int_a^b}_{\text{limits.}} \underbrace{f(x)}_{\text{integrand}}\, \overbrace{dx}^{\text{variable of integration}} = \lim_{n\to\infty} S_n = I$$

p.s. $\|P\|$ is the widest width in a partition.

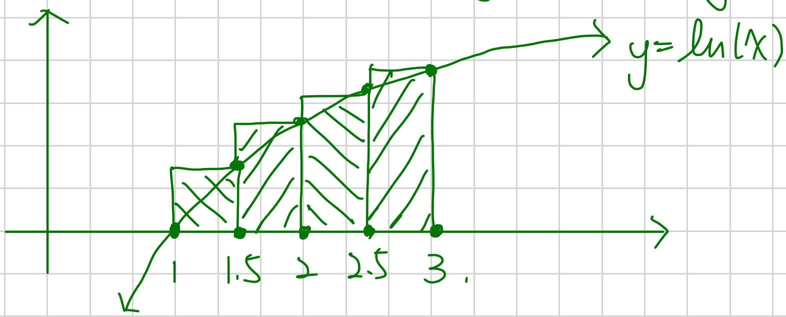
Thm: Integrability Theorm for Continuous Functions.

Let f be continuous on $[a.b]$, then f is integrable on $[a.b]$.

Defn: regular n-partition For interval $[a,b]$, the regular n-partition is the partition where all subintervals have same length.

$$\Delta t = \frac{b-a}{n} \quad \text{and} \quad t_i = t_0 + i\Delta t$$

ex, Approximate A under $y = \ln(x)$ over interval $[1, 3]$, using Reimann sum with 4 rectangles and right endpoints.



$$\Delta x = \frac{3-1}{4} = \frac{1}{2}$$

$$\text{area} \approx \sum_{i=1}^{4} f(x_i)\Delta x = [\ln(1.5) + \ln(2) + \ln(2.5) + \ln(3)](\tfrac{1}{2})$$

≈ 1.557 sq. units. $\Rightarrow$ Overestimat.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

ex: Find A for $y = x+1$ between $x=0$ & $x=2$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$f(x_i) = x_i + 1 = \frac{2i}{n} + 1 \quad , \Delta x = \frac{2}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n} + 1\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n^2} + \frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \left[\frac{4}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{2}{n}\left(\sum_{i=1}^{n} 1\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n}n\right]$$

$$= 2 + \lim_{n\to\infty} \left(\frac{4n^2+4}{2n^2}\right) = 2 + 2\lim_{n\to\infty}\left(\frac{n^2+1}{n^2}\right) = 4 \#$$

Defn : Right hand Reimann Sum taking $c_i = t_i \ \forall i$

$$R_n = \sum_{i=1}^{n} f(t_i)\,\Delta t = \sum_{i=1}^{n} \underbrace{f(t_i)}\left(\frac{b-a}{n}\right)$$

$$= \sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

Defn : Left hand Reimann Sum taking $c_i = t_{i-1} \ \forall i$

$$R_n = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \sum_{i=1}^{n} \underbrace{f(t_{i-1})}\left(\frac{a-b}{n}\right)$$

$$= \sum_{i=1}^{n} f\left(a + (i-1)\left(\frac{a-b}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

Thm

Properties of definite integral 只寫不會的 pg.18

- (iii) If $m \leq f(x) \leq M \ \forall t \in [a,b]$

  then $$m(b-a) \leqslant \int_a^b f(t)\,dt \leqslant M(b-a)$$

- (vi) function $|f|$ is integrable on $[a,b]$ and

$$\left| \int_a^b f(t)\,dt \right| \leqslant \int_a^b |f(t)|\,dt$$

Thm

1) If $f(a)$ is defined, then $\int_a^a f(x)\,dx = 0$
2) If is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3) If is integrable on interval I containing a, b, c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Defn: Average value of a function.

If f is cts on $[a,b]$, the average value of f on $[a,b]$ is $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Thm: Average Value theorem (MVT for integrals)

Assume f is cts on $[a,b]$,

$\exists\ a \le c \le b \ni \quad f(c) = \frac{1}{a-b}\int_a^b f(x)\,dx$

ex: If $\int_0^1 e^{-x^2}dx = k$, find $\int_{-1}^1 (xe^{-x^2} + e^{1-x^2} + x\cos x + 1)\,dx$ in terms of k.

Sol: $\int_{-1}^1 xe^{-x^2}dx + \int_{-1}^1 e^{1-x^2}dx + \int_{-1}^1 x\cos x\,dx + \int_{-1}^1 1\,dx$

by properties of definite integral.

- $\int_{-1}^1 xe^{-x^2}dx = 0$ (since it is a odd fcn, symetrical)

  ★ how to know if odd?

  $g(x) = xe^{-x^2}$

  $\underline{g(-x)} = -xe^{-x^2} = \underline{-g(x)}$

- $\int_{-1}^1 x\cos x\,dx = 0$ (a odd fcn)

  $g(x) = x\cos x$

  $g(-x) = -x\cos(-x)$
  $= -x\cos(x) = g(x)$

- $\int_{-1}^1 1\,dx = 2$ (rectangle)

- $\int_{-1}^1 e^{1-x^2}dx = \int_{-1}^1 e\cdot e^{-x^2}dx = e\int_{-1}^1 e^{-x^2}dx$ by prop. of def int.

  $= e(2\int_0^1 e^{-x^2}dx = 2ek$

  even fcn.

thus $= 0 + 2ek + 0 + 2 = 2 + 2ek = 2(1+ek$

ex: Express $\lim_{n\to\infty} \frac{4}{\sqrt{n^3}}\left(\sqrt{n+4} + \sqrt{n+8} + \sqrt{n+12} \dots + \sqrt{n+4n}\right)$ as definite integral.

Sol: $\frac{4}{\sqrt{n^3}}\left(\sqrt{n+4} + \sqrt{n+8} + \sqrt{n+12} \dots \sqrt{n+4n}\right)$

$= \frac{4}{n\sqrt{n}}\left(\sqrt{n+4} + \sqrt{n+8} + \sqrt{n+12} \dots \sqrt{n+4n}\right)$

$= \frac{4}{n}\left(\sqrt{\frac{n+4}{n}} + \sqrt{\frac{n+8}{n}} + \sqrt{\frac{n+12}{n}} \dots \sqrt{\frac{n+4n}{n}}\right)$

$= \frac{4}{n}\underbrace{\left(\sqrt{1+\frac{4}{n}} + \sqrt{1+\frac{8}{n}} + \sqrt{1+\frac{12}{n}} \dots \sqrt{1+n\left(\frac{4}{n}\right)}\right)}_{\sum_{i=1}^{n}\sqrt{1+i\left(\frac{4}{n}\right)}}$

概念, Reimann Sum

$\int_a^b f(x)\,dx$

$= \lim_{n\to\infty} \sum_{i=1}^{n} (f(c_i)\,\Delta x_i$

$\frac{b-a}{n}$

$= \sum_{i=1}^{n}\sqrt{1+i\left(\frac{4}{n}\right)}\left(\frac{4}{n}\right)$

$\therefore \Delta x = \frac{4}{n}$, $x_i = 1 + \frac{4i}{n}$ or $x_i = \frac{4i}{n}$,

if $x_i = \frac{4i}{n}$ $i = 0, 1 \dots n$.

$\left.\begin{array}{l} \\ a = x_0 = 0,\ b = x_i = 4\end{array}\right\}$ $f(x) = \sqrt{1+x_i} \Rightarrow \int_0^4 \sqrt{1+x}\,dx$

ex: Use Right reimann sum to calculate $\int_0^1 \left(1 - \frac{4x}{3} - x^2\right) dx$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{1}{n}\right)$, $f(x_i) = \left(1 - \frac{4x_i}{3} - x_i^2\right)$, $x_i = \frac{i}{n}$

$f(x_i) = \left(1 - \frac{4}{3n}i - \frac{1}{n^2}i^2\right)$

$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left(\frac{1}{n} - \frac{4}{3n^2}i - \frac{1}{n^3}i^2\right)\right] = \lim_{n\to\infty}\left[\left(\frac{1}{n}\sum_{i=1}^{n}1\right) - \left(\frac{4}{3n^2}\sum_{i=1}^{n}i\right) - \left(\frac{1}{n^3}\sum_{i=1}^{n}i^2\right)\right]$

$= \lim_{n\to\infty}\left(1 - \frac{2n^2+2}{3n^2} - \frac{2n^3+3n^2+n}{6n^3}\right) = 1 - \frac{2}{3} - \frac{1}{3} = 0$ #

1.5. FTC part I (FTC 1), If f is cts on $[a,b]$, then the fcn g define by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$ is cts on $[a,b]$ and diff'able on (a,b) and $g'(x) = f(x)$ $\left(\text{or } \frac{d}{dx} \int_a^x f(t)\,dt = f(x) \right)$

pf: by def, $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h} \left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$ by prop of def. int.

$$= \lim_{h \to 0} f(c) \text{ for } c \in [x, x+h]$$

$$= f(x)$$ since f is cts and $c \to x$ and $h \to 0$

ex: for $g(x) = \int_0^x \frac{1}{\sqrt{1+t^4}}\,dt$ for $x > 0$ find $g'(2)$

let $f(t) = \frac{1}{\sqrt{1+t^4}}$, then $f(t)$ is cts $\Rightarrow$ by FTC 1.

$g'(x) = \frac{1}{\sqrt{1+x^2}}$ for $x > 0$ by FTC 1

$\Rightarrow g'(2) = \frac{1}{\sqrt{17}}$

ex: let $p(x) = \int_1^{x^2} \frac{1}{t} e^{-t}\,dt$, find $p'(x) = \frac{dp}{dx}$

define $f(t) = \frac{1}{t} e^{-t}$, the $f(t)$ is cts $\forall t \neq 0$

let $u = x^2$ then $p(x) = \int_1^u \frac{1}{t} e^{-t}\,dt$, by FTC 1,

$\frac{dp}{du} = \frac{1}{u} e^{-u} = p'(x)\ \frac{dp}{du} \cdot \frac{du}{dx}$ by chain Rule.

$= \frac{1}{u} e^{-u} (2x) = \frac{1}{x^2} e^{-x^2} (2x) = \frac{2}{x} e^{-x^2}$

ex: $H(x)=\int_{x^2}^{e^x}\cos(t^2)\,dt$ find $H'(x)$.

Define $f(t)=\cos(t^2)$ then $f(t)$ cts $\forall t$

Let $u=x^2$, $v=e^x$, then.

$$H(x)=\int_u^v f(t)\,dt=\int_u^a f(t)\,dt+\int_a^v f(t)\,dt.$$

$$=-\int_a^u f(t)\,dt+\int_a^v f(t)\,dt \quad \text{, by FCT 1}$$

$$=-\cos(u^2)\frac{du}{dx}+\cos(v^2)\frac{dv}{dx}$$

$$H'(x)=e^x\cos(e^{2x})-2x\cos(x^4)$$

Thm: FTC 1 - Extend Version.

Assume that f is cts and $g+h$ are diff.

Let $H(x)=\int_{g(x)}^{h(x)} f(t)\,dt$.

Then $H(x)$ is diff. and $H'(x)=f(h(x))h'(x)-f(g(x))g'(x)$

FTC pt. 2: If f is cts on $[a,b]$, then

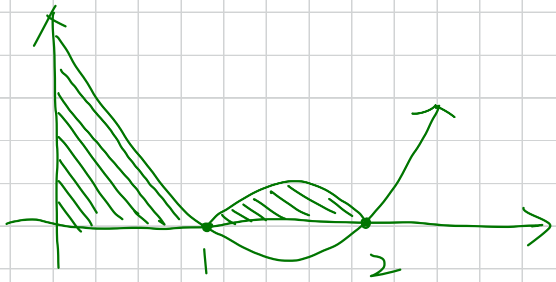
$$\int_a^b f(x)\,dx=F(b)-F(a) \quad \text{where } F'(x)=f(x).$$

ex: $\int\cos^2 x\,dx$

for even exp: $\cos^2 x=\frac{1}{2}(1+\cos 2x)$

$$=\frac{1}{2}\int 1+\cos 2x\,dx=\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C$$

ex: $\int\frac{x}{x+1}\,dx$ (long division) ex: evaluate $\int_0^2 |x^2-3x+2|\,dx$



$x^2-3x+2=(x-2)(x-1) \qquad x=1,2)$

$$\int_0^2 |x^2-3x+2|\,dx = \int_0^1 (x^2-3x+2)\,dx + \int_0^2 (-x^2+3x-2)\,dx$$

$$=\left(\frac{1}{3}x^3-\frac{3}{2}x^2+2x\right)\Big|_0^1 + \left(-\frac{1}{3}x^3+\frac{3}{2}x^2-2x\right)\Big|_1^2$$

$$=1$$

ex: $\int \tan^2 x\,dx$ $\rightarrow$ $\sin^2 x + \cos^2 x = 1$ $\div \cos^2 x$

$= \int (\sec^2 x - 1)\,dx$ $\Rightarrow$ $\tan^2 x + 1 = \sec^2 x$

$= \tan x - x + C.$

ex: $\int_0^2 2x^3 - 6x + \frac{3}{x^2+1}\,dx$

$= \frac{1}{2}x^4 - 3x^2 + 3\arctan x \Big|_0^2$

$= (8 - 12 + 3\arctan(2)) - 0$

$= 3\arctan(2) - 4$

ex: $\int \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$ t^{-2}

$= \int 2 + \sqrt{t} - \frac{1}{t^2}\,dt = 2t + \frac{2}{3}t^{\frac{3}{2}} - t^{-1} + C$

ex: Find avg v.e. of $f(x) = e^{-x} + \cos x$ on $[-\frac{\pi}{2}, 0]$.

$= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^0 e^{-x} + \cos x\,dx$

$= \frac{2}{\pi}(-e^{-x} + \sin x)\Big|_{-\frac{\pi}{2}}^0 = \frac{2}{\pi}\left[(-1+0) - (-e^{\frac{\pi}{2}} - 1)\right]$

$= \frac{2}{\pi}e^{\frac{\pi}{2}}$

$\cos 2x$
$= \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

ex: $\int_{-\pi}^{\pi} x^3 \cos x\,dx = 0$ since it's a odd fcn.

ex: $\int \sin^4 x\,dx$, $\cos 2x = 1 - \sin^2 x$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \int (\sin^2 x)^2\,dx = \int \frac{1}{4}(1 - \cos 2x)^2\,dx = \int \frac{1}{4}(1 - 2\cos 2x + \cos^2(2x))$

$= \int \frac{1}{4}[1 - 2\cos(2x) + \frac{1}{2}\cos(4x) + \frac{1}{2}]\,dx$

$= \frac{1}{4}\left(\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right) + C$

Defn: Method of Substitution, change of variable:

Recall for diff: $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$.

$\Rightarrow$ In Integral it becomes: $\int f'(g(x))\, g'(x) = f(g(x)) + C$

Formal exp: let $u = g(x)$, then $\frac{du}{dx} = g'(x)$ or $du = g' dx$.

$$\Rightarrow \int f'(g(x))\, g'(x)\, dx = \int f'(u)\, du = f(u) + C = f(g(x)) + C$$

ex:

1) $\int \frac{x}{x^2+1}\, dx$,

let $u = x^2+1$
$du = 2x\, dx$
$\frac{1}{2} du = x\, dx$

$\int \frac{x}{x^2+1} = \frac{1}{2}\int \frac{1}{u}\, du$
$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+1| + C.$

2) $\int \frac{\sin(3\ln x)}{x}$

let $u = 3\ln x$
$du = \frac{3}{x}\, dx$
$\frac{1}{3} du = \frac{1}{x}\, dx$

$\int \frac{\sin(3\ln x)}{x} = \frac{1}{3}\int \sin(u)\, du.$
$= -\frac{1}{3}\cos(3\ln x) + C.$

3) $\int x^2 e^{x^3}\, dx$

let $u = x^3$
$\frac{1}{3} du = x^2\, dx$

$\int x^2 e^{x^3}\, dx = \frac{1}{3}\int e^u\, du$
$= \frac{1}{3} e^{x^3} + C.$

4) $\int e^x \sqrt{1+e^x}\, dx$

let $u = 1+e^x$
$du = e^x\, dx$

$\int e^x\sqrt{1+e^x} = \int \sqrt{u}\, du$
$= \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C.$

5) $\int \frac{1}{x^2+4x+5}\,dx$ (complete $\square$)

$= \int \frac{1}{(x+2)^2+1}$

let $u = x+2$

$du = dx$

$\int \frac{1}{(x+2)^2+1} = \int \frac{1}{u^2+1}\,du$

$= \tan^{-1}(u)+C = \tan^{-1}(x+2)+C.$

6) $\int \frac{1}{1+e^x}\,dx$ 用不了usub, 自己造,

$\Rightarrow \int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{1+e^{-x}}\,dx$

let $u = 1+e^{-x}$

$-du = e^{-x}$

$\int \frac{e^{-x}}{1+e^{-x}}\,dx = -\int \frac{1}{u}\,du$

$= -\ln|1+e^{-x}|+C$

7) $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

$\Rightarrow \int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{e^x\sqrt{1-e^{-2x}}}\,dx = \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

let $u = e^{-x}$

$-du = e^{-x}\,dx$

$\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = -\int \frac{1}{\sqrt{1-u^2}}\,du$

$= -\sin^{-1}(u)+C = -\sin^{-1}(e^{-x})+C.$

8) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

let $u = \cos x$

$-du = \sin x\,dx$

$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$

$= -\ln|\cos x|+C = \ln|\sec x|+C$

9) $\int \sin^3 x\,dx = \int \sin x(\sin^2 x)\,dx = \int \sin x(1-\cos^2 x)\,dx$

let $u = \cos x$

$-du = \sin x\,dx$

$\int \sin x(1-\cos^2 x)\,dx = -\int (1-u^2)\,du$

$= -(u-\frac{1}{3}u^3) = -\cos x + \frac{1}{3}\cos^3 x$

(this approach works for odd powers of sin / cos)

10) $\int \sec^2\theta \tan^2\theta \, d\theta$

let $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$\int \sec^2\theta \tan^2\theta \, d\theta = \int u^2 \, du$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3\theta + C.$

11) $\int \left(\frac{x+2}{x^2+1}\right) dx = \int \frac{x}{x^2+1} dx + 2\int \frac{1}{x^2+1} dx$

let $u = x^2+1$

$\frac{1}{2} du = dx$

$= \frac{1}{2}\int \frac{1}{u} du + 2\tan^{-1}x$

$= \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}x + C.$

Thm Sub of definite integral (change of variables)

Consider definite integral:

$\int_a^b f'(g(x))\, g'(x) \, dx$, then the sub $u = g(x)$, $du = g'(x)\,dx$

transform the integral into.

$$\int_a^b f'(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

ex:

1) $\int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}} dx$

let $u = \sqrt{x+1}$

$2du = \frac{1}{\sqrt{x+1}} dx$

$\int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}} = 2\int_1^3 \cos(u)\, du$

$= 2\sin(u)\Big|_1^3 = 2(\sin 3 - \sin 1).$

2) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx$

let $u = 1+\sin x$

$du = \cos x \, dx$

$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx = \int_1^2 \frac{1}{u} du$

$= \ln|u|\Big|_1^2 = \ln|2| - \ln|1| = \ln 2$

3) $\int_{-\pi}^{\pi} \frac{t^4 \sin t}{1+t^8} dt$ $\quad f(-t) = -f(t) \Rightarrow$ odd fcn.

odd, so $\Rightarrow = 0$ #

Chapter II

Thm: Inverse Trig Sub.

① $\sqrt{a^2-b^2x^2} \Rightarrow bx = a\sin\theta \Rightarrow 1-\sin^2\theta = \cos^2\theta$

② $\sqrt{a^2+b^2x^2} \Rightarrow bx = a\tan\theta \Rightarrow 1+\tan^2\theta = \sec^2\theta$

③ $\sqrt{b^2x^2-a^2} \Rightarrow bx = a\sec\theta \Rightarrow \sec^2\theta - 1 = \tan^2\theta$

ex: Alt solu to $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3\,dx}{(4x^2+9)^{\frac{3}{2}}}$

let $u = 4x^2+9$ $\qquad$ $x^3\,dx = x^2(x\,dx)$

$du = 8x\,dx$ $\qquad$ $= \left(\frac{u-9}{4}\right)\left(\frac{du}{8}\right)$

$= \frac{1}{32}\int_9^{32} \frac{(u-9)}{u^{3/2}}\,du = \frac{1}{32}\int_9^{36}\left(\frac{1}{\sqrt{u}} - \frac{9}{u^{3/2}}\right)du$

$= \frac{1}{32}\left[2\sqrt{u} + \frac{18}{\sqrt{u}}\right]_9^{36} = \frac{3}{32}$

ex: $\int \frac{1}{\sqrt{4+2x-x^2}}\,dx$, $4+2x-x^2 = 4-(x^2-2x) = 4-[(x-1)^2-1] = 5-(x-1)^2$

$= \int \frac{1}{5-(x-1)^2}\,dx$ $\qquad$ let $x-1 = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\,d\theta$

$= \int \frac{\sqrt{5}\cos\theta}{\sqrt{5\cos^2\theta}}\,d\theta = \int \frac{\cos\theta}{|\cos\theta|}\,d\theta$, on $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2} \Rightarrow |\cos\theta| = \cos\theta$

$= \int d\theta = \theta + C = \sin^{-1}\left(\frac{x-1}{5}\right) + C$

ex: $\int_0^{\sqrt{3}} \frac{x}{(1+x^2)^2}\,dx$

let $u = 1+x^2$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int_1^4 \frac{1}{u^2}\,du = \frac{1}{2}\left[\frac{-1}{u}\right]_1^4$

$= \frac{1}{2}\left(\frac{-1}{4} - 1\right) = \frac{3}{8}$

OR

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{\tan\theta\sec^2\theta}{\sec^4\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sec^2\theta}\,d\theta$

$= \int_0^{\frac{\pi}{3}} \sin\theta\cos\theta\,d\theta$ $\qquad$ P.S. $\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin 2\theta\,d\theta = -\frac{1}{4}\left[\cos(2\theta)\right]_0^{\frac{\pi}{3}} = \frac{3}{8}$

Defn: Integration by Parts

$$\int u\,dv = uv - \int v\,du.$$

ex: $\int x e^x \, dx$ let $u = x$ $dv = e^x$
$du = dx$ $v = e^x$

$= xe^x - \int e^x dx = xe^x - e^x + C$

$+$	x^2	$\sin x$
$-$	$2x$	$-\cos x$
$+$	2	$-\sin x$
$-$	0	$\cos x$

ex: $\int x^2 \sin x \, dx$ let $u = x^2$ $dv = \sin x$
$du = 2x$ $v = -\cos x$

$= -x^2 \cos x + 2\int x \cos x \, dx$ let $u = x$ $dv = \cos x$
$du = dx$ $v = \sin x$

$= -x^2 \cos x + 2x \sin x + 2\cos x + C$

ex: $\int \ln x \, dx$ let $u = \ln x$ $dv = 1$
$du = 1/x \, dx$ $v = x$

$= x \ln x - \int dx = x \ln x - x + C$

Alt solu to $\int e^{ax}\cos bx\, dx$:

Recall Euler's formula : $e^{ix} = \cos x + i\sin x$

then $e^{ibx} = \cos(bx) + i\sin(bx)$

Consider $\int e^{ax}e^{ibx}\,dx = \int e^{(a+ib)x}\,dx$

$$= \frac{1}{a+ib}e^{(a+ib)x} + C = \frac{a-ib}{a^2+b^2}e^{ax}e^{ibx} + C$$

$$= \frac{e^{ax}(a-ib)(\cos(bx)+i\sin(bx))}{(a^2+b^2)} + C$$

$$= \frac{e^{ax}}{(a^2+b^2)}\left[(a\cos(bx) + b\sin(bx)) + i(-b\cos(bx) + a\sin(bx))\right]$$

But, $\int e^{ax}e^{ibx}\,dx = \int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx$.

Comparing we get: $\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx)) + C_R$

and $\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(-b\cos(bx) + a\sin(bx)) + C_I$

ex: $\int x^3\cos(x^2)\,dx$ let $y = x^2$ $dy = 2x\,dx$

$x^3\,dx = x^2(x\,dx) = y(\frac{1}{2}dy)$

$= \frac{1}{2}\int y\cos(y)\,dy$, now apply IBP

$u = y$, $dv = \cos(y)$

$du = dy$, $v = \sin(y)$

$$= \frac{1}{2}\left[y\sin(y) - \int \sin(y)dy\right] = \frac{1}{2}\left[y\sin(y) + \cos(y)\right]$$

$$= \frac{1}{2}y\sin(y) + \frac{1}{2}\cos(y) + c = \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C.$$

ex: $\int_1^e (\ln x)^2\, dx$ $\quad u=(\ln x)^2 \quad dv = dx$
$du = \frac{2\ln x}{x} \quad v = x$

$= x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x\, dx$ $\quad u = \ln x \quad dv = dx$
$du = \frac{1}{x} \quad v = x$

$= x(\ln x)^2\Big|_1^e - 2\left[x\ln x - \int_1^e dx\right]$

$= x(\ln x)^2 - 2x\ln x - 2x\Big|_1^e$

$= (e-0) - 2(x\ln x - x) = e-2$

ex: $\int e^{\sin x} \cos x \sin^2 x\, dx$

$= e^{\sin x}\left[\sin^2 x - 2\sin x + 2\right] + C$ $\quad$ sub → IBP

Defn: Partial Fraction

Let P and Q be polynomials with real coefficients and suppose degree $P <$ degree Q. Then,

a) $Q(x)$ can be factor into:

$$Q(x) = k(x-a_1)^{m_1} \cdots (x-a_j)^{m_j} (x^2+b_1x+c_1)^{n_1} \cdots (x^2+b_kx+c_k)^{n_k}$$

b) The rational function $P(x)/Q(x)$

(i) corresponding to each factor $(x-a)^m$ of $Q(x)$ the decomposition is $\dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_m}{(x-a)^m}$

(ii) corresponding to each factor $(x^2+bx+c)^n$ of $Q(x)$ the decomposition: $\dfrac{B_1x+C}{(x^2+bx+C)} + \dfrac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \dfrac{B_mx+C_m}{(x^2+bx+c)^m}$

ex: $\int \left(\frac{x^3 - 5x^2 - 4}{x^4 - 4x^2} \right) dx$: $\frac{x^3-5x^2-4}{x^2(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)} + \frac{D}{(x+2)}$

$$= \frac{Ax(x-2)(x+2) + B(x+2)(x-2) + Cx^2(x+2) + Dx^2(x-2)}{x^2(x+2)(x-2)}$$

To find A, B, C, D, sub $x = 0, -2, 2$

$x=0: -4B = -4 \Rightarrow B = 1$

$x=2: 16C = -16 \Rightarrow C = -1$

$x=-2: -16D = -32 \Rightarrow D = 2$

To get A, compare coeff. of x^3

$A + C + D = 1 \Rightarrow A = 0$

$$\Rightarrow \int \left[\frac{1}{x^2} + \frac{-1}{(x-2)} + \frac{2}{(x+2)} \right] dx = -\frac{1}{x} - \ln|x-2| + \ln|x+2| + C.$$

ex: $\int \left(\frac{2+3x+x^2}{x^3+x} \right) dx$: $\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$

$$= \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)}$$

Coeff: x^2: $A + B = 1$, $B = -1$

x^1: $C = 3$

x^0 $A = 2$

$$\Rightarrow \int \left[\frac{2}{x} + \frac{-x+3}{(x^2+1)} \right] dx = 2\ln|x| + \int \frac{-x}{(x^2+1)} + \frac{3}{(x^2+1)} dx$$

$$= 2\ln|x| - \frac{1}{2}\ln|x^2+1| + 3\tan^{-1}(x) + C$$

ex: let $I_n = \int_0^1 x^n e^{-x} dx$, $n = 1, 2, 3 \ldots$

a) show $I_n = n I_{n-1} - \frac{1}{e}$ (IBP)

let $u = x^n$ $dv = e^{-x}$

$du = n x^{n-1}$ $v = -e^{-x}$

$= -x^n e^{-x} \Big|_0^1 + n \int_0^1 x^{n-1} e^{-x} dx = -\frac{1}{e} - n I_{n-1}$.

Defn: Type I Improper Integrals.

Let f be integrable on $[a,b]$ for each $a \le b$ we say that:

$$\int_a^\infty \int_{-\infty}^a f(x)\,dx = \lim_{x\to\pm\infty} \int_a^b \int_b^a f(x)\,dx$$

or

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx.$$

ex: $\int_0^\infty x e^{-x}\,dx$

$\lim_{b\to\infty} \int_0^b x e^{-x}\,dx$ $\quad u = x \quad du = dx \quad dv = e^{-x} \quad V = -e^{-x}$

$\overset{\lim}{=} -xe^{-x} + \int_0 e^{-x}\,dx \overset{\lim}{=} -xe^{-x} - e^{-x}\Big|_0^b \overset{\lim}{=} -be^{-b} - e^{-b} - 0 + 1$

$= -\frac{b}{e^{-b}} - \frac{1}{e^{b}} + 1 = 1$

Thm: Comparison test:

Assume $0 \le g(x) \le f(x)$ $\forall x \ge a$ and that f & g cts on $[a,\infty)$

① If $\int_a^\infty f(x)\,dx$ converge, $\int_a^\infty g(x)\,dx$ also converge.

② If $\int_a^\infty g(x)\,dx$ diverge, $\int_a^\infty d(x)\,dx$ also diverge.

C/D? $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ for large x $\approx \frac{\sqrt{x}}{x^2} = \frac{1}{x^{\frac{3}{2}}}$

for $x \ge 1$, $x+1 \le 2x \Rightarrow \sqrt{x+1} \le \sqrt{2x} \Rightarrow \frac{\sqrt{x+1}}{x^2} \le \frac{\sqrt{2x}}{x^2} = \sqrt{2}\cdot\frac{1}{x^{\frac{3}{2}}}$

Since $\sqrt{2}\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$ converge then $\int_1^\infty \frac{\sqrt{x+1}}{x^2} \le \sqrt{2}\int_1^\infty \frac{1}{x^{\frac{3}{2}}}$ also cvg.

ex: $\int_0^\infty \frac{x}{1+x^4}\,dx$

Sol^n: $\int_0^\infty \frac{x}{1+x^4}\,dx = \lim_{b\to\infty} \int_0^b \frac{x}{1+x^4}\,dx$ let $u = x^2$, $du = 2x\,dx$

$= \frac{1}{2}\lim_{b\to\infty} \int_0^{b^2} \frac{1}{1+u^2}\,du = \frac{1}{2}\lim_{b\to\infty} \tan^{-1} u \Big|_0^{b^2} = \frac{1}{2}\lim_{b\to\infty} \tan^{-1}(b)^2 \leftarrow \frac{\pi}{2}$

$= \frac{\pi}{4}$

ex: $\int_1^\infty e^{-x^2}\,dx$ converge or diverge?

Sol^n: For $x \geq 1$, $x^2 \geq x$

$\Rightarrow e^{x^2} \geq e^x$

$\Rightarrow \frac{1}{e^{x^2}} \leq \frac{1}{e^x} \Rightarrow e^{-x^2} \leq e^{-x}$

now $\int_1^\infty e^{-x^2} = \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]\Big|_1^b = \lim_{b\to\infty}\left[-e^{-b} - (-e^{-1})\right]$

Since $\int_1^\infty e^{-x}\,dx$ converges, and

$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx \Rightarrow \int_1^\infty e^{-x^2}\,dx$ also converges by Comparison test.

ex: $\int_0^\infty \frac{\tan^{-1}(x)}{2+e^x}$ converge or diverge?

Sol^n: For $x \geq 0$, $0 \leq \frac{\tan^{-1}(x)}{2+e^x} < \frac{\frac{\pi}{2}}{2+e^x} < \frac{\frac{\pi}{2}}{e^x}$

Since $\int_0^\infty e^{-x}\,dx$ converges. and $\int_0^\infty \frac{\tan^{-1}(x)}{2+e^x}\,dx < \frac{\pi}{2}\int_0^\infty e^{-x}\,dx$

It follows that $\int_0^\infty \frac{\tan^{-1}(x)}{2+e^x}\,dx$ also converges by comparison test.

ex: $\int_1^\infty \frac{x}{x^3+1}\,dx$ converges..

C/D ? $\int_1^\infty \frac{1}{\sqrt{4x^2-1}}\,dx$

for large $x \Rightarrow \frac{1}{\sqrt{4x^2}} = \frac{1}{2x}$

for $x \geq 1$ $\quad 4x^2-1 \leq 4x^2 \Rightarrow \sqrt{4x^2-1} \leq \sqrt{4x^2} \Rightarrow \frac{1}{\sqrt{4x^2-1}} \geq \frac{1}{2x}$

Since $\int_1^\infty \frac{1}{2x}\,dx$ diverges and $\int_1^\infty \frac{1}{\sqrt{4x^2-1}} \geq \int_1^\infty \frac{1}{2x}\,dx$

thus fcn. diverges.

Defn : Improper type II

① let f integrable on $[t, b]$ for every $t \in [a, b]$,
with either $\lim_{x \to a^+} f(x) = \infty$ or $-\infty$, we say $\int_a^b f(x)\,dx$ cvg if
$\lim_{t \to a^+} \int_t^b f(x)\,dx$ exist.

② let f integrable on $[a, t]$ for every $t \in [a, b)$
with either $\lim_{x \to b^-} f(x) = \infty$ or $-\infty$, we say $\int_a^b f(x)\,dx$ cvg if
$\lim_{t \to b^-} \int_a^t f(x)\,dx$ exist.

C/D? $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ $\qquad \frac{1}{\sqrt{x}}$ converge

for $0 < x \le 1$ $\qquad \frac{1}{e} \le e^{-x} < 1 \Rightarrow \frac{1}{e\sqrt{x}} \le \frac{e^{-x}}{\sqrt{x}} < \frac{1}{\sqrt{x}}$

Since $\int_0^1 \frac{1}{\sqrt{x}}$ cvg $\Rightarrow$ cvg

ex $\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$= \int_1^e u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}}\Big|_1^e = 2\ln x\Big|_1^e = 2-0 = 2$

ex: $\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \lim_{t\to 0^+}\lim_{b\to\infty}\int_t^b \frac{1}{\sqrt{x}+x\sqrt{x}}$ $\qquad u = \sqrt{x}$ $\qquad du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$2u = dx$

$= \lim\lim\int_t^b \frac{2\cancel{u}}{\cancel{u}+u^3}\,du = \int_t^b \frac{2}{1+u^2}\,du = 2\tan^{-1}(u)\Big|_t^b$

$= 2\frac{\pi}{2} - 0 = \pi$