lecture 1: intro to vector spaces.

Defn: A vector space over $\mathbb{F}$ is a set of V together with operation "vector addition" and "scalar multiplication."

$\forall \vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} \in V$ and. $c\vec{x} \in V$

lecture 2:

Vector space axioms:

① $\forall \vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

② $\forall \vec{x}, \vec{y}, \vec{z} \in V$, $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$

③ $\forall \vec{x}, \vec{y} \in V$, $c \in \mathbb{F}$, $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$

④ $\forall \vec{x} \in V$, $c, d \in \mathbb{F}$, $(c+d)\cdot\vec{x} = c\vec{x}+d\vec{x}$

⑤ $\forall \vec{x} \in V$, $c, d \in \mathbb{F}$, $(cd)\vec{x} = c(d\vec{x})$

⑥ $\forall \vec{x} \in V$, $1 \cdot \vec{x} = \vec{x}$

⑦ $\exists\ \vec{0} \in V$ s.t. $\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}$

⑧ $\forall \vec{x} \in V$, $\exists -\vec{x}$ s.t. $\vec{x}+(-\vec{x}) = (-\vec{x})+\vec{x} = 0$

Prop: Let V be vector space over $\mathbb{F}$. then

① $\vec{0}$ is unique.

② let $\vec{x} \in V$. then $-\vec{x}$ is unique.

③ $\vec{0} \cdot \vec{x} = \vec{0}$, $\forall \vec{x} \in V$.

④ $(-1)\cdot\vec{x} = -\vec{x}$, $\forall \vec{x} \in V$.

★ main things to check for vector space $\overset{①}{+}$, $\overset{②}{\cdot}$ and $\overset{③}{\vec{0}} \in V$?

↳ ④ and additive inverse

lecture 3:

Defn: Let V be vector space over $\mathbb{F}$. let $W \subseteq V$ be a subset of V. If W is itself a vector space, then we say W is a subspace of V.

Theorem: (Subspace-test) let $W \subseteq V$. then W is a subspace of V iff.

① $W \neq \emptyset$ or $\vec{0} \in W$ (non-empty)

② $\forall \vec{x}, \vec{y} \in W$, $\vec{x} + \vec{y} \in W$ (closed under addition)

③ $\forall c \in \mathbb{F}$, $\vec{x} \in W$, $c\vec{x} \in W$ (closed under scalar multi)

Defn: let $S = \{\vec{v}_1 \ldots \vec{v}_k\}$ be a finite set of vectors in V

the span of S is $\text{Span}(S) = \{c_1\vec{v}_1 + \ldots c_k\vec{v}_k : c_i \in \mathbb{F}\}$

Prop: Span(S) is a subspace of V.

Defn: let W be a subspace of V. If $W = \text{span}(S)$ for some subset $S \subseteq V$ then we say W is spanned by S or, S is a spanning set for W.

↳ that's said, if you want to show for ex W is a subspace of V. you can instead show for some $S \subseteq V$, $\text{span}(S) = W$.

lecture 4

last time: $\vec{v}_1 \dots \vec{v}_k$ LI if no $\vec{v}_i$ is a linear comb of the others:

$$\vec{v}_i \neq c_1\vec{v}_1 \dots c_k\vec{v}_k \quad , \text{ if not LI} \; , \rightarrow \text{LD}$$

Prop: $\vec{v}_1 \dots \vec{v}_k$ are LI iff the only soln. to
$c_1\vec{v}_1 + c_2\vec{v}_2 \dots + c_k\vec{v}_k = \vec{0}$ * is the trivial soln.

Proof: (sketch)

($\Rightarrow$) Assume $\vec{v}_1 \dots \vec{v}_k$ are LI, Consider eq * If there is a soln w/ $c_i \neq 0$, then.

$$\frac{c_1\vec{v}_1}{c_i} + \frac{c_2\vec{v}_2}{c_i} + \dots \frac{c_k\vec{v}_k}{c_i} = \cancel{c_i}\vec{v}_i$$

thus $\vec{v}_i$ is a linear comb, contradiction.

ex: $V = P_2(\mathbb{R})$, $S = \{1-x, 1+x, 1-x^2, 1+x^2\}$, is S LI?

$\hookrightarrow$ $\underline{a} + \underline{b}x + \underline{c}x^2 \Rightarrow$ "$\dim V = 3$" $\qquad 0+0x+0x^2$

Sol: consider equation: $c_1(1-x) + c_2(1+x) + c_3(1-x^2) + c_4(1+x^2) = \vec{0}$

Equate coeff of $1, x, x^2$:

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0 \\ -c_1 + c_2 \qquad\qquad = 0 \\ \qquad\qquad -c_3 + c_4 = 0 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

Thus, S is LI $\Leftrightarrow$ the only soln is the trivial soln.

$\Leftrightarrow$ nullity $(A) = 0$ $\quad$ (nullity = dim(nullspace))

$\Leftrightarrow$ rank $(A) = 4$ (n)

Since A is 3×4 rank$(A) \leq 3$, thus S is not LI

p.s. nullity = non-pivot columns in ~~RREF~~ (# of cols − rank) ex: $V = P_2(\mathbb{R})$, $S = \{1+x, 1+x^2\}$, does S span V?

$\hookrightarrow \dim(V) = 3$

Sol: to show $V = \text{Span}(S)$ we must show $V \subseteq \text{Span}(S)$ ①, $\text{Span}(S) \subseteq V$ ②.

② is trivial

$\Rightarrow$ to show $V \subseteq \text{span}(S)$, we must show that

$a + bx + cx^2 \in \text{span}(S)$, $\forall a, b, c$

equivalently, we want to find soln. to

$a + bx + cx^2 = c_1(1+x) + c_2(1+x^2)$

$$\Rightarrow \ast \begin{cases} a = c_1 + c_2 \\ b = c_1 \\ c = \quad\quad c_2 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{array}\right] \leftarrow \text{Inconsistent.}$$

thus, $\text{span}(S) = V \Leftrightarrow \ast$ have a soln $\forall a, b, c$

$\Leftrightarrow [a, b, c]^T \in \text{Col}(A)$ $\forall a, b, c$

$\Leftrightarrow \text{Col}(A) = \mathbb{R}^3$

$\Leftrightarrow \dim(\text{Col}(A)) = \dim(\mathbb{R}^3)$

$\Leftrightarrow \text{rank}(A) = 3$

Since A is 3×2 $\text{rank}(A) \leq 2$, $\text{Span}(S) \neq V$

Generalization:

If $\dim(V) = n$ then,

- a set of $> n$ must be LD
- a set of $< n$ cannot span V

Defn: Let V be a vector space, A basis for V is a set $B \subseteq V$ is LI and spans V.

lecture 6

ex: standard basis for $\mathbb{F}^n$ is $B = \{\vec{e_1}, \vec{e_2} \dots \vec{e_n}\}$

ex. Standard basis for $P_n(\mathbb{F})$ is $B = \{1, x, x^2 \dots x^n\}$

ex: standard basis for $M_{m\times n}(\mathbb{F})$ is $B = \left\{ \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right\}$

There's no standard basis for arbitrary V.

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ex: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

ex: $\{1+2x, 3+3x\}$ is a basis for $P_1(\mathbb{R})$

Show that every vector space has a basis.

ex: $V = C([0,1]) = \{f: [0,1] \to \mathbb{R}, \text{cts}\}$, basis?

$\Rightarrow$ impossible to write out, the basis is itself.

Defn: We say V is finite dimensional if is has a finite spanning set. Else, it's infinite dimensional.

ex: $\mathbb{F}^n$, $M_{m\times n}(\mathbb{F})$, $P_n(\mathbb{F})$, finite dimensional.

ex: $C([0,1])$ is infinite dimensional.

Theorem: every finite dimensional vector space V has a basis.

Proof: Assume $S = \{\vec{v_1}, \dots \vec{v_k}\}$ spans V.

① If S is LI — it's done! it's a basis.

② If S is LD than some vector $\vec{v_k}$ is linear comb of other.

Let $S' = \{\vec{v_1} \dots \vec{v_{k-1}}\}$, than $\text{Span}(S') = \text{Span}(S)$

If S' is LI, done, other wise repeat.

Connection. : $\emptyset$ (empty) is LI and basis for $\{\vec{0}\}$

<u>Defn</u>: Dimension. If V is a finite-dim vector space then we define $\dim(V)$ to be the size of any basis for V

lecture 7

Prop^n: Suppose $\dim V = n$ let $S \subseteq V$ have size k.

① If $k > n$ then S cannot be LI

② If $k < n$ then S cannot span V

③ If $k = n$ then $\underline{S \text{ is LI}} \Leftrightarrow \underline{\text{Span}(S) = V}$.

Prove either

☆ important general proof.

ex: $V = P_2(\mathbb{R})$, $S = \{1+x, 1-x^2, 2x+x^2\}$

↳ $\dim V = 3$.

Claim: S is a basis for V.

Proof ① S is LI:

Consider $c_1(1+x) + c_2(1-x^2) + c_3(2x+x^2) = \vec{0}$ $(0+0x+0x^2)$

Equate coeff $1, x, x^2$:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 2c_3 &= 0 \\ -c_2 + c_3 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$\therefore$ S is LI $\Leftrightarrow$ only soln to this system is trivial soln.

$\Leftrightarrow$ Nullspace$(A) = \{\vec{0}\}$

$\Leftrightarrow$ Nullity$(A) = 0$

Proof ② S spans V.

Given $\vec{v} \in V$, check if we can express it as linear comb. of vectors in S

$\Rightarrow$ Solve $c_1(1+x) + c_2(1-x^2) + c_3(2x+x^3) = a + bx + cx^2$

↳ same coeff matrix.

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\quad\quad\quad}^{(A)}} & \\ 1 & 1 & 0 & a \\ 1 & 0 & 2 & b \\ 0 & -1 & 1 & c \end{array}\right]$$

$\therefore$ S spans V $\Leftrightarrow$ system has a soln $\forall\, [a, b, c]^T \in \mathbb{R}^3$

$$\Leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in Col(A), \ \forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

$$\Leftrightarrow Col(A) = \mathbb{R}^3$$

sketchy $\xrightarrow{(1)}$ $\Leftrightarrow \dim(col(A)) = \dim(\mathbb{R}^3)$

$$\Leftrightarrow rank(A) = 3$$

thus S is a basis. #

We just saw: ① S is LI $\Leftrightarrow$ nullity $(A) = 0$
② S spans V $\Leftrightarrow$ rank $(A) = 3$ } are the same.

rank(A) + nullity (A) = # of columns. (n)

So, rank (A) = 3 iff nullity (A) = 0

S span V iff S is LI

Problem: Suppose V is a n-dim vector space.
Suppose U is a subset of V

Prove: ① $\dim U \leq \dim V$

and ② $\dim U = \dim V \Leftrightarrow U = V$

Proof: ① Let B be a basis for U, then $\dim(U) = |B|$

Since $B \subseteq U$, $B \subseteq V$.

Since B is LI, $|B| \leq \dim(V)$

② Assume now $\dim U = \dim V$, So $\dim(V) = |B|$

Since B is a basis for U, $U = span\ B$

Since $|B| = \dim V$ and B is LI

$\Rightarrow B$ spans V $\Rightarrow V = span\ B = U$

Coordinates: In $\mathbb{R}^3$, we have std basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$

and every $\vec{v} \in \mathbb{R}^3$ looks like $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$

- In a vector space B w/ basis $B = \{\vec{b_1}, \dots \vec{b_n}\}$

  then every $\vec{v} \in V$ can be written as $\vec{v} = a_1\vec{b_1} + \dots + a_n\vec{b_n}$

* Fact: (*unique representation) these a_n are unique.

  We call them the coordinates of $\vec{v}$

  We create the B-coord vectors for $\vec{v}$

$$[\vec{v}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

lecture. 8

If V is a vector space w/ ordered basis $\mathcal{B}$, then every $\vec{v} \in V$ can be written uniquely in the form

$$\vec{v} = a_1\vec{b_1} + a_2\vec{b_2} + \cdots a_n\vec{b_n} \quad \text{for some } a_i \in \mathbb{F}$$

Defn: We record the $\mathcal{B}$-coordinate vector of $\vec{v}$ $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

ex: In $\mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$

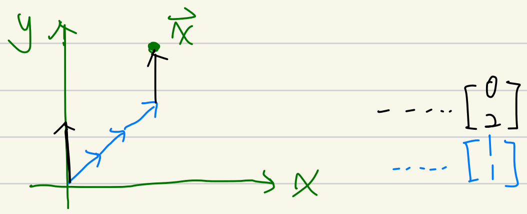
$$[\vec{x}]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \vec{x} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \vec{x} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

ex: now let $\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \in \mathbb{R}^2$. Then $[\vec{x}]_S = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

determine $[\vec{x}]_{\mathcal{B}}$.

$$\vec{x} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a=3,\ b=1,\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



ex: $V = \{ A \in M_{2\times 2}(\mathbb{F}) : tr(A) = 0 \}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

let $A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} \in V$, determine $[A]_{\mathcal{B}}$

Soln: $\begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

by inspection, $b=3$, $a=1$, $c=5$

ex: $V = P_2(\mathbb{R})$, $B = \{-1+2x+2x^2,\ 2+x^2,\ -3+x\}$

Prove that B is a basis and determine $[1+2x+3x^2]_B$

<u>Solu</u>: Since $|B| = 3$ and $\dim V = 3$,

It's sufficient to prove one of ① B is LI, ② B spans V.

↖ pick ②

To show $\text{Span}(B) = V$, take an arbitrary $a+bx+cx^2 \in V$

and find c_1, c_2, c_3 such that

$$c_1(-1+2x+2x^2) + c_2(2+x^2) + c_3(-3+x) = a+bx+cx^2$$

coeff's:
$$\begin{aligned} -c_1 + 2c_2 - 3c_3 &= a \\ 2c_1 + 0c_2 + c_3 &= b \\ 2c_1 + c_2 + 0c_3 &= c \end{aligned}$$

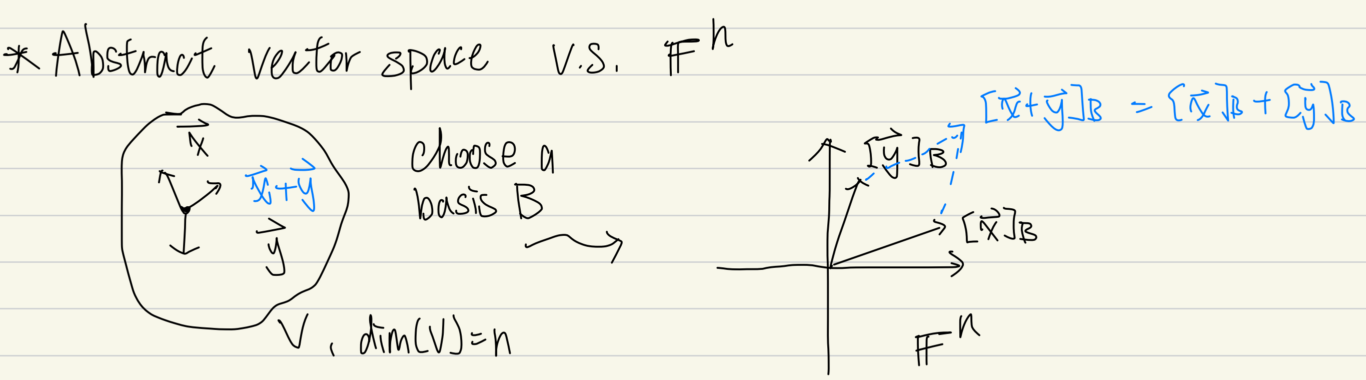
$$\left[\begin{array}{ccc|c} -1 & 2 & -3 & a \\ 2 & 0 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a+3b-2c \\ 0 & 1 & 0 & -2a-6b+5c \\ 0 & 0 & 1 & -2a-5b+4c \end{array}\right] \begin{array}{l} \to c_1 \\ \to c_2 \\ \to c_3 \end{array}$$

So $\text{span}(B) = V$ hence B is a basis for V.

Now, $[1+2x+3x^2]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1+6-6 \\ -2-12+15 \\ -2-10+12 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ #

↳ $a = 1$, $b = 2$, $c = 3$

* Abstract vector space v.s. $\mathbb{F}^n$



Theorem: let V be a n-dim vector space
let B be a basis for V.

Then: (a) $[\vec{x}+\vec{y}]_B = [\vec{x}]_B + [\vec{y}]_B \quad \forall \vec{x}, \vec{y} \in V$

(b) $[c\vec{x}]_B = c[\vec{x}]_B$

Lecture 9

Last time: Linear maps.

function: $L: V \to W$ is linear if

a) $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$

b) $L(c\vec{x}) = cL(\vec{x})$

$\Big\}\ \forall\ \vec{x}, \vec{y} \in V,\ v \in \mathbb{F}$

Remarks:

① Properties (a) and (b) is equivalent to

$$L(c\vec{x}+\vec{y}) = cL(\vec{x}) + L(\vec{y}),\ \forall\ \vec{x}, \vec{y}, c$$

② Linear map respect + and $\cdot$ What else do they respect.

③ Does L (linear map) respect $\vec{0}$?

$\Leftrightarrow L(\vec{0}_V) = \vec{0}_W$, yes.

④ Does L respect $-$?

$\Leftrightarrow L(-\vec{x}) = -L(\vec{x})$, yes.

⑤ What else does linear map respect?

- Bases? If $\{b_1 \dots b_n\}$ is a basis for V,

  is $\{L(b_1) \dots L(b_n)\}$ ________ W?

- LI? if ________ is LI in V ________ LI in V?

ex: of linear maps.

① The zero map $L: V \to W$ define by $L(\vec{v}) = \vec{0}_W,\ \forall\ \vec{v} \in V$

$\vec{v} \mapsto \vec{0}_W$

② The identity map $L: V \to V$ define by $L(\vec{v}) = \vec{v},\ \forall\ \vec{v} \in V$

$\vec{v} \mapsto \vec{v}$

ex: Fundamental examples.

① Let V be n-dimensional, Let B be order basis for V.

The B-coord map $[\ ]_B : V \to \mathbb{F}^n$ $(\vec{v} \longmapsto [\vec{v}]_B)$

Special case: $V = P_3(\mathbb{F})$, $B = \{1, x, x^2, x^3\}$

Then $[\ ]_B = P_3(\mathbb{F}) \to \mathbb{F}^4$ $(a+bx+cx^2+dx^3 \longmapsto [a+bx+cx^2+dx^3]_B)$

☆ ② Let $A \in M_{m\times n}(\mathbb{F})$, we can define linear map.

$L_A : \mathbb{F}^n \to \mathbb{F}^m$ $(\vec{x} \longmapsto A\vec{x})$

ex: Various examples:

① Differentiation: $D : P_n(\mathbb{F}) \longrightarrow P_{n-1}(\mathbb{F})$ $(p(x) \longmapsto p'(x))$

check $D(c\vec{x}+\vec{y}) = D(c\vec{x}) + D(\vec{y}) = cD(\vec{x}) + D(\vec{y})$.

② Integration: $I : P_n(\mathbb{F}) \to P_{n+1}(\mathbb{F})$ $(p(x) \longmapsto \int p(x)\,dx)$

③ Evaluation: Let $\alpha \in \mathbb{F}$

Define $ev_\alpha : P_n(\mathbb{F}) \to \mathbb{F}$ $(p(x) \longmapsto p(\alpha))$

④ Transpose: $L : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ $(A \longmapsto A^T)$

ex: is $L : P_3(\mathbb{F}) \to M_{2\times2}(\mathbb{F})$

$p(x) \longmapsto \begin{bmatrix} p(1) & p'(1) \\ p''(1) & p'''(1) \end{bmatrix}$ linear ?

yes.

ex: Non-linear map.

① Determinant $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

if $n=1$, det is linear (identity map)

↓

② $L: M_{2\times 2}(\mathbb{F}) \to P_2(\mathbb{F})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a+b) + (c+d)x + x^2$$

L is non-linear cause $L(\vec{0}) \neq \vec{0}$

The most important feature of linear map are:

① what it destroys.

② what it creates

Defn: Let $L: V \to W$ be linear

① The kernel of L is $\ker(L) = \{\vec{v} \in V : L(\vec{v}) = \vec{0}\}$

② The range of L is $\text{Range}(L) = \{L(\vec{v}) : \vec{v} \in V\}$

lecture 10

Defn: Let $L: V \to W$ be a LT. The Kernel (nullspace) of L is the set of all vectors in V that are mapped to zero vector of W. That is,

$$\ker(L) = \{\vec{v} \in V : L(\vec{v}) = \vec{0}_W\}$$

and the Range is the set of all outputs.

$$\text{Range}(L) = \{L(\vec{v}) : \vec{v} \in V\}$$

Theorem: Let $L: V \to W$ be a LT, then

① $\ker(L)$ is a subspace of V

② $\text{Range}(L)$ is a subspace of W.

Defn: Let $L: V \to W$ be a LT. then,

① $\text{Rank}(L) = \dim(\text{Range}(L))$

② $\text{nullity}(L) = \dim(\ker(L))$

Theorem: Let $L: V \to W$ be a LT, where $\dim(V) = n$, then $\text{rank}(L) + \text{nullity}(L) = n$

$\Rightarrow$ dim of range + dim of ker = dim of domain.

lecture 11

last time: If $L: V \to W$ is a linear map, then

$\text{rank}(L) = \dim(\text{Range}(L))$

$\text{nullity}(L) = \dim(\ker(L))$ (# non-pivot columns)

ex: warmup. $L: \mathbb{R}^3 \to P_1(\mathbb{R})$ $\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto (a+b+c)x \right)$

what are rank(L) and nullity(L)?

↳ seems like 1 ↳ guess 2.

Sol: , proof need $\text{Range}(L) = \left\{ L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \right\}$

$= \left\{ (a+b+c)x : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \right\}$

$\subseteq \text{Span}\{x\}$.

On the other hand, $\text{span}\{x\} \subseteq \text{Range}(L)$:

$\alpha x = L\left(\begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}\right) \; \forall \alpha \in \mathbb{R}$, So every $\alpha x \in \text{Range}(L)$

thus, $\text{Range}(L) = \text{span}\{x\}$.

$\Rightarrow \text{rank}(L) = \dim(\text{Range}(L)) = 1$ #

Now, Let's determine ker(L)

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker(L) \Leftrightarrow L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = 0 + 0x$

$\Leftrightarrow (a+b+c)x = 0 + 0x$

$\Leftrightarrow a+b+c = 0 \Leftrightarrow a = -b-c$

$\Leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So $\ker(L) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{nullity}(L) = \dim(\ker(L)) = 2$ #

Observation: $L: \mathbb{R}^3 \to P_1(\mathbb{R})$

$$\text{rank}(L) + \text{nullity}(L) = 1 + 2 = 3 \quad (= \dim(\mathbb{R}^3))$$

Fundemental theorem of linear alg. (Rank-nullity theorem):

If $L: V \to W$ is linear then $\underset{\text{domain dim}}{\underset{\downarrow}{\dim(V)}} = \underset{\text{codomain dim}}{\underset{\downarrow}{\text{rank}(L)}} + \underset{\text{dim lost.}}{\underset{\downarrow}{\text{nullity}(L)}}$

Proof: Let $\{\vec{k}_1, \vec{k}_2 \dots \vec{k}_n\}$ be a basis for $\ker(L)$

(so $n = \dim(\ker(L)) = \text{nullity}(L)$

Extend to basis for V — say $B = \{\vec{k}_1, \vec{k}_2 \dots \vec{k}_n, \vec{v}_1, \vec{v}_2 \dots \vec{v}_r\}$

(so $\dim(V) = n + r$)

let $C = \{L(v_1), L(v_2) \dots L(v_r)\} \subseteq W$

Claim, C is a basis for Range(L).

Proof • spanning Need to show span(C) = Range(L)

$\subseteq$: let $\vec{w} \in$ Span(C)

then $\vec{w} = a_1 L(v_1) + \cdots + a_r L(v_r)$

$= L(a_1 v_1 + \cdots a_r v_r)$

So $\vec{w} = L(\text{sth in } V)$, thus, $\vec{w} \in$ Range(L).

$\supseteq$: let $\vec{w} \in$ Range(L). Then $\vec{w} = L(\vec{v})$ for some $\vec{v} \in V$

Since $\vec{v} \in V$, and B is a basis for V, we have

$$V = a_1 \vec{k} + \cdots + a_n \vec{k}_n + b_1 \vec{v}_1 + \cdots b_r \vec{v}_r$$

$$\Rightarrow L(\vec{v}) = a_1 \cancelto{0}{L(\vec{k}_1)} + \cdots + a_n \cancelto{0}{L(\vec{k}_n)} + b_1 L(\vec{v}_1) + \cdots b_n L(\vec{v}_r)$$

$= \sum b L(\vec{v})$, thus $\vec{w} = L(\vec{v}) \in$ Span(C)

ex: $tr: M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

Let $W = \ker(tr) = \{A \in M_{n\times n}(\mathbb{F}) : tr(A) = 0\}$, determine $\dim W$.

Sol: note $\dim W = \dim(\ker(tr))$

$= \text{nullity}(tr) = \dim(M_{n\times n}(\mathbb{F})) - \text{rank}(tr)$

$= n^2 - \text{rank}(tr)$.

What is rank(tr)?

$tr: M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ $= \cancel{0}$ or 1

because tr is not zero map.

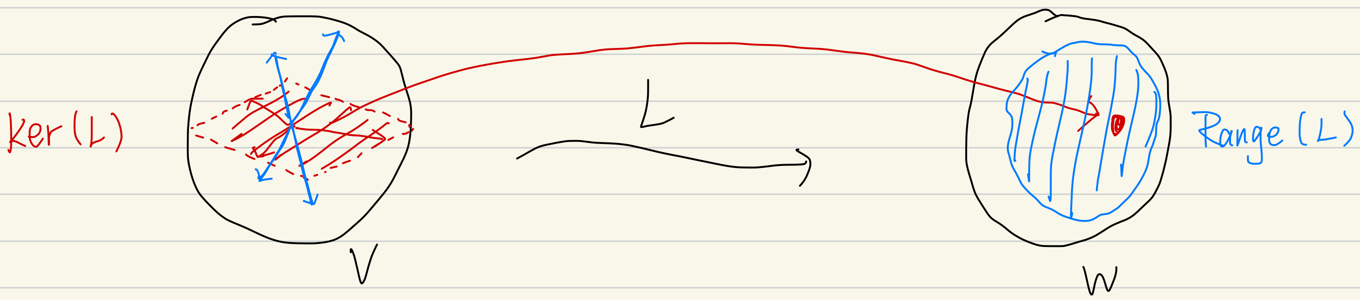
thus, $\dim W = n^2 - 1$

lecture 12.

Last time we see that if $L: V \to W$ is linear map,

then $\dim(V) = \text{rank}(L) + \text{nullity}(L)$

This is fairly intuitive:



In ideal situation, L "preserves" V and just place it in W somehow (maybe stretch or rotate). In this case, we'd expect. $\dim(V) = \dim(\text{Range}(L))$. However, in general L might squash away some dimensions (it destroys ker(L)), so we need to correct this: we lose $\dim(\ker(L))$ degrees of freedom in passage from V to Range(L). So we should have:

$$\dim(V) - \dim(\ker(L)) = \dim(\text{Range}(L))$$

↳ also rank & nullity thm.

this suggest:

Defn: Let $L: V \to W$ be linear

(a) If $\ker(L) = \{\vec{0}\}$ ($\Leftrightarrow$ nullity $= 0$)

then we say L is injective. (one-to-one)

(b) If $\text{Range}(L) = W$ ($\Leftrightarrow$ $\text{rank}(L) = \dim(W)$)

then we say L is surjective (onto)

ex: Let $L: P_1(\mathbb{F}) \to P_2(\mathbb{F})$ be defined by $L(a+bx) = bx^2$

① is L injective?

Let's compute $\ker(L) = \{a+bx \in P_1(\mathbb{F}) : L(a+bx) = \vec{0}\}$

$= \{a+bx \in P_1(\mathbb{F}) : bx^2 = 0 + 0x + 0x^2\}$

$= \{a+bx \in P_1(\mathbb{F}) : b = 0\}$

$= \{a : a \in \mathbb{F}\}$ (constant polynomial)

$\ker(L) \neq \{\vec{0}\}$, thus not injective.

② is L surjective?

Let's compute $\text{Range}(L) = \{L(a+bx) : a+bx \in P_1(\mathbb{F})\}$

$= \{bx^2 : b \in \mathbb{F}\}$

$= \text{span}\{x^2\}$

$\text{Range}(L) \neq P_2(\mathbb{F})$, thus not surjective.

ex: (Exercise)

Determine if each is injective / surjective.

(a) identity map: $id: V \to V$ defined by $id(\vec{v}) = \vec{v}$.

- injective, since $\ker(id) = \{\vec{0}\}$

- surjective, since obviously Range(L)

Prep: ① $L: V \to W$ if $\dim(V) > \dim(W) \Rightarrow L$ can't be injective.

② $L: V \to W$ if $\dim(W) > \dim(V) \Rightarrow L$ can't be injective.

*Prep: if $\dim(V) = \dim(W)$, L injective $\Leftrightarrow$ L surjective.

lecture 13.

ex: Consider $L: P_3(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ defined by

$$L(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p'(0) & p''(0) \end{bmatrix}$$

Sol: Since $\dim(P_3) = 4$ and $\dim(M_{2\times 2}(\mathbb{F})) = 4$

thus Injective $\Leftrightarrow$ surjective

Now, injective $\Leftrightarrow$ nullity $(L) = 0$

or, surjective $\Leftrightarrow$ rank$(L) = 4$

Instead of doing directly, let's observe.

$\to$ $L: P_3(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ is given by:

$$L(a+bx+cx^2+dx^3) = \begin{bmatrix} a & a+b+c+d \\ b & 2c \end{bmatrix}$$

this looks like:

$T: \mathbb{F}^4 \to \mathbb{F}^4$ given by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a \\ a+b+c+d \\ b \\ 2c \end{bmatrix}$

* The connection is coordinates! we are taking coordinate vector $[p(x)]_B$ and $[L(p(x))]_C$ with standard basis $B = \{1, x, x^2, x^3\}$ and $C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$\Rightarrow$ we'll show how every $L: V \to W$ can look like $T: \mathbb{F}^n \to \mathbb{F}^m$ where $n = \dim(V)$ and $m = \dim(W)$.

why?, easier to study.

ex: let $L: \mathbb{F}^2 \to \mathbb{F}^3$ defined $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x \end{bmatrix}$

then we claim there is a matrix A s.t.

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ? \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 2x \end{bmatrix},$$

in this case, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$ works.

In general, given a linear map $L: \mathbb{F}^n \to \mathbb{F}^m$, we define its standard matrix to be $[L] = [L(\vec{e}_1) \mid L(\vec{e}_2) \mid \dots L(\vec{e}_n)]$ where $\{\vec{e}_1, \vec{e}_2 \dots \vec{e}_n\}$ is the standard basis

* key fact: matrix $[L]$ "know everything" abt map L

Here is how to create a matrix A that performs $L: V \to W$

- Step 1: choose ordered basis $B = \{\vec{v}_1 \cdots \vec{v}_n\}$ (so $n = \dim(V)$)

  $C = \{\vec{w}_1 \cdots \vec{w}_m\}$ (so $m = \dim(W)$)
- Step 2: describe the effect of L on coordinate vector.

  This gives a linear map $\mathbb{F}^n \to \mathbb{F}^m$
- Step 3: find the standard matrix of the map $\mathbb{F}^n \to \mathbb{F}^m$

  $\Rightarrow$ We can streamline this process.

Defn: let $L: V \to W$ with ordered basis $B = \{\vec{v}_1 \dots \vec{v}_n\}$ $C = \{\vec{w}_1 \dots \vec{w}_n\}$

then the matrix of L wrt B and C is the $m \times n$ matrix.

$${}_C[L]_B = \big[\, [L(\vec{v}_1)]_C \;\; [L(\vec{v}_2)]_C \;\; \cdots \;\; [L(\vec{v}_n)]_C \,\big]$$

* key property: $[L(\vec{x})]_C = {}_C[L]_B [\vec{x}]_B$

ex: Let's look at differentiation map $D: P_3(\mathbb{R}) \to P_2(\mathbb{R})$

Let B std basis $= \{1, x, x^2, x^3\}$

and C std basis $= \{1, x, x^2\}$

$$ {}_C[D]_B = \big[\ [D(1)]_C \ \ [D(x)]_C \ \ [D(x^2)]_C \ \ [D(x^3)]_C \ \big] $$

$$ = \big[\ [0]_C \ \ [1]_C \ \ [2x]_C \ \ [3x^2]_C \ \big] $$

$$ = \big[\ [0+0x+0x^2]_C \ \ [1+0x+0x^2]_C \ \ [0+2x+0x^2]_C \ \ [0+0x+3x^2]_C \ \big] $$

$$ = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} $$

* The idea here is that the matrix "performs" D 概念

i.e. we can multiply it by a polynomial.

BUT, we can multiply it by the coordinate vect. $[p(x)]_B$ ☆

Here's what we get:

$$ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ d \end{bmatrix} $$

this basically says that $D(a+bx+cx^2+dx^3) = b+(2c)x+(3d)x^2$

BUT what it actually says is ${}_C[D]_B\,[p(x)]_B = [D(p(x))]_C$

* Takeaway: The key property tells us that we can "perform" D by multiplying coordinate vectors by ${}_C[D]_B$

lecture 14.

<u>Prop</u>: let $L: V \to W$ and let B & C be bases for V & W,

let $A = {}_{C}[L]_{B}$, then,

(a) $\vec{v} \in \ker(L) \Leftrightarrow [\vec{v}]_B \in \text{Null}(A)$: $A\vec{x} = \vec{0}$

(b) $\vec{w} \in \text{Range}(L) \Leftrightarrow [\vec{w}]_C \in \text{Col}(A)$.

Corollary: let $A = {}_{C}[L]_{B}$

(a) $\dim(\ker(L)) = \dim(\text{Null}(A))$
(i.e nullity(L) = nullity(A))

(b) $\dim(\text{range}(L)) = \dim(\text{Col}(A))$
(i.e rank(L) = rank(A))

lecture 15: Iso

Defn: An isomorphism is a linear map $L: V \to W$ that is injective and surjective. we say V is isomorphic to W.

Useful comments: ☆

- L injective means it place a copy of V in W without destroying anything.
- L surjective means the copy of V fill up all of W.

So, in essense, L "morphs" V into W perfectly.

Defn: let $L: V \to W$ be an isomorphism. The inverse of L is the unique linear map $T: W \to V$ that satisfies.

$L(T(\vec{w})) = \vec{w}$, $\forall \vec{w} \in W$ and $T(L(\vec{v})) = \vec{v}$, $\forall \vec{v} \in V$.

We describe it by L^{-1}.

lecture 16

Defn: change of coord:

$$_C[I]_B = \begin{bmatrix} | & | & & | \\ [\vec{v}_1]_C & [\vec{v}_2]_C & \cdots & [\vec{v}_n]_C \\ | & | & & | \end{bmatrix}$$ from B to C coord.

where $B = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$

Thm:

(a) $_C[I]_B [\vec{x}]_B = [\vec{x}]_C$

(b) $(_C[I]_B)^{-1} = {_B[I]_C}$

(c) $_D[I]_C \ {_D[I]_B} = {_D[I]_B}$