Max Chung

MATH

lecture 1 (missed)

Defn: a scalar function $f(x_1 \dots x_n)$ of n-variables is a fcn. whose domain & Range. is a subset of $\mathbb{R}^n$

Geometric Intepretations:

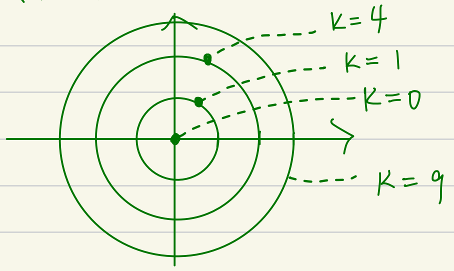
level curves: the curves $f(x,y) = k$ where $k \in$ Range (f)

ex: consider $f(x,y) = x^2+y^2$

$\Rightarrow D(f) = \mathbb{R}^2$ and $R(f) = \{z \in \mathbb{R} : z \geq 0\} \Rightarrow k \geq 0$

inspection: level curves are circles $x^2+y^2=k$, $c=(0,0)$

(note: $k = z$, "levels")



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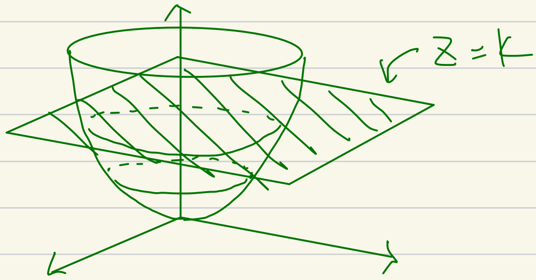


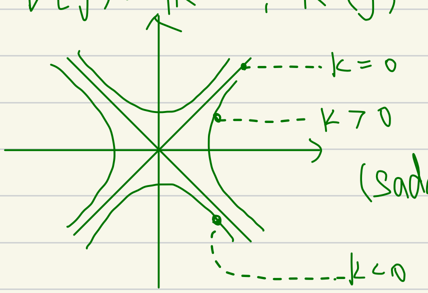
exceptional level curve

paraboloid.

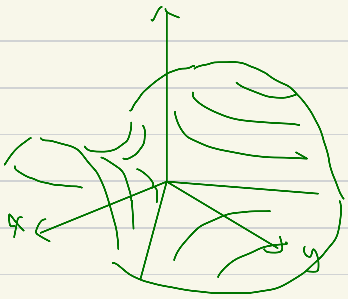
Review : $(x-a)^2+(y-b)^2 = k$, $c=(a,b)$, $r=\sqrt{k}$

ex: consider $g(x,y) = x^2-y^2$ (hyperbola)

$\Rightarrow D(g) = \mathbb{R}^2$, $R(g) = \mathbb{R} \Rightarrow k \in \mathbb{R}$

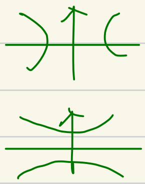


(saddle surface)

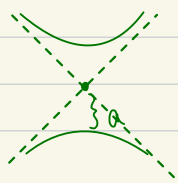


← Pringles

Review: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-v)^2}{b^2} = 1$, $a>b$, $(x,y) \Rightarrow$ horizontal, $(y,x) \Rightarrow$ vertical

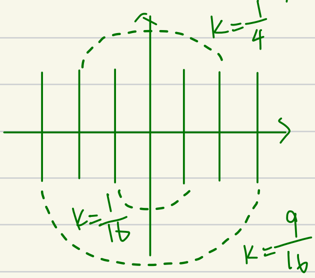


$C=(h,v)$

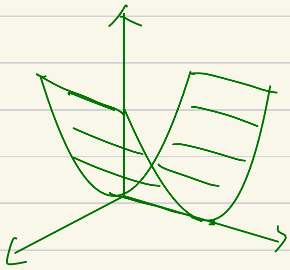


ex: consider $h(x,y) = x^2$

$\Rightarrow D(h) = \mathbb{R}^2, \ R(h) = \{z \in \mathbb{R} : z \geq 0\} \Rightarrow k \geq 0$



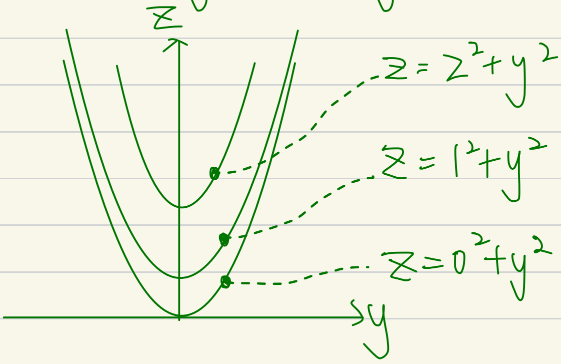
$\Rightarrow$



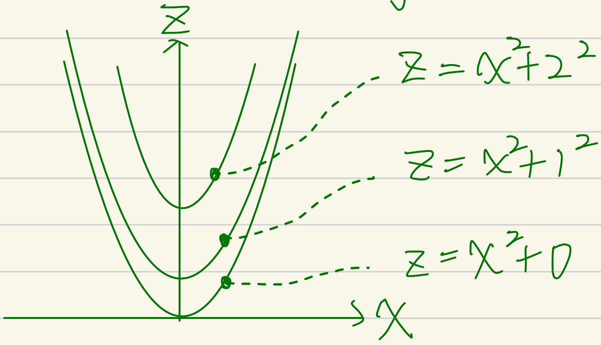
Cross sections of a surface $z = f(x,y)$ is the intersect of the plane ex: vertical planes $x=c$ or $y=d$.

ex: let $f(x,y) = x^2+y^2$ cross section with $x=c$? $y=d$?

x:



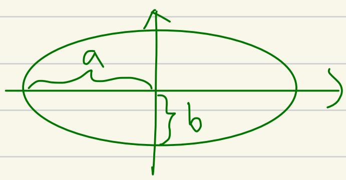
y:



Generalization:

- level surface is defined by $f(x,y,z)=k$, $k \in R(f)$
- level set of $f(x)$, $x \in \mathbb{R}^n$ is defined by $f(x)=k$, $k \in R(f)$

note: ellipse equ: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



lecture 2.

<u>Defn</u>: an r-neighborhood of a point $(a,b) \in \mathbb{R}^2$ is a set

$$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 : \|(x,y)-(a,b)\| < r\}$$

<u>Defn</u>: limit (review)

Assume $f(x,y)$ is in neighborhood of (a,b), expect possibly at (a,b)

If for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$0 < \|(x,y)-(a,b)\| < \delta$ implies $|f(x,y) - L| < \varepsilon$

then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Thm: limit are Unique.!

lecture 3

ex: $\lim_{(x,y)\to(a,b)} \frac{2x^4 y^{\frac{2}{3}}}{x^6+y^2}$, Prove lim DNE

Sol: try $y=mx$, $m\in\mathbb{R}$

$$\lim_{x\to 0} \frac{2x^4(mx)^{\frac{2}{3}}}{x^6+(mx)^2} = \lim_{x\to 0} \frac{2m^{\frac{3}{2}}x^{\frac{8}{3}}}{x^4+m^2} = \frac{0}{m^2} = 0 \quad (m\neq 0)$$

If $m=0 \Rightarrow \lim_{x\to 0} \frac{0}{x^6} = 0$ $\Rightarrow$ lim $=0$ along all $y=mx$

does not imply $\lim_{(x,y)\to(a,b)} = 0$

take $y=x^3$, $\Rightarrow \lim_{x\to 0} \frac{2x^4(x^3)^{\frac{2}{3}}}{x^6+x^6} = \frac{2x^6}{2x^6} = 1 \neq 0$

Thus, lim DNE.

ex: $\lim_{(x,y)\to(1,0)} \frac{x^2-y-1}{x+y-1}$ $\Rightarrow$ Prove DNE

Sol: try lines $y=m(x-1)$ $\to (1,0)$

Shortcut: try $y=0 \Rightarrow \lim_{x\to 1} \frac{x^2-0-1}{x+0-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)} = 2.$

try $x=1 \Rightarrow \lim_{x\to 1} \frac{1^2-y-1}{1+y-1} = -1$ $\leftarrow$ different

Thus, lim DNE

Proving that a limit exists.

Squeeze theorem: if $\exists$ a fcn. $B(x,y)$ s.t. (Bound.)

$|f(x,y)-L| \le B(x,y)$ $\forall$ (x,y) in some neighborhood of (a,b), expect possibly at (a,b),

and $\lim_{(x,y)\to(a,b)} B(x,y)=0$ then $\lim_{(x,y)\to(a,b)} f(x,y)=L$

__Proof:__ let $\varepsilon>0$. Since $\lim_{(x,y)\to(a,b)} B(x,y)=0$

we know $\exists$ $\delta>0$ s.t.

$|B(x,y)-0|<\varepsilon$ whenever $0<\|(x,y)-(a,b)\|<\delta$

Now, $\underset{\text{hypo}}{|f(x,y)-L| \le B(x,y)} \underset{\text{by above.}}{<} \varepsilon$ whenever $0<\|(x,y)-(a,b)\|<\delta$

So, $\lim_{(x,y)\to(a,b)} f(x,y)=L$ by def.

ex.: Show $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}=0$

__Sol:__ consider $\left|\frac{2x^2y}{x^2+y^2}-0\right| = \frac{2x^2|y|}{x^2+y^2} \le \frac{2(x^2+y^2)|y|}{x^2+y^2}$

$=2|y| = B(x,y)$,

Clearly $\lim_{(x,y)\to(0,0)} 2|y|=0$ so by squeez thm. $\lim=0$.

__comment:__ $\frac{2x^2|y|}{x^2+y^2} \le \frac{2x^2|y|}{x^2}$ $\leftarrow$ nope.

$\to (x,y)\neq(0,0)$ $\to x\neq 0$

$\Downarrow$

ex: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^4+x^2+y^4+y^2}{x^2+y^2}$ or show DNE.

Sol: Along $y=mx$, we have.

$$\Rightarrow \lim_{x\to 0} \frac{x^4+x^2+(mx)^4+(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{\cancel{x^2}(x^2+1+m^4x^2+m^2)}{\cancel{x^2}(1+m^2)}$$

$= \frac{1+m^2}{1+m^2} = 1$, so limit $f(x,y)$ must $=1$ if it exist.

Try squeez $\Rightarrow$ $|f(x,y)-L| = \left|\frac{x^4+x^2+y^4+y^2}{x^2+y^2} - 1\right|$

$$= \left|\frac{x^4+y^4}{x^2+y^2}\right| \leq \frac{(x^2+y^2)^2+(x^2+y^2)^2}{x^2+y^2} = \frac{2(x^2+y^2)^2}{x^2+y^2}$$

$= 2(x^2+y^2) = B(x,y)$, $\to 0$ as $(x,y)\to(0,0)$

so $\lim f(x,y)=1$ by sq thm.

lecture 4

Last time: Sqz thm $|f(x,y) - L| \leq B(x,y) \to 0$

- Common inequality tricks (pg. 19)

$$|x+y| \leq |x| + |y| \quad (\triangle \text{ ineq})$$

$$(x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4 \Rightarrow x^4 + y^4 \leq (x^2+y^2)^2$$

$$|x||y| \leq \frac{x^2+y^2}{2} \quad (\text{cosine ineq})$$

Continuity:

Defn: A fcn. $f(x,y)$ is continuous at (a,b) means.

$$\lim_{(x,y)\to(a,b)} = f(a,b)$$

f is continuous in $D \subseteq \mathbb{R}^2$ if it is cts. at every point in D.

ex: $f(x,y) = \dfrac{x^4+x^2+y^4+y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$

we showed $\lim\limits_{\to(0,0)} f(x,y) = 1$

if we define $f(0,0)=1$ then f will be cts. at $(0,0)$.

ex: Let $f(x,y) = \begin{cases} \dfrac{\sin(x^2+2y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ k, & (x,y) \neq (0,0) \end{cases}$

Can we find k so that f is cts. at $(0,0)$

Sol: Consider $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+2y^2)}{x^2+y^2}$

P.S. try along "simple" line.

$- y=0$: $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x^2} = 1$ (prev)

$- x=0: \lim_{y\to 0} \frac{\sin(2y^2)}{y^2} = 2\lim_{y\to 0}\frac{\sin(2y^2)}{2y^2} = 2.$ (different)

Since $1 \neq 2$, $\lim f(x,y)$ DNE.

Continuity Theorem:

let $f(x,y)$, $g(x,y)$ be cts. at (a,b). Then

① $f+g$ and fg are also cts. at (a,b)

② f/g is cts at (a,b), provided $g(a,b) \neq 0$

Continuity composition theorem:

let f be a single var. fcn. and g be a 2-var. fcn.

If g is cts. at (a,b) and f is cts at $g(a,b)$,

then $f(g(x,y))$ is cts. at (a,b)

ex: $\frac{y\sin x - \cos y}{x^2+y^2}$ is cts. $\forall (x,y)$ except for $(0,0)$ by cty. thm..

ex: $e^{x^3-\sin(x,y)}$ is cts on $\mathbb{R}$ by cty. thm(s).

ex: $\lim_{(x,y)\to(0,0)} \frac{e^{x^2}+\ln(2+y^2+x^4)}{(x-1)^2+y^4} = ?$

by cty. thm. $f(x,y)$ is cts. at $(0,0)$

thus, plug the fuck in $(0,0) = 1+\ln(2)$

ex: determine where $f(x,y) = \begin{cases} \frac{x^4y^6}{x^6+y^{12}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$ is cts.

Sol: for $(x,y) \neq (0,0)$, f is cts. by cty. thm.

for $(x,y) = (0,0)$, use def of continuity.

$\Rightarrow \lim_{\to(0,0)} f(x,y) \overset{?}{=} f(0,0)$

$\lim_{\to(0,0)} \frac{x^4y^6}{x^6+y^{12}}$, along $y=mx$: $\lim_{x\to 0} \frac{m^6 x^{\cancel{10}\,4}}{\cancel{x^6}(1+m^{12}x^6)} = 0$

- try a sqz: $\left|\frac{x^4y^6}{x^6+y^{12}} - 0\right| = \frac{x^4y^6}{x^6+y^{12}} = \frac{(x^6+y^{12})^{\frac{4}{6}}(x^6+y^{12})^{\frac{1}{2}}}{x^6+y^{12}}$

$= (x^6+y^{12})^{\frac{1}{6}} \to$ as $(x,y)\to(0,0)$

by sqz thm, $\lim f(x,y) = 0$, which $= f(0,0)$

so f is cts. at $(0,0)$

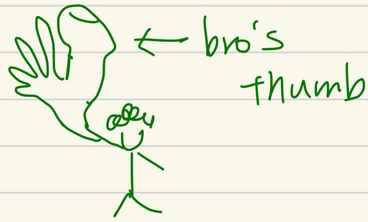
thus f is cts of $\mathbb{R}^2$.

lecture 6.

Two ways to differenciate $f(x,y)$:

① Treat y as fixed x: $\frac{\partial f}{\partial x}$

② Treat x as fixed y: $\frac{\partial f}{\partial y}$



ex: $f(x,y) = y^2 \sin(xy)$

$$\frac{\partial f}{\partial x} = y^3 \cos(xy), \quad \frac{\partial f}{\partial y} = 2y \cdot \sin(xy) + y^2 \cos(xy) \cdot x$$

at a point: $\frac{\partial f}{\partial x}(\pi, 1) = \left.\frac{\partial f}{\partial x}\right|_{(\pi,1)} = 1^3 \cdot \cos(\pi) = -1.$

Subscript notation: $\frac{\partial f}{\partial x} = f_x$

$\frac{\partial f}{\partial y} = f_y$

Formal Defn: the partial derivatives of $f(x,y)$ at (a,b) are:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

ex: find "the partial" at $(0,0)$ of

$$f(x,y) = \begin{cases} \frac{x^3 + y^4}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol: use defn.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3+0}{h^2+0} - 0}{h} = 1$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^4}{h^2} - 0}{h} = 0$$

Comment: MUST use defn. when "usual" rules (product, power, quotient, ...) don't apply

ex. page 31~32.

$f(x,y) = (x^3+y^3)^{\frac{1}{3}}$

$$\frac{\partial f}{\partial x} = \frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}(3x^2), \quad \text{for } x^3+y^3 \neq 0$$

what if $x^3+y^3=0$? i.e. if $y=-x$? i.e. at point $(a,-a)$

above formula not allowed, must use defn.

ex: volume of a ideal gas.

$$V = \frac{82.06T}{P} \quad \begin{matrix} \leftarrow \text{temp. (K)} \\ \leftarrow \text{pressure (a.t.m)} \end{matrix}$$

Find rate of change of volume:

(a) w.r.t T

(b) w.r.t P

when $T = 300$ k and $P = 5$ a.t.m.

Sol: (a) $\frac{\partial V}{\partial T} = \frac{82.06}{P} \Rightarrow \left.\frac{\partial f}{\partial T}\right|_{(300,5)} \approx 16.41 \frac{cm^3}{K}$

(b) $\frac{\partial V}{\partial P} = (82.06T)\cdot(-P^{-2}) \Rightarrow \left.\frac{\partial f}{\partial P}\right|_{(300,5)} \approx -98472 \frac{cm^3}{a.t.m.}$

Second Derivatives: of $f(x,y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1D_1f = D_1^2 f$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2D_1f$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1D_2f$$

$D_1 = \frac{\partial}{\partial x}$

$D_2 = \frac{\partial}{\partial y}$

$\downarrow$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = D_2 D_2 f$$

ex: $f(x,y) = xe^{-xy}$

$f_x = 1 \cdot e^{-xy} - yxe^{-xy} = e^{-xy}(1-xy)$

$f_y = -x^2 e^{-xy}$

$f_{xx} = (-y)e^{-xy}(1-xy) + e^{-xy}(-y)$

$f_{xy} = -xe^{-xy}(1-xy) + e^{-xy}(-x) = e^{-xy}(-2x+x^2y)$

$f_{yx} = -2xe^{-xy} + e^{-xy}(x^2y) = e^{-xy}(-2x+x^2y)$ equal!

$f_{yy} = \ldots\ldots$

Clairaut's theorem: If f_x, f_y, f_{xy}, and f_{yx} are all defined in some neighborhood of (a,b), and f_{xy} and f_{yx} are cts. on (a,b) then, $f_{xy} = f_{yx}$.

Lecture 7 : Tangent Plane, Linear approximation (43, 44)

But first, more on partial derivatives.

3rd partials of $f(x,y)$, f_{xxx}, f_{xxy}, f_{xyy} ... (8 ways)

For $f(x,y,z)$: $f_x = \frac{\partial f}{\partial x}$

<u>$f \in C^k$</u> : "f is of class C-k"

... means that the k^{th} partial derivatives are continuous.

e.g. $f(x,y) \in C^2$ means that f_{xx}, f_{xy}, f_{yx}, f_{yy} are all cts.

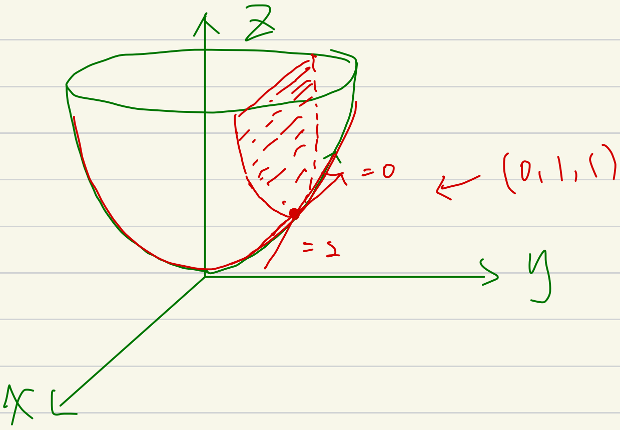
<u>Geometric interp.</u> of f_x and f_y:

ex: $f(x,y) = x^2+y^2$

slope of cross sections.

a) $f_x(0,1) = (2x+0)\big|_{(0,1)} = 0$

$f_y(0,1) = (0+2y)\big|_{(0,1)} = 2.$



Equation of tangent plane to $z=f(x,y)$ at (a,b).

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

ex: find equation of tangent plane to the surface

$z = \frac{xy}{x^2+y^2}$ at $(x,y) = (1,2)$, $f(1,2) = 2/5$

$f_x = \frac{(x^2+y^2)\cdot y - xy(2x)}{x^2+y^2} \Rightarrow \cdots \Rightarrow f_x(1,2) = \frac{6}{25}$

$f_y(1,2) = \frac{-3}{25}$

plugin: $z = \frac{2}{5} + \frac{6}{25}(x-1) - \frac{3}{25}(y-2)$

Defn: Linearization of $f(x,y)$ at (a,b) is

$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Defn: Linear approximation.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

↖ for (x,y) "near" ab

ex: approximate. $\sqrt{3.01^2 + 3.98^2}$

Sol: let $f(x,y) = \sqrt{x^2+y^2}$, and approximate near $(3,4)$

$$f(x,y) \approx f(3,4) + \frac{\partial f}{\partial x}(3,4)(x-3) + \frac{\partial f}{\partial y}(3,4)(y-4)$$

Now, $f(3,4) = 5$,

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)\Big|_{(3,4)} = \frac{x}{\sqrt{x^2+y^2}}\Big|_{(3,4)} = \frac{3}{5}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\Big|_{(3,4)} = \frac{4}{5}$$

So, linear approximation becomes.

$$f(x,y) \approx 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \text{ for } (x,y) \text{ near } (3,4)$$

$$\Rightarrow \sqrt{3.01^2+3.98^2} \approx 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.98-4)$$

$$= 5 + 0.006 - 0.016 = 4.99 \quad (\text{actual value } 4.99004)$$

lecture 8

Last time :

$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) \longrightarrow$ ① linear approx

$\Rightarrow \underbrace{f(x,y)-f(a,b)}_{\Delta f} \approx \underbrace{\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)}_{\nabla f(a,b)} \cdot \underbrace{(x-a, y-b)}$ —— ②

"gradient vector"

$= (x,y)-(a,b)$
$= \mathbf{x} - \mathbf{a}$
$= \Delta \mathbf{x}$

$\mathbf{x} = (x,y)$
$\mathbf{a} = (a,b)$

$\Rightarrow \underline{\Delta f \approx \nabla f(x,y) \cdot \Delta \mathbf{x}}$ "Increment form of linear approx"

Higher dimensions :

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$$

$$\Rightarrow f(x,y,z) - f(a,b,c) \approx \left(f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\right) \cdot (x-a, y-b, z-c)$$

$\Rightarrow \Delta f \approx \nabla f(\mathbf{a}) \cdot \Delta \mathbf{x}$ where $\mathbf{a} = (a,b,c)$, $\mathbf{x} = (x,y,z)$
$\Delta \mathbf{x} = \mathbf{x} - \mathbf{a}$,

ex: Let $h(x,y,z) = x,y,z$, and $\mathbf{a} = (1,2,3)$

what is the approx. change in h if x increase by 0.01,

y increase by 0.02 and z decreases by 0.01 ?

Sol: $\Delta h \approx \nabla h(1,2,3) \cdot (0.01, 0.02, 0.03)$

$\nabla h = (h_x, h_y, h_z) = (yz, xz, xy)$

$= (6, 3, 2) \cdot (0.01, 0.02, 0.03)$

$= 0.1 \Rightarrow \Delta h \approx 0.1$

ex: try to approx $f(0.1, 0.1)$ where,

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol: use linear approx.

$$f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

now $f(0,0) = 0$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

by symmetry, $f_y(0,0) = 0$.

$$\Rightarrow f(x,y) \approx 0 + 0 + 0 = 0$$

$$\Rightarrow f(0.1, 0.1) \approx 0 \qquad (\text{Actual value}: 0.5)$$

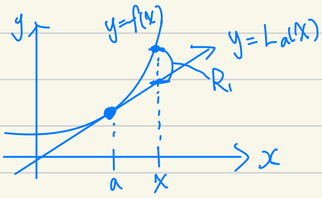
comment: the fcn. is not cts. at $(0,0)$, yet $f_x(0,0)$ and $f_y(0,0)$ exist.

- Need a careful defn. for differentiability.

Flashback: math 137/138.

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{or} \quad f(x) = \underbrace{f(a) + f'(a)(x-a)}_{L_a(x)\text{ 1st degree taylor poly.}} + \overbrace{R_{1,a}(x)}^{\text{Remainder}}$$

Theorem: If $f'(a)$ exists then $\displaystyle\lim_{x \to a} \frac{|R_{1,a}(x)|}{x-a} = 0$



Key idea: Remainder $\to 0$ faster than the distance from x to a

Proof: $\left|\frac{R_{1,a}(x)}{x-a}\right| = \left|\frac{f(x)-\left(f(a)+f'(a)(x-a)\right)}{x-a}\right|$

$= \left|\frac{f(x)-f(a)}{x-a} - \frac{f'(a)(x-a)}{(x-a)}\right| = \left|f'(a)-f'(a)\right| = 0$ as $a \to 0$.

<u>Defn</u>: $f(x,y)$ is differentiable at (a,b) <u>means</u>.

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

where $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

$= f(x,y) - \left(f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)\right)$

and $\|(x,y)-(a,b)\| = \|(x-a,\ y-b)\| = \sqrt{(x-a)^2+(y-b)^2}$

lecture 9, 10

Theorem: If a fcn satisfies

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y)-f(a,b)-c(x-a)-d(y-b)|}{\|(x,y)-(a,b)\|} = 0$$

then, $c = f_x(a,b)$ and $d = f_y(a,b)$

Defn: **tangent Plane** of surface $z = f(x,y)$ at $(a,b,f(a,b))$

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

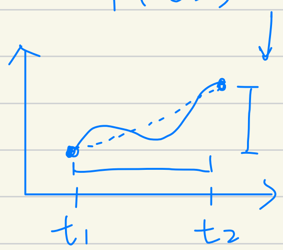
Theorem: if $f(x,y)$ is diff'able at (a,b), then f is cts. at (a,b).

diff'able $\Rightarrow$ cty.

Theorem: **MVT**.

If $f(t)$ is cts on $[t_1, t_2]$ and f is diff'able on (t_1, t_2), then $\exists\ t_0 \in (t_1, t_2)$ s.t.

$$f(t_2) - f(t_1) = f'(t_0)(t_2 - t_1)$$



Theorem: If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are cts at (a,b), then $f(x,y)$ diff'able at (a,b).

partial cts $\Rightarrow$ diff'ability

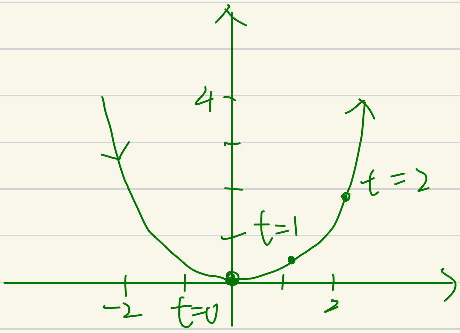
lecture 11

Review of parameter / vector curves

ex: $\mathbf{x}(t) = (t, t^2)$

Parametric form:

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$$



eliminate $y = t^2 = (x)^2$ since $x = t \Rightarrow y = x^2$ (lies on $y = x^2$)

If $\mathbf{x}(t)$ = position of particle at time t

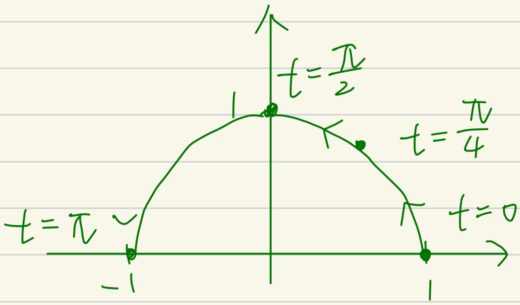
then $\mathbf{x}'(t) = (1, 2t)$ is velocity at time t. e.g. $\mathbf{x}'(1) = (1, 2)$

and $\mathbf{x}''(t) = (0, 2)$ is acceleration at time t.

ex: $\mathbf{x}(t) = (\underset{x(t)\nearrow}{\cos(t)}, \underset{y(t)\nearrow}{\sin(t)})$ $\quad 0 \le t \le \pi$

eliminate t: $\quad x^2 + y^2 = \cos^2 t + \sin^2 t$
$= 1$

so curve lies on circle.

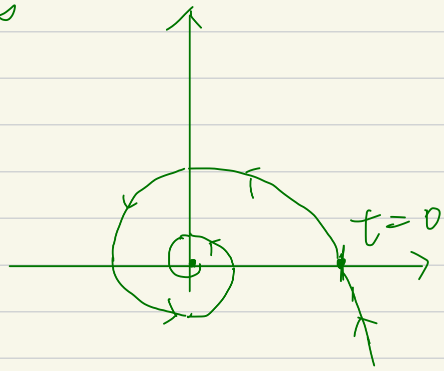


ex: $\mathbf{x}(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$, $-\infty < t < \infty$

try to eliminate t:

$x^2 + y^2 = e^{-2t}(\sin^2(t) + \cos^2(t))$

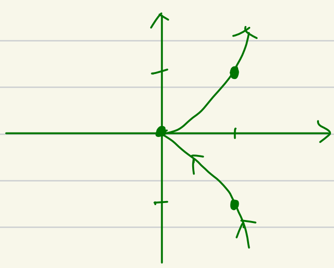
$= e^{-2t} \Rightarrow$ "circle" with dec' radius.



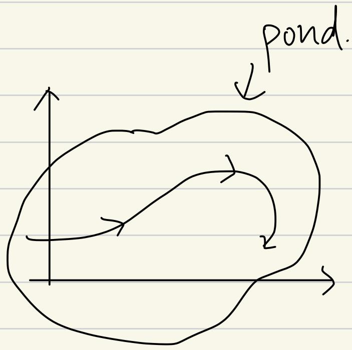
ex: $\mathbf{x}(t) = (t^2, t^3)$

$y = t^3 = (\pm\sqrt{x})^3 = \pm x^{\frac{3}{2}}$

or $x = t^2 = (y)^{\frac{2}{3}}$



Chain Rule:



Imagine a duck position: $\mathbf{x}(t) = (x(t), y(t))$

Let $f(x,y)$ = temperature of pond at (x,y)

find r.o.c of temperature experienced by duck.

$\Rightarrow$ in time Δt, duck's position changes by

$$\Delta x = x(t+\Delta t) - x(t)$$

$$\Delta y = y(t+\Delta t) - y(t)$$

by increment form of linear approx.

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \Rightarrow \frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{x(t+\Delta t)-x(t)}{\Delta t} + \frac{\partial f}{\partial y}\frac{y(t+\Delta t)-y(t)}{\Delta t}$$

Let $t \to 0$: $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$ ☆ chain Rule for $f(x(t), y(t))$

Comments:

- $\frac{df}{dt}$ means $\frac{d}{dt} f(x(t), y(t))$
- this is not proof ↗

lecture 12 & 13 . (missed)

Thm: Chain Rule.

let $G(t) = f(x(t), y(t))$ and $a = x(t_0)$, $b = y(t_0)$.

If f is diff'able at (a,b) and $x'(t_0), y'(t_0)$ exist.

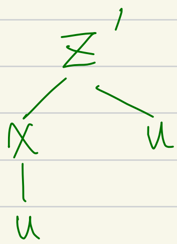
$$G'(t_0) = f_x(a,b)\, x'(t_0) + f_y(a,b)\, y'(t_0)$$

technique: draw a chain of dependence diagram.
and work upward.

now, chain rule of second partials.

ex: let $z = f(x)$, $x = e^u$, verify $z''(u) = x^2 f''(x) + x f'(x)$

$\Rightarrow z'(u) = f'(x)\, x'(u) = \underline{f'(x)}\, \underline{e^u}$, then



$$z''(u) = \frac{\partial z'}{\partial x}\frac{dx}{du} + \frac{\partial z'}{\partial u}$$

$\hookrightarrow \frac{\partial z'}{\partial x} = \frac{\partial}{\partial x}\left(f'(x) e^u\right) = f''(x) e^u$

$\hookrightarrow \frac{\partial z'}{\partial u} = \frac{\partial}{\partial u}\left(f'(x) e^u\right) = f'(x) e^u$

$$\Rightarrow z''(u) = f''(x) e^{2u} + f'(x) e^u = x^2 f''(x) + x f'(x)$$

☆ ex: $g(u,v) = f(x,y)$, where $x = u\cos v$, $y = u\sin v$.

Given: $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cos v + \frac{\partial f}{\partial y}\sin v$ & $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}(-u\sin v) + \frac{\partial f}{\partial y}(u\cos v)$

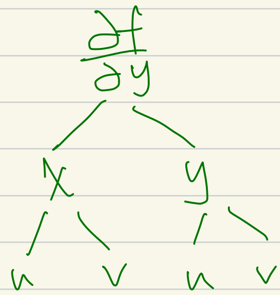
find. $\frac{\partial^2 g}{\partial u^2}$.

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\cos v + \frac{\partial f}{\partial y}\sin v\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\cos v + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\sin v.$$

$$\hookrightarrow \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 f}{\partial x^2}\cos v + \frac{\partial^2 f}{\partial y \partial x}\sin v$$



$$\hookrightarrow \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 f}{\partial x \partial y}\cos v + \frac{\partial^2 f}{\partial y^2}\sin v.$$

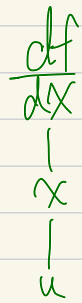
$$\frac{\partial^2 g}{\partial u^2} = (\cos v)\left(f_{xx}\cos v + f_{xy}\sin v\right)$$

$$+ (\sin v)\left(f_{xy}\cos v + f_{yy}\sin v\right)$$

$$= f_{xx}\cos^2 v + 2 f_{xy}\cos v \sin v + f_{yy}\sin^2 v. \quad \#$$

ex: Method 2 for ex 1:

$$z'(u) = \frac{df}{dx}\frac{dx}{du} = \frac{df}{dx}e^u$$



$$z''(u) = \frac{\partial}{\partial u}\left(\frac{df}{dx}e^u\right) = \frac{\partial}{\partial u}\left(\frac{df}{dx}\right)e^u + \frac{df}{dx}\frac{\partial}{\partial u}(e^u)$$

$$= \left(\frac{d^2 f}{dx^2}\frac{dx}{du}\right)e^u + \frac{df}{dx}e^u$$

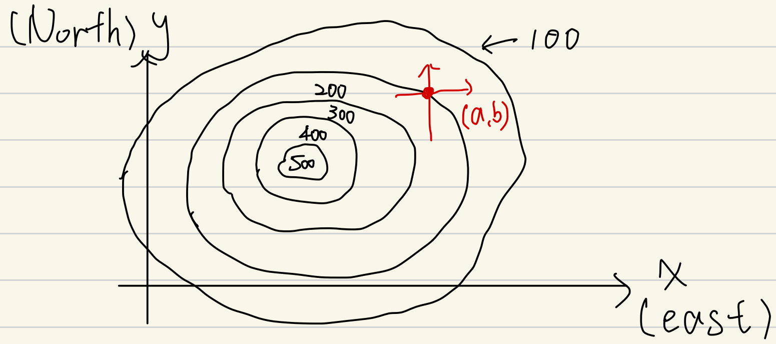
$$= f''(x)e^{2u} + f'(x)e^u$$

$$= f''(x)x^2 + f'(x)x \quad \Rightarrow \text{same } \#$$

lecture 14.

Directional Derivatives:

motivation : mountain climbing



Recall:

$\frac{\partial f}{\partial x}(a,b)$ = slope of cross-sec in x-direc as you walk east.

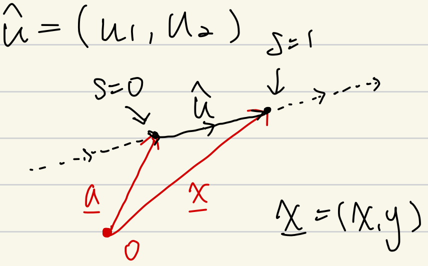
$\frac{\partial f}{\partial y}(a,b)$ = slope of cross-sec in y-direc as you walk north.

Q: What about slope in other direction? i.e. northeast?

given a point $\underline{a} = (a,b)$ and direction vector $\hat{u} = (u_1, u_2)$

a vector equation of a line through $\underline{a}$

in direction $\hat{u}$ is : $\underline{x} = \underline{a} + s\hat{u}$, $s \in \mathbb{R}$



Defn: the directional derivative of $f(x,y)$ at point $\underline{a} = (a,b)$

in the direction of a unit vector $\hat{u} = (u_1, u_2)$ is

$$D_{\hat{u}} f(\underline{a}) = \frac{d}{ds} f(\underline{a} + s\hat{u}) \Big|_{s=0}$$

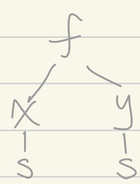
or $\frac{d}{ds} f(a+su_1, b+su_2) \Big|_{s=0}$

Theorem: If $f(x,y)$ is diff'able at $\underline{a}$,

then $D_{\hat{u}} f(\underline{a}) = \nabla f(\underline{a}) \cdot \hat{u}$, $\hat{u}$ is a unit vector

Proof: by defn,

$$D_{\hat{u}} f(\underline{a}) = \frac{d}{ds} f(\underbrace{a+su_1}_{x(s)}, \underbrace{b+su_2}_{y(s)}) \Big|_{s=0} \Rightarrow$$



$$= \frac{\partial f}{\partial x}(a+su_1, b+su_2) \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y}(a+su_1, b+su_2) \cdot \frac{dy}{ds} \Big|_{s=0}$$

$$= \frac{\partial f}{\partial x}(a,b) \cdot u_1 + \frac{\partial f}{\partial y}(a,b) \cdot u_2$$

$$= \left(\frac{\partial f}{\partial x}(\underline{a}), \frac{\partial f}{\partial y}(\underline{a})\right) \cdot (u_1, u_2) = \nabla f(\underline{a}) \cdot \hat{u}$$

#

Comments:

① if $\hat{u} = (1,0)$, then $D_{\hat{u}} f(\underline{a}) = \nabla f(a,b) \cdot (0,1)$

$$= (f_x(a,b), f_y(a,b)) \cdot (0,1) = f_x(a,b)$$

$\hat{u} = (0,1)$, $D_{\hat{u}} f(\underline{a}) = \dots = f_y(a,b)$

② If direction is given as non-unit vector $\underline{u}$, must normalize: $\frac{\underline{u}}{\|\underline{u}\|}$

③ Defn. and thm. generalize easily for fcn. of n variables.

④ if $f(x,y)$ not diff'able at (a,b), then must use defn.

ex: $f(x,y) = \frac{x}{x^2+y^2}$, calculate directional deriv. at $(2,0)$

in direction vec. $\underline{u} = (1,1)$

Sol: $\frac{\partial f}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = -x(x^2+y^2)^{-2}(2y) = \frac{-2xy}{(x^2+y^2)^2}$

cts. at $(2,0)$ by cty. thm.

$\Rightarrow$ diff'able.

So by theorem,

Aside: $\hat{u} = \frac{(1,1)}{\|(1,1)\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$D_{\hat{u}} f(2,0) = \nabla f(2,0) \cdot \hat{u}$$

$$= \left(\frac{0^2 - 2^2}{(2^2+0^2)^2}, 0\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{-1}{4\sqrt{2}} + 0 = \frac{-1}{4\sqrt{2}}$$

ex: find directional derivative of $f(x,y,z) = x^2 \cos z + e^y$

in direction $(-1,1,-1)$ at $(1, \ln 2, 0)$

Sol: $f_x = 2x \cos z$

$f_y = e^y$

$f_z = -x^2 \sin z$

clearly all cts. $\Rightarrow$ diff'able by theorem, $D_{\hat{u}}f(1, \ln 2, 0) = \nabla f(1, \ln 2, 0) \cdot \hat{u}$

$$= (2, 2, 0) \cdot \frac{(-1, 1, -1)}{\|(-1, 1, -1)\|} \leftarrow \sqrt{3}$$

$$= 0.$$

Theorem: If f diff'able at (a, b) and $\nabla f(a, b) \neq (0, 0)$, then the largest value of $D_{\hat{u}}f(a, b)$ is $\|\nabla f(a, b)\|$, and occuss when $\hat{u}$ is in direction of $\nabla f(a, b)$

Theorem: If $f(x, y) \in C^1$ in a neighborhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthogonal to level curve $f(x, y) = k$ through (a, b)

same for 3-dim.

p.s. tangent plane $\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$

lecture (didn't go) 😊

Review: one-dimensional taylor-poly.

$f(x)$ degree 2 $\Rightarrow P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2$.

$= L_a(x) + \frac{1}{2!} f''(a)(x-a)^2$

and notice $P''_{2,a}(a) = f''(a)$

now consider 2-D case.

$P_{2,(a,b)}(x,y) = L_{(a,b)}(x,y) + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$

similarly, we can see that. $\frac{\partial^2 P_{2,(a,b)}}{\partial x^2} = 2A$

similarly, $\frac{\partial^2 P_{(a,b)}}{\partial x \partial y} = B$ and $\frac{\partial^2 P_{(a,b)}}{\partial y^2} = 2C$

Defn: Second degree taylor $P_{2,(a,b)}$ of $f(x,y)$ at (a,b) is

$P_{2,(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$+ \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \right]$

ex: Use taylor degree 2 to approximate $\sqrt{(0.95)^3 + (1.98)^3}$

$f(x,y) = \sqrt{x^3+y^3} \Rightarrow \nabla f(1,2) = (\frac{1}{2}, 2)$

$Hf(1,2) = \begin{bmatrix} \frac{11}{12} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ p.s. $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$P_{2,(1,2)}(x,y) = 3 + \frac{1}{2}(x-1) + 2(y-2) + \frac{1}{2}\left[\frac{11}{12}(x-1)^2 - \frac{2}{3}(x-1)(y-2)\right.$

$\left. + \frac{2}{3}(y-2)^2\right]$

thus, $P_{2,(1,2)}(0.95, 1.98) \approx 2.9359$.

lecture (still skipped)

Review 1-D taylor remainder:

Theorem: if $f''(x)$ exist on $[a, x]$ $\exists$ $c \in (a, x)$ s.t.

$$f(x) = f(a) + f'(a)(x-a) + R_{1,a}(x)$$

where $R_{1,a}(x) = \frac{1}{2} f''(c)(x-a)^2$.

For 2-D case:

Theorem:

If $f(x,y) \in C^2$ in some neighborhood $N(a,b)$ of (a,b)

then $\forall (x,y) \in N(a,b)$ $\exists$ a point (c,d) on the line segment joining (a,b) and (x,y) such that

$$\Rightarrow f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$

where $R_{1,(a,b)}(x,y) = \frac{1}{2}\left[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\right]$

$\hookrightarrow$ Corollary: If $f(x,y) \in C^2 \dots$ $\exists$ constant M s.t.

$$|R_{1,(a,b)}(x,y)| \le M \|(x,y) - (a,b)\|^2, \quad \forall (x,y) \in N(a,b)$$

Generalization:

Theorem:

If $f(x,y) \in C^{k+1}$ at each point on the line segment joining (a,b) and (x,y) then there exist a point (c,d) on the line between (a,b) and (x,y) s.t.

$$f(x,y) = P_{k,(a,b)}(x,y) + R_{k,(a,b)}(x,y), \quad \text{where}$$

$$R_{k,(a,b)}(x,y) = \frac{1}{(k+1)!}\left[(x-a)D_1 + (y-b)D_2\right]^{k+1} f(c,d).$$

corollary 1 : $\exists M > 0$ s.t. $|f(x,y) - P_{k,(a,b)}(x,y)| \leq M \|(x,y)-(a,b)\|^{k+1}$

corollary 2 : $$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_{k,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|^{k}} = 0$$

lecture. 19.