



## lecture 3

ex:  $\lim_{(x,y) \rightarrow (a,b)} \frac{2x^4 y^{\frac{2}{3}}}{x^6 + y^2}$ , Prove  $\lim$  DNE

Sol: try  $y = mx$ ,  $m \in \mathbb{R}$

$$\lim_{x \rightarrow 0} \frac{2x^4 (mx)^{\frac{2}{3}}}{x^6 + (mx)^2} = \lim_{x \rightarrow 0} \frac{2m^{\frac{2}{3}} x^{\frac{8}{3}}}{x^6 + m^2} = \frac{0}{m^2} = 0 \quad (m \neq 0)$$

If  $m=0 \Rightarrow \lim_{x \rightarrow 0} \frac{0}{x^6} = 0 \Rightarrow \lim = 0$  along all  $y=mx$

does not imply  $\lim_{(x,y) \rightarrow (a,b)} = 0$

$$\text{take } y=x^3, \Rightarrow \lim_{x \rightarrow 0} \frac{2x^4 (x^3)^{\frac{2}{3}}}{x^6 + x^6} = \frac{2x^6}{2x^6} = 1 \neq 0$$

thus,  $\lim$  DNE.

ex:  $\lim_{(x,y) \rightarrow (1,0)} \frac{x^2 - y - 1}{x + y - 1} \Rightarrow \text{Prove DNE}$

Sol: try lines  $y = m(x-1) \rightarrow (1,0)$

Shortcut: try  $y=0 \Rightarrow \lim_{x \rightarrow 1} \frac{x^2 - 0 - 1}{x + 0 - 1} = \lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{(x-1)} = 2$ .

try  $x=1 \Rightarrow \lim_{y \rightarrow 1} \frac{1^2 - y - 1}{1 + y - 1} = -1 \leftarrow \text{different}$

thus,  $\lim$  DNE

Proving that a limit exists.

Squeeze theorem : If  $\exists$  a fcn.  $B(x,y)$  s.t.

$|f(x,y) - L| \leq B(x,y)$  if  $(x,y)$  in some neighborhood of  $(a,b)$ , except possibly at  $(a,b)$ ,

and  $\lim_{(x,y) \rightarrow (a,b)} B(x,y) = 0$  then  $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$

Proof: let  $\varepsilon > 0$ . Since  $\lim_{(x,y) \rightarrow (a,b)} B(x,y) = 0$

we know  $\exists \delta > 0$  s.t.

$$|B(x,y) - 0| < \varepsilon \text{ whenever } 0 < \|(x,y) - (a,b)\| < \delta$$

Now,  $|f(x,y) - L| \leq B(x,y) \underset{\substack{\leftarrow \text{hypo} \\ \uparrow \text{by above.}}}{\leq} \varepsilon$  whenever  $0 < \|(x,y) - (a,b)\| < \delta$

So,  $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$  by def.

Ex: Show  $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2+y^2} = 0$

$$\text{Sol: consider } \left| \frac{2xy}{x^2+y^2} - 0 \right| = \frac{2xy|y|}{x^2+y^2} \leq \frac{2(x^2+y^2)|y|}{x^2+y^2}$$
$$= 2|y| = B(x,y),$$

Clearly  $\lim_{(x,y) \rightarrow (0,0)} 2|y| = 0$  so by squeeze thm.  $\lim = 0$ .

comment:  $\frac{2xy|y|}{x^2+y^2} \leq \frac{2x^2|y|}{x^2}$   
 $\rightarrow (x,y) \neq (0,0) \rightarrow x \neq 0 \downarrow \text{nope.}$



Ex: Evaluate  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 + x^2 + y^4 + y^2}{x^2 + y^2}$  or show DNE.

Sol: Along  $y = mx$ , we have.

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^4 + x^2 + (mx)^4 + (mx)^2}{x^2 + (mx)^2} = \lim_{x \rightarrow 0} \frac{x^2(x^2 + 1 + m^4 x^2 + m^2)}{x^2(1 + m^2)}$$

$$= \frac{1+m^2}{1+m^2} = 1, \text{ so limit } f(x,y) \text{ must } = 1 \text{ if it exists.}$$

$$\begin{aligned} \text{try squeeze} \Rightarrow |f(x,y) - 1| &= \left| \frac{x^4 + x^2 + y^4 + y^2}{x^2 + y^2} - 1 \right| \\ &= \left| \frac{x^4 + y^4}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2)^2 + (x^2 + y^2)^2}{x^2 + y^2} = \frac{2(x^2 + y^2)^2}{x^2 + y^2} \\ &= 2(x^2 + y^2) = B(x,y), \rightarrow 0 \text{ as } (x,y) \rightarrow (0,0) \end{aligned}$$

so  $\lim f(x,y) = 1$  by sqz thm.

## lecture 4

Last time: Sqz thm  $|f(x,y) - L| \leq B(x,y) \rightarrow 0$

- Common inequality tricks (Pg. 19)

$$|x+y| \leq |x| + |y| \quad (\Delta \text{ ineq})$$

$$(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4 \Rightarrow x^4 + y^4 \leq (x^2 + y^2)^2$$

$$|x||y| \leq \frac{x^2 + y^2}{2} \quad (\text{cosine ineq})$$

Continuity:

Defn: A fn.  $f(x,y)$  is continuous at  $(a,b)$  means.

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

$f$  is continuous in  $D \subseteq \mathbb{R}^2$  if it is cts. at every point in  $D$ .

$$\text{ex: } f(x,y) = \frac{x^4 + x^2 + y^4 + y^2}{x^2 + y^2}, \quad (x,y) \neq (0,0)$$

$$\text{we showed } \lim_{\rightarrow (0,0)} f(x,y) = 1$$

If we define  $f(0,0) = 1$  then  $f$  will be cts. at  $(0,0)$ .

$$\text{ex: Let } f(x,y) = \begin{cases} \frac{\sin(x^2 + 2y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ k, & (x,y) = (0,0) \end{cases}$$

Can we find  $k$  so that  $f$  is cts. at  $(0,0)$

$$\text{Sol: Consider } \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + 2y^2)}{x^2 + y^2}$$

P.S. try along "simple" line.

$$-y=0 : \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = 1 \quad (\text{prev})$$

$$-\infty = \lim_{y \rightarrow 0} \frac{\sin(2y^2)}{y^2} = \lim_{y \rightarrow 0} \frac{\sin(2y^2)}{2y^2} = 2. \text{ (different)}$$

Since  $1 \neq 2$ ,  $\lim f(x,y)$  DNE.

Continuity Theorem:

Let  $f(x,y), g(x,y)$  be cts. at  $(a,b)$ . Then

- ①  $f+g$  and  $fg$  are also cts. at  $(a,b)$
- ②  $f/g$  is cts at  $(a,b)$ , provided  $g(a,b) \neq 0$

Continuity composition theorem:

Let  $f$  be a single var. fcn. and  $g$  be a 2-var. fcn.

If  $g$  is cts. at  $(a,b)$  and  $f$  is cts at  $g(a,b)$ ,

then  $f(g(x,y))$  is cts. at  $(a,b)$

Ex:  $\frac{y \sin x - \cos y}{x^2 + y^2}$  is cts.  $\forall (x,y)$  except for  $(0,0)$   
by cty. thm..

Ex:  $e^{x^3 - \sin(x,y)}$  is cts on  $\mathbb{R}$  by cty. thm (s).

$$\text{Ex: } \lim_{(x,y) \rightarrow (0,0)} \frac{e^{x^2} + \ln(2+y^2+x^4)}{(x-1)^2 + y^4} = ?$$

by cty. thm.  $f(x,y)$  is cts. at  $(0,0)$

thus, plug the fuck in  $(0,0) = 1 + \ln(2)$

Ex: determine where  $f(x,y) = \begin{cases} \frac{x^4 y^b}{x^b + y^12}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$



So for  $(x,y) \neq (0,0)$ ,  $f$  is cts. by cty. thm.

for  $(x,y) = (0,0)$ , use def of continuity.

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) \stackrel{?}{=} f(0,0)$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y^b}{x^b + y^{12}}, \text{ along } y = mx : \lim_{x \rightarrow 0} \frac{m^b x^{10} 4}{x^b (1 + m^b x^b)} = D$$

- try a sqz:  $\left| \frac{x^4 y^b}{x^b + y^{12}} - 0 \right| = \frac{x^4 y^b}{x^b + y^{12}} = \frac{(x^b + y^{12})^{\frac{4}{b}} (x^b + y^{12})^{\frac{1}{b}}}{x^b + y^{12}}$

$$= (x^b + y^{12})^{\frac{1}{b}} \rightarrow \text{as } (x,y) \rightarrow (0,0)$$

By sqz thm,  $\lim f(x,y) = 0$ , which  $= f(0,0)$

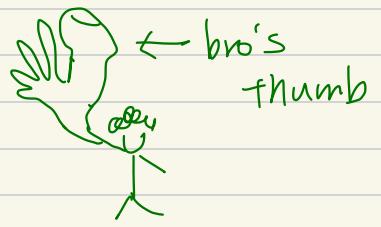
so  $f$  is cts. at  $(0,0)$

thus  $f$  is cts. at  $\mathbb{R}^2$ .

## lecture 6.

Two ways to differentiate  $f(x,y)$ :

- ① Treat  $y$  as fixed  $x$ :  $\frac{\partial f}{\partial x}$
- ② Treat  $x$  as fixed  $y$ :  $\frac{\partial f}{\partial y}$



$$\text{ex: } f(x,y) = y^2 \sin(xy)$$

$$\frac{\partial f}{\partial x} = y^3 \cos(xy), \quad \frac{\partial f}{\partial y} = 2y \cdot \sin(xy) + y^2 \cos(xy) \cdot x$$

$$\text{at a point: } \frac{\partial f}{\partial x}(\pi, 1) = \left. \frac{\partial f}{\partial x} \right|_{(\pi, 1)} = 1^3 \cdot \cos(\pi) = -1.$$

Subscript notation:  $\frac{\partial f}{\partial x} \equiv f_x$   
 $\frac{\partial f}{\partial y} \equiv f_y$

Formal Defn: the partial derivatives of  $f(x,y)$  at  $(a,b)$  are:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

ex: find "the partial" at  $(0,0)$  of

$$f(x,y) = \begin{cases} \frac{x^3 + y^4}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol: use defn.

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h^3 + 0^4}{h^2 + 0^2} - 0}{h} = 1$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0^3 + h^4}{0^2 + h^2} - 0}{h} = 0$$

Comment: MUST use defn. when "usual" rules

(product, power, quotient, ...) don't apply

ex. page 31~32.

$$f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{3} (x^3 + y^3)^{-\frac{2}{3}} (3x^2), \text{ for } x^3 + y^3 \neq 0$$

what if  $x^3 + y^3 = 0$ ? i.e. if  $y = -x$ ? i.e. at point  $(a, -a)$

above formula not allowed, must use defn.

ex! volume of a ideal gas.

$$V = \frac{82.06T}{P} \quad \begin{matrix} \leftarrow \text{temp. (K)} \\ \leftarrow \text{pressure (a.t.m)} \end{matrix}$$

Find rate of change of volume:

(a) w.r.t T

when  $T = 300\text{K}$  and  $P = 5\text{ a.t.m.}$

(b) w.r.t P

$$\text{Sol: (a)} \quad \frac{\partial V}{\partial T} = \frac{82.06}{P} \Rightarrow \frac{\partial f}{\partial T} \Big|_{(300,5)} \approx 16.41 \frac{\text{cm}^3}{\text{K}}$$

$$\text{(b)} \quad \frac{\partial V}{\partial P} = (82.06T) \cdot (-P^{-2}) \Rightarrow \frac{\partial f}{\partial P} \Big|_{(300,5)} \approx -98472 \frac{\text{cm}^3}{\text{a.t.m.}}$$

Second Derivatives: of  $f(x,y)$

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1 D_1 f = D_1^2 f \quad \left| \begin{array}{l} D_1 = \frac{\partial}{\partial x} \end{array} \right.$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2 D_1 f \quad \left| \begin{array}{l} D_2 = \frac{\partial}{\partial y} \end{array} \right.$$

$$\left( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1 D_2 f \right)$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = D_2 D_2 f$$

ex:  $f(x,y) = x e^{-xy}$

$$f_x = 1 \cdot e^{-xy} - y x e^{-xy} = e^{-xy}(1 - xy)$$

$$f_y = -x^2 e^{-xy}$$

$$f_{xx} = (-y)e^{-xy}(1 - xy) + e^{-xy}(-y)$$

$$f_{xy} = -xe^{-xy}(1 - xy) + e^{-xy}(-x) = e^{-xy}(-2x + x^2 y)$$

$$f_{yx} = -2xe^{-xy} + e^{-xy}(x^2 y) = e^{-xy}(-2x + x^2 y) \leftarrow \text{equal!}$$

$$f_{yy} = \dots$$

Clairaut's theorem: If  $f_x, f_y, f_{xy}$ , and  $f_{yx}$  are all defined in some neighborhood of  $(a,b)$ , and  $f_{xy}$  and  $f_{yx}$  are cts. on  $(a,b)$  then,  $f_{xy} = f_{yx}$ .

## Lecture 7, Tangent Plane, Linear approximation (43, 44)

But first, more on partial derivatives.

3rd partials of  $f(x, y)$ ,  $f_{xx}$ ,  $f_{xxx}$ ,  $f_{xyy}$  ... (8 ways)

For  $f(x, y, z)$ :  $f_x = \frac{\partial f}{\partial x}$

$f \in C^k$  :  $\Rightarrow$   $f$  is of class  $C-k'$

... means that the  $k^{th}$  partial derivatives are continuous.

e.g.  $f(x, y) \in C^2$  means that  $f_{xx}, f_{xy}, f_{yx}, f_{yy}$  are all GTS.

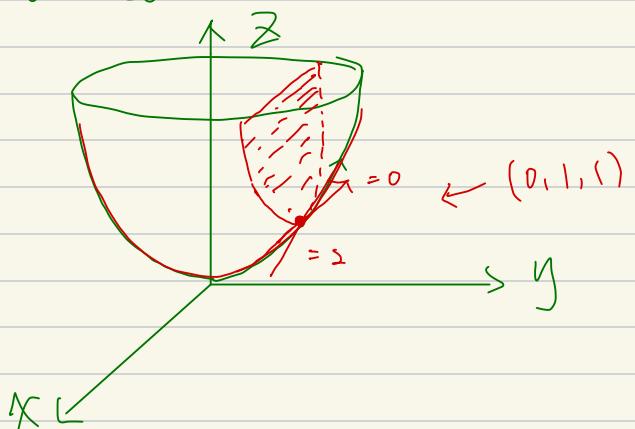
Geometric interp. of  $f_x$  and  $f_y$ :

$$\text{ex: } f(x, y) = x^2 + y^2$$

↓ Slope of cross sections.

$$\text{a) } f_x(0, 1) = (2x + 0)|_{(0, 1)} = 0$$

$$f_y(0, 1) = (0 + 2y)|_{(0, 1)} = 2.$$



Equation of tangent plane to  $z = f(x, y)$  at  $(a, b)$ .

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Ex: find equation of tangent plane to the surface

$$z = \frac{xy}{x^2 + y^2} \text{ at } (x, y) = (1, 2), \quad f(1, 2) = \frac{2}{5}$$

$$f_x = \frac{(x^2 + y^2) - y - xy(2x)}{x^2 + y^2} \Rightarrow \dots \Rightarrow f_x(1, 2) = \frac{6}{25}$$

$$f_y(1, 2) = \frac{-3}{25}$$

$$\text{plugin: } z = \frac{2}{5} + \frac{6}{25}(x-1) - \frac{3}{25}(y-2)$$

Defn: Linearization of  $f(x,y)$  at  $(a,b)$  is

$$L(a,b)(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Defn: Linear approximation.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

for  $(x,y)$  "near"  $ab$

Ex: approximate  $\sqrt{3.01^2 + 3.98^2}$

Sol: let  $f(x,y) = \sqrt{x^2+y^2}$ , and approximate near  $(3,4)$

$$f(x,y) \approx f(3,4) + \frac{\partial f}{\partial x}(3,4)(x-3) + \frac{\partial f}{\partial y}(3,4)(y-4)$$

$$\text{Now, } f(3,4) = 5,$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) \Big|_{(3,4)} = \frac{x}{\sqrt{x^2+y^2}} \Big|_{(3,4)} = \frac{3}{5}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \Big|_{(3,4)} = \frac{4}{5}$$

So, linear approximation becomes.

$$f(x,y) \approx 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \text{ for } (x,y) \text{ near } (3,4)$$

$$\Rightarrow \sqrt{3.01^2 + 3.98^2} \approx 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.98-4)$$

$$= 5 + 0.006 - 0.016 = 4.99 \quad (\text{actual value } 4.99004)$$

## lecture 8

Last time :

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) \rightarrow \text{① linear approx}$$

$$\Rightarrow f(x,y) - f(a,b) \approx \left( \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right) \cdot (x-a, y-b) \quad \text{②}$$

Δf      ∇f(a,b)  
= gradient vector

$$\begin{aligned}
 &= (x,y) - (a,b) \\
 &= x-a \\
 &= \Delta x
 \end{aligned}
 \quad \left| \begin{array}{l} x=(x,y) \\ a=(a,b) \end{array} \right.$$

$$\Rightarrow \Delta f \approx \nabla f(x,y) \cdot \Delta x \quad \text{Increment form of linear approx'}$$

Higher dimensions :

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$$

$$\Rightarrow f(x,y,z) - f(a,b,c) \approx (f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)) \cdot (x-a, y-b, z-c)$$

$$\Rightarrow \Delta f \approx \nabla f(a) \cdot \Delta x \quad \text{where } a = (a,b,c), x = (x,y,z) \\ \Delta x = x-a.$$

Ex: Let  $h(x,y,z) = x,y,z$ , and  $a = (1,2,3)$

what is the approx. change in  $h$  if  $x$  increase by 0.01,

$y$  increase by 0.02 and  $z$  decreases by 0.01?

$$\text{Sol: } \Delta h \approx \nabla h(1,2,3) \cdot (0.01, 0.02, 0.03)$$

$$\left( \nabla h = (h_x, h_y, h_z) = (yz, xz, xy) \right)$$

$$= (6, 3, 2) \cdot (0.01, 0.02, 0.03)$$

$$= 0.1 \Rightarrow \Delta h \approx 0.1$$

Ex: try to approx  $f(0.1, 0.1)$  where.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Sol: use linear approx.

$$f(x, y) \approx f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0)$$

$$\text{now } f(0, 0) = 0$$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

$$\text{by symmetry, } f_y(0, 0) = 0.$$

$$\Rightarrow f(x, y) \approx 0 + 0 + 0 = 0$$

$$\Rightarrow f(0.1, 0.1) \approx 0 \quad (\text{Actual value: } 0.5)$$

comment: the fcn. is not cts. at  $(0, 0)$ , yet  $f_x(0, 0)$  and  $f_y(0, 0)$  exist.

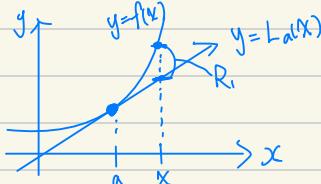
- Need a careful defn. for differentiability.

Flashback: math 137/138.

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{or} \quad f(x) = \underbrace{f(a) + f'(a)(x-a)}_{L_a(x)} + \overbrace{R_{1,a}(x)}^{\text{Remainder}}$$

$L_a(x)$  1st degree taylor poly.

Theorem: If  $f'(a)$  exists then  $\lim_{x \rightarrow a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$



Key idea: Remainder  $\rightarrow 0$  faster than the distance from  $x$  to  $a$

$$\text{Proof: } \left| \frac{R_{1,a}(x)}{x-a} \right| = \left| \frac{f(x) - (f(a) + f'(a)(x-a))}{x-a} \right|$$

$$= \left| \frac{f(x) - f(a)}{x-a} - \frac{f'(a)(x-a)}{(x-a)} \right| = |f'(a) - f'(a)| = 0 \text{ as } a \rightarrow 0.$$

Defn:  $f(x,y)$  is **differentiable** at  $(a,b)$  means.

$$\lim_{(x,y) \rightarrow (a,b)} \frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

$$\text{where } R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$$

$$= f(x,y) - (f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b))$$

$$\text{and } \|(x,y)-(a,b)\| = \|(x-a, y-b)\| = \sqrt{(x-a)^2 + (y-b)^2}$$

## lecture 9, 10

**Theorem:** If a fn satisfies

$$\lim_{(x,y) \rightarrow (a,b)} \frac{|f(x,y) - f(a,b) - c(x-a) - d(y-b)|}{\|(x,y) - (a,b)\|} = 0$$

then,  $c = f_x(a,b)$  and  $d = f_y(a,b)$

Defn, **tangent Plane** of surface  $z = f(x,y)$  at  $(a,b, f(a,b))$

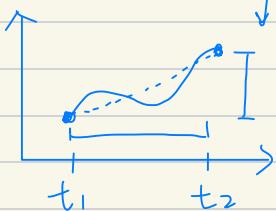
$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

**Theorem:** If  $f(x,y)$  is diff'able at  $(a,b)$ , then  
 $f$  is cts. at  $(a,b)$ .

**Theorem:** MVT.

If  $f(t)$  is cts on  $[t_1, t_2]$  and  $f$  is diff'able on  $(t_1, t_2)$ , then  $\exists t_0 \in (t_1, t_2)$  s.t.

$$f(t_2) - f(t_1) = f'(t_0)(t_2 - t_1)$$



**Theorem:** If  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are cts at  $(a,b)$ , then  
 $f(x,y)$  diff'able at  $(a,b)$ .

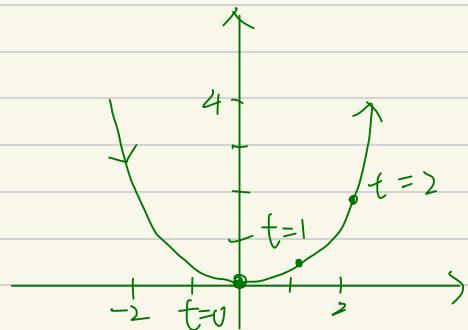
## Lecture 11

### Review of parameter / vector curves

ex:  $\mathbf{X}(t) = (t, t^2)$

Parametric form:

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$$



eliminate  $y = t^2 = (x)^2$  since  $x = t \Rightarrow y = x^2$  (lies on  $y = x^2$ )

If  $\mathbf{X}(t)$  = position of particle at time  $t$

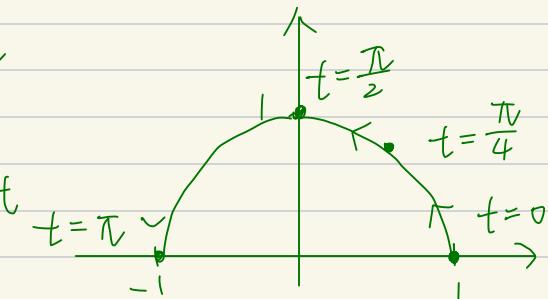
then  $\mathbf{X}'(t) = (1, 2t)$  is velocity at time  $t$ , e.g.  $\mathbf{X}'(1) = (1, 2)$

and  $\mathbf{X}''(t) = (0, 2)$  is acceleration at time  $t$ .

ex:  $\mathbf{X}(t) = (\cos(t), \sin(t)) \quad 0 \leq t \leq \pi$   
 $x(t) \uparrow \quad y(t) \uparrow$

eliminate  $t$ :  $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

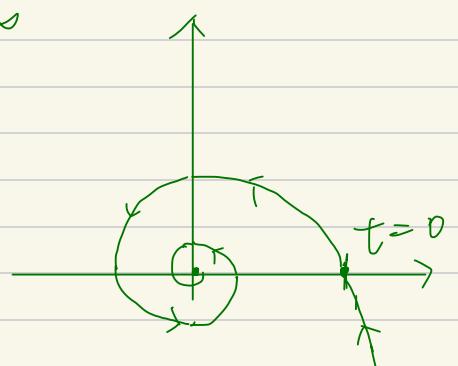
so curve lies on circle.



ex:  $\mathbf{X}(t) = (e^{-t} \cos(t), e^{-t} \sin(t)) \quad -\infty < t < \infty$

try to eliminate  $t$ :

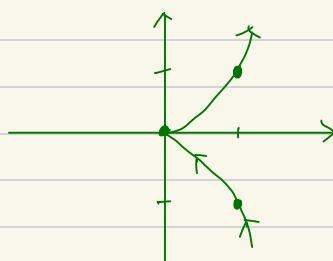
$$\begin{aligned} x^2 + y^2 &= e^{-2t} (\sin^2(t) + \cos^2(t)) \\ &= e^{-2t} \Rightarrow \text{"circle" with dec' radius.} \end{aligned}$$



ex:  $\mathbf{X}(t) = (t^2, t^3)$

$$y = t^3 = (\pm \sqrt{x})^3 = \pm x^{3/2}$$

$$\text{or } x = t^2 = (y)^{2/3}$$

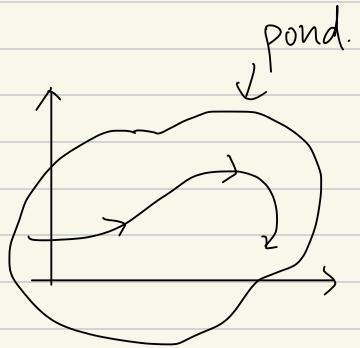


Chain Rule:

Imagine a duck position:  $\mathbf{x}(t) = (x(t), y(t))$

Let  $f(x, y) =$  temperature of pond at  $(x, y)$

find r.o.c of temperature experienced by duck.



$\Rightarrow$  in time  $\Delta t$ , duck's position changes by

$$\Delta \mathbf{x} = \mathbf{x}(t + \Delta t) - \mathbf{x}(t)$$

$$\Delta y = y(t + \Delta t) - y(t)$$

by increment form of linear approx.

$$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \Rightarrow \frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x} \frac{x(t + \Delta t) - x(t)}{\Delta t} + \frac{\partial f}{\partial y} \frac{y(t + \Delta t) - y(t)}{\Delta t}$$

$$\text{Let } t \rightarrow 0 : \frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \quad \text{chain Rule for } f(x(t), y(t))$$

Comments :

- $\frac{df}{dt}$  means  $\frac{d}{dt} f(x(t), y(t))$

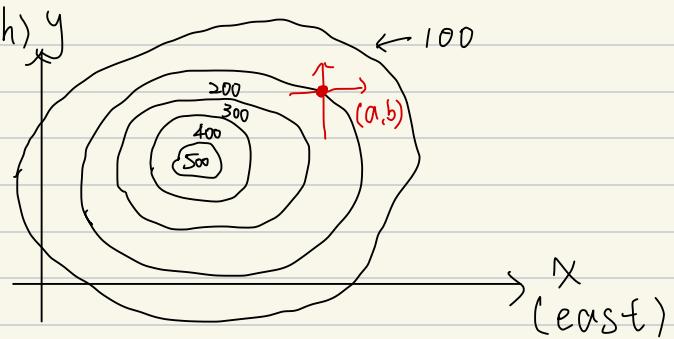
- this is not proof  $\rightarrow$

# lecture 14.

## Directional Derivatives:

motivation: mountain climbing

Recall:



$\frac{\partial f}{\partial x}(a, b)$  = slope of cross-sec in x-direc as you walk east.

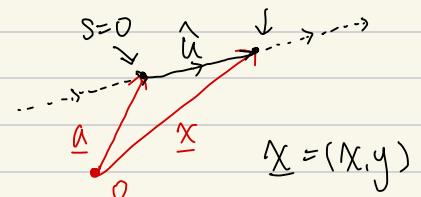
$\frac{\partial f}{\partial y}(a, b)$  = slope of cross-sec in y-direc as you walk north.

Q: What about slope in other direction? i.e. northeast?

given a point  $\underline{a} = (a, b)$  and direction vector  $\hat{u} = (u_1, u_2)$

a vector equation of a line through  $\underline{a}$

in direction  $\hat{u}$  is:  $\underline{x} = \underline{a} + s\hat{u}, s \in \mathbb{R}$



Defn: the directional derivative of  $f(x, y)$  at point  $\underline{a} = (a, b)$

in the direction of a unit vector  $\hat{u} = (u_1, u_2)$  is

$$D_{\hat{u}} f(\underline{a}) = \left. \frac{d}{ds} f(\underline{a} + s\hat{u}) \right|_{s=0}$$

$$\text{or } \left. \frac{d}{ds} f(a + su_1, b + su_2) \right|_{s=0}$$

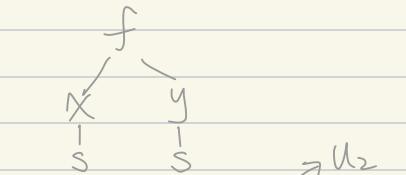
Theorem: If  $f(x, y)$  is diff'able at  $\underline{a}$ ,

then  $D_{\hat{u}} f(\underline{a}) = \nabla f(\underline{a}) \cdot \hat{u}$ ,  $\hat{u}$  is a unit vector

Proof: by defn,

$$D_{\hat{u}} f(\underline{a}) = \left. \frac{d}{ds} f(a + su_1, b + su_2) \right|_{s=0}$$

$$\begin{aligned} &= \frac{\partial f}{\partial x}(a + su_1, b + su_2) \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y}(a + su_1, b + su_2) \cdot \frac{dy}{ds} \Big|_{s=0} \\ &= \frac{\partial f}{\partial x}(a, b) \cdot u_1 + \frac{\partial f}{\partial y}(a, b) \cdot u_2 \\ &= \left( \frac{\partial f}{\partial x}(\underline{a}), \frac{\partial f}{\partial y}(\underline{a}) \right) \cdot (u_1, u_2) = \nabla f(\underline{a}) \cdot \hat{u} \end{aligned}$$



$u_2$

Comments:

① If  $\hat{u} = (1, 0)$ , then  $D_{\hat{u}} f(a) = \nabla f(a, b) \cdot (0, 1)$

$$= (f_x(a, b), f_y(a, b)) \cdot (0, 1) = f_x(a, b)$$

$\hat{u} = (0, 1)$ ,  $D_{\hat{u}} f(a) = \dots = f_y(a, b)$

② If direction is given as non-unit vector  $u$ , must normalize:  $\frac{u}{\|u\|}$

③ Defn. and thm. generalize easily for fcn. of  $n$  variables.

④ If  $f(x, y)$  not diff'able at  $(a, b)$ , then must use defn.

Ex:  $f(x, y) = \frac{x}{x^2+y^2}$ , calculate directional deriv. at  $(2, 0)$

in direction vec.  $u = (1, 1)$

$$\begin{aligned} \text{Sol: } \frac{\partial f}{\partial x} &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \left. \begin{array}{l} \text{cts. at } (2, 0) \\ \text{by} \\ \text{cty. thm.} \end{array} \right\} \\ \frac{\partial f}{\partial y} &= -x(x^2+y^2)^{-2}(2y) = \frac{-2xy}{(x^2+y^2)^2} \quad \Rightarrow \text{diff'able.} \end{aligned}$$

So by theorem,

$$\begin{aligned} D_{\hat{u}} f(2, 0) &= \nabla f(2, 0) \cdot \hat{u} \\ &= \left( \frac{0^2-2^2}{(2^2+0^2)^2}, 0 \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \\ &= \frac{-1}{4\sqrt{2}} + 0 = \frac{-1}{4\sqrt{2}} \end{aligned}$$

$$\left. \begin{array}{l} \text{Aside: } \hat{u} = \frac{(1, 1)}{\|(1, 1)\|} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \end{array} \right\}$$

ex. find directional derivative of  $f(x, y, z) = x^2 \cos z + e^y$

In direction  $(-1, 1, -1)$  at  $(1, \ln 2, 0)$

$$\text{Sol: } f_x = 2x \cos z$$

$$f_y = e^y$$

$$f_z = -x^2 \sin z$$

$\left. \begin{array}{l} \text{clearly all cts.} \Rightarrow \text{diff'able} \end{array} \right\}$

$$\begin{aligned} \text{by theorem, } D_{\hat{u}} f(1, \ln 2, 0) &= \nabla f(1, \ln 2, 0) \cdot \hat{u} \\ &= (2, 2, 0) \cdot \frac{(-1, 1, -1)}{\|(-1, 1, -1)\|} \leftarrow \sqrt{3} \\ &= 0. \end{aligned}$$

