Linear Algebra

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- We know how to solve $Ax = 0$ —> Elimination converted the problem to $Rx = 0$
  - The free variables were given special values (1 and 0)
  - then the pivot variables were found by back substitution
- We didn't care about the right side b because it is 0 —> the solution was in the nullspace of A
- Now b is not 0 —> row operations on the left side must act also on the right side
- One way to organize that is to add b as an extra column of the matrix

The Complete Solution to $Ax = b$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{6} \\ \mathbf{7} \end{bmatrix} \quad \text{has the augmented matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 1 & 3 & 1 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$$

- When we apply the usual elimination steps to A, we also apply them to b
- In this example we subtract row 1 from row 3 and then subtract row 2 from row 3
- This produces a complete row of *zeros*:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{6} \\ \mathbf{0} \end{bmatrix} \quad \text{has the augmented matrix} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}.$$

The Complete Solution to $Ax = b$

- The very last zero is crucial —> it means that the equations can be solved; the third equation has become $0 = 0$
- In the original matrix A, the first row plus the second row equals the third row
- If the equations are consistent —> this must be true on the right side of the equations also
- Here are the same augmented matrices for a general $b = (b_1, b_2, b_3)$

$$\begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} = [R \ d]$$

- The third equation is $0 = 0$ only if $b_3 = b_1 + b_2$

One Particular Solution

- Choose the free variables to be $x_2 = x_4 = 0$ —> then the equations give the pivot variables $x_1 = 1$ and $x_3 = 6$
- Our particular solution to $Ax = b$ is $x_p = (1, 0, 6, 0)$
- The process starts with reducing $[A \;\; b]$ to $[R \;\; d]$

For a solution to exist

Zero rows in R must also be zero in d. Since I is in the pivot rows and pivot columns of R, the pivot variables in x_p come from d

- After row reduction we are just solving $Ix = d$
- Notice how we choose the free variables (as zero) and solve for the pivot variables

The Complete Solution to $Ax = b$

- After the row reduction to R, these steps will be quick
- When the free variables are zero —> the pivot variables for x_p are already seen in the right side vector d

The particular solution solves	$Ax_p = b$
The $n - r$ special solution solves	$Ax_n = 0$

- The two special (nullspace) solutions to $Rx = 0$ come from the two free columns of R, by reversing signs of $3, 2$, and 4.
- $x = x_p + x_n$ is known as the complete solution to $Ax = b$

Complete solution
one x_p
many x_n

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Q: Suppose A is a square invertible matrix, $m = n = r$, what are x_p and x_n?

A:

- The particular solution is the one and only solution $x_p = A^{-1}b$
- There are no special solutions or free variables —> $R = I$ has no zero rows
- The only vector in the nullspace is $x_n = 0$
- The complete solution is $x = x_p + x_n = A^{-1}b + 0$

The Complete Solution to $Ax = b$

- If A was invertible, then the nullspace $N(A)$ contained only the zero vector
- Reduction went from $[A \quad b]$ to $[I \quad A^{-1}b]$
- The matrix A was reduced all the way to I
- Then $Ax = b$ became $x = A^{-1}b = d$

Ex 1: Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Reduction:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & \boldsymbol{b_3 + b_1 + b_2} \end{bmatrix}$$

The Complete Solution to $Ax = b$

- The last equation is $0 = 0$ provided $b_3 + b_1 + b_2 = 0$
- This is the condition to put b in the column space —> then $Ax = b$ will be solvable
- There is no free variables since $n - r = 2 - 2$ —> no special solutions
- The nullspace solution is $x_n = 0$
- The particular solution to $Ax = b$ and $Rx = d$ is at the top of the final column d:

$$\text{Only solution to } Ax = b \text{ is } x = x_p + x_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The previous example is typical of the extremely important case when A has full column rank —> every column has a pivot
- The rank is $r = n$ —> the matrix is tall and thin $(m \geq n)$
- Row reduction puts I at the top, when A is reduced to R:

$$\textbf{Full column rank} \quad R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix}$$

- There are no free columns or free variables —> the nullspace is zero vector

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Every matrix A with full column rank $(r = n)$ has all these properties:

1. All columns of A are pivot columns
2. There are no free variables or special solutions
3. The nullspace $N(A)$ contains only the zero vector $x = 0$
4. If $Ax = b$ has a solution (it might not) then it has only one solution

The Complete Solution

- The other extreme case is full row rank
- Now either has one or infinitely many solutions
- In this case A is short and wide $(m \leq n)$ —> the number of unknowns is at least the number of equations
- A matrix has full row rank if $r = m$ —> every row has a pivot

Ex 2: Given the following systems of linear equations

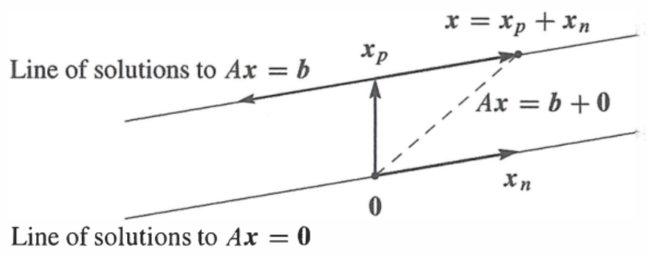
$$\begin{array}{rcrcrl} x & + & y & + & z & = 3 \\ x & + & 2y & - & z & = 4 \end{array}$$

Find the rank of coefficient matrix. Discuss about the solution of the given system

- There are two planes in xyz space
- The planes are not parallel, so they intersect in a line
- This line of solutions is what elimination will find
- The particular solution will be one point on the line
- Adding the nullspace vectors x_n will move us along the line in the Figure

The Complete Solution

- $x = x_p + x_n$ gives the whole line of solutions:



The Complete Solution

- We find x_p and x_n by elimination on $[A \ \ b]$

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{3} \\ 1 & 2 & -1 & \mathbf{4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & \mathbf{3} \\ 0 & 1 & -2 & \mathbf{1} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & \mathbf{2} \\ 0 & 1 & -2 & \mathbf{1} \end{bmatrix} = \left[\ R \quad \boldsymbol{d} \ \right]$$

- The particular solution (x_p) has free variable $x_3 = 0$
  - x_p comes directly from d on the right side: $x_p = (2, 1, 0)$
- The special solution (s) has $x_3 = 1$
  - s comes directly from the third column (free column) of R: $s = (-3, 2, 1)$
- It is wise to check that x_p and s satisfy the original equations $Ax_p = b$ and $As = 0$
- The nullspace solution x_n is any multiple of s —> it moves along the line of solutions

The Complete Solution

- Please notice how to write the answer:

Complete solution $$x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

- Any point on the line could have been chosen as the particular solution —> we choose the point with $x_3 = 0$
- The particular solution is NOT multiplied by an arbitrary constant

The Complete Solution

Every matrix A with full row rank $(r = m)$ has all these properties:

1. All rows have pivots, and R has no zero rows
2. $Ax = b$ has a solution for every right side b
3. The column space is the whole space $\mathbb{R}^m$
4. There are $n - r = n - m$ special solutions in the nullspace of A

- In this case with m pivots, the rows are "linearly independent"
- So the columns of A^T are "linearly independent" —> The nullspace of A^T is the zero vector

The Complete Solution

The four possibilities for linear equations depend on the rank r :

$\boldsymbol{r}=\boldsymbol{m}$	and	$\boldsymbol{r}=\boldsymbol{n}$	*Square and invertible*	$A\boldsymbol{x}=\boldsymbol{b}$	has 1 solution
$\boldsymbol{r}=\boldsymbol{m}$	and	$r<n$	*Short and wide*	$A\boldsymbol{x}=\boldsymbol{b}$	has ∞ solutions
$r<m$	and	$\boldsymbol{r}=\boldsymbol{n}$	*Tall and thin*	$A\boldsymbol{x}=\boldsymbol{b}$	has 0 or 1 solution
$r<m$	and	$r<n$	*Not full rank*	$A\boldsymbol{x}=\boldsymbol{b}$	has 0 or ∞ solutions

- The reduced R will fall in the same category as the matrix A
- In case the pivot columns happen to come first, we can display these four possibilities for R as well:

Four types for R	$\begin{bmatrix} I \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$r=m=n$	$r=m<n$	$r=n<m$	$r<m, r<n$

- There are n columns in an m by n matrix, but the true "dimension" of the column space is not necessarily n
- The dimension is measured by counting independent columns
- We will see that the true dimension of the column space is the rank r
- The idea of independence applies to any vectors $v_1, \ldots, v_n$ in any vector space
- Most of this section concentrates on the subspaces that we know and use —> the column space in $\mathbb{R}^m$ and the nullspace in $\mathbb{R}^n$

- Previously, we study "vectors" that are not column vectors
  - they can be matrices and functions
  - they can be linearly independent or dependent
- The goal is to understand a basis for a vector space
- A basis contains independent vectors that "span the space"

The four essential ideas in this section are:

1. Independent vectors (not too many)
2. Spanning a space (not too few)
3. Basis for a space (not too many or too few)
4. Dimension of a space (the right number of vectors)

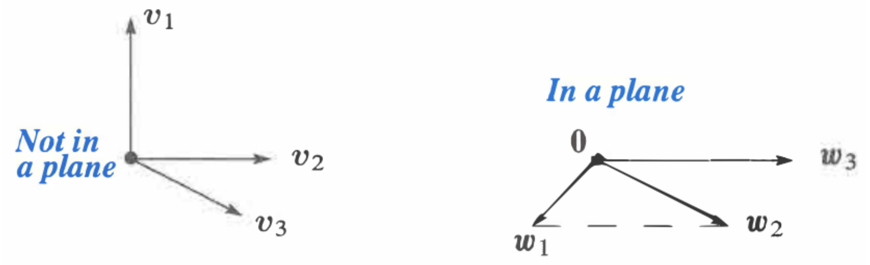
Linear Independence

Definition

The columns of A are linearly independent when the only solution to $Ax = 0$ is $x = 0$. —> No other combination Ax of the columns gives the zero vector

- With linearly independent columns, the nullspace $N(A)$ contains only the zero vector
- Consider a example of linear independence (and dependence) with three vectors in $\mathbb{R}^3$:
  - If three vectors are NOT in the same plane, they are independent. No combination of v_1, v_2, v_3 (see the following figure)
  - If three vectors w_1, w_2, w_3 are in the same plane, they are dependent

Linear Independence



- Suppose the vectors are the columns of A, and independent —> the nullspace only contains $x = 0$
- The following definition of independence will apply to any sequence of vectors in any vector space (they would mean the same thing as the previous definition)

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Definition

The sequence of vectors $v_1, \ldots, v_n$ is linearly independent if the only combination that gives the zero vector is $0v_1 + 0v_2 + \ldots + 0v_n$.
—> thus linear independent means that
$x_1 v_1 + x_2 v_2 + \ldots + x_n v_n = 0$ only happens when all x's are zero.

- If a combination gives 0, when the x's are not all zero —> the vectors are dependent
- **Correct language:** "The sequence of vectors is linearly independent"
  - **Acceptable**: "The vectors are independent"
  - **Unacceptable**: "The matrix is independent"

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- The key question is: Which combinations of the vectors give zero?
- Some small examples in $\mathbb{R}^2$:
  (a) The vector $(1,0)$ and $(0,1)$ are *independent*
  (b) The vectors $(1,0)$ and $(1,0.00001)$ are *independent*
  (c) The vector $(1,1)$ and $(-1,-1)$ are *dependent*
  (d) The vector $(1,1)$ and $(0,0)$ are *dependent* because of the zero vector

Ex 3: Given the matrix A, show that the columns of A are dependent.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

- The columns of A are dependent. —> $Ax = 0$ has a nonzero solution.

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ is } -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The rank is only $r = 2$

Q: How to find that solution to $Ax = 0$?

A: The systematic way is elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \text{ reduces to } \quad R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution $x = (-3, 1, 1)$ was exactly the special solution.

Full column rank

- The columns of A are independent exactly when the rank is $r = n$
- there are n pivots and no free variables
- Only $x = 0$ is in the nullspace

- Important fact
  - Suppose seven columns have five components each ($m = 5$ less than $n = 7$)
  - Then the columns must be dependent —> any seven vectors from $\mathbb{R}^5$ are dependent
  - The rank of A cannot be larger than 5 —> there cannot be more than five pivots in five rows
  - The system has at least $7 - 5 = 2$ free variables —> so it has nonzero solutions

Any set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$

- If $n \leq m$, the columns might be *dependent* or might be *independent*

Vectors that Span a Subspace

- The first subspace discussed previously was the column space
- Starting with columns $v_1 + ... + v_n$ —> the subspace was filled out by including all combinations $x_1 v_1 + ... + x_n v_n$
- The column space consists of all combinations Ax of all the columns
- The column space is spanned by the columns

Definition

A set of vectors spans a space if their linear combination fill the space

Vectors that Span a Subspace

- The columns of a matrix span its column space. They might be dependent:
  - $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full two-dimensional space $\mathbb{R}^2$
  - $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbb{R}^2$
  - $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbb{R}^2$
- Think of two vectors coming out from $(0, 0, 0)$ in $3-$dimensional space —> generally, they span a plane
- Other possibilities: (which might not independent)
  - two vectors span a line
  - three vectors span all of $\mathbb{R}^3$, or only a plane
  - three vectors span a line

- Here is a new subspace —> the combinations of the rows produce the "row space"

Definition

The row space of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.
The row space of A is $C(A^T)$ —> it is the column space of A^T

Ex 4: Given the matrix A, find the column space and row space of A

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}$$

- The column space of A is the plane in $\mathbb{R}^3$ spanned by the two columns of A
- The row space of A is spanned by the three rows of A in $\mathbb{R}^2$