A contribution on the problem of practical ergodicity of parametric roll in longitudinal long crested irregular sea

Gabriele BULIAN(*) (gbulian@units.it), Claudio LUGNI(**) (c.lugni@insean.it), Alberto FRANCESCUTTO(*) (francesc@units.it)
(*) DINMA, University of Trieste - (**) INSEAN, Rome

Abstract
This paper presents the preliminary results of an analytical, numerical and experimental study regarding the problem of possible non-ergodicity of parametric roll in longitudinal irregular long crested waves. The problem is tackled bearing in mind the concept of "practical ergodicity", that is the possibility of obtaining reliable information about ensemble averages by using temporal averages. Some general analytical tools are given to address the problem of accuracy of temporal averages. A series of numerical simulations are then performed by means of an analytical 1.5-DOF model of parametrically excited roll motion. The outcomes of such simulations are analysed to show the effect of ship speed and sea spectrum shape. The effect of wave grouping phenomenon is discussed with particular attention to the Doppler effect. Qualitative indications given by the numerical simulations are then compared with experimental tests showing a good agreement. Practical-ergodicity of generated sea in towing tank is also briefly addressed.

1. Introduction
In present seakeeping tests in irregular waves, the probabilistic qualities of ship motions are usually estimated by means of what are called "temporal averages". Although the statistical quantities of a stochastic process should be estimated, in principle, by means of "ensemble averages", the assumption of ergodicity allows a theoretically correct substitution of ensemble averaging with temporal averaging. The possibility of assuming ergodicity derives from the main assumptions of linear seakeeping theory: ship modelled as a linear system excited by correlated gaussian ergodic processes. When pitch and heave are of concern in a longitudinal long crested sea, the aforementioned assumptions can be considered quite good, and ergodicity of motion can be assumed, provided that sea elevation process is gaussian and ergodic (Belenky et al. 2001) . When roll, instead, is of concern, nonlinear effects come into play both regarding restoring and damping. Moreover, if we restrict our attention to the longitudinal sea, the build up of roll motion is due to a parametric excitation, and the build-up mechanism of roll motion is very different from that present in beam sea (at least when there is no presence of multiple solutions). Roll motion in longitudinal sea arises because of the unstabilization of the upright position due to a sufficiently large parametric excitation (large enough to exceed the stability threshold). Roll motion is therefore due to a continuous bifurcation process of stabilization/unstabilization of the upright condition with consequent inception of large amplitude rolling. The resulting stochastic roll motion arising from irregular parametric excitation is largely non-gaussian.

The problem of possible non-ergodicity of roll has been recently posed by Belenky et al. (Belenky et al. 1998; Belenky et al. 2001; Belenky et al. 2003). In principle, a process has to be considered as not ergodic until a proof of ergodicity is given. However, in order to give such a proof, the analytical knowledge of high-order correlation functions is needed, even in the frame of correlation theory, where only the stationarity of the mean and autocorrelation function are requested (Parkus 1969). In general such proof cannot be given for nonlinear systems that, in principle, should be considered as not ergodic (Belenky et al. 1998; Belenky et al. 2001; Belenky et al. 2003). However, even if a proof of ergodicity cannot be given, this does not mean that the process is necessarily non-ergodic.

Although ergodicity plays a foundational philosophical and still very debated role in the field of statistical mechanics (Badino 2003; Van Lith 2001), it is to be said that, in the field of seakeeping, the assumption of ergodicity can only be elevated to the rank of a more or less good and unavoidable approximation: "more or less good" due to the unavoidable lack of stationarity (Crandall 1973) for processes of interest of ship motion analysis,

"unavoidable" due to practical reasons related to numerical and/or experimental limitations.
We have thus decided to analyse the problem of ergodicity from a practical point of view, so dealing with what could be named "practical ergodicity" (Belenky et al. 2001), that is the possibility of obtaining reliable statistical information from physical/numerical experiments by means of temporal averages.
This paper presents a series of preliminary results concerning the attempt to give an answer to the question *"how good is the approach based on temporal averages?"* bearing in mind the practical limitations arising in real and numerical experiments. Although the parametrically excited roll motion is not gaussian, hereafter we will concentrate our attention on the analysis of the roll standard deviation estimator only as a measure of dispersion.
We will report some examples showing how the shape of the spectrum could influence the variance of the variance estimator for a gaussian process, using as a reference, the work of Naito and Kihara (1993).
The briefly described general theoretical approach reported at the beginning of the paper is applied to numerical simulations of parametric roll motion in longitudinal long crested irregular waves. In order to be able to develop a 1.5-DOF analytical model of roll motion, the Grim's effective wave concept (Grim 1961) will be used, as it is very useful when combined with preliminary hydrostatic calculations in a Monte Carlo approach.
Experiments confirm that, when time averages are used, as usual, as a means to obtain information about ensemble averages, the width of the intervals of confidence of time averages (connected with their standard deviation) can be very large and strongly depending on sea spectrum characteristics.

2. Theoretical background

2.1 Basic concepts

When the responses of a ship to a stochastic excitation (as it is the irregular sea) are of concern, we are dealing with stochastic processes. To be more precise, we are dealing with continuous stochastic processes, that is, stochastic processes that are continuous functions of the time (although we, usually, measure their value at discrete time instants).
A generic random process X is considered "determined" if its n-dimensional (n=1,2,3,...) probability density functions $f_n(x_1,...,x_n;t_1,...,t_n)$ are given.
In general, such probability density functions depend on the time lags, but a special type of processes exists that are called "stationary processes". For such special cases, the joint pdfs remain unchanged under a time shift, that is:

$$f_n(x_1,...,x_n;t_1,...,t_n) = f_n(x_1,...,x_n;.t_1+\Delta t,..,t_n+\Delta t)\ \forall \Delta t \tag{1}$$

A process for which (1) holds is said to be stationary in the strict sense, or "strictly stationary". The class of strictly stationary processes is a sub-class of a larger group of processes that are called "weakly stationary" processes. A process is said to be "weakly stationary" if (Parkus 1969):

- the mean value $E\{x(t)\}$ of the process is independent of the time t;
- the autocorrelation function $R(t,s) = E\{x(t)\cdot x(s)\}$ depends only on the time difference $\tau = s - t$;

The operator $E\{.\}$ is called "expectation" or "expected value". Let $(x_1, x_2,...,x_n)$ be a generic set of n random variables whose joint pdf is $f_n(x_1,...,x_n)$, and let $g(x_1,...,x_n)$ be a generic scalar function of the aforementioned random variables. The expected value of g, denoted as $E\{g(x_1,...,x_n)\}$ is defined as:

$$E\{g(x_1,...,x_n)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} g(x_1,...,x_n)\cdot f_n(x_1,...,x_n)dx_1 dx_2 ... dx_n \tag{2}$$

A strictly stationary process is always weakly stationary, but not vice-versa.
Very often it is almost impossible to have information on all the joint pdfs of the process, so we have to be satisfied with the knowledge of the mean and the autocorrelation function: the correlation theory is that part of the analysis of stochastic processes that deals only with mean and autocorrelation function. It must be noted here a very important property of gaussian processes, that are processes exhibiting joint gaussian pdfs (Parkus 1969): a gaussian process is completely determined if $E\{x(t)\}$ and $R(t,s)$ are known, since then all the joint pdfs $f_n(x_1,...,x_n;t_1,...,t_n)$ can be determined. This means that correlation theory is sufficient for the complete analysis of gaussian processes.

Unfortunately, many random processes are not gaussian, so, in principle, the correlation theory is not sufficient to fully describe such phenomena.

2.2 Estimation of statistical properties

In most cases we do not have the a-priori knowledge of the statistical properties of a process but we have to estimate them from a set of measurements. Let Y be the process under analysis and let $y_i(t)$ be the *i*-th realization of the process (e.g., the i-th experimental measure of the roll motion in irregular sea). It is possible to define two types of statistical estimators:

- estimators based on ensemble averages;
- estimators based on temporal averages;

When ensemble averages are used, the time instant is fixed, and estimation of statistical averages is performed in the ensemble domain, that is, by using the values of the process from each outcome at the same, fixed, time instant. Conversely, when temporal averages are used the realization index *i* is fixed, and estimation is performed over the time record.
As an example, the temporal and ensemble mean and mean square estimators are reported for a process Y, for which N realizations have been measured in a time interval $[0,T]$:

$$\begin{aligned}
&m(t)=\frac{1}{N}\sum_{i=1}^{N} y_i(t) \quad \text{Ensemble Mean}\\
&\mathrm{v}(t)=\frac{1}{N}\sum_{i=1}^{N}[y_i(t)]^2 \quad \text{Ensemble Mean Square}\\
&\langle \mathrm{y_i}\rangle=\frac{1}{T}\int_0^T y_i(t)dt \quad \text{Temporal Mean}\\
&\langle \mathrm{y_i^2}\rangle=\frac{1}{T}\int_0^T [y_i(t)]^2\,dt \quad \text{Temporal Mean Square}
\end{aligned} \tag{3}$$

In principle only ensemble averages can estimate statistical properties of the stochastic process under analysis, and temporal averages have little or no meaning if the process is not stationary. However, as mentioned in the introduction, there is a special class of processes for which temporal averages and ensemble averages lead to the same result: the so-called “ergodic” processes.

2.3 Ergodicity

A process is said to be ergodic when all the joint pdfs can be estimated from one single realization of the process having infinite time length. To be ergodic, a process must necessarily be stationary. If we stay in the framework of the correlation theory, it is sufficient that the mean value and the autocorrelation function can be obtained with infinite accuracy from an infinitely long realization. So, a process X is said to be ergodic in the framework of correlation theory if, given a particular realization $x(t)$, it is:

$$\begin{aligned}
&\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = E\{x\}\\
&\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot x(t+\tau)dt = E\{x(t)\cdot x(t+\tau)\} = R(\tau)
\end{aligned} \tag{4}$$

If (4) hold for a gaussian process, then all the joint pdfs can be obtained and the process if fully determined. A particular case of gaussian ergodic process is the output of a damped linear dynamic system with constant coefficients excited by an ergodic gaussian input: this is (very) approximately the case of ideal linear ship motions as pitch and heave.
Unfortunately the theoretical definition (4) poses some problems when the analysis of a real process has to be carried out:

- the realization of a real process must start and stop, so a real process cannot be truly stationary;
- even if we assume that the process has started well before the starting of measurement (so the process can be

treated as stationary), the saved record length will necessarily be limited, and thus we are sure that, in general, the results of temporal averages will not coincide with the true values of $E\{x\}$ and $R(\tau)$;

Thus from a practical point of view, the knowledge (or assumption) of ergodicity property for a particular process, tells us that, in principle, we can use temporal averages as theoretically correct estimators of the ensemble averages, but it does not give us any information on the accuracy of temporal estimates.

2.4 Accuracy of ensemble analysis

Before going to the analysis of the accuracy of temporal averages, let us introduce the case of the statistical moments estimation of a real random variable A. Suppose that A is a real random variable with a given probability density function $f_A(a)$, and suppose that we have measured N outcomes of this random variable with infinite accuracy. Each outcome a_i $(i = 1,\dots,N)$ is supposed to be independent on the other outcomes. Using the set of outcomes we want to estimate, for example, the mean value of A and its variance. The following unbiased estimators can be used:

$$\begin{aligned} &\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} a_i \quad \text{Estimator of the mean} \\ &\hat{v} = \frac{1}{N-1}\sum_{i=1}^{N}(a_i - \hat{\mu})^2 \quad \text{Estimator of the variance} \end{aligned} \tag{5}$$

Both $\hat{\mu}$ and $\hat{v}$ are random variables characterised by their own probability density functions. When the number of elements N is large, thanks to the central limit theorem, the pdfs of $\hat{\mu}$ and $\hat{v}$ approach gaussian distributions, so we can be satisfied with the knowledge of their expected values and standard deviations. From, e.g. Ang and Tang (1975), it is:

$$\begin{aligned} &E\{\hat{\mu}\} = \mu \\ &Var\{\hat{\mu}\} = \frac{\sigma^2}{N} \\ &E\{\hat{v}\} = \sigma^2 \\ &Var\{\hat{v}\} = \frac{\sigma^4}{N}\left(\frac{\mu_4}{\sigma^4} - \frac{N-3}{N-1}\right) \end{aligned} \tag{6}$$

being

$$\begin{aligned} &\mu = E\{A\} \\ &\sigma^2 = Var\{A\} = E\{(a-\mu)^2\} \\ &\mu_4 = E\{(a-\mu)^4\} \end{aligned}$$

A quantity called the "coefficient of variation" (not to be confused with the "covariance") can be defined as a measure of the variability of a random variable with non-zero expected value. Such coefficient is defined as the ratio between the standard deviation and the expected value of the variable under analysis. In the case of the variance estimator $\hat{v}$ such coefficient becomes:

$$CoV_{\hat{v}} = \sqrt{\frac{Var\{\hat{v}\}}{E^2\{\hat{v}\}}} = \sqrt{\frac{1}{N}\left(\frac{\mu_4}{\sigma^4} - \frac{N-3}{N-1}\right)} \tag{7}$$

In the special case of A being a gaussian process the fourth central moment can be expressed in terms of variance and so expression (7) reduces to:

$$\mu_4 = 3\sigma^4 \Rightarrow \mathrm{cov}_{\hat{v}} = \sqrt{\frac{2}{N-1}} \tag{8}$$

It can be seen from (6)-(8) that increasing the sample population N, leads to a decrease of the standard deviations associated to $\hat{\mu}$ and $\hat{v}$. Such decrease, for large N, is proportional, in both cases, to $N^{-1/2}$: the larger the population, the higher the accuracy of the estimators. In the limit of $N \to +\infty$, it is $\hat{\mu} \to \mu$ and $\hat{v} \to \sigma^2$ in a probabilistic sense.

When ensemble averages of a set of realizations of a stochastic process Y are carried out, the aforementioned principles can be used if a_i is replaced by $y_i(t)$, being i the realization index and t the particular time instant for which the analysis is carried out. If the process is assumed to be stationary in the measured time window with expected value $E\{y\}$ and variance $Var\{y\}$, the variance for the estimated mean and for the estimated variance at time t can be obtained using (6). From (6) it can be seen that, in order to reduce the standard deviation of the mean by a factor of 2, the number of realizations must be increased by a factor of 4. Assuming N to be large enough, the same approximately holds for the standard deviation of the estimator of the variance.

2.5 Accuracy of temporal averages for a zero-mean process

Let $y_i(t)$ be a generic realization of a process Y in a time interval $[0,T]$. We now analyse the cases of temporal mean and temporal mean square as already defined in (3). The process Y is supposed to be weakly stationary, with zero-mean, standard deviation σ_Y and autocorrelation function $R_{YY}(\tau)$. From the definition of temporal mean $\langle y\rangle$ and temporal mean square $\langle y^2\rangle$ it follows that their expected values are

$$
\begin{aligned}
&E\{\langle y\rangle\}=0\\
&E\{\langle y^2\rangle\}=\sigma_Y^2
\end{aligned}
\tag{9}
$$

From (9) it follows that the temporal mean and the temporal mean square are unbiased estimators for the expected value of Y and Y^2 respectively. We are now interested in the degree of variation of the estimators. To analyse this aspect, we can calculate the variance of the estimators.

The variance of the mean and of the mean square, bearing in mind the assumptions done in this paragraph, can be expressed as follows (Ochi 1990; Naito and Kihara 1993; Parkus 1969):

$$
\begin{cases}
Var\{\langle y\rangle\}=\dfrac{2}{t^2}\displaystyle\int_0^t\left\{\int_0^\xi R_{YY}(\tau)d\tau\right\}d\xi=\dfrac{2}{t}\displaystyle\int_0^t\left(1-\dfrac{\tau}{t}\right)\cdot R_{YY}(\tau)d\tau\\
Var\{\langle y^2\rangle\}=\dfrac{2}{t^2}\displaystyle\int_0^t\left\{\int_0^\xi\left(R_{Y^2Y^2}(\tau)-\sigma_Y^4\right)d\tau\right\}d\xi=\dfrac{2}{t}\displaystyle\int_0^t\left(1-\dfrac{\tau}{t}\right)\cdot\left(R_{Y^2Y^2}(\tau)-\sigma_Y^4\right)d\tau
\end{cases}
\tag{10}
$$

being

$$R_{Y^2Y^2}(\tau)=E\{y^2(t)\cdot y^2(t+\tau)\}\tag{11}$$

If the process Y, in addition, is supposed to be gaussian the following property holds (Ochi 1990):

$$R_{Y^2Y^2}(\tau)=\sigma_Y^4+2\cdot R_{YY}^2(\tau)\tag{12}$$

And hence the variance of the temporal mean square for a zero-mean gaussian process becomes (Naito and Kihara 1993):

$$Var\{\langle y^2\rangle\}=\frac{4}{t^2}\int_0^t\left\{\int_0^\xi R_{YY}^2(\tau)d\tau\right\}d\xi=\frac{4}{t}\int_0^t\left(1-\frac{\tau}{t}\right)\cdot R_{YY}^2(\tau)d\tau\tag{13}$$

The zero-mean process Y can be considered ergodic concerning the mean and the variance respectively when

$$
\begin{aligned}
&\lim_{t\to+\infty}Var\{\langle y\rangle\}=0\Leftrightarrow\lim_{t\to+\infty}=\frac{2}{t}\int_0^t\left(1-\frac{\tau}{t}\right)\cdot R_{YY}(\tau)d\tau=0\\
&\lim_{t\to+\infty}Var\{\langle y^2\rangle\}=0\Leftrightarrow\lim_{t\to+\infty}\frac{2}{t}\int_0^t\left(1-\frac{\tau}{t}\right)\cdot R_{Y^2Y^2}(\tau)d\tau=\sigma_Y^4
\end{aligned}
\tag{14}
$$

From (14) it can be said that, if we analytically know the statistical properties of the process Y, we can prove whether or not it is ergodic.

Unfortunately, usually, we do not have such knowledge of the process under analysis, because it is the result of an experiment. For this latter reason, the length of the time history is limited, and will never be infinite. Due to the limited record length, every temporal estimation will be associated to a certain level of uncertainty, that is, it will be associated to a given confidence interval. Finally, with the assumption in mind that the process under analysis is stationary, we cannot answer the question "*is the process ergodic?*", but we can only try to give an answer to the question *"how good are our temporal estimators?"*.

The faster the variances of the estimators $\langle y\rangle$ and $\langle y^2\rangle$ decrease, the smaller is the confidence interval associated to the values of the estimators, and the closer are the estimators to their expected values. If the process is ergodic, in the limit of $t \to +\infty$, the variances of the estimators go to zero. If the process is not ergodic the variances do not tend to zero. However, from a limited experimental time record, we are never in the position to definitely judge about the actual ergodicity of the process.

2.6 Some analytical results for the special case of stationary zero-mean gaussian processes

In the special case of a stationary zero-mean gaussian process, expression (13) can be used for the variance of the estimator of the variance. An usual description of a stationary zero-mean gaussian process is based on the definition of the process by means of its power spectral density $S(\omega)$. The spectral density can be defined starting from the Wiener-Kintchine equations for a real function, where the power spectral density is defined as the Fourier transform of the autocorrelation function $R(\tau)$ as follows:

$$\begin{cases} S(\omega) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} R(\tau) \cdot e^{-j\omega\tau} d\tau = \dfrac{1}{\pi} \displaystyle\int_{0}^{+\infty} R(\tau) \cdot \cos(\omega\tau) d\tau \\ R(\tau) = \displaystyle\int_{-\infty}^{+\infty} S(\omega) \cdot e^{j\omega\tau} d\omega = 2 \displaystyle\int_{0}^{+\infty} S(\omega) \cdot \cos(\omega\tau) d\omega \end{cases} \tag{15}$$

$R(\tau)$ and $S(\omega)$ form a Fourier pair, and the spectral density defined in (15) is the so-called "double sided spectrum".

Because of the symmetry of the double-sided spectral density, in the field of Naval Architecture, the "single sided spectrum" is more often used as defined below:

$${}_1S(\omega) = 2 \cdot S(\omega) \quad \omega \geq 0 \tag{16}$$

In the following, for sake of simplicity and clarity, the double sided spectrum will be used unless otherwise stated by means of the subscript "1".

Expressions reported up to here allow the theoretical analysis of estimators' accuracy when the autocorrelation function or the spectrum is given. As an example of application, a spectrum is chosen that has already been used in the past by Naito and Kihara (1993), to investigate the problem of data measuring in irregular waves, with particular attention to the problem of added resistance. Such spectrum has the peculiarity to allow the determination of $Var\{\langle y \rangle\}$ and $Var\{\langle y^2 \rangle\}$ in analytical closed form. Most of the results presented below are already reported in Naito and Kihara (1993), although in a slightly different form. The particular Fourier pair used as an example is, thus:

$$\begin{cases} R_{YY}(\tau) = \sigma_Y^2 \cdot \exp(-q \cdot |\chi|) \cdot \cos(\chi) \\ S_{YY}(\omega) = \dfrac{\sigma_Y^2}{\pi \cdot \omega_0} \cdot q \cdot \dfrac{1 + \Lambda^2 + q^2}{\left[(1+\Lambda)^2 + q^2\right] \cdot \left[(1-\Lambda)^2 + q^2\right]} \\ \text{being} \quad \omega_0 > 0; q > 0; \chi = \omega_0 \cdot \tau; \Lambda = \dfrac{\omega}{\omega_0} \end{cases} \tag{17}$$

The following expressions are found for the variance of temporal estimators:

$$\begin{aligned} &Var\{\langle y \rangle\} = \frac{2 \cdot \sigma_Y^2}{\chi^2 \cdot (1+q^2)^2} \cdot \left\{\chi \cdot q \cdot (1+q^2) + 1 - q^2 - e^{-q\cdot\chi} \cdot \left[(1-q^2) \cdot \cos(\chi) + 2 \cdot q \cdot \sin(\chi)\right]\right\} \\ &Var\{\langle y^2 \rangle\} = \frac{\sigma_Y^4}{2 \cdot \chi^2} \cdot \left\{\frac{2 \cdot \chi}{q} \cdot \frac{1 + 2 \cdot q^2}{1+q^2} - \frac{1 + q^2 \cdot (1 + 2 \cdot q^2)}{q^2 \cdot (1+q^2)^2} - \frac{e^{-2\cdot q\cdot\chi}}{(1+q^2)^2} \cdot F\right\} \\ &F = \left[(1-q^2) \cdot \cos(2 \cdot \chi) + 2 \cdot q \cdot \sin(2 \cdot \chi) - \left(\frac{1+q^2}{q}\right)^2\right] \end{aligned} \tag{18}$$

In particular, from the definition of coefficient of variation, it follows that:

$$CoV_{\langle y^2 \rangle} = \frac{1}{\chi} \cdot \sqrt{\frac{1}{2} \cdot \left\{\frac{2 \cdot \chi}{q} \cdot \frac{1 + 2 \cdot q^2}{1+q^2} - \frac{1 + q^2 \cdot (1 + 2 \cdot q^2)}{q^2 \cdot (1+q^2)^2} - \frac{e^{-2\cdot q\cdot\chi}}{(1+q^2)^2} \cdot F\right\}} \tag{19}$$

It can be shown that, provided $q \neq 0$, it is

$$\lim_{\chi \to +\infty} Var\{\langle y \rangle\} = 0$$
$$\lim_{\chi \to +\infty} Var\{\langle y^2 \rangle\} = 0 \quad (20)$$

The latter expressions lead to the conclusion that the process is ergodic concerning the expected value and the expected squared value.

However, it is the aim of this section to show how big is the influence of the parameter q on the coefficient of variation of the temporal mean square. The parameter q is a bandwidth coefficient: the smaller the bandwidth parameter, the narrower the spectrum. Fig.1 shows the shape of the spectrum when the parameter q is varied. In Fig.2 the theoretical results for $\mathrm{cov}_{\langle y^2 \rangle}$ are reported for different values of q. The big influence of the spectrum bandwidth is clearly noticeable: the narrower the spectrum, the longer the correlation time and the larger the coefficient of variation of $\langle y^2 \rangle$ for a given nondimensional time χ.

Nevertheless, ergodicity has been proved for these two estimators, but the rate of reduction of the variance of the estimators can be very small if the spectrum is very narrow.

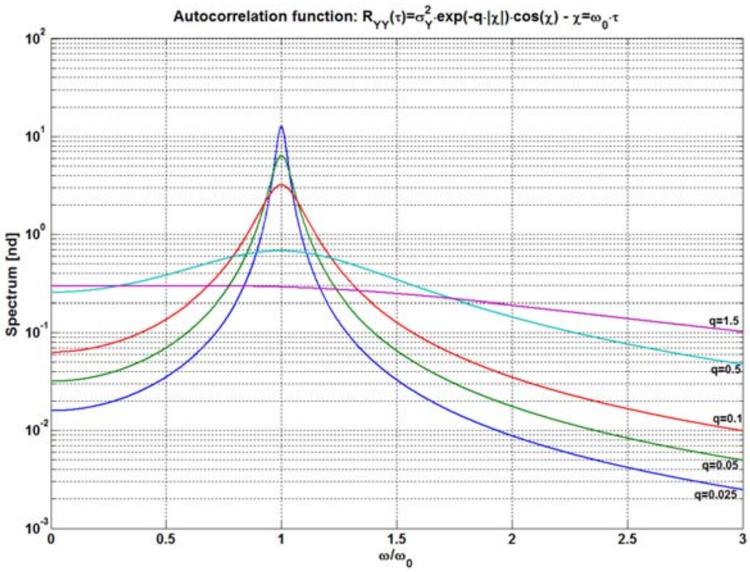


Fig. 1: Effect of parameter q on the spectrum shape. $\sigma_Y = 1$.

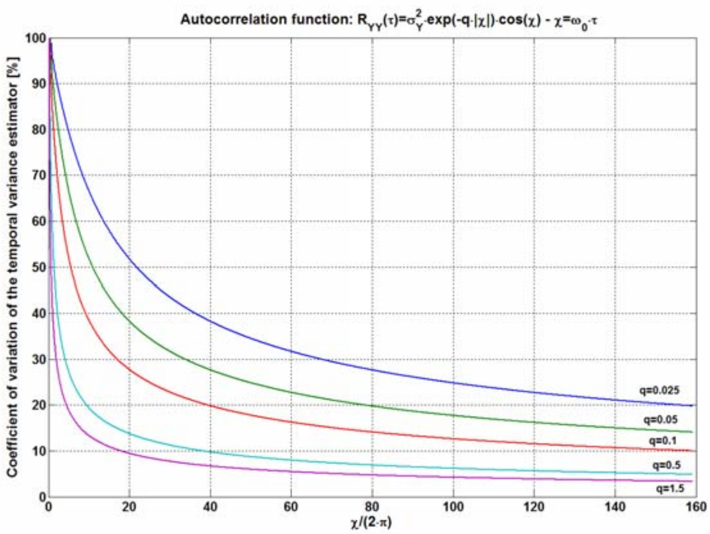


Fig. 2: Effect of parameter q on the coefficient of variation of the temporal mean square $\langle y^2 \rangle$.

We now imagine an ideal experiment in which a realization of the process described by (17) is measured. Let us suppose that the process is measured from $\chi = 0$ to $\chi = 2\pi \cdot 160$ (this means that we are approximately working on 160 oscillation periods). By means of time averaging, σ_Y^2 is estimated from the measured time history. If $q = 1.5$ (a very broad spectrum) the coefficient of variation for the estimator of the variance is found to be about 3%, this leading to a quite good accuracy in the estimation. If, on the other hand, the process is characterised by $q = 0.025$ (very narrow spectrum), a coefficient of variation of 20% is expected, that leads to a very large confidence interval for the estimator, that is, a very inaccurate estimation.

From this simple example a conclusion can be drawn, that was, in principle, already present in the work of Naito et al. Naito and Kihara (1993),: ergodicity is not sufficient to assure accurate estimations of the statistical moments by means of temporal averages. The shape of the correlation function (or, it is the same, of the spectrum) should be taken into account when the measuring time record length is chosen for the analysis of a stochastic process.

3. Numerical simulation of parametric roll

3.1 Analytical modelling overview

In order to numerically simulate the parametrically excited rolling motion of a ship, the main ideas already developed in the past for the case of regular waves (Bulian et al. 2003; Francescutto 2002; Francescutto and Bulian 2003; Bulian 2004a) have been used and adapted to the irregular sea condition. The main assumptions are the quasi-static approach for heave with constant trim (fix-trim) or quasi-static assumption for both heave and pitch (free-trim). Both assumptions lead to the possibility of developing a 1.5-DOF model for the roll motion. In order to analytically describe the nonlinear restoring lever in irregular waves, the "Grim's effective wave" concept has been used (Grim 1961). The effective wave concept has been used in the past by Umeda and Yamakoshi (1993) to predict the capsize probability for a coastal trawler in quartering and following sea. It is the Grim's idea to substitute, by means of a least square fitting, the instantaneous irregular long crested sea surface along the ship with a regular wave having length equal to the ship length and a crest (or a trough) amidship: such a wave is called "effective wave". Formally applying such procedure, it is possible to obtain the effective wave amplitude η as a stochastic process, the spectrum of which can be determined from the sea spectrum using, in deep water, the following transfer function:

$$\begin{cases} f_\eta(\omega) = 2 \cdot \dfrac{Q \cdot \sin(Q)}{\pi^2 - Q^2} \\ Q = \dfrac{\omega^2 \cdot L}{2 \cdot g} \cdot \cos(\chi) \end{cases} \tag{21}$$

In (21) L is the reference length (the ship length) and χ is the encounter angle (0deg in following sea, 180deg in head sea). Finally, the single-sided effective wave amplitude spectrum ${}_1S_\eta(\omega)$ (or, shortly, the "effective wave spectrum"), is obtained from the single-sided sea spectrum ${}_1S_Z(\omega)$ as

$${}_1S_\eta(\omega) = f_\eta^2(\omega)\,{}_1S_Z(\omega) \tag{22}$$

The same expression can be used for the double-sided spectra. The effective wave transfer function (21) acts on the sea spectrum as a band-pass filter with a modal frequency corresponding to the real frequency of the wave having a length equal to about $0.888 \cdot L$ in longitudinal sea.

The $\overline{GZ}$ surface for the ship under analysis is calculated, in a pre-processing stage, for different effective wave amplitudes (positive and negative) by means of a standard hydrostatic software, considering the trim either fix or free. The effective wave length is set equal to L_{BP} with the crest, or the trough, amidship: positive effective wave amplitudes correspond to crests amidship, whereas negative effective wave amplitudes correspond to troughs amidship. At the end of these calculations, the $\overline{GZ}$ is known for a set of couples heeling angle/effective wave amplitude. From this set of data, the $\overline{GZ}$ surface is approximated by means of the following polynomial surface:

$$\overline{GZ}(\phi,\eta) = \sum_{n=0}^{N_G} \sum_{j=0}^{N_k} Q_{jn} \cdot \eta^j \cdot \phi^n \tag{23}$$

The coefficients Q_{jn} are determined from a least square fitting. The equation of motion can thus be written in the following form

$$\begin{cases} \ddot{\phi} + d(\dot{\phi}) + \omega_0^2 \cdot \dfrac{\overline{GZ}(\phi,\eta)}{\overline{GM}} = 0 \\ d(\dot{\phi}) = 2\mu \cdot \dot{\phi} + \beta \cdot \dot{\phi}\left|\dot{\phi}\right| + \delta \cdot \dot{\phi}^3 \\ \overline{GZ}(\phi,\eta) = \sum_{n=0}^{N_G} \sum_{j=0}^{N_k} Q_{jn} \cdot \eta^j \cdot \phi^n \end{cases} \tag{24}$$

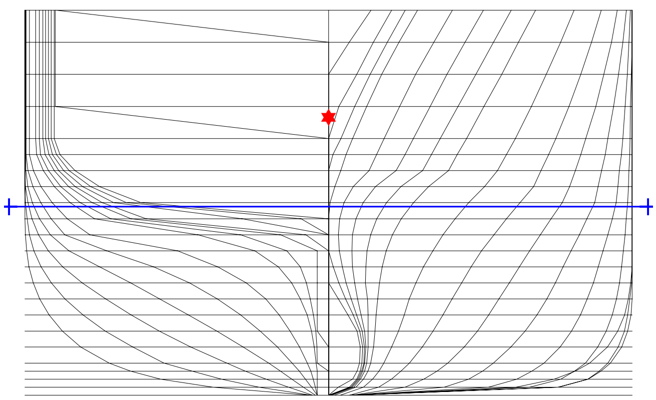
The effective wave process can be generated in time domain from the effective wave spectrum S_η. For each realization $\eta_i(t)$, equation (24) is numerically integrated and the corresponding realization $\phi_i(t)$ of the roll process can be obtained.

Before starting the analysis of the Monte Carlo simulation, two considerations have to be reported:

- the Grim's effective wave concept is not intended to be a means for obtaining extremely accurate quantitative results, but it is used here as a powerful tool for qualitative analysis, that allows a quite simple analytical description of the restoring lever in irregular waves;
- the analytical description (23) can be used to obtain a lot of useful statistical information on the statistical properties of the restoring lever in waves (however, this point is not discussed in this paper)

3.2 Ship and environmental conditions

Table 1: Data of the ship used for simulation. Bare hull condition.

RoRo Pax TR2 – C73-97	
Full scale data: Δ=7715 t_f $\overline{GM}$=0.865 m $\overline{KG}$=8.660 m T=5.875 m x_G=-0.464 m L_{BP}=132.2 m ω_0=0.396 rad/s μ/ω_0=0.012 β=0 $\delta \cdot \omega_0$=0.841 **Model Scale 1:50**	

Simulations are carried out using a ship for which experimental data are available regarding parametric roll in both regular and irregular long crested sea (Francescutto 2002; Francescutto and Bulian 2003). Data of the used ship are reported in Table 1. Damping coefficients and natural frequency have been obtained from sallying experiments (Francescutto 2002; Francescutto and Bulian 2003; Bulian 2004b).

Two different types of sea spectra are used in the simulation.

- a Bretschneider spectrum

$${}_1S_Z(\omega) = \frac{A}{\omega^5} \cdot \exp\left(-\frac{B}{\omega^4}\right) \tag{25}$$

  having a modal frequency corresponding to a wave length equal to the length between perpendiculars, that is, $\omega_m = 0.683 rad/s$.
- an analytical spectrum obtained by means of linear filtering of white noise (Francescutto and Bulian 2003; Bulian et al. 2003):

$$
{}_1S_Z(\omega) = \frac{2\cdot\gamma\cdot\left(\omega_m^2 + \frac{\gamma^2}{2}\right)\cdot S_0}{\left(\omega_m^2 + \frac{\gamma^2}{2} - \omega^2\right) + \omega^2\cdot\gamma^2} \tag{26}
$$

The spectral bandwidth sbw defined as follows

$$
sbw = \sqrt{\frac{m_0 \cdot m_2}{m_1^2} - 1} \tag{27}
$$

is used as characterizing parameter for spectra of type (26) because of its influence on the wave grouping phenomenon (Panjaitan 1998). Different significant wave heights $H_{1/3}$ are used. All the simulations are based on $\overline{GZ}$ calculations carried out using a fix-trim approach, being the fix-trim fluctuations larger than the free-trim ones.

3.3 Sample results

3.3.1 Long term behaviour of variance estimator

In order to conjecture something about ergodicity, the long term behaviour of the estimator has to be analysed. A set of simulations has been carried out for two conditions:

- Bretschneider sea spectrum
- Narrow band spectrum NB (sbw=0.1 in equation (26))

In both cases the following parameters have been used

- 100 simulations
- 10 hours for each simulation
- $H_{1/3}/L_{BP} = 1/75$
- modal wave frequency $\omega_m = 0.683 rad/s$
- ship speed 2m/s in head sea

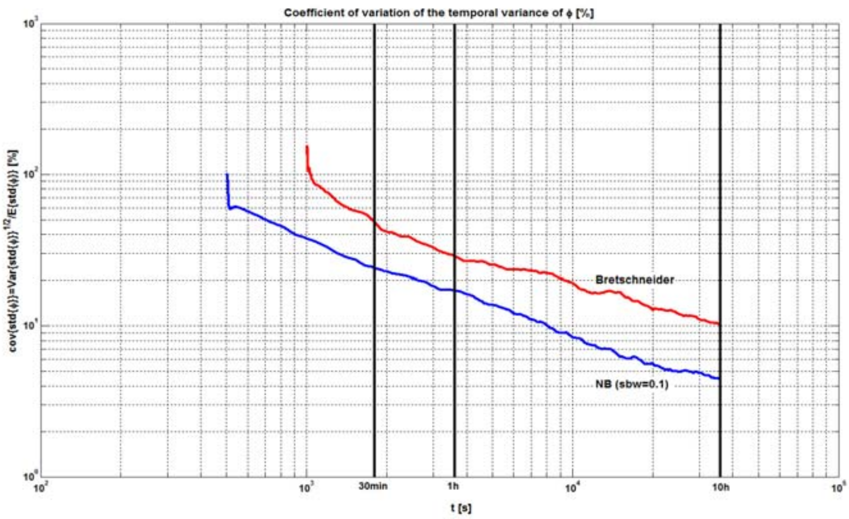


Fig. 3: Long term behaviour of the coefficient of variation of temporal variance estimator.

From a preliminary analysis, a transient of about 1000s has been considered for the Bretschneider spectrum, and the initial 500s have been cut in the case of NB spectrum. Results of analysis are reported in Fig.3. It can be clearly seen that the variance of the estimator progressively reduces in both cases as the simulation time is increased. Moreover, it is important to note that the coefficient of variation of the temporal variance for the two cases is very different, this meaning that the shape of the sea spectrum has an important influence on the roll motion characteristics. Finally, three time instants have been underlined in the figure: 30 minutes, 1 hour and 10 hours. In the case of 30min simulations, the coefficient of variation of the time variance is estimated as about 50% in the case of Bretschneider

sea spectrum, and about 23% in the case of NB spectrum: this meaning that the confidence intervals are very large. When 10 hours simulations are considered, the coefficients of variation reduce to about 5% and about 10% for the NB and Bretschneider sea spectrum respectively: a reduction of a factor of about 5 has been achieved increasing the simulation time by a factor of 20. Finally it is to be said that such results are just indicative as numerical values strongly depend on the particular conditions (ship speed, sea spectrum, significant wave height, etc.).
As a general trend, it can be said that the coefficient of variation decreases as the simulation time increases: such behaviour supports the idea that the process could be ergodic with respect to the variance. Even if not shown here, the same occurs for the mean. However, very often, experimental tests in towing tank last 30min in full scale: the coefficient of variation has been shown to be, for such simulation time, very large. From a practical point of view, especially in the case of Bretschneider spectrum, the process should be considered as “practically not ergodic” if experiments are limited to 30min, due to the large confidence interval associated to the estimation.

3.3.2 The influence of ship speed

Due to the Doppler effect, ship speed influences the roll response behaviour because, consistently with the modelling proposed in this paper, it modifies the effective wave spectrum.
Fig.4 shows a summarising plot in the case of a Bretschneider spectrum with $H_{1/3}/L_{BP}=1/50$. The sea spectrum and the corresponding effective wave spectrum are shown without Doppler effect together with the calculated zero crossing and mean effective wave frequencies and the calculated spectral bandwidth as functions of the ship speed. Spectrum is truncated for a wave frequency corresponding to a wave length of $L_{BP}/10$.
If only the tuning ratio is taken into account, the most dangerous speed would be judged to be about 1.3m/s. However, it is known that parametric roll is strongly influenced by the wave grouping phenomenon (Francescutto and Bulian 2003; Bulian 2002), and the grouping phenomenon is stronger when the spectral bandwidth parameter is small (Goda 1985); Panjaitan 1998; Bulian 2002). From Fig.4 it can be seen, as already underlined by Panjaitan (1998), that sbw decreases as the ship moves towards the following sea condition, this leading to an increasing of the grouping phenomenon for the effective wave process. On the other hand, the tuning ratio moves far from the synchronism condition; there are, thus, two competitive behaviours as the ship moves towards the following sea condition:

- the grouping phenomenon increases due to the decreasing of the spectral bandwidth parameter (dangerous effect);
- the tuning ratio moves away from the synchronism condition (mitigating effect);

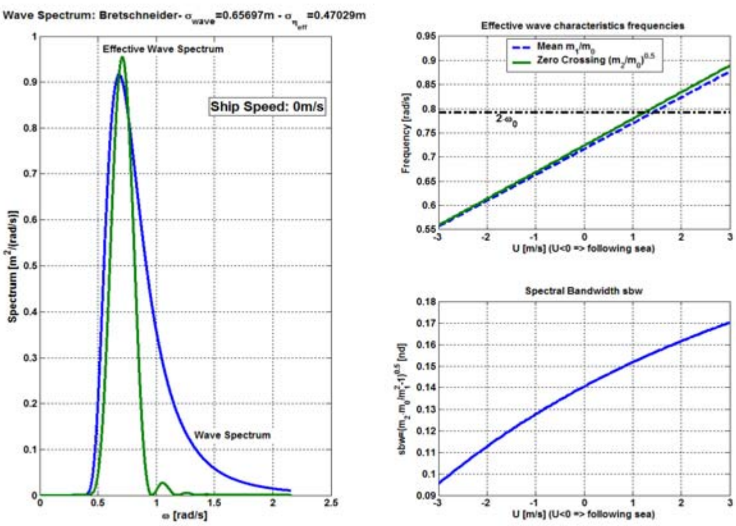


Fig. 4: Forward speed effect on characteristic parameters of effective wave spectrum.

When grouping phenomenon is strong and tuning ratio is far from the exact synchronism, a strongly grouped response can be found in the roll motion: such a behaviour for the roll motion usually leads to very large coefficients of variation for the variance estimator, especially when the condition is close to the stability limiting boundary. An example is shown in Fig.5 in the case of the spectrum reported in Fig.4 and for a ship speed of 1.75m/s in following sea. The strong grouping of the roll response is clearly noticeable: it can be understood that such behaviour could render the parametric roll a "rare" phenomenon for which temporal averages are almost useless.

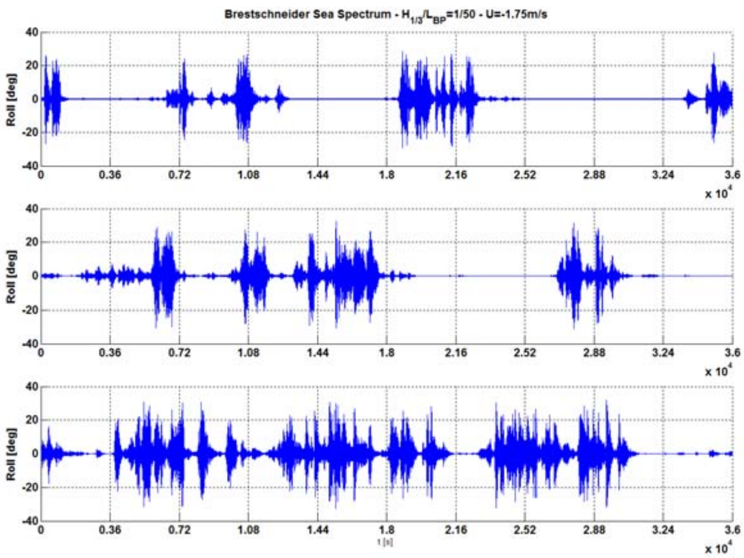


Fig. 5: Examples of roll realizations characterised by strong grouping.

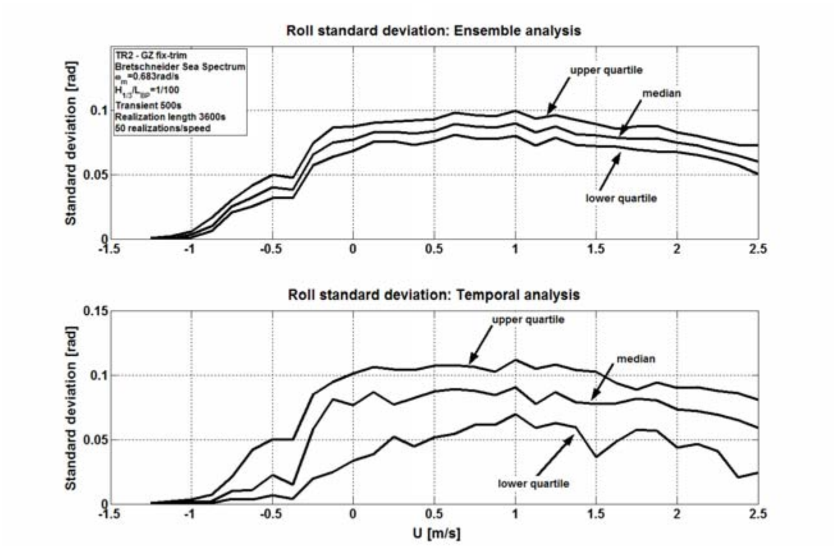


Fig. 6: Comparison between temporal and ensemble analysis.

In general it is, however, always possible (even if sometimes useless) to produce the roll response curve in terms of roll standard deviation taking into account the variability of the estimator: for different ship speeds a set of realizations is simulated and ensemble and temporal analysis are carried out. The distributions of the ensemble standard deviation and of temporal standard deviation are estimated, so that the lower quartile, the median and the upper quartile can be found for the two types of analyses. Finally, the corresponding curves are plotted as functions of the ship speed. An example is given in Fig.6 for a Bretschneider sea spectrum with $H_{1/3}/L_{BP} = 1/100$. Each simulation has been carried out for 3600s and the first 500s have been neglected in order to eliminate the effect of transient. Bearing in mind that the lower and upper quartile curves delimitate 50% of data, from figure it can be seen that a large spreading is present when temporal analysis is used. Large spreading indicates a lack of "practical ergodicity".

3.3.3 Estimation of roll probability density function

When gaussian processes are of concern, the process mean and the process standard deviation are sufficient to fully describe the instantaneous process pdf. In the case of non-gaussian processes the first and second moments are, in general, not sufficient for the description of the pdf, unless a two parameters analytical pdf is assumed. For this reason the pdf of the process is usually estimated from the process simulated time histories. If the process is ergodic, its pdf can be estimated with infinite accuracy from a single infinitely long realization. However, as already stated in previous paragraphs, in practical applications the time histories are always limited, this leading to an inherent variability in the estimated process pdf (or, equivalently, in its first integral: the cumulative distribution function (cdf)). A question thus arises: *"Do we estimated 'exactly the same' pdf if we estimate it from two different realizations?"*. Given two different realizations of a stochastic process the probability of estimating two exactly equal cdfs is, in general, zero (this does not mean that it is impossible, but that it is "practically impossible"). The aforementioned question is, thus, to be substituted by the following one: *"Is the confidence interval of the empirically estimated pdf (cdf) reasonably small?"*. A clear definition of "reasonably small" is of course dependent on the level of uncertainty in the estimation that we consider acceptable, and it is a "practical" problem. In order to assess the influence of the time record length on the confidence interval of the estimated cumulative distribution function, two different conditions have been used regarding ship speed and sea spectrum, that were previously used in 0. The cdf of roll motion and of the roll peaks have been estimated from the simulated time series cutting 1000s of transient in both cases and truncating the time histories at 30min, 1 hour, 2 hours, 5 hours and 10 hours: the results are reported in Fig.7 and Fig.8. As can be seen from the figures, the accuracy of the cdf estimation increases as the simulation time increases, however it is very poor even after 1hour of simulation in both cases. This behaviour leads to two reasonable remarks:

- the process could be ergodic with regard to the estimation of the pdf;
- care should be taken concerning the estimation of roll pdf (cdf) from a limited number of realizations, especially when the simulation time is relatively short;

Especially when the occurrence of rare events is of concern, the influence of a too rough estimation for the cdf of the roll envelope, could lead to results that are very far from the actual ship behaviour.

As a final comment it can be clearly seen from the figures that the roll cdf strongly depends on the particular sea spectrum under analysis, this leading to the necessity of using non-gaussian/non-Rayleigh distributions to analytically describe the process behaviour (Ochi 1978).

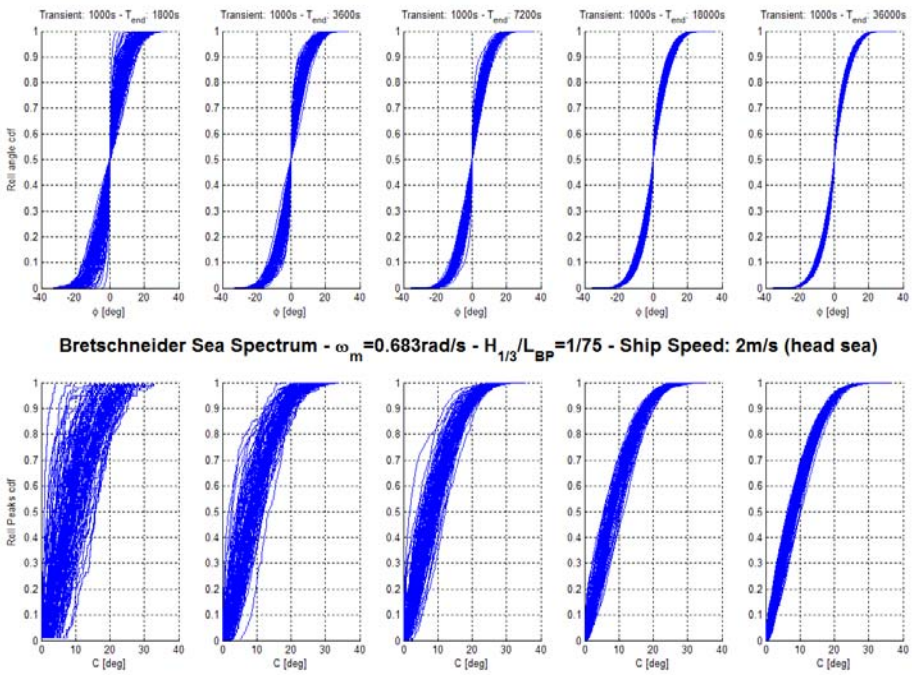


Fig. 7: Influence of the length of time history on the estimated cdf. Bretschneider spectrum.

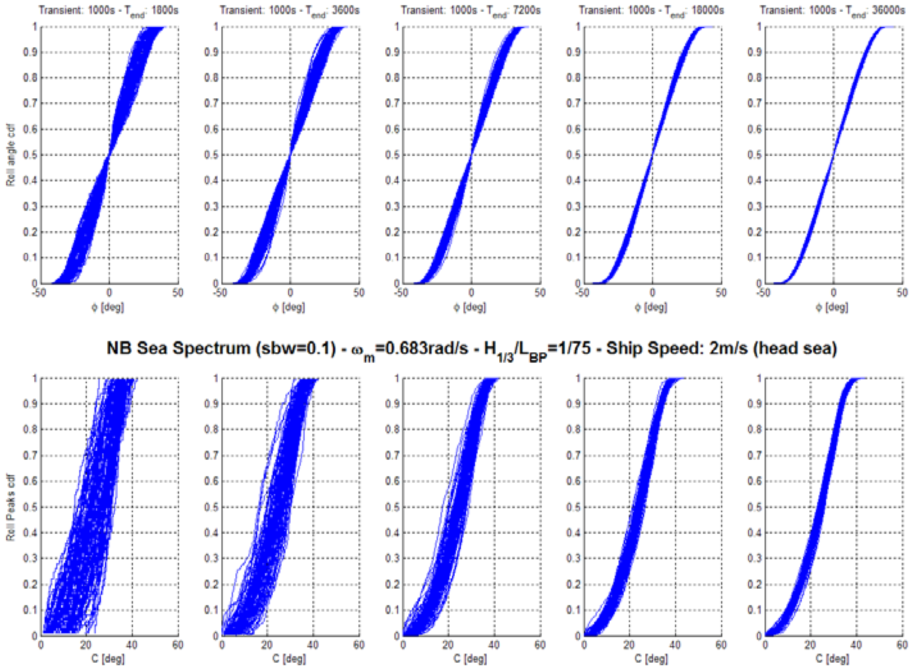


Fig. 8: Influence of the length of time history on the estimated cdf. NB spectrum.

4. Examples of experimental results

All the aforementioned discussion is based on the results of a series of numerical simulations based on a simplified 1.5-DOF analytical model: although results are physically sound and confirm the observations of a previous test campaign (Francescutto 2002; Francescutto and Bulian 2003; Bulian et al. 2003), a new series of experiments has been carried out at the INSEAN Towing Tank No.2 (220mx9mx3.6m) in order to verify the outcomes of simulations. Data of the tested model are reported in Table 1. Aim of the tests was the assessment of the confidence interval for temporal averages as a means for the estimation of ensemble averages. The model (positioned about 115m from the wavemaker) was slightly restrained in yaw and surge by means of elastic ropes connected at bow and stern. Tests have been carried out at zero forward speed with waves coming from the stern: from some preliminary tests it has been noticed that parametric roll was more easily excited with waves from the stern than in the case of waves from the bow. Two wave spectra have been used, both having $H_{1/3}/L_{BP}=1/50$: a narrow band (NB) sea spectrum described by equation (26) with spectral bandwidth equal to 0.1 and a Bretschneider sea spectrum. In both cases the modal wave frequency was chosen as that frequency corresponding to a wave length equal to the length between perpendiculars. In the range of frequencies used for the generation of the irregular sea the coefficient of reflection of the beach is less than 5%. 16 realizations for each spectrum have been recorded. Each recording was started a few seconds after an initial manual perturbation of the upright position. No further perturbation was given to the model during each test. Roll motion was recorded for about 475s at model scale in each test, leading to a total simulation time of about 15 hours at full scale. It has been noted, however, that the effect of waves reflected from the beach could be non negligible approximately after 200s from the beginning of each test, especially in the case of NB spectrum.

The first analysis of experimental results has been devoted to the recorded sea elevation processes. In Fig.9 the so-called running temporal standard deviation

$$\sigma_{z,k}(t_N)=\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left[z_k(t_i)-\frac{1}{N}\sum_{i=1}^{n}z_k(t_i)\right]^2} \quad (28)$$

is reported for both tested sea spectra and for each realization in the upper plots. On the same plots the mean value among the sixteen realizations of the running temporal standard deviation is plotted as a black thick line. Due to the limited number of realizations, initial transient stage cannot be accurately detected, and thus the whole record is used in the analysis. In the lower plot the estimated standard deviation of the temporal running standard deviation is reported for both spectra. It can be seen that:

- the dispersion of $\sigma_Z(t)$ is larger in the case of NB spectrum;
- the standard deviation of $\sigma_Z(t)$ in both cases decreases as the time record length is increased.

Assuming the sea elevation process approximately as a gaussian process, and bearing in mind the indications given by Fig. 2, such results were expected. From the aforementioned observations it can be conjectured that the sea elevation process is ergodic. It is however to be noted that the influence of wave reflections seems to be non negligible after about 200s in the case of NB sea spectrum.

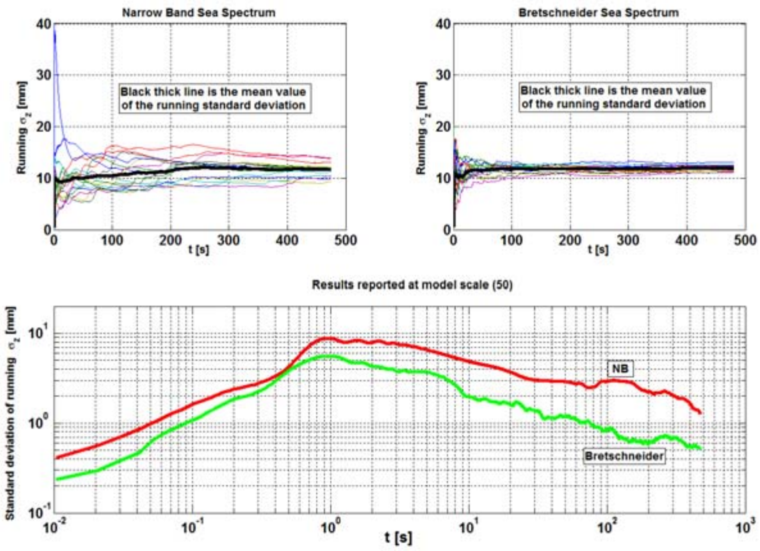


Fig. 9: Analysis of measured sea elevation processes.

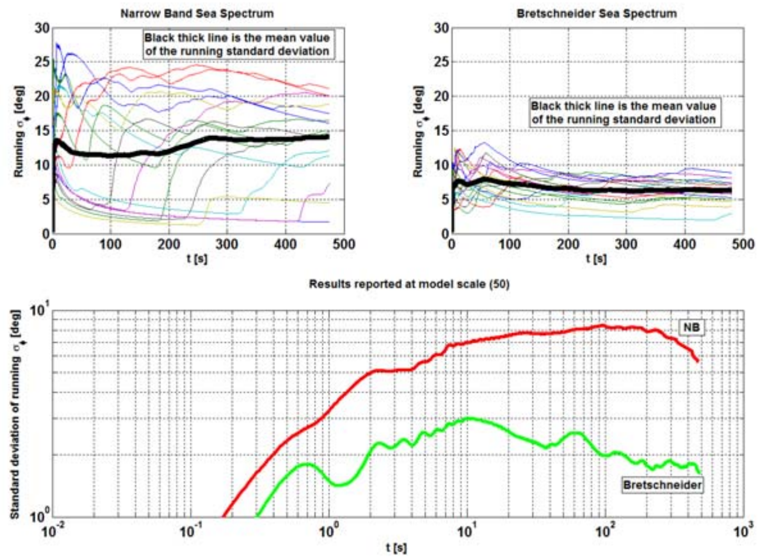


Fig. 10: Analysis of measured roll motion.

After this preliminary analysis regarding the input process, the recorded roll motion has been analysed in an identical way as was done for the sea elevation: results are reported in Fig. 10. Roll motion running standard deviation has been calculated, after substitution of Z with Φ, according to (28). The estimations of roll motion standard deviation show a very large dispersion in the case of NB sea spectrum. In the case of Bretschneider sea spectrum the dispersion, in absolute value, is smaller, but non negligible. The marked increasing in Fig. 10 of the mean value of running standard deviation visible in the case of NB sea spectrum after about 200s could be due to wave reflection. The lower plot in Fig. 10, reporting the standard deviation of the temporal running standard deviation, shows a clear tendency to decrease in the case of Bretschneider sea spectrum. In the case of NB sea spectrum such decreasing is seen only in the final part of the graph.

Finally, it can be said that experimental results confirm the outcomes of numerical simulations. Moreover, it can be conjectured that parametrically excited roll motion, in the tested conditions, is ergodic but, unfortunately, not "practically ergodic" due to the large dispersion noticed in the temporal averages. Although we are quite confident that an infinitely long realization of the roll process would allow exact estimates of the process statistics, the effort needed to obtain reliable statistical averages from real experiments seems to be quite large, especially when the sea spectrum is very narrow banded (as happens, in following sea at moderate speeds, due to the Doppler effect). In order to limit the unavoidable effects of reflections, it seems better to perform several different realizations of the process instead of carrying out less, longer, tests. As a final comment, it has to be said that the analytical modelling and the experiments have been carried out in quite unrealistic conditions: exactly longitudinal long crested sea without any external perturbation due to rudder, propeller, wind, etc. In these conditions the coefficient of variation of temporal averages has been found to be very large, but it could be guessed that in a more realistic environment, under the action of several additional non-parametric forcing, such coefficient could be smaller.

5. Final remarks

In this paper some preliminary results have been reported regarding the problem of ergodicity of stochastic processes, with particular attention to the case of parametrically excited roll motion in longitudinal irregular long crested waves. The main outcome of this work is the (numerical and experimental) proof that temporal averages can be associated to very large coefficients of variation, even though their expected values are theoretically correct estimators of ensemble averages. The dispersion of the temporal averages is larger when grouping phenomenon is strongly present. It has been shown that, in some cases, the analysis of time series of typical length (30min full scale) could be almost useless if carried out on a single realization. In some conditions, parametrically excited roll motion shows strong grouping characteristics, this supporting the idea of treating it as a rare event. On the other hand, in some other conditions, the same process shows a less marked grouping, and roll motion is well developed: in such conditions the "rare event" assumption is no longer justified.

The phenomenon of parametric roll is very different from motions described by linear seakeeping, and should be thus tackled from a new viewpoint: both the time dependence of motion equation's coefficients and nonlinearities are to be taken into account. Parametric roll process is strongly non-gaussian and the pdfs of motion and envelope depend on the particular exciting sea spectrum: this meaning that the usual characterisation based on the rms value is absolutely not sufficient for practical purposes. The estimation of roll and envelope pdfs has been found to be associated with a very large level of uncertainty in the tested conditions.

On the basis of the reported results it seems that usual standard procedures used in seakeeping analysis are not sufficient for a satisfactory description of parametric roll: there is a need for research efforts in the field of theoretical analysis based on simplified analytical models, in the field of numerical simulation of parametric roll and in the field of experimental procedures for the analysis of such phenomenon.

A final proof of ergodicity or not ergodicity for parametric roll cannot be given on the basis of a numerical/experimental Monte Carlo approach. However it can be said that, although we are confident the process in itself is ergodic, in the tested conditions, the parametric roll process should be considered as "practically not ergodic", this meaning that the run length needed to achieve a sufficiently small confidence interval for temporal averages is very long, and probably impracticable for many facilities due to wave reflections: for this reason a combination of real experiments and numerical prediction tools could help in solving practical limitations.

6. References

- Ang, A.H.S., Tang, W.H. (1975), "Probability Concepts In Engineering Planning And Design – Volume I – Basic Principles", John Wiley & Sons.
- Badino, M. (2003), "The Foundational Role of Ergodic Theory", 2003, http://philsci-archive.pitt.edu/ archive /00001757/
- Belenky, V. L., Degtyarev, A. B., Boukhanovsky, A. V. (1998), "Probabilistic Qualities of Nonlinear Stochastic Rolling", Ocean Engineering, Vol. 25, No. 1, pp. 1-25.
- Belenky, V., Suzuki, S., Yamakoshi, Y. (2001), "Preliminary Results of Experimental Validation of Practical Non-ergodicity of Large Amplitude Rolling Motion", Proceeding 5th International Workshop on Ship Stability and Operational Safety, Trieste, September, pp. 4.1.1-4.1.8.
- Belenky, V., Weems, K.M., Lin, W.M., Paulling, J.R. (2003), "Probabilistic Analysis of Roll Parametric Resonance in Head Sea", Proc. 8th International Conference on Stability of Ships and Ocean Vehicles (STAB2003), Madrid, September, pp. 325-340.
- Bulian, G. (2002), "Rollio parametrico in mare regolare e stocastico (Parametric Rolling in Regular and Stochastic Sea)", (in Italian), Graduate Thesis, Università degli Studi di Trieste, October.
- Bulian, G. (2004a), "Approximate Analytical Response Curve for a Parametrically Excited Highly Nonlinear 1-DOF System with an Application to Ship Roll Motion Prediction", Nonlinear Analysis: Real World Applications, Vol. 5, No. 4, pp. 725-748.
- Bulian, G. (2004b), "Estimation of nonlinear roll decay parameters using an analytical approximate solution of the decay time history", International Shipbuilding Progress , Vol.51, No. 1, pp. 5-32.
- Bulian, G., Francescutto, A., Lugni, C. (2003), "On the Nonlinear Modeling of Parametric Rolling in Regular and Irregular Waves", Proc. 8th International Conference on Stability of Ships and Ocean Vehicles (STAB2003), Madrid, September, pp. 305-323.
- Crandall, S. H., Mark, W. D. (1973), "Random Vibration in Mechanical Systems", Academic Press, New York.
- Francescutto, A. (2002), "Theoretical Study of the Roll Motion in Longitudinal Waves", Technical Report, (in Italian), Dept. DINMA, University of Trieste.
- Francescutto, A., Bulian, G. (2003), "Theoretical and Experimental Study of Roll Motion in Longitudinal Irregular Waves", Technical Report, (in Italian), Dept DINMA, University of Trieste.
- Goda, Y. (1985), "Random Seas and Design of Maritime Structures", University of Tokyo Press.
- Grim, O. (1961), "Beitrag zu dem Problem der Sicherheit des Schiffes im Seegang", Schiff und Hafen, No. 6, pp. 490-497.
- Naito, S., Kihara, H. (1993), "Mutual Relation between Record Length and Accuracy of Measuring Data in Irregular Waves", J. of the Society of Naval Architects of Japan, Vol. 174, pp. 397-408.
- Ochi, M. K. (1978), "Generalization of Rayleigh Probability Distribution and its Application", Journal of Ship Research, Vol. 22, No.4, December, pp. 259-265.
- Ochi, M. K. (1990), "Applied Probability & Stochastic Processes in Engineering & Physical Sciences", John Wiley & Sons.
- Panjaitan, J. P. (1998), "A Study on Ship Motions and Capsizing in Severe Astern Seas", Ph.D. Thesis, Department of Naval Architecture and Ocean Engineering, Faculty of Engineering, Osaka University.
- Parkus, H. (1969), "Random Processes in Mechanical Sciences", C.I.S.M., Udine.
- Umeda, N., Yamakoshi, Y. (1993), "Probability of Ship Capsizing due to Pure Loss of Stability in Quartering Seas", Selected Papers from the Society of Naval Architects of Japan, Vol. 30, pp. 73-85.
- Van Lith, J. (2001), "Ergodic Theory, Interpretations of Probability and the Foundations of Statistical Mechanics", http://philsci-archive.pitt.edu/archive/00000367/