PIECEWISE LINEAR APPROACH TO NONLINEAR SHIP DYNAMICS

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ABSTRACT

The paper considers piecewise linear dynamical system as a model of ship nonlinear rolling and capsizing. Main advantage of the model is a possibility to describe capsizing directly: as a transition to oscillations near upside down stable equilibrium. Such a transition can be expressed in analytical functions that allows to derive symbolic solutions for both regular and irregular seas. Practical application of the model concerned estimation of beam seas ship capsizing probability per unit of time.

The proposed paper, however, does not deal with stochastic matters. The following consequence was reproduced. Free undamped roll motions were studied, a dependence of free period vs. amplitude was derived. This figure was used as a backbone curve to obtain approximate solution for steady state forced roll motion by equivalent linearization. Then, using previous result as the first expansion an exact steady state solution was derived. The last figure allows to analyze motion stability; it was and found that the system is capable for both fold and flip bifurcations. Deterministic chaos was observed as a result of period doubling sequence. Also it was found that safe basin of the piecewise linear system experiences erosion as conventional nonlinear system. So, general behavior of piecewise linear system was found identical to conventional nonlinear one: even two ranges of piecewise linear term (triangle GZ curve in positive stability range) was enough for observe above nonlinear phenomena and all the solutions were expressed in analytical form without any simulation procedures.

INTRODUCTION

There is a dual purpose for ship capsizing study: developing rational stability regulation and physical knowledge of phenomenon. The second one is necessary to be ready for new types of ship and ocean vehicles.

It seems that probabilistic approach might be the most important for future regulation development, having in mind stochastic character of wind / wave environment. Physical nature of capsizing as «a transition to motion near another stable (upside down) equilibrium» is a nonlinear phenomenon and can be studied by means of nonlinear dynamics.

A capsizing model is an outcome of this study. Adequacy is the main requirement. Another requirement is a usability for regulation purposes, including possibility of application of probabilistic approach.

MODELS OF CAPSIZING

Mathematical models of capsizing can be classified as follows:

- Energetic approach that is the background of weather criteria;
- Motion stability of steady-state rolling (Wellicom,1975), (Ananiev, 1981), (Nayfeh, 1986), Virgin (1987)
- Classical ship stability definition or separatrix crossing model, (Sevastianov,1979), (Umeda 1990)
- Safe basin or transient behavior approach (Rainey, 1990), (Falzarano, 1990)
- Piecewise linear approach, (Belenky, 1989)

The recent development of the last one is a subject of our consideration. We will be concentrated mainly on adequacy, advantages and drawbacks of this model.

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BACKGROUND OF PIECWISE LINEAR MODEL

We consider the simplest model that contains capsizing as «...a transition to stable equilibrium, dangerous from practical point of view» (Sevastianov 1982), so we should have at least two stable equilibria: upright and upside down, see fig.1:

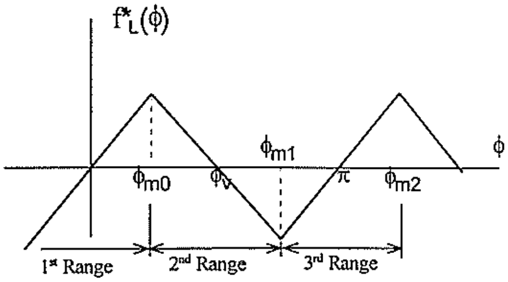

Figure 1 - Piecewise linear model of GZ curve

A differential equation of ship rolling, corresponding to this model:

$$\ddot{\phi} + 2\delta\dot{\phi} + \omega_\phi^2 f_L^*(\phi) = \alpha_E \omega^2 \sin(\omega t + \varphi_E) \qquad (1)$$

Here: ϕ-roll angle, δ-damping coefficient, ω_ϕ-natural frequency, ω-excitation frequency, α_E-effective excitation amplitude, φ_E-initial phase angle. The motion is described by two linear solution linked by initial conditions at junction points:

$$\phi = \begin{cases} \phi_{0a} e^{-\delta t} \sin(\omega_0 t + \varepsilon) + q_a \sin(\omega t + \beta_q) \\ A e^{\lambda_1 t} + B e^{\lambda_2 t} + p_a \sin(\omega t + \beta_p) \end{cases} \qquad (2)$$

where A, B, ϕ_{0a}, ε - arbitrary constants depending on initial conditions at junction points, λ_1, λ_2 - eigenvalues, ω_0 - frequency of initial free damped roll motions.

There is no way to simplify this model further, otherwise we shall get rid of capsizing as we defined it above.

What is good about this model, besides its simplicity?

ADVANTEGES AND DRAWBACKS

Capsizing Description

Early study of piecewise linear model of capsizing (Belenky 89) and (Belenky 93) showed that capsizing will happen if the value of arbitrary constant A (if λ_1 is positive) becomes positive. The solution at the 2nd linear range becomes unbounded and reaches the 3rd linear range that describes oscillation near upside down position of equilibrium. If this value is negative, then the solution will be back to the 1st range, and capsizing will not happen at leas at this semi-period of rolling. So the piecewise linear model offers clear criteria for immediate capsizing - sign of arbitrary constant A that is determined by formula:

$$A = \frac{(\dot{\phi}_1 - \dot{p}_1) - \lambda_2(\phi_1 - p_1)}{\lambda_1 - \lambda_2} , \qquad (3)$$

where $\phi_1, \dot{\phi}_1$ - are initial condition at junction point ϕ_{m0} and $p_1, \dot{p}_1$ are values of partial solution at the moment of crossing level ϕ_{m0}.

Probabilistic Approach

Another important advantage of the model is a very easy way to apply probabilistic approach: probability of capsizing is just a probability of upcrossing (that is very well studied in stochastic processes) with positive value of the arbitrary constant:

$$P_T(X) = P_T(\phi > \phi_{m0}) P(A > 0) . \qquad (4)$$

The probability of upcrossing is connected with time (since upcrossings are Poisson flow), which makes entire probability of capsizing dependent on time of exposure. The last figure is essential for correct probabilistic approach to ship stability regulation see (Sevastianov, 1982) or (Sevastianov ,1994)

Accuracy Control

Piecewise linear model can be used not for analytical study but as simulation tool as well. Algorithm contains the following steps:

1. Find the range where given point is;
2. Calculate next point with the given time step.
3. If the next point is within the same range, repeat step 2 until changing range or end of simulation.
4. If the next point in within the next range, find crossing time and crossing initial conditions. Then calculate the next point using the solution on the next range.
5. Repeat step 2 until changing range or end of simulation.

The algorithm has only one iteration procedure - crossing time search. It is numerical solution of nonlinear algebraic equation. Its accuracy can be checked easily - we just substitute crossing time into solution (2). All other steps involve calculation of elementary trigonomentric and exponential functions that can be done really accurate nowdays (at least error is known in advance).

So, even working with high amplitudes, we are still able to control accuracy and error accumulation.

Practical Applicability

Practical using of the piecewise linear model for calculation of capsizing probability in beam seas and wind was found to be possible as well, see details in (Belenky 94) and (Belenky 95). Practical applicability was reached by using of «combine» model of the GZ curve, see fig2.

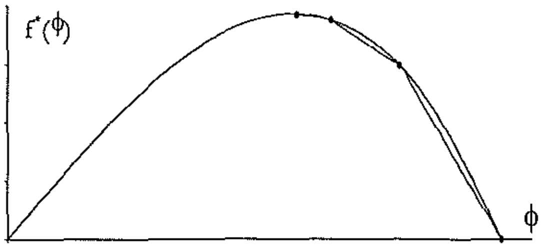

Figure 2 - «Combined» model

Known Problems

Stability in beam seas is just a part of the problem In order to obtain a real practical solution it is necessary to take into account changing GZ curve in following and quartering seas. «Brute force» attempt is senseless: if linear coefficients of piecewise linear term are dependent on time harmonically (the simplest case of regular following seas), the expression (1) becomes Mathew equation range wise, that does not have general solution expressed in elementary functions. So fully analytical solution will be not available, and all other advantages of piecewise linear model would vanish.

The problem can be solved in pure stochastic manner, using whole probability formula, see (Belenky 1997). However, this solution cannot satisfy, because of difficulties of inclusion of other degrees of freedom. In author's opinion, the most perspective way is to consider two- and multy- dimensional piecewise linear term.

Another problem is general adequacy. It is clear, that behavior of «combined system» (fig. 2) approximates real ship rolling. However it is not clear if the system (1) is capable for the same nonlinear phenomena that conventional rolling equation. So far only capsizing behavior was similar.

So we cannot proceed with the simplest piecewise linear model (that only makes a sense because of simplicity) unless we make sure that all nonlinear phenomena, rolling has been known for, could be found in the piecewise linear one too.

NONLINEAR STUDY OF PIECEWISE LINEAR SYSTEM

To prove adequacy of piecewise linear model, we should show, that it is capable for the *same nonlinear phenomena that are known for mathematical model of nonlinear rolling: It would be good if we could reproduce the same sequence of study that was applied to nonlinear rolling since early 1950ies:

- Free rolling is not isohronic: period of free undamped roll motion depends on initial amplitude;
- Response curve of nonlinear forced rolling contains non-functionality: there is an area with several amplitudes corresponding to the same excitation frequency;
- Stability analysis of steady state nonlinear roll motion shows that some regimes are unstable (Wellicome 1975), (Ananiev, 1981);
- This instability leads either to fold (escape through positive real direction) or flip bifurcation (escape through negative real direction) and consecutive period doubling may lead to chaotic response(Nayfeh 1986) (Virgin 1987).
- Dangerous combination of parameters of external excitation looks like erosion of safety basin area (Rainey 1990)

FREE OSCILLATION OF PIECWISE LINEAR SYSTEM

Let's consider first the most simple case of free motion: if there is no bias. If initial amplitude lies within the first range (see fig.1), we have pure linear oscillations: period does not depend on initial conditions - system is isohronic. If the initial amplitude is located within the second range, the period is described by formula from (Belenky 95-a):

$$T(\phi_a) = \frac{4}{\omega_\phi}\left\{\frac{1}{\sqrt{k_2}}\operatorname{arccosh}\frac{\phi_V-\phi_{m0}}{\phi_V-\phi_a} + \frac{1}{\sqrt{k_1}}\arctan\frac{\phi_{m0}\sqrt{k_1}}{\sqrt{k_2}\sqrt{(\phi_V-\phi_{m0})^2-(\phi_V-\phi_a)^2}}\right\} \quad (5)$$

here k_1 and k_2 are angle coefficients of the first and the second range correspondingly, ϕ_V is angle of vanishing stability. As it could be clearly seen, from (5), the period depends on initial amplitude, so the system is not isohronic, if initial amplitude equals to angle of vanishing stability, the period becomes infinite.

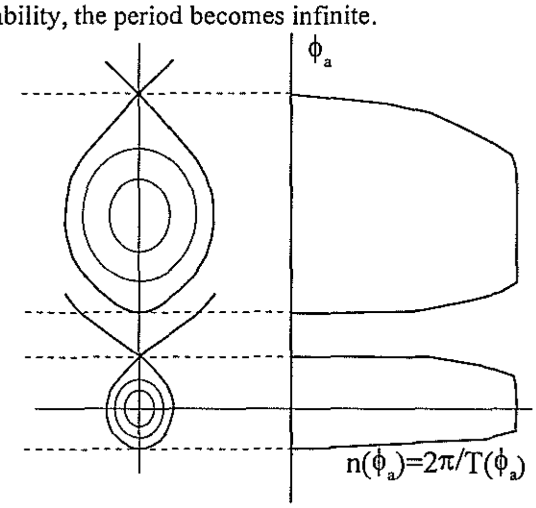

Figure 3 - backbone curve: undamped frequency dependence on initial amplitude and phase plane of free motion of piecewise linear system with bias

If there is bias, the period is expressed by a formulae that are simple but rather bulky see (Belenky 98). Here we show only image of the backbone curve, along with phase plane (build for another equilibrium as well) see fig 3. All phase trajectories can be expressed analytically, however, formulae are rather bulky as well and could be found in (Belenky 98).

The system is definitely not isohronic.

STEADY STATE FORSED MOTION

Equivalent Linearization

Since we have the backbone curve, the next step is evident, we can get approximate solution for steady state force motion using equivalent linearization. It means that we substitute real system by linear one that has the same period of free oscillation. We can do the same procedure with piecewise linear system (Belenky 95-a)

$$\phi_a = \frac{\alpha_E}{\sqrt{\left[\left(\omega_\phi(\phi_a)\right)^2 - \omega^2\right]^2 + 4\delta^2\omega^2}} \qquad (6)$$

Appearance of the approximate response curve is show in fig 4. Phase curve can be calculated analogously.

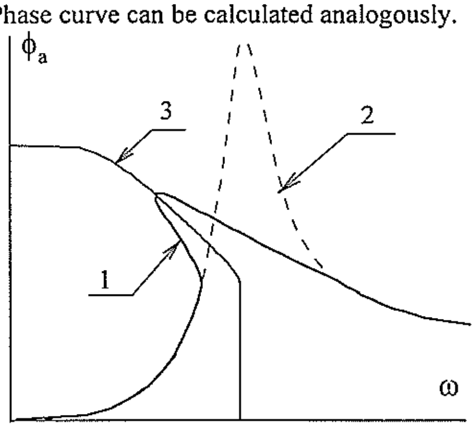

Figure 4- Response curve of piecewise linear system by equivalent linearization (1), response curve of linear system at the first range (2) and backbone curve (3) α_E=0.2, ϕ_{m0}=0.5, δ=0.1 s^{-1}, k_1= k_2=1 s^{-2}, ϕ_v=1, bias 0.05

Exact Steady State Solution

Steady state piecewise linear solution consists form fragments of linear solutions (2) as well as transition one. The only difference between transition and steady state solutions are crossing velocities and periods of time spent in different ranges of piecewise linear term. So if we find such figures that provide periodic solution with excitation frequency, steady state problem will be solved, (Belenky 97-a). These conditions can be formalized as a system of simultaneous algebraic equations, if we look at unbiased case first, it is enough to consider just half of period:

$$\begin{cases} f_0(T_0, \dot\phi_0, \varphi_0) = \phi_{m0} \\ \dot f_0(T_0, \dot\phi_0, \varphi_0) = \dot\phi_1 \\ f_1(T_1, \dot\phi_1, \varphi_0 + \omega T_0) = \phi_{m0} \\ \dot f_1(T_1, \dot\phi_1, \varphi_0 + \omega T_0) = -\dot\phi_0 \\ T_0 + T_1 = \pi \cdot \omega^{-1} \end{cases} \qquad (7)$$

Here functions f_0 and f_1 are solutions (2) at the first and second ranges of the piecewise linear term correspondingly, see also fig.5.

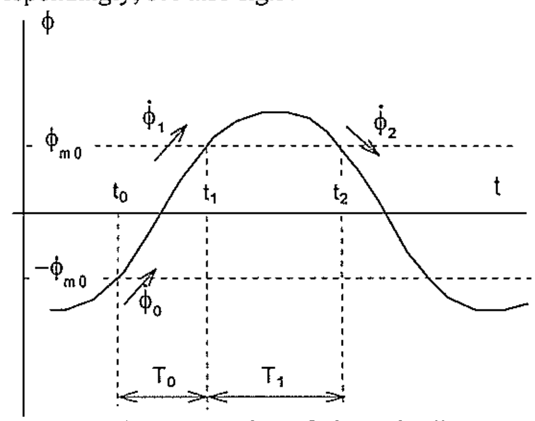

Figure 5-Steady state motion of piecewise linear system

The system (7) can be solved relative to unknown values T_0, T_1, $\dot\phi_0, \dot\phi_1$, and φ_0 using any appropriate numerical method. Results of equivalent linearization can be used for calculation of initial values of the unknown values, that makes calculations more fast and simple.

Biased case is more complicated. Main difficulty here is not only to consider entire period of motion, here we meet so-called «cross mode problem» Asymmetry caused by bias may affect on number of crossings per period: it is not necessary four as fig. 5 shows; period can contain two crossings as well. So considering biased steady state motions (that is especially important for further bifurcation analysis), we need to know in advance how much crossing will be hosted by one period. It can be done by searching frequency that provides «two crosses and one touch»; we use the system analogous to (7), that has excitation frequency as unknown value as well, see details in (Belenky 98). Resulting response curve is shown in fig. 6

As it could be seen from figure 6, response curve has quite conventional form, including hysteresis area, where three amplitudes corresponds to one excitation frequency.

We call this steady state solution exact despite numerical method was used to calculate crossing characteristics; accuracy is still controllable: we always can substitute these figures into system (7) and check how solution turns equations into equalities.

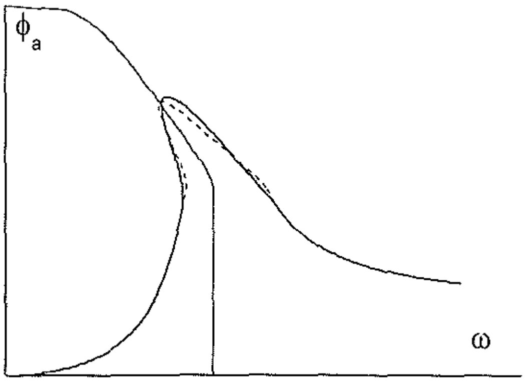

Figure 6- Exact response curve of piecewise linear system. (dotted line shows response of linearized solution) α_E=0.2, ϕ_{m0}=0.5, δ=0.1 s^{-1}, k_1= k_2=1 s^{-2}, ϕ_v=1, bias 0.05

We can call this steady state solution analytical (or, at least semi-analytical), despite the numerical method was used; solution still defined by formulae (2), so we still can manipulate it analytically.

MOTION STABILITY AND BIFURCATION ANALYSIS

The next conventional step is the motion stability determination. Analogous problem was considered in (Murashige 1998) for piecewise nonlinear system. We also will search stability indicators as characteristics of Jacobian matrix (eigenvalues and trace-determinant)

So we calculate Jacobian matrix for each range. The resulting Jacobian can be calculated as a product of the above, with regard on cross mode:

$$J_{4cross} = J_3 \cdot J_2 \cdot J_1 \cdot J_0 \qquad (8)$$

$$J_{2cross} = J_1 \cdot J_0 \qquad (9)$$

Partial derivatives of Jacobian matrix should be calculated numerically. Analytical expression for these figures are not available, because formulae (2) cannot be inverted in elementary functions.

Results of motion stability calculation are shown in figures 7 and 8.

Figures 7 and 8 indicates presence of unstable steady state regimes, fold and flip bifurcations. Let's examine them more close.

We get three responses in the hysteresis area, one of them is pure linear or trivial, so it is definitely stable. Two piecewise linear response were obtained from the same system of equation (like (7) depending on cross mode) using two different initial points. One of these initial points corresponds to high amplitude response of equivalently linearized solution; another one is from the middle one. The middle solution is unstable, the high one - stable see fig 9.

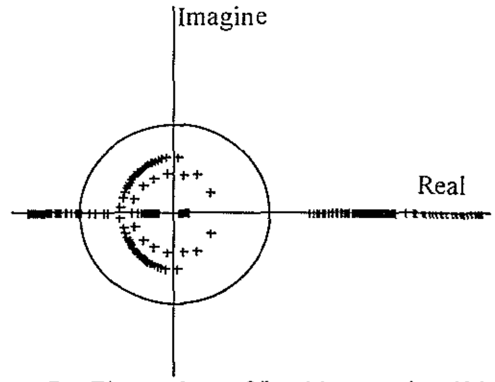

Figure 7. - Eigenvalues of Jacobian matrix of biased piecewise linear system

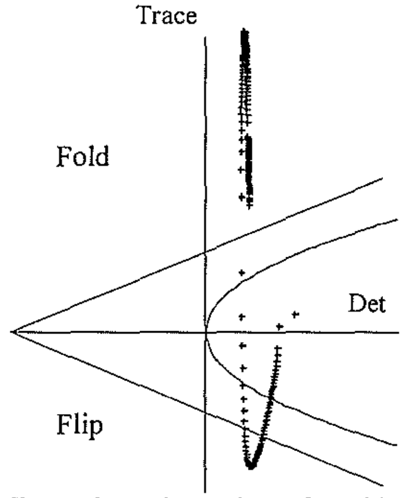

Figure 8 - Trace -determinant plane of Jacobian matrix of biased piecewise linear system

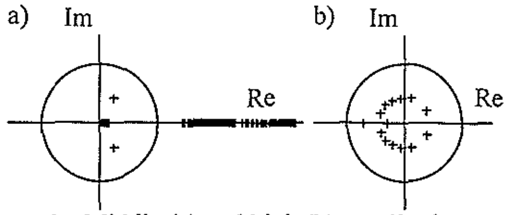

Figure 9 - Middle (a) and high (b) amplitude response stability indexes

Eigen values escape unit circle through positive direction, what is an indication of fold bifurcation. To see it of phase plane, we should reproduce unstable steady state regime and then disturb it in eigen vectors direction: the system will «jump» towards to stable mode, see figure 10. Such type of behavior exactly the same in piecewise linear and nonlinear systems.

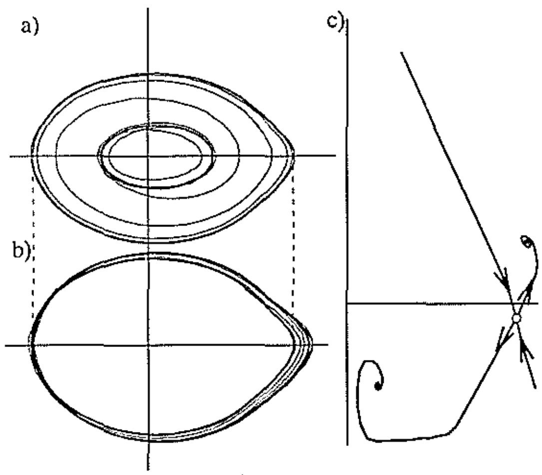

Figure 10- Fold bifurcation in piecewise linear system: «jump down» (a), «jump up» (b), invariant manifold (c), ω=0.77

Another possible type of nonlinear behavior is flip bifurcation: sequence of period doubling, see fig.11 leading to deterministic chaos, fig 12.

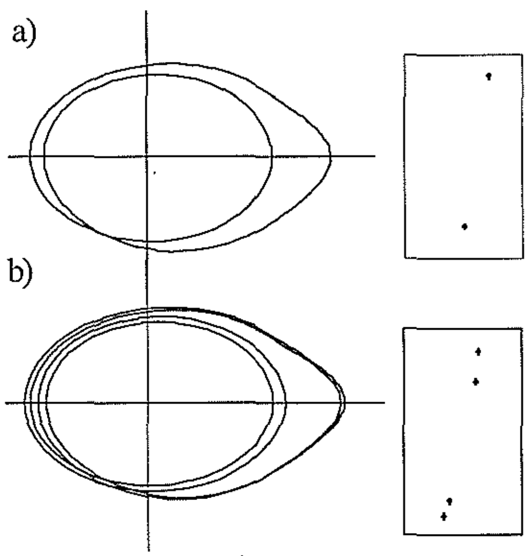

Figure 11 - Flip bifurcation in biased piecewise linear system: phase trajectories and Poincare maps, (a) ω=0.99 (b) ω=0.97.

Form of phase trajectories of piecewise linear system is very similar to conventional nonlinear ones: nothing indicates piecewise linear origin of figures 10-12.

Concluding motion stability and bifurcation analysis we can state that piecewise linear system is capable for such nonlinear type of behavior as motion instability leading to bifurcations; moreover, it is possible to use conventional tools for nonlinear analysis of piecewise linear system.

Flip bifurcation was also found for unbiased piecewise linear system (Belenky 98).

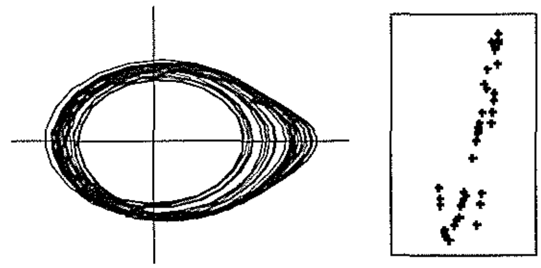

Figure 12 Deterministic chaos in piecewise linear system. ω=0.92439

SAFE BASIN ERROSION

Another important nonlinear quality of severe ship rolling is erosion of safe basin (Rainey ,1990), (Falzarano, 1990). Relative area of the safe basin was considered as stability criteria in (Rainey 1990).

We check if the piecewise linear system is capable for this type of behavior. Calculation were carried with the resolution 90x90 for a square: $\pm 1.5 \cdot \phi_V \times \pm 1.5 \cdot \phi_V \cdot \omega_\phi$.

Excitation frequency was used as a control parameter. All other parameters were the same as for above samples. Some of the results are shown on fig.12.

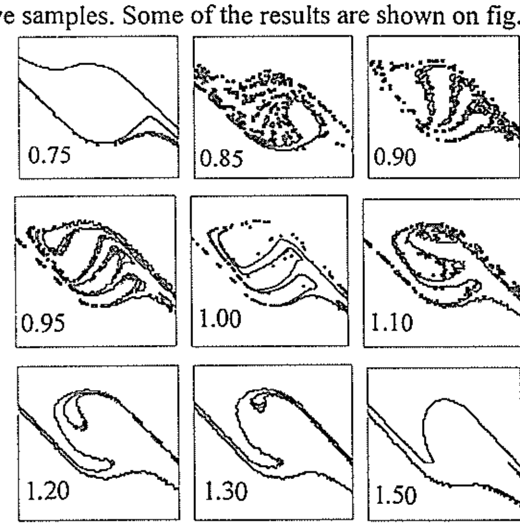

Figure 12 - Erosion of safe basin of piecewise linear system

The results are summarized in a form of dependence of the safe basin relative area on excitation frequency:

$$S_R(\omega) = \frac{A_{SB}(\omega)}{A_{SB}\big|_{\alpha_E=0}} \qquad (10)$$

Here: $A_{SB}(\omega)$ - area of the safe basin at given frequency, $A_{SB}\big|_{\alpha_E=0}$ - area of safe basin of free damped motion; see fig.13.

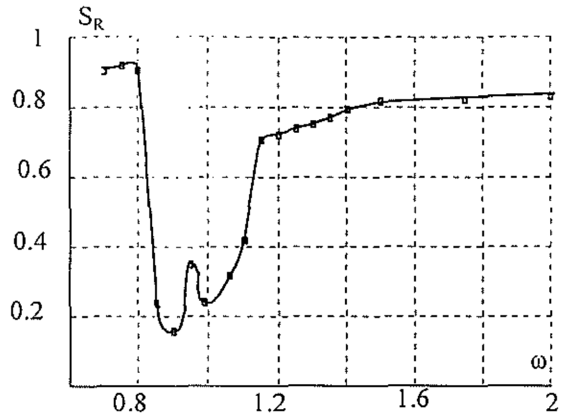

Figure 13- Relative safe basin area vs. excitation frequency.

As it could be clearly seen from figures 12 and 13, the safe basin of piecewise linear system experiences erosion, that leads to decreasing of its area. This behavior is similar to conventional nonlinear one.

FINAL COMMENTS: ADEQUACY OF PIECWISE LINEAR MODEL

The above study has shown that:

- Free motion of piecewise linear system is not isohronic; period of undamped free oscillations depends on initial amplitude (initial heel angle);
- Equivalent linearization can be applied to approximate characteristics of steady state motion of the piecewise linear system;
- Exact characteristics of steady state motion can be calculated using the only assumption that this motion is periodic; exact steady state motion can be expressed by elementary functions, however, some of parameters are results of numerical solution of a system of simultaneous algebraic equations
- Response curve of a piecewise linear system (both approximate and exact) contains non-functionality (or hysteresis): an area where three amplitudes correspond to one frequency;
- Conventional stability analysis can be applied to a piecewise linear system.
- Piecewise linear system is capable for fold and flip bifurcations, consecutive flip bifurcation leads to deterministic chaos in piecewise linear system.
- Safe basin of piecewise linear system experiences erosion, when waves become dangerous.

Since a piecewise linear system is capable for conventional nonlinear behavior it is adequate tool for qualitative analysis of nonlinear ship dynamics.

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