On Risk Evaluation at Extreme Seas

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Abstract

The paper revisits some problems, solution and available methods related to probabilistic aspects of heavy roll motion in extremely high seas, including parametric resonance.

With the increasing power of computers, numerical solutions become the standard tool to solve nonlinear differential equations. The paper revisits the conventional technique of reconstruction of irregular waves with inverse Fourier transformation focusing on particular subject of the representative length of the realization.

The necessity to consider behavior of a nonlinear system under stochastic excitation poses a number of new problems: many assumptions acceptable for linear or linearized systems are not applicable for the dynamical system where nonlinear terms are essential. The response of a nonlinear dynamical system is not necessarily ergodic, even if the excitation is proven to be ergodic. The paper considers implications of ergodicity assumption for accuracy of statistical variance estimates of the nonlinear response.

The intention of this paper is to instigate a discussion, then offer a practical solution.

Introduction: Numerical Simulations and Statistics of Stochastic Processes

A word of praise on virtues of numerical simulation would be more than trivial nowadays. A statement on the importance of interpretation problems is also quite commonplace. At the same time, a combination of non- linearity of a dynamical system with severe irregular waves poses a challenge exactly for interpretation of results.

When it comes to numerical simulation with stochastic processes involved, the result is approximate because of assumptions involved in formulation of the dynamical system and limited accuracy of the numerical solution. If one could imagine that the dynamical system is perfectly modeled and the numerical method yields a correct result till the very last digit, the resulting number would still be just an estimate, because of the finite volume of statistical data available for processing.

A conventional way to measure error caused by the finite volume of statistical data usually involves the confidence interval and the confidence probability. Generally, any result obtained with a finite volume of statistics is a random number that estimates the true probabilistic figure. Assuming a certain value for the confidence probability, it is possible to calculate the boundaries of the confidence interval, where this true figure lays. As the volume of statistics increase, these boundaries tend to shrink, as the statistical accuracy increases.

A technique for calculation of the confidence interval is well established for random numbers. However, it cannot be used directly for stochastic processes as such a technique usually assumes independence of realizations, which is true for a random number, but not necessarily true for a stochastic process. The technique for calculation of confidence interval for stochastic processes is discussed further in this paper.

Most stochastic processes possess a certain inertia; their current reading cannot change at once. It means that the current reading depends on the previous one. The autocorrelation function is a way to quantify this dependence; it is defined as:

$$R(t_1, t_2) = M\{(X(t_1) - m(t_1)) \cdot (X(t_2) - m(t_2))\} \qquad (1)$$

Here and further in the text the symbol $M\{..\}$ defined as the averaging operator, $X(t_1)$ is the value of the stochastic process at the moment t_1 (or section at t_1). As the stochastic process is meant to have an infinite number of realizations, the figure $X(t_1)$ is a random number that is also meant to have an infinite number of realizations; $m(t_1)$ is the theoretical average value of this random number and, simultaneously, a value of average of stochastic process X calculated at the moment t_1. Quantities $X(t_2)$ and $m(t_2)$ have the same meaning as above, but correspond to the moment t_2. The correlation function is defined as the correlation moment between two sections of a stochastic process and it is a function of two variables.

Considering short-term problems, the sea state is usually assumed constant. As a result of this assumption, the stochastic process of wave elevation is stationary which means that its average, standard deviation, variance and other similar characteristics are independent of time:

$$m(t_1) = m(t_2) = m_X \;; \quad V(t_1) = \sigma^2(t_1) = V(t_2) = \sigma^2(t_2) = V_X = \sigma_X^2 \tag{2}$$

Where σ and V are used for standard deviation and variance respectively. The correlation function of the stationary process depends only on the time difference between the sections and does not depend in particular on where these sections were taken; as a result it becomes a function of one variable only:

$$R(t_1, t_2) = R(t_2 - t_1) = R(\tau)\,; \qquad \tau = t_2 - t_1 \tag{3}$$

The probability density distribution (as well as cumulative distribution) also does not depend on time: their definition at any section will be identical:

$$f(X,t) = f(X(t_1)) = f(X(t_2)) = f(X) \tag{4}$$

Some stationary processes also possess a property of ergodicity; it means that their probabilistic characteristics could be defined with only one realization, for example:

$$m_X = M\{X(t)\} = \int_0^\infty X f(X)dX = M\{x(t)\} = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^T x(t)dt\right] \tag{5}$$

Here the term $x(t)$ is used to identify one particular realization of the entire process $X(t)$, which theoretically is meant to contain an infinite number of realizations like $x(t)$. Variance and standard deviation is defined in the same way:

$$\begin{aligned}\sigma_X^2 = V_X = M\{(X(t) - m_X)^2\} &= \int_0^\infty (X - m_X)^2 f(X)dX = \\ &= M\{(x(t) - m_X)^2\} = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^T (x(t) - m_X)^2 dt\right]\end{aligned} \tag{6}$$

Analogous expressions exist for autocorrelation function of on ergodic process:

$$\begin{aligned}R(\tau) &= M\{(X(0) - m_X)\cdot(X(\tau) - m_X)\} = \\ &= \int_0^\infty\int_0^\infty (X(0) - m_X)(X(\tau) - m_X) f(X(0), X(\tau))\, dX(0)dX(\tau) = \\ &= M\{(x(0) - m_X)(x(\tau) - m_X)\} = \\ &= \lim_{T\to\infty}\left[\frac{1}{T-\tau}\int_0^{T-\tau} (x(t) - m_X)(x(t+\tau) - m_X)\, dt\right]\end{aligned} \tag{7}$$

Here $f(X(0), X(\tau))$ is a combined distribution of sections of the process at the moments 0 and τ.

The autocorrelation function also has an important relation with the spectral density $S(\omega)$:

$$R(\tau) = \int_0^\infty s(\omega)\cos\omega\tau\, d\tau\,; \quad s(\omega) = \frac{2}{\pi}\int_0^\infty R(\tau)\cos\omega\tau\, d\omega \tag{8}$$

Reconstruction of the Irregular Waves: Length of Realization

Following [St. Denis and Pierson 1953], the inverse Fourier transformation is the most common way to present irregular waves defined with their spectral density $s(\omega)$:

$$\zeta_W(t) = \sum_{i=1}^{N_\omega} r_{Wi}\cos(\omega_i t + \varphi_i) \tag{9}$$

Here, the set of initial phases φ_i is comprised of random numbers, distributed uniformly from 0 to 2π; this is the only stochastic figure in the model. The amplitude of the component r_{Wi} is calculated with the spectral density:

$$r_{Wi} = \sqrt{2S_{Wi}} = \sqrt{2\int_{\omega_i - 0.5\Delta\omega_i}^{\omega_i + 0.5\Delta\omega_{i+1}} s(\omega)d\omega}\,; \qquad \Delta\omega_i = \omega_i - \omega_{i-1} \tag{10}$$

The set of frequencies ω_i has to cover a significant part of the spectrum. However, it is not the only consideration when it comes to numerical simulation. It is known from the simulation experience that it is preferred not to distribute these frequencies evenly to avoid generation of the self-repeating process. Strictly speaking, the function presented with formula (9) is periodical, as the number of frequencies is finite. Its period, however, is quite a large number: as it is not difficult to see that this period equals the least common multiple of all the periods of the components. Figure 1 shows a realization of the wave elevation process calculated with formula (9) using a Bretschneider type of spectrum for a significant wave height of 7 meter; 21 evenly distributed frequencies were used.

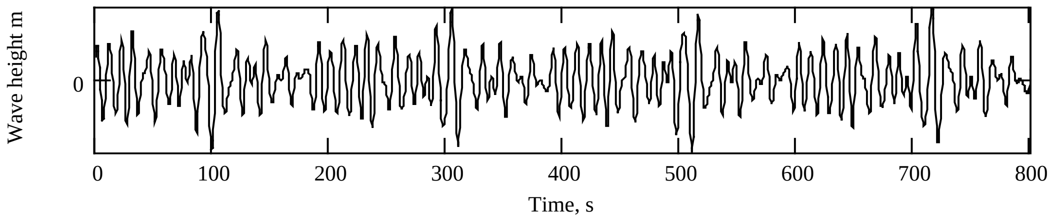

Figure 1. Time history of wave elevation

A closer look at the time history shown in Fig. 1 reveals that there are four wave groups looking suspiciously similar, these groups are shown separately in Figures 2a, 2b and 2c

Figure 2 shows that the first wave group in Figure 2a is very close to the third wave group in Figure 2c, the second wave group in Figure 2b resembles the first group taken but with the sign inverted. Figure 3a shows the first and the second wave group. The latter one is taken with the sign inverted and is superimposed over the first one with a 4 time step shift. The third curve in Figure 3a is a difference between them: it can be clearly seen that these fragments are similar, but not the same. The smaller solid curve shown in Fig. 3a indicates the difference between them. Figure 3b shows a similar picture for the first and the third wave group, with the latter group shifted 4 time steps for better visualization. The smaller solid curve in Fig. 3b shows the difference between the first and the third wave groups.

The similarities between the wave groups are reflected in a character of the autocorrelation function, calculated with formula (7); taking into account that average wave elevation equals zero and the discrete character of the data it can be presented in the form:

$$R_j = \frac{1}{N_t - j}\sum_{i=1}^{N_t - j} x_i x_{i+j} \tag{11}$$

The result of calculations of autocorrelation function is presented in Figure 4. The autocorrelation function in this figure has several characteristic features. Intervals with the high amplitude oscillation are followed with low-amplitude intervals in the form of repeating patterns. The patterns are repeated as the group similarities are encountered. The second and the fourth patterns however, have an inverted sign in comparison with the first and the third one. Amplitude of oscillations in low-amplitude patterns rises: slightly for most of the duration and significantly at the end. The latter feature is eventually the result of decreasing volume of available data for formula (9) at the end of realization.

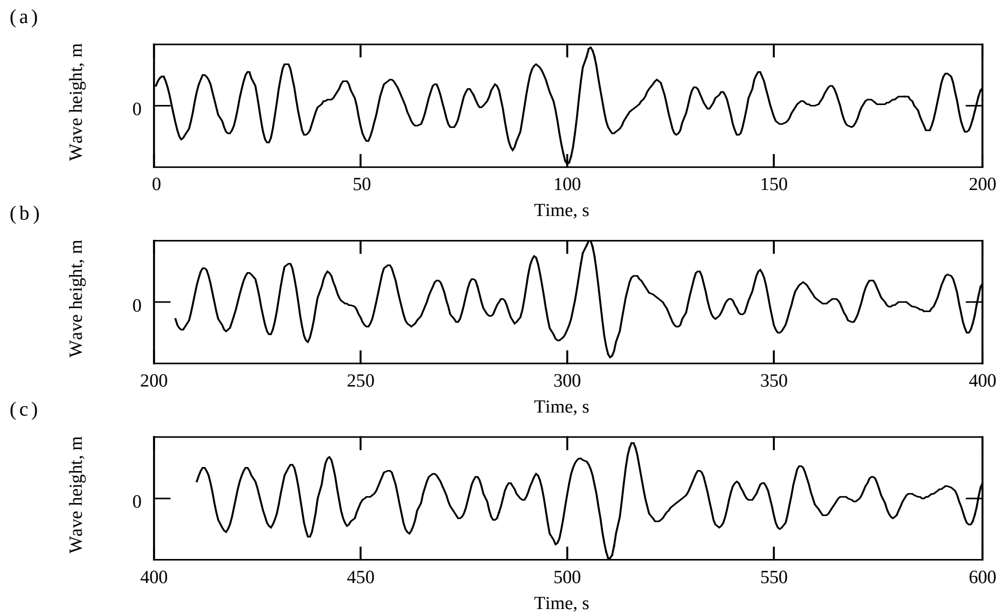

Figure 2 Wave groups in detail scale

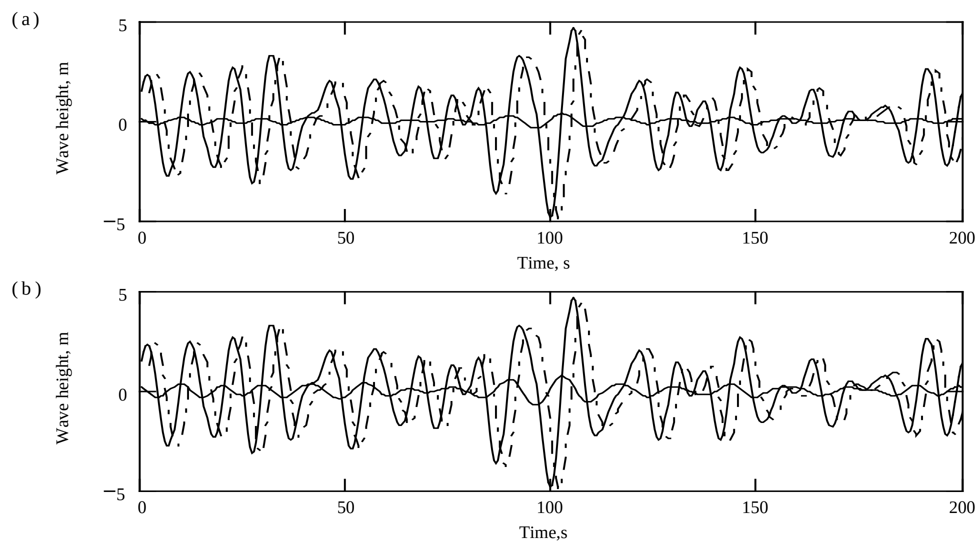

Figure 3 Wave groups and difference between them (a) the first wave group (solid) and inverted second group(dashed) (b) The first wave group (solid) and the third wave group (dashed)

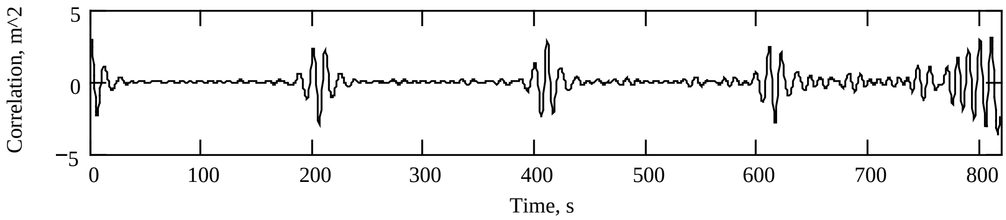

Figure 4 Autocorrelation function for irregular waves calculated by formula (11) with realization in Fig. 1

Formula (11) however is not the only way to calculate autocorrelation function for the process presented with the Inverse Fourier Transformation (9). As the spectral density of this process is known, the relationship between spectrum and autocorrelation function can be used, as it is presented below, using methods of rectangles for numerical integration:

$$R_j = \sum_{i=1}^{N_t} S_{Wi} \cos(\omega_i t_j)\,; \qquad t_j = \Delta t \cdot j \tag{12}$$

Here S_{Wi} is the spectrum value as defined by formula (10), t_j is current moment of time and Δt is a time step. The results are presented in Figure 5. The autocorrelation function presented there is generally similar to the previous one, but does not experience an increase of amplitude at low-amplitude oscillation interval, as the statistics-related accuracy consideration is no longer relevant.

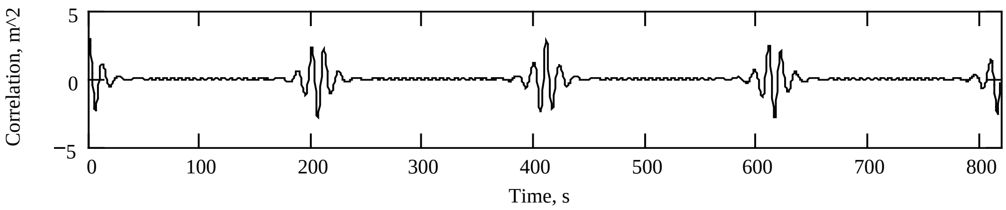

Figure 5 Autocorrelation function for irregular waves calculated by formula (12)

Using the adaptive quadrature method for numerical integration dramatically changes the form of the autocorrelation function, see Figure 6. The Romberg method yields results identical to those shown in Figure 6.

Repeating patterns have disappeared completely from the autocorrelation function, shown in Figure 6. As the only difference between the results in Figure 5 and Figure 6 is the method of integration, it would be logical to state sensitivity of the calculation result of autocorrelation function by formula (8) is due to method of numerical integration applied.

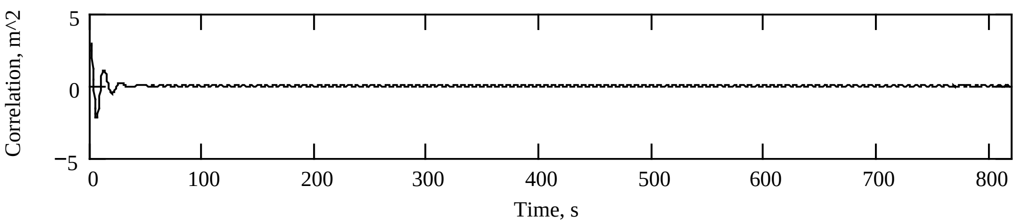

Figure 6 Autocorrelation function for irregular waves calculated by formula (8) using adaptive method

The presence of very similar wave groups and related repeating patterns of the autocorrelation function is a known fact. It is also known (see, for example, [Belenky and Sevastianov 2003]) that the period of these self-repeating patterns depends on frequency step $\Delta\omega$ as:

$$T_R = \frac{2\pi}{\Delta\omega} \tag{13}$$

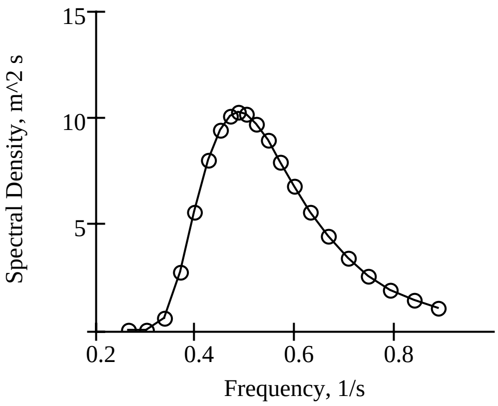

Figure 7. Spectral density with uneven frequency set

The periodicity of the autocorrelation function compromises adequacy of the model when the length of realization is longer than the time T_R. To avoid such inadequacy for evenly distributed frequencies, their number should be increased with corresponding decreasing of frequency step $\Delta\omega$; it solves the problem but increases computational cost, as more components have to be dealt with at every time step.

Another way to resolve the problem is to use non-evenly distributed frequencies: for example, to increase the frequency step by 10% for each following (subsequent) frequency outward of the spectrum peak. A similar algorithm is implemented in LAMP (Large Amplitude Motion Program, for more info see [Shin, *et al* 2003]). This technique results in the spectrum shown in Figure 7 with the realization time history in Figure 8.

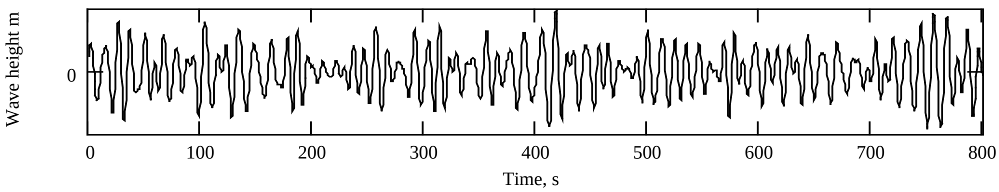

Figure 8 Time history calculated with unevenly distributed frequencies

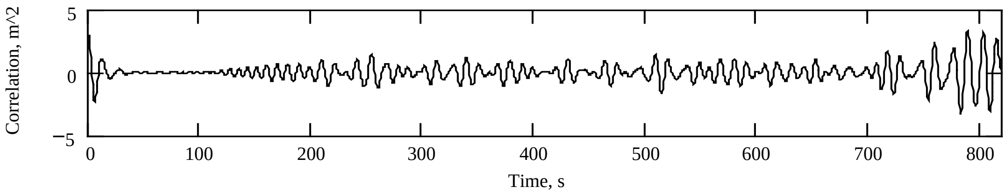

Figure 9 Autocorrelation function for irregular waves calculated by formula (11) with realization in Fig. 8

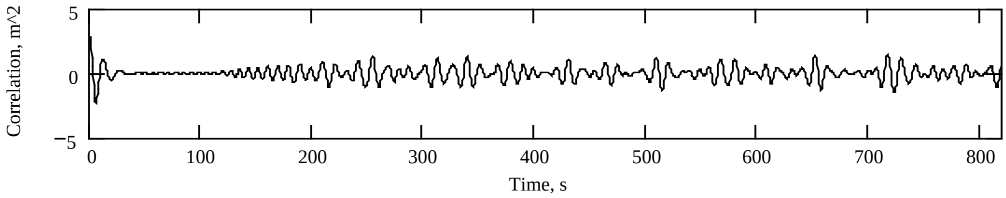

Figure 10 Autocorrelation function for irregular waves calculated by formula (12)

As can be seen from Figure 8, the similar wave groups are no longer present, which is also confirmed by the autocorrelation function calculated with this realization by formula (11) and shown in Figure 9. This autocorrelation function, however, exhibits a noticeable amplitude increase, past the time 110 seconds; these oscillations become visually stationary after approximately 200 seconds and exhibit significant increases of amplitude after approximately 700 seconds. The latter increase is probably caused by diminishing volume of statistics closer to the end of the time range. Similar behavior was observed in the previous case, see Fig. 4. Also, as in the previous case the end-of-the range amplitude increase is not reproduced when the correlation function is calculated directly from the spectrum with formula (12), see Fig. 10.

The noticeable increase of the amplitude of oscillation observed after 110 seconds in the case with unevenly distributed frequencies could be considered as the similar kind of error that has been seen in the first case. The similarity between these two cases could be seen as different discretizations to present the same integral in formula (8) or Fourier integral in rectangular discretization (9). Application of a different method of numerical integration (adaptive method with the result shown in Fig. 6) eliminates both errors. The difference is that the first one is seen as a repeating pattern, while in the second case the error is "spread" along most of the time interval.

Accuracy of Statistical Estimates of Stochastic Process

As it was noted already, any statistical estimate is approximate. Moreover, as a statistical estimate is a result of performing a numerical operation with finite quantity of random numbers, it is a random number, too. As with any other random number, the statistical estimate is characterized with its own probability distribution, mean value, variance, etc.

Gaussian distribution is a conventional assumption for statistical estimates. Evaluation of a statistical estimate involves summation of a large number of random values or deterministic functions of random values. Distribution of a sum of a large number of random values tends towards Gaussian distribution while their number is increasing and if their contribution is comparable as it is established by the Central Limit Theorem. Once a Gaussian distribution is assumed for the statistical estimates of a stochastic process, only two characteristics are to be found: mean value and variance.

The mean value of statistical average equals itself, as it is an unbiased estimate:

$$M\{m^*\} = m^* \tag{14}$$

Here m^* is a mean value estimated for the given finite realization of a stochastic process. Variance of the mean value estimate is determined through the correlation function of the process [Livshitz and Pugachev 1963]:

$$V\{m^*\} = \frac{1}{T^2}\int_0^T\int_0^T R(t_2 - t_1)dt_1 dt_2 \tag{15}$$

Since the process is presented with discrete data points, the above integral has to be rewritten as:

$$V\{m^*\} = \frac{1}{N^2}\sum_i \sum_j R(t_i - t_j) \tag{16}$$

The equation (16) could be optimized by taking into account repeating terms:

$$V\{m^*\} = \frac{1}{N}R(0) + \frac{2}{N}\sum_{i=1}^{N-1} R(t_i) - \frac{2}{N^2}\sum_{i=1}^{N-1} iR(t_i) \tag{17}$$

As it is known the estimate of variance is biased, for random values it could be corrected with the coefficient $N/(N-1)$, which is small when the volume of statistics N is large. Further, N is assumed large and the estimate of variance will be considered as an unbiased estimate.

$$M\{V^*\} = V^* = R(0) = \frac{1}{N}\sum_i (x_i - m^*)^2 \tag{18}$$

Mean value for roll motions as well as for wave elevations could be assumed known and equal to zero – it will significantly simplify further consideration and make perfect sense from a physical point of view as constant bias normally could be evaluated exactly using hydrostatics. As a results:

$$M\{V^*\} = V^* = \frac{1}{N}\sum_i x_i^2 \tag{19}$$

Variance of the variance could be estimated as [Livshitz and Pugachev 1963]

$$V\{V^*\} = \frac{4}{T}\int_0^T R^2(\tau)\left(1-\frac{\tau}{T}\right)d\tau \tag{20}$$

The above equation was derived with the assumption that the process $x(t)$ has the same relationship between the second and the fourth statistical moment as a process with a Gaussian distribution. This formula also should be re-written for the discretized data:

$$V\{V^*\} = \frac{4}{N}\sum_i (R(t_i))^2\left(1-\frac{i}{N}\right) \tag{21}$$

The equations above along with an assumption of Gaussian distribution of statistical characteristics estimated with the finite volume of statistics allow estimating accuracy with confidence intervals. Confidence probability is accepted as

$$P_\alpha = 0.9973 \tag{22}$$

Such confidence probability is conventional for technology-related problems; it almost covers entire range of possible values. It is also known as “six-sigma” probability:

$$P\left(x \in [m_A - 3\sigma_A; m_A + 3\sigma_A]\right) = P_\alpha \tag{23}$$

Where σ_A is the standard deviation of the estimate A of the stochastic process x, while m_A is its mean value, which is equal to estimate A if it is unbiased. Finally, the estimate A could be presented along with its accuracy as:

$$A = m_A \pm 3\sigma_A \tag{24}$$

Further, this technique is applied for evaluation of accuracy of the variance estimate of nonlinear roll motions.

Statistical Accuracy and Ergodic Assumption

If a dynamical system is linear and the stochastic process of excitation is ergodic, the response is also ergodic – its probabilistic characteristics could be evaluated with one realization if it is long enough. If the dynamical system is nonlinear, the response is not necessarily ergodic and evaluation of probabilistic characteristics with only one realization might introduce an error. To avoid this error a sufficient number of realizations has to be used. Two practical questions could be asked: how large is the error and how many realizations provide sufficient accuracy?

Absence or presence of ergodicity is determined relatively to certain probabilistic characteristics. If mean values estimated over different realizations are essentially the same, it means that the process is ergodic in the sense of mean values. It does not guarantee, however, that the variances estimated over different realizations will be the same: the same process might not be ergodic in the sense of variances.

The previous works [Belenky *et al*, 1998, 2001 & 2003] did not indicate an absence of ergodicity in the sense of mean values: further consideration here will be focused on possible non-ergodicity in the sense of variances.

Error due to non-ergodicity in the sense of variance estimates could be quantified as the probability that the difference between two variances estimated over different realizations are caused by purely statistical reasons due to the finite amount of statistical data; such a technique is quite commonly used for evaluation of significance of difference between two mean values, see Fig. 11.

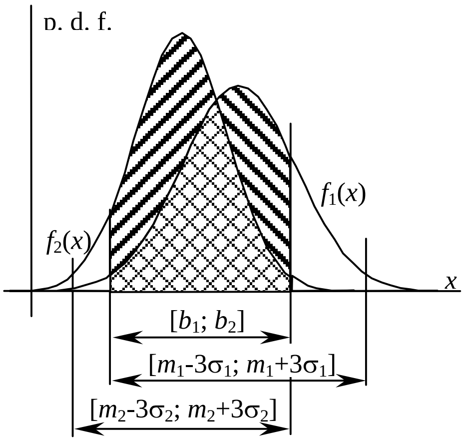

Figure 11 On the definition of probability that both difference between two estimates is caused by statistical factors

Consider variance V_1 estimated over the realization 1 and variance V_2 estimated over the realization 2 with their respective confidence intervals defined with the confidence probability (22):

$$P\left(V_1 \in [m_1 - 3\sigma_1; m_1 + 3\sigma_1]\right) = P_\alpha \tag{25}$$

$$P\left(V_2 \in [m_2 - 3\sigma_2; m_2 + 3\sigma_2]\right) = P_\alpha \tag{26}$$

The probability that the difference between these two estimates is caused by statistical factors only (with the given confidence probability) is to be calculated over the subset of the confidence intervals of these estimates, which is defined as:

$$\begin{aligned}[b_1; b_2] = [m_1 - 3\sigma_1; m_1 + 3\sigma_1] \cap \\ \cap [m_2 - 3\sigma_2; m_2 + 3\sigma_2]\end{aligned} \tag{27}$$

The probability that true variances belong to the above interval is calculated with a Gaussian distribution assumed above for the estimates:

$$P_1 = P(V_1 \in [b_1, b_2]) = \frac{1}{\sqrt{2\pi}} \int_{b_1}^{b_2} \exp\left(-\frac{(x - m_1)}{\sigma_1^2}\right) dx \tag{28}$$

$$P_2 = P(V_2 \in [b_1, b_2]) = \frac{1}{\sqrt{2\pi}} \int_{b_1}^{b_2} \exp\left(-\frac{(x - m_2)}{\sigma_2^2}\right) dx \tag{29}$$

The probability that the true value of both belong to the interval $[b_1, b_2]$ is calculated with simple multiplication of the probabilities P_1 and P_2 as the realizations 1 and 2 are assumed to be independent:

$$P_\beta = P\left((V_1 \in [b_1; b_2]) \cap (V_2 \in [b_1; b_2])\right) = P_1 P_2 \tag{30}$$

If the difference between the variances estimated over the realizations 1 and 2 caused by statistical factors only, the probability P_β should not differ significantly from the square of confidence probability P_α.

$$P_{err} = P_\alpha^2 - P_\beta \tag{31}$$

The above formula expresses the probability that the difference between two estimates is caused not by a statistical insufficiency, but by some systematic error including absence of ergodicity. This probability is defined in the similar way to what is known in statistics as a significance level. Typically the significance level is set up somewhere between 1% and 5%.

To achieve more stable results it is not unreasonable to use more realizations then two. All calculation results which are presented further were performed for 50 realizations. In order to evaluate probability of error P_{err}, the above derivations could be repeated for a chosen number of realizations. Alternatively, P_β could be calculated for each pair of realizations and further averaged over all values:

$$P_\beta = \frac{1}{N_r (N_r - 1)} \left(\sum_i \sum_j P_{\beta ij} - \sum_i P_{\beta ii} \right) \tag{32}$$

Where N_r is the number of realizations available for analysis and $P_{\beta ij}$ is a probability value (30) calculated for pair of realizations with numbers i and j respectively.

Sample Results for the Linear System

As it is known, the response of a linear system on ergodic excitation is also ergodic, the probability of error due to non-ergodicity is expected to be very small. Consider an isolated linear equation of rolling:

$$\ddot{\phi} + 2\delta\dot{\phi} + \omega_0^2\phi = f_W(t) \tag{33}$$

Here, ω_0 is natural roll frequency (accepted numerical value 0.5 rad/s), δ is damping coefficient (accepted as 5% of critical damping), $f_W(t)$ is wave excitation process, accepted as:

$$f_W(t) = \sum_{i=1}^{N_f} \omega_0^2 \frac{\omega_i^2}{g} r_{Wi} \sin(\omega_i t + \varphi_i) \tag{34}$$

Here, g is gravity acceleration, r_{Wi} is amplitude of wave component defined conventionally with wave spectrum see formula (10), φ_i set of random phases with uniform distribution on the range $[0; 2\pi]$ (each realization of the process requires its own independent set of random phases); ω_i frequency of wave component; a set of 500 frequencies was used in this calculation to represent the spectrum in Figure 7 with the frequency step 0.0013 rad/s, which provides a statistically representative realization length of 4875 seconds.

As the solution for the linear differential equation is known as an elementary function, the roll response can be expressed analytically:

$$\phi(t) = \sum_{i=1}^{N_f} \frac{\omega_0^2}{\sqrt{(\omega_i^2 - \omega_0^2)^2 + 4\omega_i^2\delta^2}} \frac{\omega_i^2}{g} r_{Wi} \sin(\omega_i t + \varphi_i) \tag{35}$$

Stochastic process (35) was discretized with a time step of 1 second and reproduced in 50 different realizations with 50 statistically independent phase sets. The autocorrelation function of each realization was calculated with formula (11) and its sample for one of the realizations is shown in Fig. 12a. As can be seen from this Figure, the correlation function contains "statistical noise" that does not make physical sense and can be filtered out with the weight function as shown in the formula below.

$$R_{fi} = \left((N_t - i)/N_t\right)^2 \cdot R_i \tag{36}$$

The "filtered" correlation function is presented in Fig.12b

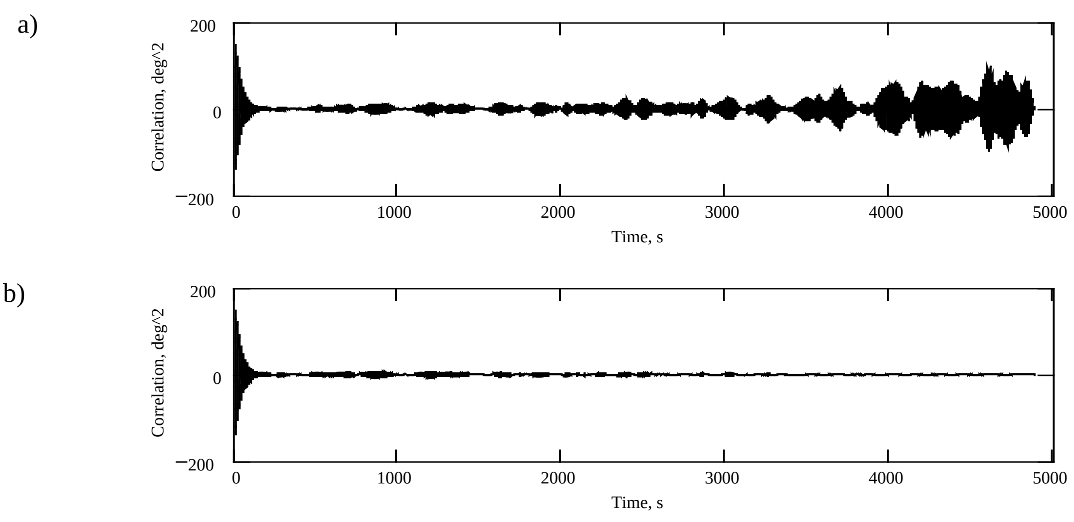

Figure 12 Sample of correlation function of response of linear system (a) original, calculated with (11) and (b) "weighted" as defined by (36)

Figure 13 shows variance estimates for all the 50 realizations presented with their respective confidence intervals. Probability of error due to possible non-ergodicity calculated by formula (32) was equal to 0.0823%, which is a very small probability, the response process can be considered as ergodic, so the method described yielded a correct answer.

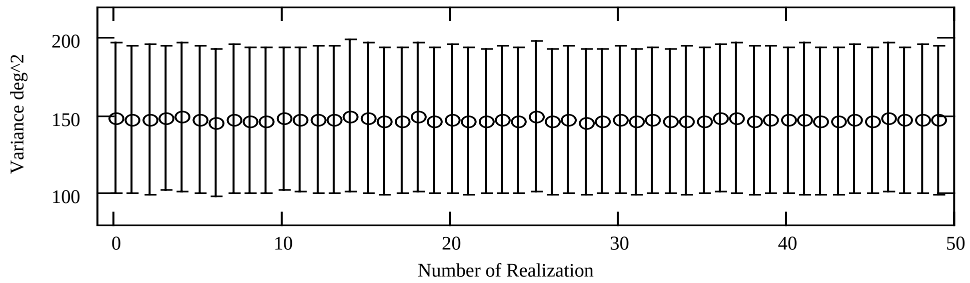

Figure 13 Variance estimate with confidence intervals of response of linear system

Sample Results for the Mild Nonlinear System

The response of a nonlinear system on ergodic excitation is not necessarily ergodic, however a possible error may depend on how nonlinear the system is; [Belenky, *et al*, 1998] demonstrated the confidence interval width increases with increasing influence of nonlinear factors. Consider the isolated equation of rolling with the restoring term in the form of a cubic parabola, see Fig. 14:

$$\ddot{\phi} + 2\delta\dot{\phi} + \omega_0^2\phi(1 - a_3\phi^2) = f_W(t) \qquad (38)$$

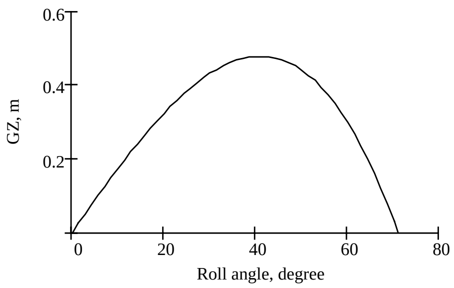

Figure 14 Restoring term for the system with mild nonlinearity (in terms of GZ curve)

Here a_3 is a third-power coefficient; accepted value is 0.65. Other terms are the same as they were in the linear case. The system (38) could be characterized as a mild nonlinear in the statistical sense, because smaller angles have more statistical weight, while extremely high roll angles, where nonlinearity is significant, are relatively rare. Figure 15 shows variance estimates for all the 50 realizations presented with their respective confidence intervals. The probability of error due to possible non-ergodicity calculated by formula (32) was equal to 0.795%, which is still quite a small probability. However, this figure is approximately 9.7 times larger than in the linear case, which shows how significant even mild nonlinearity could be. At the same time, practically the error is not large and the ergodicity assumption can be used.

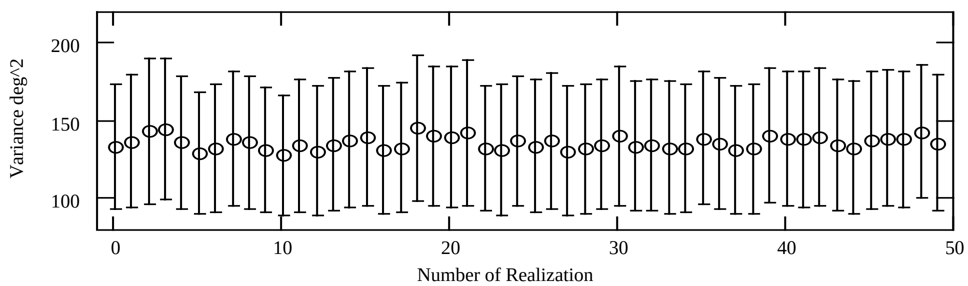

Figure 15 Variance estimate with confidence intervals of response of mild nonlinear system

Sample Results for the Highly Nonlinear System

To create nonlinearity, which will be statistically significant, the GZ curve should have an inflection point within the range of mild roll angles. This makes an “S-shape” GZ curve that is typical for high freeboard vessels such as container carriers. As it is difficult to use a single polynomial to control all the vital GZ curve parameters, a blended polynomial was used: three polynomials were defined for three ranges of roll angles:

$$GZ(\phi) = \begin{cases} a_1\phi + a_3\phi^3 \; ; \quad \phi \in [0, \phi_f\,] \\ b_0 + b_1\phi + b_2\phi^2 + b_3\phi^3 \; ; \quad \phi \in]\phi_f\,, \phi_{\max}\,] \\ c_0 + c_1\phi + c_2\phi^2 \; ; \quad \phi \in]\phi_{\max}\,, \phi_v\,] \end{cases} \tag{39}$$

Here ϕ_f is the roll angle corresponding to the inflection point of the GZ curve, $\phi_{\max}$ is angle of maximum, ϕ_v is angle of vanishing stability (see Fig. 16), coefficients *a*, *b* and *c* are used for the first, second and third range polynomials respectively and their indexes correspond to the power of the variable.

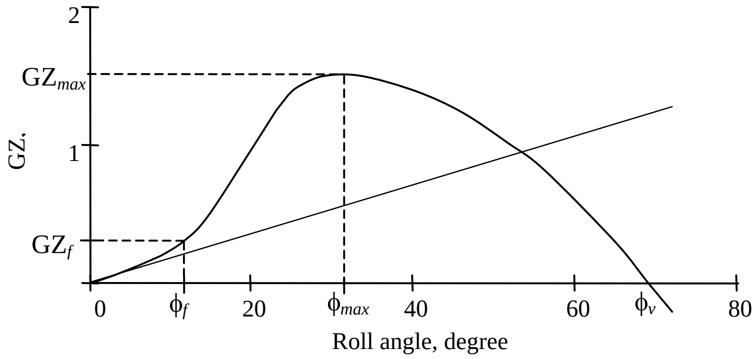

Figure 16 GZ curve presented with blended polynomials

These coefficients can be determined through the procedure, analogous to spline interpolation that requires equality of the first derivatives. The first-range coefficients a_1 and a_3 can be calculated directly:

$$a_1 = GM \; ; \quad a_3 = \left(GZ_f - a_1\phi_f\right) / \phi_f^3 \tag{40}$$

Where GZ_f is the GZ curve value corresponding to the inflection point, see Fig. 16.

The second-range coefficients are to be found through the conditions in boundary points for the function itself (first two equations of the system below) and for its first derivatives at the same point (the third and the fourth equation of the system below):

$$\begin{cases} b_0 + b_1\phi_f + b_2\phi_f^2 + b_3\phi_f^3 = GZ_f \\ b_0 + b_1\phi_{\max} + b_2\phi_{\max}^2 + b_3\phi_{\max}^3 = GZ_{\max} \\ b_1 + 2b_2\phi_f + 3b_3\phi_f^2 = a_1 + 3a_3\phi_f^2 \\ b_1 + 2b_2\phi_{\max} + 3b_3\phi_{\max}^2 = 0 \end{cases} \tag{41}$$

The system of equations (41) is linear relative to the searched coefficients *b* and its solution is trivial. The third range-coefficients are calculated through the conditions at the maximum of the GZ curve formulated both for the function and its first derivative (the first and the third equations in the system below) and through the condition in the angle of vanishing stability used for the function only (the second equation of the system below):

$$\begin{cases} c_0 + c_1\phi_{\max} + c_2\phi_{\max}^2 = GZ_{\max} \\ c_0 + c_1\phi_v + c_2\phi_v^2 = 0 \\ c_1 + c_2\phi_{\max} = 0 \end{cases} \tag{42}$$

The system (42) is linear and finding the coefficients *c* does not pose a problem. Finally, the restoring term can be expressed as:

$$f_R(\phi) = sign(\phi)\frac{\omega_0^2}{GM}GZ(|\phi|) \tag{43}$$

The dynamical system is described by the following differential equation:

$$\ddot{\phi} + 2\delta\dot{\phi} + f_R(\phi) = f_W(t) \tag{44}$$

The system (44) can be characterized as highly nonlinear in a statistical sense, because its restoring term has noticeable nonlinearity in the range of frequently encountered values of roll angles. Figure 17 shows variance estimates for all the 50 realizations presented with their respective confidence intervals. Probability of error due to possible non-ergodicity calculated by formula (32) was equal to 5.74%, which is no longer small and exceeds the conventional level of significance. The process could now be characterized as non-ergodic; however, ergodic assumptions can still be considered for some applications. At the same time the confidence probability that was used for these calculations is very high; a smaller confidence probability being accepted will lead to a larger probability of error due to non-ergodicity.

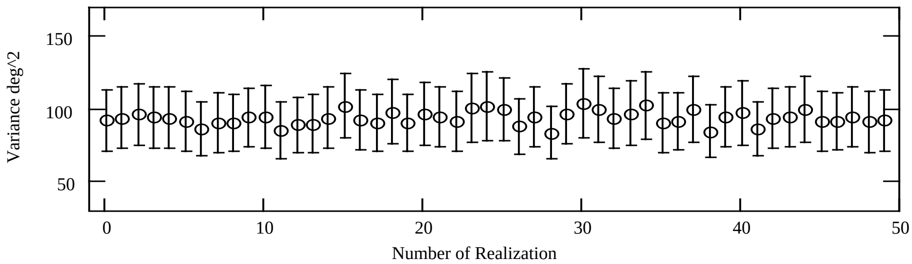

Figure 17 Variance estimate with confidence intervals of response of highly nonlinear system

Sample Results for Parametric Roll

Ergodicity analysis of irregular roll in the regime of parametric resonance in head seas was attempted in [Belenky, *et al* 2003], [Shin, *et al* 2004]. Statistical data was generated by LAMP – Large Amplitude Motion Program, simulation system based on potential body-nonlinear hydrodynamics [Shin, *et al* 2003].

Figure 18 shows variance estimates for all the 50 realizations presented with their respective confidence intervals. The probability of error due to possible non-ergodicity calculated by formula (32) was equal to 26.6%, which is significant and indicates that the process of parametric roll is not ergodic.

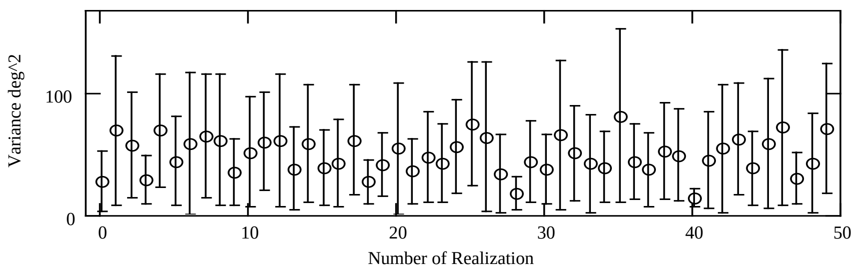

Figure 18 Variance estimate with confidence intervals of response in the regime of parametric resonance in head seas

Figure 18 differs from analogous figures in [Belenky, *et al* 2003] and [Shin, *et al* 2004] by significantly wider confidence intervals. This is a result of application of formula (21) that takes into account the correlation function, while the calculation in the above sources employed a more approximate technique, using the method of calculation of confidence intervals of random variables instead of a stochastic process. As a result, it was only possible to make a relative judgment on what process is less ergodic. The outcome, however, still holds: non-ergodicity of parametric roll has to be addressed while carrying out numerical simulation [ABS 2004].

Behind Non-ergodicity: Notes for the Further Discussion

Among the analyzed processes, the parametric roll allows the largest probability of error due to non-ergodicity. It was noted in [Belenky, *et al* 2003], that because of the group structure of waves, parametric roll may come and go, as the wave group capable to generate and keep this mode is encountered. The number of such groups per wave realization might vary, as their appearance may be dependent on the phase set. If one of the realizations contains more groups capable of generating parametric resonance, the corresponding realization of roll will have a larger variance. The real picture may be even more complicated, because not only the number of wave groups, but the sequence of their encounter may also have an influence, [Belenky and Sevastianov 2003], also [Degtyarev and Boukhanovsky, 2000].

The presence of nonlinearity at the range of relatively small and therefore statistically frequent roll angles has a significant influence on non-ergodicity error. Comparing non-ergodicity errors of responses of dynamical systems with blended curve vs. cubic parabolas as the restoring term has demonstrated ia vivid enough. Generally, the presence of any nonlinearity has an influence, but if the nonlinearity is not statistically significant, the resulting non-ergodicity error is also small. What is the physical reason?

Rare occurrences of fold bifurcation were considered as a possible reason in [Belenky, *et al* 1998]. So far, this possibility has not been ruled out. As the occurrence of fold bifurcation depends on initial conditions, which in their turn might be encountered a different number of times in different realizations, depending on the particular set of phases; as a result one realization may contain oscillations with larger amplitudes and have larger variances than the other. Model tests [Belenky, *et al* 2001], however, have shown that even in the absence of fold bifurcations non-ergodicity is stronger for the nonlinear GZ curve and weaker for almost linear GZ curve. Still, the reason for non-ergodicity of roll in beam seas is not clear…or, maybe, it is more than one reason?

Concluding Comments

The intention to evaluate risk of events in relation with extreme roll motion using numerical simulation poses a series of challenges. In particular, this paper discusses correct simulation of wave realization of given length and statistical accuracy of ergodic assumption for nonlinear roll responses.

The correlation function of wave realizations re-constructed with an evenly and non-evenly distributed frequency set was considered. It was observed that correctness of autocorrelation function might be dependent of the method of numerical integration. Presence of repeating patterns in autocorrelation function is a result of integration with rectangles over the constant frequency step. Using non-evenly distributed frequencies eliminates repeating patterns, but generates statistical "noise" over a significant part of the time range.

The error due to an ergodic assumption could be quantified as the probability that the difference between variances evaluated over two different realizations is not caused by statistical reasons. The method was tested on a linear system - it is known that the response of a linear system is ergodic if the excitation is ergodic. Sample calculations for two nonlinear systems demonstrated significant influence of the nonlinearity in the range of relatively small, but statistically significant roll angles: the error due to possible non-ergodicity increased from 0.7 to 5.7% evaluated for the confidence probability 99.73%. The highest error due to possible non-ergodicity was observed for heavy parametric roll with the probability more than 26%.

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