A perspective on theoretical estimation of stochastic nonlinear rolling

Atsuo Maki, *Osaka University,* maki@naoe.eng.osaka-u.ac.jp
Yuuki Maruyama, *Osaka University,* yuuki_maruyama@naoe.eng.osaka-u.ac.jp
Naoya Umeda, *Osaka University,* umeda@naoe.eng.osaka-u.ac.jp
Yuu Miino, *Tokyo University of Technology*, miinoy@stf.teu.ac.jp
Toru Katayama, *Osaka Prefecture University*, katayama@marine.osakafu-u.ac.jp
Masahiro Sakai, *Osaka University,* sakai@naoe.eng.osaka-u.ac.jp
Tetsushi Ueta, *Tokushima University*, ueta@tokushima-u.ac.jp

ABSTRACT

Development of the probabilistic assessment method for nonlinear ship motion in irregular seas is essential. Particularly, theoretical based method is useful for practical and wider application such as stability evaluation in preliminary design stage. In this research, the method in which Monte Carlo Simulation and theoretical method were combined is newly proposed. Here, from the limited number of Monte Carlo Simulation (MCS) realizations, the unknown coefficients containing in the theoretically obtained non-Gaussian probability density function (PDF) were successfully determined. In this research, the results on roll motion in beam sea condition and parametric rolling in longitudinal waves were shown. Then, it was found that this approach is capable of determining the PDF, and the tail of the obtained PDF shows good agreement with the theoretical results.

Keywords: *Extreme Roll Motion; Capsizing; Parametric Rolling; Probability Density Function* (*PDF*).

1. INTRODUCTION

A reliable and practical probabilistic assessment method for detecting nonlinear ship motion in irregular seas is essential in preliminary ship design stage. In our research field, the subsequent research from Haddara (1974 and 1975), a considerable amount of research from the viewpoint of the modern probabilistic theory has been carried out by Roberts (1982a), Roberts and Vasta (2000), Francescutto and Naito (2004), Kougioumtzoglou et al. (2014), Maki (2017) and Maki et al. (2018). Some of these papers were reviewed in detail in our previous research (Maki, 2017). Besides, Dostal (2012, 2014) established an energy-based stochastic averaging method to predict the ship motion in beam conditions. Results based on his theory were compared with those of the path integration (PI) method proposed by Naess (2000) in the paper of Chai et al. (2017). Chai et al. showed good agreement of Dostal's theory with the result of PI method and Monte Carlo simulation (MCS) results. This kind of theoretical approach has great advantage for saving computational time. Therefore, it is worthy of further development for practical application to an early design stage.

In our previous researches on roll motion in beam seas (Maki, 2017), the authors have proposed the estimation method of the joint probability density function (PDF) of instantaneous roll and roll rate based on the methodology proposed by Sakata et al. (1979 and 1980) and Kimura et al. (1980, 1995, 1998 and 2000). Furthermore, based upon the split time approach proposed by Belenky (1993 and 1994), the estimation method for the probability of capsizing was proposed (Maki, 2017). In this calculation method, the non-Gaussian PDF of roll response was determined by combining the moment method with equivalent linearization technique. As shown in the previous research, the results obtained from this method showed good agreement with MCS results. In the calculation process, the steady solution of the moment equation was obtained by an iterative calculation method such as the Newton method. However, in the Newton method, the choice of the initial value and step size to perform robust convergence to the solution was not easy tasks. In each iterative step, time consuming double-integral of the joint PDF for roll and roll rate is required. Considering these issues, the proposed method is not necessarily universal for practical use.

On the other hand, probabilistic assessment of parametric rolling in longitudinal waves is also important. So far, since Price (1970), Haddara (1975), Muhuri (1980)), Roberts (1982b), a considerable number of researches had been carried out. More recently, Mohamad and Sapsis (2016) provided the methodology to estimate probabilistic response for a Mathieu-type equation under parametric excitation. They successfully captured the non-Gaussianity in the PDF. The theoretical approach for parametric excitation system is still considered to be one of the difficult problems in our field, and the robust methodology for identifying the rare event in longitudinal waves should be established for practical uses. Therefore, in this paper, with use of the same approach for roll motion problem in beam seas, the parametric rolling probability is theoretically obtained via the limited number of MCS realizations.

2. MCS-BASED THEORETICAL PDF APPROACH FOR SHIP MOTION IN BEAM SEA

As stated in the introduction, the authors (Maki et al., 2018) have been conducted by using the equivalent linearization method combining with moment equation method utilized by Sakata et al. (1979 and 1980) and Kimura et al. (1980, 1995, 1998 and 2000). In the beginning, this method is briefly reviewed in this paper.

The equation of motion is shown in Eq. (1). Here, t: time, ϕ: ship roll angle, α: linear damping coefficient, β: quadratic damping coefficient, W: ship weight, I_{xx}: moment of inertia in roll (including added moment of inertia), GM: metacentric height, GZ_i: i-th component of GZ polynomial fit, $M_{wave}(t)$: time-dependent roll moment induced by wind (normalized by I_{xx}), and $M_{wind}(t)$: steady roll moment induced by wind (normalized by I_{xx}). In this study, the overdot denotes the differentiation with respect to time t.

$$\begin{aligned}&\ddot{\phi}+\alpha\dot{\phi}+\beta\dot{\phi}|\dot{\phi}|+\frac{W}{I_{xx}}\left(GM\phi+GZ_2\phi^2\right.\\&\left.+GZ_3\phi^3+GZ_4\phi^4+GZ_5\phi^5\right)\\&=M_{wind}+M_{wave}(t)\end{aligned}\tag{1}$$

In Eq. (1), setting $\omega_0=\sqrt{W\cdot GM/I_{xx}}$, $G_1=\omega_0^2$ and $G_i=\omega_0^2\,GZ_i/GM$ ($i=2,3,4,$ and 5) results in the following:

$$\begin{aligned}&\ddot{\phi}+\alpha\dot{\phi}+\beta\dot{\phi}|\dot{\phi}|\\&+G_1\phi+G_2\phi^2+G_3\phi^3+G_4\phi^4+G_5\phi^5\\&=\sum_{N_{wave}}a_n\cos(\omega_n t+\varepsilon_n)\end{aligned}\tag{2}$$

Here, a_n, ω_n and ε_n describe amplitude, frequency and phase of wave induced roll moment for each discretized wave component.

In our previous research, the authors utilized the following PDF.

$$\begin{aligned}&p_1(\phi,\dot{\phi};d)=C_1\exp\left\{-d_1\left[\alpha H+\frac{8\beta}{9\pi}(2H)^{3/2}\right]\right\}\\&\text{where } H=\frac{1}{2}\dot{\phi}^2+\sum_i\int_0^{\phi}G_i\phi^i\,d\phi\end{aligned}\tag{3}$$

Here, C_1 and d_1 included in Eq. (3) were determined by following integral form conditions.

$$\begin{aligned}&\int_{-\infty}^{\infty}d\dot{\phi}\int_{\phi_{VN}}^{\phi_{VP}}p_1(\phi,\dot{\phi};d)\,d\phi=1\\&\int_{-\infty}^{\infty}d\dot{\phi}\int_{\phi_{VN}}^{\phi_{VP}}\phi^2 p_1(\phi,\dot{\phi};d)\,d\phi=\mathrm{E}[\phi^2]\end{aligned}\tag{4}$$

The range of these integrations was bounded as $[\phi_{VN},\phi_{VP}]$ where they represent the angles of vanishing stability in negative and positive sides. The upper condition in Eq. (3) is a normalization condition of the PDF. In the lower condition, $\mathrm{E}[\phi^2]$ is the variance of roll motion, and it is iteratively determined from a set of moment equation by the Newton method. In this paper, the detailed explanation is omitted for the sake of brevity.

As introduced in Section 1, the Newton method is not always robust and easy to use. Furthermore, in each iterative step, time consuming double-integral scheme is essential. On the other hand, the equation of motion dealt with in the previous paper (Maki, 2017 and Maki et al., 2018) was 1 DoF one, and the calculation amount is not so large. Therefore, the limited number of MCS realizations can be completed in a short time. If small amount (10 or 20 times) of MCS realizations provided the necessary and sufficient information on the form of the PDF, the Newton iteration to solve the moment equation could be bypassed. Of course, the limited amount of MCS realizations cannot detect the tail of the PDF and rare event such as capsizing on its own. However, by using the theoretically obtained PDF for this function fit, the tail behavior of the PDF can be approximately detected. Therefore, even probability of capsizing can be calculated from the obtained approximate PDF.

The calculation conditions are summarized in Table 1, and the utilized GZ curve is shown in Figure1. Figure 2 shows the PDF obtained from MCS. Here, this MCS contains 20 realizations in which each duration is one hour. Since the number of realizations is limited, the obtained points of the PDF is only distributed around the origin. Figure 3 shows the fitted PDF from the results in Figure 2. The utilized PDF form is Eq. (3), and unknown parameters were C_1 and d_1. In this paper, these unknowns can be determined by the Covariance Matrix Adaptation-Evolution Strategy (CMA-ES) (Akimoto et al., 2012, Sakamoto and Akimoto, 2017a and Sakamoto and Akimoto, 2017b). On the other hand, Figure 4 shows the theoretically obtained PDF based upon our previous method (Maki et al., 2018). Although the utilized MCS information in PDF fitting is quite limited as shown in Figure 2, the obtained PDF shape well resembles the theoretically obtained PDF. As long as the theoretical based PDF form is utilized, the PDF is likely to be successfully extrapolated even for tail behaviors.

Table 1: Calculation conditions.

Items	*Values*
Ship displacement (W)	1,500 ton
Natural roll period (T_φ)	10.0 s
Effective wave slope coef. (γ)	0.9
Metacentric height (GM)	1.00 m
Roll damping coefficient (α)	0.03
Roll damping coefficient (β)	**0.00**

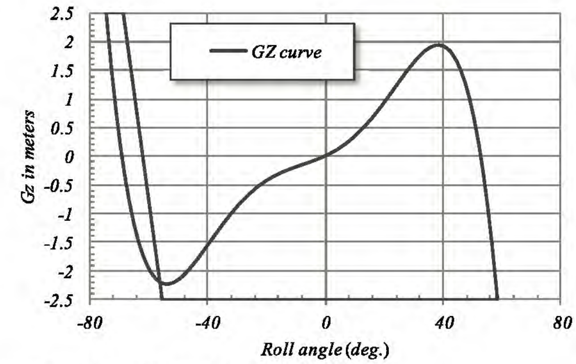

Figure 1: The utilizaed GZ curve.

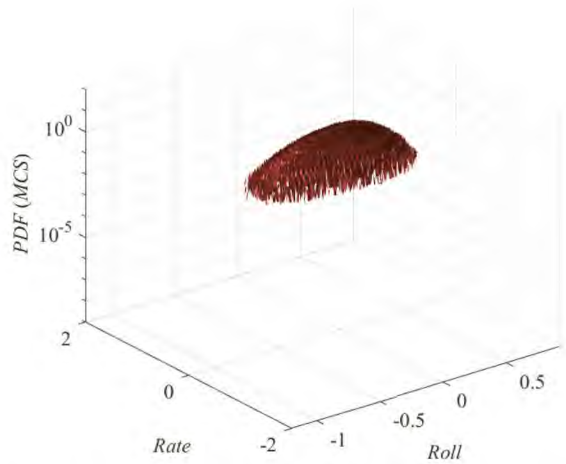

Figure 2: PDF obtained from MCS (3600 s × 20 realizations).

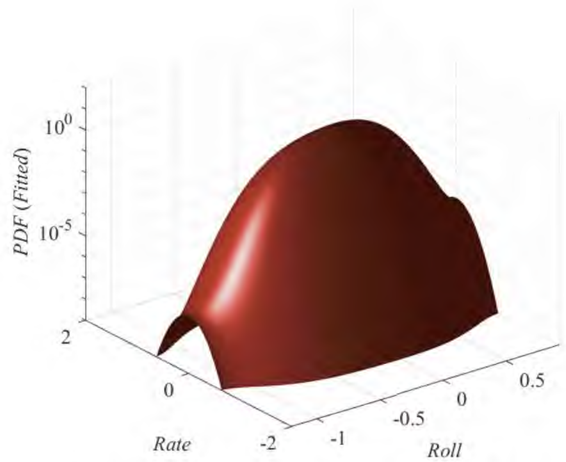

Figure 3: PDF obtained from the present method for MCS.

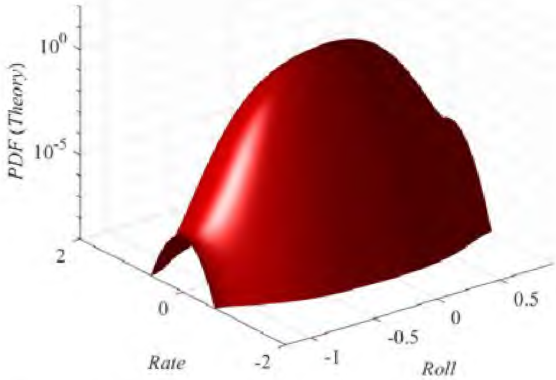

Figure 4: PDF obtained from the theory.

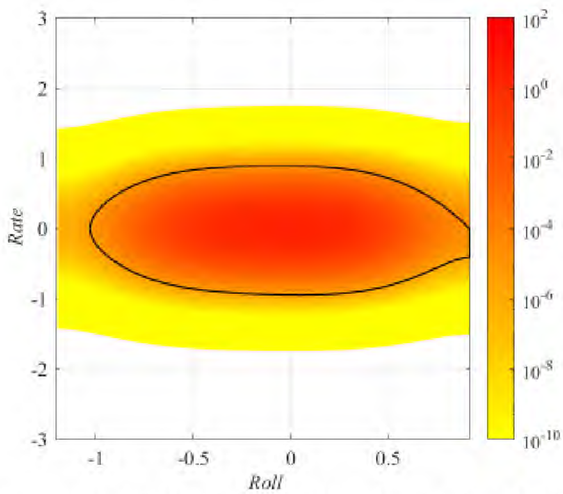

Figure 5: PDF obtained from the theory and safe basin (capsizing boundary in calm water).

Next, let us consider the probability of capsizing. Here, two methodologies for calculating the probability of capsizing are examined. One of them is the method based upon the split time approach proposed by Belenky (1993 and 1994). Belenky divided the restoring range into three ranges. Main range includes a stable upright equilibrium point. Here, this range is called range 0. The other range does unstable saddle type equilibrium point, and these range is called range 1. Range 2 includes another stable equilibrium point, that is capsizing. Since there does not exist the resonance in range 1, the capsizing condition can be approximately determined as a threshold of roll rate. Then, from the up-crossing exceedance probability and roll rate probability at the border between range 0 and 1, the probability of capsizing can be calculated. The previous research (Maki, 2017) illustrated the validity of this split time approach. On the other hand, Umeda et al. (1990 and 1994) proposed the another calculation technique for probability of capsizing due to pure loss of stability in astern waves. In their framework, capsizing probability could be estimated by integrating the joint PDF of roll and roll rate outside the safe basin when a ship meets a wave crest. In the case assumed here, this safe basin is calculated for calm water condition. The example of this safe basin is shown in Figure 5. In this figure, the contour of the joint PDF is also plotted. The final results of capsizing rate are shown in Figure 6. In this figure, "*Method 1*" means the capsizing rates per one second based on the split time approach whereas "*Method 2*" means those based on the double-integral on the phase plane. Blue and red color correspond with the results based upon the MCS-based PDF and the theoretically obtained PDF, respectively. Besides, MCS results of the capsizing rate is also plotted on this figure. This MCS results are obtained from 50,000 realizations of one hour simulation. In this figure, horizontal axis is the threshold (boundary) roll angle of range 0.

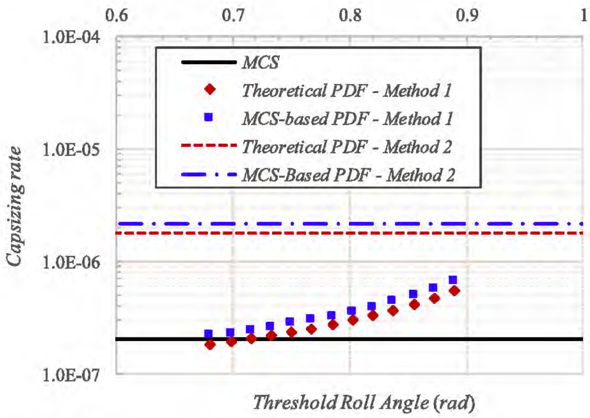

Figure 6: Capsizing rate per one second

First of all, from this figure, only small deference can be found between the theoretical PDF results and the MCS-based PDF results. It demonstrates the validity of the proposed MCS-based theoretical PDF approach. On the other hand, there exists visible discrepancy of capsizing rates between the split time approach and the double-integral method within safe basin on phase plane. The reason of this discrepancy is considered to be came from the complexity of safe basin erosion due to heteroclinic bifurcation. Figure 7 shows the stable and unstable invariant manifolds on the Poincaré map. This is an interesting example of this fractal erosion on the Poincaré map. In this figure, the utilized system is no longer equation (1), but equation (5) having cubic restoring component with regular external roll moment (Duffing oscillator).

$$\ddot{\phi}+\alpha\dot{\phi}+G_1\phi+G_3\phi^3=B\cos\omega t \tag{5}$$

The reason why the equation used here is switched to Duffing oscillator is that instability of the saddle-type fixed point of the original system (5) is too strong for numerical computation. Therefore, the use of original system to the analysis is skipped. However, the qualitative characteristic, such as softening characteristics, is considered to be almost same between the two. These manifolds are calculated by the method proposed by Miino et al. (2019). By using this method, the manifolds are accurately obtained for this stiff system. In this figure, heteroclinic points of stable and unstable manifolds can be found. Furthermore, the stable manifolds of ${}_1D_1^1$ and ${}_1D_2^1$ correspond the shape of safe basin erosion. As illustrated in this figure, the shape of the safe basin greatly changes in accordance with the external moment amplitude. Therefore, the method relying on the double-integral within the safe basin for zero external moment could not necessarily evaluate the probability of capsizing in this case once heteroclinic bifurcation occurs. This problem should be further explored in our future work (Maki et al., to be submitted).

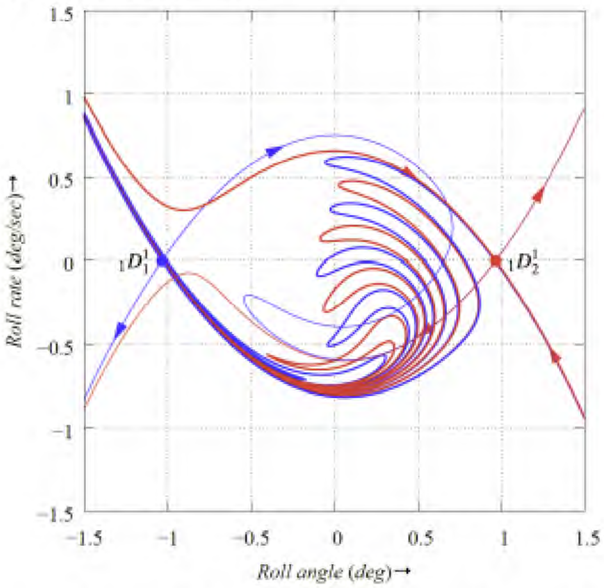

Figure 7: Manifolds of ${}_1D_1^1$ and ${}_1D_2^1$ for Duffing equation on Poincaré map ($\kappa=0.04455$, $G_1=1$ and $G_3=-1$)

As long as surveyed here, the obtained results illustrate that even the limited number of MCS realization provides the sufficient information to determine the unknown parameter included in the theoretically obtained PDF. Furthermore, since the theoretical PDF is utilized, even tail behavior characteristics can be captured within sufficient accuracy. Therefore, as shown in this chapter, the probability of capsizing was successfully calculated. Considered the robustness of the proposed method, this method could be applicable to practical uses such as in the first design stage.

3. MCS-BASED THEORETICAL PDF APPROACH FOR SHIP MOTION IN LONGITUDINAL WAVES

Among many researches on the estimation of parametric rolling probability, the paper written by Roberts (1982b) was the beginning of the research challenge for this topic. Roberts utilized the stochastic averaging technique using the SK limit theorem (Stratonovitch, 1964 and Khasminskii, 1966), then he provided the PDF of parametric rolling. More recently, Dostal showed the extension of the application of the stochastic averaging method (2012).

In this paper, the equation of ship roll motion in longitudinal waves can be represented as Eq. (6). Concerning with the roll damping term, not only linear and quadratic components but also the cubic component is taken into account from the practical viewpoint.

$$\begin{aligned}&\ddot{\phi}+2\zeta\omega_0\dot{\phi}+\beta|\dot{\phi}|\dot{\phi}+\frac{\nu}{\omega_0}\dot{\phi}^3\\&+\omega_0{}^2\left(\phi+k_3\phi^3+k_5\phi^5\right)\\&+\omega_0{}^2\phi q(t)=m(t)\end{aligned} \tag{6}$$

Here, ν: cubic damping coefficient, k_3: 3rd order restoring coefficient, k_5: 5th order restoring coefficient. Based on the stochastic averaging technique proposed by Roberts (1982b), the probability of parametric rolling amplitude can be obtained as:

$$p_s(A)=\frac{C}{A^{1+2\lambda}}\exp\left(-\frac{A}{\mu}-\frac{A^2}{\xi}\right) \tag{7}$$

Here, C is a normalization constant of the PDF, and

$$\lambda=\frac{\zeta\omega_0-S'}{S'},\ \mu=\frac{3\pi S'}{8\beta\omega_0},\ \xi=\frac{8S'}{3\nu\omega_0^2} \tag{8}.$$

As can be seen in Eq. (8), all coefficients include S' as defined in Eq. (9).

$$S' = \frac{\pi S_p(2\omega_0)}{4\omega_0^{\ 2}} \qquad (9)$$

where S_p indicates the spectral density of metacentric height in waves.

Since stochastic averaging method is the approximation technique, there could exists the error in the estimations of the drift and diffusion in FPK equation. Therefore, in this paper, the authors consider S' as unknown parameter, and then S' is determined in the meaning of the least square fit. The obtained results are shown in Figures 8-9. The subject ship is the C11 class containership which is utilized in our previous probabilistic approach (Maki et al., 2011). Figure 8 represents the results with the Froude number (Fn) of 0.000 whereas Figure 9 does those with Fn of 0.207. In these results, "stochastic averaging" represents the results obtained by Roberts's method (1982) whereas "Fitting" does those done by the proposed method in this paper. Besides, MCS results are also plotted.

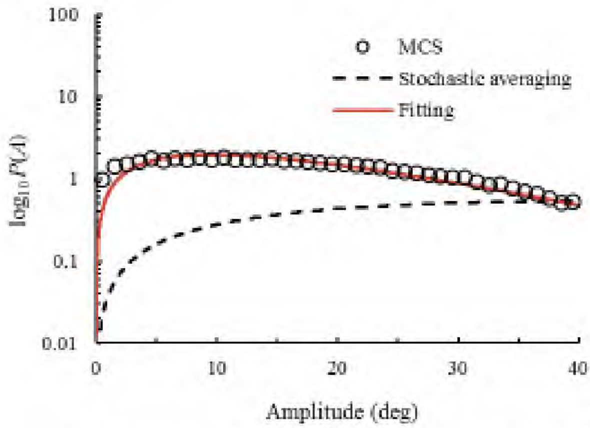

Figure 8: Comparison of roll amplitude PDF between MCS and theory with *Fn* = 0.000 in head seas.

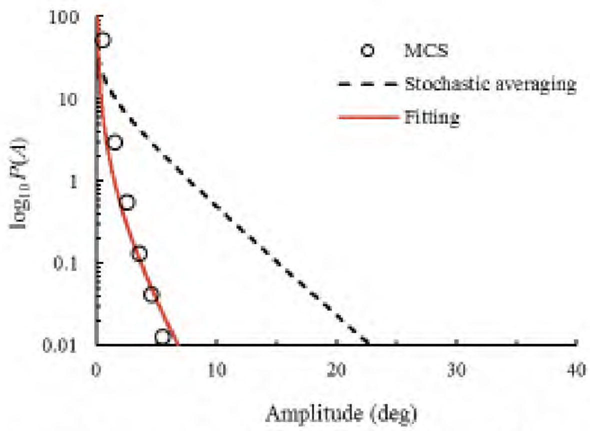

Figure 9: Comparison of the roll amplitude PDF between MCS and theory with *Fn* = 0.207 in head seas.

From these figures, it is found that the results of original averaging method results by Roberts do not show the good agreement with MCS results. These discrepancies are considered to be came from the estimation error of the coefficients in the PDF. On the other hand, the proposed fitting method using non-Gaussian PDF shows satisfactory good agreement with MCS results. This results illustrate the validity of the proposed method, and further exploration could be one of our future tasks.

4. CONCLUDING REMARKS

In this paper, the new methodology to estimate the joint PDF of roll and roll rate from the limited number of MCS realizations was proposed. Here, the unknown coefficients containing in the analytically obtained non-Gaussian PDF were successfully determined. The results obtained for roll motion in beam sea condition and parametric rolling in longitudinal waves show satisfactory agreement with MCS results. This results illustrate the validity of the proposed method. This methodology could be useful for reducing the computation time of level 3 calculation in new generation intact stability criteria.

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