Uncertainty Analysis for Parametric Roll Using Non-intrusive Polynomial Chaos

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ABSTRACT

In this paper, uncertainty analysis is carried out on both a simple parametric roll model which can be modeled as a Mathieu equation and a 1.5 degree-of-freedom parametric roll model in regular seas. For both cases, the uncertainty is brought into the system due to the error in predicting the damping coefficients. The non-intrusive polynomial chaos method has been used to assess how the parameters' uncertainty affects the prediction of parametric roll. The principle aim of this work is to demonstrate computational efficiency without loss of accuracy in probing the effects of parametric uncertainty on ship dynamical systems modeled when using polynomial chaos as compared to the more traditional Monte Carlo method.

KEYWORDS

Parametric roll; uncertainty; polynomial chaos; response surface; ship stability

INTRODUCTION

Uncertainty exists in almost all mathematical models we use to simulate the real world. Uncertainty is typically separated into aleatory uncertainty and epistemic uncertainty (Helton and Davis, 2003). The aleatory uncertainty is due to the presence of some inherent non-deterministic characteristics of the process under analysis and it is also often called stochastic or irreducible uncertainty. The epistemic uncertainty, often called subjective uncertainty or reducible uncertainty is an uncertainty representing our lack of knowledge of (some of) the characteristics of the system under analysis. The role of uncertainty analysis is the study of the propagation of uncertainty from the input to the output through the system shown in Fig. 1. The level of uncertainty in the output, whatever the source, has direct consequences on the accuracy of numerical simulations. An appropriate quantification of uncertainty propagation is therefore of primary importance in the assessment of the quality of the employed mathematical models.

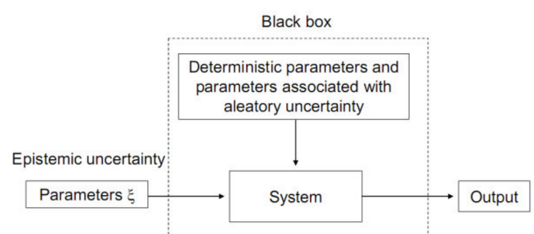


Fig. 1 Diagram of uncertainty propagation in a system

Since relatively little work has been done on the propagation of uncertainty in ship dynamics systems as compared to other fields, in this work, the uncertainty propagation in a parametric roll problem using non-intrusive polynomial chaos (NIPC) is demonstrated. Further comparison with the response surface method (RSM) is also presented.

THE POLYNOMIAL CHAOS METHOD

To analyze uncertainty propagation, one of the most commonly used methods is the Monte Carlo method. While the Monte Carlo method is straightforward to carry out, it is always time consuming due to the large number of samplings required. Polynomial chaos is a method that has been broadly studied recently due to its ability to generate an analytical function for the uncertain random variables. This method is based on the concept of homogeneous chaos proposed by Wiener (1938). Follow Hosder *et al.* (2006), the basic idea of polynomial chaos is to represent a random process (α^*), as a function of uncertain parameters, using orthogonal polynomials.

$$\alpha^*\left(t,\xi\right)=\sum_{i=0}^{P}\alpha_i\left(t\right)\Psi_i(\xi) \qquad (1)$$

where $\alpha_i(t)$ is the deterministic part and $\Psi_i(\xi)$ is the orthogonal basis function of ith mode, in which $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)$ is a n-dimensional random vector with specific probability distribution. A set of orthogonal basis functions means the inner product of any two different basis functions are zero, i.e.

$$\left\langle\Psi_i,\Psi_j\right\rangle=\begin{cases}0, i\neq j\\ \Psi_i^2, i=j\end{cases}$$

where $\langle.\rangle$ is the inner product defined as

$$f(\xi),g(\xi)=\int_{-\infty}^{\infty}f(\xi)g(\xi)p(\xi)d\xi \qquad (2)$$

$p(\boldsymbol{\xi})$ is the weight function that based on different basis of polynomial chaos, for example $p(\xi)=\frac{1}{\sqrt{(2\pi)^n}}e^{-\frac{1}{2}\xi^T\xi}$ for the Hermite Polynomials. Note that, Eq.(1) is the truncated form of a infinite polynomial, in which

$$P+1=\frac{(n+p)!}{n!p!} \qquad (3)$$

where *n* is the number of random variables and *p* is the order of polynomial chaos. According to the original work of Wiener (1938), Hermite polynomials are a set of orthogonal polynomials which span the random space with Gaussian distribution. Cameron and Martin (1947) proved that the Fourier-Hermite polynomial converges to any functional in L_2 in the L_2 sense. This means that in the stochastic processes, "the homogeneous chaos expansions converge to any processes with finite second order moments" (Xiu and Karniadakis, 2002). Hermite Polynomials can be used to expand any arbitrary random processes, but the convergence rate with the order of polynomial chaos is optimal for Gaussian processes (Lucor, *et al.* 2001). For other types of processes, the convergence rate may be slower. Xiu and Karniadakis (2002) have shown other polynomials to construct polynomial chaos expansions. For example, Charlier polynomials with the Poisson distribution, Laguerre polynomials with the gamma distribution, etc. In this work, the random variables are all assumed to be Gaussian; therefore the Hermite polynomials in the form of Eq. (4) are used.

$$H_k=e^{\frac{1}{2}\xi^T\xi}\left(-1\right)^k\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}(e^{-\frac{1}{2}\xi^T\xi}) \qquad (4)$$

where $k=0,1,\ldots,p$.

Once the polynomial chaos in Eq.(1) is fully defined, the statistical moments of the random processes can be easily calculated. The mean $\overline{\alpha^*}(t)=E\left\{\alpha^*\left(t,\xi\right)\right\}$ of the random process is:

$$\overline{\alpha^*}(t)=\alpha_0(t) \qquad (5)$$

And the variance $Var\left\{\alpha^*\left(t,\xi\right)\right\}$ of the random process is:

$$Var\left\{\alpha^*\left(t,\xi\right)\right\}=\sum_{i=1}^{P}\alpha_i^2\left(t\right)\left\langle\Psi_i^2\right\rangle \qquad (6)$$

One can perform either intrusive polynomial chaos, which requires modification for the original codes or the non-intrusive polynomial chaos (NIPC) (Hosder *et al.*, 2006), which treats the original code as a black box. With non-intrusive polynomial chaos, the output variables can be represented as polynomial expansions. A number of samplings of the random variables are used to calculate the response function $\alpha^*(t,\xi)$, then they are projected to the basis functions through the inner product. The deterministic coefficients $\alpha_i(t)$ can be calculated using Eq.(7).

$$\alpha_i(t) = \frac{\langle \alpha^*, \Psi_i \rangle}{\langle \Psi_i^2 \rangle} \tag{7}$$

Based on Hosder *et al.* (2006), there are several ways to calculate $\alpha_i(t)$ in Eq.(7). Since the definition of the inner product is actually integration, one of the ways to calculate them is numerical integration. The quadrature method is used to numerically calculate the numerator in Eq.(7). This method is called quadrature-based NIPC. The other method is called collocation method based on the definition in Eq.(1). One can find the coefficients $\alpha_i(t)$ by solving the following system of linear equations

$$\begin{pmatrix} \Psi_0(\xi_0) & \Psi_1(\xi_0) & \dots & \Psi_P(\xi_0) \\ \Psi_0(\xi_1) & \Psi_1(\xi_1) & \dots & \Psi_P(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0(\xi_m) & \Psi_1(\xi_m) & \dots & \Psi_P(\xi_m) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_P \end{pmatrix} = \begin{pmatrix} \alpha^*(\xi_0) \\ \alpha^*(\xi_1) \\ \vdots \\ \alpha^*(\xi_m) \end{pmatrix} \tag{8}$$

Note that for both NIPC methods, the solutions are evaluated based on the arbitrarily selected random variables, therefore the solutions are not unique. To quote Hosder *et al.* (2006) the current NIPC "should be interpreted as a model for estimating the projected coefficients."

MODELS AND APPLICATIONS

Mathieu Equation

Parametric roll has been a subject of tremendous study as of late, particularly in light of the APL China incident. One simple reduced model used to represent parametric roll is the damped Mathieu equation (see for example Paulling, 2006). The equation of motion of an uncoupled roll motion can be expressed as

$$I_{44}\frac{d^2\phi}{dt^2} + B_{44}\frac{d\phi}{dt} + B_{44C}\left(\frac{d\phi}{dt}\right)^3 + \Delta\phi(GM_0 + CGM_0 cos\omega t) = 0 \tag{9}$$

where I_{44} is the moment of inertia in roll (the added mass is included); ϕ is the roll angle; ; Δ is the ship displacement. The damping term is

$$B_{44}\frac{d\phi}{dt} + B_{44C}\left(\frac{d\phi}{dt}\right)^3$$

Eq.(9) can be rearranged as Eq. (10) by changing of variable $\tau=\omega t$

$$\frac{d^2\phi}{d\tau^2} + \mu\frac{d\phi}{d\tau} + \delta\left(\frac{d\phi}{d\tau}\right)^3 + \left(p + qcos\tau\right)\phi = 0 \tag{10}$$

in which

$$\mu = \frac{B_{44}}{I_{44}\omega};\ \delta = \frac{B_{44c}\omega}{I_{44}};$$

$$p = \frac{\Delta GM_0}{I_{44}\omega^2} = \frac{\omega_n^2}{\omega^2}; q = \frac{C\Delta GM_0}{I_{44}\omega^2} = C\frac{\omega_n^2}{\omega^2}$$

It is well known that for the undamped Mathieu equation, the boundaries for the bounded and unbounded solutions depend on the values of *p* and *q*. When the damping terms are added, there will be threshold boundaries for the stability of the upright position and, due to the presence of a nonlinear damping term, the roll motion will always converge towards zero amplitude (when *(p, q)* are below threshold) or towards a steady state amplitude. In this work, *p* and *q* are set to be some fixed values within the thresholds which corresponds to steady state amplitude (p=0.25, q=0.1) The linear and nonlinear damping coefficients μ and δ are assumed to be the uncertain variables:

$$\begin{aligned} \mu &= \mu_{mean}\left(1 + 0.2\xi_1\right); \\ \delta &= \delta_{mean}\left(1 + 0.2\xi_2\right) \end{aligned} \tag{11}$$

in which $\mu_{mean} = 0.05, \delta_{mean} = 1.2$. ξ_1 and ξ_2 are independent random variables with Gaussian distributions, which both have zero mean and unit standard deviation.

The uncertainty propagation in the damped Mathieu equation is studied using both Monte Carlo and NIPC methods. For the NIPC method, both the quadrature and collocation methods are carried out. Hosder *et al.* (2010) pointed out that for the *pth* degree NIPC quadrature method with *n* random varibles, at least *p+1* Gaussian quadrature points need to be used on each random varible. This is the minimum number of quadrature points needed to evaluate the integral of *p-th* order polynomial which gives total $(1+p)^n$ samplings for the simulation. For the *p-th* order NIPC collocation method, $2(n+p)!/n!p!$ random samplings are used for calculation, which is twice the number of polynomial expansion terms. This number is given by Hosder *et al.* (2010) to be the minimum number required to give better statistics.

The simulation is carried out by integrating Eq.(10) for 1000s for every sampling of parameters. A sample time history of the roll angle ϕ has been plotted in Fig. 2. The steady state is assumed to be reached when time is greater than 600s. The roll amplitude of steady state has been recorded for every sampling. The mean and standard deviation of steady state roll amplitude are calculated in Table 1 using Eqs.(5) and (6). In Table 1, 3rd order polynomial chaos has been calculated which means $p=3, n=2$. Therefore, for the quadrature method, $(1+3)^2=16$ samplings are used. While for the collocation method, the number of sampling used is

$$2\frac{(2+3)!}{2!3!}=20 .$$

For the Monte Carlo method, 50 groups of samples are used to calculate the confidence interval. For each group, 10,000 samplings are calculated. The 95% confidence interval is obtained from a standard error estimate using the data of these 50 groups. The mean and standard deviation values from NIPC methods all fall in the 95% confidence interval in Table 1.

The cumulative distribution function (CDF) has been calculated for different methods. For the Monte Carlo method, the data from one group of the samples are used to calculate the CDF. For the NIPC methods, since the response roll amplitude can be expressed as a polynomial expansion using Eq.(1), the samplings of ξ_1 and ξ_2 from the previous group of Monte Carlo method are substituted into Eq.(1) to calculate the roll amplitude, of which the CDFs are obtained.

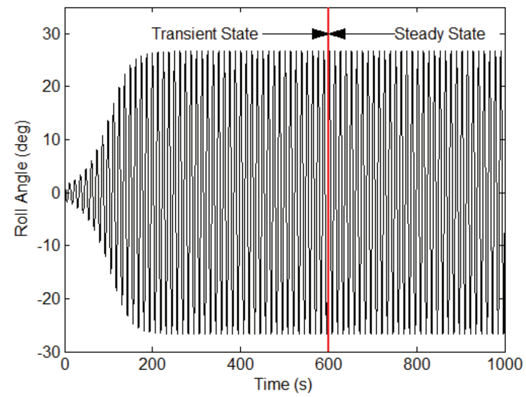


Fig. 2: Time history of roll angle.

Table 1: The mean and standard deviation of steady state roll amplitude (ϕ) calculated using NIPC method. The 95% confidence interval is calculated using Monte Carlo method.

Statistics of roll angle	95% CI of MC	3rd degree NIPC (collocation)	3rd degree NIPC (quadrature)
Mean (deg)	[27.2825, 27.4188]	27.3052	27.3463
Standard Deviation	[4.0840, 4.2413]	4.1527	4.1544

The response surface method (RSM) has also be carried out using the Response Surface Modelling tool in Matlab. Similar to the NIPC method, the RSM method also uses a polynomial to represent the stochastic response. A common polynomial model used in RSM is the quadratic model shown in Eq.(12), which is also the model used in this work.

$$\alpha^*(t,\xi)=\beta_1+\beta_2\xi_1+\beta_3\xi_2+\beta_4\xi_1\xi_2+\beta_5\xi_1^2 +\beta_6\xi_2^2 \qquad (12)$$

in which, ξ_1 and ξ_2 are random variables as shown in Eq.(11). $\{\beta_1,..,\beta_6\}$ are coefficients that need to be determined. One first need to calculate the response based on a couple of samplings. Then the determinstic coefficients $\{\beta_1,..,\beta_6\}$ can be calculated in a Least Square way. The difference between NIPC and RSM is the basis functions in NIPC are orthogonal polynomials, while the basis functions in RSM could be polynomials, trigonometric functions or even exponential functions. To choose the right basis function is one of the difficuties when using RSM.

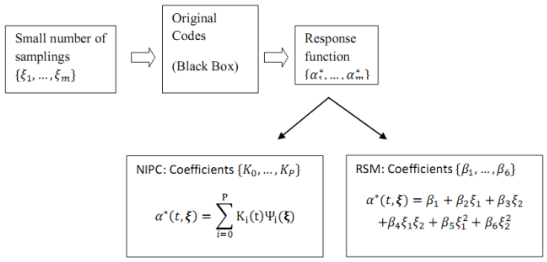


Fig. 3 Schematic diagram of comparison between non-intrusive polynomial chaos (NIPC) and response surface method(RSM).

As shown in Fig.4 (a), 2nd order NIPC collocation has been carried out using 12 random samplings, which result in 12 roll response amplitudes. This data is used as input to the RSM to calculate the coefficients $\{\beta_1,..,\beta_6\}$. The roll angles are then calculated by substituting the samplings from Monte Carlo method into Eq.(12). The CDF of these roll amplitudes can be determined. The CDF from Monte Carlo method, RSM and the NIPC collocation agree well. In Fig.4(b), 3rd oder NIPC has been carried out. In this case, 20 samplings are needed to calculate NIPC collocation. This data is also used to calculate roll angle CDFs using the RSM. In this case, these four methods match even better.

Due to the small number of samples required, the computational time for 3rd order NIPC is only a few seconds using one processor. For the one group of Monte Carlo simultion (10,000 samplings), it took 36 minutes using 32 2.26GHz processors. The NIPC method is much more computational efficient compared to the Monte Carlo method.

1.5 DOF Parametric Roll in Regular Seas

A more complex 1.5 degree-of-freedom parametric roll model was developed by Bulian (2006) to model the roll motion in longitudinal seas. Some assumptions have been made to develop the model:

- The sway and yaw motions are assumed to be neglected;
- The ship speed is assumed to be constant (surge motion is neglected);
- The heave and pitch motions are assumed to be quasi-static.

Under all these assumptions, the equation of roll motion can be written as

$$\ddot{\phi}+d\left(\dot{\phi},\phi\right)+\omega_0^2\frac{\overline{GZ}}{\overline{GM}}=0 \tag{13}$$

in which ϕ is the roll angle; $\overline{GM}$ is the metacentric height in still water; $d\left(\dot{\phi},\phi\right)$ is the damping term; ω_0 is the roll natural frequency in still water. $\overline{GZ}$ is the restoring moment, which is a function of roll angle (ϕ), wave crest position along the ship (x_c) pitch angle and heave displacement. If the pitch angle and heave displacement are related to the roll angle and wave crest position, the restoring moment becomes a function which only depends on the latter two. Bulian (2006) established the 1.5 DOF model by first obtaining the $\overline{GZ}$ data in waves from the hydrostatic software. Then $\overline{GZ}$ has been approximated using the following polynomial:

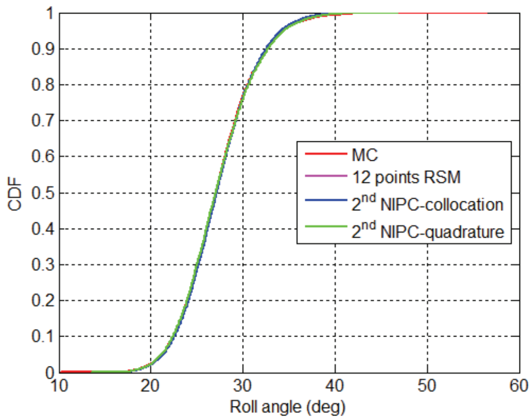
$$\overline{GZ}=\sum_{j=0}^{N}A_j(x_c)\phi^j .$$

The coefficients $A_j(x_c)$ are calculated by Fourier expansions. Finally, the model was recast in the time domain, which results in the $\overline{GZ}$ in the form of a polynomial with roll angle and time dependence. Numerical simulations can be easily carried out for this model. More details about this model are available in Bulian (2006).

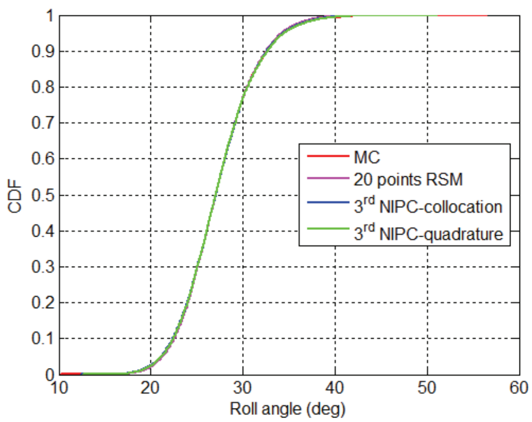
In this work, the uncertainty in the damping term has been considered. Available data for the RoRo ship TR2 (Bulian, 2006) has been used in this work for simulation. The damping term can be expressed as

$$\mu\frac{d\phi}{d\tau}+\delta\left(\frac{d\phi}{d\tau}\right)^3,$$

where the uncertainty in μ and δ is modeled by Eq.(11). The wave length is assumed to be equal to ship length, and wave steepness is 0.01. The hydrostatic data has been calculated with the ship free to trim and sinkage. The uncertainty



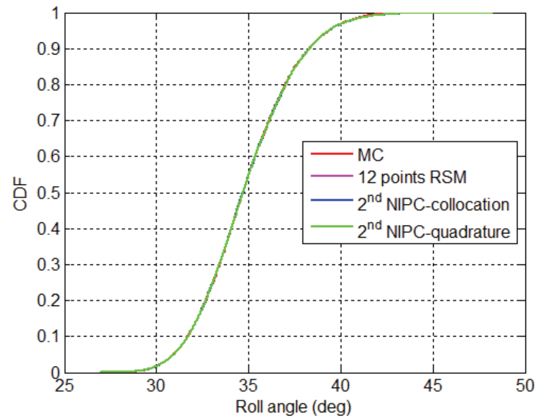
(a)



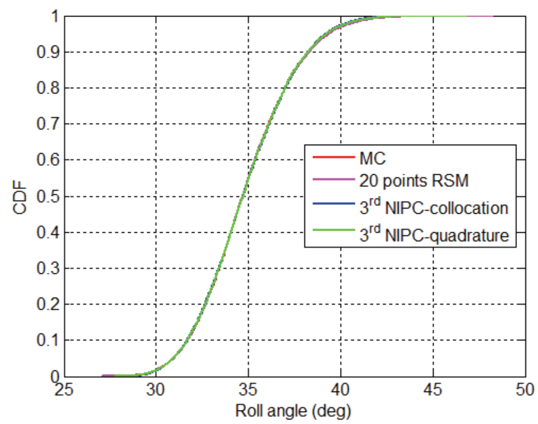
(b)

Fig. 4: Comparison of cumulative distribution function for the roll angle for different methods: MC=Monte Carlo method; RSM=Response Surface Method; NIPC=Non-Intrusive Polynomial Chaos.

propagation has been carried out using both the Monte Carlo method and NIPC method. Similar to the damped Mathieu equation case, the cumulative distribution function has been plotted in Fig. 5 for different methods. The Monte Carlo simulation was carried out using 10,000 samplings. Same quadratic models in Eq.(12) are also used for the response surface method. In Fig. 5(a), 2nd order NIPC has been carried out. For additional comparison, the data of the 12 samplings from the collocation method is used to predict the response surface. The CDF of the four methods agree well in this figure. In Fig. 5(b), 3rd order NIPC method has been carried out. And data of 20 samplings from the collocation method is used to calculate the response surface. In this case, the CDFs also agree well.



(a)



(b)

Fig. 5: Comparison of cumulative distribution function for the roll angle for different methods. MC=Monte Carlo method; RSM=Response Surface Method; NIPC=Non-Intrusive Polynomial Chaos.

Similar to the damped Mathieu case, the time needed to carry out the NIPC calculation is only a few seconds for one processor. But for the Monte Carlo simulation with 10,000 sampling, it took 9 hours using 8 2.67GHz processors.

CONCLUSIONS AND REMARKS

In this paper, the non-intrusive polynomial chaos method has been applied to understand uncertainty propagation in the damped Mathieu equation and a 1.5 DOF parametric roll model. In both models, the NIPC method provides good results compared to the Monte Carlo method. And the NIPC method shows great computational efficiency compared to the Monte Carlo method. For comparison, the response surface method has also been applied out using full quadratic models.

For the models used in this work, the RSM and NIPC predict quite similar results.

Note that in this work, the random variables are assumed to be independent with Gaussian distribution. This assumption is used here to simplify the situation. In more realistic cases, if the variables are not Gaussian, one can either transform them in such a way that they can be approximated as Gaussian variables (eg. Box-Cox transformations) or one can simply use other forms of basis functions as mentioned earlier. Additionally, as this was intended as a demonstration of concept, for simplicity, regular excitation was used in this work. Future work will examine random excitation.

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REFERENCES

Bulian, G. (2006) "Development of analytical nonlinear models for parametric roll and hydrostatic restoring variations in regular and irregular waves", PhD dissertation, University of Trieste, Italy

Cameron, R.H. and Martin, W.T. (1947) "The orthogonal development of nonlinear functional in series of Fourier-Hermite functionals", Annals of Mathematics, Vol. 48, No. 2, pp. 385-392

Helton,, J.C. and Davis, F.J. (2003) "Latin hypercube sampling and the propagation of uncertainty in analyses of complex systems", Reliability Engineering and System Safety, Vol. 81, pp. 23-69

Hosder, S., Walters, R.W. and Perez, R. (2006) "A non-intrusive polynomial chaos method for uncertainty propagation in CFD simulations", 44th AIAA Aerospace Sciences Meeting and Exhibit, Reno, Nevada

Hosder, S., Walters, R.W. and Balch, M. (2010) "Non-intrusive polynomial chaos methods for uncertainty quantification in fluid dynamics", 48h AIAA Aerospace Sciences Meeting including the New Horizon Forum and Aerospace Exhibition, Orlando, Florida

Lucor, D., Xiu, D. and Karniadakis, G.E. (2001) "Spectral representations of uncertainty in simulations: Algorithms and applications", Proc of the Intl Conf on Spectral and High Order Methods, Uppsala, Sweden

Paulling, J.R. (2006) "Parametric rolling of ships – then and now", Proc of the 9th Intl Conf on Stability of Ships and Ocean Vehicles (STAB 2006), Rio de Janeiro, Brazil

Wiener, N. (1938) "The homogeneous chaos", American J. of Math, Vol. 60, No. 4, pp. 897-936

Xiu, D. and Karniadakis, G.E. (2002) "The Wiener-Askey polynomial chaos for stochastic differential equations", SIAM J. on Scientific Computing , Vol. 24, No. 42, pp. 619-644