ON THE EXCITATION OF COMBINATION MODES ASSOCIATED WITH PARAMETRIC RESONANCE IN WAVES

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SUMMARY

Non-linear equations of ship motions in waves describing the couplings between heave, roll and pitch are investigated. These couplings of the resonant modes should be taken into consideration in order to describe the whole spectrum of possible resonant conditions. In order to investigate the occurrence of combination resonance in addition to auto-parametric resonance, the paper takes into consideration some general theoretical results regarding the possibility of amplification of motion in these conditions. Stability analysis is performed taking the linear solutions as a basis function. An idealized system is considered, which corresponds to the linear variational equation of the complete non-linear system when damping is not included. It can be shown that a system of non-linear equations as defined above may be reduced to a set of coupled Mathieu equations when the linear variational equation is taken. The set of coupled Mathieu equations then describes the essential aspects of the stability of the dynamic system when small perturbations are imposed on the basic periodic motions. This set of equations may face instabilities for more frequencies than the uncoupled Mathieu equation. A matrix of parametric excitation is defined. The theoretical conditions obtained for the occurrence of combination modes are in the form of non-symmetry conditions regarding elements of the matrix of parametric excitation. Explicit expressions for boundaries of stability are given.

NOMENCLATURE

a	Wave amplitude
A_0	Waterplane area at average hull position
∇_0	Volume at average hull position
∇_1	Volume at instantaneous hull position
∇	Incremental volume for displaced hull
x_{f0}	Centroid of waterplane at average hull position
z_{b0}	Vertical position of hull volume centroid
T_0	Ship draft
m	Ship mass
J_{xx}	Transversal mass moment of inertia
J_{yy}	Longitudinal mass moment of inertia
I_{xx0}	Transversal 2nd moment of waterplane area
I_{yy0}	Longitudinal 2nd moment of waterplane area
h	Height of elemental prisms
k	Wave number
$\hat{I},\hat{J},\hat{K}$	Unit vectors along axes of inertial frame
$\hat{i},\hat{j},\hat{k}$	Unit vectors along axes fixed in the ship
χ	Wave incidence
ω_e	Encounter frequency
η	Wave elevation
ρ	Density of water

1. INTRODUCTION

Parametric rolling of ships has continuously received wide attention of researchers and designers as a relevant instabilizing mechanism, see Paulling and Rosenberg (1959), De Kat and Paulling (1989), Munif and Umeda (2000). Much of such attention has been devoted to the particular configuration of longitudinal regular waves, either with or without speed, bow or stern waves. Roll motion has usually been modeled as an uncoupled Mathieu type equation. Considering the well-known existence of the Mathieu resonant frequencies, focus has been concentrated on the first region, the one defined by the proximity of encounter frequency to twice the roll natural frequency.

The author has proposed that complete non-linear coupling of the resonant modes should be taken into consideration in order to describe the whole spectrum of possible resonant conditions, Neves and Valerio (2000). By employing Taylor series expansions up to second order, it was possible to express restoring actions in the heave, roll and pitch modes in a completely coupled way. Wave action was taken into consideration not only in the Froude-Krilov plus diffraction first order forcing functions, but also in second order terms resulting from volumetric changes of submerged hull due to wave passage effects.

In order to investigate the occurrence of combination resonance, the present paper takes into consideration some general theoretical results regarding the possibility of amplification of motion in coupled systems, Cesari (1959), Gambill (1954). For this purpose, an idealized system will be considered, which corresponds to the linear variational equation of the complete non-linear system when damping is not included. It can be shown that a system of non-linear equations as defined above may be reduced to a set of coupled Mathieu equations when the linear variational equation is taken. The set of coupled Mathieu equations then describes the essential aspects of the stability of the dynamic system when small perturbations are imposed on the basic linear periodic motions.

A matrix of parametric excitation will then be defined. The theoretical conditions obtained for the occurrence of combination modes are in the form of non-symmetry conditions regarding elements of the matrix of parametric excitation. These conditions are important in the sense that they define the level of energy transfer from the vertical modes to the roll mode and *vice-versa* when co-parametric resonance is in effect. The theoretical approach allows interesting analysis of hull parameters associated with more or less asymmetry conditions (intensification of excitation). Newly derived coefficients are presented, which establish clearly the existence of the necessary asymmetry conditions for the excitation of combination modes. It will also be demonstrated that combination modes cannot be excited in longitudinal waves, their excitation being dependent on the occurrence of direct excitation of the roll mode. This is a relevant and interesting theoretical conclusion regarding the dynamics of intact ships in waves associated with parametric resonance. Explicit expressions defining the boundaries of stability are derived for regions where co-parametric resonance related to roll motion may occur.

Approximated limits of stability obtained by Stoker (1950) for single Mathieu equations and Hsu (1963) for systems with multiple degrees of freedom are invoked to discuss the *openness* of regions of stability of simple geometrical forms. It is pointed out that limits of stability associated with combination parametric resonances may, under some circumstances, be wider than second order limits of the single Mathieu equation.

2. EQUATIONS OF MOTION

Two right-handed co-ordinate systems are employed to describe the motions. An inertial reference frame (C,*x,y,z*) is assumed to be fixed at the mean ship motion. Regular waves are assumed to travel forming an angle χ with ship course. Another reference frame $(O,\bar{x},\bar{y},\bar{z})$ is fixed at the ship having the $\overline{xy}$ plane coinciding, for the ship at rest, with the undisturbed sea surface, $\bar{z}$ -axis passing through the vertical that contains the center of gravity. The two systems coincide when excitations are absent. See Figure 1.

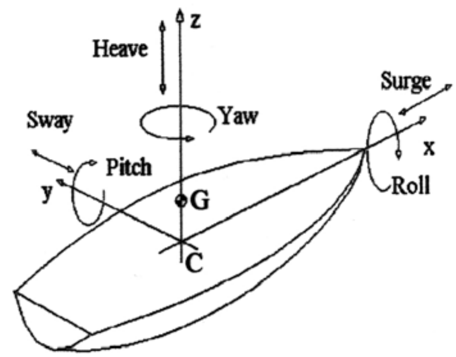

Figure 1: Co-ordinate axis and definition of motions

Non-linear equations of motion considering the three restoring degrees of freedom may be expressed in matrix form using a displacement vector:

$$\vec{q}(t) = [\; z(t) \;\; \phi(t) \;\; \theta(t) \;]^T$$

defining the heave translational mode together with the roll and pitch angular modes, see Figure 1:

$$(\widetilde{M} + \widetilde{A})\ddot{\vec{q}} + \widetilde{B}(\dot{\phi})\dot{\vec{q}} + \vec{C}(z,\phi,\theta,\eta) = \vec{Q}_W(\chi, \mathrm{a}, \omega_e, t) \quad (1)$$

Hull inertia $\widetilde{M}$ is a diagonal 3X3 matrix. Its elements are: *m*, the ship mass, J_{xx}, J_{yy} the mass moments of inertia in the roll and pitch modes, respectively, taken with reference to center O. Elements in matrix $\widetilde{A}$ represent hydrodynamic added masses, moments and products of inertia terms. Damping terms $\widetilde{B}(\dot{\phi})$ may incorporate non-linear terms in the roll equation and describe hydrodynamic reactions dependent on ship velocities, Himeno (1981). The hydrodynamic inertia and damping matrices are expressed respectively, as:

$$\widetilde{A} = \begin{bmatrix} Z_{\ddot{z}} & 0 & Z_{\ddot{\theta}} \\ 0 & K_{\ddot{\phi}} & 0 \\ M_{\ddot{z}} & 0 & M_{\ddot{\theta}} \end{bmatrix} ; \widetilde{B} = \begin{bmatrix} Z_{\dot{z}} & 0 & Z_{\dot{\theta}} \\ 0 & K_{\dot{\phi}}(\dot{\phi}) & 0 \\ M_{\dot{z}} & 0 & M_{\dot{\theta}} \end{bmatrix}$$

and vector $\vec{C}(z,\phi,\theta,\eta)$ describes non-linear positional forces and moments due to relative motions between ship hull and wave elevation $\eta(t)$. To second order, restoring terms due to ship motions in calm water and wave passage terms may be split into two separate actions that can be summed to obtain the complete relative vertical displacement. Thus, positional forces and moments are taken as:

$$\vec{C}(z,\phi,\theta,\eta) = \begin{bmatrix} Z_H \\ K_H \\ M_H \end{bmatrix} + \begin{bmatrix} Z_{WP} \\ K_{WP} \\ M_{WP} \end{bmatrix} \quad (2)$$

where, on the right hand side, the first vector represents purely hydrostatic reactions, whereas the second vector describes second order wave actions. On the right hand side of equation (1), generalized vector $\vec{Q}_w$ represent linear wave external excitation, usually referred to as the Froude-Krilov (first order wave actions) plus diffraction wave force terms, dependent on wave heading χ, encounter frequency ω_e, wave amplitude a and time *t*:

$$\vec{Q}_w = \begin{bmatrix} Z_w(\chi,a,\omega_e)cos(\omega_e t+\sigma_z) \\ K_w(\chi,a,\omega_e)cos(\omega_e t+\sigma_\phi) \\ M_w(\chi,a,\omega_e)cos(\omega_e t+\sigma_\theta) \end{bmatrix}$$

Second order wave actions, proportional to hull displacements, will be considered on the left hand side of the equations of motion, being named wave passage effects. Due to its mathematical affinity with the purely hydrostatic terms, these are written as a second vector contribution to $\vec{C}$ in equation (2).

3. HYDROSTATIC RESTORING TERMS

3.1 RESTORING FORCE

Restoring force is given by the difference between weight and instantaneous buoyancy in calm water. Vectorially:

$$\vec{F}_H = \vec{W} + \vec{E}_1$$

where ship weight:

$$\vec{W} = -\rho g \nabla_0 \hat{K}$$

is directed vertically ($\hat{I},\hat{J},\hat{K}$ are unit vectors defined in the inertial reference frame). Analogously, instantaneous buoyancy force is given as:

$$\vec{E}_1 = \rho g \nabla_1 \hat{K} = \rho g(\nabla_0 - \nabla)\hat{K}$$

such that

$$\vec{F}_H = -\rho g \nabla \hat{K} \tag{3}$$

In the above expressions, ∇_1 is the instantaneous submerged volume, ∇_0 is the submerged volume in the average upright condition, and ∇ is the instantaneous incremental volume due to hull displacements in heave, roll and pitch.

3.2 RESTORING MOMENTS

The vector representing the restoring moment is defined with respect to O (see Figure 2). The weight (equal to buoyancy in the average condition) is applied in G and buoyancy force in B_1, the instantaneous centroid of hydrostatic pressures. Restoring moment is then given as:

$$\vec{M} = \rho g[-(\nabla_0 - \nabla)\hat{K} \times \overrightarrow{OB_1} + \nabla_0 \hat{K} \times \overrightarrow{OG}]$$

Noting that B_0 is the volume centroid at average position and $\hat{k}$ is the $\bar{z}$ oriented unit vector (defined in the coordinate system fixed in the body) then:

$$\overrightarrow{OB_1} = \overrightarrow{OB_0} + \overrightarrow{B_0B_1}$$

$$\overrightarrow{OG} = (\overline{KG} - T_0)\hat{k}$$

$$\overrightarrow{OB_0} = (\overline{KB_0} - T_0)\hat{k}$$

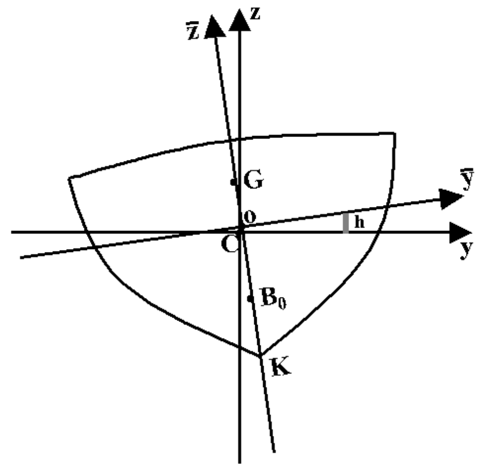

Figure 2: Hull section in a general displaced waterline

where T_0 is the ship draft in the average upright position. Taking the projections of $\hat{k}$ on the inertial reference frame, and considering, for simplicity, small angles between the reference frames, the restoring moment may then be rewritten as:

$$\vec{M} = -\rho g[\nabla_0(\overline{KB_0} - \overline{KG})\phi\,\hat{I} + \nabla_0(\overline{KB_0} - \overline{KG})\theta\,\hat{J}] - $$
$$- \rho g[\nabla\hat{K} \times (\overrightarrow{OB_0} + \overrightarrow{B_0B_1}) + \nabla_0\hat{K} \times \overrightarrow{B_0B_1}]$$

It is convenient to define:

$$\vec{M} = \vec{M}_0 + \delta\vec{M} \tag{4}$$

where:

$$\vec{M}_0 = -\rho g \nabla_0(\overline{KB_0} - \overline{KG})(\phi\,\hat{I} + \theta\,\hat{J}) \tag{5}$$

$$\delta\vec{M} = -\rho g[\nabla\hat{K} \times (\overrightarrow{OB_0} + \overrightarrow{B_0B_1}) + \nabla_0\hat{K} \times \overrightarrow{B_0B_1}]$$
$$= -\rho g(\nabla\hat{K} \times \overrightarrow{OB_0} + \nabla_0\hat{K} \times \overrightarrow{B_0B_1} + \nabla\hat{K} \times \overrightarrow{B_0B_1}) \tag{6}$$

Firstly, one notes that the first vector product in the above expression is given as:

$$\nabla\hat{K} \times \overrightarrow{OB_0} = \nabla\hat{K} \times (\overline{OB_0})\hat{k} = \nabla(\overline{KB_0} - T_0)(\phi\,\hat{I} + \theta\,\hat{J}) \tag{7}$$

and secondly that the last term in the incremental moment is of a lower order of magnitude than the others, since both $\nabla\hat{K}$ and $\overrightarrow{B_0B_1}$ are small quantities.

Next step in the derivation will be to find adequate expressions for $\nabla\hat{K}$ and $\overrightarrow{B_0B_1}$ as functions of the relevant variables, $z(t), \phi(t)$ and $\theta(t)$, that is, the displacements in heave, roll and pitch, respectively.

4. HYDROSTATIC ACTIONS UP TO SECOND ORDER

It should be noted that in order to retain all couplings between the restoring modes, the characteristics of a generic waterplane must be considered. A generic waterplane does not correspond to equivolumetric inclinations. If the hull is not wall-sided, the application of a pure roll angle will also introduce a net vertical force, due to immersion and emergence of different volume wedges.

Referring again to Figure 2, vertical elemental prisms may be employed to describe the volumetric changes introduced by a general displacement. The height of each elemental prism has three contributions:

$$h = z + y_A\phi - x_A\theta \tag{8}$$

where (x_A, y_A) is a point of the generic waterplane. The total volumetric change may be obtained by integrating all vertical prisms over the complete instantaneous waterplane area:

$$\nabla = \iint_A h dA = z\iint_A dA + \phi\iint_A y_A dA - \theta\iint_A x_A dA \tag{9}$$

The double integrals describe some well-known geometric properties of the waterplane area:

$A(z,\phi,\theta) = \iint_A dA$ - waterplane area,

$Ax_f(z,\phi,\theta) = \iint_A x_A dA$ -longitudinal first moment,

$Ay_f(z,\phi,\theta) = \iint_A y_A dA$ - transversal first moment.

With this nomenclature, the total volumetric change is then rewritten as:

$$\nabla = A(z,\phi,\theta)z + Ay_f(\phi)\phi - Ax_f(z,\phi,\theta)\theta \tag{10}$$

and it is noted that in fact displacements z and θ give no contribution to Ay_f. Yet, this function does change with roll angle for hulls with inclined sidewalls.

Consider now the volumetric moments associated with the roll and pitch modes. These will be expressed as:

$$\nabla_0\hat{K} \times \overrightarrow{B_0B_1} = -\hat{I}\iint_A y_A h dA + \hat{J}\iint_A x_A h dA$$

Substituting h defined in equation (8) into the above integrals, and recalling that the transversal static moment of waterplane area is dependent only on roll displacements, the volumetric moment is expressed as:

$$\begin{aligned}\nabla_0\hat{K} \times \overrightarrow{B_0B_1} &= \hat{I}[-I_{xx}(z,\phi,\theta)\phi] + \\ &+ \hat{J}[Ax_f(z,\phi,\theta)z + I_{xy}(\phi)\phi - \\ &- I_{yy}(z,\phi,\theta)\theta]\end{aligned} \tag{11}$$

where the following functions have been defined for the generic waterplane area:

$I_{xx} = \iint_A y_A^2 dA$ -transversal moment of inertia,

$I_{yy} = \iint_A x_A^2 dA$ -longitudinal moment of inertia,

$I_{xy} = \iint_A x_A y_A dA$ -product of inertia.

It is pointed out that the moment $I_{xy}(\phi)\phi$ in equation (11) exists whenever the centroid longitudinal coordinate of the incremental volume $Ay_f(\phi)\phi$ appearing in equation (10) does not coincide with the longitudinal centroid of the average volume ∇_0.

Compiling equations (3 to 7, 10 and 11) and multiplying equations (3) and (4) by (-1), the complete expressions for the first components of vector $\vec{C}$ given in equation (2) in heave, roll and pitch, representing purely hydrostatic reactions, result in:

$$\begin{aligned}Z_H = \rho g\{&A(z,\phi,\theta)z + Ay_f(\phi)\phi - \\ &- Ax_f(z,\phi,\theta)\theta\} = \rho g\nabla(z,\phi,\theta)\end{aligned} \tag{12}$$

$$\begin{aligned}K_H = \rho g\{&\nabla_0(\overline{KB_0} - \overline{KG}) + \\ &+ \nabla(z,\phi,\theta)(\overline{KB_0} - T_0) + \\ &+ I_{xx}(z,\phi,\theta)\}\phi\end{aligned} \tag{13}$$

$$\begin{aligned}M_H = \rho g\{&[\nabla_0(\overline{KB_0} - \overline{KG}) + \\ &+ \nabla(z,\phi,\theta)(\overline{KB_0} - T_0) + I_{yy}(z,\phi,\theta)]\theta - \\ &- Ax_f(z,\phi,\theta)z - I_{xy}(\phi)\phi\}\end{aligned} \tag{14}$$

Noting that $y_f(\phi)$ and $I_{xy}(\phi)$ defined for inclined waterplanes are odd functions of the roll angle, the following multivariable series expansions (to second order force and moments) are obtained:

$$A = A_0 + \left.\frac{\partial A}{\partial z}\right|_0 z + \left.\frac{\partial A}{\partial \theta}\right|_0 \theta$$

$$Ay_f = \left.\frac{\partial(Ay_f)}{\partial \phi}\right|_0 \phi$$

$$Ax_f = A_0 x_{f0} + \left.\frac{\partial(Ax_f)}{\partial z}\right|_0 z + \left.\frac{\partial(Ax_f)}{\partial \theta}\right|_0 \theta$$

$$I_{xx} = I_{xx0} + \left.\frac{\partial I_{xx}}{\partial z}\right|_0 z + \left.\frac{\partial I_{xx}}{\partial \theta}\right|_0 \theta$$

$$I_{yy} = I_{yy0} + \left.\frac{\partial I_{yy}}{\partial z}\right|_0 z + \left.\frac{\partial I_{yy}}{\partial \theta}\right|_0 \theta$$

$$I_{xy} = \left.\frac{\partial I_{xy}}{\partial \phi}\right|_0 \phi$$

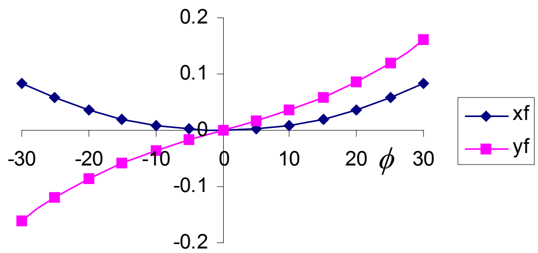

Figure 3: Different trends for centroids of inclined waterplanes: x_f and y_f against roll angle ϕ.

Figure 3 illustrates the difference in trend between the longitudinal and transversal position of the centroid of a generic transversally inclined waterplane. Clearly, the roll derivative of x_f is zero, whereas the roll derivative of y_f is not.

Introducing the series expansions in the given expressions of Z_H, K_H and M_H results in:

$$\begin{aligned} Z_H = \rho g\{ & A_0 z - A_0 x_{f0}\theta + \left.\frac{\partial A}{\partial z}\right|_0 z^2 + \\ & + \left.\frac{\partial (Ay_f)}{\partial \phi}\right|_0 \phi^2 - \left.\frac{\partial (Ax_f)}{\partial \theta}\right|_0 \theta^2 + \\ & + (\left.\frac{\partial A}{\partial \theta}\right|_0 - \left.\frac{\partial (Ax_f)}{\partial z}\right|_0) z\theta \} \end{aligned} \tag{15}$$

$$\begin{aligned} K_H = \rho g\{ & (I_{xx0} + \nabla_0 . \overline{B_0 G})\phi + \\ & + (\left.\frac{\partial I_{xx}}{\partial z}\right|_0 + A_0 z_{b0}) z\phi + \\ & + (\left.\frac{\partial I_{xx}}{\partial \theta}\right|_0 - A_0 x_{f0} z_{b0})\phi\theta \} \end{aligned} \tag{16}$$

$$\begin{aligned} M_H = \rho g\{ & -A_0 x_{f0} . z + (I_{yy0} + \nabla_0 . \overline{B_0 G})\theta + \\ & - \left.\frac{\partial (Ax_f)}{\partial z}\right|_0 z^2 + \left.\frac{\partial I_{xy}}{\partial \phi}\right|_0 \phi^2 + \\ & + (\left.\frac{\partial I_{yy}}{\partial \theta}\right|_0 - A_0 x_{f0} z_{b0})\theta^2 + \\ & + (\left.\frac{\partial I_{yy}}{\partial z}\right|_0 - \left.\frac{\partial (Ax_f)}{\partial \theta}\right|_0) z\theta \} \end{aligned} \tag{17}$$

where $z_{b0} = \overline{KB_0} - T_0$ is the vertical coordinate of the volume in the average position (a negative quantity in the assumed reference axis system).

In this case, the equations of motion are then given as:

$$\begin{aligned} & (\tilde{M} + \tilde{A})\ddot{\vec{q}} + \tilde{B}(\dot{\phi})\dot{\vec{q}} + \\ & + [\tilde{C}_0 + \tilde{C}_1(z,\phi,\theta) + \tilde{C}_{wp}(t)]\vec{q} = \vec{Q}_w(t) \end{aligned} \tag{18}$$

noting that in the above equation, $\tilde{C}_0$ is the linear restoring matrix, $\tilde{C}_1(z,\phi,\theta)$ and $\tilde{C}_{wp}(t)$ represents second order wave actions (wave passage correction), respectively. With the nomenclature employed in the present paper for derivatives, matrix $\tilde{C}_0$ is given as:

$$\tilde{C}_0 = \begin{bmatrix} Z_z & 0 & Z_\theta \\ 0 & K_\phi & 0 \\ M_z & 0 & M_\theta \end{bmatrix}$$

where the linear restoring coefficients are:

$$Z_z = \rho g A_0 \; ; \; Z_\theta = -\rho g A_0 x_{f0} = M_z$$

$$K_\phi = \rho g \nabla_0 (\frac{I_{xx0}}{\nabla_0} + \overline{B_0 G})$$

$$\mathrm{M}_\theta = \rho g \nabla_0 (\frac{I_{yy0}}{\nabla_0} + \overline{B_0 G})$$

The second order restoring terms, when grouped in matrix form become:

$$\tilde{C}_1 = \begin{bmatrix} Z_{zz} z + Z_{z\theta}\theta & Z_{\phi\phi}\phi & Z_{\theta\theta}\theta \\ 0 & K_{z\phi} z + K_{\phi\theta}\theta & 0 \\ M_{zz} z + M_{z\theta}\theta & M_{\phi\phi}\phi & M_{\theta\theta}\theta \end{bmatrix}$$

where, taking into consideration the above expressions, equations (15, 16 and 17) of Z_H, K_H and M_H, it may be recognized that:

$$Z_{zz} = \rho g \frac{\partial A}{\partial z} \quad ; \quad Z_{\phi\phi} = \rho g \frac{\partial (Ay_f)}{\partial \phi}$$

$$Z_{\theta\theta} = -\rho g \frac{\partial(Ax_f)}{\partial\theta}$$

$$Z_{z\theta} = \rho g\left[\frac{\partial A}{\partial\theta} - \frac{\partial(Ax_f)}{\partial z}\right]$$

$$K_{z\phi} = \rho g\left[\frac{\partial I_{xx}}{\partial z} + A_0 z_{b0}\right]$$

$$K_{\phi\theta} = \rho g\left[\frac{\partial I_{xx}}{\partial\theta} - A_0 x_{f0} z_{b0}\right]$$

$$M_{zz} = -\rho g \frac{\partial(Ax_f)}{\partial z} \quad ; \quad M_{\phi\phi} = \rho g \frac{\partial I_{xy}}{\partial\phi}$$

$$M_{z\theta} = \rho g\left[\frac{\partial I_{yy}}{\partial z} - \frac{\partial(Ax_f)}{\partial\theta} + A_0 x_{f0} z_{b0}\right]$$

$$M_{\theta\theta} = \rho g\left[\frac{\partial I_{yy}}{\partial\theta} - A_0 x_{f0} z_{b0}\right]$$

To simplify the notation, the vertical bar in the derivatives has been dispensed. This simplification in notation will be extended to the remaining sections of the paper.

It is pointed out that coefficients $K_{z\phi}$ and $K_{\phi\theta}$ given above describe (to second order) the internal transfer of energy from the vertical modes to the roll motion. They are both composed of two terms. The first ones represent the variation of the roll restoring moment due to changes in the moment of inertia of the waterplane area. These terms are zero for wall-sided hulls. The second terms represent the variation of the restoring moment due to the change of submerged volume induced by the vertical oscillations. These terms are effective even in the case of wall sided hulls. Complementarily, the derivative $Z_{\phi\phi}$ regulates the transfer of energy from the roll mode to the heave mode and $M_{\phi\phi}$ mediates energy transfer from roll to pitch.

5. EXPRESSIONS FOR COMPUTING THE DERIVATIVES

It is possible and relevant to relate the derivatives given above to longitudinal distributions of hull characteristics. Paulling and Rosenberg (1959) presented similar derivations, but as mentioned before, they did not consider all the possible coupling terms. Specifically, it is important in the present context to take into account the influence of roll motion upon the heave and pitch modes.

When deriving these expressions with respect to z, ϕ and θ, it will be observed that the following rules of derivation are to be applied in the case of angular displacements in roll and pitch:

$$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial z}\frac{\partial z}{\partial\phi} \quad ; \quad \frac{\partial}{\partial\theta} = \frac{\partial}{\partial z}\frac{\partial z}{\partial\theta}$$

and from the general transformation matrix between a rotated and a fixed frame of reference it can be deduced that in the case of roll:

$$z = \bar{y} sin\phi + \bar{z}\, cos\,\phi \;\therefore\; \frac{\partial z}{\partial\phi} = \bar{y}\, cos\,\phi - \bar{z} sin\phi$$

such that:

$$\left.\frac{\partial z}{\partial\phi}\right|_0 = \bar{y} \quad \Rightarrow \frac{\partial}{\partial\phi} = \bar{y}\frac{\partial}{\partial\bar{z}} \tag{19}$$

and in the case of pitch:

$$z = -\bar{x} sin\theta + \bar{z}\, cos\,\theta \;\therefore\; \frac{\partial z}{\partial\theta} = -\bar{x}\, cos\,\theta - \bar{z} sin\theta$$

$$\left.\frac{\partial z}{\partial\theta}\right|_0 = -\bar{x} \quad \Rightarrow \frac{\partial}{\partial\theta} = -\bar{x}\frac{\partial}{\partial\bar{z}} \tag{20}$$

and it is also observed that $\left.\frac{\partial z}{\partial\bar{z}}\right|_0 = 1$.

With these results, the expressions for the non-vanishing second order hydrostatic derivatives can be derived. With the integrations along the ship length being defined from after to fore perpendicular:

$$\frac{\partial A}{\partial z} = \int_{AP}^{FP}\int_{-y(\bar{x})}^{y(\bar{x})}\frac{\partial}{\partial\bar{z}} d\bar{y} d\bar{x} =$$

$$= \int_{AP}^{FP}\frac{\partial}{\partial\bar{z}}[\, y\,]_{-y(\bar{x})}^{y(\bar{x})} d\bar{x} = 2\int_{AP}^{FP}\frac{\partial\bar{y}}{\partial\bar{z}} d\bar{x}$$

$$\frac{\partial A}{\partial\theta} = \int_{AP}^{FP}(-\bar{x})\frac{\partial}{\partial z}\int_{-y(\bar{x})}^{y(\bar{x})} d\bar{y} d\bar{x} =$$

$$= -\int_{AP}^{FP}\bar{x}\frac{\partial}{\partial z}[\, 2y\,]_{-y(\bar{x})}^{y(\bar{x})} d\bar{x} = -2\int_{AP}^{FP}\bar{x}\frac{\partial\bar{y}}{\partial\bar{z}} d\bar{x}$$

where use was made of equation (20). With this result $\frac{\partial A}{\partial\theta}$ is expressed as the longitudinal distribution of $\frac{\partial A}{\partial z}$.

It should be noticed that at the average hull position the terms Ay_f and I_{xy} are zero. Yet, for a hull inclined of an angle ϕ these terms will not be zero. Their derivatives with respect to ϕ taken at the origin are not zero. In fact, given the definition of the lateral static moment of water-plane area, and taking its derivative with respect to roll angle, results in:

$$\frac{\partial(Ay_f)}{\partial\phi} = \frac{\partial}{\partial\phi}\iint_A y_A dA = \int_{AP}^{FP}\frac{\partial}{\partial\phi}\int_{-y(\bar{x})}^{y(\bar{x})}\bar{y}\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{z}d\bar{x}$$

$$= \int_{AP}^{FP}\int_{-y(\bar{x})}^{y(\bar{x})}\frac{\partial}{\partial z}\bar{y}\frac{\partial\bar{y}}{\partial\bar{z}}\bar{y}d\bar{z}d\bar{x} = \int_{AP}^{FP}\bar{y}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

where use was made of equation (19) describing derivation with respect to roll. Similar derivation may be applied to the case of I_{xy} and the result is:

$$\frac{\partial I_{xy}}{\partial\phi} = -\int_{AP}^{FP}\bar{x}.\bar{y}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

which may be understood as a first moment of the longitudinal distribution of $\frac{\partial(Ay_f)}{\partial\phi}$ with respect to the origin. Applying the integration procedure used above, all other second order derivatives may then expressed in terms of the longitudinal distribution of flare at average waterline, $\frac{\partial\bar{y}}{\partial\bar{z}}$. Hence, the following expressions may be established:

$$\frac{\partial(Ax_f)}{\partial z} = 2\int_{AP}^{FP}\bar{x}\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

$$\frac{\partial(Ax_f)}{\partial\theta} = -2\int_{AP}^{FP}\bar{x}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

$$\frac{\partial I_{xx}}{\partial z} = 2\int_{AP}^{FP}\bar{y}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

$$\frac{\partial I_{xx}}{\partial\theta} = -2\int_{AP}^{FP}\overline{xy}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

$$\frac{\partial I_{yy}}{\partial z} = 2\int_{AP}^{FP}\bar{x}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

$$\frac{\partial I_{yy}}{\partial\theta} = -2\int_{AP}^{FP}\bar{x}^3\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}$$

Following these results, it can be observed that some identities do exist between some of the derivatives previously derived:

$$\frac{\partial A}{\partial\theta} = -\frac{\partial(Ax_f)}{\partial z} \quad ; \quad \frac{\partial(Ax_f)}{\partial\theta} = -\frac{\partial I_{yy}}{\partial z}$$

$$\frac{\partial(Ay_f)}{\partial\phi} = \frac{1}{2}\frac{\partial I_{xx}}{\partial z} \quad ; \quad \frac{\partial I_{xy}}{\partial\phi} = \frac{1}{2}\frac{\partial I_{xx}}{\partial\theta}$$

such that there are in fact only six second order derivatives to be computed.

As a very simple example, in the case of a transversally inclined triangular constant cross-section prism, it can be easily demonstrated that the difference in area of the immersed and emerged wedges is:

$$\delta A = \frac{\bar{y}^2(d\bar{y}/d\bar{z})\tan^2\phi}{1-(d\bar{y}/d\bar{z})^2\tan^2\phi}$$

where $\bar{y}$ is the half-breadth of the prism, and $\frac{d\bar{y}}{d\bar{z}} = \frac{\bar{y}}{T_0}$.

This exact expression should be compared with the simplification resulting from a Taylor series expansion based on the cylinder average position. Applying a Taylor series expansion to the above expression and retaining terms up to second order results in:

$$\delta A = \bar{y}^2(d\bar{y}/d\bar{z})\phi^2$$

and this expression demonstrates that in the case of a triangular prismatic body the vertical hydrostatic force due to roll motion is an even function of the roll angle, and this verifies the previous general expression derived for a general hull form. In fact, on applying the general formulation, equation (15), to the case of a cylinder with constant cross-section inclined of an angle ϕ, integration is easily performed, resulting in:

$$Z_H(\phi) = \rho g\left.\frac{\partial(A.y_f)}{\partial\phi}\right|_0 .\phi^2 = \rho g\int_{AP}^{FP}\bar{y}^2\frac{\partial\bar{y}}{\partial\bar{z}}d\bar{x}.\phi^2 =$$

$$= \rho g l_p\bar{y}^2(d\bar{y}/d\bar{z})\phi^2$$

where l_p is the length of the cylinder. Thus, this is a particular case of the more general expression given previously for a generic hull form.

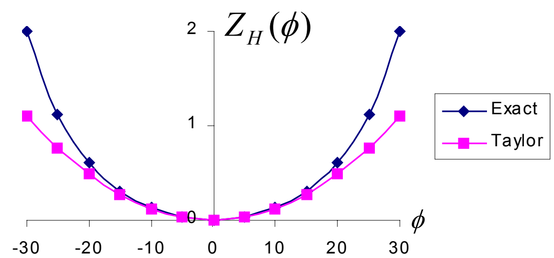

Figure 4: Hydrostatic heave force due to roll for prismatic body with triangular cross-section.

Figure 4 illustrates the vertical hydrostatic force in the case of a prism with triangular cross-section inclined of an angle ϕ, an even function passing through the origin. The figure shows the curves for the exact solution and for the derivative approximate solution up to second order, for breadth $2\bar{y} = 2.0m$ and flare equal to 45 degrees.

6. WAVE PASSAGE

Wave passage of arbitrary direction along the hull is modeled as a change of the hull average submerged volume defined by the instantaneous position of the wave. Employing a similar nomenclature to the one used for the hydrostatic terms and considering the restoring modes only, the following terms are adopted, Neves and Valerio (2000):

$$\begin{bmatrix} Z_{WP} \\ K_{WP} \\ M_{WP} \end{bmatrix} = \tilde{C}_{wp}(t)\vec{q}(t) = \begin{bmatrix} Z_{z\eta} & Z_{\phi\eta} & Z_{\theta\eta} \\ K_{z\eta} & K_{\phi\eta} & K_{\theta\eta} \\ M_{z\eta} & M_{\phi\eta} & M_{\theta\eta} \end{bmatrix} \begin{bmatrix} z \\ \phi \\ \theta \end{bmatrix} \quad (21)$$

where the time-dependent elements are defined as:

$$Z_{z\eta} = -\rho g \int_{AP}^{FP} \frac{dy}{dz} (\eta_{PS} + \eta_{SB}) dx$$

$$K_{\phi\eta} = -\rho g \int_{AP}^{FP} \bar{y}^2 \frac{dy}{dz} (\eta_{PS} + \eta_{SB}) dx$$

$$M_{\theta\eta} = -\rho g \int_{AP}^{FP} \bar{x}^2 \frac{dy}{dz} (\eta_{PS} + \eta_{SB}) dx$$

$$Z_{\phi\eta} = -\rho g \int_{AP}^{FP} \bar{y} \frac{dy}{dz} (\eta_{PS} - \eta_{SB}) dx = K_{z\eta}$$

$$Z_{\theta\eta} = \rho g \int_{AP}^{FP} \bar{x} \frac{dy}{dz} (\eta_{PS} + \eta_{SB}) dx = M_{z\eta}$$

$$K_{\theta\eta} = \rho g \int_{AP}^{FP} \bar{x}\bar{y} \frac{dy}{dz} (\eta_{PS} - \eta_{SB}) dx = M_{\phi\eta}$$

where subscripts PS and SB stand for portside and starboard wave elevation given by:

$$\eta_{PS,SB} = a\cos(k(\bar{x}\cos(\chi) \pm \bar{y}\sin(\chi)) + \omega_e t)$$

and k represents wave number.

7. VARIATIONAL EQUATION

Stability analysis is carried out by taking the linear variational equation of the system defined by equation (18). The linear variational equation is obtained by considering that the motion may be defined as being the sum of a steady function plus a perturbation imposed to this basis function:

$$\vec{q}(t) = \vec{\bar{q}}_0(t) + \vec{u}(t)$$

such that $\vec{\bar{q}}_0(t)$ is the solution of the linear system associated with the complete non-linear problem, equation (18). In addition to this, $\vec{u}(t)$ is a perturbation superimposed to the linear solution. If the solution of the linear system is considered stable, then the solution of the non-linear system near this solution will be stable if a solution of $\vec{u}(t) \to 0$ when $t \to \infty$, and is unstable otherwise.

The solution of the linear equations is well known. In this case, the anti-symmetric equations are uncoupled from the other three symmetric equations. On deriving the linear variational equation for the non-linear system given above, a coupled linear system with time-dependent coefficients is obtained:

$$[\tilde{M}+\tilde{A}]\ddot{\vec{u}} + [\tilde{B}+\tilde{B}_l(t)]\dot{\vec{u}} + [\tilde{C}_0+\tilde{C}_p(t)+\tilde{C}_{wp}(t)]\vec{u} = 0 \quad (22)$$

In the present mathematical model matrix $\tilde{B}_l(t)$, representing internal damping, contains one single time-dependent element representing dissipation of energy associated with the second order term in the roll damping moment. Matrix $\tilde{C}_p(t)$ is composed of oscillatory terms at the excitation frequency and may be expressed as:

$$\tilde{C}_p = \begin{bmatrix} \begin{pmatrix} 2Z_{zz}\hat{z} \\ + \\ Z_{z\theta}\hat{\theta} \end{pmatrix} & 2Z_{\phi\phi}\hat{\phi} & \begin{pmatrix} Z_{z\theta}\hat{z} \\ + \\ 2Z_{\theta\theta}\hat{\theta} \end{pmatrix} \\ K_{z\phi}\hat{\phi} & \begin{pmatrix} K_{z\phi}\hat{z} \\ + \\ K_{\phi\theta}\hat{\theta} \end{pmatrix} & K_{\phi\theta}\hat{\phi} \\ \begin{pmatrix} 2M_{zz}\hat{z} \\ + \\ M_{z\theta}\hat{\theta} \end{pmatrix} & 2M_{\phi\phi}\hat{\phi} & \begin{pmatrix} M_{z\theta}\hat{z} \\ + \\ 2M_{\theta\theta}\hat{\theta} \end{pmatrix} \end{bmatrix}$$

where:

$$\hat{z}(t) = z_0\cos(\omega_e t + \nu_3)$$

$$\hat{\phi}(t) = \phi_0\cos(\omega_e t + \nu_4)$$

$$\hat{\theta}(t) = \theta_0\cos(\omega_e t + \nu_5)$$

are oscillatory functions representing the linear responses of the vessel in heave, roll and pitch. It is well known that at this level the equations of the groups of symmetric and anti-symmetric modes are not coupled to each other.

The matrix given above represents the contributions to parametric excitation resulting from different couplings of modes. In addition to this, matrix $\tilde{C}_{wp}(t)$ represents the contributions to parametric amplification due to volumetric changes occurring as a consequence of wave passage along the hull. The sum of these two matrices gives the final matrix of parametric excitation:

$$\tilde{D}_0(t) = \tilde{C}_p(t) + \tilde{C}_w(t)$$

which is dependent on the wave amplitude *a*. Equation (8) may be rewritten as:

$$[\tilde{M}+\tilde{A}]\ddot{\vec{u}} + [\tilde{B}+\tilde{B}_l(t)]\dot{\vec{u}} + [\tilde{C}_0+\tilde{D}_0(t)]\vec{u} = 0 \quad (23)$$

This is a linear system with time-dependent coefficients. It will be noticed that damping has influence on limiting amplifications resulting from resonances. As interest here is centered on demonstrating the existence of sub or super harmonics, damping need not be considered in the following theoretical developments.

8. COUPLED LINEAR SYSTEMS WITH PERIODIC COEFFICIENTS

The equation under study here is, in matrix form:

$$(\widetilde{M}+\widetilde{A})\ddot{\vec{u}}+[\widetilde{C}_0+\varepsilon\widetilde{D}_3(t)]\vec{u}=0 \tag{24}$$

where ε is a small parameter and:

$$\varepsilon\widetilde{D}_3(t)=\widetilde{D}_0(t) \tag{25}$$

is a square matrix.

This system may be conveniently expressed in diagonal form. Pre-multiplying equation (24) by the inverse matrix $(\widetilde{M}+\widetilde{A})^{-1}$ and defining a new set of variables obtained from $\vec{u}=\widetilde{T}\vec{y}$, where $\widetilde{T}$ is a linear transformation matrix, the following canonical form may be given, using indicial notation:

$$\ddot{y}_j+\sigma_j^2 y_j+\varepsilon\sum_{i=1}^{n=3}\psi_{ji}(t).y_j=0;\ j=1,2,3 \tag{26}$$

where the coefficients σ_j are the eigenvalues of system (24) for $\varepsilon=0$ and:

$$\psi_{ij}(t)=p_{ij}\cos\omega_e t+q_{ij}\sin\omega_e t$$

Note that for the case where the time-dependent matrix is a diagonal matrix, equation (26) reduces to three uncoupled Mathieu equations, in heave, roll and pitch. Now, with general systems like the one described by equation (26) for $\varepsilon\neq 0$, the situation may be quite distinct.

Hale (1954), Gambill (1954) and Cesari (1959) have studied $n\times n$ systems of the type of equation (26). It has been demonstrated that such systems may present many more resonant frequencies than the case $n=1$, i. e. the Mathieu equation. Gambill (1954) has given explicit criteria for parametric instability and for boundedness of the solutions:

If:

$$m\omega_e\neq\sigma_j\pm\sigma_i \quad \text{(i, j =1, 2, 3 ; m=1,2 ...)} \tag{27a}$$

together with either:

$$\psi_{ij}(t)=\psi_{ij}(-t) \quad \text{(i, j =1, 2, 3)} \tag{27b}$$

or:

$$\psi_{ij}(t)=\psi_{ji}(t) \quad \text{(i, j=1, 2, 3)} \tag{27c}$$

then all solutions of the matrix equation (26) are bounded in $[0,+\infty]$ for sufficiently small $|\varepsilon|$.

Condition (27a) assures that no resonance occurs between the small periodic restoring terms $\varepsilon\sum_{i=1}^{n=3}\psi_{ji}y_i$ and the harmonic oscillations of the differential equations:

$$\ddot{y}_j+\sigma_j^2 y_j=0 \quad ; \quad j=1,2,\cdots,3$$

Condition (27a) should be compared with the one obtained when only one degree of freedom is assumed: $m\omega_e\neq 2\sigma$ (m = 1, 2, ...) see Stoker (1950), Paulling and Rosenberg (1959). Conditions (27b) and (27c) are essentially conditions of symmetry assuring the boundedness of all solutions of equation (26) under condition (27a). If these additional conditions are not satisfied, it is likely that equation (26) with n>1 will have unbounded solutions in $[0,+\infty]$ for small $|\varepsilon|$.

9. CO-PARAMETRIC RESONANCE

Equation (26) represents a set of coupled Mathieu equations. According to conditions (27), the occurrence of co-parametric resonance depends on the structural form of matrix $\widetilde{\psi}(t)$. Considering the derivatives obtained previously in Sections 5 and 6 it may be noticed that in general the matrix of parametric excitation is non-symmetric. Although $\widetilde{C}_{wp}(t)$, the matrix of time-dependent wave passage terms is symmetric, matrix $\widetilde{C}_P(t)$ is not. This may be concluded from comparison of the off-diagonal terms of matrix $\widetilde{C}_P(t)$, obtained in Section 7. Considering the structure of this matrix it is seen that in general for a hull all elements of $\widetilde{\psi}(t)$ are different from zero. Elements along the main diagonal intervene in auto-parametric resonance, whereas off-diagonal elements define the level of excitation of combination modes (co-parametric resonance). This is indicated in Figure 5, which illustrates the fact that the heave/roll and roll/pitch co-parametric resonances derive from the product of pairs of oscillatory functions having the first order roll motion as common multiplier.

For the co-parametric resonances associated with the roll mode and considering the off-diagonal terms in matrix $\widetilde{C}_P(t)$ it may be seen that there is no co-parametric resonance for wall-sided hulls. In this case no vertical force would exist and the second order coupling of roll with the vertical modes would disappear. Additionally, there is no co-parametric resonance for longitudinal waves, since in this case there is no (first order) roll motion $\widehat{\phi}(t)$. In other words, roll co-parametric resonance depends on the occurrence of direct excitation of the roll motion.

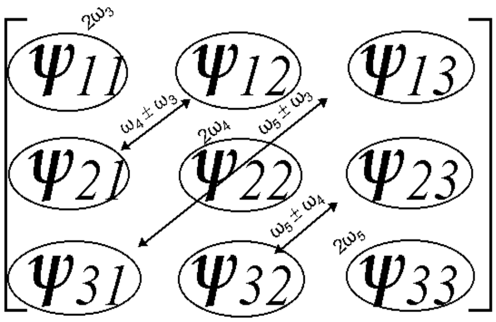

Figure 5: Role of elements of parametric resonance matrix on the development of harmonics.

In general, the algorithms for obtaining the limits of stability of dynamic systems like equation (26) may be complicate, see Hsu (1963). But taking into account that for the ship hull problem the elements $\psi_{12}(t)$, $\psi_{21}(t)$, $\psi_{23}(t)$ and $\psi_{32}(t)$ have the same phase difference with respect to the exciting waves as the roll motion, employing first-approximation analysis, simple, explicit expressions may be given for the limits of stability corresponding to the co-resonances of the type $\omega_e = \omega_k \pm \omega_4$, $k = 3,5$, see Hsu (1963):

a) The system is unstable if the condition

$$\left|\omega_e - (\omega_{k+2} + \omega_4)\right| < \frac{\varepsilon}{2}\sqrt{\frac{p_{2k}\,p_{k2}}{\omega_{k+2}\omega_4}} \quad , \quad k = 1,3 \qquad (28)$$

for the sum type of co-parametric resonance, stable otherwise;

b) The system is unstable if the condition

$$\left|\omega_e - (\omega_{k+2} - \omega_4)\right| < \frac{\varepsilon}{2}\sqrt{-\frac{p_{2k}\,p_{k2}}{\omega_{k+2}\omega_4}} \quad , \quad k = 1,3 \qquad (29)$$

for the difference type of co-resonance, stable otherwise.

Hence, an important feature of the coupled Mathieu system obtained for the ship problem derives from the fact that for the co-parametric regions related to roll, where the time-dependent functions have the same phase. It is then observed that either sum or difference type may occur, depending on the structure of the off-diagonal terms.

In the above expressions the stability boundaries for the two types are obtained as linear functions of the smallness parameter. For the uncoupled Mathieu equation the well known Stoker's approximation, see Stoker (1950), of stable and unstable regions defines, for m=1 (the first region of instability), the limits of stability. This is considered the most important region. How wide is the angle inside the unstable area is governed, for the ship problem, by the heave and pitch motions transfer functions and hull parameters. For m>1, following Stoker power expansions, limits of stability corresponding to the tuning $m\omega_e = \omega_4$ are obtained as higher order functions of the small parameter ε. Yet, the limits of stability for the co-parametric resonances $\omega_e = \omega_i \pm \omega_j$ will be defined as straight lines, see equations (28, 29), as suggested in Figure 6 for the difference type frequencies. In Figure 6 shaded areas correspond to unstable regions. It is schematically indicated that the $2\omega_4$ region has a large opening angle.

Considering again the simple example of a prismatic triangular shape, for the heave/roll coupling, it can be demonstrated that the angle defining the width of the area for the auto-parametric resonance ($2\omega_4$ region) is defined by the following law of proportionality:

$$\alpha_{44} \propto \left(\frac{z_0}{a}\right)\frac{\overline{\psi}_{22}}{\omega_4} = \left(\frac{z_0}{a}\right)\frac{\partial I_{xx}}{\partial z}\left(1 - \frac{1}{3(d\overline{y}/d\overline{z})^2}\right)\frac{1}{\omega_4}$$

On the other hand, the same angle for the co-parametric resonance ($\omega_3 - \omega_4$ region) will be given by:

$$\alpha_{34} \propto \left(\frac{\phi_0}{a}\right)\sqrt{\frac{\overline{\psi}_{12}\overline{\psi}_{21}}{\omega_3\omega_4}} = \left(\frac{\phi_0}{a}\right)\frac{\partial I_{xx}}{\partial z}\sqrt{\frac{\left|1 - \dfrac{1}{3(d\overline{y}/d\overline{z})^2}\right|}{\omega_3\omega_4}}$$

In the above expressions $\dfrac{\partial I_{xx}}{\partial z} = 2\overline{y}^2\dfrac{d\overline{y}}{d\overline{z}}$, and $\overline{\psi}_{ij}$ represent amplitude of harmonic functions $\psi_{ij}(t)$.

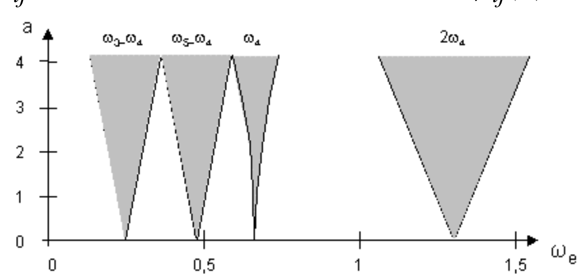

Figure 6: Limits of stability for auto and co-parametric resonance.

It may be observed that these two angles are dependent on hull characteristics and transfer functions of pertinent motions. It is conceivable that for ships with large roll motions (e.g., near $\omega_e = \omega_4$ in oblique seas) the angle α_{34} may be smaller, but of comparable order to α_{44}, if hull forms and wave incidence retain some characteristics defined above, mainly the flare. Additionally, co-parametric area of instability (linear limits) may be more *open* than the m=2 region defined for the uncoupled Mathieu equation, which in fact has width zero for small values of ε.

10. CONCLUSIONS

The effects of non-linear couplings occurring between heave, roll and pitch modes have been examined in this paper. Complete expressions for the hydrostatic actions up to second order have been implemented in the form of derivatives taken with respect to the independent variables. In addition, simplified expressions have been derived for these derivatives in terms of longitudinal distributions of relevant geometric hull properties.

For hull forms that are not wall-sided, angular displacements in roll produce non-equivolumetric inclinations. The vertical force produced by this roll inclination, and the pitch moment produced by a non-symmetric longitudinal distribution of vertical forces along the different cross-sections of the ship, will then completely couple, to second order, the vertical modes to the roll mode through hydrostatic effects. Expressions for these non-linear couplings have been derived. The analytical methodology employed in the derivation of the equations of motion clearly facilitates interpretation of the physics behind the mechanism and the disclosure of instability of the non-linear motion.

It is demonstrated that the linear variational equation of the original non-linear system may be reduced to a set of coupled Mathieu equations. The nature of the couplings in relation to wave direction is retained in the analytical procedure. Criteria for the occurrence of parametric resonance and boundedness of solutions are given. It is then concluded that there are more possible resonant conditions in this case than when the uncoupled Mathieu equation is invoked to represent the dynamics of ship roll stability. The combined co-resonant modes defined by sums and differences between the system's eigenvalues are then qualitative characteristics very distinct from the known auto-parametric resonance.

It is also demonstrated that the time-dependent off-diagonal terms associated with co-parametric resonance of the roll motion have all the same phase. This feature implies that either sum or difference types of combined resonance may take place, depending on the structure of the matrix of parametric excitation. Basic features of these instability regions are discussed for a simple geometrical form. It is shown that limits of stability for co-parametric resonance are obtained for first order values of the smallness parameter. At the same time it is pointed out that, in contrast, limits of stability corresponding to $m\omega_e = \omega_4$, $m = 2,3,...$ for the uncoupled Mathieu equation can be obtained as perturbed solutions only at higher levels of approximation of the smallness parameter.

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