On Long Numerical Simulations at Extreme Seas

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Abstract

The paper is focused on the previously known effect of "statistical self-repetition" encountered when a wave record is reconstructed from spectrum with a constant frequency step. A correlation function calculated for such a record shows a repetitive pattern of high values. It means an unrealistically strong probabilistic dependence between certain time sections of the process. The effect remains even if the autocorrelation function is calculated directly from spectrum without actual reconstruction of the wave record. An attempt to work the effect around using a variable frequency step leads to a spread of the error. It appears as a series of unrealistically high values instead of discrete peaks. The previous work of the author also found that the presence or absence of the effect strongly depends on the method of numerical integration.

The paper concludes that the nature of this effect is a numerical error caused by the highly oscillatory character of an integrand in Laplace transform (relation between spectrum and autocorrelation function) for large values of time. The paper also discusses the practical implications of this conclusion.

Introduction: Effect of Self-repetition of Stochastic Processes

This paper is a direct continuation of the previous work [Belenky, 2004]. Nevertheless the most relevant parts of it are repeated here for the sake of convenience.

Autocorrelation function $R(\tau)$ of a stationary ergodic stochastic process is defined as:

$$R(\tau) = M\{(X(0) - m_X) \cdot (X(\tau) - m_X)\} = \lim_{T \to \infty} \left[\frac{1}{T - \tau} \int_0^{T-\tau} (x(t) - m_X)(x(t+\tau) - m_X)\, dt \right] \quad (1)$$

here $X(t)$ is a value from the whole ensemble of records, while $x(t)$ is a value belonging to a certain record. M{} is an averaging operator and m_X is a theoretical average, which equals to zero as the only stochastic process considered here is wave elevation. For a record presented in a series of discrete values, the estimate of autocorrelation function is expressed as:

$$R_j = \frac{1}{N_t - j} \sum_{i=1}^{N_t - j} x_i x_{i+j} \quad (2)$$

Autocorrelation function is related with spectrum density $s(\omega)$ through Laplace transform:

$$R(\tau) = \int_0^\infty s(\omega) \cos \omega\tau \, d\omega; \quad s(\omega) = \frac{2}{\pi} \int_0^\infty R(\tau) \cos \omega\tau \, d\tau \quad (3)$$

The self-repetition effect could be seen as a periodical "surging" of the autocorrelation function (see Figure 1) where the autocorrelation function is calculated from a reconstructed time-domain record. These "surges" indicated an unrealistically tight probabilistic dependence between wave elevations at a certain instance of time.

Growing oscillation around 800 second is a result of accumulation statistical error caused by insufficient statistics. This error becomes irrelevant if the autocorrelation function is computed directly from spectrum, however, the self-repetition effect remains see Figure 2.

An attempt to "work out" the problem by using an unequal frequency step "spreads" this "surge" over time, but does not eliminate it. See Figure 3 [Belenky 2004].

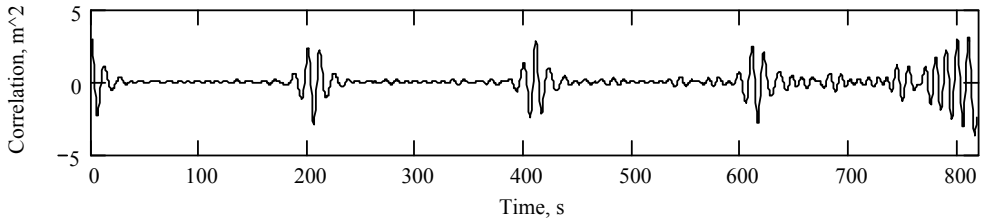

Figure 1 Autocorrelation function calculated from time-domain record

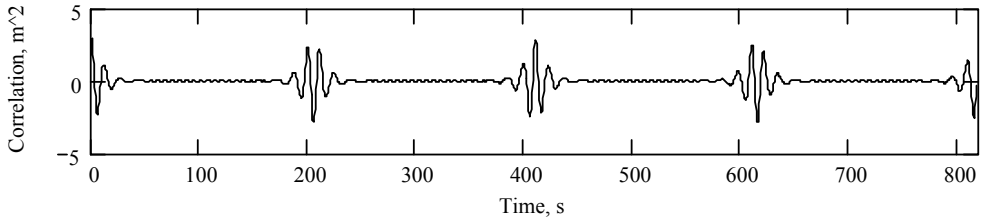

Figure 2 Autocorrelation function for calculated by Laplace Transform

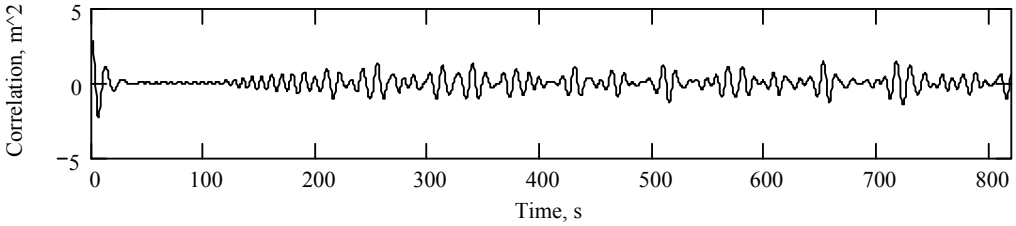

Figure 3 Autocorrelation function for calculated by Laplace Transform with unequal frequency step

Self-repeating Effect: Dependence of Integration Method

It was shown in [Belenky 2004] that using the adaptive quadrature method for numerical integration in Laplace Transform completely eliminates repeating patterns. See Figure 4.

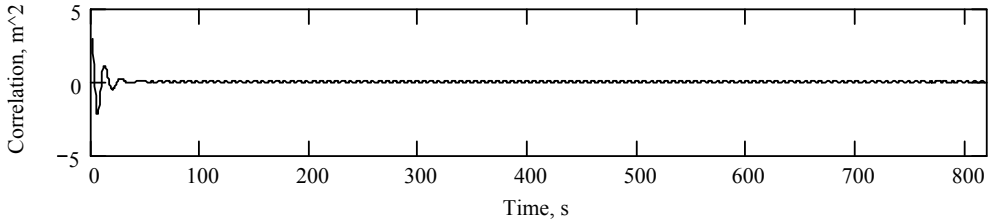

Figure 4 Autocorrelation function calculated through Laplace Transform method with Adaptive method

Rectangular numerical integration was used in both cases to calculate the autocorrelation function through Laplace Transform. The only difference between these cases was the frequency step, constant for Figure 2 and variable for Figure 3.

Inverse Fourier Transform that is conventionally used to reconstruct wave elevation could be also considered as Fourier Integral calculated numerically with the rectangular method:

$$\zeta_W(t) = \int_0^{\infty} \sqrt{2s(\omega)} \cos(\omega t + \varphi(\omega)) d\omega \approx \sum_{i=1}^{N_\omega} r_{Wi} \cos(\omega_i t + \varphi_i) \qquad (4)$$

Adaptive Method and Gauss-Legendre Quadrature

The adaptive quadrature method of numerical integration is based on the idea that smooth function requires less point to integrate accurately while quickly varying (especially oscillatory) would take more point to reach the same accuracy [Heath, 2002].

In the first step, the integration is performed on the whole range (Simpson method typically is used). Then the range is divided in half and the integration is performed for both halves. The sum of these integrals then is compared with the result of integration over the whole range. If the difference is below the given threshold, the procedure is complete.

To achieve better accuracy, Gauss-Legendre Quadrature [Bronshtein and Semendyayev, 1997] was used instead of Simpson rule.

$$\int_a^b f(x)dx \approx \sum_{i=1}^{7} 0.5 \cdot A_i \cdot (b-a) \cdot f(y_i) \qquad (5)$$

where A_i are weights for the values of function f, calculated in points (nodes) specified as:

$$y_i = 0.5\left((b-a)z_i + a + b\right) \qquad (6)$$

Numerical values are given in Table 1 [Booth 1957]:

Table 1. Numerical Values for Gauss-Legendre Quadrature

Ordinates, z	Weights, A
-0.9491079	0.129485
-0.74153119	0.27970539
-0.40584515	0.38183005
0	0.41795918
0.40584515	0.38183005
0.74153119	0.27970539
0.9491079	0.129485

The Gauss-Legendre quadrature was used along with adaptive algorithm primary because the former is known for its outstanding accuracy with only a few points. Therefore, a combination of the Gauss-Legendre quadrature with the adaptive technique may be expected to yield the best accuracy. However, the number of points required to maintain accuracy rises quite significantly, as shown in Figure 5.

Actually, if a number of frequencies will be increased, so will the time while the wave reconstruction is still valid. To achieve a duration of simulation of one hour, a conventional method with rectangular integration would require about 1150 frequencies. An adaptive method would need only about 600, which is also quite significant number.

It means that the better accuracy of the autocorrelation function in Figure 4 requires an increasing amount of frequencies, but due to the specific way adaptive algorithm works, the number of points is simply not known in advance.

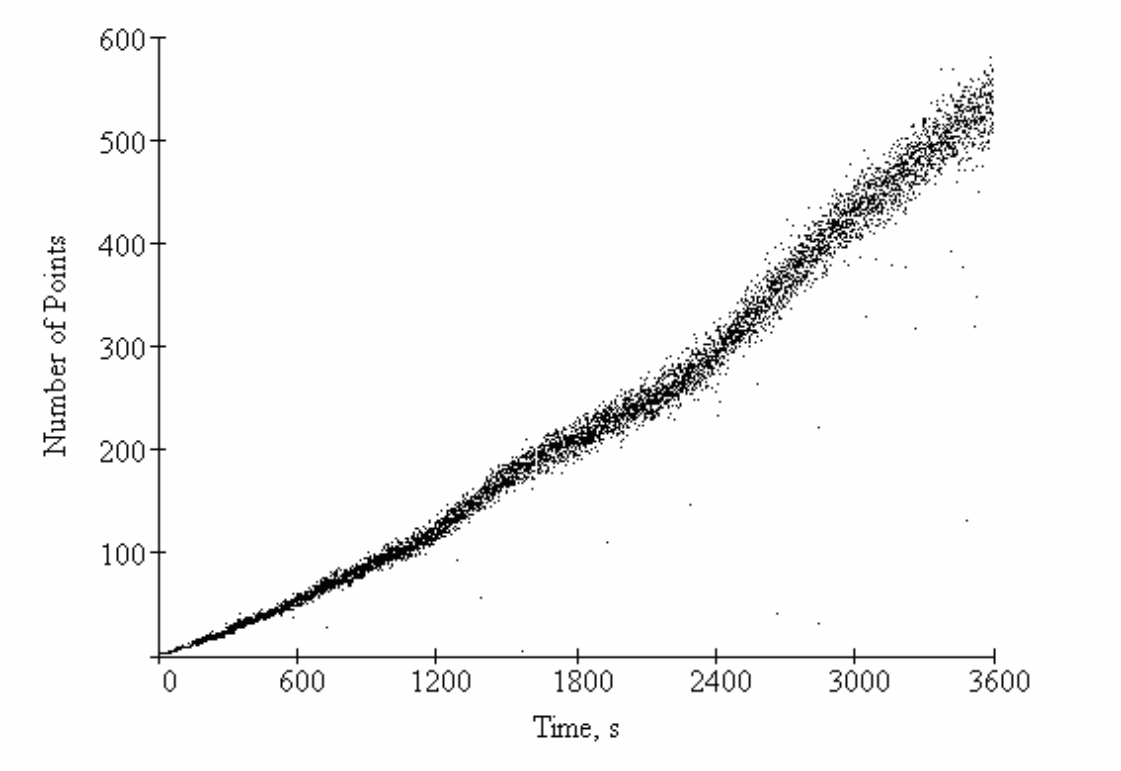

Figure 5 Number of points of Adaptive / Gauss Legendre Quadrature vs. time of integration for autocorrelation function

Oscillatory Character of the Integrand in Laplace Transform

The increasing number of points needed to keep a given accuracy of the adaptive numerical integration procedure is a clear indication of the oscillatory character of the integrand in Laplace Transform. Actually, it could be seen from the structure of the formula of Laplace Transform:

$$R(\tau) = \int_0^{\infty} s(\omega)\cos\omega\tau\, d\omega \qquad (7)$$

Here, time τ plays a role of "frequency", while the real physical frequency ω is a variable of integration. Once the time increases, the integrand becomes quite oscillatory, which is illustrated in Figure 6.

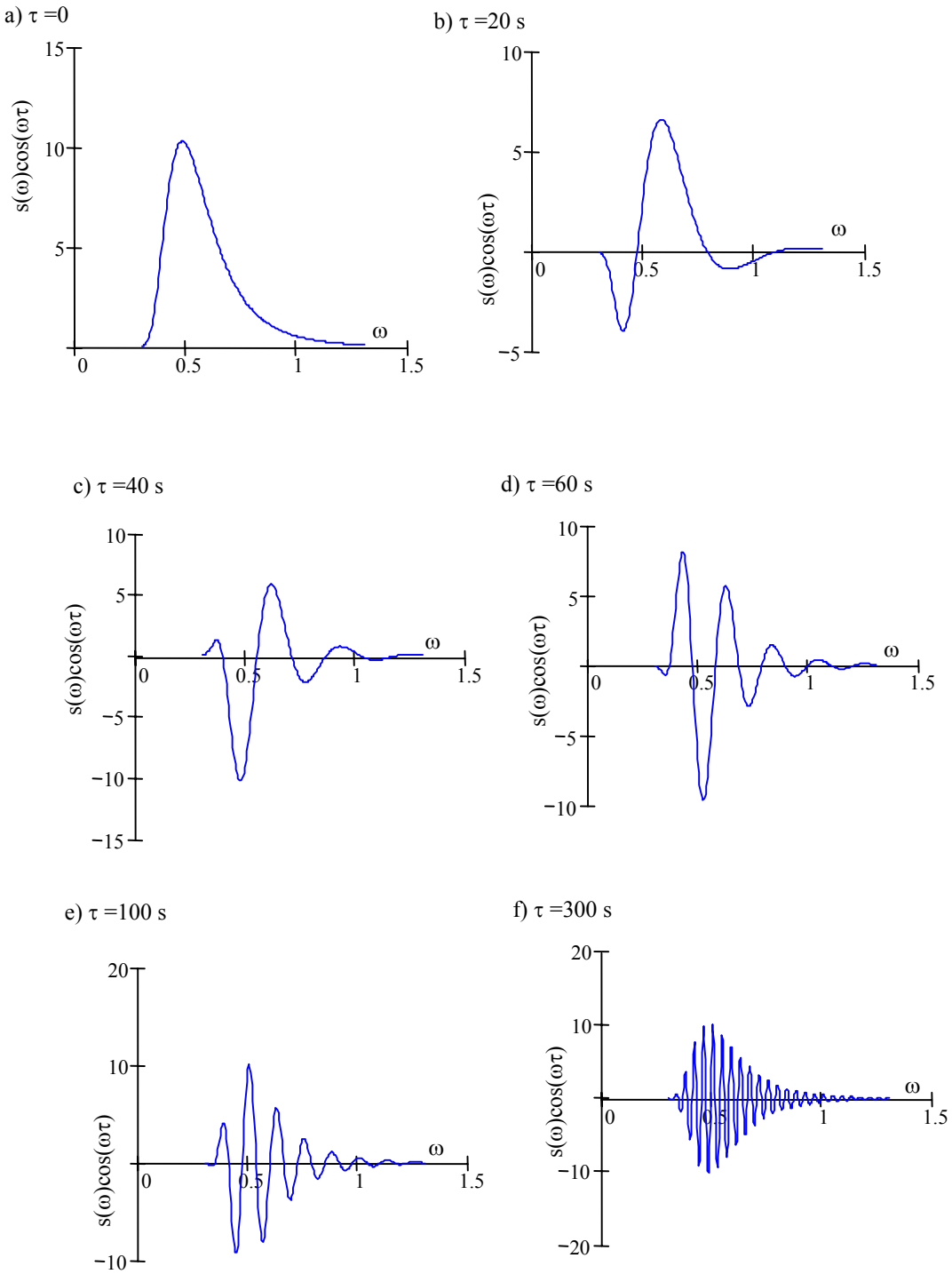

Figure 6 Integrand of Laplace Transform

Plots on Figure 6 also show how quickly the integrand becomes oscillatory. Integration of such an oscillatory function, indeed, presents a challenge and requires using many points on the frequency axis.

Numerical errors, which are inevitable when time is large and the number of frequencies is limited, will cause unrealistic close probabilistic dependence of an otherwise distant section of the stochastic process. As it has been demonstrated in [Belenky 2004] and above, these errors will be transferred to the time history records, making results of the simulation questionable from a statistical point of view.

Relation with Fourier Series

The numerical errors appearing in the autocorrelation function, when the method of rectangulars with constant frequency step is applied, have quite a regular character.

In the case of the constant frequency step, the operation of Fourier transform coincides with expansion into Fourier series. Formal definition of Fourier series and its coefficients:

$$f(t) = \sum_i A_i \sin\left(\frac{i\pi t}{T}\right) + B_i \cos\left(\frac{i\pi t}{T}\right) \tag{8}$$

$$A_i = \frac{1}{T}\int_{-T}^{T} f(t)\sin\left(\frac{i\pi t}{T}\right)dt \quad ; \quad B_i = \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{i\pi t}{T}\right)dt \tag{9}$$

These formula are written for a function $f(t)$, which is assumed to be periodical with period $2T$.

When Fourier series are used for presentation of a stochastic process, they are re-written in the following form:

$$f(t) = \sum_i C_i \sin\left(\omega_i t + \varphi_i\right) \tag{10}$$

Where figures C_i and φ_i are related with coefficients A_i and B_i:

$$C_i = \sqrt{A_i^2 + B_i^2} \; ; \quad \varphi_i = \arctan\left(\frac{A_i}{B_i}\right) \tag{11}$$

Following [St. Denis and Pierson 1953], coefficient C_i is considered as amplitude of the component r_{Wi} calculated with the spectral density:

$$C_i = r_{Wi} = \sqrt{2S_{Wi}} = \sqrt{2\int_{\omega_i - 0.5\Delta\omega_i}^{\omega_i + 0.5\Delta\omega_{i+1}} s(\omega)d\omega} \; ; \qquad \Delta\omega_i = \omega_i - \omega_{i-1} \tag{12}$$

The other value related with Fourier coefficient is a set of initial phases φ_i comprised of random numbers, distributed uniformly from 0 to 2π; this is the only stochastic figure in the model.

Nevertheless, if the frequency step is constant, the formulae (11) and (10) have the same meaning, therefore:

$$\omega_i = \frac{\pi i}{T}; \quad i = 1, 2, ..., N \tag{13}$$

This relationship inevitably leads to repetition after period of $2T$, as Fourier coefficients (9) originally supposed periodicity with $2T$.

Formula (13) supposes that frequency set starts only $\Delta\omega$ from zero, while, in fact, the first frequency value usually is set further. This circumstance does not change anything as spectral values are very small close to zero, so components with zero amplitudes always can be added to presentation (10).

It was noted in the previous work [Belenky 2004] that a sum of trigonometric function is a periodic function. Its period, however, is quite a large number. It equals the least common multiple of all the periods of the components. However, Fourier series is not just any sum of trigonometric functions.

Concluding Comments

In general, the self-repetition effect of a stochastic process presented with the Inverse Fourier Transform is related with numerical error. This error is caused by the highly oscillatory nature of the integrand in Laplace Transform and therefore cannot be eliminated by rearranging frequencies.

In the case of a constant frequency step, the Inverse Fourier Transform is equivalent to Fourier series and the self-repetition effect is also a direct result of periodicity of Fourier series defined at the finite interval.

What is the practical meaning of the above statement?

It means that if a long period of simulation is needed, a large number of frequencies must be used. For example, for the Bretschinder spectrum for significant wave height 12.5 m with zero crossing period 11.5 s, about 2050 frequencies would be needed to reconstruct a six hours wave record. It means that 2050 components have to be summed at every time step of the simulation which significantly increases the computational cost of such a simulation.

Alternatively, only 80 frequencies are necessary for the correct reconstruction of a 12 minute long wave record. Keeping in mind that only one minute is needed for complete decay of wave autocorrelation function (in this sample), it would be quite practical to use 30 such wave records instead of one six-hour-long. Each of these wave records must have its own independent set of initial phases. It would be reasonable, however, to check and compare confidence intervals for main statistical characteristics of on six-hour-long record with the set of 30 twelve-minutes records.

References

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