# Analysis of Oklahoma's Voting Power within the U.S. Electoral College

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# 1 Introduction

There is a trend among young American voters to believe that their vote in the general presidential election amounts to nothing. This can be true or false, depending on one's viewpoint and interpretation of voting itself. It's reasonable to imagine a Democrat living in Oklahoma might believe their vote is worthless - in fact, Donald Trump (R) won every single county of the state in the latest 2020 General Presidential Election.[1] This might present itself as obvious; Oklahoma is a heavily Republican-leaning state when it comes to elections. Looking at the data, however, shows that 33% of the state's population are registered Democrats.[2] It is also reasonable to imagine someone that believes elections are decided beforehand or that the outcome doesn't matter; thus, they believe voting is inconsequential.

This paper is interested in the actual, mathematical power that an Oklahoman has when it comes to influencing the electoral college. The reasoning behind why someone may *think* their right to vote means nothing is interesting but means little without data. This paper will establish the mathematical power of of the 51 members of the electoral college using a known method. That information will then be applied and interpreted to the individual Oklahoma voter. The real-world outcomes of states will be ignored for time being (such as the fact that Oklahoma is nearly guaranteed to never vote for the same candidate as California) Furthermore, this paper will discuss in detail the method for the power computation and problems encountered when applying it.

# 2 State Power Calculations

In order to determine the voting power of an Oklahoman, the power of Oklahoma itself must first be determined. The Banzhaf power index, named after and developed by American lawyer John Banzhaf, will be used to define each state's power. In any voting block, a group of aligned voters is a *coalition*. A coalition that has enough votes to equal the *quota* (the number of votes needed to pass a motion) is a *winning coalition*. A member whose presence in a coalition determines its ability to win is *critical*. In other words, if a winning coalition would lose if they lost a specific member, that member is critical to the coalition. This will also be referred to as a swing because of the member's ability to "swing" a vote. The Banzhaf power index is defined as the simple ratio between the number of coalitions in which a member is critical to the sum of coalitions in which each individual member is critical. In simple terms, a voter's Banzhaf power index is the number of swings they cause divided by the total number of swings that every member causes. The works of Brian Heger[3], V. Yakuba[4], and Mark Livingston[5] were immensely helpful in the formulation of this project. Equations 1 and 3 are adapted from their writings on this topic.

$$B_i = \frac{\eta_i}{\sum_{j \in N} \eta_j} \tag{1}$$

Where $B_i$ is the Banzhaf power index of voting member $i$, $\eta_i$ is the number of swings caused by $i$, and $N$ is the set of voting members. In the electoral college the quota a coalition of states must reach is 270 votes. Oklahoma has 7 electoral votes. Imagine a presidential race where a coalition of states has accrued 263 votes. Should Oklahoma join this coalition, it would be a critical voter. Now realize that there is an incredible number of combinations of states that can reach 263 votes; Oklahoma is critical in every single one of them. Even further, imagine coalitions summing to 264, 265, 266, 267, 268, or 269 votes. Should Oklahoma join any of these proposed coalitions (keeping in mind the large number of combinations to reach each sum) it would be critical.

Now, the idea behind the computation of state's power should be clear. The number of swings that each state causes must be found. Summing those will provide the total number of swings. Dividing each state's swings by the total will result in each state's Banzhaf power index. This computation can be done simply (though tediously) by hand with a sufficiently small voting block. However, the complexity of this computation grows tremendously as the number of voting members grows.

A traditional analysis of algorithmic complexity utilizes *Big-O Notation* which describes the type of function that "limits" an algorithm. Given constants $c > 0$ and $n_0 >= n$, there is a function $g$ such that for algorithm $f, |cg(n_0)| >= |f(n)|$. In other words, if a function if Big-O of the algorithm, that function will always grow larger than the algorithm will past a certain point $n_0$. Conventionally, constants can be ignored and only the fastest-growing function is considered. The Banzhaf computation is $O(2^n)$ meaning computation time doubles with every member added. This is acceptable for small data sets, but will become too difficult to compute past around 25 members. The traditional Banzhaf calculation would not complete within a lifetime because it has 51 members.

One method initially considered would be to perform a Monte Carlo simulation on electoral college. A Monte Carlo simulation involves generating a set of random data points and computing an outcome. After doing this a very large number of times an estimate of final values can be reached. There are considerable problems with a Monte Carlo in this application. First, there are $2^{51}$ outcomes possible and even running several billion simulations (which can take some time) will be nowhere near a statistically significant portion of potential outcomes. Next, Monte Carlo simulations rely on random number generation which isn't perfectly possible using a digital computer. This means the generated data can't be guaranteed to be truly random. Furthermore, this data should be uniformly distributed throughout the possible outcomes (again, difficult to guarantee with a computer). There are solutions to these problems but they require more effort than the chosen solution.

Instead of the traditional method or a simulation, this approach uses combinatorics to quickly and efficiently compute the number of swings a state causes. The traits of *generating functions* can be cleverly used to overcome the computational challenge that this problem poses. An explanation and example of generating functions and their implementation follows.

In the sport of basketball, points can be scored in 1's (free throws), 2's (slam dunks), and 3's (three-pointers). Think of how many ways 6 points can be earned with combinations of these points. The answer is 7 (6x1's; 4x1's + 1x2's; 2x1's + 2x2's; 3x1's + 1x3's; 3x2's; 2x3's; 1x1's + 1x2's + 1x3's;). Now expand the power series

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 10x^8 + ... \quad (2)$$

Looking closely, it can be noticed that the coefficient of $x^6$ is 7. Further analysis will show that for any sum of points $n$, the coefficient of $x^n$ will be the number of ways that sum can be reached. This is a simple generating function; this idea can be extended further to simplify the swing-vote calculations.

In the electoral college there are 51 ways to "score points". By winning a state, its electoral votes are also won and added to the sum towards a set quota: 270. Because there is a finite number of states a power series is no longer necessary. Instead, each state can be represented as a binomial $(x^0 + x^n)$ where $n$ is the number of electoral votes the state has. This binomial demonstrates the two outcomes a state can have: (1) it votes against the candidate and contributes 0 votes (shown with $x^0$) or (2) it votes for the candidate and contributes $n$ votes. Oklahoma, for example, is represented as $(x^0 + x^7) = (1 + x^7)$. The $x^0$ will be simplified to 1 moving forwards.

By multiplying these binomials together, a generating function can be found. Within that function, the quota is represented as the exponent and the number of ways to reach that quota is represented as the coefficient. Think back to the example of Oklahoma joining coalitions of 263, 264, 265 ... votes. In order to use a generating function to calculate swing votes a single state must be left out of computation. Because of this, the generating function now shows how many combinations of states *excluding one* can reach specific quotas. The range of quotas that the excluded state can influence is found as the difference between the quota and the state's electoral votes. Then, the coefficient of each term within this range can be summed to find the number of swings the excluded state causes.

Let $q$ be the quota, $k$ be the sum of votes of the coalition, $b_i(k)$ be the number of coalitions excluding $i$ that sum to $k$ (the coefficient of the term with exponent $k$), $v_i$ be the number of votes member $i$ has, and let $\eta_i$ be the number of swings that $i$ causes.

$$\eta_i = \sum_{k=q-v_i}^{q-1} b_i(k) \quad (3)$$

Equation 3 forms the basis for the functionality of the algorithm used to compute each state's relative power.

# 3 The Algorithm

First and foremost, the algorithm involves computing the product of many binomials. This proved to be the second challenge of the project (after utilizing generating functions). This is accomplished by manipulating some of the properties of polynomial multiplication.

The function first receives two polynomials (one binomial and the previously computed product). It then begins looping through every term of the second for each term of the first. In this way, every combination of terms can be compared. For each combination, their exponents are added and coefficients are multiplied. The function then places the newly computed term into a temporary polynomial. It does this by looking within the temporary polynomial to see if a term of the same exponent exits. If so, the new term and existing term are added together. If not, the new term is added to the end of the temporary polynomial. Any exponent beyond 270 is thrown out as it is no longer useful. The function returns the temporary polynomial and it will become the second polynomial of the next function call.

This is computed for every state. After each state's generating function is found, the algorithm sorts the polynomial's exponents in ascending order. It then computes what range of exponents the state would swing (remember 263 - 269 for OK) and sums those coefficients. That sum is added to a total and stored separately for later use.

After every state's sum of swings has been computed, their Banzhaf power indices are computed by dividing individual swings by the total. This proves to be another challenge due to the sheer size of the values. In C++, 128-bit integers are required to store values up to $2^{64}$ in size. However, they are not standard data types. The Boost multiprecision library was used to provide support for 128-bit integer usage. Furthermore, a high-precision float data type from the multiprecision library was used to compute the ratios between the 128-bit integers.

# 4 Results

Table 1 displays the results from the computation. There are four columns of data, each describing a different set of conditions. The first is if every state uses "Block Voting" or Winner-Take-All. In this system, each state commits every elector to whoever wins the vote of the state. Maine and Nebraska, however, do not use this system. Instead, they allocate two electors to the state winner and one elector to the winner of each congressional district. In this way, they distribute electors proportionally. This is shown in the second column. The third column shows the results if only Oklahoma were to use proportional voting and every other state were to use block voting. The fourth shows Oklahoma, Nebraska, and Maine all using proportional voting.

Table 1: Computed Banzhaf Power Indices

| State | Block Voting | NE, ME Proportional | OK Proportional | OK, NE, ME Proportional |
|---|---|---|---|---|
| California | 0.1145 | 0.1154 | 0.1156 | 0.1164 |
| Texas | 0.0726 | 0.0731 | 0.0732 | 0.0737 |
| Florida | 0.0544 | 0.0548 | 0.0549 | 0.0553 |
| New York | 0.0544 | 0.0548 | 0.0549 | 0.0553 |
| Illinois | 0.0370 | 0.0373 | 0.0374 | 0.0376 |
| Pennsylvania | 0.0370 | 0.0373 | 0.0374 | 0.0376 |
| Ohio | 0.0333 | 0.0335 | 0.0335 | 0.0338 |
| Georgia | 0.0295 | 0.0297 | 0.0298 | 0.0300 |
| Michigan | 0.0295 | 0.0297 | 0.0298 | 0.0300 |
| North Carolina | 0.0276 | 0.0278 | 0.0279 | 0.0281 |
| New Jersey | 0.0258 | 0.0260 | 0.0260 | 0.0262 |
| Virginia | 0.0239 | 0.0241 | 0.0241 | 0.0243 |
| Arizona | 0.0202 | 0.0204 | 0.0204 | 0.0205 |
| Tennessee | 0.0202 | 0.0204 | 0.0204 | 0.0205 |
| Indiana | 0.0202 | 0.0204 | 0.0204 | 0.0205 |
| Massachusetts | 0.0202 | 0.0204 | 0.0204 | 0.0205 |
| Washington | 0.0184 | 0.0185 | 0.0185 | 0.0186 |
| Minnesota | 0.0184 | 0.0185 | 0.0185 | 0.0186 |
| Missouri | 0.0184 | 0.0185 | 0.0185 | 0.0186 |
| Wisconsin | 0.0184 | 0.0185 | 0.0185 | 0.0186 |
| Maryland | 0.0184 | 0.0185 | 0.0185 | 0.0186 |
| Colorado | 0.0165 | 0.0166 | 0.0167 | 0.0168 |
| Alabama | 0.0165 | 0.0166 | 0.0167 | 0.0168 |
| South Carolina | 0.0165 | 0.0166 | 0.0167 | 0.0168 |
| Kentucky | 0.0147 | 0.0148 | 0.0148 | 0.0149 |
| Oregon | 0.0128 | 0.0129 | 0.0129 | 0.0130 |
| Oklahoma | 0.0128 | 0.0129 | 0.0043 | 0.0043 |
| Connecticut | 0.0128 | 0.0129 | 0.0129 | 0.0130 |
| Nevada | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| Utah | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| Kansas | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| Iowa | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| Arkansas | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| Louisiana | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| Mississippi | 0.0110 | 0.0111 | 0.0111 | 0.0112 |
| New Mexico | 0.0092 | 0.0092 | 0.0092 | 0.0093 |
| Nebraska | 0.0092 | 0.0046 | 0.0092 | 0.0046 |
| West Virginia | 0.0092 | 0.0092 | 0.0092 | 0.0093 |
| Idaho | 0.0073 | 0.0074 | 0.0074 | 0.0074 |
| Rhode Island | 0.0073 | 0.0074 | 0.0074 | 0.0074 |
| New Hampshire | 0.0073 | 0.0074 | 0.0074 | 0.0074 |
| Maine | 0.0073 | 0.0049 | 0.0074 | 0.0050 |
| Hawaii | 0.0073 | 0.0074 | 0.0074 | 0.0074 |

Table 1: Computed Banzhaf Power Indices

| State | Block Voting | NE, ME Proportional | OK Proportional | OK, NE, ME Proportional |
|---|---|---|---|---|
| Montana | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| Wyoming | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| South Dakota | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| North Dakota | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| Washington D.C. | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| Delaware | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| Vermont | 0.0055 | 0.0055 | 0.0055 | 0.0056 |
| Alaska | 0.0055 | 0.0055 | 0.0055 | 0.0056 |

The results of the block voting trial closely match the results of Brian Heger (Using Generating Functions). According to the proportional data, the voting power of both Maine and Nebraska decrease significantly using proportional representation. This is likely because the states are no longer voting as a block and therefore "compete" with themselves. The same phenomenon occurs with Oklahoma.

As for computing the proportional data: rather than use the simple binomial for the states like before, the algorithm detects them and instead uses a preset polynomial. Nebraska's is $(1 + x^3 + x^4 + x^5)$ and Maine's is $(1 + x^3 + x^4)$. Oklahoma's is $(1 + x^3 + x^4 + x^5 + x^6 + x^7)$.

# 5 Oklahoman Power

According to the computed data, Oklahoma holds approximately 1.28% of the power within the electoral college. The U.S. Census Bureau July 1, 2019 estimate places Oklahoma's population at 3,956,971 people.[6] The same source places the United State's total population at 328,239,523 (as of 2019). That gives Oklahoma approximately 1.21% of the countries population - remarkably close to the state's electoral power. Compare that to a very high population state like California (12.04% of population) or Texas (8.83% of population) which have powers of 11.5% and 7.3% respectively. Now compare that to Wyoming, the lowest population state in the Union which has 0.176% of the population and 0.55% of electoral power. The electoral college is not based off of pure population statistics, but the powers of each state seem to quite closely match them. The data also shows the well-known fact that higher population states are underrepresented and lower population states are over represented due to the nature of the Senate.

Dividing Oklahoma's power by its population gives a rough idea of how much power an individual Oklahoman has over the electoral college; it comes out to approximately $3.23{*}10^{-7}$%. Interestingly, moving Oklahoma to a district-proportional system like Maine and Nebraska would decrease its electoral power by over half. The state actually drops to be weaker than Maine and Nebraska when all three vote proportionally. It is possible that an error exists in the algorithms evaluation of proportional votes, but due to time constraints that will not be explored.

# 6 Conclusion

This paper has laid out the Banzhaf power index and a computer algorithm to evaluate it for the United States Electoral College. Due to the computationally time-intensive nature of traditional methods to find Banzhaf power, an alternative is used. The alternative takes advantage of generating functions to efficiently determine the number of ”swing” votes a state can cause using binomial multiplication. Some of the problems encountered while building the algorithm were discussed, as well as some problems that may still exist in the calculation of power using proportional representation.

The United States has such a large voting base that each individual voter is likely to be drowned out. It’s fair to think that an individual voter has very little power. The data seems to support that idea. But that does not mean voting itself is futile. However, there have been numerous cases that show how every vote can matter in an election. According to the National Public Radio (NPR) at least 23 elections in the last 20 years have been decided by a single vote.[7] The math may support the idea that an individual’s ballot is a drop in the bucket, but with so many elections throughout the country and traditionally low voter turnout, anything can happen.

The code for the algorithm can be found on my GitHub at `https://github.com/MaxDeSantis/state_voting_power`

# References

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