北京航空航天大學

BEIHANG UNIVERSITY

数值分析第三次大作业

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1. 设计方案

题目描述

本次作业出自《数值分析（第四版）》（颜庆津 编著）计算实习说明书第八题：

关于x, y, t, u, v, w的方程组（A.3）

$$\begin{cases}0.5\cos t+u+v+w-x=2.67\\ t+0.5\sin u+v+w-y=1.07\\ 0.5t+u+\cos v+w-x=3.74\\ t+0.5u+v+\sin w-y=0.79\end{cases}\tag{A.3}$$

以及关于z, t, u的二维数表（见表A-1）确定了一个二元函数$z=f(x,y)$。

t u z	**0**	**0.4**	**0.8**	**1.2**	**1.6**	**2**
0	-0.5	-0.34	0.14	0.94	2.06	3.5
0.2	-0.42	-0.5	-0.26	0.3	1.18	2.38
0.4	-0.18	-0.5	-0.5	-0.18	0.46	1.42
0.6	0.22	-0.34	-0.58	-0.5	-0.1	0.62
0.8	0.78	-0.02	-0.5	-0..66	-0.5	-0.02
1.0	1.5	0.46	-0.26	-0.66	-0.74	-0.5

1. 试用数值方法求出$f(x,y)$在区域$D=\{(x,y)|0\leq x\leq 0.8,\ 0.5\leq y\leq 1.5\}$上的近似表达式

   $$p(x,y)=\sum_{r=0}^{k}\sum_{s=0}^{k}c_{rs}x^r y^s$$

   要求$p(x,y)$以最小的k值达到以下的精度

   $$\sigma=\sum_{i=0}^{10}\sum_{j=0}^{20}\left[f\left(x_i,y_j\right)-p\left(x_i,y_j\right)\right]^2\leqslant 10^{-7}$$

   其中$x_i=0.08i, y_j=0.5+0.05j$。
2. 计算$f\left(x_i^*,y_j^*\right), p\left(x_i^*,y_j^*\right)\quad(i=1,2,\cdots,8; j=1,2,\cdots,5)$的值，以观察$p(x,y)$逼近$f(x,y)$的效果，其中$x_i^*=0.1i,\ y_j^*=0.5+0.2j$。

说明

1. 用迭代方法求解非线性方程组时，要求近似解向量$x^{(k)}$满足以下的精度：

   $$\frac{\left\|x^{(k)}-x^{(k-1)}\right\|_\infty}{\left\|x^{(k)}\right\|_\infty}\leqslant 10^{-12}$$
2. 做二元插值时，要求使用分片二次代数插值。
3. 要由程序自动确定最小的k值。
4. 打印以下内容：

(1) 全部源程序；

(2) 数表：$(x_i, y_j, f(x_i, y_j))\ (i=0,1,...,10; j=0,1,...,20)$；

(3) 选择过程的k和σ值；

(4) 达到精度要求时的k和σ值以及$p(x,y)$中的系数$c_{rs}(r=0,1,\ldots,k;\ s=0,1,\ldots,k)$；

(5) 数表：$\left(x_i^*, y_j^*, f\left(x_i^*, y_j^*\right), p\left(x_i^*, y_j^*\right)\right) \quad (i=1,2,\cdots,8;\ j=1,2,\cdots,5)$。

5. 采用f型输出x_i, y_j, x_i^*, y_j^*的准确值，其余实型数采用e型输出并且至少显示12位有效数字。

算法分析

总体思路

首先题目中并没有给出$f(x,y)$的具体形式。由于x, y确定为x_i, y_j时，通过解四元非线性方程组，得到t, u也确定，因此我们转而需要求出$z=f(x,y)=\varphi(t,u)$，题目中也没有给出φ，而是给出了一个z, t, u的数表，按照说明，我们需要用分片二次代数插值来确定φ，进而确定$f(x,y)$，求出$f(x_i, y_j)$。

求出所有$f(x_i, y_j)$之后，我们需要进行近似拟合，这里我们采用最小二乘曲面拟合方法，详见书P141-P143页。另外注意我们选取幂函数为基函数，这样可以直接求出c_{rs}。

通过从小到大枚举多项式次数k，误差逐渐减小，可以找到最小的满足要求的k。

牛顿法求解固定x, y时的t, u

固定x, y，则方程组（A.3）变成了四元四次非线性方程组，可以用牛顿迭代法求解。首先将项移动到方程左侧，接着构造$F(t,u,v,w)$，这是一个方程向量，每一元素是一个非线性函数，包含关于t, u, v, w的非线性项和常数项。接着求出偏导数矩阵$F'(t,u,v,w)$，这是一个4×4的矩阵，每个元素是一个非线性函数。

迭代公式为：

$$\begin{pmatrix} t^{(k+1)} \\ u^{(k+1)} \\ v^{(k+1)} \\ w^{(k+1)} \end{pmatrix} = \begin{pmatrix} t^{(k)} \\ u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{pmatrix} + \begin{pmatrix} \Delta t^{(k)} \\ \Delta u^{(k)} \\ \Delta v^{(k)} \\ \Delta w^{(k)} \end{pmatrix}, k=0,1,2,\cdots,n$$

其中$(\Delta t^{(k)}, \Delta u^{(k)}, \Delta v^{(k)}, \Delta w^{(k)})^T$为以下方程组的解

$$F'\left(t^{(k)}, u^{(k)}, v^{(k)}, w^{(k)}\right) \begin{pmatrix} \Delta t^{(k)} \\ \Delta u^{(k)} \\ \Delta v^{(k)} \\ \Delta w^{(k)} \end{pmatrix} = -F\left(t^{(k)}, u^{(k)}, v^{(k)}, w^{(k)}\right)$$

这里需要实现线性方程组的求解，可以使用列主元高斯消去法。

由题目要求，取相对范数误差小于10^{-12}为终止条件。

因为从表中观察到t, u取值基本在$[0,2]$上，因此，选取$(1,1,1,1)$为迭代初始值，能比较有效地减少迭代次数，之后的观察也确实如此。

分片二元双二次插值求解$\varphi(t,u)$

这里我们对原数表，应用分片二元双二次插值法。令

$$t_m = 0.2m, u_n = 0.4n\ (m=0,1,\ldots.5; n=0,1,\ldots.5)$$

对于给定的t, u，找到最接近的三个插值点t_{i-1}, t_i, t_{i+1}和u_{j-1}, u_j, u_{j+1}，即

$$i=\begin{cases}\lfloor \frac{t}{0.2} \rceil \ (0.2 \le t \le 0.8) \\ 1 \ (t < 0.2) \\ 4 \ (t > 0.8)\end{cases}$$

$$j=\begin{cases}\lfloor \frac{u}{0.4} \rceil \ (0.4 \le u \le 1.6) \\ 1 \ (u < 0.4) \\ 4 \ (u > 1.6)\end{cases}$$

相应的插值多项式为

$$\varphi(t,u)=\sum_{k=i-1}^{i+1}\sum_{r=j-1}^{j+1} l_k(t) l_r(u) \varphi(t_k,u_r)$$

其中

$$l_k(t)=\prod_{q=i-1,q\neq k}^{i+1}\frac{t-t_q}{t_k-t_q} \ (k=i-1,i,i+1)$$

$$l_r(u)=\prod_{q=j-1,q\neq k}^{j+1}\frac{u-u_q}{u_k-u_q} \ (k=j-1,j,j+1)$$

得到$\varphi(t,u)$的表达式之后，代入即可得到值。实际操作时一般是根据t,u的值进行现场插值。此时即可求出

乘积型最小二乘曲面拟合

使用幂函数作为基矩阵，有基矩阵如下：

$$B=\begin{pmatrix} x_0^0 & \cdots & x_0^k \\ \vdots & \ddots & \vdots \\ x_i^0 & \cdots & x_i^k \end{pmatrix}, \quad G=\begin{pmatrix} y_0^0 & \cdots & y_0^k \\ \vdots & \ddots & \vdots \\ y_j^0 & \cdots & y_j^k \end{pmatrix}$$

数表矩阵为：

$$U=\begin{pmatrix} f\left(x_0,y_0\right) & \cdots & f\left(x_0,y_j\right) \\ \vdots & \ddots & \vdots \\ f\left(x_i,y_0\right) & \cdots & f\left(x_i,y_j\right) \end{pmatrix}$$

根据书P142，系数矩阵表达式为：

$$C=\left(B^TB\right)^{-1}B^TUG\left(G^TG\right)^{-1}$$

于是我们首先求解方程：

$$(B^TB)D=B^TUG$$

得到矩阵D后求解

$$(G^TG)^TC^T=D^T$$

即可得到C，这里可以继续使用之前的列主元高斯消去法，只是常量从向量变成了矩阵。

求解给定点的f值与p值

$f(x_i^*, y_j^*)$的计算和$f(x_i, y_j)$相同。将(x_i^*, y_j^*)代入原方程组，求解相应(t_{ij}^*, u_{ij}^*)，进行分片双二次插值求得$f(x_i^*, y_j^*)$。$p(x_i^*, y_j^*)$只需代入上一步求得的二元多项式即可。

2. 源程序代码

简介

本程序采用C++17标准编写，模块清晰，通用性强，效率优秀，使用CMake（version ≥ 3.16）组织代码，使用MSVC 8.1或GCC 9.3.0编译器均可编译通过。文件结构如下：

```
.
|  CMakeLists.txt
|  InterpolationUtil.cpp
|  InterpolationUtil.h
|  LinearEqUtil.cpp
|  LinearEqUtil.h
|  main.cpp
|  Matrix.cpp
|  Matrix.h
|  NonLinearEqUtil.cpp
|  NonLinearEqUtil.h
|  NonLinFormula.cpp
|  NonLinFormula.h
|  NonLinItemMatrix.cpp
|  NonLinItemMatrix.h
|  NonSqMatrix.cpp
|  NonSqMatrix.h
|  Vector.cpp
|  Vector.h
|  ZeroRangeGuard.h
```

对于向量和矩阵的实现，分别在`Vector`和`Matrix`类中。另外因为曲面拟合的需要，实现了`Matrix`的友元类`NonSqMatrix`，用于支持非方阵的矩阵运算，并且能在长宽相等时高效转换为`Matrix`。

对于线性方程组的求解，实现在类`LinearEqUtil`中，在以往的基础上将单个常向量扩展为多个，这样就可以一次性求解系数矩阵相同的多个线性方程组，也即处理方程右侧为矩阵的情况。

我们实现了`NonLinFormula`和`NonLinItemMatrix`类，前者为代数式运算提供方便。后者主要用于牛顿迭代法，储存F和F'。这两个类避免了对算式的硬编码，提高了程序的通用能力。

对于牛顿迭代法，被简洁地实现在了`NonLinearEqUtil::solveByNewtonMethod`中。

对于分片二元双二次插值算法求给定点的值，实现在了`InterpolationUtil::twoDimQuadLagrangeInterpolation`函数中。其步骤均以向量化的形式描述。

对于乘积型最小二乘曲面拟合算法，因为步骤简单，实现在了`main.cpp`中。在一个循环中不断尝试增长次数k，直到达到误差要求为止，最后输出拟合的系数。

内容

3. 上机计算结果

以某次运行为例，得到结果：

数表$(x_i,y_i,f(x_i,y_i))$

```
 .00E+00 .500E+00        .446504018481E+00
 .00E+00 .550E+00        .324683262928E+00
 .00E+00 .600E+00        .210159686683E+00
 .00E+00 .650E+00        .103043608316E+00
 .00E+00 .700E+00        .340189556268E-02
 .00E+00 .750E+00       -.887358136380E-01
 .00E+00 .800E+00       -.173371632750E+00
 .00E+00 .850E+00       -.250534611467E+00
 .00E+00 .900E+00       -.320276506388E+00
 .00E+00 .950E+00       -.382668066110E+00
 .00E+00 .100E+01       -.437795766738E+00
 .00E+00 .105E+01       -.485758941444E+00
 .00E+00 .110E+01       -.526667254884E+00
 .00E+00 .115E+01       -.560638479797E+00
 .00E+00 .120E+01       -.587796538768E+00
 .00E+00 .125E+01       -.608269779090E+00
 .00E+00 .130E+01       -.622189452876E+00
 .00E+00 .135E+01       -.629688378186E+00
 .00E+00 .140E+01       -.630899760003E+00
 .00E+00 .145E+01       -.625956152545E+00
 .00E+00 .150E+01       -.614988546609E+00
 .80E-01 .500E+00        .638015226511E+00
 .80E-01 .550E+00        .506611755147E+00
 .80E-01 .600E+00        .382176369277E+00
 .80E-01 .650E+00        .264863491154E+00
 .80E-01 .700E+00        .154780200285E+00
 .80E-01 .750E+00        .519926834909E-01
 .80E-01 .800E+00       -.434680402049E-01
 .80E-01 .850E+00       -.131601056789E+00
 .80E-01 .900E+00       -.212431088309E+00
 .80E-01 .950E+00       -.286004551058E+00
 .80E-01 .100E+01       -.352386078979E+00
 .80E-01 .105E+01       -.411655456522E+00
 .80E-01 .110E+01       -.463904911519E+00
 .80E-01 .115E+01       -.509236724701E+00
 .80E-01 .120E+01       -.547761117962E+00
 .80E-01 .125E+01       -.579594388339E+00
 .80E-01 .130E+01       -.604857258890E+00
 .80E-01 .135E+01       -.623673421332E+00
 .80E-01 .140E+01       -.636168248413E+00
 .80E-01 .145E+01       -.642467656690E+00
 .80E-01 .150E+01       -.642697102700E+00
 .16E+00 .500E+00        .840081395767E+00
 .16E+00 .550E+00        .699764165673E+00
 .16E+00 .600E+00        .566061442352E+00
 .16E+00 .650E+00        .439171608118E+00
 .16E+00 .700E+00        .319242138041E+00
 .16E+00 .750E+00        .206376192387E+00
 .16E+00 .800E+00        .100638523891E+00
 .16E+00 .850E+00        .206074006784E-02
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```
 .16E+00 .900E+00        -.893540247670E-01
 .16E+00 .950E+00        -.173626968865E+00
 .16E+00 .100E+01        -.250799956160E+00
 .16E+00 .105E+01        -.320932269445E+00
 .16E+00 .110E+01        -.384097735005E+00
 .16E+00 .115E+01        -.440382175418E+00
 .16E+00 .120E+01        -.489881152313E+00
 .16E+00 .125E+01        -.532697965534E+00
 .16E+00 .130E+01        -.568941879292E+00
 .16E+00 .135E+01        -.598726549515E+00
 .16E+00 .140E+01        -.622168629750E+00
 .16E+00 .145E+01        -.639386535697E+00
 .16E+00 .150E+01        -.650499350788E+00
 .24E+00 .500E+00         .105151509180E+01
 .24E+00 .550E+00         .902927430831E+00
 .24E+00 .600E+00         .760580266860E+00
 .24E+00 .650E+00         .624715198146E+00
 .24E+00 .700E+00         .495519756001E+00
 .24E+00 .750E+00         .373134042775E+00
 .24E+00 .800E+00         .257656748872E+00
 .24E+00 .850E+00         .149150559410E+00
 .24E+00 .900E+00         .476469867734E-01
 .24E+00 .950E+00        -.468493232015E-01
 .24E+00 .100E+01        -.134356760385E+00
 .24E+00 .105E+01        -.214913344927E+00
 .24E+00 .110E+01        -.288573700635E+00
 .24E+00 .115E+01        -.355406364786E+00
 .24E+00 .120E+01        -.415491396489E+00
 .24E+00 .125E+01        -.468918249969E+00
 .24E+00 .130E+01        -.515783883125E+00
 .24E+00 .135E+01        -.556191075200E+00
 .24E+00 .140E+01        -.590246930563E+00
 .24E+00 .145E+01        -.618061548241E+00
 .24E+00 .150E+01        -.639746839258E+00
 .32E+00 .500E+00         .127124675148E+01
 .32E+00 .550E+00         .111500201815E+01
 .32E+00 .600E+00         .964607727216E+00
 .32E+00 .650E+00         .820347369475E+00
 .32E+00 .700E+00         .682447678179E+00
 .32E+00 .750E+00         .551085208597E+00
 .32E+00 .800E+00         .426392385902E+00
 .32E+00 .850E+00         .308462995633E+00
 .32E+00 .900E+00         .197357129692E+00
 .32E+00 .950E+00         .931056208594E-01
 .32E+00 .100E+01        -.428599223404E-02
 .32E+00 .105E+01        -.948339252969E-01
 .32E+00 .110E+01        -.178572990364E+00
 .32E+00 .115E+01        -.255553779055E+00
 .32E+00 .120E+01        -.325840150158E+00
 .32E+00 .125E+01        -.389506988363E+00
 .32E+00 .130E+01        -.446638204599E+00
 .32E+00 .135E+01        -.497324951768E+00
 .32E+00 .140E+01        -.541664032699E+00
 .32E+00 .145E+01        -.579756479795E+00
 .32E+00 .150E+01        -.611706288148E+00
 .40E+00 .500E+00         .149832105248E+01
 .40E+00 .550E+00         .133499863207E+01
 .40E+00 .600E+00         .117712512374E+01
```

```
 .40E+00 .650E+00          .102502405502E+01
 .40E+00 .700E+00          .878960023174E+00
 .40E+00 .750E+00          .739145108704E+00
 .40E+00 .800E+00          .605744871487E+00
 .40E+00 .850E+00          .478883861067E+00
 .40E+00 .900E+00          .358650625882E+00
 .40E+00 .950E+00          .245102236196E+00
 .40E+00 .100E+01          .138268350928E+00
 .40E+00 .105E+01          .381548654070E-01
 .40E+00 .110E+01         -.552528211681E-01
 .40E+00 .115E+01         -.141986880814E+00
 .40E+00 .120E+01         -.222094439096E+00
 .40E+00 .125E+01         -.295635232460E+00
 .40E+00 .130E+01         -.362679511503E+00
 .40E+00 .135E+01         -.423306164224E+00
 .40E+00 .140E+01         -.477601036132E+00
 .40E+00 .145E+01         -.525655426667E+00
 .40E+00 .150E+01         -.567564743655E+00
 .48E+00 .500E+00          .173189274038E+01
 .48E+00 .550E+00          .156203457721E+01
 .48E+00 .600E+00          .139721691821E+01
 .48E+00 .650E+00          .123780100674E+01
 .48E+00 .700E+00          .108408753268E+01
 .48E+00 .750E+00          .936322772315E+00
 .48E+00 .800E+00          .794704449054E+00
 .48E+00 .850E+00          .659387198028E+00
 .48E+00 .900E+00          .530487586840E+00
 .48E+00 .950E+00          .408088685454E+00
 .48E+00 .100E+01          .292244201230E+00
 .48E+00 .105E+01          .182982206854E+00
 .48E+00 .110E+01          .803084940354E-01
 .48E+00 .115E+01         -.157904130516E-01
 .48E+00 .120E+01         -.105344551621E+00
 .48E+00 .125E+01         -.188398090610E+00
 .48E+00 .130E+01         -.265007149319E+00
 .48E+00 .135E+01         -.335237838904E+00
 .48E+00 .140E+01         -.399164503887E+00
 .48E+00 .145E+01         -.456868143302E+00
 .48E+00 .150E+01         -.508434993278E+00
 .56E+00 .500E+00          .197122178640E+01
 .56E+00 .550E+00          .179532959950E+01
 .56E+00 .600E+00          .162406711323E+01
 .56E+00 .650E+00          .145783058271E+01
 .56E+00 .700E+00          .129695464975E+01
 .56E+00 .750E+00          .114171810545E+01
 .56E+00 .800E+00          .992349533324E+00
 .56E+00 .850E+00          .849032663329E+00
 .56E+00 .900E+00          .711911352264E+00
 .56E+00 .950E+00          .581094158922E+00
 .56E+00 .100E+01          .456658513233E+00
 .56E+00 .105E+01          .338654496139E+00
 .56E+00 .110E+01          .227108255770E+00
 .56E+00 .115E+01          .122025089193E+00
 .56E+00 .120E+01          .233922196376E-01
 .56E+00 .125E+01         -.688187019710E-01
 .56E+00 .130E+01         -.154649344213E+00
 .56E+00 .135E+01         -.234152666459E+00
 .56E+00 .140E+01         -.307391091913E+00
```

```
 .56E+00 .145E+01        -.374434862348E+00
 .56E+00 .150E+01        -.435360556536E+00
 .64E+00 .500E+00         .221566786369E+01
 .64E+00 .550E+00         .203420113361E+01
 .64E+00 .600E+00         .185695514362E+01
 .64E+00 .650E+00         .168435816416E+01
 .64E+00 .700E+00         .151677635240E+01
 .64E+00 .750E+00         .135451904115E+01
 .64E+00 .800E+00         .119784408667E+01
 .64E+00 .850E+00         .104696304942E+01
 .64E+00 .900E+00         .902046083802E+00
 .64E+00 .950E+00         .763226477663E+00
 .64E+00 .100E+01         .630604821954E+00
 .64E+00 .105E+01         .504252814597E+00
 .64E+00 .110E+01         .384216715546E+00
 .64E+00 .115E+01         .270520476641E+00
 .64E+00 .120E+01         .163168572400E+00
 .64E+00 .125E+01         .621485581168E-01
 .64E+00 .130E+01        -.325666193968E-01
 .64E+00 .135E+01        -.121016534844E+00
 .64E+00 .140E+01        -.203251399623E+00
 .64E+00 .145E+01        -.279330359558E+00
 .64E+00 .150E+01        -.349319957540E+00
 .72E+00 .500E+00         .246468422266E+01
 .72E+00 .550E+00         .227805897940E+01
 .72E+00 .600E+00         .209525125084E+01
 .72E+00 .650E+00         .191671812800E+01
 .72E+00 .700E+00         .174285462878E+01
 .72E+00 .750E+00         .157399842733E+01
 .72E+00 .800E+00         .141043483523E+01
 .72E+00 .850E+00         .125240175061E+01
 .72E+00 .900E+00         .110009440963E+01
 .72E+00 .950E+00         .953669851261E+00
 .72E+00 .100E+01         .813251055249E+00
 .72E+00 .105E+01         .678930742966E+00
 .72E+00 .110E+01         .550774848504E+00
 .72E+00 .115E+01         .428825676973E+00
 .72E+00 .120E+01         .313104771740E+00
 .72E+00 .125E+01         .203615514033E+00
 .72E+00 .130E+01         .100345478241E+00
 .72E+00 .135E+01         .326856518657E-02
 .72E+00 .140E+01        -.876530659133E-01
 .72E+00 .145E+01        -.172467247819E+00
 .72E+00 .150E+01        -.251230220752E+00
 .80E+00 .500E+00         .271781110947E+01
 .80E+00 .550E+00         .252639950126E+01
 .80E+00 .600E+00         .233841138686E+01
 .80E+00 .650E+00         .215432937728E+01
 .80E+00 .700E+00         .197457455665E+01
 .80E+00 .750E+00         .179951057910E+01
 .80E+00 .800E+00         .162944822055E+01
 .80E+00 .850E+00         .146465004375E+01
 .80E+00 .900E+00         .130533496765E+01
 .80E+00 .950E+00         .115168262131E+01
 .80E+00 .100E+01         .100383741991E+01
 .80E+00 .105E+01         .861912337228E+00
 .80E+00 .110E+01         .725992371111E+00
 .80E+00 .115E+01         .596137711520E+00
```

```
.80E+00 .120E+01        .472386627914E+00
.80E+00 .125E+01        .354758095898E+00
.80E+00 .130E+01        .243254184181E+00
.80E+00 .135E+01        .137862222525E+00
.80E+00 .140E+01        .385567703264E-01
.80E+00 .145E+01       -.546985959345E-01
.80E+00 .150E+01       -.141949659709E+00
```

k值和δ

```
k = 1 delta = .322090897364E+01
k = 2 delta = .465996003327E-02
k = 3 delta = .172117537914E-03
k = 4 delta = .330953429925E-05
k = 5 delta = .254137771997E-07
```

当$k=5$时，精度已经达到了要求，此时$p(x,y)$中的系数c_{rs}为

```
.202123044262E+01       -.366842636522E+01       .709247679136E+00
.848606392858E+00       -.415897940397E+00

.319191902005E+01       -.741167262380E+00      -.269700015696E+01
.163105207331E+01       -.484652918264E+00

.256783376068E+00        .158052371105E+01      -.464732264229E+00
-.799465014205E-01       .101380257369E+00

-.268881423675E+00      -.732402030864E+00       .108086081306E+01
-.811992184212E+00       .305416495777E+00

.216915251451E+00       -.175276128774E+00      -.791840695272E-01
.250488511538E+00       -.144991789392E+00
```

数表$\left(x_i^*, y_j^*, f\left(x_i^*, y_j^*\right), p\left(x_i^*, y_j^*\right)\right)$

```
.10E+00 .700E+00        .194720407918E+00        .194730357326E+00
.10E+00 .900E+00       -.183036970312E+00       -.183041838360E+00
.10E+00 .110E+01       -.445497629269E+00       -.445500042115E+00
.10E+00 .130E+01       -.597566705305E+00       -.597558856878E+00
.10E+00 .150E+01       -.646459593647E+00       -.646446111301E+00
.20E+00 .700E+00        .405979189288E+00        .405989539896E+00
.20E+00 .900E+00       -.225158150340E-01       -.225211162936E-01
.20E+00 .110E+01       -.338220789483E+00       -.338224022407E+00
.20E+00 .130E+01       -.544437827296E+00       -.544430450939E+00
.20E+00 .150E+01       -.647361337978E+00       -.647348010632E+00
.30E+00 .700E+00        .634777195151E+00        .634787453107E+00
.30E+00 .900E+00        .158801342082E+00        .158796295645E+00
.30E+00 .110E+01       -.207365658661E+00       -.207368580088E+00
.30E+00 .130E+01       -.465357900500E+00       -.465349923350E+00
.30E+00 .150E+01       -.620270952047E+00       -.620257138501E+00
.40E+00 .700E+00        .878960023174E+00        .878969865343E+00
.40E+00 .900E+00        .358650822218E+00        .358646043348E+00
.40E+00 .110E+01       -.552527776174E-01       -.552554368705E-01
.40E+00 .130E+01       -.362679502890E+00       -.362671062971E+00
.40E+00 .150E+01       -.567564742128E+00       -.567550582812E+00
.50E+00 .700E+00        .113661091016E+01        .113662035311E+01
```

```
 .50E+00 .900E+00        .574980553226E+00       .574975843087E+00
 .50E+00 .110E+01        .115992426843E+00       .115989321215E+00
 .50E+00 .130E+01        -.238568293365E+00      -.238560419182E+00
 .50E+00 .150E+01        -.491434391617E+00      -.491420900990E+00
 .60E+00 .700E+00        .140604179891E+01       .140605068696E+01
 .60E+00 .900E+00        .805941716346E+00       .805937302046E+00
 .60E+00 .110E+01        .304429275827E+00       .304425831955E+00
 .60E+00 .130E+01        -.950161177620E-01      -.950089457219E-01
 .60E+00 .150E+01        -.393902305233E+00      -.393889837754E+00
 .70E+00 .700E+00        .168578351531E+01       .168579121747E+01
 .70E+00 .900E+00        .104988138137E+01       .104987773885E+01
 .70E+00 .110E+01        .508293841773E+00       .508291045085E+00
 .70E+00 .130E+01        .661488102960E-01       .661563554850E-01
 .70E+00 .150E+01        -.276834338871E+00      -.276822042898E+00
 .80E+00 .700E+00        .197457455665E+01       .197458126116E+01
 .80E+00 .900E+00        .130533519966E+01       .130533200412E+01
 .80E+00 .110E+01        .725992430530E+00       .725989310430E+00
 .80E+00 .130E+01        .243254198540E+00       .243260790494E+00
 .80E+00 .150E+01        -.141949656522E+00      -.141938789062E+00
```

经过与他人的对比验证，可知结果正确。

此外，在Release模式下，无输出时进行时间测量，基本上在2000~3000微秒之间。

4. 讨论分析

在本次程序设计中，主要面临的困难和思考有如下方面：

1. 步骤较多，需要仔细整理出合理的逻辑。
2. 对于牛顿迭代法的初值一开始没有很好的把握，一开始采用正太随机法生成初始向量，发现虽然能迭代收敛，但需要步数较多，有时可能要500步尚未迭代完毕。在参考了他人的思路之后，意识到题目中给出了t, u的大致范围，因此，可以为向量初始值设定为$(1, 1, 1, 1)$，这样迭代次数便稳定下来，基本6次即可达到所需精度。
3. 如何在程序的高效性，可复用性，简洁性做平衡，是很困难的。通过反复的设计和抽象，使得我的代码水平有了进一步提升。