Semester Master Project

# $F$-Thompson's Group

# amenability and idempotent means on free magmas

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June 5, 2020

# Introduction

Abstract:
Our goal is to study J. T. Moore's ideas and construction in his demonstration attempt at answering the question "Is the $F$-Thompson's group amenable or not ?"
This paper is a gateway into this 40 years old question and doesn't require any prior knowledge of amenability theory.

In this paper, we will consider the use of paradoxical decompositions onto trees in order to solve sustainably mankind future energy crisis.[1]
Mathematically speaking, we will work on the $F$-Thompson's group. Defined in 1965 by Richard Thompson, this group is a rather singular one. It is non-abelian, finitely generated yet infinite and doesn't contain any non-abelian free group. It as been conjectured in 1979 by Geoghegan that $F$ is not amenable, but interestingly, if the question is still open after 40 years, there is no consensus in the mathematical community.
In either case, proving that $F$ is amenable or not would grant us an example of finitely generated group amenable (or not) that doesn't contain the free group on 2 elements.

To jump into this problem, we will use as a guideline the demonstration attempt that $F$ is amenable by Justin Tatch Moore, professor at Cornell University.
This work is divided into three parts, building on each others:

- **Magmas** is a part principally centred on giving the fundamental family of tree we will study a proper definition, because they are the central object of this work.
- **$F$-Thompson's Group** is where we will properly defined $F$ and prove many of its very interesting properties.
- **Amenability on $F$-Thompson's Group** is the fun part. It will study very fundamental notions of amenability (means) and idempotence on then.

Have a nice reading.

[1] No, I take no shame in this blatant green-washing

# Contents

# Chapter 1

# Magmas

## 1.1 Definitions

**Definition 1** (Magmas).
*Let $M$ be any set and let $* : M \times M \to M$ be any function.*
*We call the couple $(M, *)$ a **magma** or a **binary system**.*
*Furthermore, we call $*$ the **composition law of the magma**.*

**Remark 2.** A magma $(M, *)$ can be seen as a group, whose composition law doesn't follow any of the group axioms.

**Definition 3** (Magma homomorphisms).
*Let $(M, *)$ and $(N, \star)$ be any two magmas. Let also $f : M \to N$ be any function.*
*We call $f$ a **magma homomorphism** if $\forall a, b \in M$ :*

$$f(a * b) = f(a) \star f(b)$$

*Furthermore, if $f$ is bijective such that the inverse function $f^{-1} : N \to M$ is also a magma homomorphism. we call $f$ a **magma isomorphism**.*

**Definition 4** (Homomorphism sets)**.**
*Let $(M,*)$ and $(N,\star)$ be any two magmas. We define the set of magma homomorphism*

$$Hom(M,N) = \{f : M \to N \mid f \text{ is a magma homomorphism}\}$$

**Proposition 5.**
*Let $(M,*)$, $(N,\star)$ and $(P,\wedge)$ be any 3 magmas. Let also $f \in Hom(M,N)$, $g \in Hom(N,P)$. Then, we have the following properties :*

- $\forall (M,*)$, $id_M \in Hom(M,M)$
- $g \circ f \in Hom(M,P)$

*Proof of Proposition* 5:

- $\forall a,b \in M$, $id(a*b) = a*b = id(a)*id(b)$
- $\forall a,b \in M$, $g \circ f(a*b) = g\big(f(a) \star f(b)\big) = g\big(f(a)\big) \wedge g\big(f(b)\big) = g \circ f(a) \wedge g \circ f(b)$

□ 5

**Remark 6.** Using the previous definition, we get that the magmas form a category, $\underline{\mathbf{Mag}}$

$$Ob(\underline{\mathbf{Mag}}) = \{(M,*) \text{ a magma}\}$$

$$\underline{\mathbf{Mag}}\Big((M,*),(N,\star)\Big) = \ \mathrm{Hom}(M,N)$$

## 1.2 Free magmas

**Definition 7** (Free magmas).
*Let $X$ be any set. We define $(M_X, *)$ the **free magma** on the set $X$ inductively as follows:*

$$X_1 = X, \;\; X_n = \bigsqcup_{p+q=n} X_p \times X_q$$

$$M_X = \bigsqcup_{n \in \mathbb{N}^*} X_n$$

*Consider $a \in X_p, b \in X_q$ the composition law on $M_X$ is given by*

$$* : M_X \times M_X \to M_X$$

$$a * b = (a, b) \in X_{p+q}$$

*We also define the standard injection $i_X : X \hookrightarrow M_X$, $i_X(x) = x \in X_1$*

**Definition 8** (width of an element).
*We define the **width** of an element in $(M_X, *)$ as follow*

$$w : M_X \to \mathbb{N}^*$$

$$w(x) = p \text{ such that } x \in X_p$$

*Furthermore, using the definition of $M_X$, we have that $w \in Hom(M_X, \mathbb{N}^*)$.*

**Proposition 9** (Universal freedom property).
*Let $X$ be any set and $(M_X, *)$ the free magma on the set $X$ with $i_X$ the standard injection. Then $(M_X, *)$ follows the universal freedom property:*

*$\forall (N, \star)$ a magma and $f : X \to N$, then $\exists! \tilde{f} \in Hom(M_X, N)$ such that $\tilde{f} \circ i_X = f$. (i.e the following diagram commute)*

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & M_X \arrow[dl, dashed, "\exists! \tilde{f}"] \\ N & \end{tikzcd}$$

*Proof of Proposition* 9:
Let's define $\tilde{f}$ recessively :

$$\forall a, b \in M_X : \tilde{f}|_{X_1} = f \quad \text{and} \quad \tilde{f}(a * b) = \tilde{f}(a) \star \tilde{f}(b),$$

Then, due to the structure of $M_X$, this is a magma homomorphism and $\tilde{f} \circ i_X = \tilde{f}|_{X_1} = f$.

Now, let's show uniqueness the of $\tilde{f}$. Consider $g : M_X \to N$, $g \circ i_X = f$, $g \neq \tilde{f}$. So $\exists x \in M_X, g(x) \neq \tilde{f}(x)$. Let's consider $\mathbf{x}$ such that $w(\mathbf{x}) = q$ is minimal for the property $g(x) \neq \tilde{f}(x)$. Then:

- if $q > 1$: Then $\mathbf{x} = a * b$, $a \in X_n$, $b \in X_m$, $m + n = q$. Due to minimality of $q$, $\tilde{f}(a) = g(a)$, $\tilde{f}(b) = g(b)$ and thus $g(\mathbf{x}) = g(a * b) = g(a) \star g(b) = \tilde{f}(a) \star \tilde{f}(b) = \tilde{f}(a * b) = \tilde{f}(\mathbf{x})$. This is a contradiction.
- if $q = 1$: then $\mathbf{x} \in X$ so $g(\mathbf{x}) = g \circ i_X(\mathbf{x}) = f(\mathbf{x}) = \tilde{f} \circ i_X(\mathbf{x}) = \tilde{f}(\mathbf{x})$. This is also a contradiction.

Thus, $\tilde{f} = g$ and we have proved the uniqueness of $\tilde{f}$. Thus, the free magma is indeed free
$\square$ 9

**Corollary 10** (Unicity of free magma)**.**
*Let $(M, \star)$ be a magma, $i : X \hookrightarrow M$ be an injection such that $(M, \star)$ follows the universal freedom property. Then $(M, \star)$ is isomorphic to $(M_X, *)$*

*Proof of Corollary* 10:
Consider the following diagrams:

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "i"'] & M_X \arrow[dl, dashed, bend left, "\exists ! \tilde{i}"] \\
M \arrow[ur, dashed, "\exists ! \tilde{i_X}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "j"'] & N \\
N \arrow[ur, "id_N"] \arrow[ur, dashed, bend right, "\exists ! \tilde{j}"'] &
\end{tikzcd}$$

For the first diagram, $\tilde{i_X} \circ i = i_X$, $\tilde{i} \circ i_X = i$. Thus $\tilde{i_X} \circ \tilde{i}$ is such that $\tilde{i_X} \circ \tilde{i} \circ i_X = i_X$. Then, as it is showed in the second diagram, because $id_{M_X}$ also has this property and due to uniqueness, this imply that $\tilde{i} \circ \tilde{i_X} = id_{M_X}$. Similarly, $\tilde{i} \circ \tilde{i_X} = id_M$.
Thus $\tilde{i_X}$ is an isomorphism between $M_X$ and $M$.

$\square$ 10

## 1.3 Representation of free magmas

### 1.3.1 Ordered Rooted Binary Trees

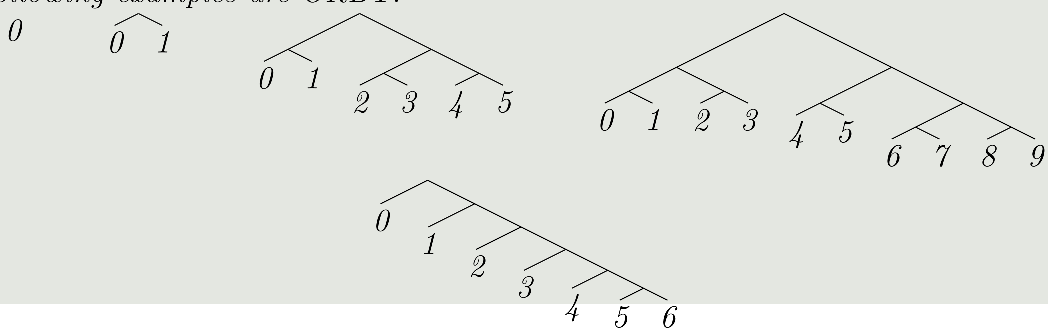
**Definition 11** (Ordered Rooted Binary Trees)**.**
*We define a **Rooted Binary Tree** $T = (V(T), E(T))$ as a finite tree (i.e a connected and acyclic finite graph) such that:*

1. *$T$ is a simple vertex tree (i.e $T = (\{v_0\}, \emptyset)$). OR*
2. (a) *$T$ has a vertex $v_0 \in V(T)$ such that $d(v_0) = 2$. Its edges are called **left edge of** $v_0$, $e^l_{v_0}$ and the **right edge of** $v_0$, $e^r_{v_0}$.*
   (b) *$\forall v \in G(T), v \neq v_0$, $d(v) > 1$, then $d(v) = 3$. We name the edge that is in $p_{v,v_0}$ the unique path between $v$ and $v_0$ the **top edge of** $v$ $e^t_v$. The other two are named **left edge of** $v$ $e^l_v$ and the **right edge of** $v$ $e^r_v$.*
   (c) *$\forall l \in G(T), l \neq v_0$, $d(l) = 1$, we call $l$ a leaf and its only edge is called the **top edge of** $v$ $e^t_v$.*

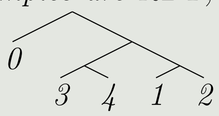
*We say that a leaf $l_1$ is at the left of the leaf $l_2$ (or $l_2$ is at the right of $l_1$) if when considering $p_{v_0,l_1}$ and $p_{v_0,l_2}$, the first edges $e_1 \in p_{v_0,l_1}, e_2 \in p_{v_0,l_2}$ that differ are such that $e_1 = e^l_v$ and $e_2 = e^r_v$.*
*Furthermore, we say that $T$ is an **Ordered Rooted Binary Tree** if there exists a left-to-right ordering $<_{l,r}$ on the leaves of $T$, $Leaf\ (T)$.*

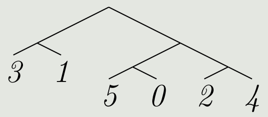
**Remark 12.** If $T$ is a simple vertex tree, we also defined the root $v_0$ as the leaf.

**Example 13.**
*The following examples are ORBT:*

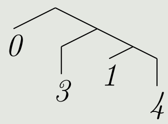


*The following examples are RBT, but not ordered:*

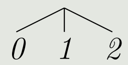



*The following example is not rooted:*




*The following examples are not binary:*


**Proposition 14.**
*Every rooted binary tree $T$ can be ordered.*

*Proof of Proposition* 14:
For all $\mathrm{l} \in \mathrm{Leaf}(T)$, consider $p_{v_0,\mathrm{l}}$ the unique path between $v_0$ and l.
Then $p_{v_0,l} = (e_{v_0}^{a_0}, e_{v_1}^{a_1}, \cdots, e_{v_n}^{a_n})$ with $a_i$ being either $l$ (left) or $r$ (right) and this sequence is unique for l (because $T$ is a tree).
Consider $m = \max_{\mathrm{l}\in\mathrm{Leaf}(T)} \mathrm{length}(p_{v_0,\mathrm{l}})$. We define the following injective function

$$p: \text{ Leaf } (T) \hookrightarrow \{0,1,2\}^m$$

$$p(\mathrm{l}) = (\delta_i)_{i=0}^{m-1}$$

$$\delta_i = \begin{cases} 0 & \text{if } a_i = l \\ 1 & \text{if } a_i = r \\ 2 & \text{if } a_i \text{ is not defined because } p_{v_0,\mathrm{l}} \text{ is shorter} \end{cases}$$

We now equip $\{0,1,2\}^m$ with the lexicographical well ordering $<_{lex}$ and consider $<_{l,r}$ its retraction using $p$ on Leaf$(T)$.
(i.e $p(l_1) <_{lex} p(l_2) \iff l_1 <_{l,r} l_2$).

Then, $<_{l,r}$ is a left-to-right well ordering on Leaf$(T)$. Indeed, if $l_1$ is at the left of $l_2$, then $p(l_1) = (\delta_i^1)_{i=0}^{m-1}$, $p(l_2) = (\delta_i^2)_{i=0}^{m-1}$ such that the (j-1)-th first terms are the same and then $\delta_j^1 = 0, \delta_j^2 = 1$. Thus $p(l_1) <_{lex} p(l_2)$.

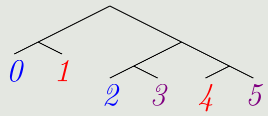
□ 14

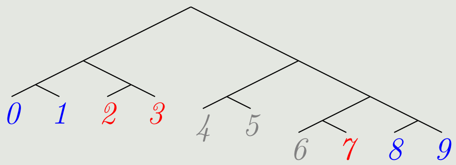
### 1.3.2 Colored Ordered Rooted Binary Trees

**Definition 15** (Colored Ordered Rooted Binary Trees)**.**
*Let $T$ be an ORBT and let $C$ be any color set. A* ***Colored Ordered Rooted Binary Tree*** *is the binome $(T, \rho)$ with*

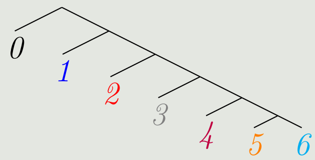
$$\rho : \ \textit{Leaf}\,(T) \to C$$

**Example 16.**
*The following exemples are 3 colors CORBT:*




*The following exemple is a 6 colors CORBT:*




**Remark 17.** We can see every ORBT as a CORBT on 1 color.

**Definition 18** (Useful notions on CORBT)**.**
*We will define a few useful notions on CORBT. Let $(T, \rho)$ be any CORBT on the color set $X$.*

- *The* ***length*** *of a vertex. Consider $v \in V(T)$*

$$l(v) = \ \textit{length of } p_{v_0, v}$$

- *The **height** of $T$*

$$h(T) = \ max\ _{v \in V(t)} l(v)$$

- *The **width** of $T$*

$$w(T) = |Leaf(T)|$$

- *The **right** of $T$*

$$R(T) = \{v \in T |\ p_{v_0,v} \textit{ is constructed only with right edges}\}$$

- *The **left** of $T$*

$$L(T) = \{v \in T |\ p_{v_0,v} \textit{ is constructed only with left edges}\}$$

*We also denote the set of all CORBT on the color set $X$ as $CORBT(X)$ and $ORBT = CORBT(1)$*

- *We define the standard bijection of the color set $X$ in $CORBT(X)$ as*

$$\nu_X : X \cong CORBT(X)$$

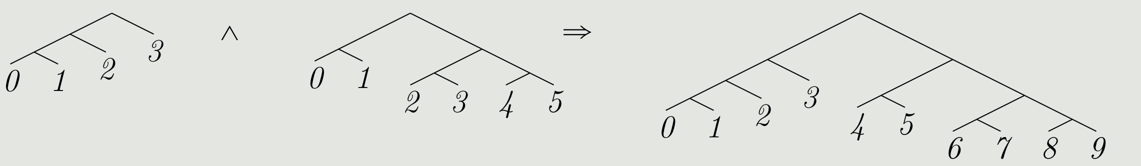
$$\nu_X(x) = \Big(\big(\{v_0\}, \varnothing\big), \rho(v_0) = x\Big)$$

- *We will also define the following binary function on $CORBT(X)$. Consider $(T_1, \rho_1), (T_2, \rho_2) \in CORBT(X)$ with respective roots $v_0^1$ and $v_0^2$.*

$$\wedge : CORBT(X) \times CORBT(X) \to CORBT(X)$$

$$(T_1, \rho_1) \wedge (T_2, \rho_2) = \Bigg(\Big(V(T_1) \cup V(T_2) \cup \{w\}, E(T_1) \cup E(T_2) \cup \big\{(w, v_0^1), (w, v_0^2)\big\}\Big) p_1 \sqcup p_2\Bigg)$$

**Example 19.**



**Proposition 20.**

- $\wedge$ *is well defined*
- $h(T_1 \wedge T_2) = \ max\ \{h(T_1, T_2)\} + 1$
- *Leaf* $(T_1 \wedge T_2) =$ *Leaf* $(T_1) \sqcup$ *Leaf* $(T_2)$
- $L(T) \cap R(T) = \{v_0\}$
- $L(T_1 \wedge T_2) = L(T_1) \cup \{w\},\ R(T_1 \wedge T_2) = R(T_2) \cup \{w\}$

*Proof of Proposition* 20:
We can easily see that $T_1 \wedge T_2$ is still a tree.

- Let's consider the case where $w(T_1), w(T_2) > 1$

  Consider $v \in T_1 \wedge T_2$.

  - If $v \in V(T_1) \cup V(T_2) \backslash \{v_0^1, v_0^2\}$. Then $d(v)$ in $T_1 \wedge T_2$ is equal to $d(v)$ in $T_i$
  - If $v = v_0^i$, then in $T_1 \wedge T_2$, $d(v) = 3$
  - If $v = w$, then d(v)=2 and thus $w$ is the root of $T_1 \wedge T_2$

  From this, we get that Leaf $(T_1 \wedge T_2) = \{v \in V(T_1 \wedge T_2) | d(v) = 1\} = \{v \in V(T_1) | d(v) = 1\} \sqcup \{v \in V(T_2) | d(v) = 1\} =$ Leaf $(T_1) \sqcup$ Leaf $(T_2)$ Thus $\rho_1 \sqcup \rho_2$ is well defined.

- Now, suppose $w(T_1) = 1$. Then $T_1 = \{v_0^1\}, d(v_0^1) = 0$ in $T_1$ and thus $d(v_0^1) = 1$ in $T_1 \wedge T_2$ and thus $v_0^1$ is still a leaf. we conserve the property that Leaf $(T_1 \wedge T_2) =$ Leaf $(T_1) \sqcup$ Leaf $(T_2)$ and thus $\rho_1 \sqcup \rho_2$ is well defined.

Finally, the ordering is given by 14. Thus $\wedge$ is well defined and the other properties are direct consequences of it. □ 20

**Remark 21.** $\wedge$ is not associative, nor commutative and it doesn't have a neutral element. So $(CORBT(X), \wedge)$ is a magma.

**Proposition 22.**
*Let* $(T, \rho) \in CORBT(X)$*,* $w(T) > 1$*. Then* $\exists!\ T_1, T_2 \in CORBT(X)$ *such that* $T_1 \wedge T_2$

*Proof of Proposition* 22:
Let $v_0$ be the root of $T$. We defined $T_l = \{v \in T | p_{v_0,v}$ start with the left edge $\}$ and $T_r = \{v \in T | p_{v_0,v}$ start with the right edge $\}$. Those are still trees.
Indeed, because they are subset of a tree, they are still acyclique.
Furthermore, consider $v_l \in T_l$, $e^l_{v_0} = (v_0, v_l)$ and take $x, y \in T_l$, then $p_{v_0,x}$ and $p_{v_0,y}$ pass by $v_l$. By taking the concatenation of $p_{x,v_0}$ and $p_{v_0,y}$, we get a path in $T_l$. Same for $T_r$
We can also see that $v_l$ (respectively $v_r$) are the new roots of $T_l$, (respectively $T_r$) and those tree are still binary. We use as color of $T_l$ (respectively $T_r$) $\rho_l = \rho|_{T_l}$ ($\rho_r = \rho|_{T_r}$)
Then, it is easy to see that $T_l \wedge T_r = T$

$\square$ 22

**Consequence 23.** Consider $CORBT(X)_p = \{(T, \rho) \in CORBT(X) | w(T) = p\}$, $p > 1$ . Then $\forall n > 1\ p, q \in \mathbb{N}^*$

$$CORBT(X)_n = \bigsqcup_{n=p+q} CORBT(X)_p \wedge CORBT(X)_q$$

**Lemma 24.**
*Let* $X$ *be any set.*

$$(M_X, *) \cong (CORBT(X), \wedge)$$

*Proof of Lemma* 24:
Consider $\tilde{\nu}_X : M_X \to CORBT(X)$ given by the freedom property. We define recursively

$$j_X : CORBT(X) \to M_X$$

$$j_X\Big(\{v_0\}, p(v_0) = x\Big) = x, \quad j_X(T_1 \wedge T_2) = (J_X(T_1), J_X(T_2))$$

Let's show by recursion that $\tilde{\nu}_X \circ j_X = id_{CORBT(X)}$, $j_X \circ \tilde{\nu}_X = id_{M_X}$

- – $\forall x \in X_1$, $j_X \circ \tilde{\nu}_X(x) = j_X\big(\{v_0\}, p(v_0) = x\big) = x$, so $j_X \circ \tilde{\nu}_X|_{X_1} = id_{X_1}$.

- $\forall\big(\{v_0\}, p(v_0) = x\big) \in CORBT(X)_1$ $\tilde{\nu}_X \circ j_X\big(\{v_0\}, p(v_0) = x\big) = \tilde{\nu}_X\big(x\big) = (\{v_0\}, p(v_0) = x)$, so $\tilde{\nu}_X \circ j_X|_{CORBT(X)_1} = id$.

- - $x \in X_n$, then $x = a * b$, $a \in X_p, b \in X_q$, $p + q = n$. Then $j_X \circ \tilde{\nu}_X(x) = j_X \circ \tilde{\nu}_X(a) * j_X \circ \tilde{v_X}(b) = a * b = x$
  - $T \in CORBT(X)_n$, then $T = T_1 \wedge T_2$, $T_1 \in CORBT(X)_p, T_2 \in CORBT(X)_q$, $p + q = n$. Then $\tilde{\nu}_X \circ j_X(T) = \tilde{\nu}_X \circ j_X(T_1) \wedge \tilde{\nu}_X \circ j_X(T_2) = T_1 \wedge T_2 = T$

□ 24

# Chapter 2

# $F$-Thompson Group

This part is inspired from the [1] first, second and third chapters, although the proves have been extended.

## 2.1 Group

**Definition 25** (Dyadic number).
*Let $a \in \mathbb{R}$, we say that $a$ is* ***dyadic*** *if $\exists\ p, k \in \mathbb{Z},\ a = \frac{p}{2^k}$*

*We can easily see that the sum and the product of dyadic numbers are still dyadic numbers.*

$$\frac{a}{2^p} * \frac{b}{2^q} = \frac{ab}{2^{p+q}}$$

$$\frac{a}{2^p} + \frac{b}{2^q} = \frac{2^{m-p}.a + 2^{m-q}.b}{2^m} \quad \text{with } m = \quad max\{p, q\}$$

**Definition 26** ($F$-Thompson set).
*Let $F$ be the set of all function $f : [0, 1] \to [0, 1]$ such that:*

1. *$f$ is an homeomorphism $\big($bijective and continuous (for standard topology)$\big)$.*
2. *$f$ is finitely piecewise linear. $\big($i.e $[0, 1] = \bigcup_{j=1}^{n} I_j$, $I_j$ are closed intervals such that $f|_{I_j}(x) = a_j.x + b_j\big)$*

3. *$f$ is differentiable except on finitely many dyadic numbers and such that on intervals of differentiability, the derivate is equal to $2^k$, $k \in \mathbb{Z}$*

*From this, we can see that $f$ is increasing and $f(0) = 0$, $f(1) = 1$*

**Example 27.**
*The following functions are in $F$*

$$id_{[0,1]} \ ; A(x) = \begin{cases} \frac{x}{2} & 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leqslant x \leqslant 1 \end{cases} \ ; B(x) = \begin{cases} x & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \leqslant x \leqslant \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \leqslant x \leqslant 1 \end{cases}$$

**Proposition 28.**

$$(F, \circ) \text{ is a subgroup group of } (Homeo([0,1]), \circ)$$

*We name $(F, \circ)$ the $F$**-Thompson's group***

*Proof of Proposition* 28:
Consider $f \in F$ and let $0 = x_0 < x_1 < \cdots < x_n = 1$ be the points at which $f$ is not differentiable. Consider $I_i = [x_i, x_{i+1}]$. Then $f|_{I_i}$ is differentiable and thus it muss be linear on $I_i$.

Furthermore, let's show that $\forall i = 0, \cdots, n, f(x_i)$ is a dyadic number.

- $f(x_0) = 0$ a dyadic number.
- Suppose $f(x_n)$ is a dyadic number. $x_n$ is dyadic and knowing that $f|_{[x_n, x_{n+1}]} = 2^k.x + b_n$, we get that $b_n = f(x_n) - 2^k x_n$ a sum and a product of dyadic numbers. Thus, $b_n$ is dyadic.

  Now, knowing that $x_{n+1}$ is dyadic, $f(x_{n+1}) = 2^k.x_{n+1} + b_n$ and is thus also dyadic.

Now, let's show that $(F, \circ)$ is indeed a subgroup.

- $id_{[0,1]} \in F$.

- Let $f \in F$ with respective non-differentiable points $\{x_i\}_{i=0}^n$. Then consider $f^{-1}$ it's inverse function. This is an homeomorphism. Because $(f^{-1}(x))' = \frac{1}{f' \circ f^{-1}(x)}$, it's non-differentiable points are the dyadic point $\{f(x_i)\}_{i=0}^n$. Then

$$f^{-1}|_{[f(x_i),f(x_{i+1})]} = (f|_{[x_i,x_{i+1}]})^{-1} = (2^{k_i}.x + b_i)^{-1} = (\frac{1}{2^{k_i}}.x - \frac{b_i}{2^{k_i}})$$

  So $f^{-1}$ is finitely piecewise linear.

  Thus $f^{-1} \in F$

- Let $f, h \in F$ with respective non-differentiable points $\{x_i\}_{i=0}^n, \{y_j\}_{j=0}^m$. $f \circ h$ is still an homeomorphism. $(f \circ h)'(x) = f'(h(x)).h'(x)$. So $f \circ h(x)$ is not differentiable $\Rightarrow x = y_j$ or $x = h^{-1}(x_i)$, all dyadic numbers. We order $\{y_j\}_{j=0}^m$ and $\{h^{-1}(x_i)\}_{i=0}^n$ from smallest to biggest into $\{z_l\}$.

$$f \circ h|_{[z_l,z_{l+1}]} = (2^{k_i}.(2^{k_j}.x + d_j) + b_i) = (2^{k_i+k_j}.x + (2^{k_i}d_j + b_i))$$

  So $f \circ h$ is still is finitely piecewise linear.

  Thus $f \circ h \in F$

□ 28

**Example 29.**
*We define* $\{X_n\}_{n\in\mathbb{N}} \in F$*:*

$$X_0 = A \ , \ X_n = A^{-(n-1)} \circ B \circ A^{(n-1)}$$

## 2.2 Standard dyadic partitions & $\mathbb{T}$-Trees

**Definition 30** (Standard dyadic intervals)**.**
*We define a* ***standard dyadic interval in*** $[0,1]$ *to be a interval of the form* $[\frac{a}{2^n}, \frac{a+1}{2^n}]$ *with* $a, b \in \mathbb{N}^*$, $a < 2^n - 1$

**Definition 31** (standard dyadic partitions)**.**
*Let* $0 = x_0 < x_1 < \cdots < x_n = 1$ *be a partition* $[0,1]$*. We name* $[x_{i-1}, x_i]$ *the* ***intervals of the partition****.*
*We say that* $0 = x_0 < x_1 < \cdots < x_n = 1$ *is a* ***standard dyadic partition of*** $[0,1]$ *if all intervals of the partition are standard dyadic interval in* $[0,1]$*.*

**Example 32.**
*The following partitions are standard dyadic partitions of* $[0,1]$*:*

$$0 < 1, \quad 0 < \frac{1}{2} < \frac{3}{4} < 1, \quad 0 < \frac{1}{8} < \frac{5}{32} < \frac{3}{16} < \frac{2}{8} < 1$$

*The following partitions are NOT standard dyadic partitions of* $[0,1]$*:*

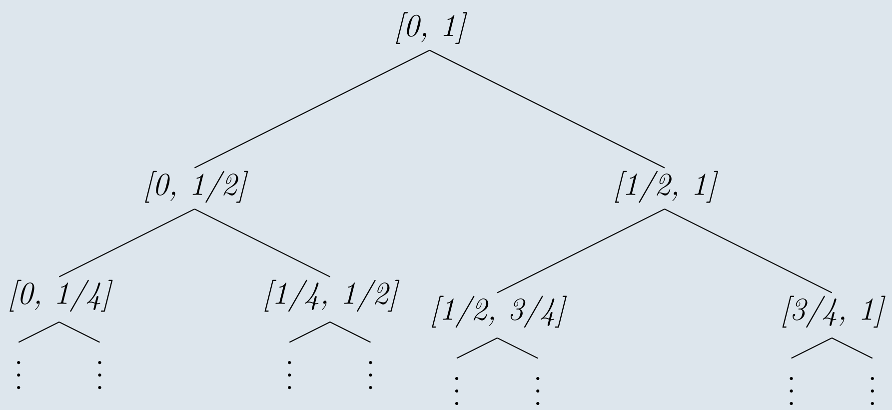
$$0 < \frac{1}{4} < \frac{3}{4} < 1, \quad 0 < \frac{1}{8} < \frac{5}{32} < \frac{2}{8} < 1$$

**Definition 33** (Tree of standard dyadic intervals and $\mathbb{T}$-tree)**.**
*Let's consider* $V = \{I \subset [0,1] | I$ *is a standard dyadic interval in* $[0,1]\}$*. We define the* ***tree of standard dyadic intervals*** *as*

$$\mathbb{T} = (V, E)$$

$$(I,J) \in E \iff I \textit{ is the left half of } J (\textit{ then we say that } (I,J) \textit{ is a left edge})$$
$$\textit{OR } I \textit{ is the right half of } J (\textit{ then we say that } (I,J) \textit{ is a right edge})$$

*The tree* $\mathbb{T}$ *is as such:*

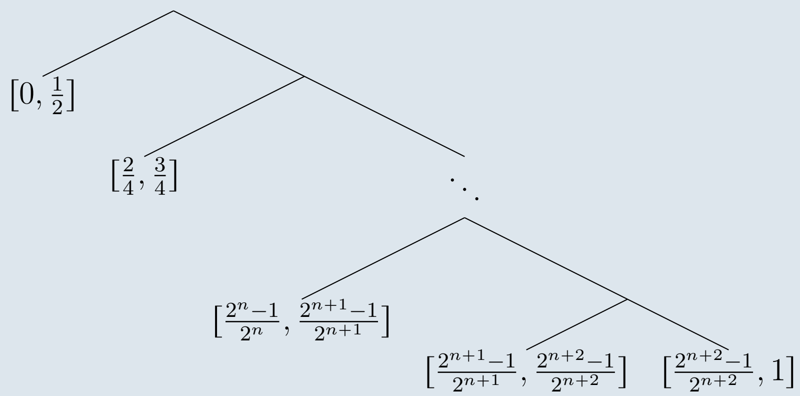


*$\mathbb{T}$ is an infinite rooted binary tree.*

*We define a $\mathbb{T}$-**tree** $T$ as a finite ordered rooted binary sub-tree of $\mathbb{T}$ with root $[0, 1]$. As a matter of fact, they is an equality between ORBT and the set of all $\mathbb{T}$-tree.*
*We call the $\mathbb{T}$-tree with one vertex the trivial $\mathbb{T}$-tree.*

**Definition 34** ($\mathbb{T}_n$ Trees)**.**
*We define a family of $\mathbb{T}$-Tree $\{\mathbb{T}_n\}_{n\in\mathbb{N}}$*

$\mathbb{T}_n =$




**Proposition 35.**
*Let $T$ be a $\mathbb{T}$-tree, $v \in T$. then the corresponding interval $I_v = [\frac{a}{2^{l(v)}}, \frac{a+1}{2^{l(v)}}], a \in \mathbb{Z}$*

*Proof of Proposition* 35:
This is a proof by induction on $l(v)$

- If $l(v) = 0$, then $v = v_0$ and $I_v = [0, 1]$
- $l(v) = n$. Consider $w \in T$ such that $(v, w) = e_v^t$. Then $l(w) = n - 1$, so $I_w = [\frac{b}{2^{l(w)}}, \frac{b+1}{2^{l(w)}}]$. Because there exists $(v, w)$, this means that $I_v$ is the left or the right half of $I_w$ and thus

$$I_v = [\frac{b}{2^{l(w)+1}}, \frac{b+1}{2^{l(w)+1}}] \text{ or } I_v = [\frac{b+1}{2^{l(w)+1}}, \frac{b+2}{2^{l(w)+1}}]$$

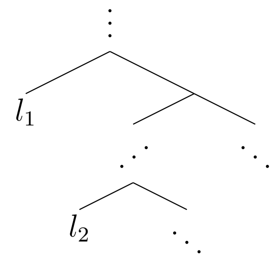
  So $I_v = [\frac{a}{2^{l(v)}}, \frac{a+1}{2^{l(v)}}]$

$\square$ 35

**Proposition 36.**
*Let $T$ be a $\mathbb{T}$-tree, $l_1, l_2 \in$ Leaf $(T)$ such that $l_1$ is the left neighbour of $l_2$. Consider $I_1 = [\frac{a}{2^k}, \frac{a+1}{2^k}], I_2 = [\frac{b}{2^u}, \frac{b+1}{2^u}]$ the corresponding standard dyadic interval.*
*Then $\frac{a+1}{2^k} = \frac{b}{2^u}$*

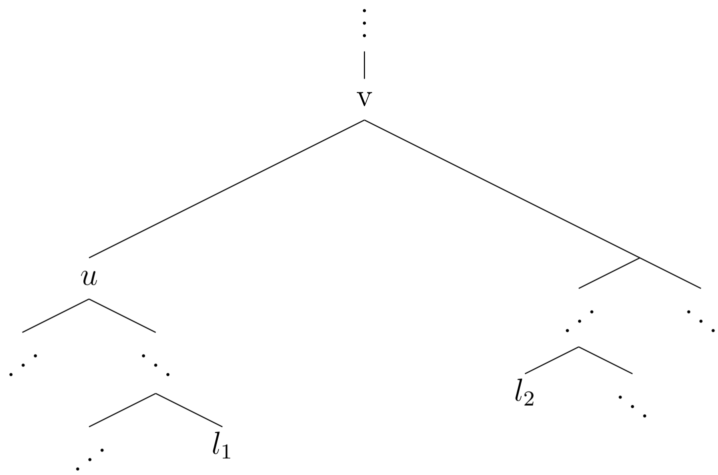
*Proof of Proposition* 36:
This proof is done by disjointing of cases:

- Case 1: $l_1$ is a left leaf. i.e:




  with $l_2$ the leftmost possible. Then take $0 \leqslant k = l(l_2) - l(l_1)$. We have $I_1 = [\frac{a}{2^{l(l_1)}}, \frac{a+1}{2^{l(l_1)}}]$. By definition, $I_2$ is the smallest $2^k$-subdivision of $[\frac{a+1}{2^{l(l_1)}}, \frac{a+2}{2^{l(l_1)}}]$. So $I_2 = [\frac{a+1}{2^{l(l_1)}}, \frac{a+1}{2^{l(l_1)}} + \frac{1}{2^{l(l_2)}}]$ which proves the proposition.

- Case 2: $l_1$ is a right leaf. i.e:




  where $u$ is the vertex in $T$ such that the edge $(u, v) \in p_{l_1, v_0}$ is the first left edge. Such a vertex exists because otherwise $l_1$ would be the rightmost leaf, but it is left to $l_2$.

Then, by forgetting all graph under $u$, we find ourselves in the case 1, with $u$ a left leaf and thus $I_u = [\frac{a}{2^u}, \frac{a+1}{2^u}], I_2 = [\frac{b}{2^v}, \frac{b+1}{2^v}]$, $\frac{a+1}{2^u} = \frac{b}{2^v}$. Now, we consider $0 < k = l(l_1) - l(u)$. $I_1$ is the biggest $2^k$ subdivision of $I_u$, so $I_1 = [\frac{a+1}{2^u} - \frac{1}{2^{l(l_1)}}, \frac{a+1}{2^u}]$ and that proves the proposition.

$\square$ 36

**Lemma 37.**
*Consider $S \in \mathbb{T}$-Trees, $Q$ a standard dyadic partitions of $[0,1]$. There is a bijection:*

$$\mathbb{T}\text{-}\textit{Trees} \cong \textit{standard dyadic partitions of } [0,1]$$
$$S \to P_S$$
$$T_Q \leftarrow Q$$

*Proof of Lemma* 37:

- Let $T$ be any $\mathbb{T}$-Tree. Consider $\{I_i\}_{i=0}^n$ the dyadic interval corresponding to the leaves of $T$, order from left to right. Then using the previous proposition, we get $\forall i, I_i < I_{i+1}, I_i \cap I_{i+1} = \{x_i\}$. Furthermore, considering $I_0$, because it is the leftmost leaf, then $I_0 = [0, x_1]$. Similarly because $I_n$ is the rightmost leaf, then $I_n = [x_n, 1]$. Thus $0 < x_1 < \cdots < x_n < 1 = P_T$ is a standard dyadic partition of $[0,1]$.
- Consider $0 < x_1 < \cdots < x_n < 1$ is a standard dyadic partition of $[0,1]$. Then consider $\{l_i\}_{i=0}^n$ the vertices of $\mathbb{T}$ corresponding to $\{I_i = [x_i, x_{i+1}]\}_{i=0}^n$. We then define $T = \bigcup_{i=0}^n p_{[0,1],l_i}$. $T$ is connected by definition, acyclique by restriction and finite due to the nature of paths. Thus $T$ is still a finite tree. Now, let's show that $\text{Leaf}(T) = \{l_i\}_{i=0}^n$ and that $T$ is binary.

  - If $T$ is the trivial $\mathbb{T}$-tree, then it is binary and the leaves are $[0,1]$ by definition.
  - Suppose $l_i$ is not a leaf. Then there is $l_j$ such that $l_i \in p_{[0,1],l_j}$. But, by definition of $\mathbb{T}$, this mean that $I_j \subset I_i$, which contradict the fact that this is a partition. Now, consider $v \notin \{l_i\}_{i=0}^n$. Then $v \in p_{[0,1],l_k}$ and it is not in either extremity. Thus $d(v) \geqslant 2$ and it is not a leaf.
  - Consider $[0,1] \in T$. consider $l_0$ and $l_n$. because $0 \in I_0, 1 \in I_n$, $p_{[0,1],l_0}$ consist only of left edges and $p_{[0,1],l_n}$ consist only of right edges. Thus $d([0,1]) = 2$. Now consider $v \in T, v \notin \{l_i\}_{i=0}^n, v \neq [0,1]$ Then $d(v) \geqslant 2$ with a top edge and either a left or a right edge $e$

- $*$ if $e$ is right. Then consider $l_i$ the leftmost leaf such that $e \in p_{[0,1],l_i}$. $l_i$ is not leftmost leaf of $T$ because it take a right turn, so there exists $l_{i-1}$ the left neighbour of $l_i$. Let $m$ be the middle of $I_v$, then $m < I_i$. Now, suppose $v \notin p_{[0,1],l_{i-1}}$, then $I_{i-1} \cap I_v = \{x\}$ or $\varnothing$. But this mean that $x_i < m \leqslant x_i$. Thus $v \in p_{[0,1],l_{i-1}}$. But $e \notin p_{[0,1],l_{i-1}}$ due to $l_i$ been the leftmost leaf such that $e \in p_{[0,1],l_i}$. So $p(v) \geqslant 3 \Rightarrow p(v) = 3$.
- $*$ if $e$ is left. Then consider $l_i$ the rightmost leaf such that $e \in p_{[0,1],l_i}$. $l_i$ is not rightmost leaf of $T$ because it take a left turn, so there exists $l_{i+1}$ the right neighbour of $l_i$. Let $m$ be the middle of $I_v$, then $m > I_i$. Now, suppose $v \notin p_{[0,1],l_{i+1}}$, then $I_{i+1} \cap I_v = \{x\}$ or $\varnothing$. But this mean that $x_i \geqslant m > x_i$. Thus $v \in p_{[0,1],l_{i+1}}$. But $e \notin p_{[0,1],l_{i+1}}$ due to $l_i$ been the rightmost leaf such that $e \in p_{[0,1],l_i}$. So $p(v) \geqslant 3 \Rightarrow p(v) = 3$.

Thus $T = T_P$ is a $\mathbb{T}$-tree

Now, It is easy to see that those two ways are mutual inverses.

□ 37

## 2.3 $F$-Thompson's group and tree diagrams

**Lemma 38.**
*Let $f \in F$. Then there exists a standard dyadic partition $0 = x_0 < \cdots < x_n = 1$ such that $f$ is linear on every interval of the partition and $0 = f(x_0) < \cdots < f(x_n) = 1$ is also a standard dyadic partition.*

*Proof of Lemma* 38:
Let $0 = a_0 < a_1 < \cdots < a_n = 1$ be the partition $P$ of $[0,1]$ by the non-differentiable point of $f$. Consider $f|_{[a_i,a_{i+1}]}(x) = 2^{-k}.x + b_i$. Because $a_i, a_{i+1}, f(a_i), f(a_{i+1})$ are dyadic numbers, $\exists m \in \mathbb{N}, m+k \geqslant 0$ such that $2^m a_i,\ 2^m a_{i+1},\ 2^{m+k} f(a_i) \in \mathbb{Z}$ and $2^{m+k} f(a_{i+1}) \in \mathbb{Z}$.

Then, we define the standard dyadic partition $P_i'$ of $[a_i, a_{i+1}]$ as

$$a_i < a_i + \frac{1}{2^m} < a_i + \frac{2}{2^m} < \cdots < a_{i+1}$$

Now, because $f(x) = 2^{-k}.x + b_i$, $f(a_i + \frac{p}{2^m}) = f(a_i) + \frac{p}{2^{m+k}}$. Thus the partition given by $f$,

$$f(P_i') = f(a_i) < f(a_i) + \frac{1}{2^{m+k}} < f(a_i) + \frac{2}{2^{m+k}} < \cdots < f(a_{i+1})$$

$f(P_i')$ is also a standard dyadic partition on $[f(a_i), f(a_{i+1})]$.

Consider $P' = \bigcup_{i=0}^{n-1} P_i'$ and $f(P')$. Those are standard dyadic partitions proving the lemma.

$\square$ 38

**Definition 39** (Tree diagrams)**.**
*Let $(T, S)$ be a pair of $\mathbb{T}$-trees such that $w(T) = w(S)$. We call $(T, S)$ a* ***tree diagram*** *and denoted it*

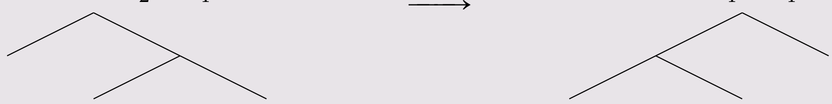
$$T \to S$$

*We call $T$ the* ***domain tree*** *and $S$ the* ***range tree*** *of the diagram.*

**Consequence 40.** Let $f \in F$. Then, by using the lemma 38, there exists $P, Q$ standard dyadic partitions of $[0, 1]$, $f(P) = Q$. Then there exists $T_P \to T_Q$ a tree diagram that represent $f$.
We denote such a tree diagram $T_f \to S_f$

**Remark 41.** Those standard dyadic partitions are not unique. Consider $A(x)$, then

$$P_1 = 0 < \tfrac{1}{2} < \tfrac{3}{4} < 1 \quad \xrightarrow{A(x)} \quad A(P_1) = 0 < \tfrac{1}{4} < \tfrac{2}{4} < 1$$



$$P_2 = 0 < \tfrac{1}{4} < \tfrac{2}{4} < \tfrac{3}{4} < 1 \quad \xrightarrow{A(x)} \quad A(P_2) = 0 < \tfrac{1}{8} < \tfrac{1}{4} < \tfrac{1}{2} < 1$$

**Lemma 42.**
*In the other way, let $T \to S$ be any tree diagram. Then, $\exists! f \in F$ such that $T \to S$ represent $f$ (i.e. $T \to S = T_f \to S_f$).*
*We denote such a function $f_{T,S}$*

*Proof of Lemma* 42:
Consider $P_T$ and $P_S$ two standard dyadic partitions, $P_T = 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = 1$, $P_S = 0 = \beta_0 < \beta_1 < \cdots < \beta_n < \beta_{n+1} = 1$. Because those are standard dyadic partitions, $\alpha_{i+1} - \alpha_i = 2^{p_i}$, $\beta_{i+1} - \beta_i = 2^{q_i}$
Then we construct the function $f_{T,S}$ on $[\alpha_i, \alpha_{i+1}]$.

$$f_{T,S}|_{[\alpha_i,\alpha_{i+1}]} = \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i}x + \frac{\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1}}{\alpha_{i+1} - \alpha_i} = 2^{q_i - p_i}.x + \frac{\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1}}{2^{p_i}}$$

Then $f_{T,S}(\alpha_i) = \beta_i$ and $f$ is increasing. $f_{T,S}$ is a homeomorphism piecewise linear whose derivate are power of 2. Then $f_{T,S} \in F$. because $f_{T,S}(P_T) = P_S$ then $f_{T,S}$ is represented by $T_{P_T} = T \to S = T_{P_S}$

For the uniqueness, consider $g \in F$ such that $T \to S$ represent $g$. Then $g(P_T) = P_S$, so $g$ linear on $[\alpha_i, \alpha_{i+1}]$, $g(\alpha_i) = \beta_i = f_{T,S}(\alpha_i)$. Then

$$g'|_{[\alpha_i,\alpha_{i+1}]} = \frac{g(\alpha_{i+1}) - g(\alpha_i)}{\alpha_{i+1} - \alpha_i} = \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i} = f'|_{[\alpha_i,\alpha_{i+1}]}$$

$$\text{Thus } \forall i,\ g|_{[\alpha_i,\alpha_{i+1}]} = f|_{[\alpha_i,\alpha_{i+1}]} \Rightarrow g = f$$

□ 42

**Proposition 43.**
*Let $T \to S, S \to R$ be any tree diagrams. Then we have the following properties*

$$f_{T,T} = id$$

$$f_{T,S}^{-1} = f_{S,T}$$

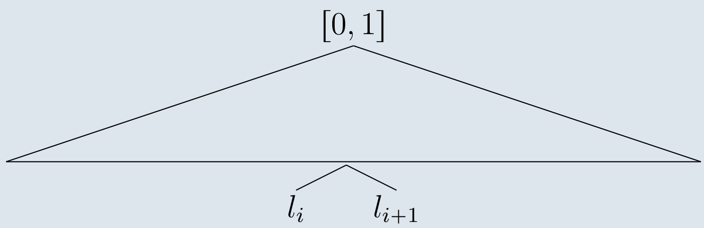
$$f_{S,R} \circ f_{T,S} = f_{T,R}$$

*Proof of Proposition* 43:
Consider $P_T, P_S, P_R$ the standard dyadic partitions equivalent to $T, S, R$.

- $\forall x \in [0,1], f'_{T,T}(x) = 1$. So $f_{T,T}$ is linear on $[0,1]$. $f_{T,T}(x) = 2^k.x + b, f_{T,T}(0) = 0, f_{T,T}(1) = 1 \Rightarrow f_{T,T} = id$
- $f_{T,S}(P_T) = P_S$, so $f^{-1}_{T,S}(P_S) = P_T$, so $f^{-1}_{T,S}$ can be represented as $S \to T$, so $f_{S,T} = f^{-1}_{T,S}$
- $f_{T,S}(P_T) = P_S$ and thus $f_{S,R} \circ f_{T,S}(P_T) = f_{S,R}(P_S) = P_R$. Thus $f_{S,R} \circ f_{T,S}$ can be represented as a tree diagram as $T \to R$, so $f_{S,R} \circ f_{T,S} = f_{T,R}$

$\square$ 43

**Definition 44** (Carets)**.**
*Let $T$ be any $\mathbb{T}$-tree and let $l_1, l_2$ be 2 neighbour leaves of $T$. We say that $(l_1, l_2)$ form a **caret** writen $C(l_1, l_2)$ if*




**Definition 45** (Reduced tree diagrams)**.**
*Consider $T \to S$ a tree diagram. We say that $T \to S$ is a **reduced tree diagram** if $\forall t_i \in Leaf(T)$, $\forall s_i \in Leaf(S)$ both ordered from left to right, we have not the following case:*

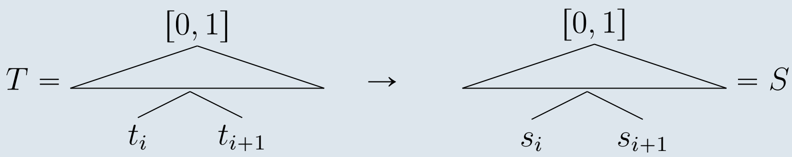
$$(t_i, t_{i+1}) \text{ form a caret in } T \text{ AND } (s_i, s_{i+1}) \text{ also form a caret in } S$$

*Otherwise, we say that $T \to S$ is a **reducible tree diagram**.*

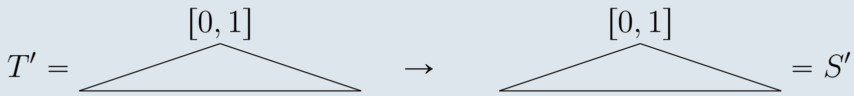
**Remark 46.** If $T \to S$ is reduced, then so is $S \to T$

**Definition 47** (the reduced of a tree diagram)**.**
*Let $T \to S$ be any tree diagram. Then we define $\overline{T \to S}$ **the reduced** of $T \to S$ using the following algorithm :*

- ***IMPUT*** *: $(T \to S)$*
- ***ALGO:***
  - ***While*** *$(T \to S)$ is not reduced, consider $(t_i, t_{i+1})$ and $(s_i, s_{i+1})$ such that they both form caret :*




  - *Define $T' \to S'$*
$$T' = T \backslash C(t_i, t_{i+1}), S' = S \backslash C(s_i, s_{i+1})$$




    *Note that $w(T') = w(T) - 1 = w(S) - 1 = w(S')$*
  - *$T \to S$ becomes $T' \to S'$*
- ***OUTPUT:*** *$T \to S$*

*Note that this algorithm end at some point because $w(T)$ is finite.*

**Lemma 48.**
*Let $f \in F$ and $T_f \to S_f$ the corresponding tree diagram. Let $\overline{T_f \to S_f}$ be the reduced of $T_f \to S_f$. Then:*
$$f_{\overline{T_f, S_f}} = f$$

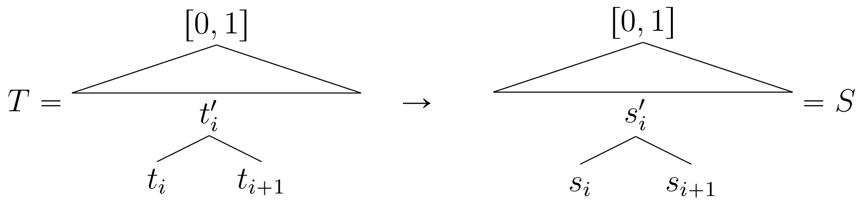
*Proof of Lemma* 48:
To prove this lemma, it suffice to show that:

$$f_{T'_f,S'_f} = f_{T_f,S_f}$$

The only difference between $f_{T'_f,S'_f}$ and $f_{T_f,S_f}$ can lie on $I_i$, $I_{i+1}$ and $I'_i = I_i \cup I_{i+1}$ the interval referring respectively to $t_i$, $t_{i+1}$ and $t'_i$, the $i$-th and $i+1$-th leaf of $T_f$ and the $i$-th of $T'_f$. They are such that $\exists C(t_i, t_{i+1})$ We denote note the corresponding leaves in $S$ and $S'$ $s_i, s_{i+1}, s'_i$ ($\exists C(s_i, s_{i+1})$) and intervals $J_i, J_{i+1}, J'_i = J_i \cup J_{i+1}$.

$$I_i = [\alpha_i, \alpha_{i+1}], I_{i+1} = [\alpha_{i+1}, \alpha_{i+2}], I'_i = [\alpha_i, \alpha_{i+2}],$$

$$J_i = [\beta_i, \beta_{i+1}], J_{i+1} = [\beta_{i+1}, \beta_{i+2}], J'_i = [\beta_i, \beta_{i+2}]$$




$$f_{T_f,S_f}|_{I_{i+1}} = \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i}x + \frac{\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1}}{\alpha_{i+1} - \alpha_i}$$

$$f_{T_f,S_f}|_{I_{i+2}} = \frac{\beta_{i+2} - \beta_{i+1}}{\alpha_{i+2} - \alpha_{i+1}}x + \frac{\alpha_{i+2}\beta_{i+1} - \alpha_{i+1}\beta_{i+2}}{\alpha_{i+2} - \alpha_{i+1}}$$

Because $\exists\ C(t_i, t_{i+1}), C(s_i, s_{i+1})$ Then :

$$(\alpha_{i+1} - \alpha_i = \alpha_{i+2} - \alpha_{i+1}), \quad (\beta_{i+2} - \beta_{i+1} = \beta_{i+1} - \beta_i)$$

$$\text{Thus} \quad \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i} = \frac{\beta_{i+2} - \beta_{i+1}}{\alpha_{i+2} - \alpha_{i+1}} = \frac{\beta_{i+2} - \beta_i}{\alpha_{i+2} - \alpha_i}$$

$$\frac{\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1}}{\alpha_{i+1} - \alpha_i} = \frac{\alpha_{i+2}\beta_i - \alpha_i\beta_{i+2}}{\alpha_{i+2} - \alpha_i}$$

Thus $f_{T_f,S_f}$ is derivable on $\alpha_{i+1}$ So it is linear on $I_i \cup I_{i+1} = I'_i$

$$f_{T_f,S_f}|_{I'_i} = \frac{\beta_{i+1} - \beta_i}{\alpha_{i+1} - \alpha_i}x + \frac{\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1}}{\alpha_{i+1} - \alpha_i} = \frac{\beta_{i+2} - \beta_i}{\alpha_{i+2} - \alpha_i}x + \frac{\alpha_{i+2}\beta_i - \alpha_i\beta_{i+2}}{\alpha_{i+2} - \alpha_i} = f_{T'_f,S'_f}|_{I'_i}$$

So $f_{T_f,S_f} = f_{T'_f,S'_f}$ which prove the lemma

$\square$ 48

**Theorem 49.**
*There exists a bijection*

$$\overline{(-)_T} : F \to \textit{ reduced tree diagram}$$
$$\overline{(-)_T}(f) = \overline{T_f \to S_f}$$

*with inverse fonction*

$$f_- : \textit{ reduced tree diagram } \to F$$
$$f_-(T \to S) = f_{T,S}$$

*Proof of Theorem* 49:
First, if $f_-$ is well defined, we still have to prove that $\overline{(-)_T}$ is well defined. This mean consider $P_1$, $Q_1$, $P_2$ and $Q_2$ standard dyadic partition of $[0,1]$ such that $f(P_1) = Q_1$, $f(P_2) = Q_2$. We have $\overline{T_{P_1} \to T_{Q_1}} = \overline{T_{P_2} \to T_{Q_2}}$.

Let's show that if $T \to S$ is a reduced tree diagram for $f$ ( i.e $f_{T,S} = f$), then $T \to S$ is unique.

- Let $I$ be a standard dyadic interval in $\mathbb{T}$ such that $I$ is either a leaf or not in $T$. Then $\exists I_i \in \text{Leaf}(T)$ such that $I \subset I_i$. Then $f = f_{T,S}$, so $f$ is linear on $I$ because and it is on $I_i$ with derivate $2^k$. Then $I = [a,b]$, $b - a = 2^{-p}$ $f(I) = [f(a), f(b)]$ with $f(a), f(b)$ dyadic numbers and $f(b) - f(a) = 2^{k-p}$. So $f(I)$ is a dyadic interval.
- Consider $I_1 = [x_1, x_2] = [\frac{a}{2^{p_1}}, \frac{a+1}{2^{p_1}}]$ and $I_2 = [x_2, x_3] = [\frac{b}{2^{p_2}}, \frac{b+1}{2^{p_2}}]$, 2 neighbour leaves of $T$ and there image by $f$ $J_1 = [f(x_1), f(x_2)]$ and $J_2 = [f(x_2), f(x_3)]$. suppose $f$ is linear on $I_1 \cup I_2$. This implies that $f'|_{I_1} = f'|_{I_2} = 2^k$ and thus
$$f(x_2) - f(x_1) = 2^k(x_2 - x_1), f(x_3) - f(x_2) = 2^k(x_3 - x_2)$$
Now suppose $I_1 \cup I_2$ is a standard dyadic interval, thus $\exists q \in \mathbb{N}, \frac{1}{2^d} = x_3 - x_1 = (x_3 - x_2) + (x_2 - x_1) = \frac{1}{2^{p_1}} + \frac{1}{2^{p_2}}$ and this is possible only if $p_1 = p_2$ and thus, due to the nature of $\mathbb{T}, l(I_1) = l(I_2)$. This mean that $x_3 = \frac{a+2}{2^{p_1}}$ and $[\frac{a}{2^{p_1}}, \frac{a+2}{2^{p_1}}]$ is a standard dyadic interval, so $a$ is even and thus $I_1$ is a left edge. because $I_2$ is neighbour and of same length, we get that $\exists C(I_1, I_2) \subset T$. This means that
$$I_1 \cup I_2 \text{ is a standard dyadic interval } \iff \exists C(I_1, I_2)$$
Using the same proof, we get that $J_1 \cup J_2$ is a standard dyadic interval $\Rightarrow \exists C(J_1, C_2) \subset S$. We can also see that if $f$ is linear on $I_1 \cup I_2$ is a standard dyadic interval, then $J_1 \cup J_2$ is also a standard dyadic interval.

- Now, let $I$ be a standard dyadic interval in $\mathbb{T}$ such that $f$ is linear on $I$ and $f(I)$ is still a standard dyadic interval. Then either $I \in$ Leaf$(T)$ or $I \notin T$. Indeed otherwise, consider $l_1$ the leaf of $T$ under $I$ ($I_1 \subset I$) such that $l(I_1)$ is maximal among other leaves under $I$. Because $\tilde{T} = \{J \in T | J \subset I\}$ is a ORBT, we get that $\exists l_2 \in$ Leaf$(T)$ such that $\exists C(I_1, I_2) \subset T$ xor $C(I_2, I_1) \subset T$. (wlog, we can admit that $I_1 < I_2$)

  Then, $I_1 \cup I_2$ is a standard dyadic interval and $f$ is linear on $I_1 \cup I_2 \Rightarrow J_1 \cup J_2$ is a standard dyadic interval $\Rightarrow \exists C(J_1, C_2)$. So $\exists C(I_1, I_2) \subset T$ and $C(J_1, J_2) \subset S$ But this contradicts the fact that $T \to S$ is reduced.

  Thus, $I \in$ Leaf$(T)$ or $I \notin T$.

Thus, Let's consider $T \to S$, $T' \to S'$ 2 reduced tree diagrams such that $f_{T,S} = f_{T',S'} = f$. Consider Leaf$(T) = \{I_i\}_{i=0}^{n}$, Leaf$(T') = \{J_j\}_{j=0}^{m}$ with $n < m$. Then for the standard dyadic set $I_0$, $f$ is linear. Thus using the previous point and because $0 \in I_0$, we have that $I_0 \subseteq J_0$. Similarly, $J_0 \subseteq I_0$, so $I_0 = J_0$. Now consider $I_1$, because $I_1 \nsubseteq I_0$ and $I_1 \cap I_0 \neq \emptyset$, we get that $I_1 \subset J_1 \cdots$
At the end, we get $n = m$ and $\{I_i\}_{i=0}^{n} = \{J_j\}_{j=0}^{m} \Rightarrow$ Leaf$(T) =$ Leaf$(T') \Rightarrow T = T'$ and Leaf$(S) =$ Leaf$(S') \Rightarrow T \to S = T' \to S'$. So a reduced tree diagram for $f$ is unique.

Now, we have finally proven that $\overline{(-)_T} : F \to$ reduced tree diagram is well defined.

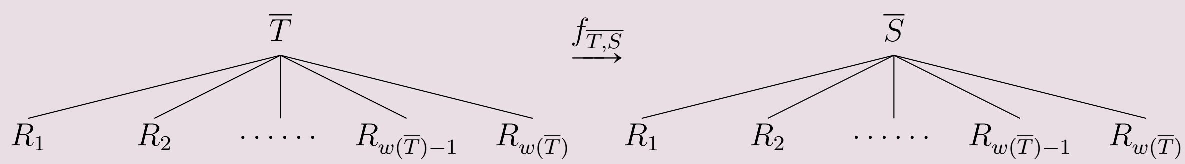
- $\forall\ T \to S$ a reduced tree diagram, $\overline{(-)_T} \circ f_-(T \to S) = \overline{(-)_T}(f_{T,S}) = \overline{T \to S} = T \to S$
- By the previous lemma $\forall f \in F$, $f_- \circ \overline{(-)_T}(f) = f_-(\overline{T_f \to S_f}) = f_{\overline{T,S}} = f$

□ 49

**Corollary 50.**
*Consider*

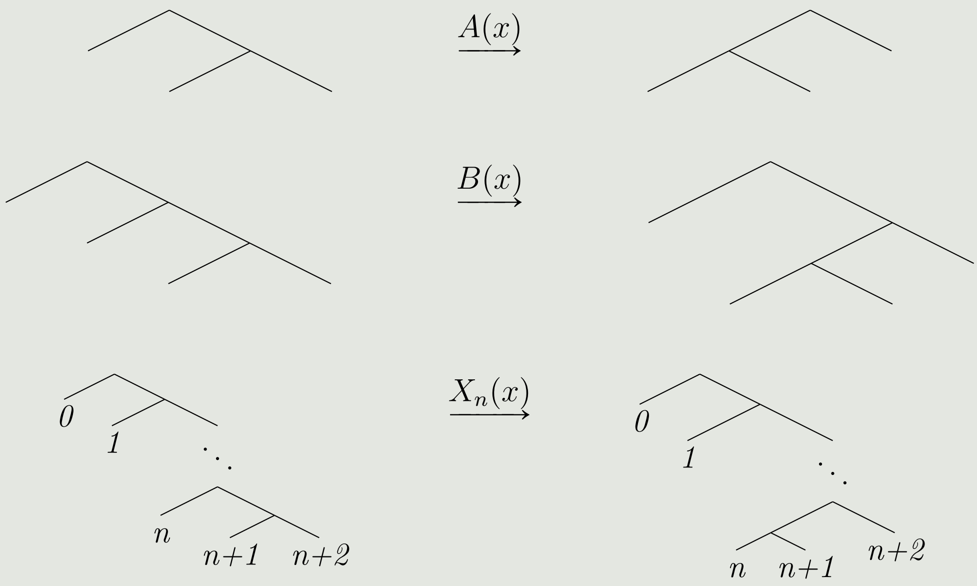
$$\overline{(-)} : \text{ Tree diagrams } \twoheadrightarrow \text{ Reduced tree diagrams}$$
$$\overline{(-)}(T \to S) = \overline{T \to S}$$

*Then it's fibers $\overline{(-)}^{-1}(\overline{T \to S})$ are of the following form:*



*with* $\{R_i\}_{i=1}^{w(\overline{T})}$ *being* $ORBT$.

**Examples 51.** *A few examples of this bijection :*




## 2.4 Representation of $F$

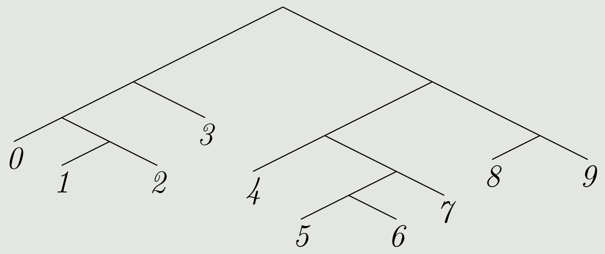
**Definition 52** (The exponents of an $\mathbb{T}$-tree)**.**
*Let* $T$ *be a* $\mathbb{T}$*-tree, Leaf* $(T) = \{I_i\}_{i=0}^{n}$, $p$ *be a path of* $T$ *and let* $P_i$ *be the following set:*

$$P_i = \{p \in T|\ p \textit{ start in } I_i, \textit{ is composed only of left edges and doesn't end in } R(T)\}$$

$\forall\ 0 \leqslant k \leqslant n$, *we define* $a_k$, *the exponent of* $I_k$ *as follows*

$$a_k = \ max_{p \in P_k}(l(p)), \qquad a_k = 0 \textit{ if } P_k = \emptyset$$

**Example 53.**



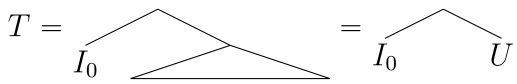

*Its exponents are* $2, 1, 0, 0, 1, 2, 0, 0, 0, 0$.
*In fact, due to their definition, we have that for any tree, the last 2 exponents are always 0.*

**Theorem 54.**
*Let* $T \to S$ *be any tree diagram with* $w(T) = n + 3$. *Let* $a_0, \cdots, a_{n+2}$, *be the exponents of* $T$, $b_0, \cdots, b_{n+2}$ *be the exponents of* $S$. *Then*

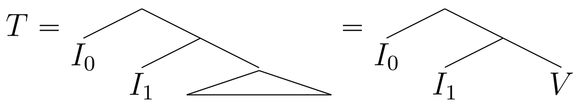
$$f_{T,S} = X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$$

*Proof of Theorem* 54:
Consider $T \to \mathbb{T}_n$ and $\mathbb{T}_n \to S$. Then $f_{T,S} = f_{\mathbb{T}_n,S} \circ f_{T,\mathbb{T}_n}$
Let's show $f_{T,\mathbb{T}_n} = X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$. We will prove this by induction on $a = \sum_{i=0}^{n} a_i$

- If $a = 0$, then $\forall i, a_i = 0$. Then consider $I_0 \in T$. $I_0 \in L(T)$, It is link to $T$ using $e_0 = (I_0, v)$ a left edge. but because $a_0 = 0$, then $v \in R(T) \cap L(T) = v_0$. So

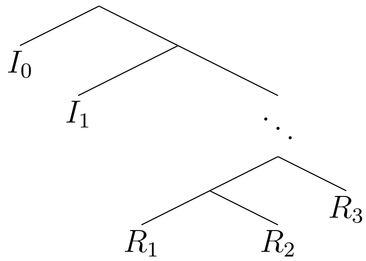
  Then consider $I_1$. It is link to $U$ using $e_1 = (I_1, w)$ also a left edge. Then $I_1 \in L(U) \Rightarrow w \in L(U)$ and $a_1 = 0 \Rightarrow w \in R(U)$. So

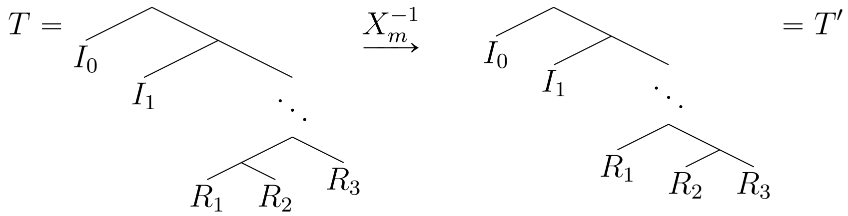
  We use the same argument up until $n$. Then $I_{n+2} \in R(T)$ and because $I_n$ was left, then $I_{n+1}$ muss be left too. So $T = \mathbb{T}_n$. Then $f_{\mathbb{T}_n,\mathbb{T}_n} = id = X_n^0 \cdots X_2^0 X_1^0 X_0^0$

- $a > 0$. Let $m = \min\{n \in \mathbb{N} | a_n > 0\}$. Then using the same argument than $a = 0$, we get that

$T =$




with $R_1, R_2, R_3$ being ORBT, $I_m$ is the leftmost leaf of $R_1$. We apply $X_m^{-1}$ on $T$




Consider $a'_i$ the exponents of $T'$.

  - If $i < m$, then $a'_i = 0 = a_i$.
  - If $i = m$, then we have shorten the maximal path by 1, so $a'_m = a_m - 1$.
  - If $i > m$, then if $I_i \in R_j$, its maximal path was lying in $R_j$ and it still the case in $T'$. Thus $a'_i = a_i$.

Using the induction hypothesis, we have that

$$f_{T',\mathbb{T}_n} = X_n^{-a'_n} \cdots X_2^{-a'_2} X_1^{-a'_1} X_0^{-a'_0} = X_n^{-a'_n} \cdots X_m^{-a'_m}$$

$$f_{T,\mathbb{T}_n} = f_{T',\mathbb{T}_n} \circ f_{T,T'} = X_n^{-a'_n} \cdots X_m^{-a'_m} \circ X_m^{-1} = X_n^{-a_n} \cdots X_m^{-a_m} = X_n^{-a_n} \cdots X_1^{-a_1} X_0^{-a_0}$$

$$f_{\mathbb{T}_n,S} = {}^{1}(f_{S,\mathbb{T}_n})^{-1} = (X_n^{-b_n} \cdots X_1^{-b_1} X_0^{-b_0})^{-1} = X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n}$$

$$f_{T,S} = f_{\mathbb{T}_n,S} \circ f_{T,\mathbb{T}_n} = X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$$

$\square$ 54

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**Corollary 55.**
*$F =< X_i|_{i\in\mathbb{N}} >$.*

*But because $X_n = A^{-(n-1)}BA^{n-1}$ , $F =< A, B >$.*

*Proof of Corollary* 55:
Let $f \in F$. Consider $\overline{T_f \to S_f}$. If $w(\overline{T}) < 3$, then $w(\overline{T}) = 1$ and $f = id_{[0,1]}$.
Otherwise, using 54, we get that $f = f_{\overline{T_f,S_f}} = X_0^{b_0}X_1^{b_1}\cdots X_n^{b_n}X_n^{-a_n}\cdots X_2^{-a_2}X_1^{-a_1}X_0^{-a_0}$.
□ 55

## 2.5 Partial action of $F$

**Definition 56** (Partial function)**.**
*Let $X, Y$ be any sets. A* ***partial fonction*** *$f$ from $X$ to $Y$ written*

$$f : X \rightharpoonup Y$$

*is a extension on the domain of a function. i.e. $\exists H \subset X$, $f|_H : H \to Y$ is a function.*

**Definition 57** (Partial group action)**.**
*Let $G$ be any group with neutral element $e$ and let $X$ be any set. A* ***partial action*** *of $G$ on $X$ is a partial function*

$$\circlearrowright_p : G \times X \rightharpoonup X$$

*such that:*

1. *$\forall x \in X$, $\exists e \circlearrowright_p x$ and $e \circlearrowright_p x = x$*
2. *$\forall g \in G$, IF $\exists g \circlearrowright_p x$, THEN $\exists\ g^{-1} \circlearrowright_p (g \circlearrowright_p x)$ and $g^{-1} \circlearrowright_p (g \circlearrowright_p x) = x$*
3. *$\forall f, g \in G$, IF $\exists g \circlearrowright_p (f \circlearrowright_p x)$, THEN $\exists (g\cdot f) \circlearrowright_p x$ AND $g \circlearrowright_p (f \circlearrowright_p x) = (g\cdot f) \circlearrowright_p x$*

**Definition 58** (Another definition of partial group action)**.**
*Let $G$ be any group with neutral element $e$ and let $X$ be any set. A* ***partial action*** *of $G$ on $X$ is a family of set $\{X_g\}_{g\in G}$ and a family of bijections $\{\delta_g : X_{g^{-1}} \cong X_g\}_{g\in G}$ such that:*

- *$X_e = X$ and $\delta_e = id_X$.*
- *$\forall g \in G$ $\delta_g$ and $\delta_{g^{-1}}$ are mutual inverses.*
- *$\forall x \in X_g \cap X_{h^{-1}}$, then $\delta_g^{-1}(x) \in X_{(hg)^{-1}}$ and $\delta_h(x) = \delta_{hg}(\delta_g^{-1}(x))$.*

**Remark 59.** Every standard group action can also be considered as a partial group action

**Definition 60** (Standard partial action of $F$ on $M_1$)**.**
*Consider $M_1$ the free magma on 1 element. We know that $M_1 \cong CORBT(1) = ORTP =$ $\mathbb{T}$-tree. So consider $x \in M_1$, $x \sim T_x$, its equivalent as a $\mathbb{T}$-tree.*
*We define the* ***standard partial action of $F$ on*** *$M_1$ as follows:*

$$F \circlearrowleft M_1 \rightharpoonup M_1$$
$$f \circlearrowleft x = y \sim f(T_x) \text{ if } \overline{T_f} \subset T_x$$

*with $f(S) = T_{f(P_S)}$.*

**Proposition 61.**
*$F \circlearrowleft M_1$ is indeed a partial action.*

*Proof of Proposition* 61:
We can see that our action is well defined.

- Let $f = id$. Then $f$ is linear on $[0, 1]$. so it's reduced tree diagram is the trivial tree diagram $(\cdot \to \cdot)$ with $\cdot$ being the trivial tree. Then, $\forall T$, $\cdot \subseteq T$ and $id(T) = T$. so $\forall x \in M_1, id \circlearrowleft x = x$

- Consider $f \in F$, $f = f_{\overline{T,S}}$, $R$ a $\mathbb{T}$-tree such that $\overline{T} \subset R$, then $\overline{S} \subset f(R)$ and because $f = f_{\overline{T,S}}$, $f^{-1} = f_{\overline{S,T}}$ and thus $f^{-1}(f(R)) = R$ is well defined.

- Consider $f \in F$, $f = \overline{T \to S}$, $R$ a $\mathbb{T}$-tree. Then:

$$\overline{T} \subset R \iff P_R \text{ and } f(P_R) \text{ are standard dyadic partitions of } [0,1]$$

Thus, consider $g \in F$, $g = \overline{U \to V}, g \circ f = \overline{P \to Q}$.

$\overline{T} \subset R$ and $\overline{U} \subset f(R) \iff P_R, f(P_R)$ and $g \circ f(P_R)$ are standard dyadic partitions $\Rightarrow$

$$\Rightarrow P_R \text{ and } g \circ f(P_R) \text{ are standard dyadic partitions} \iff \overline{P} \subset R$$

So, $\exists\ g \circlearrowright (f \circlearrowright x) \Rightarrow \exists (g \circ f) \circlearrowright x$ and $g \circlearrowright (f \circlearrowright x) = g \circ f(T_x) = (g \circ f) \circlearrowright x$

$\square$ 61

**Definition 62.**
*Consider $G \circlearrowright_p X$ a partial group action, we define the following notions:*

- $\forall x \in X$ *, we define* ***the domain of*** $x$*,* $Dom(x) = \{g \in G | \exists g \circlearrowright_p x\}$
- $\forall x \in X$*, we define* ***the orbits of*** $x$*,* $O_x = \{y \in X |\ \exists g \in G \text{ s.t. } \exists g \circlearrowright_p x = y\}$*. Furthermore, they still form a partition of $X$*
- *We say that $G \circlearrowright_p X$ is a* ***partial free group action*** *if $\forall x \in X, \forall g \neq e$, if $\exists g \circlearrowright_p x$, then $g \circlearrowright_p x \neq x$*

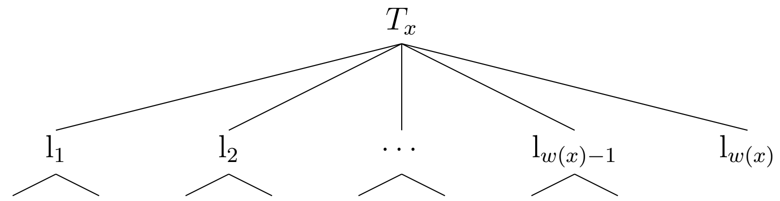
**Proposition 63.**
*Consider $F \circlearrowright M_1$ the standard partial action of $F$ on $M_1$. Then:*

- *$F \circlearrowright M_1$ is a free partial group action*
- $\forall x \in M_1, O_x = X_{w(x)}$
- $\forall x \in M_1,\ Dom(x) \neq F$

*Proof of Proposition* 63:

- Consider $f \in F$, $f \neq id$ and $x \in M_1$ such that $\overline{T_f} \subset T_x$. because $f \neq id$, $\overline{T_f}$ is not the simple $\mathbb{T}$-tree and $\overline{S} \neq \overline{T}$. So $F(T_x) \neq T_x$
- Consider $x, y \in X_n$. Then $w(T_x) = n = w(T_y)$ so $T_x \to T_y$ is a tree diagram and $f_{T_x,T_y}(T_x) = T_y$. So $f_{T_x,T_y} \circlearrowright x = y$. conversely $f \in F$ only send tree to other tree of same width.
- Consider $T_x$ the equivalent of $x$ as a $\mathbb{T}$-tree. Because $\mathrm{Dom}(\cdot) = \{id\} \neq F$, we can assume $w(x) \geqslant 2$ We define $\tilde{T_x}$ by adding to each leaves of $T_x$ exempt $l_{w(x)}$ a caret:

  $\tilde{T_x} =$

  We can see that by construction $\tilde{T_x} \nsubseteq T_x$ and also by construction, we get that

  $$\tilde{T_x} \to \mathbb{T}_{2w(x)-4} \text{ is a reduced tree diagram.}$$

  Indeed, the only leaves in $\mathbb{T}_{2w(x)-4}$ that form a caret are the 2 rightmost ones but they don't form a caret in in $\tilde{T_x}$. Defining $g = f_{\tilde{T_x},\mathbb{T}_{2w(x)-4}}$, we get that $g \notin \mathrm{Dom}(x)$

$\square$ 63

**Definition 64** ($G$-equivariant maps)**.**
*Let $G \circlearrowright_p X$ and $G \circlearrowright_p Y$ be two partial G-actions. Consider the following map:*

$$f : X \to Y$$

*$f$ is called $G$-**equivariant** if:*

1. $\forall g \in G,\ f(X_g) \subseteq Y_g$
2. $\forall x \in X_g,\ f(g \circlearrowright_p x) = g \circlearrowright_p f(x)$

*Furthermore, if $f$ is bijective and $f^{-1}$ is also $G$-equivariant, then $f$ is called an **equivalence of partial actions***

We sadly won't have the time to prove that the free group $F_2$ is not a subgroup of $F$ (see chapter 4 of [1]). This result is interesting because it open $F$ to be a counterexample of Von-Neumann conjecture (disproved in 1980 by Aleksandr Olshansky), that stated that

$$\text{a group } G \text{ is non-amenable} \iff F_2 \subseteq G.$$

# Chapter 3

# Amenability on $F$-Thompson group

## 3.1 Definitions

This part is mostly a repetition of the course of Analysis on Group given by N.Monod at EPFL.

**Definition 65** (Means)**.**
*Let $X$ be any set. We define* ***a mean*** *on $X$ as follows*

$$\mu : \mathcal{P}(X) \to [0, 1]$$
$$\mu(X) = 1$$
$$A, B \subset X, A \cap B = \emptyset \Rightarrow \mu(A \sqcup B) = \mu(A) + \mu(B)$$

*We also define the* ***set of means on X*** *$\mathcal{M}(X)$*

**Remark 66.** Means are finitely additive probability measure.

**Proposition 67.**
*$\mathcal{M}(X)$ is a compact Hausdorff convex set.*

*Proof of Proposition* 67:

- $\mathcal{M}(X)$ is convex: Consider $t_1, \cdots, t_n \geqslant 0$ with $\sum_{i=1}^n t_i = 1$ and $\mu_1, \cdots, \mu_n \in \mathcal{M}(X)$. Then:

$$\sum_{i=1}^{n} t_i \mu_i \in \mathcal{M}(X)$$

  - $\sum_{i=1}^n t_i \mu_i(X) = \sum_{i=1}^n t_i = 1$
  - $\sum_{i=1}^n t_i \mu_i(A \sqcup B) = \sum_{i=1}^n t_i \big(\mu_i(A) + \mu_i(B)\big) = \sum_{i=1}^n t_i \mu_i(A) + \sum_{i=1}^n t_i \mu_i(B)$

- $\mathcal{M}(X)$ is Hausdorff: Because $[0,1]$ is Hausdorff, we get by product conservation that $[0,1]^{\mathcal{P}(X)}$ is Hausdorff. because $\mathcal{M}(X) \subset [0,1]^{\mathcal{P}(X)}$, we get $\mathcal{M}(X)$ is Hausdorff.

- $\mathcal{M}(X)$ is compact: Because $[0,1]$ is compact, we get using Tychonoff's theorem that $[0,1]^{\mathcal{P}(X)}$ is compact. Now, we have to show that $\mathcal{M}(X)$ is a closed subset of $[0,1]^{\mathcal{P}(X)}$. But

$$\mathcal{M}(X) = \left\{\mu \in [0,1]^{\mathcal{P}(X)} | \, \mu(X) = 1\right\} \cap \bigcap_{A \cap B = \emptyset} \left\{\mu \in [0,1]^{\mathcal{P}(X)} | \, \mu(A \sqcup B) = \mu(A) + \mu(B)\right\}$$

  and all those parts are closed. So $\mathcal{M}(X)$ is compact.

$\square$ 67

**Definition 68** (Dirac mass)**.**
*Let $X$ be any set. we define the* ***Dirac mass***

$$\delta : X \hookrightarrow \mathcal{M}(X)$$

$$\delta(x) = \delta_x$$

$$\delta_x(A) = \begin{cases} 1 & \textit{if } x \in A \\ 0 & \textit{otherwise} \end{cases}$$

**Definition 69** (Ultra-filters).
*Let $X$ be any set. An **ultra-filter** on $X$ is $\mathcal{F} \subset \mathcal{P}(X)$ such that:*

1. $\emptyset \notin \mathcal{F}$
2. $A \subset B$ *and* $A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$
3. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
4. $A \in \mathcal{F} \iff X \backslash A \notin \mathcal{F}$

**Lemma 70.**
*Let $X$ be any set. Consider $\beta X = \overline{\delta(X)}$ the Stone-Cech compactification. Then, there exists a bijection*

$$\textit{Ultra-filters on } X \cong \beta X$$

*Proof of Lemma* 70:
First, let's show that $\beta X = \Big\{\mu \in \mathcal{M}(X) | \ \mu(A) = 0 \text{ or } 1\Big\}$.
Consider $U$ an open in the topological basis of $[0,1]^{\mathcal{P}(X)}$. $U = \prod_{A \in \mathcal{P}(X)} U_A$, $U_A$ open in $[0,1]$ and $U_A = [0,1]$ except on finitely many cases. We will be using that

$$x \in \overline{V} \iff \forall O \text{ in the topological basis}, x \in O \Rightarrow O \cap V \neq \emptyset$$

- $\supseteq$: Consider $\mu$, $\mu(A) = 0$ or 1. Let $U$ be an open in the topological basis of $[0,1]^{\mathcal{P}(X)}$, $\mu \in U$. Let $\{U_{A_i}\}_{i=0}^n$ be the non trivial open part of $U$. Because $\mu \in U$, then either 0 or 1 is in $U_{A_i}$ (because $\mu(A_i) = 0$ or 1).

  Then consider $J = \{i = 0, \cdots, n \mid \mu(A_i) = 1\}$ and let $J' = \{0, \cdots, n\} \backslash J$. At least one of them is non-empty. We define the sets

  $$A = \bigcap_{j \in J} A_j, \quad B = \bigcup_{j \in J'} A_j$$

  whenever $J$ or $J'$ is non empty. Otherwise, they are respectively $X$ and $\emptyset$.

Because it is a finite intersection, $\mu(A) = 1$. Similarly, $\mu(B) = 0$ and thus $\mu(X\backslash B) = 1$ so $\mu(A\backslash B) = \mu(A \cap X\backslash B) = 1$, so $A\backslash B \neq \emptyset$.

Take $x \in A\backslash B$. Then $\delta_x \in U$. Indeed $x \in A_j \iff j \in J$.

- $\subseteqq$ Consider $\mu \in \mathcal{M}(X)$ such that $\mu(Y) = \alpha \neq 0, 1$. Then consider the open set $U = \prod_{A\in\mathcal{P}(X)} U_A$, $U_Y = (\frac{\alpha}{2}, \frac{1+\alpha}{2}), U_A = [0,1]$ otherwise. Then $\mu \in U$, but if $\delta_x \in U$, then $\delta_x(Y) \neq 0, 1$. So $U \cap \delta(X) = \emptyset$

Now, consider $\mu \in \beta X$, We define the following ultra-filter

$$\mathcal{F}_\mu = \{A \in \mathcal{P}(X) \mid \mu(A) = 1\}$$

This is indeed an ultra-filter[1]

1. $\mu(\emptyset) = 0 \Rightarrow \emptyset \notin \mathcal{F}_\mu$
2. $A \subseteq B$ and $A \in \mathcal{F}_\mu$. Thus $\mu(A) = 1$, so $\mu(B) = 1$, so $B \in \mathcal{F}_\mu$
3. $A, B \in \mathcal{F}_\mu$. i.e. $\mu(A) = \mu(B) = 1$ and thus
$$\mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cup B) = 2 - 1 = 1$$
and therefore $A \cap B \in \mathcal{F}_\mu$
4. $A \in \mathcal{F}_\mu \iff \mu(A) = 1 \iff \mu(X\backslash A) = 0 \iff$ [2]$X\backslash A \notin \mathcal{F}_\mu$

Now, let $\mathcal{F}$ be any ultra-filter on $X$. We define the following mean in $\beta X$

$$\mu_{\mathcal{F}}(A) = \begin{cases} 1 & \iff & A \in \mathcal{F}(A) \\ 0 & \iff & A \notin \mathcal{F}(A) \end{cases}$$

This is indeed a mean.

1. $X \in \mathcal{F} \Rightarrow \mu(X) = 1$
2. Let $A, B \subseteq X$, $A \cap B = \emptyset$. then $A \in \mathcal{F} \Rightarrow B \notin \mathcal{F}$ and inversely. So if either $A$ or $B \in \mathcal{F}$, then $\mu(A \cup B) = 1 = \mu(A) + \mu(B)$.

   Now, let $A, B \notin \mathcal{F}$, then so does $A \cup B$ and thus $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$.

[1]In fact, this construction is generalisable to any $\mu \in \mathcal{M}(X)$ and we would get a filter (i.e. an ultra-filter without the 4th axiom)

[2]using the fact that $\mu(A) \neq 1 \iff \mu(A) = 0$

Then, by definition $\mathcal{F}_{\mu_{\mathcal{F}}} = \mathcal{F}$ and $\mu_{\mathcal{F}_\mu} = \mu$. So we have proven the bijection.

□ 70

**Definition 71** (Fonction on means)**.**
*Let $f : X \to Y$ be any fonction. This induce a fonction*

$$f_* : M(X) \to M(Y)$$
$$f_*(\mu)(A) = \mu(f^{-1}(A))$$

**Definition 72** (Action on means)**.**
*Let $G \circlearrowright X$ be any group action. This induce a action on $M(X)$*

$$G \circlearrowright \mathcal{M}(X)$$
$$g \circlearrowright \mu(A) = \mu\big(g^{-1} \circlearrowright A\big)$$

**Definition 73** (Amenability)**.**
*Let $G \circlearrowright X$ be any group action. This action is said* ***amenable*** *if $G \circlearrowright \mathcal{M}(X)$ admits an orbit of cardinality 1.*

$$\exists \mu \in \mathcal{M}(X), \forall g \in G, g \circlearrowright \mu = \mu$$

*If $G \circlearrowright G$ the standard action by multiplication is amenable, we say that $G$ is amenable*

**Proposition 74.**
*Let $m \in l^\infty(X)^*$. Any two of those propositions imply the third:*

1. $m \geqslant 0$
2. $\|m\| = 1$
3. $m(1_X) = 1$

*Proof of Proposition* 74:

- $(1) + (2) \Rightarrow (3)$: Using (2), we know that $m(1_X) \leqslant 1$.

  Now, suppose $m(1_X) < 1$. Then, because (2), $\exists f \in l^\infty(X), f \leqslant 1$ s.t. $|m(f)| > |m(1_X)|$. Furthermore, using (1), we can take $f > 0$. Then $1 \geqslant 1_X - f > 0$ and thus

  $$m(f) > m(1_X) = m(1_X - f) + m(f) \geqslant m(f)$$

- $(2) + (3) \Rightarrow (1)$: Suppose $\exists f \in l^\infty(X), 0 < f \leqslant 1$, $m(f) < 0$. Then $1_X - f < 1$ and $m(1_X - f) = m(1_X) - m(f) > 1$ which contradicts (2).

- $(1) + (3) \Rightarrow (2)$: Using (1), we have that $\|m\| = \sup_{0<f\leqslant 1} m(f)$ and $\forall 0 < f \leqslant 1$, $f \leqslant 1_X$, thus $m(f) \leqslant m(1_X)$. So $\|m\| = m(1_X) = 1$

$\square$ 74

**Definition 75** (Linearisation on $\mathcal{M}(X)$)**.**

*Let $X$ be any non-empty set*

$$\mathcal{M}'(X) = \left\{ m \in l^\infty(X)^* |\ m > 0, \|m\| = 1, m(1_X) = 1 \right\}$$

*We give $\mathcal{M}'(X)$ a topology as a restriction of the weak* topology on $l^\infty(X)^*$*

**Theorem 76.**

$$\mathcal{M}(X) \cong \mathcal{M}'(X)$$

## 3.2 Extension of amenability to partial actions

**Definition 77** (Partial action on means)**.**
*Let $G \circlearrowright_p X$ be any partial action. This induce a partial map on $\mathcal{M}(X)$*

$$\star : G \times \mathcal{M}(X) \rightharpoonup \mathcal{M}(X)$$

$$g \star \mu(A) = \frac{1}{\mu(X_{g^{-1}})}\mu\big(g^{-1} \circlearrowright_p (A \cap X_g)\big) \text{ if } \mu(X_{g^{-1}}) \neq 0$$

*with $g \circlearrowright_p : X_{g^{-1}} \cong X_g$ and $g^{-1} \circlearrowright_p : X_g \cong X_{g^{-1}}$ mutual inverses given by the partial action*

**Proposition 78.**
*$- \star -$ is well defined*

*Proof of Proposition* 78:
$g \star \mu \in \mathcal{M}(X)$

- $g \star \mu(X) = \frac{1}{\mu(X_{g^{-1}})}\mu\big(g^{-1} \circlearrowright_p (X_g)\big) = \frac{1}{\mu(X_{g^{-1}})}\mu(X_{g^{-1}}) = 1$
- $A, B \subseteq X, A \cap B = \emptyset$. Then [3]

$$\begin{aligned}
g \star \mu(A \cup B) &= \tfrac{1}{\mu(X_{g^{-1}})}\mu\big(g^{-1} \circlearrowright_p ((A \sqcup B) \cap X_g)\big) \\
&= \tfrac{1}{\mu(X_{g^{-1}})}\mu\big(g^{-1} \circlearrowright_p ((A \cap X_g) \sqcup (B \cap X_g))\big) \\
&= \tfrac{1}{\mu(X_{g^{-1}})}\mu\big(g^{-1} \circlearrowright_p ((A \cap X_g) \sqcup g^{-1} \circlearrowright_p (B \cap X_g))\big) \\
&= \tfrac{1}{\mu(X_{g^{-1}})}\Big(\mu\big(g^{-1} \circlearrowright_p ((A \cap X_g))\big) + \mu\big(g^{-1} \circlearrowright_p (B \cap X_g)\big)\Big) \\
&= g \star \mu(A) + g \star \mu(B)
\end{aligned}$$

□ 78

[3] We are using here the fact that the image of a disjoint union by a bijection is still disjoint

**Proposition 79.**

- *If* $\exists g \star \mu$*, then* $\exists g^{-1} \star (g \star \mu)$*.*

$$g^{-1} \star (g \star \mu)(A) = \frac{1}{\mu(X_{g^{-1}})}\mu(A \cap X_{g^{-1}})$$

- *If* $\exists g \star \mu$ *and* $\exists h \star (g \star \mu)$*, then* $\exists (h.g) \star \mu$ *but we don't always have the equality*

$$\begin{aligned}
(h.g) \star \mu(A) &= \tfrac{1}{\mu(X_{(hg)^{-1}})}\mu((hg)^{-1} \circlearrowleft_p (A \cap X_{(hg)}))\\
h \star (g \star \mu)(A) &= \frac{1}{\mu\left(g^{-1} \circlearrowleft_p (X_h^{-1} \cap X_g)\right)}\mu\left(g^{-1} \circlearrowleft_p \left(h^{-1} \circlearrowleft_p (A \cap X_h)\ \cap X_g\right)\right)\\
&= \frac{1}{\mu\left(g^{-1} \circlearrowleft_p (X_h^{-1} \cap X_g)\right)}\mu\left(\left((hg)^{-1} \circlearrowleft_p A\right) \cap g^{-1} \circlearrowleft_p \left(X_{h^{-1}} \cap X_g\right)\right)
\end{aligned}$$

- *If* $G \circlearrowleft_p X$ *is a standard group action, then* $\star$ *coincide with the induce action on means.*

*Proof of Proposition* 79:
This comes from the definition. See that :

- $g \star \mu(X_g) = \frac{\mu(X_{g^{-1}})}{\mu(X_{g^{-1}})} = 1$

- $\exists g \star \mu$ and $\exists h \star (g \star \mu)$ means that $\mu(X_{g^{-1}}) \neq 0$ and $g \star \mu(X_{h^{-1}}) = \frac{\mu(g^{-1} \circlearrowleft_p (X_{h^{-1}} \times X_g))}{\mu(X_{g^{-1}})} \neq 0$. Because $g^{-1} \circlearrowleft_p (X_{h^{-1}} \times X_g) \subset X_{(hg)^{-1}}$, then $\mu(X_{(hg)^{-1}}) \neq 0$ and thus $\exists (hg) \star \mu$

□ 79

**Definition 80** ($G$-stable means)**.**
*Let* $G \circlearrowleft_p X$ *be any partial action,* $\mu \in \mathcal{M}(X)$*. We say that* $\mu$ *is a* $G$*-stable means if and only if* $\forall g \in G$

$$\mu(X_g) \neq 0 \Rightarrow g \star \mu = \mu$$

**Proposition 81.**
*Let $\mu$ being a $G$-stable mean. Then*

1. $\mu(X_g) \neq 0 \Rightarrow \mu(X_{g^{-1}}) \neq 0$
2. $\mu(X_g) \neq 0 \Rightarrow \mu(X_g) = 1$
3. $A \subseteq X$ *and* $\mu(X_g) \neq 0 \Rightarrow \mu(A) = \mu(A \cap X_g)$
4. $\mu(X_g) \neq 0, \mu(X_h) \neq 0 \Rightarrow \mu(X_{hg}) \neq 0$

*Proof of Proposition* 81:

1. Let $\mu(X_g) \neq 0$. Then
$$\begin{array}{rcl} \mu(X_{g^{-1}}) & = & g^{-1} \star \mu(X_{g^{-1}}) \\ & = & \frac{1}{\mu(X_g)}\mu\big(g \circlearrowright_p (X_{g^{-1}} \cap X_{g^{-1}})\big) \\ & = & \frac{1}{\mu(X_g)}\mu(X_g) = 1 \end{array}$$
So $\mu(X_{g^{-1}}) \neq 0$
2. From the previous point, $\mu(X_g) \neq 0 \Rightarrow \mu(X_{g^{-1}}) = 1$, so by taking $g^{-1}$, $\mu(X_g) = g^{-1} \star \mu(X_g) = 1$
3. Let $A \subset X$ and $\mu(X_g) \neq 0$. then $\mu(A) = g^{-1} \star (g \star \mu)(A) = \mu(A \cap X_g)$ because $\mu(X_{g^{-1}}) = 1$
4. $\mu(X_g) = 1, \mu(X_h) = 1$, then $\mu(X_{h^{-1}}) = 1$ and thus $\mu(X_g \cap X_{h^{-1}}) = 1$.
$$1 = \mu(X_g \cap X_{h^{-1}}) = g \star \mu(X_g \cap X_{h^{-1}}) = \mu(g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}})$$
$g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}}) \subset X_{(hg)^{-1}}$. Therefore
$$\mu(X_{(hg)^{-1}}) = 1 \Rightarrow \mu(X_{hg}) = 1$$

$\square$ 81

**Definition 82** (Amenable partial action)**.**
*Let $G \circlearrowright_p X$ be any partial action. We say that $G \circlearrowright_p X$ is* ***amenable*** *if and only if there exists $\mu$, a $G$-stable mean such that $\forall g \in G, \mu(X_g) \neq 0$*

**Lemma 83.**
*Let $G \circlearrowright_p X$ be any partial action, $G$ finitely generated (i.e. $G =< g_1, \cdots, g_n >$ ). Then*

$$G \circlearrowright_p X \text{ is amenable} \iff \exists \mu,\ \mu\Big(\bigcap_{i=1}^{n} X_{g_i^{-1}}\Big) = 1 \text{ and } g_i \star \mu = \mu$$

*Proof of Lemma* 83:

- $\underline{\Rightarrow}$: By definition
- $\underline{\Leftarrow}$: Let $\nu \in \mathcal{M}(X)$
  - Let $g \in G$, $\nu(X_{g^{-1}}) = 1$, $g \star \nu = \nu$. Then $\nu(X_g) = g \star \nu(X_g) = \nu(X_{g^{-1}}) = 1$ and $\mu(X \backslash X_{g^{-1}}) = 0$

    Thus, $\forall A \subset X$
    $$\begin{array}{rcl} g^{-1} \star \nu(A) & = & g^{-1} \star \big(g \star \nu(A)\big) \\ & = & \nu\big(A \cap X_{g^{-1}}\big) \\ & = & \nu(A) - \nu\big(A \cap (X \backslash X_{g^{-1}})\big) = \nu(A) \end{array}$$
    So $g^{-1} \star \nu = \nu$
  - Let $g, h \in G$, $\nu(X_{g^{-1}}) = \nu(X_{h^{-1}}) = 1$, $g \star \nu = h \star \nu = \nu$. Then $\nu(X_g \cap X_{h^{-1}}) = 1$. Thus
    $$1 = g \star \nu(X_g \cap X_{h^{-1}}) = \nu\big(g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}})\big)$$
    and because $g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}}) \subseteq X_{(hg)^{-1}}$,
    $\nu(X_{(hg)^{-1}}) = 1$
    $\nu(X_{(hg)^{-1}} \backslash g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}})) = \nu(U) = 0$.

    Thus, $\forall A \subset X$
    $$\begin{array}{rcl} (h.g) \star \nu(A) & = & \nu\big((hg)^{-1} \circlearrowright_p (A) \cap X_{(hg)^{-1}}\big) \\ & = & \nu\big((hg)^{-1} \circlearrowright_p (A) \cap g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}})\big) + \nu\Big((hg)^{-1} \circlearrowright_p (A) \cap U\Big) \\ & = & \nu\big((hg)^{-1} \circlearrowright_p (A) \cap g^{-1} \circlearrowright_p (X_g \cap X_{h^{-1}})\big) = h \star \big(g \star \nu(A)\big) = \nu(A) \end{array}$$
    So $(h.g) \star \nu = \nu$

  Because $\mu\big(\bigcap_{i=1}^n X_{g_i^{-1}}\big) = 1$, this imply that for all i, $\mu(X_{g_i^{-1}}) = 1$. Using the fact that any element of $G$ can is given by a combination of $g_i$ and $g_i^{-1}$ and the previous 2 points on $\mu$, we get that $\forall g \in G$, $\mu(X_g^{-1}) = 1$ and $g \star \mu = \mu$. Thus $G \circlearrowright_p X$ is amenable

$\square$ 83

**Corollary 84.**

$$F \circlearrowright M_1 \textit{ is amenable } \iff \exists \mu \in \mathcal{M}(M_1),\ \mu(M_{A^{-1}} \cap M_{B^{-1}}) = 1,\ A \star \mu = B \star \mu = \mu$$

**Proposition 85.**
*Let $G \circlearrowright_p X$ and $G \circlearrowright_p Y$ be two $G$-partial actions and let $f : X \to Y$ be a $G$-equivariant map. Then*

$$G \circlearrowright_p X \textit{ amenable } \Rightarrow G \circlearrowright_p Y \textit{ amenable}$$

*Proof of Proposition* 85:

Let $\mu \in \mathcal{M}(X)$ such that that $\forall g \in G$, $\mu(X_g) = 1$ and $g \star \mu = \mu$. Then, consider $f_*\mu$.

- $\forall g \in G,\ f_*\mu(Y_g) = \mu\Big(f^{-1}(Y_g)\Big) \geqslant \mu\Big(f^{-1}\big(f(X_g)\big)\Big) = \mu(X_g) = 1$

  Then $f_*\mu\big(Y_g \backslash f(X_g)\big) = 0$

- $\forall g \in G,\ \forall A \subset Y$

$$\begin{aligned}
f^{-1}\Big(g \circlearrowright_p \big(A \cap f(X_{g^{-1}})\big)\Big) &= \Big\{x \in X|\ f(x) \in g \circlearrowright_p A \cap f(X_{g^{-1}})\Big\} \\
&= \Big\{x \in X|\ g^{-1} \circlearrowright_p f(x) \in A \cap f(X_{g^{-1}})\Big\} \\
&= \Big\{x \in X|\ f(g^{-1} \circlearrowright_p x) \in A \cap f(X_{g^{-1}})\Big\} \\
&= \Big\{x \in X|\ g^{-1} \circlearrowright_p x \in f^{-1}(A) \cap X_{g^{-1}}\Big\} \\
&= \Big\{x \in X|\ x \in g \circlearrowright_p (f^{-1}(A) \cap X_{g^{-1}})\Big\} \\
&= g \circlearrowright_p (f^{-1}(A) \cap X_{g^{-1}})
\end{aligned}$$

  Using this result, we get

$$
\begin{array}{rcl}
g \star (f_*\mu)(A) & = & f_*\mu\Big(g^{-1} \circlearrowleft_p (A \cap Y_g)\Big) \\
& = & f_*\mu\Big(g^{-1} \circlearrowleft_p \big(A \cap f(X_g)\big) \sqcup g^{-1} \circlearrowleft_p \big(A \cap Y_g \backslash f(X_g)\big)\Big) \\
& = & f_*\mu\Big(g^{-1} \circlearrowleft_p \big(A \cap f(X_g)\big)\Big) + f_*\mu\Big(g^{-1} \circlearrowleft_p \big(A \cap Y_g \backslash f(X_g)\big)\Big) \\
& = & f_*\mu\Big(g^{-1} \circlearrowleft_p \big(A \cap f(X_g)\big)\Big) + 0 \\
& = & \mu\bigg(f^{-1}\Big(g^{-1} \circlearrowleft_p \big(A \cap f(X_g)\big)\Big)\bigg) \\
& = & \mu\Big(g^{-1} \circlearrowleft_p (f^{-1}(A) \cap X_g)\Big) \\
& = & g \star \mu\Big(f^{-1}(A)\Big) \\
& = & \mu\big(f^{-1}(A)\big) \\
& = & f_*\mu(A)
\end{array}
$$

So $f_*\mu$ gives us that $G \circlearrowleft_p Y$ is amenable.

$\square$ 85

**Corollary 86.**

$$F \circlearrowleft M_1 \text{ is amenable } \Rightarrow F \text{ is amenable}$$

*Proof of Corollary* 86:
Consider the following map:

$$\mathbf{\Phi} : M_1 \longrightarrow F$$

$$\mathbf{\Phi}(x) = f_{\mathbb{T}_{n-3},T_x} \text{ if } w(x) = n \geqslant 3$$

$$\mathbf{\Phi}(x) = id \text{ otherwise.}$$

Let's show $\mathbf{\Phi}$ is a $F$-equivariant map.

- $\forall f \in F$, $F_f = F$. So by definition $\mathbf{\Phi}(M_f) \subseteq F$
- If $\exists\ g \circlearrowleft x$, $w(x) \geqslant 3$ then

$$
\begin{array}{rcll}
\mathbf{\Phi}(g \circlearrowleft x) & = & f_{\mathbb{T}_n,g(T_x)} & \\
& = & f_{T_x,g(T_x)} \circ f_{\mathbb{T}_n,T_x} & \text{using 43} \\
& = & g \circ f_{\mathbb{T}_n,T_x} & \text{because } T \to g(T) \text{ represent } g \\
& = & g \circlearrowleft \mathbf{\Phi}(x) &
\end{array}
$$

If $w(x) < 3$, then $\mathrm{Dom}(x) = \{id\}$ and this still works.

Thus, using 85, we get our result.

$\square$ 86

**Lemma 87.**

$$F \textit{ is amenable } \Rightarrow F \circlearrowleft M_1 \textit{ is amenable}$$

*Proof of Lemma* 87:
We can not use the same technique to prove this lemma. Indeed, there is no $F$-equivariant map $\varphi$ from $F$ to $M_1$. Indeed, because $\forall f \in F, F_f = F$, we get that $\forall f \in F, \varphi(F) \subseteq M_f$ and therefore

$$\varphi(F) \subseteq \bigcap_{f \in F} M_f = \emptyset$$

because using 63, we have that $\forall x \in X, \exists g \in F$ such that $x \notin X_g$.
So this result is not obvious and we won't have the time to do it in this report. Although, a prove of it can be found in [2]. This proof is using Følner and marginal sets.

$\square$ 87

## 3.3 Idempotent means

This part is inspired by J. T. Moore papers [3] and[4] and many theorem's proves are just extended versions of his.

### 3.3.1 General properties

**Definition 88** (expectancy by a mean)**.**
*Let $\mu \in \mathcal{M}(X)$, $f \in l^\infty(X)$. We define the* ***expectancy of** $f$ **by** $\mu$*

$$\int_{x \in X} f(x) d\mu(x)$$

1. *If $f$ is simple, that is $f = \sum_{i=0}^{n} a_i \cdot 1_{A_i}$, $a_i \geqslant 0$ then*

$$\int_{x \in X} f(x) d\mu(x) = \sum_{i=0}^{n} a_i \mu(A_i)$$

2. *If* $f > 0$. *Then*

$$\int_{x\in X} f(x)d\mu(x) = \ sup\{ \int_{x\in X} \varphi(x)d\mu(x) \mid \varphi \leqslant f \ \textit{and simple} \ \}$$

3. $\forall f \in l^\infty(X), f = f_+ - f_-$ *both positive fonction. then*

$$\int_{x\in X} f(x)d\mu(x) = \int_{x\in X} f_+(x)d\mu(x) - \int_{x\in X} f_-(x)d\mu(x)$$

**Proposition 89.**
*Let* $\mu \in \mathcal{M}(X)$, $f, h \in l^\infty(X)$, $\lambda \in \mathbb{R}$. *Here is a non exhaustive list of property of the expectancy by a mean:*

1. *Consider* $A \subseteq X$ $f = 1_A$, *then*

$$\int_{x\in X} f(x)d\mu(x) = \mu(A)$$

2. $$f \leqslant h \Rightarrow \int_{x\in X} f(x)d\mu(x) \leqslant \int_{x\in X} h(x)d\mu(x)$$

3. $$\int_{x\in X} \big(f(x) + h(x)\big)d\mu(x) = \int_{x\in X} f(x)d\mu(x) + \int_{x\in X} h(x)d\mu(x)$$

4. $$\int_{x\in X} \lambda.f(x)d\mu(x) = \lambda \int_{x\in X} f(x)d\mu(x)$$

5. $$\int_{x\in X} - \ d\mu(x) \in l^\infty(X)^*$$

6. *Let* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $\forall A \subseteq X$, $\lim \mu_n(A) = \mu(A)$. *Then*

$$\lim \int_{x\in X} - \ d\mu_n(x) = \int_{x\in X} - \ d\mu(x)$$

*Proof.* Any good book on Probability Theory, for exemple [5]. □

**Definition 90** (Induced composition law on $\mathcal{M}(X)$)**.**
*Let $(X, *)$ be any magma, $f \in l^\infty(X)$. The existence of this law induce a composition law on $\mathcal{M}(X)$*

$$* : \mathcal{M}(X) \times \mathcal{M}(X) \to \mathcal{M}(X)$$
$$\mu * \nu(f) = \int_{s\in X} \Big(\int_{t\in X} f(s*t)\ d\nu(t)\Big) d\mu(s)$$

**Proposition 91.**
*The composition law $*$ on $\mathcal{M}(X)$ is well defined.*

*Proof of Proposition* 91:

- $\mu * \nu \in l^\infty(X)^*$:

$$\begin{array}{rcl} \mu * \nu((\lambda.f) + h) & = & \int_{s\in X} \Big(\int_{t\in X} \lambda.f(s*t) + h(s*t)\ d\nu(t)\Big) d\mu(s) \\ & = & \int_{s\in X} \Big(\int_{t\in X} \lambda.f(s*t) + \int_{t\in X} h(s*t)\ d\nu(t)\Big) d\mu(s) \\ & = & \int_{s\in X} \Big(\int_{t\in X} \lambda.f(s*t)\Big) d\mu(s) + \int_{s\in X} \Big(\int_{t\in X} h(s*t)\ d\nu(t)\Big) d\mu(s) \\ & = & \lambda. \int_{s\in X} \Big(\int_{t\in X} f(s*t)\Big) d\mu(s) + \int_{s\in X} \Big(\int_{t\in X} h(s*t)\ d\nu(t)\Big) d\mu(s) \\ & = & \lambda.\mu * \nu(f) + \mu * \nu(h) \end{array}$$

$$\begin{array}{rcl} \mu * \nu(f) & = & \int_{s\in X} \Big(\int_{t\in X} f(s*t)\ d\nu(t)\Big) d\mu(s) \\ & \leqslant & \int_{s\in X} \Big(\int_{t\in X} \|f\|_\infty 1_X(s*t)\ d\nu(t)\Big) d\mu(s) \\ & = & \int_{s\in X} \|f\|_\infty 1_X(s)\ d\mu(s) \\ & = & \|f\|_\infty \end{array}$$

- $\mu * \nu \in \mathcal{M}(X)$. Let $f \geqslant 0$

$$\begin{array}{rcl} \mu * \nu(1_X) & = & \int_{s\in X} \Big(\int_{t\in X} 1_X(s*t)\ d\nu(t)\Big) d\mu(s) \\ & = & \int_{s\in X} \Big(\int_{t\in X} 1_X(s) \cdot 1_X(t)\ d\nu(t)\Big) d\mu(s) \\ & = & \int_{s\in X} 1_X(s).\nu(X)\ d\mu(s) \\ & = & \mu(X) \\ & = & 1 \end{array}$$

$$\begin{aligned}
\mu * \nu(f) &= \int_{s\in X}\Big(\int_{t\in X} f(s*t)\ d\nu(t)\Big)d\mu(s)\\
&\geqslant \int_{s\in X}\Big(\int_{t\in X} 0\ d\nu(t)\Big)d\mu(s)\\
&= 0
\end{aligned}$$

□ 91

**Proposition 92.**
*Let $(X,*)$ be any magma, $\mu,\nu \in \mathcal{M}(X)$ and $A, B \subset X$. Then*

$$\mu * \nu(A * B) = \mu(A)\nu(B)$$

*Proof of Proposition* 92:

We have that $1_{A*B}(a*b) = 1_A(a).1_B(b)$. Thus

$$\begin{aligned}
\mu * \nu(A * B) = \mu * \nu(1_{A*B}) &= \int_{s\in X}\Big(\int_{t\in X} 1_{A*B}(s*t)\ d\nu(t)\Big)d\mu(s)\\
&= \int_{s\in X}\Big(\int_{t\in X} 1_A(s)1_B(t)d\nu(t)\Big)d\mu(s)\\
&= \Big(\int_{s\in X} 1_A(s)d\mu(s)\Big)\cdot\Big(\int_{t\in X} 1_B(t)d\nu(t)\Big)\\
&= \mu(A)\nu(B)
\end{aligned}$$

□ 92

**Proposition 93.**
*Let $(X,*)$ be any magma, $A \subseteq X$, $\nu \in \mathcal{M}(X)$ and $c \in [0,1]$ such that $\forall s \in X$, $\nu\big(\{t \in X \mid s*t \in A\}\big) = c$ . Then $\forall \mu \in \mathcal{M}(X)$*

$$\mu * \nu(A) = c$$

*Proof of Proposition* 93:

$$\begin{aligned}
\mu * \nu(A) &= \int_{s\in X}\Big(\int_{t\in X} 1_A(s*t)\ d\nu(t)\Big)d\mu(s)\\
&= \int_{s\in X}\nu\big(\{t\in X \mid s*t\in A\}\big)d\mu(s)\\
&= \int_{s\in X} c\ d\mu(s)\\
&= c
\end{aligned}$$

□ 93

**Proposition 94.**
*Let $(X, *)$ be any magma, Then the restriction of the induced composition law on $\beta X$ is closed. That is $\forall \mu, \nu \in \beta X, \mu * \nu \in \beta X$*

*Proof of Proposition* 94:
Consider $\mathcal{F}_\nu, \mathcal{F}_\mu$ the ultra-filters equivalent to $\nu, \mu$. Remember that

$$\beta X = \Big\{\mu \in \mathcal{M}(X) | \forall A \subseteq X, \mu(A) = \{0,1\}\Big\}$$

$$\begin{array}{rcl}
\mu * \nu(A) & = & \int_{s \in X} \left( \int_{t \in X} 1_A(s * t) \; d\nu(t) \right) d\mu(s) \\
& = & \int_{s \in X} \nu\Big(\{t \in X \mid s * t \in A\}\Big) d\mu(s) \\
& = & \int_{s \in X} 1_{\big\{\{t \in X \ \mid \ s * t \in A\} \in \mathcal{F}_\nu\big\}}(s) \; d\mu(s) \\
& = & \mu\Bigg(\Big\{s \in X | \big\{t \in X \mid s * t \in A\big\} \in \mathcal{F}_\nu\Big\}\Bigg) \in \{0,1\}
\end{array}$$

We have therefore the form of $\mathcal{F}_{\mu * \nu}$

$$A \in \mathcal{F}_{\mu * \nu} \iff \Big\{s \in X | \big\{t \in X \mid s * t \in A\big\} \in \mathcal{F}_\nu\Big\} \in \mathcal{F}_\mu$$

Furthermore, using previous proposition, if $A \in \mathcal{F}_\mu, B \in \mathcal{F}_\nu$, then $\mu * \nu(A * B) = \mu(A)\nu(B) = 1$, so $A * B \in \mathcal{F}_{\mu * \nu}$. Thus

$$\mathcal{F}_\mu * \mathcal{F}_\nu \subset \mathcal{F}_{\mu * \nu}$$

$\square$ 94

**Definition 95** (Idempotent means)**.**
*Let $(X, *)$ be any magma, $\mu \in \mathcal{M}(X)$. We say that a mean is **idempotent** if*

$$\mu * \mu = \mu$$

**Lemma 96.**
*Let $(X, *)$ be any magma with $*$ associative (i.e. a semigroup). Then the induced composition law on $\beta X$ is also associative.*

$$\eta * (\mu * \nu) = (\eta * \mu) * \nu$$
$$\mathcal{F}_{\eta*(\mu*\nu)} = \mathcal{F}_{(\eta*\mu)*\nu}$$

*Proof of Lemma* 96: We will do this proof without any form of subtlety

$$\begin{aligned}
A \in \mathcal{F}_{\eta*(\mu*\nu)} &\Leftrightarrow \Big\{s \in X \big| \{t \in X | s * t \in A\} \in \mathcal{F}_{\mu*\nu}\Big\} \in \mathcal{F}_\eta \\
&\Leftrightarrow \Bigg\{s \in X \Bigg| \Bigg\{u \in X \Bigg| \Big\{v \in X | u * v \in \big\{t \in X | s * t \in A\big\}\Big\} \in \mathcal{F}_\nu\Bigg\} \in \mathcal{F}_\mu\Bigg\} \in \mathcal{F}_\eta \\
&\Leftrightarrow \Bigg\{s \in X \Bigg| \Bigg\{u \in X \Bigg| \Big\{v \in X | s * u * v \in A\Big\} \in \mathcal{F}_\nu\Bigg\} \in \mathcal{F}_\mu\Bigg\} \in \mathcal{F}_\eta \\
A \in \mathcal{F}_{(\eta*\mu)*\nu} &\Leftrightarrow \Big\{s \in X \big| \{t \in X | s * t \in A\} \in \mathcal{F}_\nu\Big\} \in \mathcal{F}_{\eta*\mu} \\
&\Leftrightarrow \Bigg\{u \in X \Bigg| \Bigg\{v \in X \Bigg| u * v \in \Big\{s \in X \big| \{t \in X | s * t \in A\} \in \mathcal{F}_\nu\Big\}\Bigg\} \in \mathcal{F}_\mu\Bigg\} \in \mathcal{F}_\eta \\
&\Leftrightarrow \Bigg\{u \in X \Bigg| \Bigg\{v \in X \Bigg| \Big\{t \in X | u * v * t \in A\Big\} \in \mathcal{F}_\nu\Bigg\} \in \mathcal{F}_\mu\Bigg\} \in \mathcal{F}_\eta
\end{aligned}$$

□ 96

**Theorem 97** (Ellis-Numakura lemma)**.**

*Let $(X, *)$ be a non-empty semigroup with $\mathcal{T}$ a topology such that $X$ is compact, $T_1$ and the maps $- * x$: $y \to y * x$ are continuous for all $x \in X$. Then, $(X, *)$ has a nilpotent element*

$$\exists\ p \in X \text{ such that } p * p = p$$

*Proof of Theorem* 97:
Let $\mathcal{E}$ be the following family of subset of $X$.

$$\mathcal{E} = \Big\{A \subseteq X |\ A \neq \emptyset, \text{compact and } A * A \subseteq A\Big\}$$

and let $\mathcal{E}$ be a poset under the following partial order relation $A \leqslant B \iff B \subseteq A$. $X \in \mathcal{E}$.
Now, consider any chain $\mathcal{C}$ in $\mathcal{E}$. It has an upper bound. Indeed, consider $V = \bigcap_{A \in \mathcal{C}} A$. Because $A$ are closed in a compact space and with non empty finite intersection [4], by compacity of $X$, $V$ is also a non empty compact space. Furthermore:

$$V * V = \Big(\bigcap_{A \in \mathcal{C}} A\Big) * \Big(\bigcap_{B \in \mathcal{C}} B\Big) = \bigcap_{A,B \in \mathcal{C}} A * B \subseteq \bigcap_{A \in \mathcal{C}} A * A \subseteq \bigcap_{A \in \mathcal{C}} A = V$$

Thus, $V \in \mathcal{E}$ and $\forall A \in \mathcal{C}, A \leqslant V$. So $V$ is the upper bound of $\mathcal{C}$ and thus $(\mathcal{E}, \leqslant)$ is an inductive poset.
Using **Zorn's lemma** ( a.k.a the greatest lemma), we get that there exists a maximal in $(\mathcal{E}, \leqslant)$, named $A$.

Now, let $r \in A$ and consider $A * r$. Because it is the image of a compact space by a continuous function, it is itself a non empty compact subset of $X$. Furthermore:

$$A * r \subseteq A * A \subseteq A$$
$$(A * r) * (A * r) \subseteq \big((A * r) * A\big) * r \subseteq (A * A) * r \subseteq A * r$$

Thus, $A \leqslant A * r$ so by maximality $A = A * r$. Thus there exists $p \in A,\ p * r = r$.

Now, consider $L = \Big\{a \in A |\ a * r = r\Big\}$. It is a non empty subset of $A$ and because this is the preimage of a singleton in a $T_1$ topology, it is closed and thus compact. Furthermore, if $p, q \in L$, then $(p * q) * r = p * (q * r) = p * r = r$ and thus $L * L \subseteq L$. So $A \leqslant L$ and by maximality $L = A$. Thus:

$$r * r = r$$

$\square$ 97

**Corollary 98** (Idempotent ultrafilters)**.**

*Let $(X, *)$ be a non-empty semigroup. Then $\exists \mu \in \beta X$ such that $\mu$ is idempotent.*

*Proof of Corollary* 98:

We have already proven that $(\beta X, *)$ was a non-empty semigroup. Furthermore, because it is closure in $\mathcal{M}(X)$, a compact Haudorff set, so it is itself compact and $T_1$. We just

[4] because this is a chain, in the finite case, $\exists A$ maximal for $\leqslant$ and thus the intersection is non empty.

have to show that $\forall \nu \in \beta X$, $- * \nu$ is continuous on $\beta X$.

Let's show $\forall \nu \in \mathcal{M}(X)$, $- * \nu$ is continuous on $\mathcal{M}(X)$. Using weak* topology, let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of means such that $\forall A \subseteq X$, $\lim \mu_n(A) = \mu(A)$. $\forall f \in l^\infty(X)$, we define $\Delta_f(s) = \int_{t\in X} f(s*t)d\nu(t) \in l^\infty(X)$. Then, $\forall f \in l^\infty(X)$,

$$\lim \mu_n * \nu(f) = \lim \int_{s\in X} \Delta_f(s) d\mu_n(s) = {}^5 \int_{s\in X} \Delta_f(s) d\mu(s) = \mu * \nu(f)$$

Thus, $- * \nu$ is continuous.

Then, by using Ellis-Numakura lemma on $(\beta X, *)$, we get our nilpotent ultra-filter.

□ 98

### 3.3.2 From idempotent to amenable

**Theorem 99.**

$$\exists \mu \in \mathcal{M}(M_1) \text{ s.t. } \mu \text{ idempotent} \Rightarrow F \text{ is amenable}$$

*Proof of Theorem* 99:
Let $\mu$ be idempotent, then

$$\mu(\{1\}) = \mu(M_1 \backslash (M_1 * M_1)) = \mu(M_1) - \mu(M_1 * M_1) = 1 - \mu(M_1)\mu(M_1) = 0$$

$$\mu(\{1\} * M_1) = \mu(M_1 * \{1\}) = \mu(M_1)\mu(\{1\}) = 0$$

Now, using 50, we know the forms of $M_{A^{-1}}, M_A, M_{B^{-1}}$ and $M_B$.

$$M_{A^{-1}} = M_1 * (M_1 * M_1),\ M_A = (M_1 * M_1) * M_1$$
$$M_{B^{-1}} = M_1 * \big(M_1 * (M_1 * M_1)\big),\ M_{B^{-1}} = M_1 * \big((M_1 * M_1) * M_1\big)$$

$$A \circlearrowright \Big(a * (b * c)\Big) = (a * b) * c,\quad B \circlearrowright \Big(a * \big(b * (c * d)\big)\Big) = a * \big((b * c) * d\big)$$

Then $M_1 \backslash M_{A^{-1}} = \{1\} \sqcup M_1 * \{1\}$, $M_1 \backslash M_A = \{1\} \sqcup \{1\} * M_1$, so

$$\mu(M_{A^{-1}}) = \mu(M_A) = 1.$$

[5] point 6 of 89

Furthermore, because $\mu$ is idempotent,

$$\mu = \mu * (\mu * \mu) = (\mu * \mu) * \mu$$

$$\mu = \mu * \big(\mu * (\mu * \mu)\big) = \mu * \big((\mu * \mu) * \mu\big)$$

with a bit of computation, we see that

$$\mu * (\mu * \mu)(A) = \int_{x \in M_1} \int_{y \in M_1} \int_{z \in M_1} 1_A\big(x * (y * z)\big) d\mu(z) d\mu(y) d\mu(x)$$

$$(\mu * \mu) * \mu(A) = \int_{x \in M_1} \int_{y \in M_1} \int_{z \in M_1} 1_A\big((x * y) * z\big) d\mu(z) d\mu(y) d\mu(x)$$

Thus, $\mu * (\mu * \mu)(E) = (\mu * \mu) * \mu\big(A^{-1} \circlearrowright (E \cap M_A)\big)$

And similarly, $\mu * \big(\mu * (\mu * \mu)\big)(E) = \mu * \big((\mu * \mu) * \mu\big)(B^{-1} \circlearrowright (E \cap M_B))$.

Then, $\forall E \subseteq M_1$,

$$\begin{array}{rcl} \mu(E) & = & \mu(E \cap M_{A^{-1}}) \\ & = & \mu\Big(\big\{a * (b * c) \in E |\ a, b, c \in M_1\big\}\Big) \\ & = & \mu * (\mu * \mu)\big(E\big) \\ & = & (\mu * \mu) * \mu\big(A^{-1} \circlearrowright (E \cap M_A)\big) \\ & = & \mu\big(A^{-1} \circlearrowright (E \cap M_A)\big) \\ & = & A \star \mu(E) \end{array}$$

Thus, $\mu = A \star \mu$ and by the proof of 85, $\mu = A^{-1} \star \mu$.

Now, because $M_{B^{-1}} \subset M_{A^{-1}}$, we can see that

$$M_{A^{-1}} \backslash M_{B^{-1}} = M_1 * (M_1 * \{1\})$$

and thus, $A \circlearrowright (M_{A^{-1}} \backslash M_{B^{-1}}) = (M_1 * M_1) * \{1\}$

$$\begin{array}{rcl} \mu(M_1 \backslash M_{B^{-1}}) & = & \mu\big(M_1 \backslash M_{A^{-1}}\big) + \mu\big(M_1 \backslash M_{B^{-1}}\big) \\ & = & 0 + A^{-1} \star \mu\big(M_1 \backslash M_{B^{-1}}\big) \\ & = & \mu\big((M_1 * M_1) * \{1\}\big) \\ & \leqslant & \mu\big(M_1 * \{1\}\big) = 0 \end{array}$$

Thus, $\mu(M_{B^{-1}}) = 1$ (and similarly, $\mu(M_B) = 1$). Now, $\forall E \subseteq M_1$,

$$\begin{aligned}
\mu(E) &= \mu(E \cap M_{B^{-1}}) \\
&= \mu\Big(\Big\{a * \big(b * (c * d)\big) \in E \mid\ a, b, c, d \in M_1\Big\}\Big) \\
&= \mu * \big(\mu * (\mu * \mu)\big)\Big(E\Big) \\
&= \mu * \big((\mu * \mu) * \mu\big)\Big(B^{-1} \circlearrowright (E \cap M_B)\Big) \\
&= \mu\big(B^{-1} \circlearrowright (E \cap M_B)\big) \\
&= B \star \mu(E)
\end{aligned}$$

Thus, $\mu = B \star \mu$.

Using 84 and 86, we get that $F$ is amenable.

□ 99

### 3.3.3 Idempotent means on $M_X$

**Lemma 100.** *Let $(M_X, *)$ be the free magma on $X$.*

*Then there is no idempotent ultra-filter on $M_X$*

*Proof of Lemma* 100:
Suppose $\mu$ is an idempotent ultrafilter. We define by recursively the following function

$$\# : M_X \to \mathbb{N}$$

$$\#|_{X_1} = 0 \quad , \quad \#(a * b) = \#(a) + 1$$

We define $X_e = \{m \in M_X \mid \#(m) \text{ is even}\}$,
$X_o = M_X \backslash X_e = \{m \in M_X \mid \#(m) \text{ is odd}\}$.
Due to the nature of $M_X$ and $*$, $X_e * X = X_o$. Then

$$\mu(X_o) = \mu * \mu(X_e * X) = \mu(X_e).\mu(X) = \mu(X_e) = 1 - \mu(X_o)$$

So $\mu(X_o) = \frac{1}{2}$ but this is in contradiction with $\mu \in \beta X$

□ 100

**Definition 101** ($T_n$ sets)**.**
*Let $(M_X, *)$ be the free magma on $X$. $X_n = \{m \in M_X | w(m) = n\}$. We define recursively the family of set $\{T_n^k\}_{n\in\mathbb{N}}^{k\in\mathbb{N}*}$*

$$\forall k, T_0^k = X_k$$

$$\forall n > 0, T_n^1 = \emptyset$$

$$T_{n+1}^k = \bigcup_{i=1}^{k-1} \bigcap_{j=1}^{i-1} \left(X_i \backslash X_j * T_j^{i-j}\right) * T_n^{k-i}$$

*We also befine $\{Z^k\}_{k\in\mathbb{N}*}$*

$$Z^k = \bigcup_{i=1}^{k-1} X_i * T_i^{k-i}$$

*We finally define $\{T_n\}_{n\in\mathbb{N}}$, and $Z$:*

$$T_n = \bigcup_{k\in\mathbb{N}*} T_n^k$$

$$Z = \bigcup_{k\in\mathbb{N}*} Z^k$$

**Proposition 102.**
*$\{T_n\}$ sets have a few interesting properties :*

1. *$\{T_n^k\}_{n\in\mathbb{N}}^{k\in\mathbb{N}*}$ are well defined*
2. *$\forall m \in T_n^k$ or $Z^k$, $w(m) = k$*
3. *$\forall n \in \mathbb{N}, T_{n+1} = (M_X \backslash Z) * T_n$*
4. *$Z = \bigcup_{n\in\mathbb{N}} X_n * T_n$*
5. *$\forall n \in \mathbb{N}, T_{n+1} \subseteq T_n$*

*Proof of Proposition* 102:

1. We prove by recursively on $n+k$

   - $n+k=1$. The only case is $n=0, k=1$. Then $T_0^1 = X_1 = X$
   - $n+k=2$ Then either $T_1^1 = \emptyset$ or $T_0^2 = X_2 = X * X$
   - Now, $\forall n+k \leqslant m$, $T_n^k$ is defined. Consider $n+k = m+1$. Then

   $$T_n^k = \bigcup_{i=1}^{k-1} \bigcap_{j=1}^{i-1} \left(X_i \backslash X_j * T_j^{i-j}\right) * T_{n-1}^{k-i}$$

   $T_n^k$ is well defined if the $T_v^u$ defining it are too. But $i-j+j = i < k < n+k$ and $k-i+n-1 < n+k$. So they are all inferior to $m$, so they are all well defined. So $\forall n+k = m+1$, $T_n^k$ is well defined.

2. We prove by recursively on $n+k$

   - $n+k=1$. The only case is $n=0, k=1$. Then $T_0^1 = X_1 = X$ and $\forall x \in X$, $w(x) = 1$
   - $n+k=2$. Because $T_1^1 = \emptyset$, the only case is $T_0^2 = X_2 = X * X$ and $\forall x \in X$, $w(x) = 1$
   - $\forall n+k \leqslant m$ $\forall x \in T_n^k$, $w(x) = k$. Then consider $n+k = m+1$

   $$T_n^k = \bigcup_{i=1}^{k-1} \bigcap_{j=1}^{i-1} \left(X_i \backslash X_j * T_j^{i-j}\right) * T_{n-1}^{k-i}$$

   $\forall x \in T_n^k, x = a * b, a \in X_i, b \in T_{n-1}^k - i$ then $w(x) = w(a) + w(b) = i + k - i = k$

   Similarly $m \in Z^k = \bigcup_{i=1}^{k-1} X_i * T_i^{k-i}$, so $m = a * b, a \in X_i, b \in T_i^{k-i}$. Then $w(m) = w(a) + w(b) = i + k - i = k$

3. First we can see that

$$T_{n+1}^k = \bigcup_{i=1}^{k-1} \bigcap_{j=1}^{i-1} \left(X_i \backslash X_j * T_j^{i-j}\right) * T_n^{k-i} =$$

$$= \bigcup_{i=1}^{k-1} \left(X_i \backslash (\bigcup_{j=1}^{i-1} X_j * T_j^{i-j}\right) * T_n^{k-i} = \bigcup_{i=1}^{k-1} \left(X_i \backslash Z^i\right) * T_n^{k-i}$$

$$T_{n+1} = \bigsqcup_{k\in\mathbb{N}^*} T_{n+1}^k = \bigsqcup_{k\in\mathbb{N}^*} \bigcup_{i=1}^{k-1} \left(X_i \backslash Z^i\right) * T_n^{k-i} = \bigsqcup_{k\in\mathbb{N}^*} \bigcup_{i\in\mathbb{N}} \left(X_i \backslash Z^i\right) * T_n^{k-i6} =$$

$$= \bigcup_{i\in\mathbb{N}} \bigsqcup_{k\in\mathbb{N}^*} \left(X_i \backslash Z^i\right) * T_n^{k-i} = \bigcup_{i\in\mathbb{N}} \left(X_i \backslash Z^i\right) * T_n = \left(M_X \backslash Z\right) * T_n$$

4.

$$Z = \bigsqcup_{k\in\mathbb{N}^*} Z^k = \bigsqcup_{k\in\mathbb{N}^*} \bigcup_{i=1}^{k-1} X_i * T_i^{k-i} = \bigsqcup_{k\in\mathbb{N}^*} \bigcup_{i\in\mathbb{N}^*} X_i * T_i^{k-i} =$$

$$= \bigcup_{i\in\mathbb{N}^*} \bigsqcup_{k\in\mathbb{N}^*} X_i * T_i^{k-i} = \bigcup_{i\in\mathbb{N}^*} X_i * T_i$$

5. This is proved by recursion

   - Let $n = 0$, then $T_0 = M_X$ and by definition $T_1 \subseteq M_X$, so $T_1 \subseteq T_0$
   - Consider $T_n \subseteq T_{n-1}$. Then

$$T_{n+1} = (M_X \backslash Z) * T_n \subseteq (M_X \backslash Z) * T_{n-1} = T_n$$

□ 102

**Theorem 103.** *Let $(M_X, *)$ be the free magma on $X$.*

*There exist no idempotent means on $M_X$.*

*Proof of Theorem* 103:
Let $\mu \in \mathcal{M}(M_X)$ be an idempotent mean. We define $r = \mu(Z)$. Then, it has a few properties :

- $\mu(X) = \mu(X_1) = \mu(M_X \backslash (M_X * M_X)) = 1 - \mu(M_X * M_X) = 1 - \mu(M_X)\mu(M_X) = 0$
- $\forall n \in \mathbb{N}^*, \mu(X_n) = \mu(\bigsqcup_{i+j=n} X_i * X_j) = \sum_{i+j=n} \mu(X_i)\mu(X_j)$ and thus by induction,

$$\forall n \in \mathbb{N}^*, \quad \mu(X_n) = 0$$

- $\forall n \in \mathbb{N}, \mu(T_{n+1}) = \mu\big((M_X \backslash Z) * T_n\big) = (1-r)\mu(T_n)$ and $T_0 = M_X$. Thus, by induction

$$\forall n \in \mathbb{N}, \mu(T_n) = (1-r)^n$$

[6] whenever this is defined

Now, if $r > 0$, then $\exists n \in \mathbb{N}$, $(1-r)^n < r$. Then

$$Z = \bigcup_{k<n} S_k * T_k \cup \bigcup_{k=n}^{\infty} S_k * T_k \subseteq \bigcup_{k<n} S_k * T_k \cup \bigcup_{k=n}^{\infty} S_k * T_n$$

$$\text{So } r = \mu(Z) \leqslant \mu\big(\bigcup_{k<n} S_k * T_k\big) + \mu\big(\bigcup_{k=n}^{\infty} S_k * T_m\big) \leqslant$$

$$\leqslant \sum_{k<n} \mu(S_k * T_k) + \mu(M_X * T_m) \leqslant (1-r)^n$$

Thus , $r = \mu(Z) \leqslant (1-r)^n < r$ a contradiction.
Now, if $r = 0$, then $\forall s \in M_X$

$$\mu\big(\{t \in M_X | s * t \in Z\}\big) = \mu(T_{w(s)}) = (1-r)^{w(s)} = 1$$

$$\text{Thus}, 0 = \mu(Z) = \mu * \mu(Z) = 1 \text{ using } 93$$

Thus, $Z$ can not have a value, which imply that $\mu$ can not be a idempotent mean. $\square$ 103

# Conclusion

So, we see that the idea to find an idempotent mean on $M_1$ to solve the question of the amenability on $F$ is a dead end, due to there lack of existence. However, this attempt yields interesting results, such as our extension of amenability to partial actions as well as the the amenability equivalence between $F$ and its action onto $M_1$ ; constructions that might find use in the future.
We will put here an end to this work, although it we can continue by taking a look at [1] to discover further properties of $F$ and the others Thompson's group. [2], [3] and [4] are also very interesting reading, although those require some further notions of amenability.

I would like to thank Nicolas Monod for giving me this project and for his continious support despite the covid crisis, that upset our organisation.
Oh, and to answer our opening statement, we sadly can't use paradoxical decompositions onto trees because even in case of existence of those decompositions, they are highly non-constructible, and therefore sadly impractical in our world.

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