Four Color Theorem

Maximiliano Eaton

May 2022

Contents

1 Introduction

1.1 History

The four color theorem (hereon referred to as 4CT) simply states that any map can be colored using at most 4 colors in such a way that no adjacent regions share a color. The theorem was popularized in the 1800s by Augustus DeMorgan when approached by Francis Guthrie who had determined the problem to simply be accepted as a postulate after several failures at attempted proofs. Much in the same way of Fermat's Last Theorem (which simply states that no such positive integers a, b, c satisfies $a^n + b^n = c^n$ for any integer $n > 2$), the simplicity of the 4CT with an incomparably difficult proof caught the attention of many mathematicians.

However, in 1879 Alfred Kempe derived a proof for the 4CT which had been largely accepted until it's disproof in 1890 by Percy John Heawood by means of a counterexample. However, his erroneous proof was adapted into a proof for the five color theorem (5CT; a theorem with the same principles as the 4CT except accepts 5 colors as opposed to 4) which led many to pursue proofs for the original conjecture of 4 colors in much the same vein in vain.

It was not until 1976 that a proof that has been accepted was presented by Kenneth Appel and Wolfgang Haken by relying on an exhaustive analysis of the problem by addressing every case through assistance of a computer. After proving that every planar graph can be contained into a set of 1834 unavoidable configurations which could then be reduced into a simpler graph.

Much debate was sparked by the mathematics community since it was the first major proof that was computer assisted and impossible to be checked by hand. Not just had the proof not given any new insight into the problem, the philosophical issue with the proof was that if we were to accept a proof that we ourselves could not confirm can we definitively consider it proved. More proofs have been published, further simplifying the original proof, but none that do not heavily rely on computers.

Thus the implication of the proof was two fold: it served as a precedent that paved the way for the acceptance of computer assisted proofs such as Hale's proof of the Kepler's Conjecture as well as becoming an important theorem with applications in both mathematics and computer science such as register allocation by coloring for compiler optimizations.[1]

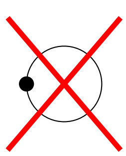
[1] Wikipedia, "Four Color Theorem", last modified 2022, https://en.wikipedia.org/wiki/Four_color_theorem.

1.2 Graphs

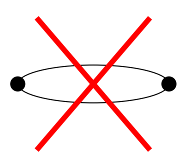
Consider the following map of South America. In this form, it is difficult to discern the relationships between regions. We can convert this into a planar graph, replacing regions for vertices and borders for edges, thus providing us with an abstracted version of the map.

[2]

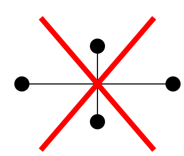
All that differs from the original theorem is that now we must show that vertices connected by an edge must be a different color instead of that of adjacent regions. Vertices connected by edges will continue to be referred to as adjacent and the number of such connections will be referred to as the degree of a vertex. To comply with the characteristics of a map, the graph must be both simple (meaning that there can be neither loops nor parallel edges) and planar (meaning that edges cannot overlap).



no loops



no parallel edges



no edge overlap

Hereon, we will interchangeably use terms referring to graphs with that of maps (i.e. vertices and regions). Furthermore, vertices, edges, and faces (which are any regions surrounded by edges including the infinite outer region) will be denoted as V, E and F respectively. With these clarifications made, we can examine the proofs.

[2]All graphics created using LaTeX.

2 Kempe's Proof

2.1 Working Backwards

Alfred Kempe had developed a proof which relied on a recursive process inducting on the number of regions. But first, we must show a property of a planar graph that will be integral to the proof.

Theorem 2.1 (Euler's Formula) *V-E+F=2 for every planar graph, where V denotes the number of vertices, E denotes the number of edges connecting vertices, and F denotes the number of faces bounded by edges.*

Proof 2.1.1 *By induction on the number of edges. For the base case, there are no edges thus leaving an isolated vertex thus V-E+F=1-0+1=2. For the inductive step, let e denote an edge in the k+1th case and consider two cases: if the edge connects 2 vertices, shorten the edge to a magnitude of 0 thereby combining vertices into 1, resulting in the number of vertices and edges decreasing while the number of faces remains constant; else if the edge connects to its original vertex, remove the edge thereby combining 2 faces, resulting in the number of vertices and faces while the number of vertices remains constant. Either way, the induction holds.*

Theorem 2.2 (Five Neighbors Theorem) *For every planar graph there exists 1 vertex that has at most a degree of 5.*

Proof 2.2.1 *Since a face must be made up of at least 3 edges, and each edge is shared between 2 faces, we can set up the following inequality:*

$$E \geq \frac{3}{2}F$$

which is equivalent to

$$F \leq \frac{2}{3}E$$

By way of contradiction, assume that every vertex has a degree greater or equal to 6. Since each edge is shared between 2 vertices, we can once again set up an inequality:

$$E \geq \frac{6}{2}V$$

which is equivalent to

$$V \leq \frac{1}{3}E$$

Then by theorem 2.1 we can combine the inequalities:

$$V - E + F \leq \frac{1}{3}E - E + \frac{2}{3}E = 0$$

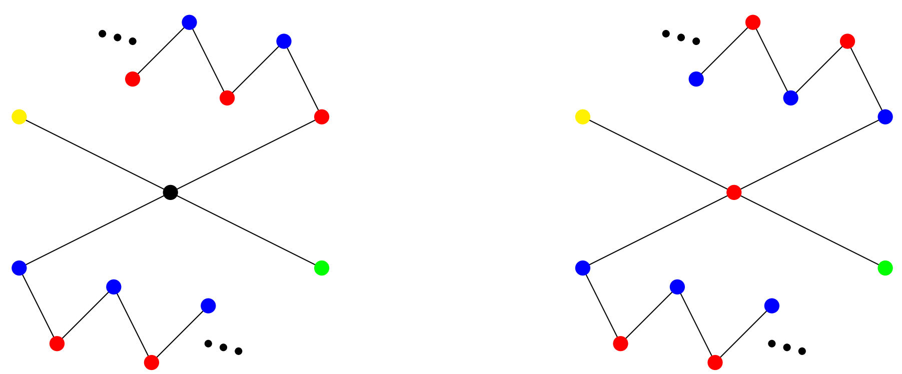
Since 2 is not included in this range, there is a contradiction thus proving the existence of a vertex with degree 5.

Based on this property, proving that every vertex of 5 degrees or less can be 4 colorable will prove the theorem.

Theorem 2.3 (Four Color Theorem) [3] *Every simple, planar graph is four colorable. Degree 1, 2, 3 are trivial since there always remains an avaiable color. Consider the remaining cases for a vertex of 5 degrees or less:*

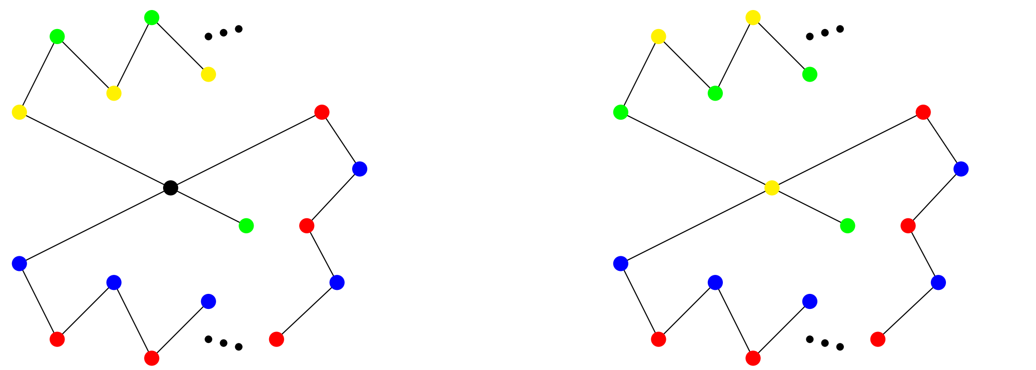
Case 2.3.1 (Degree 4) *Use remaining color if not all used in adjacent vertices. Otherwise, we will have to consider what is now known as a* Kempe Chain*. 2 vertices on opposite sides with different colors could be isolated into a subgraph consisting of an alternation of the 2 colors. These subgraphs could be disjointed or eventually connect creating a cyclical* Kempe Chain*:*

Subcase 2.3.1.1 (Not Cyclical) *Since the subgraphs never intersect, switching the colors for 1 of them would be inconsequential. In doing so, a color is available now for the center vertex to use.*



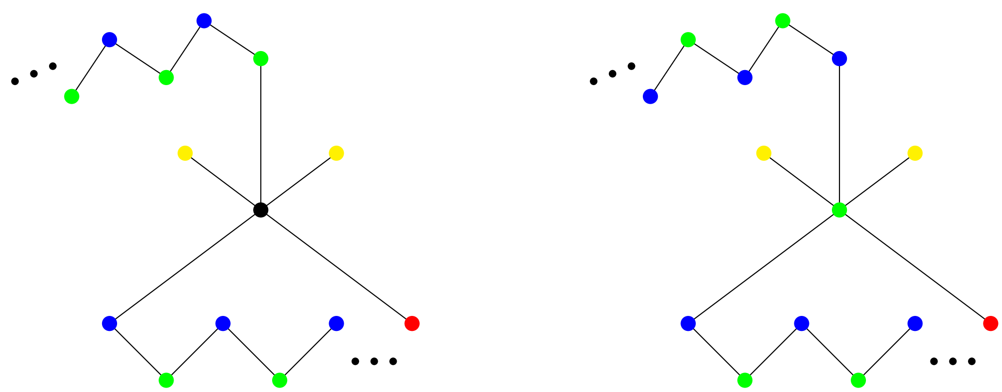
Subcase 2.3.1.2 (Cyclical) *Unlike the previous subcase, switching the colors only would result in the arrangement to change and not make a color available. However, by making a loop a vertex is enclosed within a border thus making the current subcase impossible for it. Thus it can follow the previous subcase.*

[3]Alfred Kempe, "On The Geographical Problem Of The Four Colours", American Journal of Mathematics 2, no. 3 (1879): 193-220.

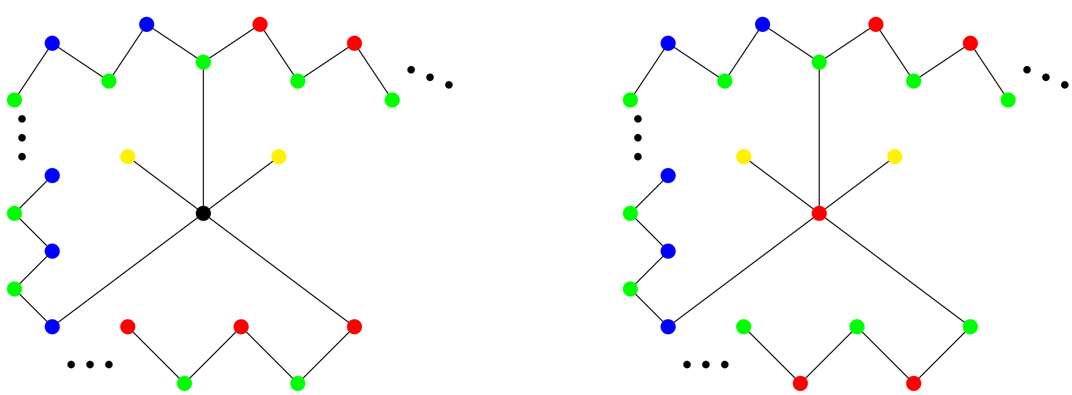


Case 2.3.2 (Degree 5) *If the newly added adjacent vector has the same color as 1 besides it, it is the exact same argument as the previous case. Otherwise, there is an additional* Kempe Chain *to consider:*

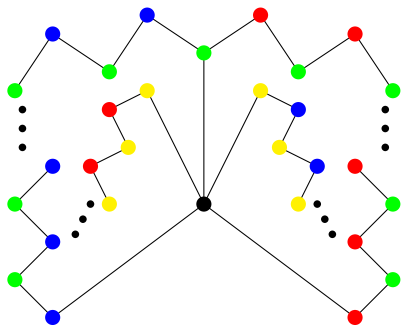
Subcase 2.3.2.1 (0 Cyclical) *Switch the colors for 1 of the* Kempe Chains.



Subcase 2.3.2.2 (1 Cyclical) *Switch the colors of 1 of the* Kempe Chains *that are not cyclical.*



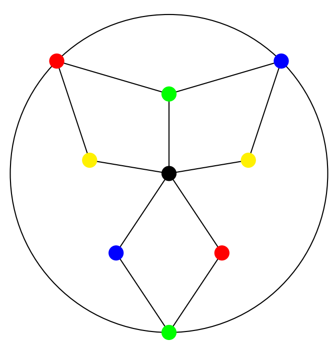
Subcase 2.3.2.3 (2 Cyclical) *Consider the 2* Kempe Chains *that have become enclosed by a cyclical* Kempe Chain *of which neither color is involved. If both of these have their colors switched, this creates a color available for the center vertex to use.*



And thus, since every vertex of degree 5 or less meets the criteria of being 4 colorable and removing the vertex will result in a different vertex to be of degree 5 or less until every vertex is accounted for, the induction holds.

2.2 The Un*kemp*t Chain

For 11 years the flaw in the proof went unnoticed until Percy John Heawood presented a counterexample.[4] While the proof accounted for most cases, the final case had yet another subcase that contradicted the rule of 4 colorability. Consider the following:



While this graph may disguise as the final subcase of the proof, attempting to switch the colors meant for this case would result in adjacent vertices to be the same color. And since the other *Kempe Chains* are cyclical, there is no way to make a color available for the center vertex. Thus the proof was made invalid.

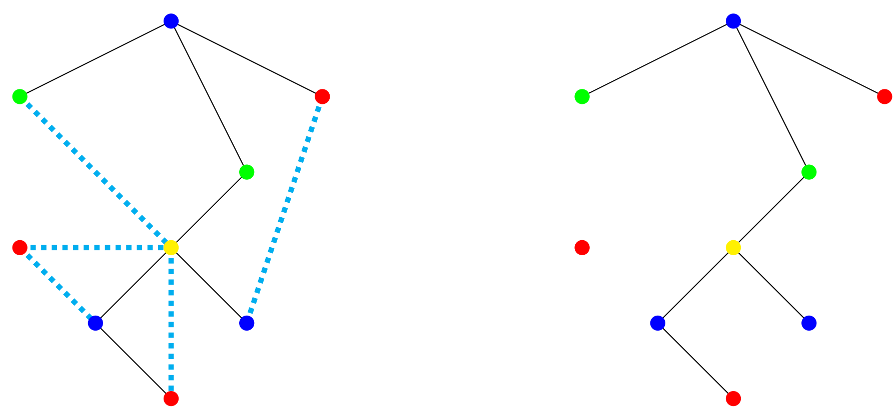
[4] Percy John Heawood, "Map-Colour Theorem", The Quarterly Journal of Pure and Applied Mathematics 24 (1890): 332-338.

3 Enter Computer

3.1 Reducibility

While Kempe's proof may not have been valid, it served as a basis for the 5CT and eventually Kenneth Appel and Wolfgang Haken's computer assisted proof published in 1976. They had proved that every graph can be constructed from a collection of reducible configurations each of which are 4 colorable thus making every graph 4 colorable.

But first, a couple important properties of graphs must be observed. Any simple, planar graph that is 4 colorable that has edges removed remains 4 colorable since the existing color orientation can be maintained. Similarly, if the removal of an edge results in the degree of a vertex to become 0, the graph still remains 4 colorable. Thus, if we add edges between vertices or add new vertices connected via new edges and show the graph to be 4 colorable, the initial graph too is 4 colorable.



Another property exhibited in all planar graphs with at least 3 vertices is the ability for them to be triangulated - that is each face is reduced to a triangle by connecting vertices with new edges in any polygon with more than three sides. This property can be derived off of Euler's Formula.

Lemma 3.0.1 *For any triangulated graph* $V \geq 3$, $E = 2V - 3$ *and* $F = V - 1$, *so by theorem 2.1* $V - E + F = V - (2V - 3) + (V - 1) = 2$ *thus the graph is planar.*

Based on these two properties, we can triangulate any planar graph, prove it to be 4 colorable, and conclude that the original graph is also 4 colorable. In order do this, Appel and Haken first showed that there is an unavoidable set of subgraphs that every graph can be reduced down to.

Theorem 3.1 [5] *If a planar graph has a minimum degree of 5, then there is at least 1 edge with endpoints of both 5 or endpoints of 5 and 6.*

Proof 3.1.1 *If the triangulation of the graph has a vertex with a degree of 5, then the same vertex on the original graph must also have had a degree of 5. For a triangulated graph, there is a clear relationship between faces and edges*

$$3F = 2E$$

Plug this into theorem 2.1 and we get

$$V - \frac{1}{3}E = 2$$

which is equivalent to

$$6V - 2E = 12$$

If we let the subscript for V denote the degree of vertices, we can find the sum of all the edges of the faces we will get

$$\sum_{k=5}^{\infty} kF_k = 2E$$

Plug this into the previous expression and we get

$$12 = 6V - 2E = 6\sum_{k=5}^{\infty} V_k - \sum_{k=5}^{\infty} kV_k = \sum_{k=5}^{\infty}(6-k)V_k$$

Thus we can see that if we assign an initial charge *of (6-k) to each vertex where k denotes degree of the vertex, the sum of* charges *will be 12. Now we can use the method of discharging to distribute the* charges *based on a certain set of rules. In this case, let each vertex of degree 5 give each vertex with a degree 7 or more* $\frac{1}{5}$ charge. *By way of contradiction, let each vertex of degree 5 have only vertices of degree 7 adjacent to them. Then, all of these vertices will have a* charge *of 0 since it loses* $\frac{1}{5}$ *of a* charge *per side. By default, vertices with degree 6 will have a charge of 0 and not lose or gain any. All remaining vertices can gain* charge *from at most* $\frac{k}{2}$ *adjacent vertices (since vertices with degree 5 cannot be adjacent to each other) thus giving it a total* charge *of* $6 - k + (\frac{1}{5})(\frac{k}{2}) = 6 - \frac{9k}{10}$. *Since* $k \geq 7$, $6 - \frac{9k}{10} \leq \frac{-3}{10}$. *Thus from distributing these charges we get that*

$$\sum_{k=5}^{\infty}(6-k)V_k \leq \frac{-3}{10}$$

which 12 is clearly not within a range of, thus a contradiction is yielded which proves that for every planar graph each vertex of degree 5 must be adjacent to a vertex with either degree 5 or 6.

[5] Wikipedia, "Discharging Method (Discrete Mathematics)", last modified 2022, `https://en.wikipedia.org/wiki/Discharging_method_(discrete_mathematics)`.

Using this as a basis, more discharging rules were considered to show that every planar graph in fact must include at least 1 subgraph within this unavoidable set of configurations. A program was implemented to then show that every graph with a ring size (the number of polygons) of 14 or less could in fact be reduced into 1 of these 1936 unavoidable configurations, and by showing that each of these were 4 colorable the theorem was proved.[6,7]

3.2 Philosophical Debate

The mathematics community was divided. This was the first major proof that relied on computation that could not be manually verified. Over the years the proof has been simplified and more importantly formalized by a Coq proof assistant program which is designed to systematically check mathematical assertions made. Still, it has been subject to much controversy.

Much of this debate lay in the philosophy of mathematics. If this proof were accepted, at what point is an unverifiable proof too far to be rejected. Thomas Tymoczko, a mathematics philosopher, notably put forth the following analogy: Let there be some mathematics genius called Simon who asserts theorems by appealing to his genius. This would allow for those who follow him to simply justify any difficult conjecture that they do not comprehend by insisting that "Simon says," thus gaining no insight into the problem at hand.[8]

While this argument is not completely analogous, there is some truth to it. Computers, unlike an entity like Simon, operate in a way that is understood by us thus making computer assisted proofs far more acceptable than blindly trusting Simon. However, Appel and Haken's proof did fit with Tymoczko's second point. All the logic behind the proof had been developed by mathematicians who had worked on the problem previously, and thus while they had proved the theorem, no new insight had really been brought forth. This was another reason for the hesitance in accepting the proof, since it would open the doors to other conjectures to be proved by an exhaustive method by computer without any of the insight.

There still is yet to be an accepted proof that does not rely on computer assistance. As such, the theorem continues to generate interest in attempted proofs without the use of computers. While controversy over the authority of computer assisted proofs remains, the proof of the 4CT paved the way for its acceptance in mathematics.

---

[6] Kenneth Appel and Wolfgang Haken, "Every Planar Map Is Four Colorable. Part I: Discharging", Illinois Journal of Mathematics 21, no. 3 (1977).

[7] Kenneth Appel, Wolfgang Haken and John Koch, "Every Planar Map Is Four Colorable. Part II: Reducibility", Illinois Journal of Mathematics 21, no. 3 (1977).

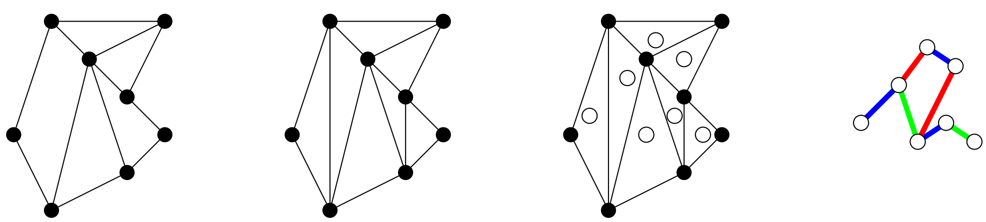
[8] E. R. Swart, "The Philosophical Implications Of The Four-Color Problem", The American Mathematical Monthly 87, no. 9 (1980): 697.

4 Potential Proof

4.1 Inspiration by Tait

Kempe was not the only mathematician who had devised a long enduring proof. Peter Guthrie Tait published a proof in 1880 which also remained without a disproof for 11 years. He had transformed the problem from a colorability problem for vertices to that of edges, showing that in fact 4 colorability was equivalent to 3 edge colorability.[9] Consider the following rules and its corresponding graphs:

1. Blue connects cyan with magenta or yellow with black.
2. Red connects cyan with yellow or magenta with black.
3. Green connects cyan with black or magenta with yellow.



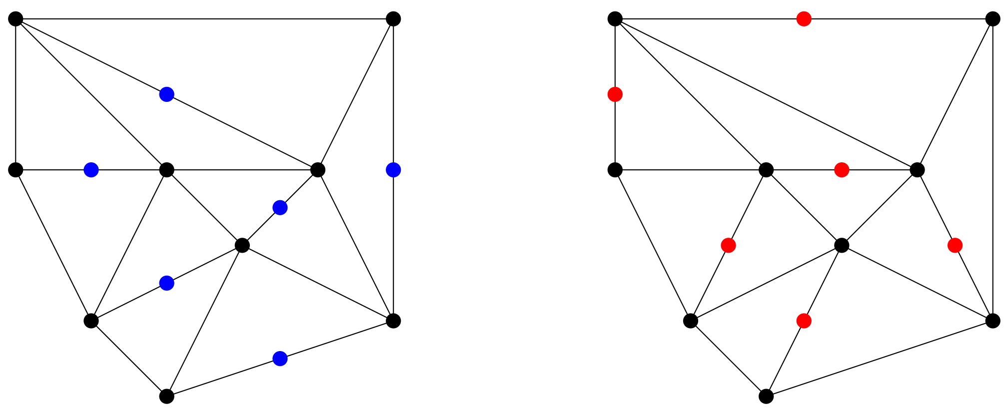
Note that in this case the graph is representative of a map where faces are regions of a map. First by triangulating the graph, we obtain a cubic graph when taking the dual (treating faces as vertices) which is 3 edge colorable. Then if any of these vertices are colored with any color, based on the rules all other vertices can be colored.

This was not Tait's only attempted proof however. Earlier in 1880, he had published another incomplete proof which he had redacted after his second proof. The earlier one had taken a very different approach, where vertices were added to the triangulation of the dual of the map.[10] The following is an original proof of mine inspired by this lesser known proof.

[9] Peter Tait, "On The Colouring Of Maps", Proceedings of the Royal Society of Edinburgh 10 (1880): 501-503.
[10] Peter Tait, "Remarks On The Previous Communication", Proceedings of the Royal Society of Edinburgh 10 (1880): 729.

4.2 Quadrilateralation

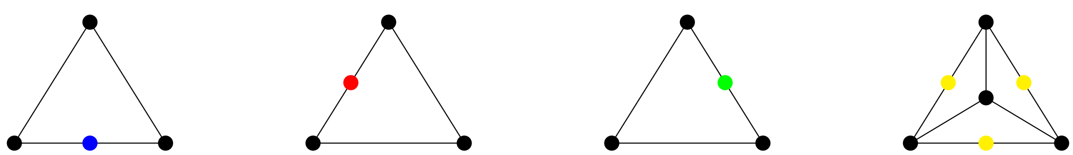
Consider adding a vertex along an edge of a triangle. The result is a quadrilateral which by definition is 2 colorable. We must now first show that any triangulated graph has two distinct ways in which vertices can be added such that each triangle becomes a quadrilateral.



Theorem 4.1 (Quadrilateralation) *Every graph has a way to be triangulated which allow for two distinct ways in which vertices can be added such that each triangle becomes a quadrilateral through a systematic method. Distinct entails that an edge that had a vertex added in one variation may not have one added in the other.*

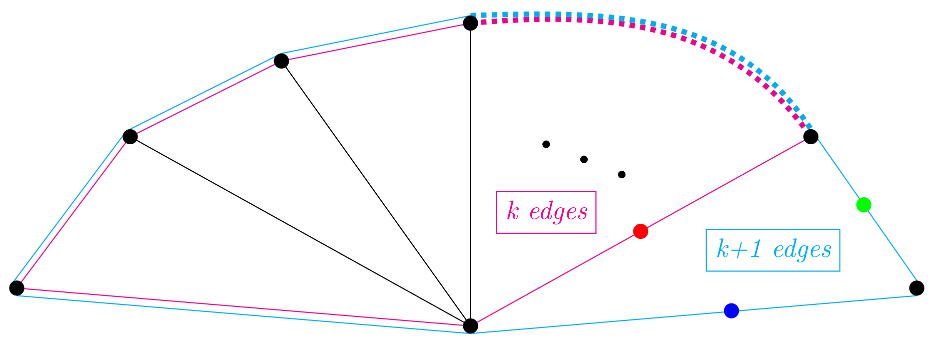
Lemma 4.1.1 *Every polygon can be triangulated in a way that will satisfy theorem 4.1.*

Proof 4.1.1 *By induction on the number of edges in the polygon. For the base case, a triangle can have 4 variations for how vertexes can be added:*



However, the fourth variation is irrelevant since if it is used in one variation of adding vertices, the other variation will not have any way to be distinct. Thus only the first three cases will be considered.

A polygon with k+1 can thus be constructed from these triangles in such a way that allows for two cases: if k+1 is even, there are an even number of vertices added to the exterior edges of the polygon between 0 and k-1; else if k+1 is odd, there are an odd number of vertices added to the exterior edges of the polygon between 1 and k-1.



Corollary 4.1.1 *The parity of the number of edges matches that of the vertices added to the exterior edges. Thus two polygons with edge numbers m and n can be concatenated to make one with m+n-2k, where k denotes the number of intersecting edges, and still satisfy theorem 4.1.*

Now there is the issue of nested polygons since they cannot be triangulated in the same way as in lemma 4.1.1. In this case the parity will not comply with corollary 4.1.1 since the overarching polygon will dictate the overall parity and thus we must show that nested polygons will continue to satisfy theorem 4.1 despite its parity.

Lemma 4.1.2 *Every polygon embedded within another can be triangulated in a way that will satisfy theorem 4.1.*

Proof 4.1.2 *Nested polygons can be considered as an overarching polygon with m sides bordering that with n sides such that n-1 edges intersect, thus satisfying the formula in corollary 4.1.1: m+n-k=(l+k)+(k+1)-2k=l+1, where l > 0.*

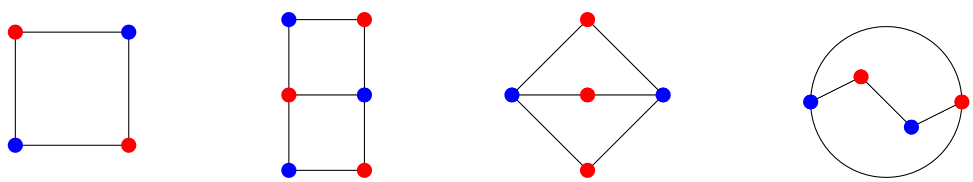
Taking into account theorem 2.1, we can see that any simple, planar graph can be triangulated in such a way that allows for two distinct ways of adding vertices. Hereon we will refer to such vertices that convert triangles into quadrilaterals as *ghost nodes*.

4.3 Do You Believe in Ghosts?

Independently a quadrilateral is 2 colorable, but we must show this to be the case for our newly created lattice of quadrilaterals.

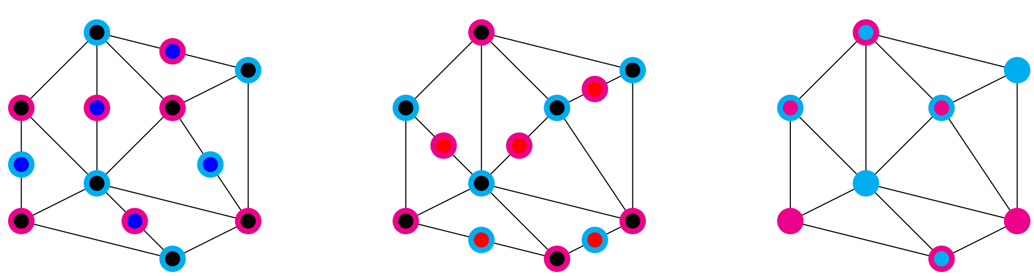
Theorem 4.2 *Every graph that has had* ghost nodes *added is 2 colorable.*

Proof 4.2.1 *By induction on the number of exterior edges. The base cases to be considered are an isolated quadrilateral as well as two quadrilaterals sharing 1 edge, sharing 2 edges, and sharing 3 edges, thus having 2, 4, or 6 exterior edges.*



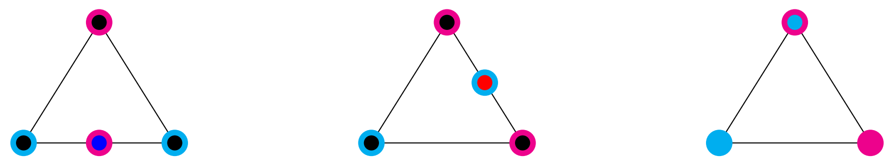
Since the exterior edges will always sum to be even, the k+2th case will be considered. The alternating pattern exhibited in the exterior edges for the kth case will result in the k+2th case to continue the pattern. If a quadrilateral is added with more than 1 intersecting edges, theorem holds by strong induction since the number of exterior edges remain constant or decrease by 2. Furthermore, it should be noted that in the case that k is 2, there cannot be 3 intersecting edges and thus the number of exterior edges cannot decrease by 2 and only remain constant or increase.

Combining this with theorem 4.1, we now have 2 distinct ways of characterizing each vertex from the triangulated graph before the *ghost nodes* were added. If we overlay the 2, each vertex now can be characterized, or colored, in 4 ways.

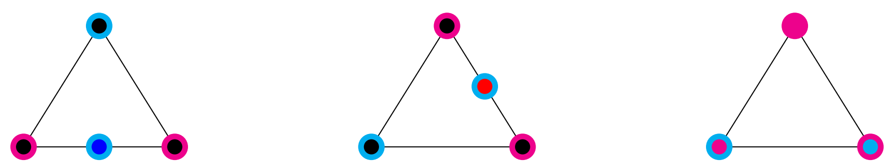


All that remains is to show that adjacent vertices will never have the same color based on the method the color was obtained.

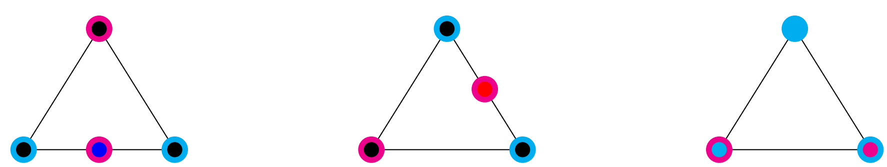
Theorem 4.3 (Four Color Theorem) *By induction on the number of triangles added. For the base case consider the overlay of the following 2 triangles with* ghost nodes*:*



Alternatively, the coloring can be shifted for 1 of the previous triangles, thus producing a different orientation of color:

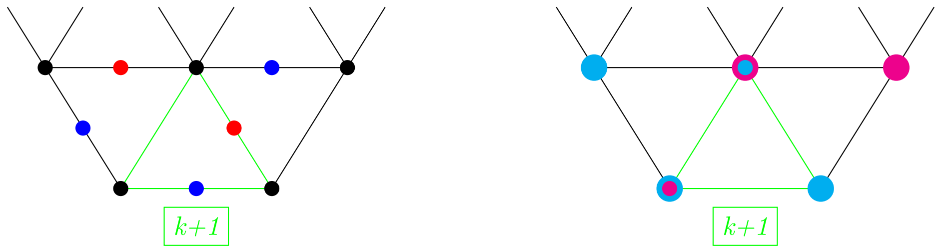


Finally, the last orientation is produced by shifting the coloring of the other triangle:

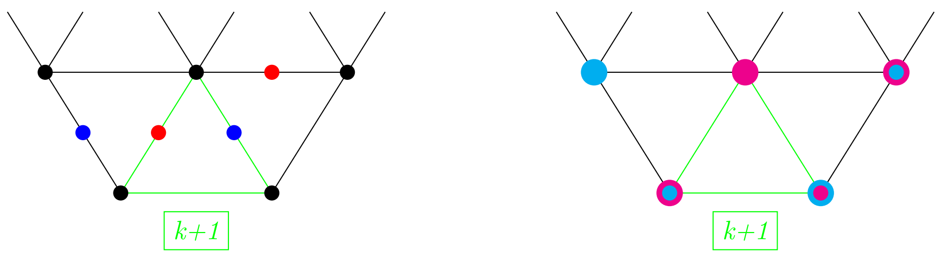


Thus there exists every combination of colors between a singular edge of a triangle. For the k+1th triangle, 3 cases must be considered. The first case in which a triangle is added so that it intersects with 1 edge is trivial. The second case in which there are 3 intersections is irrelevant since the ghost nodes *and thus color arrangement is already determined. The third case in which there are 2 intersections has 2 cases to be considered without loss of generality such that the vertices that were once disjointed can be shown to be different colors:*

Case 4.3.1



Case 4.3.2



Since the previously disjointed vertices will always be different colors, the induction holds.

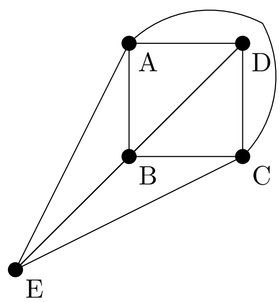
With the triangulated form of the graph proven to be 4 colorable by exploiting *ghost nodes*, the original graph is 4 colorable as previously deliberated.

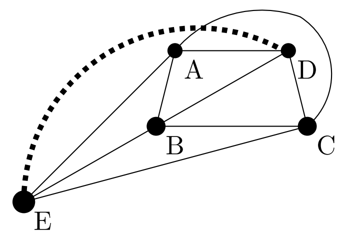
4.4 *Genera*lizing Potential

Since the 4CT only applies to planar graphs which are of genus 0, naturally the next question is whether the method of adding *ghost nodes* could be implemented for all genera. In fact, the colorability for all other genera was proven before the 4CT in 1968 by Gerhard Ringel and Ted Youngs.[11] Known as the Heawood Conjecture, it was formulated by Percy John Heawood in the same paper in which he had disproved Kempe's proof.[12]

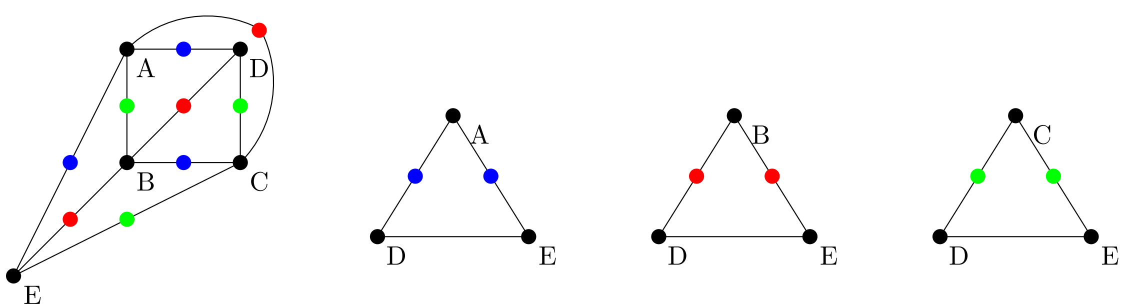
[11]Gerhard Ringel and Ted Youngs, "Solution of the Heawood Map-Coloring Problem", Proceedings of the National Academy of Sciences 60, no. 2 (1968): 438-445.
[12]Heawood, "Map-Colour Theorem".

However, it is unlikely that the method of adding *ghost nodes* can prove all genera due to two reasons, the first being that the number of colors required are not necessarily powers of 2 for higher genera meaning that binary characterization would not be suitable to determine the number of colors required. More importantly however, there is no way to add *ghost nodes* when there are overlapping edges as there would be in graphs of a higher genus. This can be seen in the most simplistic example using a graph with 5 vertices with each connected to the other (also known as a complete graph on 5 vertices denoted as K_5) which exists in genus greater than 0. Consider the following K_5 graph with the overlapping edge depicted as existing in another dimension:

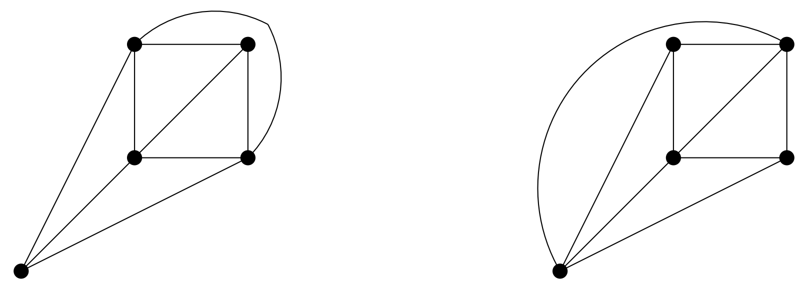




Then it can be seen that this edge constructs three triangles. If *ghost nodes* are added without the consideration of this edge, then upon inspection of the three triangles it can be seen that all three configurations of *ghost nodes* break the rules for this edge:



On the other hand, an approach in which both edges involved in the overlapping are considered to exist in another dimension could be taken. Consider the same K_5 graph that is instead decomposed into two separate graphs:



Upon adding *ghost nodes* for each and then overlaying the two separate graphs, there are 16 possible characterizations for the vertices. Thus, besides showing the trivial fact that a K_5 graph requires more than 4 colors and less than 16, this methodology seems unlikely to be able to be generalized across genera.

5 Conclusion

The four color theorem, disguised as a simplistic theorem like so many other difficult theorems, has a rich history with deep implications for the philosophy of mathematics on top of its applicability in fields such as computer science. Investigating this history and observing the evolution in the mathematics along the way was fascinating, and really gave me a better appreciation for understanding not just the math but the context behind its discovery. Nuances like that of Tait's background in knot theory which had influenced his approach to tackling the 4CT would otherwise not have been in my consideration when only learning the mathematics.

Writing this paper, I could not help but think about the extreme irony of pursuing a proof after reading paper after paper outlining the countless mathematicians who attempted to prove the theorem fruitlessly and warning others to follow in their footsteps. Strangely, upon first stumbling across Tait's proof that I had based my proof on, I had instantaneously gravitated towards focusing my paper on my proof despite the persistent warning. Unfortunately, I was unable to get it reviewed but even if the proof ends up to be false (which is most probable) the process of developing it and deliberating possible connection to other theorems was enjoyable nonetheless.

6 Bibliography

Appel, K., Haken, W. (1977). Every planar map is four colorable. Part I: Discharging. Illinois Journal Of Mathematics, 21(3). https://doi.org/10.1215/ijm/1256049011.

Appel, K., Haken, W., Koch, J. (1977). Every planar map is four colorable. Part II: Reducibility. Illinois Journal Of Mathematics, 21(3). https://doi.org/10.1215/ijm/1256049012.

Heawood, P. (1890). Map-Colour Theorem. The Quarterly Journal Of Pure And Applied Mathematics, 24, 332-338.

Kempe, A. (1879). On the Geographical Problem of the Four Colours. American Journal Of Mathematics, 2(3), 193-220. https://doi.org/10.2307/2369235.

Ringel, G., Youngs, T. (1968). Solution of the Heawood Map-Coloring Problem. Proceedings Of The National Academy Of Sciences, 60(2), 438-445. https://doi.org/10.1073/pnas.60.2.438

Swart, E. (1980). The Philosophical Implications of the Four-Color Problem. The American Mathematical Monthly, 87(9), 697. https://doi.org/10.2307/2321855.

Tait, P. (1880). On the Colouring of Maps. Proceedings Of The Royal Society Of Edinburgh, 10, 501-503. https://doi.org/10.1017/s0370164600044229.

Tait, P. (1880). Remarks on the previous Communication. Proceedings Of The Royal Society Of Edinburgh, 10, 729. https://doi.org/10.1017/s0370164600044643.

Wikipedia. (2022). Discharging method (discrete mathematics). https://en.wikipedia.org/wiki/Discharging_method_(discrete_mathematics).

Wikipedia. (2022). Four Color Theorem. https://en.wikipedia.org/wiki/Four_color_theorem.