ex.1

当矩阵$D \in R^{2n \times 4}$满秩时，寻找：

$$\min_y ||Dy||^2, st||y|| = 1 \tag{1}$$

的最小二乘解。上式等价于：

$$\min_y (Dy)^T (Dy) = \min_y y^T D^T Dy, st||y|| = 1 \tag{2}$$

对$D^T D$进行SVD分解，假设D的SVD分解为:

$$D = \sum_i \sigma_i \mathbf{u}_i \mathbf{v}_i^T \tag{3}$$

则

$$D^T D = (\sum_i \sigma_i \mathbf{v}_i \mathbf{u}_i^T)(\sum_j \sigma_j \mathbf{u}_j \mathbf{v}_j^T) = \sum_{i,j} \sigma_i \sigma_j \mathbf{v}_i (\mathbf{u}_i^T \mathbf{u}_j) \mathbf{v}_j^T \tag{4}$$

又$\mathbf{u}_i^T \mathbf{u}_j$当$i \neq j$时都等于 0，因此继续化简：

$$D^T D = \sum_i \sigma_i^2 \mathbf{v}_i \mathbf{v}_i^T \tag{5}$$

但是由于$D^T D$是对称阵，SVD后的$\mathbf{U}$阵与$\mathbf{V}$阵相同，所以有$\mathbf{u}_i = \mathbf{v}_i$,即有：

$$D^T D = \sum_i \sigma_i^2 \mathbf{u}_i \mathbf{u}_i^T \tag{6}$$

其中:

$$\sigma_1^2 \geq \cdots \geq \sigma_4^2, \mathbf{u}_l^T \mathbf{u}_m = \begin{cases} 0 & l \neq m \\ 1 & otherwise \end{cases}$$

假设$y=\mathbf{u}_4+\mathbf{v},\mathbf{v}\perp\mathbf{u}_4$,将式(2)代入式(1)中：

$$y^T(\sum_{j=1}^{4}\sigma_j^2\mathbf{u}_j\mathbf{u}_j^T)y=\sum_{j=1}^{4}y^T\sigma_j^2\mathbf{u}_j\mathbf{u}_j^Ty=\sum_{j=1}^{4}\sigma_j^2(\mathbf{u}_j^Ty)^2 \tag{7}$$

当$j=4$时有：

$$\sigma_4^2[\mathbf{u}_4^T(\mathbf{u}_4+\mathbf{v})]^2=\sigma_4^2(\mathbf{u}_4^T\mathbf{u}_4+\mathbf{u}_4^T\mathbf{v})=\sigma_4^2 \tag{8}$$

因此将式(4)重写为：

$$\sum_{j=1}^{4}\sigma_j^2(\mathbf{u}_j^Ty)^2=\sigma_4^2+\sum_{j=1}^{3}\sigma_j^2(\mathbf{u}_j^Ty)^2=\sigma_4^2+\sum_{j=1}^{3}\sigma_j^2[\mathbf{u}_j^T(\mathbf{u}_4+\mathbf{v})]^2 \tag{9}$$

由于$\mathbf{u}_j^T\mathbf{u}_4=0,j=1\ldots3$,可继续将式(6)继续化简：

$$\sum_{j=1}^{4}\sigma_j^2(\mathbf{u}_j^Ty)^2=\sigma_4^2+\sum_{j=1}^{3}\sigma_j^2(\mathbf{u}_j^T\mathbf{v})^2\geq\sigma_4^2 \tag{10}$$

由上式可知，当$\mathbf{v}=0,y=\mathbf{u}_4$时，式(4)有最小值。