ex.2

作业说明：

完整推导了离散时间下的误差动力学，结果跟VINS代码中预积分部分，矩阵V中除了前四项噪声的系数符号相反，其余全部一致。这里解释下这个负号的成立：

1.由于零均值的高斯噪声，正负不影响。

2.另外从误差的协方差传递方式：

$$P_{i+1}^{15\times 15} = FP_iF^T + VQV^T \tag{1}$$

以及误差的雅各比：

$$J_{i+1}^{15\times 15} = FJ_i \tag{2}$$

可以得出噪声传递矩阵的系数正负，不会影响最后结果。

1. 连续时间下的误差动力学

为简化起见，忽略了所有高阶项和重力的变化。参考《四元数数学基础》5.3节。

$$\dot{\delta p} = \delta v$$
$$\dot{\delta v} = -R[\hat{a}_t - b_a]_\times \delta\theta - R\delta b_a - Rn_a$$
$$\dot{\delta\theta} = -[\hat{w}_t - b_w]_\times \delta\theta - \delta b_w - w_n$$
$$\dot{b_a} = n_{b_a}$$
$$\dot{b_w} = n_{b_w} \tag{3}$$

这里δp,δv,$\delta\theta$对应IMU预积分中的$\delta\alpha$,$\delta\beta$,$\delta\gamma$分别对应位置，速度，方向的变化量。其中方向的变化量用$\delta\theta$来表示。可以写出：

$$\begin{vmatrix} \delta\dot{\alpha}_t^{b_k} \\ \delta\dot{\beta}_t^{b_k} \\ \delta\dot{\theta}_t^{b_k} \\ \delta\dot{b}_{a_t} \\ \delta\dot{b}_{w_t} \end{vmatrix} = \begin{vmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & -R_t^{b_k}[\hat{a}_t - b_{a_t}]_\times & -R_t^{b_k} & 0 \\ 0 & 0 & -[\hat{w}_t - b_{w_t}]_\times & 0 & -I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} \delta\alpha_t^{b_k} \\ \delta\beta_t^{b_k} \\ \delta\theta_t^{b_k} \\ \delta b_{a_t} \\ \delta b_{w_t} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 & 0 & 0 \\ -R_t^{b_k} & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{vmatrix} \begin{vmatrix} n_a \\ n_w \\ n_{b_a} \\ n_{b_w} \end{vmatrix} = F_t \delta z_t^{b_k} + G_t n_t \tag{4}$$

2. 离散时间下的误差动力学

这里我们同样使用中值法积分处理离散情形。根据上一节的内容，我们可以知道方向误差的导数连续形式为：

$$\delta\dot{\theta}_t^{b_k} = -[\hat{w}_t - b_{w_t}]_\times \delta\theta_t - \delta b_{w_t} - n_w \tag{5}$$

则中值法离散形式为：

$$\delta\dot{\theta}_i = -[\frac{\hat{w}_i + \hat{w}_{i+1}}{2} - b_{w_i}]_\times \delta\theta_i - \frac{n_{w_i} + n_{w_{i+1}}}{2} - \delta b_{w_i} \tag{6}$$

由此根据导数定义可得：

$$\delta\theta_{i+1} = [I - [\frac{\hat{w}_i + \hat{w}_{i+1}}{2} - b_{w_i}]_\times \delta t]\delta\theta_i - \delta t \frac{n_{w_i} + n_{w_{i+1}}}{2} - \delta t \delta b_{w_i} \tag{7}$$

令：

$$f_{11} = I - [\frac{\hat{w}_i + \hat{w}_{i+1}}{2} - b_{w_i}]_\times \delta t$$

$$v_{11} = v_{13} = -\frac{\delta t}{2}$$

$$f_{14} = -\delta t$$

速度误差的导数连续形式为：

$$\delta\dot{\beta}_t^{b_k} = -R_t^{b_k}[\hat{a}_t - b_{a_t}]_\times \delta\theta_t - R_t^{b_k}\delta b_{a_t} - R_t^{b_k} n_a \tag{8}$$

则中值法离散形式为：

$$\delta\dot{\beta}_i = -\frac{1}{2}R_i[\hat{a}_i - b_{a_i}]_\times \delta\theta_i - \frac{1}{2}R_{i+1}[\hat{a}_{i+1} - b_{a_i}]_\times \delta\theta_{i+1} - \frac{1}{2}(R_i + R_{i+1})\delta b_{a_i}$$

$$-\frac{1}{2}R_i n_{a_i} - \frac{1}{2}R_{i+1} n_{a_{i+1}} \tag{9}$$

将式（32）代入上式可得：

$$\delta\dot{\beta}_i = -\frac{1}{2}R_i[\hat{a}_i - b_{a_i}]_\times \delta\theta_i - \frac{1}{2}R_{i+1}[\hat{a}_{i+1} - b_{a_i}]_\times$$

$$\{[I - [\frac{\hat{w}_i + \hat{w}_{i+1}}{2} - b_{w_i}]_\times \delta t]\delta\theta_i - \frac{n_{w_i} + n_{w_{i+1}}}{2}\delta t - \delta b_{w_i}\delta t\}$$

$$-\frac{1}{2}(R_i + R_{i+1})\delta b_{a_i} - \frac{1}{2}R_i n_{a_i} - \frac{1}{2}R_{i+1} n_{a_{i+1}}$$

继续整理：

$$\delta\dot{\beta}_i = \{-\frac{1}{2}R_i[\hat{a}_i - b_{a_i}]_\times - \frac{1}{2}R_{i+1}[\hat{a}_{i+1} - b_{a_i}]_\times [I - [\frac{\hat{w}_i + \hat{w}_{i+1}}{2} - b_{w_i}]_\times \delta t]\}\delta\theta$$

$$+\frac{1}{4}R_{i+1}[\hat{a}_{i+1}-b_{a_i}]_\times(n_{w_i}+n_{w_{i+1}})\delta t+\frac{1}{2}R_{i+1}[\hat{a}_{i+1}-b_{a_i}]_\times\delta b_{w_i}\delta t$$

$$-\frac{1}{2}(R_i+R_{i+1})\delta b_{a_i}-\frac{1}{2}R_i n_{a_i}-\frac{1}{2}R_{i+1}n_{a_{i+1}}$$

根据导数定义可得：

$$\delta\beta_{i+1}=\delta\beta_i+f_{21}\delta\theta_i-\frac{1}{2}(R_i+R_{i+1})\delta t\delta b_{a_i}+f_{24}\delta b_{w_i}$$

$$-\frac{1}{2}R_i\delta t n_{a_i}-\frac{1}{2}R_{i+1}\delta t n_{a_{i+1}}+v_{21}n_{w_i}+v_{23}n_{w_{i+1}} \tag{10}$$

令：

$$f_{21}=-\frac{1}{2}R_i[\hat{a}_i-b_{a_i}]_\times\delta t-\frac{1}{2}R_{i+1}[\hat{a}_{i+1}-b_{a_i}]_\times[I-[\frac{\hat{w}_i+\hat{w}_{i+1}}{2}-b_{w_i}]_\times\delta t]\delta t$$

$$f_{22}=I$$

$$f_{23}=-\frac{1}{2}(R_i+R_{i+1})\delta t$$

$$f_{24}=\frac{1}{2}R_{i+1}[\hat{a}_{i+1}-b_{a_i}]_\times\delta t^2$$

$$v_{20}=-\frac{1}{2}R_i\delta t$$

$$v_{21}=v_{23}=\frac{1}{4}R_{i+1}[\hat{a}_{i+1}-b_{a_i}]_\times\delta t^2$$

$$v_{22}=-\frac{1}{2}R_{i+1}\delta t$$

位置误差导数的连续形式：

$$\delta\dot{\alpha}_t^{b_k} = \delta\beta_t^{b_k} \tag{11}$$

则中值法离散形式为：

$$\delta\dot{\alpha}_i = \frac{1}{2}\delta\beta_i + \frac{1}{2}\delta\beta_{i+1} \tag{12}$$

将式（35）代入上式可得：

$$\delta\dot{\alpha}_i = \delta\beta_i + \frac{1}{2}f_{21}\delta\theta_i - \frac{1}{4}(R_i + R_{i+1})\delta t\delta b_{a_i} + \frac{1}{2}f_{24}\delta b_{w_i}$$

$$-\frac{1}{4}R_i\delta t n_{a_i} - \frac{1}{4}R_{i+1}\delta t n_{a_{i+1}} + \frac{1}{2}v_{21}n_{w_i} + \frac{1}{2}v_{23}n_{w_{i+1}} \tag{13}$$

根据导数定义：

$$\delta\alpha_{i+1} = \delta\alpha_i + \delta t\delta\beta_i + \frac{1}{2}f_{21}\delta t\delta\theta_i - \frac{1}{4}(R_i + R_{i+1})\delta t^2\delta b_{a_i} + \frac{1}{2}f_{24}\delta t\delta b_{w_i}$$

$$-\frac{1}{4}R_i\delta t^2 n_{a_i} - \frac{1}{4}R_{i+1}\delta t^2 n_{a_{i+1}} + \frac{1}{2}v_{21}\delta t n_{w_i} + \frac{1}{2}v_{23}\delta t n_{w_{i+1}} \tag{14}$$

令：

$$v_{00} = -\frac{1}{4}R_i\delta t^2$$

$$v_{01} = v_{03} = \frac{\delta t}{2}v_{21}$$

$$v_{02} = -\frac{1}{4}R_{i+1}\delta t^2$$

$$f_{00} = I$$

$$f_{01}=\frac{\delta t}{2}f_{21}$$
$$f_{02}=\delta t$$
$$f_{03}=-\frac{1}{4}(R_i+R_{i+1})\delta t^2$$
$$f_{04}=\frac{\delta t}{2}f_{24}$$

根据式（7）可得：

$$\delta b_{a_{k+1}}=\delta b_{a_k}+\delta t n_{b_a}$$
$$\delta b_{w_{k+1}}=\delta b_{w_k}+\delta t n_{b_w}$$

令：

$$f_{33}=f_{44}=I$$
$$v_{34}=v_{45}=\delta t$$

由上可以写出离散时间下的误差动力学,这里交换了β,θ：

$$\begin{vmatrix}\delta\alpha_{i+1}\\ \delta\theta_{i+1}\\ \delta\beta_{i+1}\\ \delta b_{a_{i+1}}\\ \delta b_{w_{i+1}}\end{vmatrix}=\begin{vmatrix}f_{00}&f_{01}&f_{02}&f_{03}&f_{04}\\ 0&f_{11}&0&0&f_{14}\\ 0&f_{21}&f_{22}&f_{23}&f_{24}\\ 0&0&0&f_{33}&0\\ 0&0&0&0&f_{44}\end{vmatrix}\begin{vmatrix}\delta\alpha_i\\ \delta\theta_i\\ \delta\beta_i\\ \delta b_{a_i}\\ \delta b_{w_i}\end{vmatrix}$$

$$
+\begin{vmatrix} v_{00} & v_{01} & v_{02} & v_{03} & 0 & 0 \\ 0 & v_{11} & 0 & v_{13} & 0 & 0 \\ v_{20} & v_{21} & v_{22} & v_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 & v_{45} \end{vmatrix} \begin{vmatrix} n_{a_i} \\ n_{w_i} \\ n_{a_{i+1}} \\ n_{w_{i+1}} \\ n_{b_a} \\ n_{b_w} \end{vmatrix}
$$