Lie groups, Lie algebras, projective geometry and optimization for 3D Geometry, Engineering and Computer Vision

March 3, 2014

Chapter 1

Preliminaries

È più facile resistere all'inizio che alla fine
It is easier to resist at the beginning than the end

Leonardo da Vinci

1.1 Scalars

Scalars are things like real numbers that can be added or multiplied together. There are three types of scalars of interest in this book; Real numbers ($\mathbb{R}$), Complex numbers ($\mathbb{C}$) and Quaternions ($\mathbb{H}$). This book will assume familiarity with Real and Complex numbers which form a **field**, ie they follow these rules:

- **Addition:** They can be added together and addition is commutative: $a + b = b + a$.
- **Multiplication:** They can be multiplied together and multipliation is commutative: $a \cdot b = b \cdot a$.
- **Associativity:** Addition and multiplication are associative: $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **0:** There is a zero element that can be added to anything to give that same number: $a + 0 = a$.
- **1:** There is a one element that can be multiplied to anything to give that same number: $a \cdot 1 = a$.
- **Additive Inverse:** Every element has an additive inverse, a number that can be added to it to give zero: $a + -a = 0$.
- **Multiplicative Inverse:** Every element except 0 has a multiplicative inverse, a number that can be multiplied to it to give one: $a \cdot \frac{1}{a} = 1$.
- **Distributivity:** Multiplication distributes over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Quaterions follow the same rules as complex numbers except that instead of having one imaginary axis, they have three denoted i, j and k. As for complex numbers $i^2 = -1$ and similarly $j^2 = -1$ and $k^2 = -1$. Finally $ijk = -1$. This means that $ij = k$ but $ji = -k$, so i and j do not commute (breaking the rule about multiplication being commutative) and so quaternions are not a field.

1.2 Vector spaces

A vector space is a set of things that can be added together or multiplied by scalars following these rules:

- **Addition:** Vectors can be added together. Addition is commutative: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$, it is associative: $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$, there is a zero vector: $\mathbf{v} + \mathbf{0} = \mathbf{v}$ and vectors have additive inverses: $\mathbf{v} + -\mathbf{v} = \mathbf{0}$.
- **Multiplication:** Vectors can be multiplied by scalars. Multiplication distributes over vector addition: $a(\mathbf{v}_1 + \mathbf{v}_2) = a\mathbf{v}_1 + a\mathbf{v}_2$, and also over scalar addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

This book will mostly be concerned with vector spaces over real numbers. The most obvious example of these are elements of $\mathbb{R}^N$ which are written as a column of N real numbers in parentheses:

$$\mathbf{v} \in \mathbb{R}^3, \qquad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1}$$

Vectors are not the only things that live in a vector space, however. Matrices (see below) also satisfy the rules.

1.2.1 Basis of a vector space

A **basis** for a vector space is a set of linearly independent vectors that span the space. Linear independence means that no basis vector can be expressed as a weighted sum of other basis vectors. "Span the space" means that all vectors in the space can be expressed (uniquely) as a weighted sum of basis vectors.

1.3 Matrices

Matrices are linear transformations from one vector space to another. Linear transformations have to satisfy two rules. If M is a **linear transformation**, u and v are vectors and λ is a scalar, then:

- **Commuting with scalars:** $M(\lambda v) = \lambda(Mv)$
- **Distribution over vector addition:** $M(u + v) = (Mu) + (Mv)$

Matrices are represented as tables of numbers, for example:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \tag{1.2}$$

represents a linear transformation. This matrix transforms vectors in $\mathbb{R}^2$ into vectors in $\mathbb{R}^3$. For example, M transforms the vector $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ into the vector $\left(\begin{smallmatrix} 5 \\ 11 \\ 17 \end{smallmatrix}\right)$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 17 \end{pmatrix} \tag{1.3}$$

1.3.1 Matrices as a vector space

Note that two matrices of the same size can be added together and therefore the set of matrices of a given size form a vector space. Subsets of these can matrices also form vector spaces, e.g. the set of upper triangular 2×2 matrices form a 3-dimensional vector space, where each matrix looks like:

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \tag{1.4}$$

A basis for this vector space consists of three matrices, for example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.5}$$

form a basis for this vector space.

1.4 Groups

If both vector spaces are $\mathbb{R}^N$, then the matrices are square $N \times N$ arrays. Such matrices can be multiplied together to obtain a new matrix of the same size. Provided that matrices with zero determinant are avoided, such matrices form a **group** with the following properties:

- **Closure:** If g_1 and g_2 are members of a group then the product $g_1 g_2$ must also be in the group.
- **Identity:** A group must contain an identity element, I such that if g is in the group then $gI = Ig = g$ (the Identity matrix fulfills this requirement).
- **Inverse:** If g is a member of the group then it has a unique inverse h such that $gh = hg = I$ (the transpose of a rotation matrix always exists and fulfills this requirement).

1.5 3D Coordinates

In order to do any 3D geometry at all, it is necessary to find a way of agreeing where in 3D space something is. The most fundamental way to do this is to give the coordinates of a point, which can be expressed as a vector.

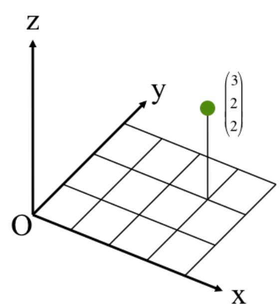

Figure 1.1: Coordinates of a point

Figure 1.1 shows how the coordinates of the centre of the green dot can be found by counting 3 units along the x axis, 2 units along y and 2 units along z. This point lives in real three dimensional space, written $\mathbb{R}^3$. In order for everyone to agree about which point we're talking about in space, we need to agree where the origin of the coordinate system is, which directions the x, y and z axes run in and what the unit of measurement is.

1.5.1 Rotations

If a different set of directions are chosen, the coordinate values of the same point will be different as seen in figure 1.2.

This happens a lot in 3D geometry because it's convenient to have many different coordinate frames for different purposes. Fortunately coordinates in one frame are simply related to those in a rotated frame.

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = R \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad \text{where } R \text{ is a } 3 \times 3 \text{ matrix} \tag{1.6}$$

In the example in figure 1.2, R is given by

$$R = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.7}$$

The coordinates in figure 1.2 can be obtained by multiplying the coordinates given in figure 1.1.

$$\begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3.535 \\ -0.707 \\ 2 \end{pmatrix} \tag{1.8}$$

Not all 3×3 matrices represent a simple rotation of the coordinate axes. In fact the constraints on R are quite strong:

1. the length of a vector must be conserved
2. the coordinate axes must be orthogonal (at right angles) to each other
3. the coordinate frame must remain right-handed

The columns of R correspond to unit vectors pointing along the coordinate axes of the old frame expressed in the new coordinate system. This can be seen because the first column of R is the result of transforming the unit vector pointing in the direction of the original x axis, $(1, 0, 0)^T$:

$$\begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \\ -0.707 \\ 0 \end{pmatrix} \tag{1.9}$$

and the other two columns can be retrieved by multiplying R by $(0, 1, 0)^T$ and $(0, 0, 1)^T$.

The three conditions can then be interpreted to provide constraints on the columns of R. Condition 1 means that the magnitude of each column of R must be 1. Condition 2 means that the three columns of R must be orthogonal to each other (their pairwise dot products must be 0). Finally, condition 3 means that the determinant of R must be 1 (not -1).

The first two conditions also means that:

$$R^T R = I \tag{1.10}$$

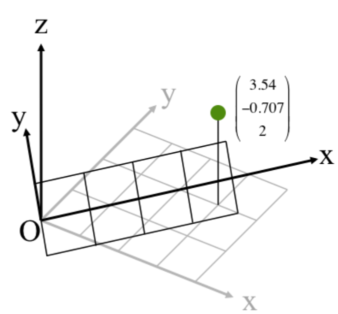

Figure 1.2: Coordinates of the same point with a rotated set of axes

since if we write R in terms of its 3 columns, r_1, r_2 and r_3, equation 1.10 becomes:

$$\begin{bmatrix} - & r_1^T & - \\ - & r_2^T & - \\ - & r_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.11}$$

since $r_i^T r_i = 1$ (the magnitude of each column is 1) and $r_i^T r_j = 0 \quad (i \neq j)$ (different columns are orthogonal).

Note that the Identity matrix, I is a valid rotation matrix.

This means that for a rotation matrix, $R^{-1} = R^T$ (the inverse is equal to the transpose). This in turn means that $RR^T = I$ since a matrix always commutes with its inverse:

$$R^{-1}R = I \quad \Longrightarrow \quad RR^{-1}R = RI = R = IR \quad \Longrightarrow \quad RR^{-1} = I \tag{1.12}$$

Consequently, this also means that the *rows* of R must also be unit length and orthogonal.

The final property of rotation matrices is that if two are multiplied together, the result must also be a rotation matrix. This can be seen because if $M = R_1 R_2$ then

$$\begin{aligned} M^T M &= R_2^T R_1^T R_1 R_2 && (1.13) \\ &= R_2^T I R_2 && (1.14) \\ &= R_2^T R_2 && (1.15) \\ &= I. && (1.16) \end{aligned}$$

This means that the columns (and rows) of M must also be unit length and orthogonal and its determinant is the product of the determinants of R_1 and R_2 $(= 1 \times 1 = 1)$.

All of these properties together mean that rotation matrices satisfy the conditions to be a group. This group is called the Special Orthogonal group in 3 dimensions, SO(3).

1.5.2 Translations

In addition to rotating axes, it is often advantageous to choose the origin to be in a different place. Figure 1.3 shows an example of the same point in a translated (shifted) coordinate frame.

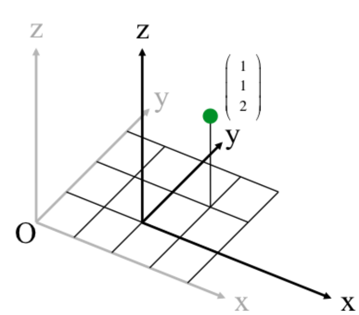

Figure 1.3: Coordinates of the same point with a translated set of axes

To convert coordinates into this new frame, it is necessary to add a constant vector to the vector of coordinates in the old frame.

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \mathbf{t} \qquad \text{where } t \text{ is a 3-vector} \tag{1.17}$$

$\mathbf{t}$ is the location of the origin of the first coordinate frame in the second coordinate frame. In the example in figure 1.3:

$$\mathbf{t} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}. \tag{1.18}$$

1.5.3 Combined rotation and translation

The general case involves both a rotation and a translation simultaneously as shown in Figure 1.4.

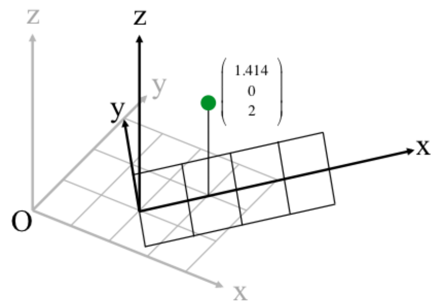

Figure 1.4: Coordinates of the same point with a general (rotated and translated) set of axes

In this case, it is necessary to apply a rotation and a translation to obtain the new coordinates:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = R \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \mathbf{t} \qquad \text{where } t \text{ is a 3-vector} \tag{1.19}$$

In the example in Figure 1.4, R and $\mathbf{t}$ are given by:

$$R = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{t} = \begin{pmatrix} -2.121 \\ 0.707 \\ 0 \end{pmatrix} \tag{1.20}$$

and the coordinate transformation for the green dot can be calculated as:

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2.121 \\ 0.707 \\ 0 \end{pmatrix} \tag{1.21}$$

$$= \begin{pmatrix} 3.535 \\ -0.707 \\ 2 \end{pmatrix} + \begin{pmatrix} -2.121 \\ 0.707 \\ 0 \end{pmatrix} \tag{1.22}$$

$$= \begin{pmatrix} 1.414 \\ 0 \\ 2 \end{pmatrix} \tag{1.23}$$

1.5.4 Homogeneous coordinates

A common trick for dealing with these kinds of coordinate transformations is to represent 3D points with a 4-vector by adding a 1 to the end of the 3 coordinates. These are called **homogeneous coordinates**. Using this, the coordinates of the green dot in its original coordinate frame can be represented as:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix} \tag{1.24}$$

This makes it possible to represent the rotation and translation with a single matrix of the form:

$$\begin{bmatrix} \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} & \begin{matrix} | \\ \mathbf{t} \\ | \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \tag{1.25}$$

Applying this matrix to the coordinates of the green dot gives:

$$\begin{bmatrix} 0.707 & 0.707 & 0 & -2.121 \\ -0.707 & 0.707 & 0 & 0.707 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.414 \\ 0 \\ 2 \\ 1 \end{pmatrix} \tag{1.26}$$

Matrices of the form given in equation 1.25 also form a group. The identity matrix has the correct form to be a member of the group and the inverse is given by:

$$\begin{bmatrix} \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} & \begin{matrix} | \\ \mathbf{t} \\ | \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \begin{bmatrix} & & \\ & R^T & \\ & & \end{bmatrix} & \begin{matrix} | \\ -R^T\mathbf{t} \\ | \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \tag{1.27}$$

So the inverse of the matrix in equation 1.26 is given by:

$$\begin{bmatrix} 0.707 & 0.707 & 0 & -2.121 \\ -0.707 & 0.707 & 0 & 0.707 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.707 & -0.707 & 0 & 2 \\ 0.707 & 0.707 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.28}$$

Engineers usually refer to this group as the Special Euclidean group in 3 dimensions, SE(3). In this context, “Euclidean” means “rigid body motions”.

Chapter 2

Lie Groups

All brontosauruses are thin at one end, much, much thicker in the middle, and then thin again at the far end.

Anne Elk (Miss)

The previous chapter introduced coordinate frames and showed how transformations between frames can be represented by matrices which form groups. Two groups of interest are SO(3), the group of rotations about the origin in 3 dimensions and SE(3), the group of rigid body motions (comprising rotations and translations) in 3 dimensions. These groups also form **manifolds**. The easiest way to visualise this is to consider all the matrices in a group like SO(3). These matrices contain 9 values and we can use this to think of these matrices as inhabiting a 9-dimensional space. It might help to think of the numbers in the matrix being unpacked and rearranged into a 9-vector (this is illustrated in figure 2.1 for the matrices corresponding to the red and blue dots).

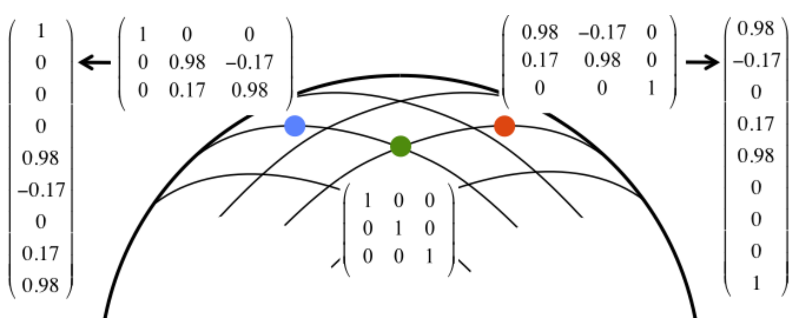

Figure 2.1: Part of the manifold of SO(3). The green dot represents the identity matrix, the blue dot is a 10 degree rotation about the x-axis and the red dot is a 10 degree rotation about the z-axis.

2.1 Neighbourhood of the Identity

Not all 3×3 matrices are members of SO(3) though, so not all of 9D space is filled by the group. In fact locally, the group is 3-dimensional and so if the immediate neighbourhood of any member of the group is examined, it looks like a 3-dimensional subspace of the 9 dimensions.

A 3-dimensional manifold embedded in 9D space is difficult to draw, so figure 2.1 shows a reduced representation of part of SO(3) as a 2-dimensional surface embedded in 3D space. Three points on this surface (manifold) are highlighted with green, blue and red dots. The rotation matrices corresponding to them are shown as well and the unpacked 9-dimensional vector is shown for the blue and red dot matrices.

What does it mean to say that the manifold of matrices in SO(3) is 3-dimensional? To understand this, it is necessary to look at a small part of the manifold. We will start by considering a small part of the manifold (a neighbourhood) near the identity and we shall see later that every small region on the manifold looks the same as this.

So the question is: "What matrices in SO(3) differ from the identity matrix by only a small amount?".

This question can be answered by writing a matrix near the identity as:

$$R = \begin{bmatrix} 1+a & b & c \\ d & 1+e & f \\ g & h & 1+i \end{bmatrix} \quad \text{where } a, b, c, d, e, f, g, h, i \text{ are small quantities} \tag{2.1}$$

This can be substituted into equation 1.10 to obtain:

$$\begin{bmatrix} 1+a & d & g \\ b & 1+e & h \\ c & f & 1+i \end{bmatrix} \begin{bmatrix} 1+a & b & c \\ d & 1+e & f \\ g & h & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

Multiplying out the left hand side gives:

$$\begin{bmatrix} 1+2a+a^2+d^2+g^2 & b+d+ab+de+gh & c+g+ac+df+gi \\ b+d+ab+de+gh & 1+2e+b^2+e^2+h^2 & f+h+bc+ef+hi \\ c+g+ac+df+gi & f+h+bc+ef+hi & 1+2i+c^2+f^2+i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

Because the variables $a, b, c, d, e, f, g, h, i$ are small (ie infinitessimal), second order terms like a^2 or ab can be discarded giving:

$$\begin{bmatrix} 1+2a & b+d & c+g \\ b+d & 1+2e & f+h \\ c+g & f+h & 1+2i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

This creates six independent constraints (since three are repeated):

$$\begin{matrix} a = 0 & e = 0 & i = 0 \\ b+d = 0 & c+g = 0 & f+h = 0 \end{matrix} \tag{2.5}$$

Which means that for infinitessimal b, c, f, matrices of the form:

$$R = \begin{bmatrix} 1 & b & c \\ -b & 1 & f \\ -c & -f & 1 \end{bmatrix} \tag{2.6}$$

are valid members of SO(3). These matrices are parameterised by three variables (b, c, f) and hence the set of members of SO(3) in the vicinity of the identity look like a three dimensional space. Expanding out equation 2.6 allows us to express any matrix R near the identity as:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - b \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

The three matrices of which b, c, f are coefficients are called **Generators**. Note that equation 2.7 has rearranged the order of the coefficients and that two of them have a negative sign; this has been chosen for later convenience. Labelling these matrices G_1, G_2 and G_3 and changing the names of the coefficients so that $\alpha_1 = -f$, $\alpha_2 = c$ and $\alpha_3 = -b$ allows equation 2.7 to be rewritten as:

$$R = I + \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3 \qquad \text{where } \alpha_1, \alpha_2, \alpha_3 \text{ are infinitessimal} \tag{2.8}$$

and

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.9}$$

2.2 The tangent space and derivatives: The Lie algebra

Allowing $\alpha_1, \alpha_2, \alpha_3$ to become non-infinitessimal creates a flat 3-dimensional space that is tangent to SO(3) at the identity. Figure 2.2 shows this illustrated with the 2-dimensional representation of SO(3) presented earlier. The tangent space is a vector space with its origin at the position of the identity matrix. This vector space (along with a bilinear operation described in section 2.9) is called the **Lie algebra** of the group. The Generator matrices are a set of vectors in this space which form a basis for it.

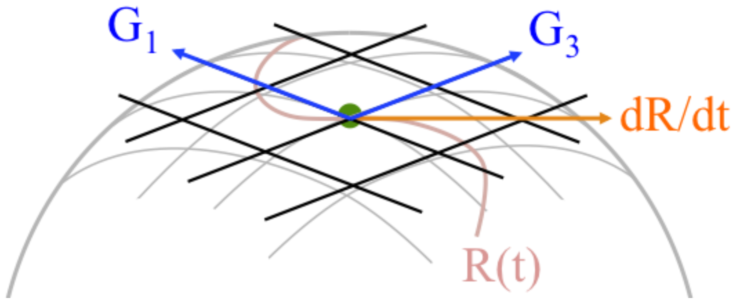

Figure 2.2: The tangent plane to SO(3) at the identity spanned by G_1 and G_3. The red line, $R(t)$, shows a path over the surface of the manifold. The orange arrow shows the derivative of this path with respect to t as it passes through the identity.

Another way of thinking about the tangent space is that it is the set of possible derivatives in SO(3) at the identity. More formally if a rotation matrix R is a function of a variable t such that at some value of t, $R(t) = I$, then the derivative of R with respect to t at this moment must lie in the tangent space and hence is some linear combination of the Generators:

$$\left.\frac{dR}{dt}\right|_{R=I} = \sum_i \alpha_i G_i \tag{2.10}$$

In fact this is another way of obtaining the Generators. If $R(t)$ is chosen to be a rotation by t radians around the x axis, then R is given by:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix} \tag{2.11}$$

and its derivative:

$$\frac{dR}{dt} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(t) & -\cos(t) \\ 0 & \cos(t) & -\sin(t) \end{bmatrix} \tag{2.12}$$

At $t = 0$, $R = I$ and the derivative is:

$$\left.\frac{dR}{dt}\right|_{t=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = G_1 \tag{2.13}$$

Thus G_1 corresponds to rotations around the x axis. Similarly G_2 and G_3 can be obtained by calculating derivatives of rotations around the y and z axes respectively.

2.3 The exponential map

If a matrix A is a linear combination of the Generators:

$$A = \sum_i \alpha_i G_i \tag{2.14}$$

The exponential of A can be calculated by using a matrix analogue of the Taylor expansion for e^x:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \tag{2.15}$$

Because A is a square (3×3) matrix, it is possible to calculate its square, cube or higher powers just by multiplying it with itself the appropriate number of times.

$$M = e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{n!}A^n \qquad \text{where } A^2 = AA \text{ and so on} \tag{2.16}$$

The magic is that if M is calculated this way, then it is guaranteed to be a member of SO(3). Furthermore *any* matrix in SO(3) can be calculated this way. In fact this is true for any group which is connected, meaning that there is a path through the manifold joining any two points in it.

To see why this is true, it is helpful to use a different expansion of e^A:

$$e^A = \lim_{n\to\infty}\left(I + \frac{1}{n}A\right)^n \tag{2.17}$$

As n grows, $\frac{1}{n}A$ shrinks and eventually becomes infinitessimal and therefore $I + \frac{1}{n}A$ becomes a member of SO(3) (recall equations 2.7 and 2.8). Since this is a member of SO(3) which is a group (and therefore closed under multiplication), raising it to the power n also gives a member of SO(3). Figure 2.3 illustrates this by showing a side view of the manifold with successive approximations to e^A with increasing n in equation 2.17.

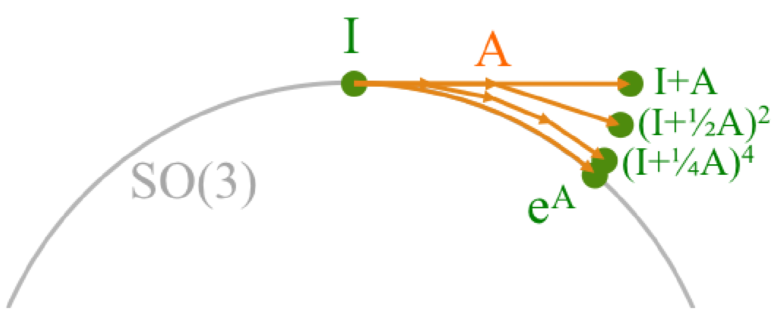

Figure 2.3: A side view of the exponential map e^A calculated according to equation 2.17

A simple example may help to illustrate how equation 2.16 works. This example uses SO(2) instead of SO(3). Matrices in SO(2) are planar rotation matrices of the form:

$$R = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} \tag{2.18}$$

These form a 1-dimensional manifold parameterised by θ. The single generator matrix is given by the derivative of R with respect to θ at $\theta = 0$:

$$G = \frac{dR}{d\theta}\Big|_{\theta=0} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \tag{2.19}$$

We can calculate the exponential of a constant times this generator matrix (αG):

$$e^{\alpha G} = I + \alpha G + \frac{1}{2}(\alpha G)^2 + \frac{1}{6}(\alpha G)^3 + \frac{1}{24}(\alpha G)^4 + \cdots \tag{2.20}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -\alpha\\ \alpha & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-\alpha^2 & 0\\ 0 & -\alpha^2\end{bmatrix} + \frac{1}{6}\begin{bmatrix}0 & \alpha^3\\ -\alpha^3 & 0\end{bmatrix} + \frac{1}{24}\begin{bmatrix}\alpha^4 & 0\\ 0 & \alpha^4\end{bmatrix} + \cdots \tag{2.21}$$

$$= \begin{bmatrix}1 - \frac{1}{2}\alpha^2 + \frac{1}{24}\alpha^4 + \cdots & -\alpha + \frac{1}{6}\alpha^3 + \cdots\\ \alpha - \frac{1}{6}\alpha^3 + \cdots & 1 - \frac{1}{2}\alpha^2 + \frac{1}{24}\alpha^4 + \cdots\end{bmatrix} \qquad \text{which are Taylor series for...} \tag{2.22}$$

$$= \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix} \qquad \text{which is the matrix for a rotation by } \alpha \text{ radians} \tag{2.23}$$

2.4 Exponentiation of a derivative

The exponential map creates a relationship between derivatives (ie velocities) at the identity and members of the group. This relationship is equivalent to maintaining the velocity for one unit of time and the group element that results is equal to the exponentiation. This can be expressed in a differential equation:

$$\text{Let} \quad A = \sum_i \alpha_i G_i \quad \text{be a velocity at the identity} \tag{2.24}$$

$$\text{Then the differential equation} \quad \frac{dR}{dt} = AR \tag{2.25}$$

$$\text{with} \quad R(0) = I \tag{2.26}$$

$$\text{gives} \quad R(t) = e^{tA} \tag{2.27}$$

$$\text{In particular} \quad R(1) = e^{A} \tag{2.28}$$

2.5 Generators of SO(3) and the cross product

In three dimensional space it is possible to take the cross product of two vectors:

$$\mathbf{v}_1 \wedge \mathbf{v}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ cd - af \\ ae - bd \end{pmatrix} \tag{2.29}$$

It is possible to think of $[\mathbf{v}_1 \wedge]$ as an operator that acts on any vector $\mathbf{v}_2$ to give $\mathbf{v}_1 \wedge \mathbf{v}_2$. This operator acts linearly on $\mathbf{v}_2$, so it can be represented by a 3×3 matrix:

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \right] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \tag{2.30}$$

Thus the cross product in equation 2.29 can be rewritten as a matrix multiplication:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ cd - af \\ ae - bd \end{pmatrix} \tag{2.31}$$

The cross product matrix is a linear sum of the generator matrices of SO(3) introduced in equation 2.7,

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \right] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} = aG_1 + bG_2 + cG_3 \tag{2.32}$$

(which is the reason for the choice of order and sign of G_i).

This means that rotation matrices in SO(3) can be represented as:

$$R = e^{[\mathbf{v}\wedge]} \tag{2.33}$$

parameterised by the elements of $\mathbf{v}$. This parameterisation creates a nice relationship between $\mathbf{v}$ and R; The rotation matrix is a rotation about the axis in the direction of $\mathbf{v}$ by an angle equal to the magnitude of $\mathbf{v}$ in a clockwise direction when looking in the direction of $\mathbf{v}$. Note that when $\mathbf{v} = 0$, the axis of rotation is undefined, but fortunately the magnitude is also zero and so there is no rotation taking place. When there's no rotation, it doesn't matter that you don't know which axis you're not rotating about!

To explain the relationship between $\mathbf{v}$ and R, recall that $[\mathbf{v}\wedge]$ is the derivative at the identity that causes R after one unit of time. $[\mathbf{v}\wedge]$ can be interpreted as causing a velocity vector field (a velocity vector at every point in space). The vector field at a point $\mathbf{p}$ is given by $[\mathbf{v}\wedge]\mathbf{p}$.

Points on the axis $\mathbf{v}$ have no velocity since $\mathbf{v} \wedge \mathbf{v} = 0$. Points $\mathbf{p}$ off the axis have a velocity proportional to the magnitude $\mathbf{v}$ times the magntitude of the component of $\mathbf{p}$ perpendicular to $\mathbf{v}$ in a direction perpendicular to $\mathbf{v}$ and $\mathbf{p}$. The velocity field is shown in figure 2.4.

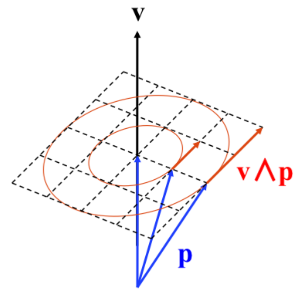

Figure 2.4: The velocity field created by $\mathbf{v}\wedge$ generates a circular rotation around $\mathbf{v}$.

2.5.1 Other parameterisations of SO(3)

Euler angles

Quaternions

2.6 The generators of SE(3)

The group of Euclidean (rigid body) transformations in three dimensions forms a six dimensional manifold. It has three generators of translations as well as three generators of rotations. The six generators of SE(3) are:

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$G_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.34}$$

G_1, G_2 and G_3 are the generators of translations in the x, y and z directions, while G_4, G_5 and G_6 are rotations about the x, y and z axes respectively.

When a translation is combined with a rotation that has an axis perpendicular to it, it has the effect of moving the axis of rotation to another point in space. This is illustrated in figure 2.5 which shows a translation in the x direction being added to a rotation about the z axis. The addition of the translation has the effect of moving the axis of rotation in the y direction. The new centre of rotation is the point where the velocity in the old rotation vector field is exactly cancelled out by the translation.

If the translation has a component in the same direction as the axis of rotation, then this remains, and when combined with the rotation causes a rotation about some axis in space combined with translation along that axis. This is exactly the motion of a bolt or a screw being twisted into a nut. For that reason a member of SE(3) is often referred to as a **screw**, while a derivative or velocity (member of the tangent space) is referred to as a **twist**. To complete the metaphor, a force on a rigid body that causes a twist (a torque about some axis plus a force along that axis) is referred to as a **wrench**.

2.7 The tangent space at non-Identity elements of the group

Section 2.1 examined the neighbourhood of the Identity element and promised that the manifold of the group looked like this everywhere. In fact the neighbourhood of the identity can be simply mapped to the

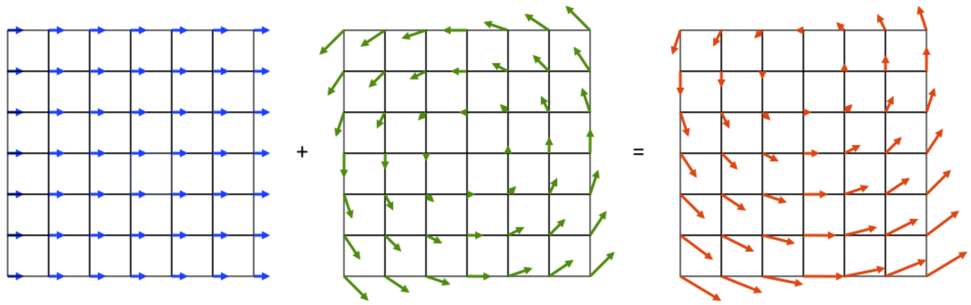

Figure 2.5: Adding the velocity fields of a translation in the x direction and a rotation about the z axis has the effect of moving the centre of rotation in the y direction.

neighbourhood of some other element of the group, R, by multiplying group elements near the identity by R.

$$I + \Delta \quad \rightarrow \quad R(I + \Delta) = R + R\Delta \tag{2.35}$$

This must be a member of the group since $I + \Delta$ is in the group and R is in the group and it must be near R because Δ is small and so $R\Delta$ must also be small. In a similar way, elements near R can be mapped to elements near the identity with the inverse map:

$$R + \Delta' \quad \rightarrow \quad R^{-1}(R + \Delta') = I + R^{-1}\Delta' \tag{2.36}$$

This process maps the neighbourhood of the identity to the neighbourhood of another member of the group, R. In the infinitessimal case, it maps the tangent space at the identity to the tangent space at R, so RG_i form a basis for this tangent space.

This mapping from the identity to another point on the group is very important for solving many problems in computer vision.

2.8 The adjoint representation of the group

Equation 2.35 maps the neighbourhood of the identity (or the tangent space) to R by left multiplication. It is also possible to use right multiplication, so for Δ in the tangent space:

$$I + \Delta \quad \rightarrow \quad (I + \Delta)R = R + \Delta R \tag{2.37}$$

Because group elements do not in general commute, this mapping is not the same as that in equation 2.35. One way to examine this difference is to use one mapping to transform the tangent space from the identity to R and the other mapping to bring it back to the identity:

$$R : \Delta \rightarrow R\Delta R^{-1} \tag{2.38}$$

So R induces a transformation on the tangent space and because 2.38 is a matrix expression which is linear in Δ, this is a linear transformation. We can work out in particular what R does to each of the generator matrices. Since equation 2.38 maps the tangent space to itself, each generator matrix must map to some linear combination of the generators:

$$R : G_i \rightarrow RG_iR^{-1} = \sum_j A_{ji}G_j \tag{2.39}$$

And because this is linear, the same coefficients a_{ji} can be used to transform any linear combination of G_i:

$$R : \sum_i \alpha_i G_i \rightarrow R\left(\sum_i \alpha_i G_i\right)R^{-1} = \sum_{ij} A_{ji}\alpha_i G_j = \sum_j \beta_j G_j \tag{2.40}$$

This means that A_{ji} can be viewed as a matrix acting on a vector of coefficients (α) to give another vector of coefficients (β). This creates a mapping from members of the group, R to matrices that act on coefficients of basis vectors in the tangent space, A. The matrices A have the same multiplication structure as R ie:

$$R_1 = R_2 R_3 \implies A_1 = A_2 A_3 \tag{2.41}$$

A mapping of the group onto matrices is called a **representation** of the group. This particular representation is called the **adjoint representation**.

2.8.1 The adjoint representation of SO(3)

Recall from equation 2.32 that the generators of SO(3) are cross product matrices, in particular, $G_i = [\mathbf{e}_i\wedge]$, where $\mathbf{e}_i$ is the unit vector in the direction of the $\mathbf{i}^{\text{th}}$ axis. Equation 2.39 becomes:

$$R : G_i \to R G_i R^{-1} = R[\mathbf{e}_i\wedge]R^{-1} = \sum_j a_{ji} G_j \tag{2.42}$$

Now $R[\mathbf{e}_i\wedge]R^{-1}$ can be calculated by noticing that the cross product operation is coordinate free:

$$\mathbf{a} = \mathbf{b} \wedge \mathbf{c} \tag{2.43}$$

$$\implies R\mathbf{a} = R\mathbf{b} \wedge R\mathbf{c} \quad \text{for any } R \in \text{SO}(3) \tag{2.44}$$

$$\implies \mathbf{a} = R^{-1}\left(R\mathbf{b} \wedge R\mathbf{c}\right) \tag{2.45}$$

$$\implies \mathbf{a} = R^{-1}[(R\mathbf{b})\wedge]R\mathbf{c} \tag{2.46}$$

$$\implies R^{-1}[(R\mathbf{b})\wedge]R = [\mathbf{b}\wedge] \tag{2.47}$$

$$\implies [(R\mathbf{b})\wedge] = R[\mathbf{b}\wedge]R^{-1} \tag{2.48}$$

Hence $R[\mathbf{e}_i\wedge]R^{-1} = [R\mathbf{e}_i\wedge]$ which means that for SO(3), the coefficients A_{ji} in equation 2.39 are the same as those in the matrix R_{ji}. This means that SO(3) is **self adjoint**

2.8.2 The adjoint representation of SE(3)

fSE(3) on the other hand cannot be self adjoint. The tangent space of SE(3) is six dimensional, so the matrices A are 6×6, whereas the matrices in SE(3) are 4×4. This means that there is a mapping from members of SE(3) parameterised by R and $\mathbf{t}$ to 6×6 matrices. To work out what this mapping is, it is necessary to look at how the members of SE(3) act on the generators by conjugation. It is easiest to consider the three translation generators and the three rotation generators as two separate groups. Considering the translation generators first:

$$\text{Let } \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{then} \quad aG_1 + bG_2 + cG_3 = \begin{bmatrix} 0 & 0 & 0 & | \\ 0 & 0 & 0 & \mathbf{v} \\ 0 & 0 & 0 & | \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.49}$$

Then the action on this by a member of SE(3) by conjugation is:

$$\begin{bmatrix} \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} & \begin{matrix} | \\ \mathbf{t} \\ | \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & | \\ 0 & 0 & 0 & \mathbf{v} \\ 0 & 0 & 0 & | \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} & & \\ & R^T & \\ & & \end{bmatrix} & \begin{matrix} | \\ -R^T\mathbf{t} \\ | \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \tag{2.50}$$

$$= \begin{bmatrix} 0 & 0 & 0 & | \\ 0 & 0 & 0 & R\mathbf{v} \\ 0 & 0 & 0 & | \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} & & \\ & R^T & \\ & & \end{bmatrix} & \begin{matrix} | \\ -R^T\mathbf{t} \\ | \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \tag{2.51}$$

$$= \begin{bmatrix} 0 & 0 & 0 & | \\ 0 & 0 & 0 & R\mathbf{v} \\ 0 & 0 & 0 & | \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.52}$$

So when rotations act on the translation generators by conjugation, they just cause a rotation of the basis of generators, the coefficients in $\mathbf{v}$ are mapped to $R\mathbf{v}$. The translation component $\mathbf{t}$ has no effect on the translation generators, which should not be a surprise because tranlations commute with each other (reversing the order of two translations has no effect).

Now consider the rotation generators. Keeping the same definition of $\mathbf{v}$ from equation 2.49, gives:

$$aG_4 + bG_5 + cG_6 = \begin{bmatrix} \begin{bmatrix} & & \\ & \mathbf{v}\wedge & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \tag{2.53}$$

And the action on this by a member of SE(3) by conjugation is:

$$\begin{bmatrix} \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} & \begin{matrix} | \\ \mathbf{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} & & \\ & \mathbf{v}\wedge & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} & & \\ & R^T & \\ & & \end{bmatrix} & \begin{matrix} | \\ -R^T\mathbf{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \tag{2.54}$$

$$= \begin{bmatrix} \begin{bmatrix} & & \\ & R[\mathbf{v}\wedge] & \\ & & \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} & & \\ & R^T & \\ & & \end{bmatrix} & \begin{matrix} | \\ -R^T\mathbf{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \tag{2.55}$$

$$= \begin{bmatrix} \begin{bmatrix} & & \\ & R[\mathbf{v}\wedge]R^T & \\ & & \end{bmatrix} & \begin{matrix} | \\ -R[\mathbf{v}\wedge]R^T\mathbf{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \tag{2.56}$$

$$= \begin{bmatrix} \begin{bmatrix} & & \\ & [(R\mathbf{v})\wedge] & \\ & & \end{bmatrix} & \begin{matrix} | \\ -(R\mathbf{v})\wedge\mathbf{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \tag{2.57}$$

$$= \begin{bmatrix} \begin{bmatrix} & & \\ & [(R\mathbf{v})\wedge] & \\ & & \end{bmatrix} & \begin{matrix} | \\ [\mathbf{t}\wedge]R\mathbf{v} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \tag{2.58}$$

This means that rotations act on the rotation generators in the same way that they acted on the translation generators. Translations act on the rotation generators to couple them to the translation generators. Combining equations 2.58 and 2.52 gives the general form for the adjoint representation $\mathrm{Adj}(E)$ for any $E \in \mathrm{SE}(3)$:

$$\begin{bmatrix} \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} & \begin{matrix} | \\ \mathbf{t} \\ | \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \xrightarrow[\mathrm{Adj}]{} \begin{bmatrix} \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} & \begin{bmatrix} & & \\ & [\mathbf{t}\wedge]R & \\ & & \end{bmatrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \end{bmatrix} \tag{2.59}$$

This means that if

$$A = \sum_{i=1}^{6} \alpha_i G_i \tag{2.60}$$

and for $E \in \mathrm{SE}(3)$,

$$EAE^{-1} = B \tag{2.61}$$

with

$$B = \sum_{i=1}^{6} \beta_i G_i \tag{2.62}$$

then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by

$$\boldsymbol{\beta} = \mathrm{Adj}(E)\boldsymbol{\alpha} \tag{2.63}$$

and also

$$Ee^A E^{-1} = e^B \tag{2.64}$$

2.9 The Lie bracket

Properly, the Lie algebra is the vector space tangent to the Identity of the group, together with a bilinear antisymmetric operator called the Lie bracket. The tangent space can be mapped onto elements of the group by exponentiation and thus captures the local structure of the group.

If the group is not commutative (in general $g_1 g_2 \neq g_2 g_1$ for $g_1, g_2 \in G$), then this is visible in the tangent space also. If A and B are matrices in the tangent space (i.e. linear combinations of generator matrices), then

$$e^A e^B \neq e^B e^A \implies AB \neq BA \tag{2.65}$$

The difference between AB and BA is called the commutator, the **Lie bracket**, denoted $[A, B]$:

$$[A, B] = AB - BA \tag{2.66}$$

It is a property of Lie groups that the Lie bracket of two matrices in the Lie algebra (the tangent space) is also in the Lie algebra. The Lie bracket is bilinear in A and B, so this can be checked by computing the Lie bracket for the Generator matrices. The Lie bracket for any linear combinations of matrices is then just a linear combination of the Lie brackets of the Generators.

For SO(3), the generators are given in 2.9 and the Lie bracket $[G_1, G_2]$ can be computed as:

$$[G_1, G_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.67}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.68}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.69}$$

$$= G_3 \tag{2.70}$$

The Lie bracket is antisymmetric, so $[G_2, G_1] = -[G_1, G_2] = -G_3$. The Lie brackets for the other generators can be calculated similarly, so $[G_3, G_1] = G2$ and $[G_2, G_3] = G_1$.

2.9.1 The Baker-Cambell-Hausdorff formula

The Lie bracket is particularly useful for combining non-infinitessimal elements of the Lie algebra. Given A and B in the tangent space, it is sometimes useful to calculate C in the tangent space such that

$$e^C = e^A e^B \tag{2.71}$$

Because matrices don't commute in general, $e^A e^B \neq e^B e^A$ and so it should not be expected that $C = A+B$ as would be the case for scalars. Instead C can be calculated by successively refining estimates of C.

$$e^A e^B = (I + A + \tfrac{1}{2}A^2 + \cdots)(I + B + \tfrac{1}{2}B^2 + \cdots) \tag{2.72}$$

$$= I + A + B + \tfrac{1}{2}A^2 + AB + \tfrac{1}{2}B^2 + \cdots \tag{2.73}$$

Estimating $C = A + B$ gives

$$e^C = I + C + \tfrac{1}{2}C^2 + \cdots \tag{2.74}$$

$$= I + A + B + \tfrac{1}{2}(A^2 + AB + BA + B^2) + \ldots \tag{2.75}$$

This is very similar to the series in equation 2.73 but instead of AB the series has $\frac{1}{2}(AB + BA)$. Thus e^{A+B} needs to have $\frac{1}{2}(AB - BA) = \frac{1}{2}[A, B]$ added to it.

This can be fixed by refining the estimate: $C = A + B + \frac{1}{2}[A, B]$. This can be substituted back into e^C expanded to more terms and the most significant (lowest power) term absorbed back into the estimate for C. This gives the Baker-Cambell-Hausdorff formula, for which the first few terms are:

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [[A, B], B]) + \frac{1}{24}[A, [[A, B], B]] + \cdots \tag{2.76}$$

The series doesn't have a discernable pattern but in essence the series generated by e^{A+B} has terms involving arbitrary sequences of As and Bs (e.g.$\frac{1}{24}BABA$) and it's the job of the Lie brackets to pump all the As to the left and the Bs to the right so that all terms are of the form $A^n B^m$ as they are in $e^A e^B$.

2.10 Other Lie groups

The groups SO(3) and SE(3) have been used extensively as examples, but much of what has been described applies to all Lie groups. In particular:

- All matrix groups are closed under matrix multiplication and must include the identity matrix.
- The set of possible derivatives at the identity form a vector space called the Lie algebra of the group.
- A basis for this vector space is called the Generators of the group.
- Exponentiating any member of the Lie algebra gives an element of the group.
- If the group is connected, any member of the group can be obtained this way.
- The tangent space at any member of the group can be obtained by multiplying the members of the Lie algebra by that member of the group.
- The adjoint representation of the group can be obtained from the action of the group on its tangent space by conjugation.

Chapter 3

Projective geometry

In section 1.5.4, homogeneous coordinates were introduced as a way of allowing translations and rotations to be combined in a single matrix. Sometimes it is convenient to relax the notation and consider a vector in homogeneous coordinates to be equivalent to any scalar multiple of that vector:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \equiv \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda \end{pmatrix} \tag{3.1}$$

Here the symbol "$\equiv$" means "equivalent up to scale" and will be treated like "$=$" for projective geometry.

It is always possible to recover the 3D position referred to by any 4-vector by dividing through by the last element (normalising the vector so that the last element equals one):

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \equiv \begin{pmatrix} a/d \\ b/d \\ c/d \\ 1 \end{pmatrix} \tag{3.2}$$

So the 3D coordinates of the point referred to by this vector are $x = a/d$, $y = b/d$ and $z = c/d$.

These equivalence classes of 4-vectors are members of **projective 3 space**, formally written $\mathbb{P}^3(\mathbb{R})$ and here abbreviated to $\mathbb{P}^3$. Note that the zero vector would be equivalent to everything in the space and hence is excluded.

3.1 Matrix multiplication

Consider what happens when a vector $\mathbf{v}_1$ in an equivalence class in $\mathbb{P}^3$ is multiplied by a matrix M giving a result $\mathbf{v}_2$:

$$\mathbf{v}_2 = M\mathbf{v}_1 \tag{3.3}$$

and then a different member of the equivalence class, $\lambda\mathbf{v}_1$, is substituted for $\mathbf{v}_1$. The multiplication becomes:

$$M(\lambda\mathbf{v}_1) = \lambda M\mathbf{v}_1 \tag{3.4}$$

because multiplying each of the elements of $\mathbf{v}_1$ by λ just makes the result λ times larger. But $M\mathbf{v}_1 = \mathbf{v}_2$, so:

$$M(\lambda\mathbf{v}_1) = \lambda\mathbf{v}_2 \equiv \mathbf{v}_2 \tag{3.5}$$

Thus matrix multiplication still works on members of projective space. In fact matrices that act on $\mathbb{P}^3$ can do more things than those that act on $\mathbb{R}^3$ since they are 4×4 matrices rather than 3×3. Importantly, though matrices that act on $\mathbb{P}^3$ only have 15 degrees of freedom, not 16, since a scalar multiple of a matrix performs the same transformation. These transformations are referred to as **homographies** in the computer vision literature and invertible homographies on $\mathbb{P}^N$ form a Lie group called PGL(N).

3.2 Points at infinity

Look again at equation 3.2. What happens when $d = 0$?. The normalisation of dividing everything by d is no longer possible and such vectors cannot refer to points in space. To understand what they do refer to, consider taking the limit as $d \to 0$. For small positive d, the real 3D coordinates x, y and z are very large (since they're obtained by dividing by d). As d goes to zero, the 3D point becomes infinitely far away. Note that for small negative d the values of x, y and z head off to infinity in the opposite direction, so there is no distinction between a point at inifinity in one direction and in the opposite direction.

When $d = 0$, the vector is controlled by just three numbers (a, b and c). The same rules about scale equivalence apply to these numbers and so the set of points for which $d = 0$ form a projective subspace of $\mathbb{P}^3$ isomorphic to $\mathbb{P}^2$. For this reason, these points are described as being on the **plane at infinity**.

3.3 Images and projective 2 space

Projective geometry arises naturally in the formation of images in a pinhole camera. A camera has a natural coordinate system with its origin at the pinhole, its x and y axes parallel to the imaging plane and its z axis perpendicular (see figure 3.1).

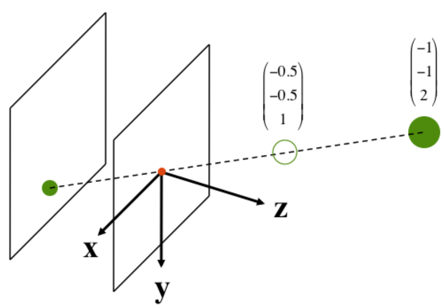

Figure 3.1: A pinhole camera with its natural coordinate system centred on the pinhole (red dot), showing the projection of a real world point (large green disc) onto the image plane (small green disc).

From the image of the green disc, it is not possible to determine how far away it is, only which line it lies on. If the x, y and z coordinates of the disc were divided by two, it would move in a straight line half way towards the pinhole and so it would appear in the same place on the image. This means that the image cannot distinguish between a point at position $\mathbf{v}$ and one at $\lambda\mathbf{v}$. In other words, as far as the image is concerned, these points are equivalent and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \tag{3.6}$$

This has the same form as 3.1 and so the act of projecting the 3 dimensional world onto the image plane creates a mapping from $\mathbb{R}^3 \to \mathbb{P}^2$.

It is convenient to use homogeneous coordinates to describe points on the image plane. For a real world point with coordinates x, y and z:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \tag{3.7}$$

u and v are called **normalised camera coordinates**

Here when $z = 0$, points project infinitely far off to the side of the image plane and following the same convention as before, these points form a subspace isomorphic to $\mathbb{P}^1$ called the **line at infinity**.

3.4 Points, lines and planes

3.4.1 Lines in $\mathbb{P}^2$

One reason that it is convenient to use homogeneous coordinates to represent points on the image plane is that it makes it easier to represent straight lines. A straight line in $u - v$ space might be represented by an equation:

$$v = au + c \tag{3.8}$$

but this is unable to represent vertical lines and so a more general expression is used:

$$au + bv + c = 0 \tag{3.9}$$

With this form, a vertical line can be represented (with the coefficient $a = 0$) or a horizontal line (with $b = 0$). Because of the use of homogeneous coordinates, this line equation can be represented as a dot product:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = 0 \tag{3.10}$$

This allows the line to be represented as a vector that has a dot product of zero with all points on the line (ie the line vector is orthogonal to the coordinate vector of any point on the line).

This representation of lines makes some geometry easy. Given two points $\mathbf{p}_1$ and $\mathbf{p}_2$, the vector of the line $\mathbf{l}$ is easy to determine because it must be orthogonal to both $\mathbf{p}_1$ and $\mathbf{p}_2$. Hence the line vector can calculated as:

$$\mathbf{l} = \mathbf{p}_1 \wedge \mathbf{p_2}. \tag{3.11}$$

Given two lines $\mathbf{l}_1$ and $\mathbf{l}_2$, the point at which they intersect is similarly easy to determine since that point must be orthogonal to both of the line vectors and can be calculated as:

$$\mathbf{p} = \mathbf{l}_1 \wedge \mathbf{l_2}. \tag{3.12}$$

Thus there is a nice duality between points and lines in $\mathbb{P}^2$.

3.4.2 Planes in $\mathbb{P}^3$

Planes in $\mathbb{P}^3$ are similarly easy to define in homogeneous coordinates with an equation of the form:

$$ax + by + cz + d = 0 \tag{3.13}$$

which can similarly be represented by a dot product:

$$\pi \cdot \mathbf{p} = 0 \quad (\pi \text{ is the usual name for a vector representing a plane}) \tag{3.14}$$

3.4.3 Lines in $\mathbb{P}^3$

Lines in $\mathbb{P}^3$ take a little more care and require a generalisation of the cross product operator to four dimensions. The generalised cross product operator $\wedge$ makes use of a tensor called the Levi-Civita symbol, written ϵ.

In four dimensions, ϵ has four indices: ϵ_{ijkl}. Each of the indices i, j, k, l has a value from 1 to 4, so it is a $4 \times 4 \times 4 \times 4$ tensor. The value of ϵ is given by:

$$\epsilon_{ijkl} = \begin{cases} 1 & \text{if } i,j,k,l \text{ are an even permutation of } 1,2,3,4 \\ -1 & \text{if } i,j,k,l \text{ are an odd permutation of } 1,2,3,4 \\ 0 & \text{if } i,j,k,l \text{ are not a permutation of } 1,2,3,4 \text{ (ie contain a repeated value)} \end{cases} \tag{3.15}$$

For example, $\epsilon_{1234} = 1$, $\epsilon_{1243} = -1$ and $\epsilon_{1233} = 0$. A similar tensor exists in 3 dimensions but only has 3 indices which have values from 1 to 3.

The line L through two points $\mathbf{p}$ and $\mathbf{q}$ is represented by an antisymmetric matrix which is the cross product of $\mathbf{p}$ and $\mathbf{q}$ in four dimensions, defined by:

$$L = [\mathbf{p} \wedge \mathbf{q}]_{ij} = \sum_{kl} \epsilon_{ijkl} \mathbf{p}_k \mathbf{q}_l \tag{3.16}$$

To see how this works, an example is helpful. To calculate row 1, column 3 of this matrix: $[\mathbf{p} \wedge \mathbf{q}]_{13}$ $(i = 1, j = 3)$, then all values of k and l have to be considered. If either k or l has the value 1 or 3 then $ijkl$ is not a permutation of 1234 and ϵ has the value 0. The only values k and l can have that make a contribution are 2 and 4. If $k = 2$ and $l = 4$ then $ijkl = 1324$. This is an odd permutation of 1234 because it can be obtained with just one transposition (2 and 3), so the contribution of $\mathbf{p}_2\mathbf{q}_4$ is negative (and similarly the contribution of $\mathbf{p}_4\mathbf{q}_2$ is positive. Thus $[\mathbf{p} \wedge \mathbf{q}]_{13} = \mathbf{p}_4\mathbf{q}_2 - \mathbf{p}_2\mathbf{q}_4$.

Writing $\mathbf{p}$ and $\mathbf{q}$ in terms of their entries gives:

$$L = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \wedge \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{bmatrix} 0 & ch - dg & df - bh & bg - cf \\ dg - ch & 0 & ah - de & ce - ag \\ bh - df & de - ah & 0 & af - be \\ cf - bg & ag - ce & be - af & 0 \end{bmatrix} \tag{3.17}$$

This matrix has some useful properties. First, each of the two vectors used to create it is in its null space:

$$\begin{bmatrix} 0 & ch - dg & df - bh & bg - cf \\ dg - ch & 0 & ah - de & ce - ag \\ bh - df & de - ah & 0 & af - be \\ cf - bg & ag - ce & be - af & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} bch - bdg + cdf - bch + bdg - cdf \\ adg - ach + ach - cde + cde - adg \\ abh - adf + bde - abh + adf - bde \\ acf - abg + abg - bce + bce - acf \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.18}$$

and

$$\begin{bmatrix} 0 & ch - dg & df - bh & bg - cf \\ dg - ch & 0 & ah - de & ce - ag \\ bh - df & de - ah & 0 & af - be \\ cf - bg & ag - ce & be - af & 0 \end{bmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} cfh - dfg + dfg - bgh + bgh - cfh \\ deg - ceh + agh - deg + ceh - agh \\ beh - def + def - ahf + afh - beh \\ cef - beg + afg - cef + beg - afg \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.19}$$

This means that any linear combination of $\mathbf{p}$ and $\mathbf{q}$ (any point on the line through them) is in the nullspace of $[\mathbf{p} \wedge \mathbf{q}]$, so this matrix represents the line through the two points in much the same way as the line vector resulting from the cross product of two points does in $\mathbb{P}^2$.

Further, for a point $\mathbf{r}$ not on the line the result of $[\mathbf{p} \wedge \mathbf{q}]\,\mathbf{r}$ represents the plane through $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. So if $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ are coplanar:

$$([\mathbf{p} \wedge \mathbf{q}]\,\mathbf{r}) \cdot \mathbf{s} = 0 \tag{3.20}$$

There is a second matrix that also represents a line. This matrix is given by the antisymmetric product of any two points on the line: $\mathcal{L} = \mathbf{p}\mathbf{q}^T - \mathbf{q}\mathbf{p}^T$.

$$\mathcal{L} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}^T - \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}^T = \begin{bmatrix} 0 & af - be & ag - ce & ah - de \\ be - af & 0 & bg - cf & bh - df \\ ce - ag & cf - bg & 0 & ch - dg \\ de - ah & df - bh & dg - ch & 0 \end{bmatrix} \tag{3.21}$$

This matrix contains the same entries as the cross product matrix, but in different locations. Figure 3.2 shows how entries in the matrix must be swapped to go from one matrix to the other. This swapping can be written mathematically by using the Levi-Civita symbol:

$$[\mathbf{p}\mathbf{q}^T - \mathbf{q}\mathbf{p}^T]_{ij} = \frac{1}{2}\sum_{kl} \epsilon_{ijkl}[\mathbf{p} \wedge \mathbf{q}]_{kl} \equiv \sum_{kl} \epsilon_{ijkl}[\mathbf{p} \wedge \mathbf{q}]_{kl} \tag{3.22}$$

$$\text{or} \quad \mathcal{L}_{ij} \equiv \sum_{kl} \epsilon_{ijkl} L_{kl} \tag{3.23}$$

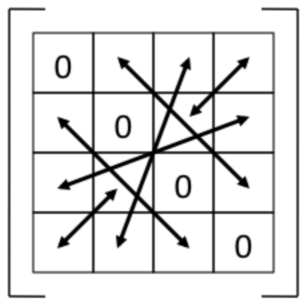

Figure 3.2: How the entries in the matrix must be swapped to go between $L = [\mathbf{p} \wedge \mathbf{q}]$ and $\mathcal{L} = \mathbf{p}\mathbf{q}^T - \mathbf{q}\mathbf{p}^T$ or vice versa.

The second form of the line matrix $\mathcal{L}$ is known as **Plücker coordinates**, while the first form L is known as **dual Plücker coordinates**.

The second line representation is also useful in a number of ways. If it is multiplied by the vector representing a plane, it gives the point of intersection between the line and the plane (the result lies on both the line and the plane):

$$\text{If} \quad \mathbf{r} = \mathcal{L}\pi \quad \text{then} \quad \pi \cdot \mathbf{r} = 0 \quad \text{and} \quad L\mathbf{r} = \mathbf{0} \tag{3.24}$$

In the above equation, if $\mathbf{r} = \mathbf{0}$, then the line $\mathcal{L}$ lies in the plane π.

A line can also be obtained by intersecting two planes π_1 and π_2 in much the same way that it can be obtained from two points, except that the equations for obtaining the two line representations have to be swapped:

$$L = \pi_1\pi_2^T - \pi_2\pi_1^T \tag{3.25}$$

$$\mathcal{L} = [\pi_1 \wedge \pi_2] \tag{3.26}$$

Finally, two lines can be tested for intersection by using one matrix of each form. The lines L and M intersect if and only if

$$\sum_{ij} L_{ij}\mathcal{M}_{ij} = 0 \qquad (L \text{ in first form and } \mathcal{M} \text{ in second form}) \tag{3.27}$$

or equivalently:

$$\mathrm{Trace}\,(L\mathcal{M}) = 0 \tag{3.28}$$

or

$$\sum_{ijkl} \epsilon_{ijkl}\, L_{ij}\, M_{kl} = 0 \tag{3.29}$$

or

$$\sum_{ijkl} \epsilon_{ijkl}\, \mathcal{L}_{ij}\, \mathcal{M}_{kl} = 0 \tag{3.30}$$

3.5 Transforming points lines and planes

Given an invertible 4×4 matrix A which might be a member of a group like SE(3), it can act on a point $\mathbf{p}$ in $\mathbb{P}^3$ to give a point $\mathbf{p}'$ by:

$$\mathbf{p}' = A\mathbf{p} \tag{3.31}$$

It is also important to work out how planes and lines transform when the matrix A acts on the space. Given a plane π,

$$\text{If} \quad \pi \cdot \mathbf{p} = 0 \quad \text{then} \quad \pi' \cdot \mathbf{p}' = 0 \tag{3.32}$$

writing the dot product using transpose notation gives:

$$\pi^T \mathbf{p} = 0 \tag{3.33}$$
$$\Longrightarrow \pi^T A^{-1} A\mathbf{p} = 0 \tag{3.34}$$
$$\Longrightarrow \pi^T A^{-1} \mathbf{p}' = 0 \tag{3.35}$$
$$\Longrightarrow \pi'^T = \pi^T A^{-1} \tag{3.36}$$
$$\Longrightarrow \pi' = A^{-T} \pi \tag{3.37}$$

Thus a plane equation transforms according to the transpose of the inverse (=the inverse of the transpose) of A. In $\mathbb{P}^2$, points and lines transform in exactly the same way.

The transformation of the line matrix $L = [\mathbf{p} \wedge \mathbf{q}]$ can also be determined by looking at the equation that gives a plane from a point and a line:

$$\pi = L\mathbf{r} \tag{3.38}$$
$$\Longrightarrow A^{-T}\pi = A^{-T} L A^{-1} A\mathbf{r} \tag{3.39}$$
$$\Longrightarrow \pi' = A^{-T} L A^{-1} \mathbf{r}' \tag{3.40}$$
$$\Longrightarrow L' = A^{-T} L A^{-1} \tag{3.41}$$

The transformation of the second form of the line matrix is also easy to determine:

$$\mathcal{L} = \mathbf{p}\mathbf{q}^T - \mathbf{q}\mathbf{p}^T \tag{3.42}$$
$$\Longrightarrow \mathcal{L}' = \mathbf{p}'\mathbf{q}'^T - \mathbf{q}'\mathbf{p}'^T \tag{3.43}$$
$$\Longrightarrow \mathcal{L}' = A\mathbf{p}\mathbf{q}^T A^T - A\mathbf{q}\mathbf{p}^T A^T \tag{3.44}$$
$$\Longrightarrow \mathcal{L}' = A\mathcal{L}A^T \tag{3.45}$$

This can be checked with the equation for obtaining the intersection between a line and a plane:

$$\mathbf{p}' = \mathcal{L}'\pi' \tag{3.46}$$
$$= A\mathcal{L}A^T A^{-T}\pi \tag{3.47}$$
$$= A\mathcal{L}\pi \tag{3.48}$$
$$= A\mathbf{p} \tag{3.49}$$

3.6 Conics

In addition to points, lines and planes, it is also convenient to represent circles, ellipses, parabolae and hyperbolae. These structures are governed by quadratic formulae. For example, a circle in 2D of radius r centred at (α, β) can be represented by the equation

$$(x-\alpha)^2 + (y-\beta)^2 = r^2 \tag{3.50}$$

This equation can be varied to obtain an ellipse by changing the coefficient of $(x-\alpha)^2$ or $(y-\beta)^2$ or by adding a term in $(x-\alpha)(y-\beta)$ to give equations like

$$A(x-\alpha)^2 + B(y-\beta)^2 + C(x-\alpha)(y-\beta) = r^2 \tag{3.51}$$

which is a general quadratic form on x and y:

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \tag{3.52}$$

Conveniently this can be written in matrix form using homogenous coordinates:

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{bmatrix} a & \frac{c}{2} & \frac{d}{2} \\ \frac{c}{2} & b & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \tag{3.53}$$

This matrix is chosen because it is symmetric and so equation 3.53 is identical to its transpose. Thus the general form of conics is

$$\mathbf{p}^T C \mathbf{p} = 0 \tag{3.54}$$

3.6.1 Transforming conics

Calculating how conics transform under coordinate frame transformations is straightforward. If $\mathbf{p}' = A\mathbf{p}$ then

$$\begin{align} 0 &= \mathbf{p}^T C \mathbf{p} \tag{3.55} \\ &= \mathbf{p}^T A^T A^{-T} C A^{-1} A \mathbf{p} \tag{3.56} \\ &= \mathbf{p}'^T A^{-T} C A^{-1} \mathbf{p}' \tag{3.57} \\ &= \mathbf{p}'^T C' \mathbf{p}' \tag{3.58} \end{align}$$

Thus

$$C' = A^{-T} C A^{-1} \tag{3.59}$$

Chapter 4

Optimization

4.1 Least squares

Many complex systems have multiple inputs and outputs. This can be represented mathematically as:

$$\mathbf{o} = F(\boldsymbol{\theta}) \tag{4.1}$$

where F represents the system, $\boldsymbol{\theta}$ represents the inputs and $\mathbf{o}$ are the outputs.

A common requirement is the need to adjust the inputs of such a system so that the outputs match a predetermined set of values $\hat{\mathbf{o}}$:

$$\text{Find } \boldsymbol{\theta} \text{ such that } F(\boldsymbol{\theta}) = \hat{\mathbf{o}} \tag{4.2}$$

This situation commonly arises in two ways:

- F is a system, $\boldsymbol{\theta}$ are the control parameters of the system and $\hat{\mathbf{o}}$ are the desired outputs from the system. Here the goal is to adjust the system into a desired state.
- F is a model of a system, $\boldsymbol{\theta}$ are the control parameters of the model and $\hat{\mathbf{o}}$ are the observed outputs from the real system. Here the goal is to determine the parameters of the model that make its output match that of the real system and hence infer something about the state of the real system.

With complex systems it is often impossible to invert F analytically and so an optimization procedure is used, starting with some plausible value for $\boldsymbol{\theta}$ and adjusting it until $F(\boldsymbol{\theta})$ has its desired value, $\hat{\mathbf{o}}$. It is also often the case that the range of F does not fill the space that it inhabits and so $\hat{\mathbf{o}}$ may be unachieveable. In this case, it is typically desired to minimize $||\hat{\mathbf{o}} - \mathbf{o}||$, the L_2 norm, or sum of squared differences.

4.1.1 Relationship between least squares and iid normal distributions

Minimizing the squared error between the vectors $\hat{\mathbf{o}}$ and $\mathbf{o}$ is equivalent to finding the most probable $\boldsymbol{\theta}$ under the assumption that the errors between $\hat{\mathbf{o}}$ and $\mathbf{o}$ are independent, identically distributed zero mean normal distributions. Such distributions have a density function given by:

$$p(x) = \tfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} \tag{4.3}$$

Writing x_i as the ith element of $\hat{\mathbf{o}} - \mathbf{o}$ means that the probability of observing $\hat{\mathbf{o}} - \mathbf{o}$ is given by:

$$p(\hat{\mathbf{o}} - \mathbf{o}) = \prod_i \tfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x_i^2}{2\sigma^2}} \tag{4.4}$$

$$= \left(\tfrac{1}{\sigma\sqrt{2\pi}}\right)^N e^{-\frac{\sum_i x_i^2}{2\sigma^2}} \tag{4.5}$$

Thus $p(\hat{\mathbf{o}} - \mathbf{o})$ is maximized if $\sum_i x_i^2$ is minimized. So maximum probability in this regime corresponds directly to least squares.

4.1.2 Solving for the least squares solution

Setting the error function

$$E = \sum_i x_i^2 \tag{4.6}$$

provides a framework for optimization in which $\boldsymbol{\theta}$ is adjusted so that E is minimized. In particular this means that at the optimum:

$$\frac{\partial E}{\partial \theta_j} = 0 \tag{4.7}$$

Substituting equation 4.6 into equation 4.7 gives:

$$\frac{\partial \sum_i x_i^2}{\partial \theta_j} = 0 \tag{4.8}$$

$$= \sum_i \frac{\partial x_i^2}{\partial \theta_j} \tag{4.9}$$

$$= \sum_i 2x_i \frac{\partial x_i}{\partial \theta_j} \tag{4.10}$$

4.2 Optimizing with Lie groups

4.3 Probability density functions over Lie groups

Chapter 5

Applications

That's wonderful. But what does it do?

Arthur Dent

5.1 Computing pose from keypoints

5.2 Computing pose from edges

5.3 Kalman Filtering with Lie groups