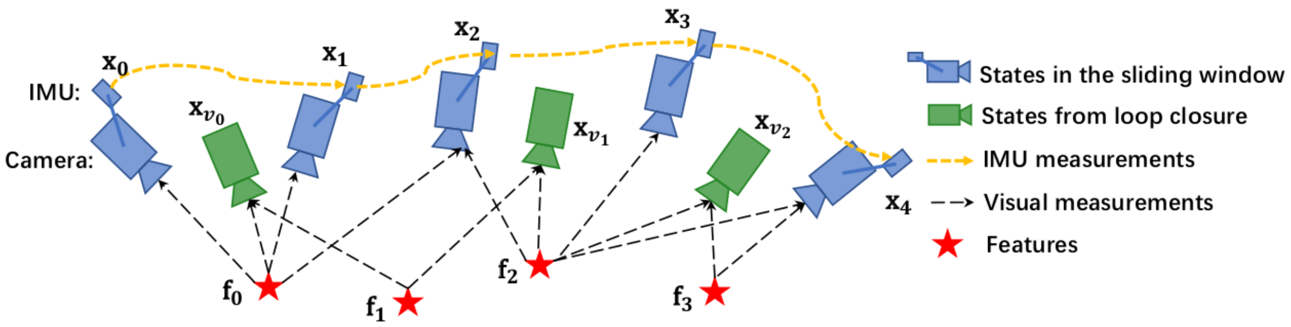
后端非线性优化

这是一个基于滑窗的紧耦合VIO。



1.状态向量

在滑窗内的所有状态向量包括$n+1$个IMU状态和相机到IMU的外参以及$m+1$个特征点的逆深度。其中每个IMU状态包括IMU在世界坐标系中的位置，速度，方向，在IMU坐标系中的加速度bias，陀螺仪bias。

$$\begin{gathered}
\mathcal{X} = [x_0, x_1, \ldots, x_n, x_c^b, \lambda_1, \lambda_2, \ldots, \lambda_m] \\
x_k = [p_{b_k}^w, v_{b_k}^w, q_{b_k}^w, b_a, b_g] \\
x_c^b = [p_c^b, q_c^b]
\end{gathered} \tag{1}$$

这里λ_l是第l个特征点第一次被观测到的逆深度。

2.目标函数

最小化先验信息和所有测量参差的和，从而得到一个最大后验估计：

$$\min_{\mathcal{X}} \{ \| r_p - H_p \mathcal{X} \|^2 + \sum_{k \in \mathcal{B}} \| r_{\mathcal{B}}(\hat{z}_{b_{k+1}}^{b_k}, \mathcal{X}) \|_{p_{b_{k+1}}^{b_k}}^2 + \sum_{(l,j)} \rho(\| r_{\mathcal{C}}(\hat{z}_l^{c_j}, \mathcal{X}) \|_{p_l^{c_j}}^2) \} \tag{2}$$

其中Huber算子定义如下：

$$\rho(s)=\begin{cases}1, & s\geq 1\\ 2\sqrt{s}-1, & s<1\end{cases} \tag{3}$$

$r_{\mathcal{B}}(\hat{z}^{b_k}_{b_{k+1}},\mathcal{X})$,$r_{\mathcal{C}}(\hat{z}^{c_j}_{l},\mathcal{X})$分别对应IMU和视觉的测量残差。$\mathcal{B}$是所有IMU测量的集合，$\mathcal{C}$是在当前滑窗中至少被观测到一次的所有特征点的集合。$\{r_p,\mathcal{H}_p\}$是来自边缘化的先验信息。3种残差都是用马氏距离表示。

根据《SLAM中的优化理论》，定义目标函数为：

$$f(\mathcal{X})=\frac{1}{2}r_i^T(\mathcal{X})r_i(\mathcal{X}) \tag{4}$$

雅各比矩阵$J(\mathcal{X})$:

$$J_{i,j}(\mathcal{X})=[\frac{\partial r_i(\mathcal{X})}{\partial x_j}] \tag{5}$$

假设目标函数f是平滑的，我们可以对其进行泰勒展开：

$$f(\mathcal{X}+\triangle\mathcal{X})=f(\mathcal{X})+\triangle\mathcal{X}^Tg(\mathcal{X})+\frac{1}{2}\triangle\mathcal{X}^T\mathcal{H}(\mathcal{X})\triangle\mathcal{X}+\mathcal{O}(\parallel\triangle\mathcal{X}\parallel^3) \tag{6}$$

其中$g(\mathcal{X})$,$\mathcal{H}(\mathcal{X})$分别对应目标函数的梯度和海森矩阵：

$$g(\mathcal{X})=J^T(\mathcal{X})r(\mathcal{X}) \tag{7}$$

$$\mathcal{H}(\mathcal{X})=J^T(\mathcal{X})J(\mathcal{X}) \tag{8}$$

高斯牛顿法是一种高效的方法，它是一种基于一阶导数推导而来的。在某些情况下，具有平方收敛的效果。高斯牛顿方法工作的基础是在工作点$\mathcal{X}$附近邻域线性化：

$$r(\mathcal{X}+\triangle\mathcal{X})=r(\mathcal{X})+J(\mathcal{X})r(\mathcal{X}) \tag{9}$$

由于要使得目标函数有极小值，目标函数的一阶导数应该为0，由此可以得出高斯牛顿法的正则方程或者叫做增量方程：

$$J^T(\mathcal{X})J(\mathcal{X})\triangle\mathcal{X}=-J(\mathcal{X})r(\mathcal{X}) \tag{10}$$

通过增量方程可以知道优化算法下一步的迭代如何进行。

对当前目标函数(2)而言，整体的增量方程可以写成：

$$
\begin{aligned}
(H_p + \sum {J_{b_{k+1}}^{b_k}}^T {P_{b_{k+1}}^{b_k}}^{-1} J_{b_{k+1}}^{b_k} + \sum {J_l^{C_j}}^T {P_l^{C_j}}^{-1} J_l^{C_j}) \triangle \mathcal{X} \\
= b_p + \sum {J_{b_{k+1}}^{b_k}}^T {P_{b_{k+1}}^{b_k}}^{-1} r_B + \sum {J_l^{C_j}}^T {P_l^{C_j}}^{-1} r_C \quad (11)
\end{aligned}
$$

其中$P_{b_{k+1}}^{b_k}$，$P_l^{C_j}$分别为IMU预积分噪声项的协方差和视觉观测的噪声协方差。协方差越大，意味着观测越不稳定。

注意，上面的雅各比矩阵虽然是误差项对状态向量的一阶导数。但在具体求解时，通常采用扰动的方式$\delta\mathcal{X}$计算，而不是增量$\triangle\mathcal{X}$:

$$
J(\mathcal{X}) = \frac{\partial r}{\partial \mathcal{X}} = \lim_{\delta\mathcal{X}\to 0} \frac{r(\mathcal{X} \oplus \delta\mathcal{X}) - r(\mathcal{X})}{\delta\mathcal{X}} \qquad (12)
$$

3.IMU约束

3.1 残差

根据《IMU预积分》中IMU预积分模型公式（18）可以定义连续两帧IMU之间预积分的测量残差为：

$$
r_{\mathcal{B}}^{15\times 1}(\hat{z}_{b_{k+1}}^{b_k}, \mathcal{X}) = \begin{vmatrix} \delta\alpha_{b_{k+1}}^{b_k} \\ \delta\theta_{b_{k+1}}^{b_k} \\ \delta\beta_{b_{k+1}}^{b_k} \\ \delta b_a \\ \delta b_g \end{vmatrix} = \begin{vmatrix} R_w^{b_k}(p_{b_{k+1}}^w - p_{b_k}^w - v_{b_k}^w \triangle t_k + \frac{1}{2} g^w \triangle t_k^2) - \alpha_{b_{k+1}}^{b_k} \\ 2[{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes ({q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w)]_{xyz} \\ R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k) - \beta_{b_{k+1}}^{b_k} \\ b_{a_{b_{k+1}}} - b_{a_k} \\ b_{w_{b_{k+1}}} - b_{w_k} \end{vmatrix} \qquad (13)
$$

其中$[]_{xyz}$表示四元数的向量部分，即可表示方向的误差，当然向量空间的方向变化是四元数方向变化角度的2倍。具体可参考《四元数数学基础》2.4.6节。

3.2 优化变量

$$
[p_{b_k}^w, q_{b_k}^w], [v_{b_k}^w, b_{a_k}, b_{w_k}], [p_{b_{k+1}}^w, q_{b_{k+1}}^w], [v_{b_{k+1}}^w, b_{a_{k+1}}, b_{w_{k+1}}]
$$

3.3 雅各比

3.3.1 残差关于$[p_{b_k}^w, q_{b_k}^w]$的雅各比

$$
J[0]^{15\times 7} = [\frac{\partial r_{\mathcal{B}}}{\partial p_{b_k}^w}, \frac{\partial r_{\mathcal{B}}}{\partial q_{b_k}^w}] = \begin{vmatrix} -R_w^{b_k} & [R_w^{b_k}(p_{b_{k+1}}^w - p_{b_k}^w - v_{b_k}^w \triangle t_k + \frac{1}{2} g^w \triangle t_k^2)]_\times \\ 0 & -\mathcal{L}[q_{b_{k+1}}^w{}^{-1} \otimes q_{b_k}^w]\mathcal{R}[\gamma_{b_{k+1}}^{b_k}] \\ 0 & [R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k)]_\times \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \tag{14}
$$

其中$\frac{\partial r_{\mathcal{B}}}{\partial q_{b_k}^w}$第一行和第三行推导完全一样，都是旋转矩阵乘上向量对四元数求导，这里推导第三行：

$$
\frac{\partial \delta \beta_{b_{k+1}}^{b_k}}{\partial q_{b_k}^w} = \lim_{\delta\theta_{b_k}^w \to 0} \frac{(R_{b_k}^w \exp([\delta\theta_{b_k}^w]_\times))^{-1}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k) - R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k)}{\delta\theta_{b_k}^w}
$$

$$
= \lim_{\delta\theta_{b_k}^w \to 0} \frac{(I - [\delta\theta_{b_k}^w]_\times) R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k)}{\delta\theta_{b_k}^w}
$$

$$
= \lim_{\delta\theta_{b_k}^w \to 0} \frac{-[\delta\theta_{b_k}^w]_\times R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k)}{\delta\theta_{b_k}^w}
$$

根据向量性质$[a]_\times b = -[b]_\times a$可得：

$$
= \lim_{\delta\theta_{b_k}^w \to 0} \frac{[R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k)]_\times \delta\theta_{b_k}^w}{\delta\theta_{b_k}^w}
$$

$$
= [R_w^{b_k}(v_{b_{k+1}}^w - v_{b_k}^w + g^w \triangle t_k)]_\times \tag{15}
$$

再推导第二行：

$$\frac{\partial \delta\theta_{b_{k+1}}^{b_k}}{\partial q_{b_k}^w} = 2\lim_{\delta\theta_{b_k}^w \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes [(q_{b_k}^w \otimes \begin{vmatrix} 1 \\ \frac{\delta\theta_{b_k}^w}{2}\end{vmatrix})^{-1} \otimes q_{b_{k+1}}^w] - {\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes [(q_{b_k}^w \otimes \begin{vmatrix}1\\0\end{vmatrix})^{-1} \otimes q_{b_{k+1}}^w]}{\delta\theta_{b_k}^w}$$

$$= 2\lim_{\delta\theta_{b_k}^w \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes (\begin{vmatrix} 1 \\ -\frac{\delta\theta_{b_k}^w}{2}\end{vmatrix} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w) - {\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes (\begin{vmatrix}1\\0\end{vmatrix} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w)}{\delta\theta_{b_k}^w}$$

$$= 2\lim_{\delta\theta_{b_k}^w \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes (\begin{vmatrix} 1 \\ -\frac{\delta\theta_{b_k}^w}{2}\end{vmatrix} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w) - {\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes (\begin{vmatrix}1\\0\end{vmatrix} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w)}{\delta\theta_{b_k}^w}$$

$$= 2\lim_{\delta\theta_{b_k}^w \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes \begin{vmatrix} 1 \\ -\frac{\delta\theta_{b_k}^w}{2}\end{vmatrix} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w}{\delta\theta_{b_k}^w}$$

根据《四元数数学基础》式（15）（16）交换律和分配律可得：

$$= 2\lim_{\delta\theta_{b_k}^w \to 0} \frac{\mathcal{R}[{q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w]\mathcal{L}[{\gamma_{b_{k+1}}^{b_k}}^{-1}]\begin{vmatrix} 1 \\ -\frac{\delta\theta_{b_k}^w}{2}\end{vmatrix}}{\delta\theta_{b_k}^w} \tag{16}$$

下面推导一个性质，令$q=[x,y,z,s]=[w,s]$，根据《四元数数学基础》式（19）可得：

$$\mathcal{R}(q)=\begin{vmatrix}0 & -w^T \\ w & -[w]_\times\end{vmatrix}+sI_{4\times4}$$

$$\mathcal{L}(q)=\begin{vmatrix}0 & -w^T \\ w & [w]_\times\end{vmatrix}+sI_{4\times4} \tag{17}$$

如果只有右下角3×3虚部部分，则有：

$$\mathcal{R}(q^{-1})_{3\times3}=sI_{3\times3}+[w]_\times=\mathcal{L}(q)_{3\times3} \tag{18}$$

将式（18）代入式（16）可得：

$$=2\lim_{\delta\theta_{b_k}^w\to0}\frac{\{\mathcal{L}[q_{b_{k+1}}^{w}{}^{-1}\otimes q_{b_k}^w]\mathcal{R}[\gamma_{b_{k+1}}^{b_k}]\begin{vmatrix}1 \\ -\frac{\delta\theta_{b_k}^w}{2}\end{vmatrix}\}_{3\times3}}{\delta\theta_{b_k}^w}$$

$$=-\mathcal{L}[q_{b_{k+1}}^{w}{}^{-1}\otimes q_{b_k}^w]\mathcal{R}[\gamma_{b_{k+1}}^{b_k}] \tag{19}$$

3.3.2 残差关于$[v_{b_k}^w,b_{a_k},b_{w_k}]$的雅各比

$$J[1]^{15\times9}=[\frac{\partial r_\mathcal{B}}{\partial v_{b_k}^w},\frac{\partial r_\mathcal{B}}{\partial b_{a_k}},\frac{\partial r_\mathcal{B}}{\partial b_{w_k}}]=\begin{vmatrix}-R_w^{b_k}\triangle t_k & -J_{b_a}^\alpha & -J_{b_w}^\alpha \\ 0 & 0 & -\mathcal{L}[q_{b_{k+1}}^{w}{}^{-1}\otimes q_{b_k}^w\otimes\gamma_{b_{k+1}}^{b_k}]_{3\times3}J_{b_w}^\gamma \\ -R_w^{b_k} & -J_{b_a}^\beta & -J_{b_w}^\beta \\ 0 & -I & 0 \\ 0 & 0 & -I\end{vmatrix} \tag{20}$$

其中$\frac{\partial r_\mathcal{B}}{\partial b_{a_k}}$, $\frac{\partial r_\mathcal{B}}{\partial b_{w_k}}$可根据《IMU预积分》中式（20）推导出残差关于bias雅各比的多数项，这里推导下$\frac{\partial r_\mathcal{B}}{\partial b_{w_k}}$的第二行：

$$\frac{\partial \delta\theta^{b_k}_{b_{k+1}}}{\partial \delta b_{w_k}} = 2\lim_{\delta b_{w_k}\to 0}\frac{\left(\gamma^{b_k}_{b_{k+1}}\otimes\begin{vmatrix}1\\ \frac{1}{2}J^{\gamma}_{b_w}\delta b_{w_k}\end{vmatrix}\right)^{-1}\otimes\left(q^{w\ -1}_{b_k}\otimes q^{w}_{b_{k+1}}\right)-\left(\gamma^{b_k}_{b_{k+1}}\otimes\begin{vmatrix}1\\ 0\end{vmatrix}\right)^{-1}\otimes\left(q^{w\ -1}_{b_k}\otimes q^{w}_{b_{k+1}}\right)}{\delta b_{w_k}}$$

$$= 2\lim_{\delta b_{w_k}\to 0}\frac{\begin{vmatrix}0\\ -\frac{1}{2}J^{\gamma}_{b_w}\delta b_{w_k}\end{vmatrix}\otimes{\gamma^{b_k}_{b_{k+1}}}^{-1}\otimes\left(q^{w\ -1}_{b_k}\otimes q^{w}_{b_{k+1}}\right)}{\delta b_{w_k}}$$

$$= 2\lim_{\delta b_{w_k}\to 0}\frac{\mathcal{R}\left[{\gamma^{b_k}_{b_{k+1}}}^{-1}\otimes\left(q^{w\ -1}_{b_k}\otimes q^{w}_{b_{k+1}}\right)\right]\begin{vmatrix}0\\ -\frac{1}{2}J^{\gamma}_{b_w}\delta b_{w_k}\end{vmatrix}}{\delta b_{w_k}}$$

$$= 2\lim_{\delta b_{w_k}\to 0}\frac{\mathcal{R}\left[{\gamma^{b_k}_{b_{k+1}}}^{-1}\otimes\left(q^{w\ -1}_{b_k}\otimes q^{w}_{b_{k+1}}\right)\right]\begin{vmatrix}0\\ -\frac{1}{2}J^{\gamma}_{b_w}\delta b_{w_k}\end{vmatrix}}{\delta b_{w_k}}$$

又根据式（18）化简：

$$= 2\lim_{\delta b_{w_k}\to 0}\frac{\left\{\mathcal{L}\left[{q^{w}_{b_{k+1}}}^{-1}\otimes q^{w}_{b_k}\otimes\gamma^{b_k}_{b_{k+1}}\right]\begin{vmatrix}0\\ -\frac{1}{2}J^{\gamma}_{b_w}\delta b_{w_k}\end{vmatrix}\right\}_{3\times 3}}{\delta b_{w_k}}$$

$$= \lim_{\delta b_{w_k}\to 0}\frac{\left\{-\mathcal{L}\left[{q^{w}_{b_{k+1}}}^{-1}\otimes q^{w}_{b_k}\otimes\gamma^{b_k}_{b_{k+1}}\right]\begin{vmatrix}0\\ J^{\gamma}_{b_w}\end{vmatrix}\right\}_{3\times 3}\delta b_{w_k}}{\delta b_{w_k}}$$

$$= -\mathcal{L}[{q_{b_{k+1}}^{w}}^{-1} \otimes q_{b_k}^{w} \otimes \gamma_{b_{k+1}}^{b_k}]_{3\times 3} J_{b_w}^{\gamma} \tag{21}$$

3.3.3 残差关于$[p_{b_{k+1}}^{w}, q_{b_{k+1}}^{w}]$的雅各比

$$J[2]^{15\times 7} = [\frac{\partial r_{\mathcal{B}}}{\partial p_{b_{k+1}}^{w}}, \frac{\partial r_{\mathcal{B}}}{\partial q_{b_{k+1}}^{w}}] = \begin{vmatrix} R_w^{b_k} & 0 \\ 0 & \mathcal{L}[{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes {q_{b_k}^{w}}^{-1} \otimes q_{b_{k+1}}^{w}] \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \tag{22}$$

这里推导下$\frac{\partial r_{\mathcal{B}}}{\partial q_{b_{k+1}}^{w}}$第二行：

$$\frac{\partial \delta\theta_{b_{k+1}}^{b_k}}{\partial q_{b_k}^{w}} = 2 \lim_{\delta\theta_{b_k}^{w} \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes [{q_{b_k}^{w}}^{-1} \otimes （q_{b_{k+1}}^{w} \otimes \begin{vmatrix} 1 \\ \frac{\delta\theta_{b_k}^{w}}{2} \end{vmatrix}）] - {\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes [{q_{b_k}^{w}}^{-1} \otimes （q_{b_{k+1}}^{w} \otimes \begin{vmatrix} 1 \\ 0 \end{vmatrix}）]}{\delta\theta_{b_k}^{w}}$$

$$= 2 \lim_{\delta\theta_{b_k}^{w} \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes [{q_{b_k}^{w}}^{-1} \otimes （q_{b_{k+1}}^{w} \otimes \begin{vmatrix} 1 \\ \frac{\delta\theta_{b_k}^{w}}{2} \end{vmatrix}）] - {\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes [{q_{b_k}^{w}}^{-1} \otimes （q_{b_{k+1}}^{w} \otimes \begin{vmatrix} 1 \\ 0 \end{vmatrix}）]}{\delta\theta_{b_k}^{w}}$$

$$= 2 \lim_{\delta\theta_{b_k}^{w} \to 0} \frac{{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes {q_{b_k}^{w}}^{-1} \otimes q_{b_{k+1}}^{w} \otimes (\begin{vmatrix} 1 \\ \frac{\delta\theta_{b_k}^{w}}{2} \end{vmatrix} - \begin{vmatrix} 1 \\ 0 \end{vmatrix})}{\delta\theta_{b_k}^{w}}$$

$$= \lim_{\delta\theta_{b_k}^w \to 0} \frac{\mathcal{L}[{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w] \begin{vmatrix} 0 \\ \delta\theta_{b_k}^w \end{vmatrix}}{\delta\theta_{b_k}^w}$$

$$= \mathcal{L}[{\gamma_{b_{k+1}}^{b_k}}^{-1} \otimes {q_{b_k}^w}^{-1} \otimes q_{b_{k+1}}^w] \tag{23}$$

3.3.4 残差关于$[v_{b_{k+1}}^w, b_{a_{k+1}}, b_{w_{k+1}}]$的雅各比

$$J[3]^{15\times 9} = [\frac{\partial r_{\mathcal{B}}}{\partial v_{b_{k+1}}^w}, \frac{\partial r_{\mathcal{B}}}{\partial b_{a_{k+1}}}, \frac{\partial r_{\mathcal{B}}}{\partial b_{w_{k+1}}}] = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_w^{b_k} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{vmatrix} \tag{24}$$

4 视觉约束

4.1 残差

这里的视觉测量残差定义在一个单位球面上而不是传统的成像平面上。考虑第l个路标点，第一次在第i帧图像被观测到，现在在第j帧也观测到后，可定义视觉测量残差为：

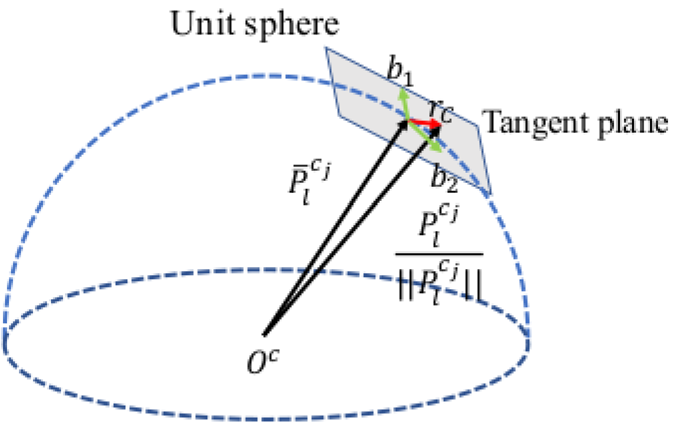
$$r_{\mathcal{C}}(\hat{z}_l^{c_j}, \mathcal{X})_{2\times 1} = \begin{vmatrix} b_1 \\ b_2 \end{vmatrix}_{2\times 3} (\frac{P_l^{c_j}}{\| P_l^{c_j} \|} - \hat{P}_l^{c_j})_{3\times 1} \tag{25}$$

其中，

$$\hat{P}_l^{c_j} = {\pi_c}^{-1}(\begin{vmatrix} \hat{u}_l^{c_j} \\ \hat{v}_l^{c_j} \end{vmatrix}) \tag{26}$$

$$P_l^{c_j} = R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\} \quad (27)$$

π_c是相机的内参矩阵。$\hat{P}_l^{c_j}$是在第j帧相机观测到第l个路标点的坐标，$P_l^{c_j}$是通过将路标点的第一次被观测到的第i帧相机坐标系转换到第j帧相机坐标系，预测其在单位球上的坐标，$b_1 b_2$是正切平面上任意的正交基。



这里推导下式（27）：

第l个路标点P_l^w在第i帧相机下的图像坐标为：

$$p_{uv}^{c_i} = \lambda_l \pi_c(T_b^c T_w^{b_i} P_l^w) \quad (28)$$

其中λ_l为第l路标点在第i帧相机中第一次被观测到的逆深度。由此可以得到路标点的世界坐标为：

$$P_l^{c_j} = R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\} \quad (29)$$

第l个路标点P_l^w在第j帧相机下的坐标为（注意这里不求图像坐标）：

$$P_l^{c_j} = T_b^c T_w^{b_j} P_l^w \quad (30)$$

由此同样可得到路标点的世界坐标为：

$$P_l^w = R_{b_j}^w(R_c^b P_l^{c_j} + p_c^b) + p_{b_j}^w \quad (31)$$

因此式（29）（31）相等：

$$R_{b_j}^w(R_c^b P_l^{c_j} + p_c^b) + p_{b_j}^w = R_{b_i}^w(R_c^b(\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix})) + p_c^b)) + p_{b_i}^w$$

$$P_l^{c_j} = R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\} \tag{32}$$

4.2 优化变量

优化变量包括第i,j两帧相机的位姿，IMU和相机之间的外参，路标点的逆深度：

$$[p_{b_i}^w, q_{b_i}^w], [p_{b_{i+1}}^w, q_{b_{i+1}}^w], [p_b^c, q_b^c], \lambda_l \tag{33}$$

4.3 雅各比

4.3.1 残差关于$[p_{b_i}^w, q_{b_i}^w]$的雅各比

$$J[0]^{2\times 7} = [\frac{\partial r_{\mathcal{C}}}{\partial p_{b_i}^w}, \frac{\partial r_{\mathcal{C}}}{\partial q_{b_i}^w}] = \begin{vmatrix} R_b^c R_w^{b_j} & -R_b^c R_w^{b_j}[R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b)]_\times \\ 0 & 0 \end{vmatrix} \tag{34}$$

这里推导一下$\frac{\partial r_{\mathcal{C}}}{\partial q_{b_i}^w}$第一行：

$$\frac{\partial r_{\mathcal{C}}}{\partial q_{b_i}^w} = \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{R_w^{b_j}[R_{b_i}^w \exp([\delta\theta_{b_i}^w]_\times)(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_i}^w}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{R_w^{b_j}[R_{b_i}^w(I + [\delta\theta_{b_i}^w]_\times)(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_i}^w}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{R_w^{b_j}[R_{b_i}^w[\delta\theta_{b_i}^w]_\times(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_i}^w}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{R_w^{b_j}[-R_{b_i}^w[R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b)]_\times\delta\theta_{b_i}^w + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_i}^w}$$

$$= -R_b^c R_w^{b_j}[R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b)]_\times \tag{35}$$

4.3.2 残差关于$[p_{b_{i+1}}^w, q_{b_{i+1}}^w]$的雅各比

$$J[1]^{2\times7} = [\frac{\partial r_{\mathcal{C}}}{\partial p_{b_{i+1}}^w}, \frac{\partial r_{\mathcal{C}}}{\partial q_{b_{i+1}}^w}] = \begin{vmatrix} -R_b^c R_w^{b_j} & R_b^c\{[R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w]]_\times\} \\ 0 & 0 \end{vmatrix} \tag{36}$$

这里推导一下$\frac{\partial r_{\mathcal{C}}}{\partial q_{b_{i+1}}^w}$第一行：

$$\frac{\partial r_{\mathcal{C}}}{\partial q_{b_{i+1}}^w} = \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{[R_{b_j}^w \exp([\delta\theta_{b_j}^w]_\times)]^{-1}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_j}^w}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{(I - [\delta\theta_{b_j}^w]_\times)R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_j}^w}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{-R_b^c\{[\delta\theta_{b_j}^w]_\times R_w^{b_j}[R_{b_i}^w(R_c^b \frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_{b_j}^w}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{R_b^c\{[R_w^{b_j}[R_{b_i}^w(R_c^b \frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w]]_\times \delta\theta_{b_j}^w - p_c^b\}}{\delta\theta_{b_j}^w}$$

$$= R_b^c\{[R_w^{b_j}[R_{b_i}^w(R_c^b \frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w]]_\times\} \quad (37)$$

4.3.3 残差关于$[p_b^c, q_b^c]$的雅各比

$$J[2]^{2\times 7} = [\frac{\partial r_{\mathcal{C}}}{\partial p_c^b}, \frac{\partial r_{\mathcal{C}}}{\partial q_c^b}] = \begin{vmatrix} R_b^c(R_w^{b_j}R_{b_i}^w - I) & -R_b^c R_w^{b_j} R_{b_i}^w R_c^b[\frac{\hat{P}_l^{c_j}}{\lambda_l}]_\times + [R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{\hat{P}_l^{c_j}}{\lambda_l} + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}]_\times \\ 0 & 0 \end{vmatrix} \quad (38)$$

这里推导一下$\frac{\partial r_{\mathcal{C}}}{\partial q_c^b}$第一行：

$$\frac{\partial r_{\mathcal{C}}}{\partial q_c^b} = \lim_{\delta\theta_{b_i}^w \to 0} \frac{[R_c^b \exp([\delta\theta_c^b]_\times)]^{-1}\{R_w^{b_j}[R_{b_i}^w(R_c^b \exp([\delta\theta_c^b]_\times)\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_c^b}$$

$$= \lim_{\delta\theta_{b_i}^w \to 0} \frac{(I - [\delta\theta]_\times)R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b \exp([\delta\theta_c^b]_\times)\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix} \hat{u}_l^{c_i} \\ \hat{v}_l^{c_i} \end{vmatrix}) + p_c^b) + p_{b_i}^w - p_{b_j}^w] - p_c^b\}}{\delta\theta_c^b}$$

$$
= \lim_{\delta\theta_{b_i}^{w} \to 0} \frac{(I-[\delta\theta_c^b]_\times)R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b(I+[\delta\theta_c^b]_\times)\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}}{\delta\theta_c^b} \tag{39}
$$

将上式分子化简：

$$
= R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b(I+[\delta\theta_c^b]_\times)\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}
$$

$$
-[\delta\theta_c^b]_\times R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b(I+[\delta\theta_c^b]_\times)\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}
$$

整理出含$\delta\theta_c^b$的项：

$$
= R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b[\delta\theta_c^b]_\times\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}
$$

$$
-[\delta\theta_c^b]_\times R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}
$$

$$
-[\delta\theta_c^b]_\times R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b[\delta\theta_c^b]_\times\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}
$$

省略掉第三项：

$$
\approx R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b[\delta\theta_c^b]_\times\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}
$$

$$-[\delta\theta_c^b]_\times R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}$$

继续整理出含$\delta\theta_c^b$的项：

$$=R_b^cR_w^{b_j}R_{b_i}^wR_c^b[\delta\theta_c^b]_\times\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})$$

$$+[R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}]_\times\delta\theta_c^b$$

继续化简：

$$=-R_b^cR_w^{b_j}R_{b_i}^wR_c^b[\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})]_\times\delta\theta_c^b$$

$$+[R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{1}{\lambda_l}\pi_c{}^{-1}(\begin{vmatrix}\hat{u}_l^{c_i}\\ \hat{v}_l^{c_i}\end{vmatrix})+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}]_\times\delta\theta_c^b$$

将式（26）代入上式可得：

$$=-R_b^cR_w^{b_j}R_{b_i}^wR_c^b[\frac{\hat{P}_l^{c_j}}{\lambda_l}]_\times\delta\theta_c^b$$

$$+[R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{\hat{P}_l^{c_j}}{\lambda_l}+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}]_\times\delta\theta_c^b \quad (40)$$

将上式（40）代入式（39）可得：

$$\frac{\partial r_{\mathcal{C}}}{\partial q_c^b}=-R_b^cR_w^{b_j}R_{b_i}^wR_c^b[\frac{\hat{P}_l^{c_j}}{\lambda_l}]_\times+[R_b^c\{R_w^{b_j}[R_{b_i}^w(R_c^b\frac{\hat{P}_l^{c_j}}{\lambda_l}+p_c^b)+p_{b_i}^w-p_{b_j}^w]-p_c^b\}]_\times \quad (41)$$