Option Pricing - Quant POV

Get ready

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Overview

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7. Conclusion

Wooclap

- Wooclap was deleted
- To have access to the notebook, please visit author's Github here

Sec 2 - Calls and Puts: Definition and Illustration

Intuitive Definition

Basic Derivatives (Financial Products)

- Call = Right to buy an asset at a given price.
- Put = Right to sell an asset at a given price.

Vocabulary

- Asset (S_t) = Stocks, indices, interest rates.
- Long position = I own the option.
- Short position = I sold the option.
- Maturity (T) = End date of the contract.
- Payoff = The amount the derivative will pay.
- Strike price (K) = The fixed price agreed upon in the contract.

Mathematical Formulation

Call Option

$$\max\left(S_T - K, 0\right) \quad \text{or} \quad \left(S_T - K\right)_+$$

Put Option

$$\max\left(K - S_T, 0\right) \quad \text{or} \quad \left(K - S_T\right)_+$$

American vs. European Options

European options can be exercised only at maturity, whereas American options can be exercised at any time before maturity.

In this presentation, we will focus on European options only. (Those of you who have worked with American options know how painful their implementation is compared to European ones...)

Illustration - 1

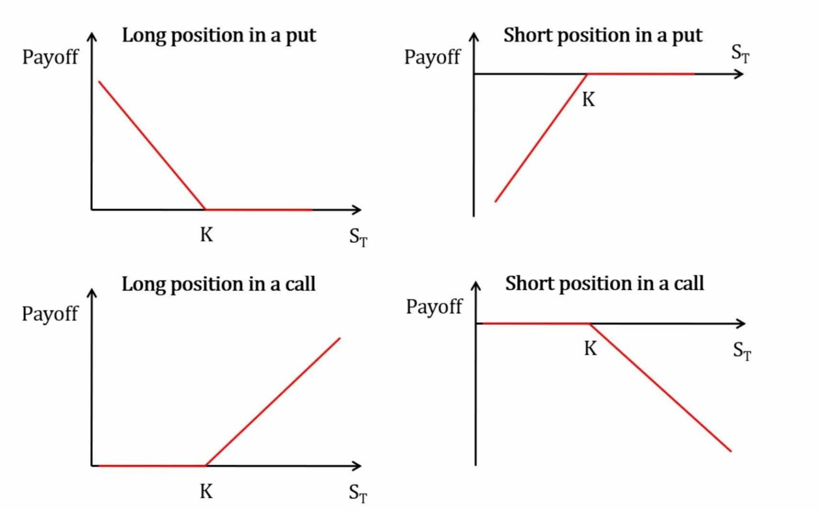
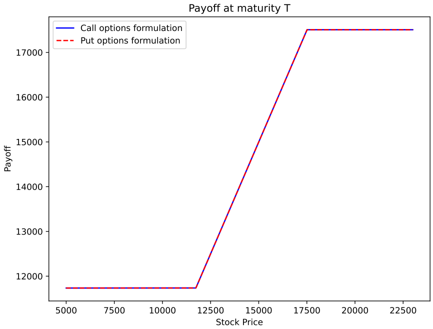


Figure: Call and Put for Long and Short Positions

Illustration - Bull Spread

$$L_T = P(T, S_T) = \begin{cases} K_1(T) + (S_T - K_1(T))_+ - (S_T - K_2(T))_+ & \text{(Using Call Options)} \\ K_2(T) + (K_1(T) - S_T)_+ - (K_2(T) - S_T)_+ & \text{(Using Put Options)} \end{cases}$$

$$K_1(T) = S_0 \cdot e^{0.02T}$$
$$K_2(T) = S_0 \cdot e^{0.07T}$$



Sec 3 - Use of Derivatives in the Real World

Different Uses of Options

Farmers at the Beginning of the Russian Invasion

The price of raw materials (particularly wheat and electricity) began to rise. To protect themselves against these price increases, farmers could purchase call options on these raw materials.

Speculation

Of course, options can also be used to bet on the rise or fall of certain stocks. For example, if you believe Tesla's stock is going to drop, you can buy a put option, which allows you to sell the stock at a higher price than the market price.

Insurance Company Contracts

Minimum Guaranteed Contracts

Individuals can purchase "minimum guaranteed contracts," which ensure a guaranteed payout under certain conditions. Some examples include:

- **GMAB** (Guaranteed Minimum Accumulation Benefit): A minimum capital is guaranteed in case of survival.
- **GMDB** (Guaranteed Minimum Death Benefit): A minimum capital is guaranteed in case of death.
- **GMIB** (Guaranteed Minimum Income Benefit): A minimum lifetime annuity is guaranteed.
- **GMWB** (Guaranteed Minimum Withdrawal Benefit): Minimum periodic withdrawals are guaranteed.

[Hainaut, 2023]

Sec 4 - Two Methods: Monte Carlo and Analytical

Option Pricing - Methods

Now that we know what options are and what we can do with them, let's price them!
You (because I'm lazy and don't want to do it) will use two methods to price options:

1. **Monte Carlo simulations** - Numerical solution
    - Pros: Adaptable to almost any derivative and easy to implement.
    - Cons: Time-consuming.
2. **Black-Scholes model** - Analytical solution
    - Pros: Precise and fast.
    - Cons: Difficult to derive and not always applicable.

$\mathbb{P}$ & $\mathbb{Q}$

Reminder: Asset price dynamics are given by:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathbb{P}}$$

Pricing occurs in the risk-neutral world $\mathbb{Q}$! We need to define the risk-free rate r with the evolution of a cash account:

$$B_t = B_0 e^{rt}$$

With some magic[1], the dynamics in the risk-neutral world become:

$$\boxed{\frac{dS_t}{S_t} = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t^{\mathbb{Q}}}$$

[1]For the curious ones: Change of measure - Radon-Nikodym derivatives - Girsanov's theorem - Itô's Lemma

Method 1 - Monte Carlo

Idea

1. Simulate a large number of potential stock price paths over time.
2. Compute the average of the outcomes.
3. Discount this value to obtain the contract's present value (at $t = 0$).

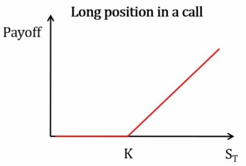
Your turn! Explore Monte Carlo pricing in the notebook with your team.

Method 2 - Black-Scholes - 1/2

The world is cruel, but don't be cruel to me—just pretend to understand what I'm saying.

Step 1: Payoff Formulation

Payoff: $L_T = K + (S_T - K)_+$



Method 2 - Black-Scholes - 2/2

Step 2: The contract value is the discounted expected value of the payoff

$$V_0 = \underbrace{e^{-rT}}_{(a)} \mathbb{E}^{\mathbb{Q}} \left[\underbrace{\mathbb{1}\{\tau_x > T\}}_{(b)} L_T \right]$$

- (a): Discount factor.
- (b): Indicator that the individual survives until contract maturity.

Step 3: Closed-form solution after some math...

$$V_0 = {}_T p_x . \left(e^{-rT}.K + e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[(S_T - K)_+ \right] \right)$$

Using Black-Scholes:

$$V_0 = {}_T p_x . \left(e^{-rT}.K + (S_t \Phi(d_1(K)) - K\Phi(d_2(K))) \right)$$

Your turn! Play with the parameters and observe how the contract value evolves.

Sec5 - Issues of the Suggested Framework and Solutions

Issues of suggested model

Issue 1 : No jumps

Even with Brownian motion, we cannot replicate sudden jumps in prices:

Issue 2 : Constant volatility

We assume constant volatility so far.
VIX is an indicator of volatility in the US financial market.This index is calculated by averaging the annual volatility of calls and puts on the S&P500.

Solutions

Solution 1: Jump-Diffusion Processes

$$X_t = \mu t + \sigma W_t + \sum_{k=0}^{N_t} Y_k$$

Some details:

- $N_t \sim \text{Poisson}(\lambda t)$
- Y_k follows either the "Merton" or "Kou" jump process. The Merton model is easier to understand but more complex to implement.

Solutions

Solution 2: The Heston Model

$$\text{Stochastic variance } (\sigma_t^2 = V_t) : dV_t = \underbrace{\kappa(\gamma - V_t)dt}_{(a)} + \sigma\sqrt{V_t}dW_t^V$$

$$\text{Asset price dynamics under } \mathbb{P} : \frac{dS_t}{S_t} = \mu dt + \sqrt{V_t}\underbrace{\left(\rho dW_t^V + \sqrt{1-\rho^2}dW_t^S\right)}_{(b)}$$

- (a) Variance oscillates around a mean (you will observe this in the notebook).
- (b) $\rho \in (-1, 1)$ is the correlation between asset price and volatility.

This framework allows us to derive almost closed-form solutions for call and put options, but it requires advanced mathematics[2].

[2] Fourier Transforms, Itô's Lemma for semi-martingales, Complex Analysis [Hainaut, 2024]

Variance Gamma (1/2)

A fascinating concept: contracting and expanding time! The idea is that trading activity is not constant throughout the day. For example, markets are typically more volatile at 9 AM (opening) than at 1 PM (lunchtime).
Consequence: Stock prices exhibit time-dependent volatility.
How do we model this?

Subordination

We define the log returns evolution as S_{T_t}, where:

- S follows a Geometric Brownian Motion (GBM):

$$dS = \mu T_t + \sigma dW_{T_t}$$

- T_t follows a Gamma process:

$$T_t \sim \text{Gamma}(\dots)$$

Variance Gamma (2/2)

During our Monte Carlo simulations, we observe the following:

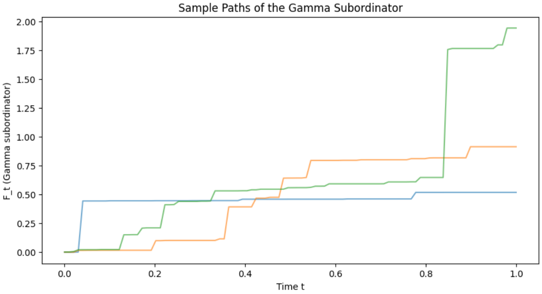


Figure: Simulated Gamma Subordinator

Interpretation:

- "Horizontal" plateaus represent periods of low trading activity.
- "Vertical" jumps represent periods of high trading activity.

Sec6 - Greeks in finance

Introduction to greeks

Let's go back to initial model : $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$
It can be shown this is strictly equal to

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

We can see appear something some of you may have already encountered in their cursus..

- $\frac{\partial V}{\partial S} = \Delta$
- $\frac{\partial^2 V}{\partial S^2} = \Gamma$

These are called "Greeks" and Quants will be tell you more about that during the semester

Sec7 - Conclusion

Conclusion

Today, you have been introduced to option pricing. We talked about :

- Some basic vocabulary (Maturity, strike, positions,etc.)
- Some real world uses (other than speculation)
- Monte-Carlo and Analytical solutions (and the importance of pricing under risk-free measure !)
- Jump-Diffusion, Heston model, Variance Gamma
- Greeks in finance

We tried to make it as interactive as possible, do not hesitate to tell us how you felt during the presentation !
See you

Thank you

References

Hainaut, D. (2023).
Stochastic finance.
LACTU2170.

Hainaut, D. (2024).
Advanced processes in life insurance engineering.
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