1. Model

1.1 Building blocks

Consider a dynamic market that amount to a set N = {1, 2, …, n} of polluters, n > 1, called firms for simplicity. For each period $t \geq 0$, let abatement of firm *i* be written as $a_{it} = q_{it}^0 - q_{it}$, where q_{it}^0 is the level of business-as-usual (BAU) emissions (i.e. the level of emissions in absence of any policy), and where q_{it} is the actual level of emissions of firm *i*. Abatement costs are determined by the abatement cost function $C(a_{it})$, that is given by,

$$C(a_{it}) = \alpha a_{it} + \frac{\beta}{2} a_{it}^2, \tag{1}$$

which satisfy $C_{it}'(a_{it}) \coloneqq \frac{\partial C_{it}(a_{it})}{\partial a_{it}} > 0$, and $\frac{\partial^2 C_{it}(a_{it})}{\partial a_{it}^2} \geq 0$, and where α and β are constants. In each period, firms choose their emissions simultaneously, and the abatement cost functions are further assumed to be common knowledge.

Consider a cap-and-trade scheme in which emissions are regulated over a total of 3 periods so that *t = {0, …, T}*, where *T = 2* is the duration of the scheme. Let s_{it} be the number of allowances supplied to firm *i* at the start of period *t*. Allowances can be traded on a secondary market at price p_t, which the firms takes as given for respective period. Let m_{it} denote the number of allowances that firm *i* buys on the secondary market in period *t*. Additionally, we assume that the total number of allowances bought must also be sold, so that:

$$\sum_i m_{it} = 0\,, \tag{2}$$

for all *t*. Allowances can be traded over time, so that unused allowances that were supplied in one period may be carried over to the next period, i.e., banking.
Banking of firm *i* during period *t* is given by $b_{it} := s_{it} + m_{it} - q_{it}$. Hence, the bank of allowances held by firm *i* at the start of period *t* is

$$B_{it} \coloneqq \sum_{s=0}^{t-1} b_{is} = B_{it-1} + b_{it-1} = B_{it-1} + s_{it-1} + m_{it-1} - q_{it-1}, \tag{3}$$

and where total amount of banking at the start of period *t* is denoted as $B_t := \sum_i B_{it}$.

Furthermore, we assume that banking is not allowed to be negative, i.e. borrowing of allowances is not allowed. In other words,

$$B_{it} \geq 0, \tag{4}$$

for all *i* and *t*.(not a necessary assumption, albeit a realistic one)
Hence, total emissions may not exceed the total supply of allowances, and thus the effective constraint on firm *i*'s emissions is given by:

$$\sum_{s=0}^{t} q_{is} \leq \sum_{s=0}^{t} s_{is} + m_{is}\,, \tag{5}$$

In this regard, allowances can only be used to cover emissions during the scheme. After the scheme ends, any leftover allowances therefore lose their value.

1.2 Demand-side: firms' problem

In any period *t*, the respective firm *i* minimizes the discounted sum of total costs, that is given by abatement a_{it} and allowances m_{it},

$$\min_{a_{it}, m_{it}} \quad \sum_{t=0}^{2} \left(\frac{1}{1+r}\right)^{t} [C_{it}(a_{it}) + p_t m_{it}], \tag{6}$$

subject to (2)-(5). The Lagrangian is for the firms' problem is specified as follows,[1]

$$\mathcal{L}_{\mathrm{i}} = \sum_{t=0}^{2} \left(\frac{1}{1+r}\right)^{t} \left[\alpha a_{it} + \frac{\beta}{2} a_{it}^{2} + p_t m_{it}\right] + \lambda_i \left[\sum_{t} q_{it} - s_{it} - m_{it}\right] + \sum_{t} \left(\frac{1}{1+r}\right)^{t} \mu_t \left[\sum_{i} m_{it}\right] + \omega_{it}[B_{it} - B_{it-1} - s_{it-1} - m_{it-1} + q_{it-1}] + \left(\frac{1}{1+r}\right)^{t} \psi_{it} B_{it}, \tag{7}$$

Taking partial derivatives of (7) with respect to a_{it}, m_{it}, and B_{it} yields the following first-order conditions:

$$\frac{\partial \mathcal{L}_i}{\partial a_{it}} = \left(\frac{1}{1+r}\right)^{t} \alpha + \left(\frac{1}{1+r}\right)^{t} \beta a_{it} - \lambda_i - \omega_{it+1} = 0, \tag{8}$$

$$\frac{\partial \mathcal{L}_i}{\partial m_{it}} = \left(\frac{1}{1+r}\right) p_t - \lambda_i + \left(\frac{1}{1+r}\right)^{t} \mu_t - \omega_{it+1} = 0, \tag{9}$$

$$\frac{\partial \mathcal{L}_i}{\partial B_{it}} = \omega_{it} - \omega_{it+1} + \left(\frac{1}{1+r}\right)^{t} \psi_{it} = 0, \tag{10}$$

[1] Note that $q_{it} = q_{it}^{0} - a_{it}$ is given by default in the lagrangian.

Rearranging (8) and (9) gives the level of abatement for firm *i* in period *t*:

$$a_{it}(p_t) = \frac{p_t + \mu_t - \alpha}{\beta},$$

Hence, the level of emissions for firm *i* in period *t* is:

$$q_{it}(p_t) = q_{it}^0 - \frac{p_t + \mu_t - \alpha}{\beta}, \tag{11}$$

For a vector of prices $p = (p_t)$, let $q_{it}(p_t)$ that is given above represent the firms' solution to the minimization problem. Convexity of abatement costs C_{it} implies

$$\frac{\partial q_{it}(p_t)}{\partial p_t} \leq 0, \tag{12}$$

for all $t \leq T$. The inequality in (12) is strict in all cases when $q_{it}(p_t)$ is not a corner solution. For a given period *t*, the cost minimizing level of emissions for firm *i* is decreasing in the allowance price for that period.

Observation 1. *In each period* $t \in \{0, \dots, T\}$, *aggregate demand for emissions* $q_{it}(p_t)$ *is decreasing in the price for emission allowances* p_t.

Observation 2. *For all* $t \in \{0, \dots, T-1\}$, *allowance prices co-move between periods:*

$$\frac{\partial p_{t+1}}{\partial p_t} > 0, \tag{13}$$

Proof of Observation 2.
Observation 2 is proven by equation (8), that is

$$\omega_{it+1} = \left(\frac{1}{1+r}\right)^t \alpha + \left(\frac{1}{1+r}\right)^t \beta a_{it} - \lambda_i,$$

which implies

$$\omega_{it} = \left(\frac{1}{1+r}\right)^{t-1} \alpha + \left(\frac{1}{1+r}\right)^{t-1} \beta a_{it-1} - \lambda_i,$$

and thereafter plugging the two expressions above into (10) and rearranging so that

$$C'_{it-1} + \left(\frac{1}{1+r}\right) \psi_{it} = \left(\frac{1}{1+r}\right) C'_{it}, \tag{14}$$

Next, from (11) one can derive the following two expressions:

$$p_t + \mu_{it} = C'_{it},$$

and

$$p_{t-1} + \mu_{t-1} = C'_{it-1},$$

Inserting these two expressions into (14) and rearranging yields the following,

$$p_t = (1+r)(p_{t-1} + \mu_{t-1}) + \psi_{it} - \mu_t, \tag{15}$$

From which (13), and thus Observation 2, are proven. For the sake of simplicity however, we will assume for the remainder of the model that allowance prices rises with the interest rate *r*, that is

$$p_{t+1} = (1+r)p_t, \tag{16}$$

The change of prices over time has been well documented in the literature of environmental evaluation. The above formulation for dynamic arbitrage has been used and supported in a large span of studies in recent years.

1.3 Supply-side

1.3.1 Price mechanism

The first class of supply mechanism considered are price mechanisms. Let supply of allowances, under price mechanism, be denoted as s_t^P. By definition, the cap and trade scheme operates a price mechanism if the supply of allowances in period *t* is increasing in the allowance price. More formally, we specify the supply of allowances in period *t* as follows:

$$s_t^P(p_{t-1}) = \overline{s_t} + \gamma p_{t-1}, \tag{17}$$

where $\overline{s_t}$ is the amount of allowances supplied to firms at the start of period *t*, and which they take as given. γ is a parameter that determine the change in supply of allowances in period *t* given allowance price in period $t-1$.

1.3.2 Quantity mechanism

The second mechanism considered under a cap-and-trade scheme are quantity mechanisms. Here we let supply of allowances be denoted as s_t^Q. By definition, a cap-and-trade scheme is operating a quantity mechanism if the supply of allowances in period *t* is decreasing in the amount of banked allowances at the start of that period. That is,

$$s_t^Q\big(B_t(p_{t-1})\big) = \begin{cases} \overline{s_t} - \delta B_t, & if \quad \overline{s_t} \geq \delta B_t \\ 0, & otherwise \end{cases}, \tag{18}$$

where B_t is banking at the start of period t, and $\delta < 1$ by definition.

1.4 Equilibrium

Equilibrium is reached when demand is equal to supply of emission allowances, of which prices adjust to bring about the equilibrium allowance price for respective period in the competitive market.

The solution of the equilibrium stage is given by the price of allowances. Equilibrium price determines affects the level of emissions demanded by firms, and hence also the level the adjustable cap on emissions.

1.4.1 Price mechanism

Under a price mechanism, the equilibrium price p_t^P is found by solving the equality

$$q_0^P(p_o) + q_1^P(p_1) + q_2^P(p_2) = s_o + s_1^P(p_0) + s_2^P(p_1), \tag{19}$$

where supply in the first period is straightly given, and not determined by the price. Next, plugging in (11) on the left-hand side, and (17) on the right-hand side of the equation yields the following,

$$q_0^0 + q_1^0 + q_2^0 + \frac{3\alpha - (\mu_0 + \mu_1 + \mu_2)}{\beta} - \frac{(p_0 + p_1 + p_2)}{\beta} = \bar{s_0} + \bar{s_1} + \bar{s_2} + \gamma p_0 + \gamma p_1$$

To make the next steps easier to follow, denote $q^0 := q_0^0 + q_1^0 + q_2^0$, $\mu := \mu_0 + \mu_1 + \mu_2$, and $\bar{s} = \bar{s_0} + \bar{s_1} + \bar{s_2}$. Additionally, use the Hotelling's rule from (16) to express all prices in terms of p_0. The above expression can hence be stated as,

$$q^0 + \frac{1}{\beta}(3\alpha - \mu) - \frac{p_0}{\beta}[(2+r) + (1+r)^2] = \bar{s} + \gamma(2+r)p_0 \tag{20}$$

Solving for equilibrium price of allowances in period $t = 0$, p_0^P:

$$p_0^P = \frac{\beta(q^0 - \bar{s}) + 3\alpha - \mu}{\beta\gamma(2+r) + (2+r) + (1+r)^2}, \tag{21}$$

By Hotelling's formula, prices in periods $t = 1, 2$ are given by,

$$p_1^P = (1+r)\frac{\beta(q^0 - \bar{s}) + 3\alpha - \mu}{\beta\gamma(2+r) + (2+r) + (1+r)^2}, \tag{22}$$

$$p_2^P = (1+r)^2 \frac{\beta(q^0 - \bar{s}) + 3\alpha - \mu}{\beta\gamma(2+r) + (2+r) + (1+r)^2}, \tag{23}$$

Assuming instead that $r = 0$, then prices are given as,

$$p_0^P = p_1^P = p_2^P = \frac{\beta(q^0 - \bar{s}) + 3\alpha - \mu}{3 + 2\beta\gamma}$$

The equilibrium price for respective time period gives the following levels for emissions $q_t^P(p_t)$ and supply of allowances s_t^P in equilibrium:

$$q_0^P = q_0^0 - \left[\frac{q^0 - \bar{s} + \frac{1}{\beta}(3\alpha - \mu)}{\beta\gamma(2+r) + (2+r) + (1+r)^2}\right] + \frac{\alpha - \mu_0}{\beta},$$

$$q_1^P = q_1^0 - (1+r)\left[\frac{q^0 - \bar{s} + \frac{1}{\beta}(3\alpha - \mu)}{\beta\gamma(2+r) + (2+r) + (1+r)^2}\right] + \frac{\alpha - \mu_1}{\beta},$$

$$q_2^P = q_2^0 - (1+r)^2\left[\frac{q^0 - \bar{s} + \frac{1}{\beta}(3\alpha - \mu)}{\beta\gamma(2+r) + (2+r) + (1+r)^2}\right] + \frac{\alpha - \mu_2}{\beta},$$

$$s_1^P = \bar{s_1} + \gamma p_0 = \bar{s_1} + \gamma\left[\frac{\beta(q^0 - \bar{s}) + 3\alpha - \mu}{\beta\gamma(2+r) + (2+r) + (1+r)^2}\right],$$

$$s_2^P = \bar{s_2} + \gamma p_1 = \bar{s_2} + (1+r)\gamma\left[\frac{\beta(q^0 - \bar{s}) + 3\alpha - \mu}{\beta\gamma(2+r) + (2+r) + (1+r)^2}\right].$$

1.4.2 Quantity mechanism

The equilibrium price when supply is determined through a quantity measure is denoted as p_t^Q, and is solved through,

$$q_0^Q(p_0) + q_1^Q(p_1) + q_2^Q(p_2) = s_0 + s_1^Q(B_1) + s_2^Q(B_2), \tag{24}$$

The left-hand side of (24) can be characterized by (11), and the right-hand side is given by (18) so,

$$q_0^0 + q_1^0 + q_2^0 + \frac{3\alpha - (\mu_0 + \mu_1 + \mu_2)}{\beta} - \frac{(p_0 + p_1 + p_2)}{\beta} = \bar{s_0} + \bar{s_1} + \bar{s_2} - \delta B_1 - \delta B_2$$

From (3), define $B_1 = \bar{s_0} - q_0(p_0)$, and $B_2 = B_1 + \bar{s_1} - q_1(p_1)$, given that the number of allowances m_{it} bought are also sold in the same period. We thus rewrite the above expression as follows,

$$q^0 + \frac{1}{\beta}(3\alpha - \mu) - \frac{p_0}{\beta}[(2+r) + (1+r)^2] = \bar{s} - \delta(2\bar{s_0} + \bar{s_1} - 2q_0 - q_1), \qquad (25)$$

$$\begin{aligned}\Leftrightarrow q^0 + \frac{1}{\beta}(3\alpha - \mu) - \frac{p_0}{\beta}[(2+r) + (1+r)^2] \\ = \bar{s} + \delta\left(2q_0^0 + q_1^0 - 2\bar{s_0} - \bar{s_1} + \frac{3\alpha - 2\mu_0 - \mu_1}{\beta}\right) - \frac{p_0\delta(3+r)}{\beta},\end{aligned}$$

$$\begin{aligned}\Leftrightarrow \beta[q^0 - \bar{s} - \delta(2q_0^0 + q_1^0 - 2\bar{s_0} - \bar{s_1})] + 3\alpha - \mu - \delta(3\alpha - 2\mu_0 - \mu_1) \\ = p_0[(2+r) + (1+r)^2 - \delta(3+r)],\end{aligned}$$

Solving for p_0^Q gives equilibrium allowance price under a quantity measure for period $t = 0$. From here a straightforward solution for p_1^Q and p_2^Q can be reached.

$$p_0^Q = \frac{\beta(q^0 - \bar{s}) - \delta\beta(2q_0^0 + q_1^0 - 2\bar{s_0} - \bar{s_1}) + 3\alpha - \mu - \delta(3\alpha - 2\mu_0 - \mu_1)}{(2+r) + (1+r)^2 - \delta(3+r)},$$

$$p_1^Q = (1+r)\left[\frac{\beta(q^0 - \bar{s}) - \delta\beta(2q_0^0 + q_1^0 - 2\bar{s_0} - \bar{s_1}) + 3\alpha - \mu - \delta(3\alpha - 2\mu_0 - \mu_1)}{(2+r) + (1+r)^2 - \delta(3+r)}\right],$$

$$p_2^Q = (1+r)^2\left[\frac{\beta(q^0 - \bar{s}) - \delta\beta(2q_0^0 + q_1^0 - 2\bar{s_0} - \bar{s_1}) + 3\alpha - \mu - \delta(3\alpha - 2\mu_0 - \mu_1)}{(2+r) + (1+r)^2 - \delta(3+r)}\right],$$

Under the assumption that $r = 0$, prices for respective period are,

$$\begin{aligned}p_0^Q = p_1^Q = p_2^Q \\ = \frac{\beta(q^0 - \bar{s}) - \delta\beta(2q_0^0 + q_1^0 - 2\bar{s_0} - \bar{s_1}) + 3\alpha - \mu - \delta(3\alpha - 2\mu_0 - \mu_1)}{3 - 3\delta}\end{aligned}$$

which thus implies equal prices for all periods.