BISYNTHETIC SPECTRA

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1. Recollections on Synthetic Spectra

Definition 1.1. Given E any spectrum and X a finite spectrum, X is said to be E-finite projective if E_*X is finitely generated and projective over E_*. We denote the full subcategory of spectra spanned by such $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$. A map $X \to Y$ in $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ is said to be a cover if it is an epimorphism after taking E-homology.

Definition 1.2. A site $\mathcal{C}$ which is additive is said to be in addition excellent if it is equipped with a symmetric monoidal structure such that all objects admit duals and such that the functors $- \otimes c$ preserve covers for all $c \in \mathcal{C}$.

Lemma 1.3. The category $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ is additive and acquires the structure of a site with the covering families given by singletons of E-epimorphisms as above. Equipped with the smash product of spectra, it is excellent.

Definition 1.4. A spectrum E is said to be Adams-type if there exists a filtered diagram X_α such that each X_α is in $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$ and such that the natural map $E^*X_\alpha \to \mathrm{Hom}_{E_*}(E_*X_\alpha, E_*)$ is an isomorphism.

Definition 1.5. A presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ on a category with finite coproducts is said to be spherical if for all $c, c' \in \mathcal{C}$ the natural map $F(c \amalg c') \to F(c) \times F(c')$ is an equivalence, i.e., if F preserves finite products as a covariant functor on $\mathcal{C}^{\mathrm{op}}$.

Definition 1.6. The category $\mathcal{S}\mathrm{yn}_E$ of synthetic spectra is the category of spherical presheaves of spectra on the excellent site $\mathcal{S}\mathrm{p}_E^{\mathrm{fp}}$.

2. The Bisynthetic Model

2.1. Synthetic finite projectives.

Definition 2.1. Given $F, X \in \mathcal{S}\mathrm{yn}_E$ we say that X is F-finite projective if it compact as a synthetic spectrum and if $F_{*,*}X := \pi_{*,*}(F \otimes X)$ is a finitely generated projective module over $F_{*,*} := \pi_{*,*}F$. We denote the full subcategory of F-finite projectives $(\mathcal{S}\mathrm{yn}_E)_F^{\mathrm{fp}}$. A map $X \to Y$ of F-finite projectives is said to be a cover if it is an epimorphism after applying taking F-homology.

Lemma 2.2. The category $\mathcal{S}\mathrm{yn}_F^{\mathrm{fp}}$ is an additive site when equipped with the covering families consisting of single $F_{*,*}$-epimorphisms.

Proof. The proof is identical to [**piotr**]. □

Lemma 2.3. Equipped with the tensor product of synthetic spectra, $(\mathcal{S}\mathrm{yn}_E)_F^{\mathrm{fp}}$ is excellent.

Proof. all these proofs look like the one in piotrs paper goes through identically, but I am going to come back to that later. □

3. Special and Generic fibers over λ and τ

3.1. The λ-generic fiber.

Theorem 3.1. The subcategory of λ-local objects in $\mathcal{B}\mathrm{isyn}$ is canonically equivalent to $\mathcal{S}\mathrm{yn}_E$.

3.2. The λ-special fiber.

Theorem 3.2. The category $\mathrm{Mod}(\mathcal{B}\mathrm{isyn}, \mathbb{S}/\lambda)$ is a full subcategory of $\mathcal{S}\mathrm{table}(\nu F_{*,*}\nu F)$ which is an equivalence if (???). Restricted to the image of ν_F, this equivalence takes an E-synthtetic spectrum to its νF-homology.

3.3. The τ-generic fiber.

Notation 3.3. We will write $(\mathcal{S}\mathrm{yn}_E)_F^{\tau-\mathrm{loc}}$ for the site $(\mathcal{S}\mathrm{yn}_E)^{\mathrm{fp}}_{\tau^{-1}F}$.

Lemma 3.4. The functor τ^{-1} induces a morphism of excellent sites $(\mathcal{S}\mathrm{yn}_E)_F^{\mathrm{fp}} \to (\mathcal{S}\mathrm{yn}_E)_F^{\tau-\mathrm{loc}}$.

Proof. Because the category of τ-local synthetic spectra is a smashing localization, inverting τ preserves compact objects. Then note that there is an equivalence $\tau^{-1}F \otimes \tau^{-1}X \simeq \tau^{-1}(F \otimes X)$, so that we can compute:

$$(\tau^{-1}F)_{*,*}X \cong F_{*,*}X[\tau^{-1}]$$

and if $F_{*,*}X$ is finitely generated and projective over $F_{*,*}$, then $F_{*,*}X[\tau^{-1}]$ will be finitely generated and projective over $F_{*,*}[\tau^{-1}] \cong (\tau^{-1}F)_{*,*}$ and this process will also preserve epimorphisms. Because the relevant pullbacs in both sites are computed in $\mathcal{S}\mathrm{yn}_E$ they are also pushouts and the left adjoint τ^{-1} will preserve them. The symmetric monoidality of τ^{-1} shows that this morphism of sites upgrades to one of excellent sites. □

Lemma 3.5. In the induced adjunction $F : \mathcal{B}\mathrm{isyn} \to \mathrm{Sh}_\Sigma((\mathcal{S}\mathrm{yn}_E)^{\{}\tau - \mathrm{loc}_F) : G$, the right adjoint G is cocontinuous, $G(X)$ is τ-local for all X, and the essential image consists of all τ-local bisynthetic spectra.

Proof. □

Proposition 3.6. The subcategory of τ-local objects in $\mathcal{B}\mathrm{isyn}$ is equivalent to the category of spherical sheaves on the site $(\mathcal{S}\mathrm{yn}_E)_{\nu F}^{\tau-\mathrm{loc}}$.

Theorem 3.7. There is an equivalence of spherical sheaves over $(\mathcal{S}\mathrm{yn}_E)_{\nu F}^{\tau-\mathrm{loc}}$ and $\mathcal{S}\mathrm{p}_F^{\mathrm{fp}}$. As a result, the category of τ-local bisynthetic spectra is equivalent to $\mathcal{S}\mathrm{yn}_F$.

3.4. The τ-special fiber.

I have no idea what to do for this at the moment, would love any ideas.