1. τ- AND λ-BOCKSTEIN SPECTRAL SEQUENCES

In this section, we prove results about the τ- and λ-Bockstein spectral sequences in $\mathcal{B}\mathrm{isyn}$. Let $\nu^2 : \mathcal{S}\mathrm{p} \to \mathcal{B}\mathrm{isyn}$ denote the composition of the functors $\mathcal{S}\mathrm{p} \xrightarrow{\nu_E} \mathcal{S}\mathrm{yn}_E \xrightarrow{\nu_F} \mathcal{B}\mathrm{isyn}$.

Remark 1.1. For this theorem, I'm using these assumptions:

- $\nu_F(\mathbb{S}_E^{k,w}) = \mathbb{S}^{k,w,k}$
- SES in $\nu_E F_{*,*}$-homology induces cofiber sequence in ν^2
- ν^2 is symmetric monoidal on projective objects
- homotopy of $\nu^2(F)$-module is λ-free
- λ^{-1} is symmetric monoidal and satisfies $\lambda^{-1} \circ \nu_F = \mathrm{id}$
- inclusion or equivalence $\mathrm{Mod}(\mathcal{B}\mathrm{isyn}; C\lambda) \simeq \mathcal{S}\mathrm{table}(\nu_E F_{*,*} \nu_E F)$

Theorem 1.2. For the bisynthetic spectrum $\nu_F(Y)$ with $Y \in \mathcal{S}\mathrm{yn}_E$, the $\nu^2(F)$-Adams spectral sequence is isomorphic to the λ-Bockstein spectral sequence.

Proof. Our proof of this follows very closely to the proofs in [BHS19, App. A]. Consider the canonical $\nu_E(F)$-Adams tower

$$\begin{tikzcd}
\cdots \arrow[r] & Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Y_0 \arrow[d] & = & Y \\
& Y_2 \otimes \nu_E(F) & Y_1 \otimes \nu_E(F) & Y_0 \otimes \nu_E(F) & &
\end{tikzcd}$$

of $Y \in \mathcal{S}\mathrm{yn}_E$. This tower is made up of cofiber sequences

$$Y_{n+1} \to Y_n \to Y_n \otimes \nu_E(F) \to \Sigma^{1,0} Y_{n+1}$$

such that the maps $Y_{n+1} \to Y_n$ induce the zero map on $\nu_E(F)_{*,*}$-homology. By SES prop. of ν_F (!!!need this fact!!!), these become cofiber sequences

$$\Sigma^{-1}\nu_F(Y_{n+1}) \to \nu_F(Y_n) \to \nu_F(Y_n \otimes \nu_E(F)) \to \nu_F(\Sigma^{1,0} Y_{n+1}).$$

in $\mathcal{B}\mathrm{isyn}$. By symmetric monoidality of ν_F on projective objects (!!!need this fact!!!), we can identify these cofiber sequences as

$$\Sigma^{0,0,1}\nu_F(Y_{n+1}) \to \nu_F(Y_n) \to \nu_F(Y_n) \otimes \nu^2(F) \to \Sigma^{1,0,1}\nu_F(Y_{n+1}).$$

Hence, the canonical $\nu^2(F)$-Adams tower for $\nu_F(Y)$ can be written as

$$\begin{tikzcd}
\cdots \arrow[r] & \Sigma^{0,0,2}\nu_F(Y_2) \arrow[r] \arrow[d] & \Sigma^{0,0,1}\nu_F(Y_1) \arrow[r] \arrow[d] & \nu_F(Y_0) \arrow[d] & = & \nu_F(Y) \\
& \Sigma^{0,0,2}\nu_F(Y_2) \otimes \nu^2(F) & \Sigma^{0,0,1}\nu_F(Y_1) \otimes \nu^2(F) & \nu_F(Y_0) \otimes \nu^2(F) & &
\end{tikzcd}$$

In the usual way, we get the Adams E_1-page

$$\begin{aligned}
{}_{\nu^2 F}E_1^{f,k,w,v} &= \pi_{k,w,v}(\Sigma^{0,0,f}\nu_F(Y_f) \otimes \nu^2(F)) \\
&\cong \pi_{k,w,v-f}(\nu_F(Y_f) \otimes \nu^2(F))
\end{aligned}$$

Because of the equivalence

$$\nu_F(Y_n \otimes \nu_E(F)) \simeq \nu_F(Y_n) \otimes \nu^2(F),$$

and the fact that the homotopy of a $\nu^2(F)$-module is λ-free (!!!need this fact!!!), we can identify the E_1-page as

$${}_{\nu^2 F}E_1^{f,k,w,v} \cong {}_{\nu_E F}E_1^{f,k,w} \otimes \mathbb{Z}[\lambda],$$

where $x \in {}_{\nu_E F}E_1^{f,k,w}$ lives in quad-degree $(f, k, w, k+f)$ (I'm pretty sure, but this is just a guess!!! double check when we know about homotopy of $\nu^2 F$-modules better).

The differentials in this spectral sequence are of the form

$$d_r : {}_{\nu^2 F}E_1^{f,k,w,v} \to {}_{\nu^2 F}E_1^{f+r,k-1,w,v}.$$

Since $\lambda^{-1} \circ \nu_F = \mathrm{id}$ and λ is symmetric monoidal, the λ-inverted $\nu^2(F)$-Adams tower is the $\nu_E(F)$-Adams tower, and, in particular, the λ-inverted differentials

$$d_{r,\lambda^{-1}} : {}_{\nu_E F}E_1^{f,k,w} \to {}_{\nu_E F}E_1^{f+r,k-1,w}$$

are exactly the differentials of the $\nu_E(F)$-Adams spectral sequence. Just as in [BHS19], this implies that if $d_{r,\lambda^{-1}}(x) = y$ is a $\nu_E(F)$-Adams differential, then

$$d_r(x) = \lambda^{r-1} y.$$

Using the equivalence (inclusion?)

$$\mathrm{Mod}(\mathcal{B}\mathrm{isyn}; C\lambda) \simeq \mathcal{S}\mathrm{table}(\nu_E F_{*,*} \nu_E F),$$

we see that the λ-Bockstein E_1-page

$${}_\beta E_1^{f,k,w,v} = \pi_{k,w,v}(\Sigma^{0,0,-f} \nu_F(Y) \otimes C\lambda)$$

and Adams E_2-page are, up to a degree shift, isomorphic. Because they have the same formula for differentials, they must be isomorphic as spectral sequences. □

Remark 1.3. We might actually need to prove a little bit more. [BHS19] cites their Thm. 9.19(1) when they prove their version of this, but I'm not sure how exactly they're using that. Probably need to assume everything is nilpotent/lambda complete too, but don't feel like thinking about this for now. They also prove that the *filtrations* are literally the same, not just that the spectral sequences are isomorphic.

Remark 1.4. For the next lemma and theorem, I need these assumptions:

- $\nu_F(\mathbb{S}_E^{k,w}) = \mathbb{S}^{k,w,k}$
- E Adams-type and ring map $E \to F$
- $\nu_E P$, with P finite E_*-projective, is "finite" in $\mathcal{S}\mathrm{yn}_E$
- ν^2 is symmetric monoidal on projective objects
- SES in $\nu_E F_{*,*}$-homology induces cofiber sequence in ν^2
- Commuting diagram of functors

$$\begin{tikzcd}
\mathcal{B}\mathrm{isyn} \arrow[r, "\tau^{-1}"] & \mathcal{S}\mathrm{yn}_F \\
\mathcal{S}\mathrm{yn}_E \arrow[u, "\nu_F"] \arrow[r, "\tau^{-1}"'] & \mathcal{S}\mathrm{p} \arrow[u, "\nu_F"']
\end{tikzcd}$$

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Lemma 1.5. The synthetic spectrum $\nu_E E \in \mathcal{S}\mathrm{yn}_E$ is a filtered colimit of finite, $\nu_E F$-projectives. In particular, for $Y \in \mathcal{S}\mathrm{yn}_E$,

$$\nu_F(Y \otimes \nu_E E) \simeq \nu_F(Y) \otimes \nu^2(E).$$

Proof. Let $E \simeq \mathrm{colim}_\alpha E_\alpha$ where each E_α is a finite, E_*-projective spectrum. Since $\nu_E(E) \simeq \mathrm{colim}_\alpha \nu_E E_\alpha$, it suffices to show that each $\nu_E E_\alpha$ is a finite, $\nu_E F$-projective synthetic spectrum. Now, $\nu_E E_\alpha$ is automatically finite (!!!make sure we define what this means!!!). For projectivity, we see that

$$\nu_E E_\alpha \otimes \nu_E F \simeq \nu_E(E_\alpha \otimes F),$$

with homotopy $\pi_{*,*}(\nu_E E_\alpha \otimes \nu_E F) \cong F_* E_\alpha[\tau]$. By a Kunneth spectral sequence argument,

$$F_* E_\alpha \cong E_* E_\alpha \otimes_{E_*} F_*.$$

In particular, since $E_* E_\alpha$ is projective over E_*, then $F_* E_\alpha$ is projective over F_*. Tensoring with $\mathbb{Z}[\tau]$ preserves projectivity, so that $F_* E_\alpha[\tau]$ is projective over $\nu_E F_{*,*} \cong F_*[\tau]$. □

Theorem 1.6. For the bisynthetic spectrum $\nu^2(X)$ with $X \in \mathcal{S}\mathrm{p}$, the $\nu^2(E)$-Adams spectral sequence is isomorphic to the τ-Bockstein spectral sequence.

Proof. Consider the canonical E-Adams tower

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[d] & = & X \\ & X_2 \otimes E & X_1 \otimes E & X_0 \otimes E & & \end{tikzcd}$$

of $X \in \mathcal{S}\mathrm{p}$. This tower is made up of cofiber sequences

$$X_{n+1} \to X_n \to X_n \otimes E \to \Sigma X_{n+1}$$

such that the maps $X_{n+1} \to X_n$ induce the zero map on E_*-homology. Applying ν_E to these cofiber sequences, as in [BHS19] we get cofiber sequences

$$\Sigma^{0,1}\nu_E(X_{n+1}) \to \nu_E(X_n) \to \nu_E(X_n) \otimes \nu_E(E) \to \Sigma^{1,1}\nu_E(X_{n+1}).$$

Since $F_*(-) \cong E_*(-) \otimes_{E_*} F_*$ and the maps $\Sigma^{0,1}\nu_E(X_{n+1}) \to \nu_E(X_n)$ induce the zero map on $\nu_E(E)_{*,*}$-homology, the maps $\Sigma^{0,1}\nu_E(X_{n+1}) \to \nu_E(X_n)$ also induce zero on $\nu_E(F)_{*,*}$-homology. This, together with Lemma 1.5, implies that there are cofiber sequences

$$\Sigma^{0,1,1}\nu^2(X_{n+1}) \to \nu^2(X_n) \to \nu^2(X_n) \otimes \nu^2(E) \to \Sigma^{1,1,1}\nu^2(X_{n+1})$$

which build up the canonical $\nu^2(E)$-Adams tower

$$\begin{tikzcd} \cdots \arrow[r] & \Sigma^{0,2,2}\nu^2(X_2) \arrow[r] \arrow[d] & \Sigma^{0,1,1}\nu^2(X_1) \arrow[r] \arrow[d] & \nu^2(X_0) \arrow[d] & = & \nu^2(X) \\ & \Sigma^{0,2,2}\nu^2(X_2) \otimes \nu^2(E) & \Sigma^{0,1,1}\nu^2(X_1) \otimes \nu^2(E) & \nu^2(X_0) \otimes \nu^2(E) & & \end{tikzcd}$$

for $\nu^2(X) \in \mathcal{B}\mathrm{isyn}$. The $\nu^2(E)$-Adams E_1-page has the form

$$\begin{aligned} {}_{\nu^2 E}E_1^{f,k,w,v} &= \pi_{k,w,v}(\Sigma^{0,f,f}\nu^2(X_f) \otimes \nu^2(E)) \\ &\cong \pi_{k,w-f,v-f}(\nu^2(X_f) \otimes \nu^2(E)). \end{aligned}$$

We aim to show that this is τ-free. $\square$

REFERENCES

[BHS19] Robert Burklund, Jeremy Hahn, and Andrew Senger. *On the boundaries of highly connected, almost closed manifolds*. 2019. arXiv: 1910.14116.