BISYNTHETIC SPECTRA

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(Peter) Here's a list of things we'll need here

- Lemma that $\nu_E F_{*,*}(-) : (\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F} \to \mathcal{C}\mathrm{omod}^{fp}_{\nu_E F_{*,*}\nu_E F}$ is morphism of ∞-sites which reflects coverings and admits a common envelope
- Other lemmas about other maps of ∞-sites? Or wait to do that later
- Definition of Adams-type E-synthetic spectrum (prolly same as spectra one)
- Lemma that $\nu_E F_{*,*}\nu_E F$ is Adams Hopf algebroid a la [Pst22, Def. 3.1]
- Lemma about when μ_F is symmetric monoidal
- Lemma about homotopy of μ_F-modules being lambda free

1. Categorical Preliminaries and Generalized Synthetic Spectra

Throughout this section we fix a presentably symmetric monoidal stable ∞-category $\mathcal{C}$. Our goal in this section is to generalize the original construction of synthetic spectra due to [Pst22] for a broader class of homology theories on stable ∞-categories. We note that in some sense this has been accomplished by [**todo**], however, the categories produced therein are subject to technical limitations, for example, we will need the existence of a nice symmetric monoidal structure on our synthetic categories which is not constructed in loc. cit.

1.1. Homological Contexts.

In order to set up a theory of synthetic spectra, we must first specify what the necessary data is. The original theory takes as input a (nice) ring spectrum E, but makes substantial use of the underlying structure present in $\mathcal{S}\mathrm{p}$ and the concomitant properties of the induced homology theory E_*. We spell out below what we believe is a suitably general theory of homological contexts, i.e., theories which act sufficiently like functors $X \mapsto \pi_*(X \otimes E)$.

Definition 1.1. A local grading on a category $\mathcal{D}$ is an auto-equivalence $-[1] : \mathcal{D} \to \mathcal{D}$. A category is said to be locally graded if it has a chosen local grading.

Example 1.2. All stable ∞-categories are locally graded by by the formal suspension $-[1] := \Sigma$.

Example 1.3. A graded category $\mathrm{Fun}(\mathbb{Z}, \mathcal{D})$ where $\mathbb{Z}$ is the discrete category on the integers is locally graded by the shift functor induced by $n \mapsto n \pm 1$ on $\mathbb{Z}$.

Both examples above frequently arise as special cases of the following:

Example 1.4. If $\mathcal{D}$ is monoidal and X is a Picard-object, i.e., it is $\otimes$-invertible, then the functor $- \otimes X$ forms a local grading on $\mathcal{D}$.

Definition 1.5. Let $\mathcal{D}$ be a presentably symmetric monoidal stable ∞-category and let $\mathcal{A}$ be an abelian 1-category equipped with a local grading. A functor $\pi_\star : \mathcal{D} \to \mathcal{A}$ is said to be a homotopy groups functor if it commutes with filtered colimits, is lax monoidal and additive, and if in addition it

1. sends cofibers in $\mathcal{D}$ to exact sequences in $\mathcal{A}$,
2. and intertwines the local gradings $\mathrm{H}(\Sigma X) = H(X)[1]$ naturally.

Remark 1.6. An important consequence of the above definition is that any such H will send a cofiber sequence $X \to Y \to Z$ in $\mathcal{D}$ to a long exact sequence in $\mathcal{A}$ as rotating the cofiber in $\mathcal{D}$ results in a local-grading-shift in $\mathcal{A}$.

The above definition is engineered not just to capture the classical examples of homotopy groups of spectra, but also the categories of (genuine) equivariant spectra, motivic spectra, and synthetic spectra. In general, we will want the extra flexibility of considering homotopy groups (and later homology theories) which are multigraded and have long exact sequences with respect to the formal suspension.

Definition 1.7. A *homological context* is the data of a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, a presentably symmetric monoidal left adjoint $\mathrm{H} : \mathcal{C} \to \mathcal{D}$, and a homotopy groups functor $\pi_\star : \mathcal{C} \to \mathcal{A}$. If G is the right adjoint to H we write $\mathrm{H}_\star$ for the composite $\pi_\star \circ \mathrm{G} \circ \mathrm{H}$.

Example 1.8. All examples we study in this paper will be given by the following data. First we fix $\mathcal{C}$ as above and define $\pi_\star$ to be some abelian enrichment of mapping out of the unit. We then let $R \in \mathrm{Alg}(\mathcal{C})$ and put $\mathrm{H} := - \otimes R$ valued in $\mathrm{Mod}(R)$ so that it is symmetric monoidal and preserves colimits by assumption. Then $\mathrm{H}_\star$ corresponds to taking the homotopy groups on the underlying $\mathcal{C}$-objects after extending by $- \otimes R$.

Definition 1.9. Note that by definition the composite $H_\star$ is lax monoidal, so that the image of the unit $H_\star(1_\mathcal{C})$, which we will refer to as the coefficient ring for H and denote by simply $H_\star$, is a commutative ring object in the 1-category $\mathcal{A}$.

The essential reason for separating the functors H and $\pi_\star$ in the definition of a homological context is to be able to make sense of the homological comonad.

Definition 1.10. The homological comonad for H is the comonad on $\mathcal{D}$ (the codomain of H) induced by the adjunction between H and its right adjoint G, i.e. it is the comonad $\mathrm{HG} : \mathcal{D} \to \mathcal{D}$.

Definition 1.11. The $\mathcal{C}$-object of H-cooperations is defined to be the object $\mathrm{Coop}(\mathrm{H}) := \mathrm{HGH}(1_\mathcal{C})$. The algebraic cooperations of $\mathrm{H}_\star$ are defined to be $\pi_\star\mathrm{Coop}(\mathrm{H})$ and we denote them by $\mathrm{H}_\star\mathrm{H}$.

Example 1.12. If $\mathcal{C} = \mathcal{S}\mathrm{p}$ and E is a commutative ring, then the spectral and algebraic cooperations associated to $H = - \otimes E$ are given by the spectrum $E \otimes E$ and $E_* E$ respectively.

Note that the object $\mathrm{Coop}(\mathrm{H})$ is acquires both a left and right module structure over $\mathrm{H}(1_\mathcal{C})$ from the structure morphisms arising from the comonad. The lax monoidality of $\pi_\star$ then preserves these structures, so that $\mathrm{H}_\star\mathrm{H}$ acquires a left and right module structure over $\mathrm{H}_\star$.

Definition 1.13 (Adams Flat)**.** A homological context is said to be Adams flat if $\mathrm{H}_\star\mathrm{H}$ is flat as a right $\mathrm{H}_\star$-module.[1]

Proposition 1.14. If our homological context is Adams flat, then the pair $(\mathrm{H}_\star, \mathrm{H}_\star\mathrm{H})$ is a Hopf algebroid in $\mathcal{A}$ and for any $X \in \mathcal{C}$ we have that $\mathrm{H}_\star(X)$ acquires a canonical comodule structure over $\mathrm{H}_\star\mathrm{H}$.

Proof. todo. Should be identical to the blue book proof. ☐

1.2. **The H-Adams Spectral Sequence.** Maybe just cite Krause?

1.3. **H-finite sites.** In this section we fix a homological context (Definition 1.7)

$$\mathcal{C} \xrightarrow{\mathrm{H}} \mathcal{D} \xrightarrow{\pi_\star} \mathcal{A}$$

whose composition is denoted $\mathrm{H}_\star$. To such a context we will assocaited a site $\mathcal{C}_\mathrm{H}^{\mathrm{fp}}$ which will encode the relevant properties of the H-Adams spectral sequence.

Definition 1.15. With notation as above, the H-projective site of $\mathcal{C}$, denoted $\mathcal{C}_\mathrm{H}^{\mathrm{fp}}$, is the full subcategory of $\mathcal{C}^\omega$ consisting of objects X such that $\mathrm{H}_\star(X)$ is finitely generated and projective as an $\mathrm{H}_\star$ module. The coverings in $\mathcal{C}_\mathrm{H}^{\mathrm{fp}}$ are the single maps $f : X \to Y$ such that $\mathrm{H}_\star(f)$ is an epimorphism, which we call H-epimorphisms for short.

Lemma 1.16. Let $Q, R, P \in \mathcal{C}_\mathrm{H}^\omega$. Suppose $f : Q \to P$ is an H-epimorphism and $g : R \to P$ is arbitrary. Let $X = Q \times_P R$ denote the pullback in $\mathcal{C}$. Then X is again in $\mathcal{C}_\mathrm{H}^\omega$ and $X \to R$ is an H-epimorphism.

Proof. First we note that the pullback may equivalently be described via the fiber sequence

$$X \to Q \oplus R \xrightarrow{f-g} P$$

and as a result is compact as the fiber of a map between compact objects. Moreover, because $\mathrm{H}(Q) \to \mathrm{H}(P)$ is an H-epimorphism and because $\mathrm{H}(Q)$ is projective, we get a lift $\tilde{g} : \mathrm{H}(R) \to \mathrm{H}(P)$ which then splits the long exact sequence. As a result, the long exact sequence breaks up into short exact sequences, and we use the 2-out-of-3 property to claim that $\mathrm{H}(X)$ is therefore finitely generated and projective. That $\mathrm{H}(X) \to \mathrm{H}(R)$ is an epimorphism follows from the splitting. ☐

Definition 1.17. We say that a homological context $\mathrm{H}_\star$ is projectively monoidal if the symmetric monoidal structure on $\mathcal{C}$ restricts to the site $\mathcal{C}_\mathrm{H}^{\mathrm{fp}}$ if in addition the restriction $\mathcal{C}_\mathrm{H}^{\mathrm{fp}} \to \mathcal{A}$ is a monoidal functor.

[1]We are interested in various notions which have been called "Adams Type" in the literature. We reserve the later term for Definition todo.

Remark 1.18. The above condition is resonably common. One way it frequently arises is the existence of a spectral sequence

$$\mathrm{Tor}_{\mathcal{A}}(\mathrm{H}_\star(X), \mathrm{H}_\star(Y)) \Rightarrow \mathrm{H}_\star(X \otimes Y)$$

where the requirement that both terms be projective causes the spectral sequence to collapse immediatley.

Convention 1.19. We will assume going forward that the fixed homological context above is in additional projectively monoidal.

Definition 1.20 ([Pst22])**.** A small ∞-site is *additive* if the coverings are provided by singletons and the underlying category is additive.

Definition 1.21 ([Pst22])**.** An additive ∞-site is said to be excellent if it is equipped with a symmetric monoidal structure in which every object has a dual and such that the functors $- \otimes P$ preserve coverings for all P in the site.

Proposition 1.22. The category $\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$ is an excellent site.

Proof. The symmetric monoidal structure is the one guaranteed by convention 1.19. Because compact objects are dualizable, it is automatic that every object has a dual in $\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$. As such, $- \otimes P$ is a right adjoint and preserves all pullbacks for all $P \in \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$. It therefore suffices to show that it takes coverings to coverings. But since the functor $H_\star$ is monoidal (again by convention), this is immediate. □

1.4. H**-Synthetic Spectra.** Again we fix a homological context $\mathcal{C} \to \mathcal{D} \to \mathcal{A}$ with notation as in all previous sections. Recall that a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is said to be spherical if for all $X, Y \in \mathcal{C}$ the natural map

$$F(X \amalg Y) \to F(X) \times F(Y)$$

is an equivalence. A sheaf is said to be spherical if the underlying presheaf is, and the sheafification functor when it exists sends spherical sheaves to spherical presheaves. Spherical presheaves are very well behaved when the category $\mathcal{C}$ is additive. In particular, in this case we get canonical lifts to grouplike commutative monoids in $\mathcal{D}$, so long as these make sense. As a result, spherical sheaves of spaces on $\mathcal{C}$ lift canonically to spherical sheaves of connective spectra.

Definition 1.23. The category of of Synthetic Spectra with respect to the context above is the category of spherical sheaves of spectra $\mathrm{Sh}^{\mathcal{S}\mathrm{p}}_{\Sigma}(\mathcal{C}^{\omega}_{\mathrm{H}})$ on the H-finite site. We will often drop much of the context data and refer to this as the category $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ of H-synthetic spectra.

Lemma 1.24. The category $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ is presentably symmetric monoidal and stable.

Proof. Stability and presentability follow from [Pst22, Corollary 2.13] and presentably symmetric monoidality follows from [Pst22, Proposition 2.30]. □

Because $\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$ is a full subcategory of $\mathcal{C}$, the yoneda embedding extends to a functor $\mathcal{C} \to \mathrm{Sh}(\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}})$ after sheafification, which we will denote by $\hat{y}$ (reserving the undecorated y for the restriction back to $\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$). Note that both y and $\hat{y}$ automatically land in the subcategory of spherical sheaves by the calculation

$$\hat{y}(c)(d \amalg d') = \mathrm{Map}_{\mathcal{C}}(d \amalg d', c) \simeq \mathrm{Map}_{\mathcal{C}}(d, c) \times \mathrm{Map}_{\mathcal{C}}(d', c).$$

There is then an adjunction ([Pst22])

$$\Sigma^\infty_+ : \mathrm{Sh}(\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}) \leftrightarrows \mathrm{Sh}^{\mathcal{S}\mathrm{p}}(\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}) : \Omega^\infty$$

which allows us to lift both to functors valued in spherical presheaves. Indeed, because spherical presheaves of spaces canonically lift to groupline $\mathbb{E}_\infty$-monoids levelwise, the functor Σ^∞_+ identifies the source with the full subcategory of the target consisting of those objects which are connective under the induced t-structure (see **??**), so that spherical sheaves of spaces admit canonical lifts to spherical sheaves of spectra.

Definition 1.25 (Synthetic Analog Functor)**.** The synthetic analog functor $\nu : \mathcal{C} \to \mathcal{S}\mathrm{yn}_{\mathrm{H}}$ is defined to be $\Sigma^\infty_+ \hat{y}$, the canonical lift of $\hat{y}$ to a spherical sheaf of spectra.

Proposition 1.26. The category $\mathcal{S}\mathrm{yn}_H$ and its synthetic analog functor enjoy the following properties:

(1) For all $P \in \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$ there is an equivalence $\mathrm{Map}(\nu P, X) \simeq \Omega^\infty X(P)$.
(2) The category $\mathcal{S}\mathrm{yn}_H$ is generated under colimits by the compact objects $\Sigma^k \nu P$ for $P \in \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$.[2]

[2] In almost all cases of interest, the synthetic category has multi-graded suspensions. Here we are only interested in the formal suspension in the stable ∞-category $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$.

(3) The functor ν preserves filtered colimits and direct sums.
(4) The functor ν is lax monoidal.
(5) The restriction $\nu : \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}} \to \mathcal{S}\mathrm{yn}_{\mathrm{H}}$ is symmetric monoidal.

Proof. Claim (1) is proven identically to [Pst22, Lemma 4.11]:

$$\mathrm{Map}(\nu P, X) \simeq \mathrm{Map}(\Sigma^\infty_+ y(P), X) \simeq \mathrm{Map}(y(P), \Omega^\infty X) \simeq \Omega^\infty X(P).$$

After noting that for $P \in \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$ the object νP is compact in $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ by [Pst22, Cor. 4.12], the result follows from (1) as these objects can detect equivalences of spherical presheaves levelwise. Both (3) and (4) are true for the same reason: we can write the synthetic analog as the composite

$$\mathcal{C} \xrightarrow{\hat{y}} \mathrm{Sh}_\Sigma(\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}) \xrightarrow{\Sigma^\infty_+} \mathcal{S}\mathrm{yn}_{\mathrm{H}}$$

wherein the second functor is a symmetric monoidal left adjoint, so it suffices to show that $\hat{y}$ is lax monoidal, preserves filtered colimits, and direct sums. Lax monoidality follows from recognizing that the functor $\hat{y}$ admits a left left adjoint which is the unique colimit preserving extension of $\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}} \hookrightarrow \mathcal{C}$ and that this left adjoint is symmetric monoidal, so that its right adjoint its automatically lax monoidal. This is already enough for direct sums, which are both finite limits and colimits. For filtered colimits, we let X_α be a filtered diagram in $\mathcal{C}$ and note that for any $Y \in \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}}$ we have

$$\mathrm{colim}_\alpha\, \hat{y}(X_\alpha)(P) \simeq \mathrm{colim}_\alpha\, \mathrm{Map}(P, X_\alpha).$$

But then P is compact, so this filtered colimit is computed levelwise. But filtered colimits of sheaves are computed levelwise, so we are done. Finally we note that the restriction in (5) is the composite $\Sigma^\infty_+ \circ y$ so that it suffices to show that $y : \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}} \to \mathrm{Sh}(\mathcal{C}^{\mathrm{fp}}_{\mathrm{H}})$ itself is symmetric monoidal, but this property characterizes the Day convolution product of sheaves. □

1.5. **The sheaf t-structure.** Recall ([**todo**]) that for an arbitrary small ∞-site $\mathcal{T}$ the category of sheaves of spectra on $\mathcal{T}$ inherits a t-structure from the standard t-structure on spectra. Explicitly, the category of coconnective objects consists of the levelwise coconnective sheaves and the category of connective objects is determined against these. We can take homotopy groups levelwise and sheafify to get functors

$$\pi_n^\heartsuit : \mathrm{Sh}^{\mathcal{S}\mathrm{p}}(\mathcal{T}) \to \mathrm{Sh}^{\mathrm{Ab}}(\mathcal{T})$$

and which provide an alternative characterization of the t-structure, the connective objects are those sheaves whose sheaf homotopy groups vanish in negative degrees.

Proposition 1.27 ([Pst22])**.** Let $\mathcal{T}$ be an additive ∞-site. The t-structure on $\mathrm{Sh}^{\mathcal{S}\mathrm{p}}(\mathcal{T})$ described above restricts to a right-complete t-structure on spherical sheaves of spectra which is compatible with filtered colimits. Moreover, the heart of this t-structure is equivalent to the category of spherical sheaves of sets $\mathrm{Sh}^{\mathcal{S}\mathrm{et}}_\Sigma(\mathcal{T})$.[3]

Convention 1.28. Our synthetic categories of interest will have multiple interesting t-structures. For easy of notation, we will refer to this t-structure as the *sheaf t-structure*.

Lemma 1.29. There is an equivalence between the 1-categories of $\mathrm{H}_\star\mathrm{H}$-comodules in $\mathcal{A}$ and the category $\mathrm{Sh}^{\mathcal{S}\mathrm{et}}_\Sigma(\mathcal{C}\mathrm{omod}^{\mathrm{fp}}_{\mathrm{H}_\star\mathrm{H}})$ where the site $\mathcal{C}\mathrm{omod}^{\mathrm{fp}}_{\mathrm{H}_\star\mathrm{H}}$ is the subcategory of finitely generated and projective comodules with single epimorphisms for covers.

Lemma 1.30.

Proposition 1.31. When $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ is equipped with the sheaf t-structure, there is a monoidal equivalence $\mathcal{S}\mathrm{yn}_{\mathrm{H}}^\heartsuit \simeq \mathcal{C}\mathrm{omod}_{\mathrm{H}_\star\mathrm{H}}$.

Proof. We claim that the functor

$$H_\star : \mathcal{C}^{\mathrm{fp}}_{\mathrm{H}} \to \mathcal{C}\mathrm{omod}_{\mathrm{H}_\star\mathrm{H}}$$

is a morphism of additive ∞-sites which induces an equivalence on categories of spherical sheaves of sets. Clearly covers are preserved by definition and finite generation and projectivity are checked on underlying $\mathrm{H}_\star$-modules, so that this is indeed a morphism of sites. In fact, it clearly reflects covers as well. TODO □

[3] Recall that spherical sheaves acquire abelian group structures levelwise so that the category of sheaves of sets is indeed abelian.

1.6. **Thread Structures and τ.** The functor ν does not preserve (co)fiber sequences in general, although we will prove eventually that it preserves certain H-exact cofibers. In particular, ν will not commute with formal suspensions. This failure is measured by a canonical comparison map

$$\tau : \Sigma \circ \nu \to \nu \circ \Sigma$$

induced by the universal property of the pushout defining Σ. We will refer to this map as the *deformation parameter* of the deformation $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ of $\mathcal{C}$.

1.7. **Recovering Categories via Spherical Sheaves.** The purpose of this section is to provide critera for when a stable ∞-category can be recovered as a category of spherical sheaves on an additive ∞-site which is a full subcategory. Let $\mathcal{C}_0$ be a small full subcategory of a stable ∞-category $\mathcal{C}$ equipped with the structure of an additive ∞-site. Then there is a functor $Y : \mathcal{C} \to \mathrm{P}(\mathcal{C}_0, \mathcal{S}\mathrm{p})$ given by the spectral Yoneda embedding.

Lemma 1.32. The functor Y factors through $\mathrm{Sh}_{\Sigma}^{\mathcal{S}\mathrm{p}}(\mathcal{C})$.

Proof. It is easy to see that the functor is spherical, it remains to show that it is a sheaf. But because $\mathcal{C}$ was assumed additive, it is enough to show that for any cover $A \to B$ in $\mathcal{C}$ with fiber F the sequence

$$Y(B) \to Y(A) \to Y(F)$$

is still a fiber. But because Y is given by mapping spectra, this is immediate. Because Y is exact, it suffices to check that it preserves filtered colimits. □

Convention 1.33. We will from here only be interested in the functor $Y : \mathcal{C} \to \mathrm{Sh}_{\Sigma}^{\mathcal{S}\mathrm{p}}(\mathcal{C}_0)$ rather than the version valued in presheaves.

Lemma 1.34. If, with notation as above, $\mathcal{C}_0 \subset \mathcal{C}^{\omega}$ is a recover pair, then the functor Y is cocontinuous.

Proof. The functor Y is exact by definition, so it suffices to check that it preserve filtered colimits. But this is immediate as we have assumed that $\mathcal{C}_0$ consists only of compact objects in $\mathcal{C}$ so that the relevant filtered colimits are computed levelwise. □

Definition 1.35. A recovery pair is a pair $(\mathcal{C}_0, \mathcal{C})$ of a presentable stable ∞-category $\mathcal{C}$ and a small full subcategory $\mathcal{C}_0 \subset \mathcal{C}^{\omega}$ equipped with the structure of an additive ∞-site satisfying additional axioms:

(1) The (de)suspensions of the objects in $\mathcal{C}_0$ generate $\mathcal{C}$ under colimits.
(2) If $X \in \mathcal{C}_0$ then $Y(X)$ is connective in $\mathrm{Sh}_{\Sigma}^{\mathcal{S}\mathrm{p}}(\mathcal{C})$.

Proposition 1.36. If $(\mathcal{C}_0, \mathcal{C})$ is a recovery pair then the functor Y is an equivalence $\mathcal{C} \simeq \mathrm{Sh}_{\Sigma}^{\mathcal{S}\mathrm{p}}(\mathcal{C})$.

Proof. We know that the functor Y is cocontinuous from Lemma **??**. To see that it is fully faithful we can therefore restrict to checking on the generating objects within $\mathcal{C}_0$. Let $Z \in \mathcal{C}$ be fixed but arbitrary. Then it suffices to check that for all $X \in \mathcal{C}_0$ the map

$$\mathrm{Map}(X, Z) \to \mathrm{Map}(Y(X), Y(Z))$$

but we have that this latter space is equivalent to $\mathrm{Map}(\Sigma^{\infty}_{+} y(X), Y(Z))$ which is itself equivalent to $\mathrm{Map}(y(X), \Omega^{\infty} Y(Z))$ which is the same as $\simeq \mathrm{Map}(y(X), y(Z))$ and we conclude by the Yoneda lemma. To see that Y is essentially surjective we recall that it is cocontinunous and the image contains a family of generators. □

2. Bisynthetic Spectra

Having established the necessary categorical preliminaries, the remainder of this article will study our category of Bisynthetic spectra. The following convention will be enforced throughout the remainder of the document unless specifically stated otherwise.

Convention 2.1. We will fix two Adams-type ring spectra E, F satisfying the following additional assumptions:

(1) The E-Adams spectral sequence for F has no nonzero differentials.
(2) Some flatness hypothesis?

Definition 2.2. The homological context for (E, F)-bisynthetic spectra is given by the pair

$$\mathcal{S}\mathrm{yn}_E \xrightarrow{-\otimes \nu_E F} \mathrm{Mod}(\nu_E F) \xrightarrow{\pi_{*,*}} \mathrm{Fun}(\mathbb{Z}^2, \mathrm{Ab})$$

and the resulting site of Definition 1.15 is denoted $(\mathcal{S}\mathrm{yn}_E)^{\mathrm{fp}}_{\nu_E F}$. It can be explicitly described as the full subcategory of $\mathcal{S}\mathrm{yn}_E$ containing those compact objects whose $\nu_E F$ homology is finitely generated and projective over $\nu_E F_{**}$.

Definition 2.3. The category of Bisynthetic spectra $\mathcal{S}\mathrm{yn}_{E,F}$ is the category of synthetic objects (Definition 1.23) with respect to the (E,F)-bisynthetic homological context, i.e., it is the category of spherical sheaves on the site $(\mathcal{S}\mathrm{yn}_E)^{\mathrm{fp}}_{\nu_E F}$. We will denote the synthetic analog functor (definition 1.25) by simply ν_F for brevity.

Proposition 2.4. The category $\mathcal{S}\mathrm{yn}_{E,F}$ is a presentably symmetric monoidal stable ∞-category. It comes equipped with a synthetic analog functor

$$\mu_F : \mathcal{S}\mathrm{yn}_E \to \mathcal{S}\mathrm{yn}_{E,F}$$

which is lax monoidal, fully faithful, and preserves filtered colimits.

Proof. todo. □

Remark 2.5. We use the notation μ_F as ν_F will be reserved for the functor $\nu_F : \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{yn}_F$. We will later construct an analogous $\mu_E : \mathcal{S}\mathrm{yn}_F \to \mathcal{S}\mathrm{yn}_{E,F}$.

Notation 2.6. We will write ν^2 for the functor $\mu_F \circ \nu_E$. Note that this functor again is fully faithful, preserves filtered colimits, and is lax monoidal as a composition of such functors. Note that the unit for $\mathcal{S}\mathrm{yn}_{E,F}$ is $\nu^2(\mathbb{S})$; we will abuse notation and write $\mathbb{S}$ for this object.

Notation 2.7. We will consider $\mathcal{S}\mathrm{yn}_{E,F}$ to be a trigraded category, with the convention

$$\mathbb{S}^{t,w,v} = \Sigma^{t-v}\nu_F\Sigma^{v-w}\nu_E\mathbb{S}^w$$

Lemma 2.8. Each sphere $\mathbb{S}^{t,w,v}$ is $\otimes$-invertible and there are equivalences $\mathbb{S}^{t,w,v} \otimes \mathbb{S}^{t',w',v'} \simeq \mathbb{S}^{t+t',w+w',v+v'}$.

Notation 2.9. The category $\mathcal{S}\mathrm{yn}_{E,F}$ can be seen to have two distinct deformation parameters. We will denote by λ the parameter which arises via todo. We will denote simply by τ the map $\mu_F(\tau)$. With our conventions above these maps have gradings:

- $\lambda : \mathbb{S}^{0,0,-1} \to \mathbb{S}^{0,0,0}$
- $\tau : \mathbb{S}^{0,-1,0} \to \mathbb{S}^{0,0,0}$

Definition 2.10. We define the trigraded mapping objects between bisynthetic spectra to be

$$[X,Y]_{t,w,v} = [\Sigma^{t,w,v}X, Y]$$

and trigraded homotopy groups $\pi_{t,w,v}X = [\mathbb{S}, X]_{t,w,v}$. As we see above, this leads to $|\tau| = (0,-1,0)$ and $|\lambda| = (0,0,-1)$ as elements of $\pi_{***}\mathbb{S}$.

Definition 2.11. Given a triple (t,w,v) we will say that it has E-Chow degree $t-w$ and F-Chow-degree $t-v$.

2.1. A Tale of Two t-Structures.

There is a natural t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$ given by the sheaf t-structure of todo.

Definition 2.12. The F-t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$ is defined to be the sheaf t-structure of todo.

The analogous t-structure plays a vital role in the original [Pst22] where it is shown that it has an equivalent definition: its connective objects are given by those synthetic spectra whose $\nu_E E$-homology is concentrated in positive chow degree. We will later show (todo) that this sheaf t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$ admits a similar description in terms of $\nu^2 F$-homology (todo). In addition, we can also use this result as a definition for an E-t-structure:

Definition 2.13. A bisynthetic spectrum is said to be E-connective if $\nu^2 E_{***}X$ is concentrated in positive E-chow degree.

Lemma 2.14. The E-chow-connectives form the connective part of a t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$.

Proof. It suffices by todo to show that these objects are closed under colimits and extensions. They are clearly closed under finite direct sums, and the compactness of the trigraded spheres shows that this extends to infinite coproducts. Finally, long exact sequence arguments show that they are closed under cofibers and extensions as desired. □

Notation 2.15. We use superscripts E, F to distinguish the t-structures, e.g., $\tau^F_{\geqslant n}$ and $\pi^{\heartsuit,F}_n$ will denote the truncations and homotopy groups of the F-t-structure and likewise replacing F with E for the E-t-structure.

We will refer to the t-structure obtained from the above lemma as the E-t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$. We will first attempt to study the F-t-structure in detail, as it closely mirrors the arguments and development of [Pst22]. We can identify the heart in a similar manner to [Pst22]:

Theorem 2.16. The heart $\mathcal{S}\mathrm{yn}_{E,F}^{F,\heartsuit}$ is equivalent to $\mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F}$. (monoidal conditions should be added too)

Proof. By Proposition **??**, the heart is equivalent to $Sh_\Sigma^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)_{\nu F}^{fp})$. By (ref. to lemma in Section 2), the morphism of ∞-sites

$$\nu_E F_{*,*}(-) : (\mathcal{S}\mathrm{yn}_E)_{\nu F}^{fp} \to \mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F}^{fp}$$

is one which reflects coverings and admits a common envelope. By [Pst22, Rem. 2.50], this induces an adjoint equivalence

$$Sh_\Sigma^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)_{\nu F}^{fp}) \rightleftarrows Sh_\Sigma^{\mathrm{Set}}(\mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F}^{fp})\,.$$

The bigraded Hopf algebroid $(\nu_E F_{*,*}, \nu_E F_{*,*}\nu_E F)$ is Adams, in the sense of [Pst22, Def. 3.1], by (Lemma in Section 2 which proves that it's Adams). By a bigraded version of **[GH05]**, [Pst22, Thm. 3.2] there is an equivalence

$$\mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F} \simeq Sh_\Sigma^{\mathrm{Set}}(\mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F}^{fp}),$$

and the result follows. ☐

Now we work towards identifying the homotopy objects $\pi_k^{F,\heartsuit}X$ in terms of $\nu^2 F$-homology.

Lemma 2.17. For $X \in \mathcal{S}\mathrm{yn}_{E,F}$, the graded components of the $\nu_E F_{*,*}\nu_E F$-comodule $\pi_k^{F,\heartsuit}X$ are described by

$$(\pi_k^{F,\heartsuit}X)_{l,m} \cong \mathrm{colim}_\alpha\, \pi_k X(\Sigma^{l,m} D\nu_E F_\alpha),$$

where $F \simeq \mathrm{colim}_\alpha F_\alpha$ is a presentation of F as a filtered colimit of F-finite projective spectra.

Proof. This is essentially a bigraded version of [Pst22, Lemma 4.17] and the proof is similar to the proof of that lemma. By [Pst22, Thm. 2.58], the sheaf $\pi_k^{F,\heartsuit}X \in Sh_\Sigma^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)_{\nu F}^{fp})$ is representable by some comodule N; i.e.

$$(\pi_k^{F,\heartsuit}X)(-) \simeq \mathrm{Hom}_{\nu_E F_{*,*}\nu_E F}(\nu_E F_{*,*}(-), N).$$

Now notice that $\nu_E F_{*,*}\nu_E F \simeq \mathrm{colim}_\alpha\, \nu_E F_{*,*}\nu_E F_\alpha$, since ν_E commutes with filtered colimits, and $E_*(D\nu_E F_\alpha) \cong \mathrm{Hom}_{\nu_E F_{*,*}\nu_E F}(\nu_E F_{*,*}\nu_E F_\alpha, \nu_E F_{*,*})$. Then by [Pst22, Lemma 3.3], as a bigraded abelian group

$$N_{l,m} \cong \mathrm{colim}_\alpha\, \pi_k^{F,\heartsuit} X(\Sigma^{l,m} D\nu_E F_\alpha).$$

By a bigraded version of [Pst22, Lemma 3.25],

$$\mathrm{colim}_\alpha\, \pi_k^{F,\heartsuit} X(\Sigma^{l,m} D\nu_E F_\alpha) \cong \mathrm{colim}_\alpha\, \pi_k X(\Sigma^{l,m} D\nu_E F_\alpha),$$

which completes the proof. ☐

Theorem 2.18. For $X \in \mathcal{S}\mathrm{yn}_{E,F}$, there is an isomorphism

$$(\pi_k^{F,\heartsuit}X)_{l,m} \cong \nu^2 F_{k+l,m,l} X,$$

In effect, $\pi_k^{F,\heartsuit}X$ captures the F-Chow-degree ℓ part of the $\nu^2 F$ homology of X.

Proof. Again, this is a similar proof to [Pst22, Thm. 4.18]. We have that

$$\begin{aligned}
\nu^2 F_{k+l,m,l} X &\cong [\mathbb{S}^{k+l,m,l}, \nu^2 F \otimes X] \\
&\cong \mathrm{colim}_\alpha [\Sigma^k \mu_F(\mathbb{S}_E^{l,m}), \nu^2 F_\alpha \otimes X] \\
&\cong \mathrm{colim}_\alpha [\Sigma^k \mu_F(\Sigma^{l,m} D\nu_E F_\alpha), X] \\
&\cong \mathrm{colim}_\alpha\, \pi_k X(\Sigma^{l,m} D\nu_E F_\alpha) \\
&\cong (\pi_k^{F,\heartsuit}X)_{l,m}.
\end{aligned}$$

The first isomorphism is by definition, the second isomorphism follows from (definition from Section 2 about trigraded spheres) and equivalence $\nu^2 F \simeq \mathrm{colim}_\alpha\, \nu^2 F_\alpha$, the fourth isomorphism follows from (lemma from Section 2 which shows that $map(\mu_F P, X) \simeq \Omega^\infty(X(P))$ for $P \in (\mathcal{S}\mathrm{yn}_E)_{\nu F}^{fp}$), and the fifth isomorphism follows from Lemma 2.17. ☐

As a corollary, we get the following analog of [Pst22, Cor. 4.19]:

Corollary 2.19. A bisynthetic spectrum $X \in \mathcal{S}\mathrm{yn}_{E,F}$ is in $(\mathcal{S}\mathrm{yn}_{E,F})_{\geqslant 0}^F$ if and only if $\nu^2 F_{k,w,v} X = 0$ for Chow degree $k - v < 0$.

Proof. In this t-structure, $X \in \mathcal{S}\mathrm{yn}_{E,F}$ is in $(\mathcal{S}\mathrm{yn}_{E,F})_{\geqslant 0}^F$ if and only if $\pi_k^{F,\heartsuit}X$ vanishes for $k < 0$. By Theorem 2.18, this happens exactly when $k - v < 0$. ☐

As a consequence, we see that the νF-synthetic analog of an E-synthetic spectrum Y is always connective.

Corollary 2.20. If $Y \in \mathcal{S}\mathrm{yn}_E$, then $\mu_F Y \in (\mathcal{S}\mathrm{yn}_{E,F})^F_{\geqslant 0}$.

Proof. Consider the homology calculation

$$\begin{aligned}\nu^2 F_{*,*,*}\mu_F Y &\cong \mu_F(\nu F \otimes Y)_{*,*,*}\\ &\cong \nu F_{*,*}Y[\lambda]\,,\end{aligned}$$

where $\nu F_{k,w}Y$ lives in tridegree (k, w, k). The first isomorphism follows from (lemma in Section 2 about when μ_F is symmetric monoidal) and the second isomorphism follows (lemma in Section 2 about homotopy of νF-module). The result then follows from Corollary 2.19. ☐

This means that for the νF-synthetic analog of an E-synthetic spectrum Y, the tensor product $\mu_F Y \otimes C\lambda$ lives in the heart $\mathcal{S}\mathrm{yn}^{F,\heartsuit}_{E,F}$.

Corollary 2.21. If $Y \in \mathcal{S}\mathrm{yn}_E$, then $\Sigma^{0,0,-1}\mu_F Y \simeq \tau^F_{\geqslant 1}(\mu_F Y)$ and $\mu_F Y \otimes C\lambda \simeq \tau^F_{\leqslant 0}(\mu_F Y)$. In particular, $\mu_F Y \otimes C\lambda \in \mathcal{S}\mathrm{yn}^{F,\heartsuit}_{E,F}$.

Proof. Again, the proof is similar to the proof of [Pst22, Lemma 4.29]. Consider the cofiber sequence

$$\Sigma^{0,0,-1}\mu_F Y \xrightarrow{\lambda} \mu_F Y \to \mu_F Y \otimes C\lambda$$

By Corollary 2.19, it's clear that $\Sigma^{0,0,-1}\mu_F Y$ is 1-connective. By using the definition of μ_F and the colimit-comparison definition of λ, it follows that $\mu_F Y \otimes C\lambda$ lives in $(\mathcal{S}\mathrm{yn}_{E,F})^F_{\leqslant 0}$. The result then follows. ☐

Remark 2.22. Similar to $\mathcal{S}\mathrm{yn}_E$, we see that $\mu_F Y \otimes C\lambda$ is lives in an algebraic category; namely the category of $\nu_E F_{*,*}\nu_E F$-comodules. In Section 4, we will show that, in fact, $\mu_F Y \otimes C\lambda$ can be identified with the comodule $\nu_E F_{*,*}Y$ and there is an embedding $\mathrm{Mod}_{C\lambda}(\mathcal{S}\mathrm{yn}_{E,F}) \hookrightarrow \mathcal{S}\mathrm{table}_{\nu_E F_{*,*}\nu_E F}$ of $C\lambda$-modules into the stable comodule category associated to the bigraded Hopf algebroid $(\nu_E F_{*,*}, \nu_E F_{*,*}\nu_E F)$.

3. τ-LOCAL BISYNTHETIC SPECTRA

The main result we are to prove in this section is the following:

Theorem 3.1. There is a symmetric monoidal equivalence of categories $\mathcal{S}\mathrm{yn}_F \simeq \tau^{-1}\mathcal{S}\mathrm{yn}_{E,F}$.

To prove this, we will show that both are equivalent to an intermediate category of spherical sheaves of spectra on a certain site. Our approach is heavily inspired by the original comparison between cellular motives and MU-synthetic spectra of [Pst22].

Definition 3.2. The subcategory $(\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})^{\mathrm{fp}}$ of τ-local finite projectives is defined to be those bisynthetic spectra $X \in \mathcal{S}\mathrm{yn}_{E,F}$ satisfying

- X is compact in $\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F}$
- and $\pi_{***}(X \otimes \nu^2 F)$ is finitely generated by generators in E-Chow degree 0 and projective over $\pi_{***}\nu^2 F$.

Our proof of Theorem 3.1 will proceed first by showing that the site described above forms recovery pair $\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F}$ and then by comparing the site in this pair to $\mathcal{S}\mathrm{p}^{\mathrm{fp}}_F$ via the double realization Re^2.

Lemma 3.3. Suppose that $X \in \mathcal{S}\mathrm{yn}_{E,F}$ is τ-local. Then there is an isomorphism

$$\pi_{t,w,v}X \cong \pi_t \mathrm{Re}^2(X)$$

whenever $t - v \geqslant 0$, i.e., in positive F-chow degree.

Proof. It follows from todo that we have an isomorphism $\pi_{t,w,v}X \cong \pi_{t,w}\mathrm{Re}^E(X)$ in positive F-Chow degree. ☐

Proof. ☐

Lemma 3.4. The full realization functor $\mathrm{Re}^2 : \mathcal{S}\mathrm{yn}_{E,F} \to \mathcal{S}\mathrm{p}$ sends $(\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})^{\mathrm{fp}}$ onto $\mathcal{S}\mathrm{p}^{\mathrm{fp}}_F$.

Proof. Suppose $X \in (\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})^{\mathrm{fp}}$. First recall that the functor Re^2 will preserve compact objects so it suffices to check that

$$\pi_*\mathrm{Re}^2(X) \otimes F \cong \pi_*\mathrm{Re}^2(X \otimes \nu^2 F)$$

is finitely generated over $\pi_* F$ and projective. But up to regrading Re^2 factors as first inverting λ and then inverting τ. But inverting classes will preserve projectivity and finite generation. To see that this restriction is essentially surjective we note that the composition

$$\mathcal{S}\mathrm{p} \xrightarrow{\nu^2} \mathcal{S}\mathrm{yn}_{E,F} \xrightarrow{\tau^{-1}} \tau^{-1}\mathcal{S}\mathrm{yn}_{E,F}$$

provides a section. □

Lemma 3.5. The restricted Re^2 functor of Lemma 3.4 is a morphism of sites which reflects covers.

Proof. A map $\pi_{***}(X \otimes \nu^2 F) \to \pi_{***}(Y \otimes \nu^2 F)$ for $X, Y \in (\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})^{\mathrm{fp}}$ will be an epimorphism if and only if it is an epimorphism in Chow degree 0 so that the result follows from Lemma **??**. □

Proposition 3.6. The pair $((\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})^{\mathrm{fp}}, \tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})$ is a recovery pair, i.e., there is a symmetric monoidal equivalence

$$\mathrm{Sh}_{\Sigma}^{\mathcal{S}\mathrm{p}}((\tau^{-1}\mathcal{S}\mathrm{yn}_{E,F})^{\mathrm{fp}}) \simeq \tau^{-1}\mathcal{S}\mathrm{yn}_{E,F}$$

induced by the Yoneda embedding.

Proof. □

Definition 3.7. We define the functor $\nu_E : \mathcal{S}\mathrm{yn}_F \to \mathcal{S}\mathrm{yn}_{E,F}$ to be the composite

$$\mathcal{S}\mathrm{yn}_F \simeq \tau^{-1}\mathcal{S}\mathrm{yn}_{E,F} \hookrightarrow \mathcal{S}\mathrm{yn}_{E,F} \xrightarrow{\tau^E_{\geqslant 0}} \mathcal{S}\mathrm{yn}_{E,F}$$

and we will refer to this ν_E is as the E-synthetic analog (of an F-synthetic spectrum).

Lemma 3.8. The functor ν_E above is lax monoidal and preserves finite direct sums and filtered colimits.

Proof. The functor is defined as a composite of functors which have the desired properties. □