1. τ- AND λ-BOCKSTEIN SPECTRAL SEQUENCES

In this section, we prove results about the τ- and λ-Bockstein spectral sequences in $\mathcal{B}\mathrm{isyn}$. Let $\nu^2 : \mathcal{S}\mathrm{p} \to \mathcal{B}\mathrm{isyn}$ denote the composition of the functors $\mathcal{S}\mathrm{p} \xrightarrow{\nu_E} \mathcal{S}\mathrm{yn}_E \xrightarrow{\nu_F} \mathcal{B}\mathrm{isyn}$.

Remark 1.1. For this theorem, I'm using these assumptions:

- $\nu_F(\mathbb{S}_E^{k,w}) = \mathbb{S}^{k,w,k}$
- SES in $\nu_E F_{*,*}$-homology induces cofiber sequence in ν^2
- ν^2 is symmetric monoidal on projective objects
- homotopy of $\nu^2(F)$-module is λ-free
- λ^{-1} is symmetric monoidal and satisfies $\lambda^{-1} \circ \nu_F = \mathrm{id}$
- inclusion or equivalence $\mathrm{Mod}(\mathcal{B}\mathrm{isyn}; C\lambda) \simeq \mathcal{S}\mathrm{table}(\nu_E F_{*,*}\nu_E F)$

Theorem 1.2. For the bisynthetic spectrum $\nu_F(Y)$ with $Y \in \mathcal{S}\mathrm{yn}_E$, the $\nu^2(F)$-Adams spectral sequence is isomorphic to the λ-Bockstein spectral sequence.

Proof. Our proof of this follows very closely to the proofs in [BHS19, App. A]. Consider the canonical $\nu_E(F)$-Adams tower

$$\begin{tikzcd} \cdots \arrow[r] & Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Y_0 \arrow[d] & = & Y \\ & Y_2 \otimes \nu_E(F) & Y_1 \otimes \nu_E(F) & Y_0 \otimes \nu_E(F) & & \end{tikzcd}$$

of $Y \in \mathcal{S}\mathrm{yn}_E$. This tower is made up of cofiber sequences

$$Y_{n+1} \to Y_n \to Y_n \otimes \nu_E(F) \to \Sigma^{1,0} Y_{n+1}$$

such that the maps $Y_{n+1} \to Y_n$ induce the zero map on $\nu_E(F)_{*,*}$-homology. By SES prop. of ν_F (!!!need this fact!!!), these become cofiber sequences

$$\Sigma^{-1}\nu_F(Y_{n+1}) \to \nu_F(Y_n) \to \nu_F(Y_n \otimes \nu_E(F)) \to \nu_F(\Sigma^{1,0}Y_{n+1}).$$

in $\mathcal{B}\mathrm{isyn}$. By symmetric monoidality of ν_F on projective objects (!!!need this fact!!!), we can identify these cofiber sequences as

$$\Sigma^{0,0,1}\nu_F(Y_{n+1}) \to \nu_F(Y_n) \to \nu_F(Y_n) \otimes \nu^2(F) \to \Sigma^{1,0,1}\nu_F(Y_{n+1}).$$

Hence, the canonical $\nu^2(F)$-Adams tower for $\nu_F(Y)$ can be written as

$$\begin{tikzcd} \cdots \arrow[r] & \Sigma^{0,0,2}\nu_F(Y_2) \arrow[r] \arrow[d] & \Sigma^{0,0,1}\nu_F(Y_1) \arrow[r] \arrow[d] & \nu_F(Y_0) \arrow[d] & = & \nu_F(Y) \\ & \Sigma^{0,0,2}\nu_F(Y_2) \otimes \nu^2(F) & \Sigma^{0,0,1}\nu_F(Y_1) \otimes \nu^2(F) & \nu_F(Y_0) \otimes \nu^2(F) & & \end{tikzcd}$$

In the usual way, we get the Adams E_1-page

$$\begin{aligned} {}_{\nu^2 F}E_1^{f,k,w,v} &= \pi_{k,w,v}(\Sigma^{0,0,f}\nu_F(Y_f) \otimes \nu^2(F)) \\ &\cong \pi_{k,w,v-f}(\nu_F(Y_f) \otimes \nu^2(F)) \end{aligned}$$

Because of the equivalence

$$\nu_F(Y_n \otimes \nu_E(F)) \simeq \nu_F(Y_n) \otimes \nu^2(F),$$

and the fact that the homotopy of a $\nu^2(F)$-module is λ-free (!!!need this fact!!!), we can identify the E_1-page as

$${}_{\nu^2 F}E_1^{f,k,w,v} \cong {}_{\nu_E F}E_1^{f,k,w} \otimes \mathbb{Z}[\lambda],$$

where $x \in {}_{\nu_E F}E_1^{f,k,w}$ lives in quad-degree $(f, k, w, k+f)$ (I'm pretty sure, but this is just a guess!!! double check when we know about homotopy of $\nu^2 F$-modules better).

The differentials in this spectral sequence are of the form

$$d_r : {}_{\nu^2 F}E_r^{f,k,w,v} \to {}_{\nu^2 F}E_r^{f+r,k-1,w,v}.$$

Since $\lambda^{-1} \circ \nu_F = \mathrm{id}$ and λ is symmetric monoidal, the λ-inverted $\nu^2(F)$-Adams tower is the $\nu_E(F)$-Adams tower, and, in particular, the λ-inverted differentials

$$d_{r,\lambda^{-1}} : {}_{\nu_E F}E_r^{f,k,w} \to {}_{\nu_E F}E_r^{f+r,k-1,w}$$

are exactly the differentials of the $\nu_E(F)$-Adams spectral sequence. Just as in [BHS19], this implies that if $d_{r,\lambda^{-1}}(x) = y$ is a $\nu_E(F)$-Adams differential, then

$$d_r(x) = \lambda^{r-1}y.$$

Using the equivalence (inclusion?)

$$\mathrm{Mod}(\mathcal{B}\mathrm{isyn}; C\lambda) \simeq \mathcal{S}\mathrm{table}(\nu_E F_{*,*}\nu_E F),$$

we see that the λ-Bockstein E_1-page

$${}_\lambda E_1^{f,k,w,v} = \pi_{k,w,v}(\Sigma^{0,0,-f}\nu_F(Y) \otimes C\lambda)$$

and Adams E_2-page are, up to a degree shift, isomorphic. Because they have the same formula for differentials, they must be isomorphic as spectral sequences. □

Remark 1.3. We might actually need to prove a little bit more. [BHS19] cites their Thm. 9.19(1) when they prove their version of this, but I'm not sure how exactly they're using that. Probably need to assume everything is nilpotent/lambda complete too, but don't feel like thinking about this for now. They also prove that the *filtrations* are literally the same, not just that the spectral sequences are isomorphic.

Remark 1.4. For the next lemmas and theorem, I need these assumptions:

- $\nu_F(\mathbb{S}_E^{k,w}) = \mathbb{S}^{k,w,k}$
- E Adams-type and ring map $E \to F$
- $\nu_E P$, with P finite E_*-projective, is "finite" in $\mathcal{S}\mathrm{yn}_E$
- ν^2 is symmetric monoidal on projective objects
- SES in $\nu_E F_{*,*}$-homology induces cofiber sequence in ν^2
- Homotopy of $\nu^2 E$-module is τ-free
- Commuting diagram of functors

$$\begin{tikzcd} \mathcal{B}\mathrm{isyn} \arrow[r, "\tau^{-1}"] & \mathcal{S}\mathrm{yn}_F \\ \mathcal{S}\mathrm{yn}_E \arrow[u, "\nu_F"] \arrow[r, "\tau^{-1}"'] & \mathcal{S}\mathrm{p} \arrow[u, "\nu_F"'] \end{tikzcd}$$

Lemma 1.5. If a spectrum $X \in \mathcal{S}\mathrm{p}$ is E_*-projective, then it is F_*-projective. In particular for any spectrum $Y \in \mathcal{S}\mathrm{p}$,

$$\nu_F(Y \otimes E) \simeq \nu_F Y \otimes \nu_F E.$$

Proof. By the assumption that E is Adams-type and F is an E-module, by a Künneth spectral sequence argument we have a natural isomorphism of functors

$$F_*(-) \xrightarrow{\cong} E_*(-) \otimes_{E_*} F_*.$$

In particular, if E_*X is projective over E_* then $F_*X \cong E_*X \otimes_{E_*} F_*$ is projective over F_*.

The second statement follows from the fact that E is a filtered colimit of finite, F_*-projective spectra and [**Pst22**]. □

Lemma 1.6. The synthetic spectrum $\nu_E E \in \mathcal{S}\mathrm{yn}_E$ is a filtered colimit of finite, $\nu_E F$-projectives. In particular, for $Y \in \mathcal{S}\mathrm{yn}_E$,

$$\nu_F(Y \otimes \nu_E E) \simeq \nu_F(Y) \otimes \nu^2(E).$$

Proof. Let $E \simeq \mathrm{colim}_\alpha E_\alpha$ where each E_α is a finite, E_*-projective spectrum. Since $\nu_E(E) \simeq \mathrm{colim}_\alpha \nu_E E_\alpha$, it suffices to show that each $\nu_E E_\alpha$ is a finite, $\nu_E F$-projective synthetic spectrum. Now, $\nu_E E_\alpha$ is automatically finite (!!!make sure we define what this means!!!). For projectivity, we see that

$$\nu_E E_\alpha \otimes \nu_E F \simeq \nu_E(E_\alpha \otimes F),$$

with homotopy groups $\pi_{*,*}(\nu_E E_\alpha \otimes \nu_E F) \cong F_* E_\alpha[\tau]$. By Lemma **??**, $F_* E_\alpha$ is projective over F_*. Tensoring with $\mathbb{Z}[\tau]$ preserves projectivity, so that $F_* E_\alpha[\tau]$ is projective over $\nu_E F_{*,*} \cong F_*[\tau]$. □

Lemma 1.7. Suppose

$$\nu_E X \to \nu_E Y \to \nu_E Z$$

is a cofiber sequence in $\mathcal{S}\mathrm{yn}_E$, induced by a cofiber sequence $X \to Y \to Z$ in $\mathcal{S}\mathrm{p}$. If the cofiber sequence in $\mathcal{S}\mathrm{yn}_E$ induces a short exact sequence in $\nu_E E_{*,*}$-homology, then it induces a short exact sequence in $\nu_E F_{*,*}$-homology.

Proof. By the assumption that E is Adams-type and F is an E-module, there are natural isomorphisms of functors

$$\begin{aligned}\nu_E F_{*,*}(\nu_E(-)) &\cong F_*(-)[\tau]\\ &\cong (E_*(-)\otimes_{E_*} F_*)[\tau]\\ &\cong \nu_E E_{*,*}(\nu_E(-))\otimes_{\nu_E E_{*,*}} \nu_E F_{*,*}\,.\end{aligned}$$

Consider a map $Z \to \Sigma X$ induced by the cofiber sequence in $\mathcal{S}\mathrm{p}$. Note that $\nu_E(\Sigma X) \simeq \Sigma^{1,1}\nu_E X$ and the map $\nu_E Z \to \Sigma^{1,0}\nu_E X$ sits in a commutative diagram

$$\begin{tikzcd} \nu_E Z \arrow[r, "f"] \arrow[dr, "g"'] & \Sigma^{1,0}\nu_E X \arrow[d, "\tau"] & \\ & \nu_E(\Sigma X) & \simeq \quad \Sigma^{1,1}\nu_E X \end{tikzcd}$$

with the horizontal and diagonal maps inducing the zero map in $\nu_E E_{*,*}$-homology. Since everything is τ-free when we apply $\nu_E F_{*,*}(-)$ and the map $\nu_E F_{*,*}(g)$ is zero by the natural isomorphism above, then $\nu_E F_{*,*}(\tau)$ is injective so that $\nu_E F_{*,*}(f)$ is also the zero map. $\square$

Theorem 1.8. For the bisynthetic spectrum $\nu^2(X)$ with $X \in \mathcal{S}\mathrm{p}$, the $\nu^2(E)$-Adams spectral sequence is isomorphic to the τ-Bockstein spectral sequence.

Proof. The proof starts off similarly to the proof for the $\nu^2(F)$-Adams SS and the λ-Bockstein SS. Consider the canonical E-Adams tower

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[d] & = & X \\ & X_2\otimes E & X_1\otimes E & X_0\otimes E & & \end{tikzcd}$$

of $X \in \mathcal{S}\mathrm{p}$. This tower is made up of cofiber sequences

$$X_{n+1} \to X_n \to X_n\otimes E \to \Sigma X_{n+1}$$

such that the maps $X_{n+1}\to X_n$ induce the zero map on E_*-homology. Applying ν_E to these cofiber sequences, as in [BHS19] we get cofiber sequences

$$\Sigma^{0,1}\nu_E(X_{n+1}) \to \nu_E(X_n) \to \nu_E(X_n)\otimes\nu_E(E) \to \Sigma^{1,1}\nu_E(X_{n+1}).$$

Since $F_*(-)\cong E_*(-)\otimes_{E_*} F_*$ and the maps $\Sigma^{0,1}\nu_E(X_{n+1}) \to \nu_E(X_n)$ induce the zero map on $\nu_E(E)_{*,*}$-homology, the maps $\Sigma^{0,1}\nu_E(X_{n+1}) \to \nu_E(X_n)$ also induce zero on $\nu_E(F)_{*,*}$-homology. This, together with Lemma 1.5, implies that there are cofiber sequences

$$\Sigma^{0,1,1}\nu^2(X_{n+1}) \to \nu^2(X_n) \to \nu^2(X_n)\otimes\nu^2(E) \to \Sigma^{1,1,1}\nu^2(X_{n+1}).$$

Regrading by letting $Z_n = \Sigma^{0,0,n}\nu^2(X_n)$, these become cofiber sequences

$$\Sigma^{0,1,0}Z_{n+1} \to Z_n \to Z_n\otimes\nu^2(E) \to \Sigma^{1,1,0}Z_{n+1}$$

which build up the canonical $\nu^2(E)$-Adams tower

$$\begin{tikzcd} \cdots \arrow[r] & \Sigma^{0,2,0}Z_2 \arrow[r] \arrow[d] & \Sigma^{0,1,0}Z_1 \arrow[r] \arrow[d] & Z_0 \arrow[d] & = & \nu^2(X) \\ & \Sigma^{0,2,0}Z_2\otimes\nu^2(E) & \Sigma^{0,1,0}Z_1\otimes\nu^2(E) & Z_0\otimes\nu^2(E) & & \end{tikzcd}$$

for $\nu^2(X) \in \mathcal{B}\mathrm{isyn}$. The $\nu^2(E)$-Adams E_1-page has the form

$$\begin{aligned}{}_{\nu^2 E}E_1^{f,k,w,v} &= \pi_{k,w,v}(\Sigma^{0,f,0}Z_f\otimes\nu^2(E))\\ &\cong \pi_{k,w-f,v}(Z_f\otimes\nu^2(E)).\end{aligned}$$

Since $Z_f\otimes\nu^2(E)$ is a $\nu^2(E)$-module, the E_1-page is τ-free and via the τ-inversion functor τ^{-1}, we have an isomorphism

$${}_{\nu^2 E}E_1^{f,k,w,v} \cong {}_{\nu_F E}E_1^{f,k,v}\otimes\mathbb{Z}[\tau]$$

$\square$

where $x \in {}_{\nu_E F}E_1^{f,k,v}$ lives in quad-degree $(f, k, k+f, v)$. The differentials in this spectral sequence are of the form
$$d_r : {}_{\nu^2 E}E_r^{f,k,w,v} \to {}_{\nu^2 E}E_r^{f+r,k-1,w,v}.$$

By Lemma **??** and the commutative diagram of functors

$$\begin{tikzcd} \mathcal{B}\mathrm{isyn} \arrow[r, "\tau^{-1}"] & \mathcal{S}\mathrm{yn}_F \\ \mathcal{S}\mathrm{yn}_E \arrow[u, "\nu_F"] \arrow[r, "\tau^{-1}"'] & \mathcal{S}\mathrm{p} \arrow[u, "\nu_F"'] \end{tikzcd}$$

the symmetric monoidal τ-inversion functor $\tau^{-1} : \mathcal{B}\mathrm{isyn} \to \mathcal{S}\mathrm{yn}_F$ sends the $\nu^2(E)$-Adams tower to a $\nu_F E$-Adams tower in $\mathcal{S}\mathrm{yn}_F$. In particular, the τ-localized differentials $d_{r,\tau^{-1}}$ are exactly the $\nu_F E$-Adams spectral sequence differentials and, so, if $d_{r,\tau^{-1}}(x) = y$, then
$$d_r(x) = \tau^{r-1} y.$$

Using the equivalence (inclusion?)
$$\mathrm{Mod}(\mathcal{B}\mathrm{isyn}; C\tau) \simeq \mathcal{S}\mathrm{table}(\nu_F E_{*,*}\nu_F E),$$

we see that the τ-Bockstein E_1-page
$${}_{\tau}E_1^{f,k,w,v} = \pi_{k,w,v}(\Sigma^{0,-f,0}\nu^2(X) \otimes C\tau)$$

and Adams E_2-page are, up to a degree shift, isomorphic. Because they have the same formula for differentials, they must be isomorphic as spectral sequences.

References

[BHS19] Robert Burklund, Jeremy Hahn, and Andrew Senger. *On the boundaries of highly connected, almost closed manifolds*. 2019. arXiv: 1910.14116.