BISYNTHETIC SPECTRA

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1. Generalizing Synthetic Spectra

Throughout this section we fix a presentably symmetric monoidal stable ∞-category $\mathcal{C}$. Our goal in this section is to generalize the original construction of synthetic spectra due to [Pst22] for a broader class of homology theories on stable ∞-categories.

1.1. Homological Contexts.

In order to set up a theory of synthetic spectra, we must first specify what the necessary data is. The original theory takes as input a (nice) ring spectrum E, but makes substantial use of the underlying structure present in $\mathcal{S}\mathrm{p}$ and the concomitant properties of the induced homology theory E_*. We spell out below what we believe is a suitably general theory of homological contexts, i.e., theories which act sufficiently like functors $X \mapsto \pi_*(X \otimes E)$.

Definition 1.1. A local grading on a category $\mathcal{D}$ is an auto-equivalence $-[1] : \mathcal{D} \to \mathcal{D}$. A category is said to be locally graded if it has a chosen local grading.

Example 1.2. All stable ∞-categories are locally graded by by the formal suspension $-[1] := \Sigma$.

Example 1.3. A graded category $\mathrm{Fun}(\mathbb{Z}, \mathcal{D})$ where $\mathbb{Z}$ is the discrete category on the integers is locally graded by the shift functor induced by $n \mapsto n \pm 1$ on $\mathbb{Z}$.

Both examples above frequently arise as special cases of the following:

Example 1.4 (Pic Grading)**.** If $\mathcal{D}$ is monoidal and X is a Picard-object, i.e., it is $\otimes$-invertible, then the functor $- \otimes X$ forms a local grading on $\mathcal{D}$.

Definition 1.5. Let $\mathcal{D}$ be a presentably symmetric monoidal stable ∞-category and let $\mathcal{A}$ be an abelian 1-category equipped with a local grading. A functor $\pi_\star : \mathcal{D} \to \mathcal{A}$ is said to be a homotopy groups functor if it is conservative, lax monoidal, and additive, and if in addition it

(1) sends cofibers in $\mathcal{D}$ to exact sequences in $\mathcal{A}$,
(2) and intertwines the local gradings $\mathrm{H}(\Sigma X) = H(X)[1]$ naturally.

I think this definition is in the Irakli-Piotr paper as a homology theory (Def. 2.8). We could just quote them on it

Remark 1.6. An important consequence of the above definition is that any such H will send a cofiber sequence $X \to Y \to Z$ in $\mathcal{D}$ to a long exact sequence in $\mathcal{A}$ as rotating the cofiber in $\mathcal{D}$ results in a local-grading-shift in $\mathcal{A}$.

Remark 1.7. Note that if $\pi_\star : \mathcal{D} \to \mathcal{A}$ satisfies all of the above except that it fails to be conservative, we may pass to the localization $\mathcal{D}^{\mathrm{cell}}$ of $\mathcal{D}$ which, among other descriptions, can be taken to be the cofiber in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ of the inclusion of the objects in $\mathcal{D}$ which are $\pi_\star$-isomorphic to $0 \in \mathcal{A}$.

Example 1.8. The above definition is engineered not just to capture the classical examples of homotopy groups of spectra, but also the categories of (genuine) equivariant spectra, cellular motivic spectra, and cellular synthetic spectra. In general, we will want the extra flexibility of considering homotopy groups (and later homology theories) which are multigraded and have long exact sequences with respect to the formal suspension.

Definition 1.9. A *homological context* is the data of two presentably symmetric monoidal stable ∞-categories $\mathcal{C}, \mathcal{D}$, a symmetric monoidal left adjoint $\mathrm{H} : \mathcal{C} \to \mathcal{D}$, and a homotopy groups functor $\pi_\star : \mathcal{C} \to \mathcal{A}$. If G is the right adjoint to H we write $\mathrm{H}_\star$ for the composite $\pi_\star \circ \mathrm{G} \circ \mathrm{H}$.

Example 1.10. All examples we study in this paper will be given by the following data. First we fix $\mathcal{C}$ as above and assume $\mathcal{C}$ has a homotopy groups functor $\pi_\star$. We then let $R \in \mathrm{CAlg}(\mathcal{C})$ and put $\mathrm{H} := - \otimes R$ valued in $\mathrm{Mod}(R)$ where it becomes symmetric monoidal (and preserves colimits by assumption). Then $\mathrm{H}_\star$ corresponds to taking the homotopy groups on the underlying $\mathcal{C}$-objects of $- \otimes R$.

1.2. **The Adams Spectral Sequence of a Homological Context.** Here we elucidate some of our assumptions on a homological context $(\mathcal{C}, \mathcal{D}, \mathrm{H}, \pi_\star)$ by explaining how to construct the relevant Adams spectral sequence which will have the form:

$$\mathrm{Ext}_{\mathcal{A}}(\mathrm{H}_\star X, H_\star Y) \Rightarrow \pi_\star \mathrm{Map}_{\mathcal{C}}$$

To construct this spectral sequence, we will use the Adams spectral sequence associated to an adjunction due to Krause [**Krause**]. Namely, let G denote the right adjoint to the functor H. Then the composition HG gives rise to an exact comonad on $\mathcal{D}$ and we may consider the category $\mathcal{C}\mathrm{omod}(\mathrm{HG})$ of comodules[1] over this comonad. Note that any object in the image of H automatically acquires the structure of an HG-comodule and thus we may factor the functor

$$\mathrm{H} : \mathcal{C} \to \mathcal{C}\mathrm{omod}(\mathrm{HG}) \xrightarrow{\mathrm{Forget}} \mathcal{D}$$

We will write $\hat{\mathrm{H}}$ for the first functor above.

Definition 1.11 ([**Definition 2.24**][Krause)**.**] An object $X \in \mathcal{C}$ is HG-complete if the induced map

$$\mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}\mathrm{omod}(\mathrm{HG})}(\hat{H}(Z), \hat{H}(X))$$

is an equivalence for all Z.

The full subcategory of HG-complete objects in $\mathcal{C}$ is a localization of $\mathcal{C}$, we refer to the completion functor as HG-completion [**Krause**]. The comonad HG allows us to resolve (the HG completions of) objects of $\mathcal{C}$ via the cobar resolution:

Definition 1.12. The cobar complex of HG is the functor $\mathrm{cb}_{\mathrm{HG}} : \mathcal{C} \to \mathcal{C}^{\Delta^{\mathrm{op}}}$ which sends an object X to the cosimplicial object $(\mathrm{GH})^{\circ n+1}$. This produces a cosimplicial object in $\mathcal{A}$ after applying $\pi_\star$ levelwise which we denote $\mathrm{cb}_{\mathrm{H}_\star}$.

Remark 1.13. The comodule structure above should be viewed as a generalization of remembering that the E-homology of a spectrum X comes equipped with the structure of a comodule over the E-cooperations.

Definition 1.14. The H-Adams Spectral sequence for the mapping object from Z to X is the Bousfield-Kan spectral sequence obtained by applying the functor $\pi_\star \mathrm{Map}_{\mathcal{C}}^{\mathcal{C}}(Z, -)$ to $\mathrm{cb}_{\mathrm{HG}}(X)$. It converges conditionally to $\pi_\star \mathrm{Map}_{\mathcal{C}}^{\mathcal{C}}(Z, X^{\wedge}_{\mathrm{HG}})$.

Note that in the above discussion we made substantial use of the fact that we had both the left adjoint $\mathcal{C} \to \mathcal{D}$ as well as the functor $\pi_\star : \mathcal{C} \to \mathcal{A}$. Our general theory of "homology theories" differs from much of the literature in that we ask for both of these data, however, we believe that most examples of interest in nature arise in this fashion anyway.

1.3. H**-finite sites.** In this section we fix a homological context (Definition 1.9)

$$\mathcal{C} \xrightarrow{\mathrm{H}} \mathcal{D} \xrightarrow{\pi_\star} \mathcal{A}$$

whose composition is denoted $H_\star$. To such a context we will assocaited a site $\mathcal{C}^\omega_{\mathrm{H}}$ which will encode the relevant properties of the H-Adams spectral sequence.

Definition 1.15. With notation as above, the H-finite site of $\mathcal{C}$, denoted $\mathcal{C}^\omega_{\mathrm{H}}$, is the full subcategory of $\mathcal{C}^\omega$ consisting of objects X such that $\mathrm{H}(X)$ is dualizable in $\mathcal{A}$. The coverings in $\mathcal{C}^\omega_{\mathrm{H}}$ are the single maps $f : X \to Y$ such that $\mathrm{H}(f)$ is an epimorphism, which we call H-epimorphisms for short.

Remark 1.16. Note that in a symmetric monoidal abelian category $\mathcal{A}$, the condition of an object P being dualizable is equivalent to be finitely generated, in the sense that there is an epimorphism $1_{\mathcal{A}}^{\oplus n} \to P$, and projective.

Lemma 1.17. Let $Q, R, P \in \mathcal{C}^\omega_{\mathrm{H}}$. Suppose $f : Q \to P$ is an H-epimorphism and $g : R \to P$ is arbitrary. Let $X = Q \times_P R$ denote the pullback in $\mathcal{C}$. Then X is again in $\mathcal{C}^\omega_{\mathrm{H}}$ and $X \to R$ is an H-epimorphism.

Proof. First we note that the pullback may equivalently be described via the fiber sequence

$$X \to Q \oplus R \xrightarrow{f-g} P$$

and as a result is compact as the fiber of a map between compact objects. Moreover, because $\mathrm{H}(Q) \to \mathrm{H}(P)$ is an H-epimorphism and because $\mathrm{H}(Q)$ is projective, we get a lift $\tilde{g} : H(R) \to H(P)$ which then splits the long exact sequence. As a result, the long exact sequence breaks up into short exact sequences, and we use the 2-out-of-3 property to claim that $\mathrm{H}(X)$ is therefore dualizable. That $\mathrm{H}(X) \to \mathrm{H}(R)$ is an epimorphism follows from the additional splitting. ☐

[1]Sometimes refered to as HG-coalgebras, but we prefer the terminology of [**Krause**] since such an object is essentially an object of $\mathcal{D}$ with a coaction of HG.

Lemma 1.18. The tensor product on $\mathcal{C}$ restricts to $\mathcal{C}^\omega_{\mathrm{H}}$ and $\mathrm{H}_\star : \mathcal{C}^\omega_{\mathrm{H}} \to \mathcal{A}$ is monoidal.

Proof. The tensor product automatically restricts to $\mathcal{C}^\omega$, so it is only necessary to show that if $X, Y \in \mathcal{C}^\omega_{\mathrm{H}}$ then $\mathrm{H}_\star(X \otimes Y)$ is dualizable, which itself would follow from the monoidality claim on $\mathrm{H}_\star$. But then even if only one of X or Y were projective, the lax-monoidality map

$$\mathrm{H}_\star(X \otimes Y) \to H_\star(X) \otimes H_\star(Y)$$

is an isomorphism via Kunneth spectral sequence:

$$\mathrm{Tor}_{\mathcal{A}}(\mathrm{H}(X), \mathrm{H}(Y)) \Rightarrow H(X \otimes Y)$$

which is concentrated in the 0-line isomorphic to $H(X) \otimes H(Y)$ due to the projectivity of either factor. □

I definitely believe this Kunneth SS exists in this generality, but is there a source for that? If not, we might want to write down a proof.

Remark 1.19. Recall from [Pst22] that a site is *additive* if the coverings are provided by singletons and the underlying category is additive.

Definition 1.20 ([Pst22])**.** An additive site is site whose underlying category is additive and whose coverings are all single maps. Such a site is in addition excellent if it is equipped with a symmetric monoidal structure in which every object has a dual and such that the functors $- \otimes P$ preserve coverings for all P in the site.

Proposition 1.21. The category $\mathcal{C}^\omega_{\mathrm{H}}$ is an excellent site.

Proof. By definition the underlying category is additive and by Lemma 1.17 we know that it forms a site which whose coverings are single maps by definition. Finally by Lemma 1.18 we already know the tensor product restricts and all objects have duals. All that remains is to show that for all $P \in \mathcal{C}^\omega_{\mathrm{H}}$ the functor $- \otimes P$ preserves coverings. But this again follows from the proof of Lemma 1.18 as the tensor product of epimorphisms is again an epimorphism. □

1.4. H**-Synthetic Spectra.** Again we fix a homological context $\mathcal{C} \to \mathcal{D} \to \mathcal{A}$ with notation as in all previous sections. Recall that a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is said to be spherical if for all $X, Y \in \mathcal{C}$ the natural map

$$F(X \amalg Y) \to F(X) \times F(Y)$$

is an equivalence. A sheaf is said to be spherical if the underlying presheaf is, and the sheafification functor when it exists sends spherical sheaves to spherical presheaves. Spherical presheaves are very well behaved when the category $\mathcal{C}$ is additive. In particular, in this case we get canonical lifts to grouplike commutative monoids in $\mathcal{D}$, so long as these make sense. As a result, spherical sheaves of spaces on $\mathcal{C}$ lift canonically to spherical sheaves of connective spectra.

Definition 1.22. The category of of Synthetic Spectra with respect to the context above is the category of spherical sheaves of spectra $\mathrm{Sh}_\Sigma(\mathcal{C}^\omega_{\mathrm{H}}, \mathcal{S}\mathrm{p})$ on the H-finite site. We will often drop much of the context data and refer to this as the category $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ of H-synthetic spectra.

Proposition 1.23. The category $\mathcal{S}\mathrm{yn}_{\mathrm{H}}$ is presentably symmetric monoidal and comes equipped with a synthetic analog functor $\nu : \mathcal{C} \to \mathcal{S}\mathrm{yn}_{\mathrm{H}}$ defined by lifting the Yoneda embedding. The functor ν preserves filtered colimits and direct sums. It is in addition lax monoidal and for any $P \in \mathcal{C}^\omega_{\mathrm{H}}$ the natural map $\nu(- \otimes X) \to \nu(-) \otimes \nu(P)$ is an equivalence.

Overall, I think this general definition of "synthetic spectra" seems to hold up. I'm surprised that it works in this generality and that this wasn't written down already by the likes of Piotr/Irakli/Robert etc.

(couldn't fit this in a todo environment) I have a few more thoughts about all this and where you could go from here:

- Like I said in some of the earlier comments, I think it would be a good idea to simplify some of the definitions and make sure we reference people (e.g. Piotr/Irakli) who have already come up with some of these notions.
- Some other Piotr/Irakli stuff we could quote from here include thread structure/tau-localization/Ctau-mod
- With these notions all set up, it should be straightforward then to start writing down all the categories we need that come out of bisynthetic spectra; e.g. Ctau/Clambda-mod, tau/lambda inverted categories, etc. and the categories these module categories are equivalent to
- Question: for this def. of $\mathrm{Sh}_\Sigma(\mathcal{C}^\omega_{\mathrm{H}}, \mathcal{S}\mathrm{p})$, is it still true that Ctau-mod embeds in some "stable comodule" category related to H? Would have to make the stable comodule category precise. This is probably related to the derived categories that Irakli/Piotr deal with

1.5. Thread Structures and τ.

1.6. Relationship to the H-Adams Spectral sequence.

2. Bisynthetic Spectra

(Peter) Here's a list of things we'll need here

- Def of bisynthetic spectra, including stuff from [Pst22] about Σ^∞_+, Ω^∞
- Properties of monoidal structure
- Definition of $\nu_{\nu F}$ and trigraded spheres
- Definition of trigraded homotopy classes of maps
- Lemma that $\nu_E F_{*,*}(-) : (\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F} \to \mathcal{C}\mathrm{omod}^{fp}_{\nu_E F_{*,*}\nu_E F}$ is morphism of ∞-sites which reflects coverings and admits a common envelope
- Other lemmas about other maps of ∞-sites? Or wait to do that later
- Definition of Adams-type E-synthetic spectrum (prolly same as spectra one)
- Lemma that $\nu_E F_{*,*}\nu_E F$ is Adams Hopf algebroid a la [Pst22, Def. 3.1]
- Lemma that $map(\nu_{\nu F} P) \simeq \Omega^\infty(X(P))$
- Lemma about when $\nu_{\nu F}$ is symmetric monoidal
- Lemma about homotopy of $\nu_{\nu F}$-modules being lambda free

3. t-structures

In the category $\mathcal{S}\mathrm{yn}_E$ of E-synthetic spectra developed by [Pst22], there is a natural t-structure which plays an important role in the structure of the category. This t-structure is a specialization of a general t-structure on spherical sheaves, whose heart can also be identified:

Definition 3.1 ([Pst22])**.** Suppose $\mathcal{C}$ is an additive ∞-category and let $Sh^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{C})$ denote the category of spectra-valued spherical sheaves on $\mathcal{C}$. An object $X \in Sh^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{C})$ is *connective* if the sheafification of the presheaf $\pi_n X$ defined by

$$c \in \mathcal{C} \mapsto \pi_n X(c)$$

satisfies $\pi_n X = 0$ for $n < 0$. An object X is *coconnective* if $\Omega^\infty X$ is a discrete sheaf of spaces.

Proposition 3.2 ([Pst22])**.** The pair $(Sh^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{C})_{\geqslant 0}, Sh^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{C})_{\leqslant 0})$ of full subcategories of connective and coconnective objects determines a right complete t-structure on $Sh^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{C})$ compatible with filtered colimits. Moreover, there is a canonical equivalence $Sh^{\mathcal{S}\mathrm{p}}_\Sigma(\mathcal{C})^\heartsuit \simeq Sh^{\mathrm{Set}}_\Sigma(\mathcal{C})$ between the heart of this t-structure and the category of spherical sheaves of sets.

When specializing to $\mathcal{C} = \mathcal{S}\mathrm{p}^{fp}_E$ for an Adams-type spectrum E, [Pst22] shows that the functor of additive ∞-sites $E_*(-) : \mathcal{S}\mathrm{p}^{fp}_E \to \mathcal{C}\mathrm{omod}^{fp}_{E_*E}$ induces an equivalence on spherical sheaves of sets. Together with work of Goerss-Hopkins, this gives a nice identification of the heart $\mathcal{S}\mathrm{yn}^\heartsuit_E$ of the t-structure on $\mathcal{S}\mathrm{yn}_E$ in terms of E_*E-comodules:

Theorem 3.3 ([GH05],[Pst22])**.** If E is an Adams-type spectrum, then the functor of additive ∞-sites $E_*(-) : \mathcal{S}\mathrm{p}^{fp}_E \to \mathcal{C}\mathrm{omod}^{fp}_{E_*E}$ induces an equivalence $Sh^{\mathrm{Set}}_\Sigma(\mathcal{S}\mathrm{p}^{fp}_E) \simeq Sh^{\mathrm{Set}}_\Sigma(\mathcal{C}\mathrm{omod}^{fp}_{E_*E})$. In particular, there are equivalences

$$\mathcal{S}\mathrm{yn}^\heartsuit_E \simeq Sh^{\mathrm{Set}}_\Sigma(\mathcal{C}\mathrm{omod}^{fp}_{E_*E}) \simeq \mathcal{C}\mathrm{omod}_{E_*E}.$$

For $X \in \mathcal{S}\mathrm{yn}_E$, [Pst22] also identifies an explicit formula for the homotopy objects $\pi^\heartsuit_k X$ in terms of synthetic E-homology:

Theorem 3.4 ([Pst22])**.** At the level of graded abelian groups, there's an isomorphism

$$(\pi^\heartsuit_k X)_l \cong \nu E_{k+l,l} X$$

In particular, X is connective if and only if $\nu E_{k,w} X$ is concentrated in non-negative Chow degree $k - w \geqslant 0$.

If $X = \nu Y$ is the synthetic analog of a spectrum Y, the calculation

$$\nu E_{*,*} \nu Y \cong \nu(E \otimes Y)_{*,*} \cong E_* Y[\tau],$$

where $E_* Y$ is concentrated in bidegree (k, k), shows that νY is always connective in this t-structure. In particular, this implies that $\nu Y \otimes C\tau$ lies in $\mathcal{S}\mathrm{yn}^\heartsuit_E$. This fact is key in relating $\mathrm{Mod}_{C\tau}(\mathcal{S}\mathrm{yn}_E)$ to $\mathcal{S}\mathrm{table}_{E_*E}$ and the E-Adams spectral sequence for Y to the τ-Bockstein spectral sequence for νY in $\mathcal{S}\mathrm{yn}_E$.

In bisynthetic spectra $\mathcal{S}\mathrm{yn}_{E,F} := Sh_{\Sigma}^{\mathcal{S}\mathrm{p}}((\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F})$, there are two parameters τ and λ. One should expect then that there are two t-structures to study: namely, one related to E (i.e. τ) and one related to F (i.e. λ). This is indeed the case.

We first study the one related to F in 3.1. This t-structure comes about in the exact same way that the t-structure in $\mathcal{S}\mathrm{yn}_E$ appears. We also prove several results about this t-structure, analogous to results in [Pst22], which will be useful later for identifying $\mathcal{S}\mathrm{yn}_{E,F}[\lambda^{-1}]$ and $\mathrm{Mod}_{C\lambda}(\mathcal{S}\mathrm{yn}_{E,F})$ in terms of more familiar categories.

We then study a t-structure related to E in 3.2. The connective objects of this t-structure are controlled by $\nu^2(E)$-homology, analogous to Theorem 3.4. We also prove several results about this t-structure, analogous to results in [Pst22], which will be useful later for identifying $\mathcal{S}\mathrm{yn}_{E,F}[\tau^{-1}]$ and $\mathrm{Mod}_{C\tau}(\mathcal{S}\mathrm{yn}_{E,F})$ in terms of more familiar categories.

3.1. **t-structure for F.** Since $\mathcal{S}\mathrm{yn}_{E,F}$ is the category of spherical sheaves on an additive ∞-structure, we immediately get a t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$ via Definition 3.1 and Proposition 3.2:

Proposition 3.5. The pair $((\mathcal{S}\mathrm{yn}_{E,F})^F_{\geqslant 0}, (\mathcal{S}\mathrm{yn}_{E,F})^F_{\leqslant 0})$ of full subcategories of connective and coconnective objects determines a right complete t-structure on $\mathcal{S}\mathrm{yn}_{E,F}$ compatible with filtered colimits. Moreover, there is a canonical equivalence $\mathcal{S}\mathrm{yn}^{F,\heartsuit}_{E,F} \simeq Sh_{\Sigma}^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F})$ between the heart of this t-structure and the category of spherical sheaves of sets.

Remark 3.6. We use the superscript F to emphasize that this t-structure is related to F and λ. This will become clearer later in the subsection when we relate the t-structure to $\nu^2 F$-homology. We will also use the notation $\tau^F_{\geqslant n}, \tau^F_{\leqslant n}$ for the associated truncation functors.

We can identify the heart in a similar manner to [Pst22]:

Theorem 3.7. The heart $\mathcal{S}\mathrm{yn}^{F,\heartsuit}_{E,F}$ is equivalent to $\mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F}$. (monoidal conditions should be added too)

Proof. By Proposition 3.5, the heart is equivalent to $Sh_{\Sigma}^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F})$. By (ref. to lemma in Section 2), the morphism of ∞-sites

$$\nu_E F_{*,*}(-) : (\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F} \to \mathcal{C}\mathrm{omod}^{fp}_{\nu_E F_{*,*}\nu_E F}$$

is one which reflects coverings and admits a common envelope. By [Pst22, Rem. 2.50], this induces an adjoint equivalence

$$Sh_{\Sigma}^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F}) \rightleftarrows Sh_{\Sigma}^{\mathrm{Set}}(\mathcal{C}\mathrm{omod}^{fp}_{\nu_E F_{*,*}\nu_E F}).$$

The bigraded Hopf algebroid $(\nu_E F_{*,*}, \nu_E F_{*,*}\nu_E F)$ is Adams, in the sense of [Pst22, Def. 3.1], by (Lemma in Section 2 which proves that it's Adams). By a bigraded version of [GH05, p. 2.1.12], [Pst22, Thm. 3.2] there is an equivalence

$$\mathcal{C}\mathrm{omod}_{\nu_E F_{*,*}\nu_E F} \simeq Sh_{\Sigma}^{\mathrm{Set}}(\mathcal{C}\mathrm{omod}^{fp}_{\nu_E F_{*,*}\nu_E F}),$$

and the result follows. ∎

Now we work towards identifying the homotopy objects $\pi_k^{F,\heartsuit} X$ in terms of $\nu^2 F$-homology.

Lemma 3.8. For $X \in \mathcal{S}\mathrm{yn}_{E,F}$, the graded components of the $\nu_E F_{*,*}\nu_E F$-comodule $\pi_k^{F,\heartsuit} X$ are described by

$$(\pi_k^{F,\heartsuit} X)_{l,m} \cong \mathrm{colim}_\alpha\, \pi_k X(\Sigma^{l,m} D\nu_E F_\alpha),$$

where $F \simeq \mathrm{colim}_\alpha F_\alpha$ is a presentation of F as a filtered colimit of F-finite projective spectra.

Proof. This is essentially a bigraded version of [Pst22, Lemma 4.17] and the proof is similar to the proof of that lemma. By [Pst22, Thm. 2.58], the sheaf $\pi_k^{F,\heartsuit} X \in Sh_{\Sigma}^{\mathrm{Set}}((\mathcal{S}\mathrm{yn}_E)^{fp}_{\nu F})$ is representable by some comodule N; i.e.

$$(\pi_k^{F,\heartsuit} X)(-) \simeq \mathrm{Hom}_{\nu_E F_{*,*}\nu_E F}(\nu_E F_{*,*}(-), N).$$

Now notice that $\nu_E F_{*,*}\nu_E F \simeq \mathrm{colim}_\alpha\, \nu_E F_{*,*}\nu_E F_\alpha$, since ν_E commutes with filtered colimits, and $E_*(D\nu_E F_\alpha) \cong \mathrm{Hom}_{\nu_E F_{*,*}\nu_E F}(\nu_E F_{*,*}\nu_E F_\alpha, \nu_E F_{*,*})$. Then by [Pst22, Lemma 3.3], as a bigraded abelian group

$$N_{l,m} \cong \mathrm{colim}_\alpha\, \pi_k^{F,\heartsuit} X(\Sigma^{l,m} D\nu_E F_\alpha).$$

By a bigraded version of [Pst22, Lemma 3.25],

$$\mathrm{colim}_\alpha\, \pi_k^{F,\heartsuit} X(\Sigma^{l,m} D\nu_E F_\alpha) \cong \mathrm{colim}_\alpha\, \pi_k X(\Sigma^{l,m} D\nu_E F_\alpha),$$

which completes the proof. ∎

Theorem 3.9. For $X \in \mathcal{S}\mathrm{yn}_{E,F}$, there is an isomorphism

$$(\pi_k^{F,\heartsuit} X)_{l,m} \cong \nu^2 F_{k+l,m,l} X,$$

where $\nu^2 F_{*,*,*}(-)$ denotes bisynthetic F-homology.

Proof. Again, this is a similar proof to [Pst22, Thm. 4.18]. We have that

$$\begin{aligned}
\nu^2 F_{k+l,m,l} X &\cong [\mathbb{S}^{k+l,m,l}, \nu^2 F \otimes X] \\
&\cong \operatorname{colim}_\alpha [\Sigma^k \nu_{\nu F}(\mathbb{S}_E^{l,m}), \nu^2 F_\alpha \otimes X] \\
&\cong \operatorname{colim}_\alpha [\Sigma^k \nu_{\nu F}(\Sigma^{l,m} D\nu_E F_\alpha), X] \\
&\cong \operatorname{colim}_\alpha \pi_k X(\Sigma^{l,m} D\nu_E F_\alpha) \\
&\cong (\pi_k^{F,\heartsuit} X)_{l,m}.
\end{aligned}$$

The first isomorphism is by definition, the second isomorphism follows from (definition from Section 2 about trigraded spheres) and equivalence $\nu^2 F \simeq \operatorname{colim}_\alpha \nu^2 F_\alpha$, the fourth isomorphism follows from (lemma from Section 2 which shows that $map(\nu_{\nu F} P, X) \simeq \Omega^\infty(X(P))$ for $P \in (\mathcal{S}\mathrm{yn}_E)_{\nu F}^{fp}$), and the fifth isomorphism follows from Lemma 3.8. ☐

As a corollary, we get the following analog of [Pst22, Cor. 4.19]:

Corollary 3.10. A bisynthetic spectrum $X \in \mathcal{S}\mathrm{yn}_{E,F}$ is in $(\mathcal{S}\mathrm{yn}_{E,F})_{\geqslant 0}^F$ if and only if $\nu^2 F_{k,w,v} X = 0$ for Chow degree $k - v < 0$.

Proof. In this t-structure, $X \in \mathcal{S}\mathrm{yn}_{E,F}$ is in $(\mathcal{S}\mathrm{yn}_{E,F})_{\geqslant 0}^F$ if and only if $\pi_k^{F,\heartsuit} X$ vanishes for $k < 0$. By Theorem 3.9, this happens exactly when $k - v < 0$. ☐

This result is what motivates naming this t-structure after F. As a consequence, we see that the νF-synthetic analog of an E-synthetic spectrum Y is always connective.

Corollary 3.11. If $Y \in \mathcal{S}\mathrm{yn}_E$, then $\nu_{\nu F} Y \in (\mathcal{S}\mathrm{yn}_{E,F})_{\geqslant 0}^F$.

Proof. Consider the homology calculation

$$\begin{aligned}
\nu^2 F_{*,*,*} \nu_{\nu F} Y &\cong \nu_{\nu F}(\nu F \otimes Y)_{*,*,*} \\
&\cong \nu F_{*,*} Y[\lambda],
\end{aligned}$$

where $\nu F_{k,w} Y$ lives in tridegree (k, w, k). The first isomorphism follows from (lemma in Section 2 about when $\nu_{\nu F}$ is symmetric monoidal) and the second isomorphism follows (lemma in Section 2 about homotopy of νF-module). The result then follows from Corollary 3.10. ☐

This means that for the νF-synthetic analog of an E-synthetic spectrum Y, the tensor product $\nu_{\nu F} Y \otimes C\lambda$ lives in the heart $\mathcal{S}\mathrm{yn}_{E,F}^{F,\heartsuit}$.

Corollary 3.12. If $Y \in \mathcal{S}\mathrm{yn}_E$, then $\Sigma^{0,0,-1} \nu_{\nu F} Y \simeq \tau_{\geqslant 1}^F(\nu_{\nu F} Y)$ and $\nu_{\nu F} Y \otimes C\lambda \simeq \tau_{\leqslant 0}^F(\nu_{\nu F} Y)$. In particular, $\nu_{\nu F} Y \otimes C\lambda \in \mathcal{S}\mathrm{yn}_{E,F}^{F,\heartsuit}$.

Proof. Again, the proof is similar to the proof of [Pst22, Lemma 4.29]. Consider the cofiber sequence

$$\Sigma^{0,0,-1} \nu_{\nu F} Y \xrightarrow{\lambda} \nu_{\nu F} Y \to \nu_{\nu F} Y \otimes C\lambda$$

By Corollary 3.10, it's clear that $\Sigma^{0,0,-1} \nu_{\nu F} Y$ is 1-connective. By using the definition of $\nu_{\nu F}$ and the colimit-comparison definition of λ, it follows that $\nu_{\nu F} Y \otimes C\lambda$ lives in $(\mathcal{S}\mathrm{yn}_{E,F})_{\leqslant 0}^F$. The result then follows. ☐

Remark 3.13. Similar to $\mathcal{S}\mathrm{yn}_E$, we see that $\nu_{\nu F} Y \otimes C\lambda$ is lives in an algebraic category; namely the category of $\nu_E F_{*,*} \nu_E F$-comodules. In Section 4, we will show that, in fact, $\nu_{\nu F} Y \otimes C\lambda$ can be identified with the comodule $\nu_E F_{*,*} Y$ and there is an embedding $\mathrm{Mod}_{C\lambda}(\mathcal{S}\mathrm{yn}_{E,F}) \hookrightarrow \mathcal{S}\mathrm{table}_{\nu_E F_{*,*} \nu_E F}$ of $C\lambda$-modules into the stable comodule category associated to the bigraded Hopf algebroid $(\nu_E F_{*,*}, \nu_E F_{*,*} \nu_E F)$.

3.2. t-structure for E.

4. Specializations by τ, λ

5. The Categorified Miller Square

References

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