$\mathbb{F}_1$OOD FOR THOUGHT

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References: This talk is almost entirely based on this survey by Oliver Lorscheid. If the talk was at all interesting to you, I strongly recommend reading this, as it subsumes all but the end of the talk. The latter end is mostly based on the theory of Connes-Consani which I learned from this paper of Beardsley and Nakamura. As always, feel free to contact me at mmj002@ucsd.edu with any questions or requests for more sources.

1. Introduction

Today I am going to be discussing briefly some elements of the theory of the so-called "field with one element" $\mathbb{F}_1$. Of course, I feel compelled to begin fundamental theorem:

Theorem 1.1 (Fundamental Theorem of the Field with One Element)**.** There is no field with one element.

Proof. Exercise. □

While this may seem like a non-starter, my hope is to convince you by the end of this talk of the following two statements:

(1) There is a good reason to study the idea around the field with one element, and
(2) these ideas are rich and interesting.

To do so, I will be giving a high-level overview of why people began to think about this, what types of applications there are to be had, and what the present state-of-the-art looks like. Of course, given the short nature of this talk, I will not be able to do any of these true justice.

2. Vector Spaces, Grassmannians, and K-Theory

The theory of $\mathbb{F}_1$ arose as a result of the following philosophy, first taken truly seriously by Jacques Tits but this came at a time when dinosaurs still roamed the earth and Grothendieck had not yet told people what a scheme was. Nonetheless, Tits, ahead of his time, observed the following pattern:

Philosophy 2.1. The theory of combinatorics is often the limit of the theory of algebraic geometry over $\mathbb{F}_q$ as $q \to 1$.

We will begin with some simple numerology to justify this claim. To be clear, we will not do this justice as the examples in this note are quite simple and there are richer things to say, but JJ refused to give me the 3 hours I originally asked for for this talk.

Dictionary 2.2. A vector space over $\mathbb{F}_1$ is a pointed finite set, and the category of vector spaces over $\mathbb{F}_1$ is the category of pointed finite sets. The space of invertible matricies $\mathrm{GL}_n(\mathbb{F}_1)$ is given by the symmetric group Σ_n.

If one thinks of a vector space as the data of a basis combined with scalar extensions, the first claim is hopefully clear. Essentially to specify an $\mathbb{F}_1$-vector space we provide a basis (or just a dimension) and then we formally extend by scalars. But there are no nontrivial scalars! As a result, a vector space over $\mathbb{F}_1$ of dimension n ought to be a formal 0 vector and then n additional elements. We have a nice canonical presentation of such a things as the sets:

$$[n]_* := \{*, 1, ..., n\}$$

where we $[n]$ will denote the same set sans $*$, which we view as a formal "basepoint" or 0 vector. Similarly, if we think about what an $\mathbb{F}_1$-linear map between these should be, the requirements are that $* \mapsto *$ and that the nonexistence scalars are preserved, i.e., its just a map of sets $[n] \to [m]$. This data naturally compiles into a category $\mathcal{F}\mathrm{in}_*$, whose objects are the sets $[n]_*$ and whose maps are $*$-preserving maps of sets.

Remark 2.3. The astute reader/listener will note that I have completely ignored the addition axioms for a vector space. The justification for this is twofold: first is that a so-called field with one element would not itself have a reasonable notion of addition and the second is that combinatorial objects need not have additivity. As a result, some people refer to $\mathbb{F}_1$-geometry as *nonadditive geometry*.

Immediate from this description is that $\mathrm{GL}_n(\mathbb{F}_1) = \Sigma_n$ is the symmetric group on n symbols as for any field $\mathbb{F}$ we always have $\mathrm{GL}_n(\mathbb{F}) = \mathrm{Aut}(\mathbb{F}^n)$. To pursure this line of thinking a bit further, recall that for a field $\mathbb{F}$ the *Grassmannian* $\mathrm{Gr}(k, n)$ is defined to be the set of all k-dimensional subspaces of an n-dimensional $\mathbb{F}$-vector space. Note that $\mathrm{Gr}(k, n)$ aquires a natural action of $\mathrm{GL}_n(\mathbb{F})$ as an invertble linear transformation will send k-dimensional subspaces to k-dimensional subspaces, and this action is easily seen to be transitive.

The transitivity of this action shows via the orbit stabilizer theorem

$$\#\mathrm{Gr}(k, n) = \frac{\#\mathrm{GL}_n(\mathbb{F})}{\#\mathrm{Stab}(V)}$$

where we write $\#$ for cardinality and V is an arbitrary element of $\mathrm{Gr}(n, k)$. One can classify the matricies in the stabilizer as those which look like:

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are $n-k \times n-k$ and $k \times k$ invertible matricies, respectively. We want to study the size of the objects involved in the orbit stabilizer as $q \to 1$. We will study the counts as $q \to 1$. First, put:

$$\langle n \rangle_q := \sum_{i=0}^{n-1} q^i, \quad \langle n \rangle_q! := \prod_{i=1}^{n} \langle i \rangle_q, \quad \text{and} \quad \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_q := \frac{\langle n \rangle_q!}{\langle n-k \rangle_q! \langle k \rangle_q!}$$

which have limits n, $n!$ and $\binom{n}{k}$ respectively as $q \to 1$. A common exercise in an abstract algebra course is to count explicitly:

$$\#\mathrm{GL}_n(\mathbb{F}_q) = \prod_{i=1}^{n} (q^n - q^{i-1}) = (q-1)^n q^{\frac{1}{2}(n^2-n)} \langle n \rangle_q!$$

and similar arguments show that:

$$\#\mathrm{Stab}(\mathbb{F}_q^n) = (q-1)^n q^{\frac{1}{2}(n^2-n)} \langle n-k \rangle_q! \langle k \rangle_q!$$

and dividing them yields:

$$\#\mathrm{Gr}(k, n) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$$

We note that this last formula already gives the correct answer as $q \to 1$, but the size of GL_n and the stabilizer are 0 in this limit. However, if one removes the obvious terms which vanish (which I guess can be thought of as some resolution of singularities) we again get the desired answers. One of the motivating reasons for the conjectural existence of $\mathbb{F}_1$-geometry was to explain this.

3. The BPQ Theorem

Shifting gears slightly, we will again think categorically about $\mathcal{F}\mathrm{in}_*$. If K-theory is completely uunfamiliar to you, bear with me, this part will be short. If you don't know what a spectrum is, for this talk you can essentially think of the following analogy:

sets : abelian groups :: spaces : spectra

i.e., they are like spaces with a commutative addition operation where everything is only forced to hold up to higher coherent homotopy. Every space as an associated "suspension spectrum" and the suspension spectrum for the sphere, denoted $\mathbb{S}$, plays the role of $\mathbb{Z}$ in spectra. Associated to every ring R, Grothendieck constructed the group $K_0(R)$ defined to be the group-completion (i.e., formally add in inverses) of the monoid of isomorphism classes of f.g. projective R-modules under direct sum. For a field $\mathbb{F}$, this group is not particularly interesting: f.g. projective $\mathbb{F}$-modules are just finite $\mathbb{F}$-vector spaces, whose isomorphism classes are given by the nonnegative integers which group complete to $\mathbb{Z}$. Later, "higher" groups were introduced, notably $K_1(R) = \mathrm{GL}(R)^{\mathrm{ab}}$ and these were shown to be naturally linked together, like a homology theory. Later, Quillen introduced a K-theory spectrum associated to rings given by:

$$\pi_*(B\mathrm{GL}(R)^+)$$

where GL is the directed limit of the inclusions $\mathrm{GL}_n \subset \mathrm{GL}_{n+1}$, the notation B denotes the classifying space of this group, and the operation $+$ is like a π_1-abelianization. The functor K_0 can easily be seen to depend only on the category of modules over R. Waldhausen later showed similarly that the K-theory spectrum could be defined in terms of the category of R-modules and could be extended to categories that were formally similar to a category of f.g. projective modules, so-called Waldhausen categories. Essentially, a Waldhausen category is a category with a good notion of injections and weak equivalences. Luckily for us, $\mathcal{F}\mathrm{in}_*$ has these: the weak equivalences are just the isomorphisms, i.e., bijections, and the injections are just injections.

Theorem 3.1 (Barratt-Priddy-Quillen)**.** There is an equivalence $K(\mathcal{F}\mathrm{in}_*) \simeq \mathbb{S}$. If we believe that $\mathcal{F}\mathrm{in}_*$ is the category of finite dimensional $\mathbb{F}_1$-vector spaces, then $K(\mathbb{F}_1) = \mathbb{S}$.

Remark 3.2. We also know by a famous result of Quillen that:

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/(q^j - 1) & j = 2i - 1 \text{ is odd.} \\ 0 & \text{otherwise.} \end{cases}$$

which are substantially more regular than the famously complicated homotopy groups $\pi_*\mathbb{S}$. To my knowledge, it is unclear if there is any expected limiting $q \to 1$ phenomena here, or if this limit is simply not respected by K-theory.

One way to motivate the above theorem is as follows: taking Quillen's original construction and recalling that $\mathrm{GL}(\mathbb{F}_1)$ is the infinite symmetric group Σ_∞, there should be an identification:

$$K(\mathcal{F}\mathrm{in}_*) = (B\Sigma_\infty)^+$$

But we can create an altnerative description of $B\Sigma_\infty = \coprod_n B\Sigma_n$ where the disjoint union of symmetric groups is given the "block" multiplication $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. While this is not the same group as Σ_∞, the compactness of S^1 causes their classifying spaces to be equivalent.

This latter construction is easier to give a name to. To see this, we recall that one way to construction BG for G a group is to give the point $*$ the trivial G action and take its *homotopy orbits* $*_{hG} \simeq BG$. But then:

$$(\coprod_n B\Sigma_n)^+ \simeq (\coprod_n *_{h\Sigma_n})^+$$

which hopefully you will trust is me is the homotopical analog of the formula for the free commutative ring on a single point. But in normal algebra, this is $\mathbb{Z}$, so I hope it is not surprising that in higher algebra this corresponds to $\mathbb{S}$.

4. The Riemann Hypothesis and the ABC Conjecture

Recall that the Riemann-Zeta function is the analytic continuation of:

$$\zeta(s) = \sum n^{-s}$$

and that the *Riemann Hypothesis* is the conjecture that its nontrivial zeroes lie on the strip of real part $1/2$. This function admits a reformulation as:

$$\zeta(s) = \prod_{p \text{ a prime}} \frac{1}{1 - p^s}$$

which is easier to generalize. We will really be intersted however in the completed zeta function:

$$\hat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

where we have essentially just removed the trivial zeroes at the negative integers of the Riemann Zeta function.

Recall that an absolute value on a field K is a function $\nu : F \to \mathbb{R}_{\geq 0}$ satisfying $\nu(ab) = \nu(a)\nu(b)$ and $\nu(a + b) \leq \nu(a) + \nu(b)$. It is nonarchimedian if it satisfies $\nu(a + b) \leq \max\{\nu(a), \nu(b)\}$.

Given a nonarchimedian absolute value ν on a field K, the valution (sub)ring is the ring:

$$\mathcal{O}_\nu := \{x \in K \mid \nu(x) \leq 1\}$$

with maximal ideal $m_\nu := \{x \in K \mid \nu(x) < 1\}$. The residue field is $\kappa_\nu := \mathcal{O}_\nu/m_\nu$.

Theorem 4.1. The nontrivial absolute values on $\mathbb{Q}$ are given by the standard absolute value ν_∞ and the p-adic valuations ν_p. The residue fields $\kappa(\nu_p)$ are isomorphic to $\mathbb{F}_p$.

We will refer to a nontrivial absolute value on K as a *place*. This leads to the following definition:

Definition 4.2. Given a nonarchimedian place ν, the associated *local zeta factor* is

$$\zeta_\nu := \frac{1}{1-(\#\kappa_\nu)^s}$$

we will also give the ad-hoc definition over $\mathbb{Q}$ that

$$\zeta_{\nu_\infty} = \pi^{-s/2}\Gamma(s/2)$$

Using the slightly ad-hoc definition above, we recover the completed zeta function:

$$\hat{\zeta} = \prod_{\nu \text{ a place in } \mathbb{Q}} \zeta_\nu$$

This definitions can be extended to more general fields.

Theorem 4.3. Every aboslute value in positive characteristic is nonarchimedian.

As a result, we can simply define without ad-hoc notation:

Definition 4.4. Given a field K in positive characteristic, the associated zeta function is:

$$\zeta_K(s) = \prod_{\nu \text{ a place in } K} \zeta_\nu$$

Definition 4.5. A function field over $\mathbb{F}_q$ of degree n is a finite extension of $\mathbb{F}_q(T_1, ..., T_n)$.

Theorem 4.6 (Hasse, Weil, Deligne)**.** The Riemann Hypothesis holds for ζ_K when K is a function field over $\mathbb{F}_q$.

Remark 4.7. The reason for the term ”function field” is that such field extensions arise as the fields-of-functions for n-dimensional algebraic varieties over $\mathbb{F}_q$.

We will only need to discuss the case where F is a 1-dimensional function field so that the associated algebraic variety is a curve. In this case, the curve C_F will encode the various places of the function field $\mathbb{F}$. Using this information, it turns out we can tie the function field Riemann Hypothesis to counting intersection points of the diagonal embedding:

$$C \hookrightarrow C \times_{\operatorname{Spec} \mathbb{F}_q} C$$

where one copy of C is twisted by the Frobenius. Using techniques from arithmetic geometry, this can then be explicitly computed.

The upshot here is that we already have a nice ”curve” which encodes most of the places of $\mathbb{Q}$, i.e., the curve $\operatorname{Spec} \mathbb{Z}$. The only issue is that (1) this is not a curve as naturally defined and (2) it is missing the place ν_∞. The proposed resolution to this is:

Philosophy 4.8 (Deninger(?))**.** The scheme $\operatorname{Spec} \mathbb{Z}$ is a curve over $\mathbb{F}_1$ and can be compactified to a scheme $\overline{\operatorname{Spec} \mathbb{Z}}$ whose point at infinity corresponds to ν_∞.

If the above is made sense of, and a suitable theory of arithmetic geometry over $\mathbb{F}_1$ established, the hope is to prove the Riemann Hypothesis using techniques analogous to curves over $\mathbb{F}_q$. It was later noticed by Smirnov that if suitable Hurwitz equalities are satisfied, then a conjectural map:

$$\overline{\operatorname{Spec} \mathbb{Z}} \to \mathbb{P}^1_{\mathbb{F}_1}$$

would imply the abc-conjecture.

5. Brave New Algebra and $\mathbb{F}_1$

As more people began to study the idea of $\mathbb{F}_1$-geometry, a large number of models for $\mathbb{F}_1$ were introduced, many of which overlapped and some of which subsumed eachother. Of particular interest to homotopy theorists is the model of Connes-Consani which at its introduction also subsumed a great deal of other theories.

Philosophy 5.1. The field with one element plays the role of $\mathbb{N}$ associated to the sphere spectrum $\mathbb{S}$.

My final goal is to make this sound at least slightly less absurd by the end of this talk.

If we allow ourselves to believe that $\mathbb{F}_1$ is not really a field, then we also allow ourselves to enlarge its category of modules. Before we can define it we need some notation. Let Spc denote one of: topological spaces with their usual model structure, simplicial sets with the Kan model structure, or simply the ∞-category of spaces. For any category $\mathcal{C}$ with terminal object $*$ let $\mathcal{C}_*$ denoted the category of objects-with-basepoints, i.e., the slice category under $*$. Finally let $\mathrm{Fun}_*(\mathcal{C}_*, \mathcal{D}_*)$ denote $*$-preserving functors. Then we define:

$$\Gamma\mathcal{C} := \mathrm{Fun}_*(\mathcal{F}\mathrm{in}_*, \mathcal{C}_*)$$

Definition 5.2. The category $\mathrm{Mod}_{\mathbb{F}_1}$ of $\mathbb{F}_1$-modules is defined to be the category $\Gamma\mathrm{Sets}$. The object $\mathbb{F}_1$ itself is the functor corresponding to including finite sets into sets.

Remark 5.3. This functor category aquires a symmetric monoidal tensor product via Day Convolution. With this structure, the unit is $\mathbb{F}_1$, justifying the notation.

Remark 5.4. Note that there is no change to $\mathrm{Aut}(\mathbb{F}_1^n) = \mathrm{GL}_n(\mathbb{F}_1)$ so that the Barratt-Priddy-Quillen theorem still holds.

An immediate consequence of this definition is that the original category $\mathcal{F}\mathrm{in}_*$ of finite dimensional $\mathbb{F}_1$-vector spaces includes into our new $\mathbb{F}_1$ by sending $[n]_*$ to the n-fold coproduct $\mathbb{F}_1^n \in \mathrm{Mod}_{\mathbb{F}_1}$. However, we now thing of these as the much smaller subcategory of f.g. free $\mathbb{F}_1$-modules. The point of enlgarging the category of $\mathbb{F}_1$-modules is to allow the theory of $\mathbb{F}_1$ to capture a vast array of combinatorial objects. The most general embedding in the literature is that of Beardsley and Nakamura which appeared on the arxiv recently.

Definition 5.5. A *plasma* is a set M equipped with a *hyperoperation* $\star : M \times M \to \mathcal{P}(M)$ and an identity element $e \in M$ such that:

(1) $a \in e \star a$ for all $a \in M$,
(2) $a \star b = b \star a$ for all $a, b \in M$.

In effect, a plasma is a set with a multi-valued commutative multiplication with a weak notion of identity. Examples of plasmas include:

(1) all commutative monoids,
(2) all matroids,
(3) and all projective geometries.[1]

Hopefully the notion of commutative monoid is familiar, but we will briefly recall the other two. A matroid is supposed to capture the notion of finding the smallest affine subspace of a vector space containing a collection of points. A projective geometry is a matroid which is determined by finite subsets and respects unions.

Remark 5.6. I would like to know why this is called a projective geometry. My guess is that these extra axioms are satisfies when one finds the smallest geometrically meaningful subspaces of projective spaces containing given points, but I have not attempted to work this out.

Definition 5.7. A *matroid* is a set M with an operation $\kappa : \mathcal{P}(M) \to \mathcal{P}(M)$ such that:

(1) $A \in \kappa(A)$
(2) $A \subset B \implies \kappa(A) \subset \kappa(B)$
(3) $\kappa\kappa = \kappa$
(4) If $x \in \kappa(A \cup y)$ but $x \notin \kappa(A)$ then $y \in \kappa(A \cup x)$.

a matroid is simple pointed if $\kappa(\emptyset) = \{0\}$ is a singleton and satisfies $\kappa(\{x\}) = \{x, 0\}$ for all x. A simple pointed matroid is a *projective geometry* if $\kappa(A)$ is the union of $\kappa(B)$ across finite subsets $B \subset A$ and if $\kappa(A \cup B)$ is the union of the $\kappa(\{x, y\})$ across $x \in \kappa(A)$ and $y \in \kappa(B)$.

Theorem 5.8 (Beardsley-Nakamura)**.** The category of plasmas embeds fully faithfully into the category of $\mathbb{F}_1$-modules. Moreover, the restriction to commutative monoids extends the Eilenberg Mac-Lane functor from abelian groups to spectra.

[1] I am told there are several competing definitions of projective geometry. The one we will discuss is equivalent to that of Faire and Frolicher in *Modern Projective Geometry*.

The above is hopefully justification enough to make Philosophy 2.1 seem plausible. The final part of this talk will be to justify our final philosophy:

Philosophy 5.9. The ring $\mathbb{F}_1$ is to the sphere spectrum $\mathbb{S}$ as $\mathbb{N}$ is to $\mathbb{Z}$. Explicitly, under the relevant notion of group completion, $\mathbb{F}_1$ is the monoid which completes to the "abelian group" $\mathbb{S}$.

This will require me to give an actual construction of the category of (connective) spectra. First, let us think more closely about the category $\mathcal{F}\mathrm{in}_*$. I want to claim that functors out of $\mathcal{F}\mathrm{in}_*$ are exactly trying to encode commutative monoid objects and that $\Gamma\mathcal{C}$ roughly encodes commutative monoids in $\mathcal{C}$ up to something called the "Segal condition".

Definition 5.10. A functor $X \in \Gamma\mathcal{C}$ satisfies the Segal condition if for all n we have $X([n]_*) = X([n])^{\times n}$. Such are called "special" Γ-objects (where "object" is replaced by whatever we call objects of $\mathcal{C}$).

Note that a special Γ-object in $\mathcal{C}$ is exactly a commutative monoid in $\mathcal{C}$: the multiplication comes from the map $[2]_* \to [1]_*$ which sends $1, 2 \mapsto 1$. The unique map $[0]_* \to [1]_*$ will encode the unit. The commutativity comes from the fact that the we have the commutative diagram:

$$\begin{tikzcd} {[2]_*} \arrow[rr, "\text{swap}"] \arrow[dr] & & {[2]_*} \arrow[dl] \\ & {[1]_*} & \end{tikzcd}$$

Objects of $\Gamma\mathcal{C}$ are then generalizations of commutative monoids wherein the "multiplication" does not need to be of the form $X \times X \to X$ but is instead a much weaker notion, which is why the theory of $\mathrm{Mod}_{\mathbb{F}_1} = \Gamma\mathrm{Sets}$ can account for objects like plasmsas.

Definition 5.11. A special Γ object is said to be very special if its monoid structure has inverses, defined diagrammatically. These should be viewed as abelian group objects in $\mathcal{C}$. In good situations, there is a group completion functor from special to very special Γ objects.

Definition 5.12. The category of (connective) spectra Sp is the ∞-category presented by the very special Γ-spaces. This category is symmetric monoidal with unit $\mathbb{S}$.

Now finally we note that there is a natural inclusion $\mathrm{Mod}_{\mathbb{F}_1} \hookrightarrow \mathrm{Sp}$ given by including sets into spaces with the discrete topology.

Theorem 5.13. The object $\mathbb{F}_1 \in \mathrm{Mod}_{\mathbb{F}_1}$ is a special Γ-object whose group completion is the sphere $\mathbb{S}$.[2]

Remark 5.14. One is led to view the category of $\Gamma\mathrm{Spc}$ as some sort of (connective) derived category of $\mathbb{F}_1$-modules. I do not think this has recieved serious study in the literature, and would be interested to understand better if this would lead to applications of homotopy theory to combinatorics in the same way that derived categories greatly enrich the study of commutative algebra and schemes.

Remark 5.15. There is a way to view the above theorem as a restatement of Barratt-Priddy-Quillen, but I will leave that for the interested reader. The point is that one can take the seriously the original definition of K_0 via Grothendieck's group completion and derive it, and for commutative monoids in spaces (of which standard commutative monoids are examples), the derived group completion is exactly the K-theory spectrum.

[2]This should follow from some monoidality statement for group completion, but I could not find a reference. I will update this if I stumble across one.