

AN INTRODUCTION TO HOMOTOPY COHERENT ALGEBRA

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1. INTRODUCTION

2. RECOLLECTIONS FROM HIGHER CATEGORY THEORY

In this section we review salient properties of the theory of ∞ -categories as put forth in [?], [?], and other sources. Although we work in the framework of such monolithic works, they are not meant to be prerequisites in their entirety. In fact, all of the material in these notes already exists in such references. The goal here is to collect these results into a shorter work, with much generality omitted for the sake of describing a particular story.

Instead, the reader is encouraged to be familiar with the theory of quasicategories at a potentially superficial level, returning to the details or constructions omitted here as necessary. In many ways, the purpose of this note is to introduce

these concepts as model-independently as is currently feasible. At their conclusion our hope is that the reader will feel comfortable doing homotopy coherent algebra with minimal reference to the details of the quasicategorical model.

That said, we *will* assume familiarity with the basic definitions found in e.g. the first chapter of [?], as well as the theory of stable ∞ -categories put forth in the first chapter of [?]. Including the material would only be

2.1. coCartesian Fibrations and the Grothendieck Construction. The goal of this section is to describe when we can view a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ as having fibers \mathcal{C}_d sitting over objects $d \in \mathcal{D}$ connected by functors induced by the morphisms in \mathcal{D} . In particular, we want to know when the data of such a functor F is equivalent to the data of a functor $\mathcal{D} \rightarrow \text{Cat}_\infty$. We will adopt the model-independent perspective developed in [?].

Let $F : \mathcal{E} \rightarrow \mathcal{C}$ be a functor between ∞ -categories.

Definition 2.1. A morphism $\phi : e_1 \rightarrow e_2$ in \mathcal{C} is said to be F -coCartesian if it induces a pullback square:

$$\begin{array}{ccc} \mathcal{E}_{e_2}/ & \xrightarrow{\phi \circ -} & \mathcal{E}_{e_1}/ \\ \downarrow F & & \downarrow F \\ \mathcal{B}_{F(e_2)}/ & \xrightarrow{f(\phi) \circ -} & \mathcal{B}_{F(e_1)}/ \end{array}$$

in Cat_∞ . In this situation we say that ϕ is an F -coCartesian¹ lift of $F(\phi)$ relative to e_1 .

Remark 2.2. The square above will commute for any morphism in \mathcal{E} . Then the condition of being coCartesian this tells us that maps $e_2 \rightarrow x$ are determined by pairs $F(e_2) \rightarrow F(x)$ and maps $e_1 \rightarrow X$ such that we get an equivalence

$$F(e_1) \rightarrow F(x) \simeq F(e_1) \xrightarrow{F(\phi)} F(e_2) \rightarrow F(x)$$

Definition 2.3. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a coCartesian fibration if, given a map $\psi : b_1 \rightarrow b_2$ in \mathcal{B} and any lift e_1 of b_1 , there is an F -coCartesian lift of ψ relative to e_1 .

Theorem 2.4. *There is a functor Gr taking a coCartesian fibration $\mathcal{E} \rightarrow \mathcal{B}$ to a functor $\mathcal{B} \rightarrow \text{Cat}_\infty$ known as the Grothendieck Construction that furnishes an equivalence $\text{coCart}(\mathcal{B}) \simeq \text{Fun}(\mathcal{B}, \text{Cat}_\infty)$.*

It is shown in [?] that this model independent approach is equivalent to the quasicategorical approach in [?] so that the result is [?, todo]. Essentially, given a coCartesian fibration $F : \mathcal{E} \rightarrow \mathcal{B}$, the functor Gr acts by sending $b \in \mathcal{B}$ to its fiber $F^{-1}(b)$ and maps $b_1 \rightarrow b_2$ to covariant functors between the fibers; the axioms of a coCartesian fibrations assure this is all well-defined.

3. ∞ -OPERADS

3.1. An ∞ -category of Finite Sets. Let $[n]$ be the set $\{1, 2, \dots, n\}$. Consider the following 1-category of finite pointed sets due to (Todo: who?). For objects, we take the pointed sets $[n]_* = [n] \cup \{*\}$, i.e., the sets $[n]$ with an appended basepoint. The morphisms are just functions of finite sets preserving the elements $*$.

¹Potentially omitting F when obvious.

Definition 3.1. Let \mathcal{F} denote the 1-category above and let $N(\mathcal{F})$ denote its ∞ -categorical nerve.

There are two particularly important classes of morphisms forming a factorization system in \mathcal{F} (and therefore in $N(\mathcal{F})$).

Definition 3.2. A morphism $\phi : [n]_* \rightarrow [m]_*$ is *inert* if every $k \in [m]$ (excluding the basepoint) has exactly one preimage. A morphism is said to be *active* if ϕ does not send any element of $[n]$ to the basepoint of $[m]_*$.

Lemma 3.3. Every morphism $\phi : [n]_* \rightarrow [m]_*$ in \mathcal{F} can be factored as $[n]_* \rightarrow [k]_* \rightarrow [m]_*$ where the first map is inert and the second is active.

Inert morphisms $\phi : [n]_* \rightarrow [m]_*$ exhibit $[m]_*$ as being obtained from $[n]_*$ by sending extraneous elements to $*$. Active morphisms are exactly those that remain well-defined after removing basepoints.

Among the inert morphisms, we will often have reason to refer to the morphisms $p^i : [n]_* \rightarrow [1]_*$, with $i \in [n]$, which send everything but i to $*$.

The definition of an ∞ -operad will involve functors valued in $N(\mathcal{F})$. As such is it prescient to extend the definitions of inert and active morphisms to all objects of $(\text{Cat}_\infty)_{/N(\mathcal{F})}$.

Definition 3.4. Given $F : \mathcal{C} \rightarrow N(\mathcal{F})$, a morphism $f : c_1 \rightarrow c_2$ in \mathcal{C} is said to be inert (active) if $F(f)$ is inert (active).

The definition of an ∞ -operad is closely related to that of a coCartesian fibration over $N(\mathcal{F})$ (Definition 2.3). Naturally we will want to consider the fibers over each $[n]_*$. For now, we will think of them only as collections of objects. If $F : \mathcal{E} \rightarrow \mathcal{B}$ is a functor and $c \in \mathcal{C}$, then let \mathcal{E}_c denote the objects lying over c . Of course, though we omit it from the notation, this depends on the choice of F .

Definition 3.5. An ∞ -operad is an ∞ -category \mathcal{O}^\otimes equipped with a functor $\gamma : \mathcal{O}^\otimes \rightarrow \mathcal{F}$ satisfying:

- (1) If $f : [n]_* \rightarrow [m]_*$ is inert, and if x is a lift of $[n]_*$, then there is a coCartesian lift of f relative to x in \mathcal{O}^\otimes .
- (2) If $x_1 \in \mathcal{O}_{[n]_*}^\otimes$ and $x_2 \in \mathcal{O}_{[m]_*}^\otimes$, let $\text{maps}^f(x_1, x_2)$ denote the union of connected components lying over $f \in \text{maps}([n]_*, [m]_*)$. Due to axiom (1) we may choose coCartesian lifts $\phi^i : x_2 \rightarrow y_i$ of p^i relative to x_2 . Then we require that

$$\text{maps}^f(x_1, x_2) \rightarrow \prod_{i=1}^n \text{maps}^{p^i \circ f}(x_1, y_i)$$

is an equivalence.

- (3) The induced functors (Definition ??) $p_!^i : \mathcal{O}_{[n]_*}^\otimes \rightarrow \mathcal{O}_{[1]_*}^\otimes$ induce an equivalence $\mathcal{O}_{[n]_*}^\otimes \simeq (\mathcal{O}_{[1]_*}^\otimes)^{\times n}$

Remark 3.6. Our choice of axiom (3) is slightly nonstandard. Usually one assumes the apparently weaker condition that for every n and every sequence of objects $x_1, \dots, x_n \in \mathcal{O}_{[1]_*}^\otimes$, we have an object $x \in \mathcal{O}_{[n]_*}^\otimes$ with coCartesian lifts $\bar{p}^i : x \rightarrow x_i$ of the p_i . Assuming (1) and (2), however, (3) and (3') are equivalent [?, Remark].

Notation 3.7. Given an ∞ -operad $\mathcal{O}^\otimes \rightarrow N(\mathcal{F})$, we will write \mathcal{O} for $\mathcal{O}_{[1]_*}^\otimes$ and think of it as the underlying category of the operad.

Definition 3.8. A morphism of ∞ -operads is a functor $\mathcal{O}^\otimes \rightarrow \mathcal{Q}^\otimes$ in $(\text{Cat}_\infty)_{/N(\mathcal{F})}$ preserving inert morphisms.

We may think of an ∞ -operad morphism $F : \mathcal{O}^\otimes \rightarrow \mathcal{Q}^\otimes$ in two ways. The first is as an \mathcal{O}^\otimes -algebra in \mathcal{Q} . Because the notion is stable under composition, however, we may also think of $\mathcal{O}^\otimes \rightarrow \mathcal{Q}^\otimes$ as inducing a functor for turning \mathcal{Q}^\otimes -algebras into \mathcal{O}^\otimes -algebras.

Notation 3.9. The full subcategory of $\text{Fun}(\mathcal{O}^\otimes, \mathcal{Q}^\otimes)$ spanned by the ∞ -operad morphisms will be denoted $\text{Alg}_{\mathcal{O}}(\mathcal{Q})$.

We will not refer to the 1-categorical notion of operad in this document. As a result, much like with the notion of ∞ -category, we will often drop the ∞ and simply say operad. We will give several examples of operads in the next few sections. We finish the current discussion by introducing the notion of a symmetric monoidal ∞ -category.

Definition 3.10. A symmetric monoidal category \mathcal{C}^\otimes is an operad $\mathcal{C}^\otimes \rightarrow N(\mathcal{F})$ that is also a coCartesian fibration.

In effect, a symmetric monoidal ∞ -category requires that we strengthen the operad axiom (1) to apply to all maps in $N(\mathcal{F})$. Then the tensor product (\otimes) is given by the map

$$\otimes : \mathcal{C} \times \mathcal{C} \simeq \mathcal{C}_{[2]}^\otimes \xrightarrow{f_1} \mathcal{C}_{[1]}^\otimes \simeq \mathcal{C}$$

where f_1 is induced by the function $f(1) = f(2) = 1$. The equivalence $X \otimes Y \simeq Y \otimes X$ is clear from the fact that f commutes with the swap map $[2]_* \rightarrow [2]_*$.

Remark 3.11. We note that via the Grothendieck construction we can view a symmetric monoidal ∞ -category as a functor $F : N(\mathcal{F}) \rightarrow \text{Cat}_\infty$ such that the functors $F(p^i)$ induces an equivalence $F([n]_*) \simeq F([1]_*)^{\times n}$.

This motivates Notation 3.9 as now given an ∞ -operad \mathcal{O}^\otimes and a symmetric monoidal ∞ -category \mathcal{C}^\otimes , we may think of the objects of $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ as \mathcal{O} -algebras with respect to the tensor product in \mathcal{C} .

3.2. Several (Deceptively) Simple Examples. We now list some of the most important operads for our purposes. First we give the following general condition:

Lemma 3.12. *Let \mathcal{E} be a 1-categorical subcategory of \mathcal{F} such that:*

- (1) *All of the objects $[n]_*$ of \mathcal{F} are in \mathcal{E} .*
- (2) *All of the inert morphisms of \mathcal{F} are in \mathcal{E} .*

Then the nerve $N(\mathcal{E}) \rightarrow N(\mathcal{F})$ is an ∞ -operad.

Proof. For axiom (1), given any inert morphism $f : [n]_* \rightarrow [m]_*$ then f will be a coCartesian lift of itself, considered as a morphism in $N(\mathcal{E})$. Namely, the condition is that a map $[m]_* \rightarrow [k]_*$ be determined by a pair $[m]_* \rightarrow [k]_*$ and a morphism $[n]_* \rightarrow [k]_*$ which is equivalent to $[n]_* \rightarrow [m]_* \rightarrow [k]_*$. But this latter requirement reduces to simply choosing a map $[m]_* \rightarrow [k]_*$.

For axiom (2), we note that a map $[n]_* \rightarrow [m]_*$ is fully determined by product of the compositions with the $p^i : [m]_* \rightarrow [1]_*$.

For axiom (3), we turn instead to the (3') of Remark 3.6. There is a unique object in $\mathcal{E}_{[1]_*}$, namely $[1]_*$ itself. By assumption we have the necessary coCartesian lifts. \square

Example 3.13. The *trivial* operad \mathbf{Triv}^\otimes is the operad associated to the subcategory \mathcal{E} of \mathcal{F} given by all objects and only the inert morphisms. This is clearly the minimal operad conforming to Lemma 3.12.

Example 3.14. The *commutative* operad \mathbf{Comm}^\otimes is the maximal application of Lemma 3.12, namely the operad associated to the identity $\mathcal{F} \rightarrow \mathcal{F}$.

Remark 3.15. Expanding on Remark 3.11, we note that a symmetric monoidal ∞ -category $\mathcal{C}^\otimes \rightarrow N(\mathcal{F})$, whose image under the Grothendieck construction is $N(\mathcal{F}) \rightarrow \mathbf{Cat}_\infty$ can be considered, via Example ?? and Definition ??, a \mathbf{Comm}^\otimes -algebra in \mathbf{Cat}_∞ .

Example 3.16. Let \mathcal{F}^{inj} denote the subcategory of \mathcal{F} consisting of all objects and those morphisms that are injective away from the basepoint. The associated operad will be denoted \mathbb{E}_0^\otimes .

There is a family of operads \mathbb{E}_n^\otimes which will be the focus of Section 3.3. In fact, these interpolate between the commutative algebra above, the associative algebra below.

Remark 3.17. There is a subtlety in our notation regarding the commutative operad. In particular, the underlying category of \mathbf{Comm}^\otimes , as defined in Notation 3.7, is equivalent to the infinity category Δ^0 with one object and one non-identity morphism. Therefore we have to be careful in distinguishing between the category $N(\mathcal{F}) \simeq \mathbf{Comm}^\otimes$ and the "underlying category" of the operad \mathbf{Comm}^\otimes .

Example 3.18. Consider the 1-category \mathcal{E} whose objects are those of \mathcal{F} but whose morphisms consist of a choice of morphism $f : [n]_* \rightarrow [m]_*$ in \mathcal{F} equipped with chosen linear orderings \leq_i on the preimages $f^{-1}(i)$ for all $1 \leq i \leq m$. Then $N(\mathcal{E}) \rightarrow N(\mathcal{F})$ is an operad we will denote \mathbf{Assoc}^\otimes .

Remark 3.19. Because we allow ourselves to choose orderings on the preimages, we note that there are now two distinct tensor product maps. If $f : [2]_* \rightarrow [1]_*$ is given as $f(1) = f(2) = 1$, then our choices are the pairs $(f, 1 < 2)$ and $(f, 2 < 1)$. These need not be equivalent in \mathbf{Assoc}^\otimes -algebras and the swap map swaps these two choices of product as well. However, the associativity equivalence can be seen when we consider $f : [3]_* \rightarrow [1]_*$ defined analogously. Thinking of the linear order as prescribing the order of multiplication, we see that, fixing a linear order on $[3]$, we may factor this map compatibly through $[2]_*$ in two different yet equivalent ways.

3.3. Little Disks.

4. ALGEBRAS OVER OPERADS

4.1. Free Algebras.

4.2. Forgetful Functors of Algebras.

5. SYMMETRIC MONOIDAL ∞ -CATEGORIES

6. STRUCTURED RINGS

6.1. The ∞ -category of Structured Ring Objects.

6.2. Ideals and Quotients of Structured Rings.

7. MODULES OVER STRUCTURED RINGS

7.1. The ∞ -category of Modules over a Structured Ring.

8. STRUCTURED ALGEBRAS

9. SOME APPLICATIONS

9.1. Adams Spectral Sequences and Deformations. We will generalize the theory of deformations developed in [?]. In particular, most of the results here can be first seen in [?]. That said, there is genuine novelty to our work. In particular, it can be used to construct new spectral sequences with fairly general inputs. To that end we will seek to state results in as much generality as possible; we will start with a stable ∞ -category \mathcal{C} and add assumptions as needed. The reader may at all times think of \mathcal{C} as being spectra, or a similarly nice symmetric monoidal stable ∞ -category.

Suppose \mathcal{C} is a stable ∞ -category and \mathcal{A} is abelian. Then a functor $F : \mathcal{C} \rightarrow \mathcal{A}$ is said to be exact if it takes cofibers $X \xrightarrow{f} Y \rightarrow Cf$ in \mathcal{C} to exact sequences $F(X) \rightarrow F(Y) \rightarrow F(Cf)$ in \mathcal{A} . For the remainder of this section, we fix the data of an exact functor $F : \mathcal{C} \rightarrow \mathcal{A}$.

Definition 9.1. Given $F : \mathcal{C} \rightarrow \mathcal{A}$ as above, the category of compact F -dualizables, denoted $[\mathcal{C}]_F$, is the full subcategory of \mathcal{C} spanned by the objects X such that:

- (1) X is compact in \mathcal{C} .
- (2) $F(X)$ is dualizable in \mathcal{A} .

We further endow $[\mathcal{C}]_F$ with the structure of an additive ∞ -site (Definition [?]) by declaring the coverings to be those maps that are sent to surjections by F .²

Lemma 9.2. *The category $[\mathcal{C}]_F$ is an excellent ∞ -site.*

Example 9.3. The category $\mathrm{Sp}_E^{\mathrm{fp}}$ from [?] is equivalently the category $[\mathrm{Sp}]_{E_*}$, where we think of E_* as being valued in graded E_* comodules.

Definition 9.4. The synthetic deformation of \mathcal{C} by F , denoted Syn_F , is the category $\mathrm{Sh}_\Sigma([\mathcal{C}]_F, \mathrm{Sp})$.

The remainder of this section will be to prove that the above generalization of Pstragowski's synthetic spectra is a good one, proving that most of the important results still hold in this setting. Many of the proofs follow the originals almost verbatim, but we provide them for the reader's ease where possible.

Proposition 9.5. *Suppose \mathcal{C} and \mathcal{A} are both presentable. Then Syn_F is presentable.*

Proposition 9.6. *The ∞ -category Syn_F is stable. If \mathcal{C} is presentable, then so is Syn_F . If \mathcal{C} is symmetric monoidal, then so is Syn_F . The tensor product in Syn_F preserves colimits in each variable.*

Proposition 9.7. *The ∞ -category Syn_F admits a t -structure whose connective part consists of what??? The heart is like \mathcal{A} or something.*

9.2. Factorization Homology.

9.3. K-Theory.

²Recall that in a general abelian category, $A \rightarrow B$ is said to be surjective if $A \rightarrow B \rightarrow 0$ is exact.

9.4. A Primer on Spectral Algebraic Geometry.

REFERENCES