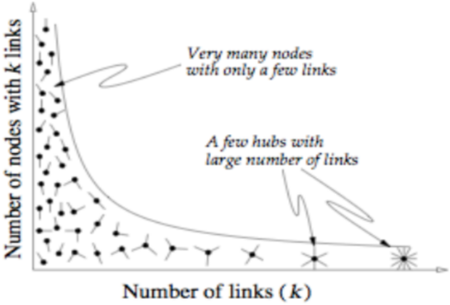
Network Science

2019. szeptember 23.



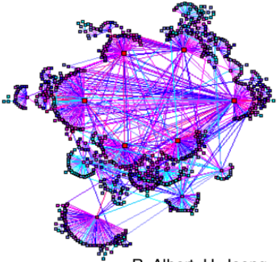
SCALE-FREE NETWORKS

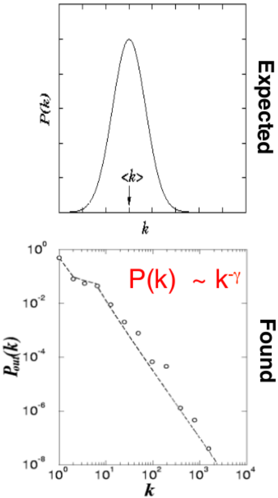
Degree distribution of real networks

Nodes: **WWW documents**
Links: **URL links**

Over 3 billion documents

ROBOT: collects all URL's found in a document and follows them recursively





R. Albert, H. Jeong, A-L Barabasi, ***Nature***, 401 130 (1999).

(from the slides of A.-L. Barabási)

Plotting power-law $p(k)$

- Usually we plot a power-laws on log-log scale. Why?

Plotting power-law $p(k)$

- Usually we plot a power-laws on log-log scale. Why?
  - Power-law: $p(k) \simeq ck^{-\gamma}$,

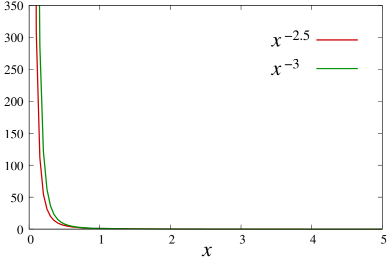
  $\rightarrow$ $\ln p(k) \simeq \ln c - \gamma \cdot \ln k$.
  - In a log-log plot

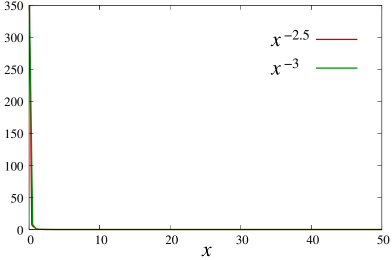
$$
\begin{aligned}
x &\rightarrow \ln k, \\
y = f(x) &\rightarrow \ln p(k), \\
\rightarrow f(x) &= \ln c - \gamma \cdot x
\end{aligned}
$$

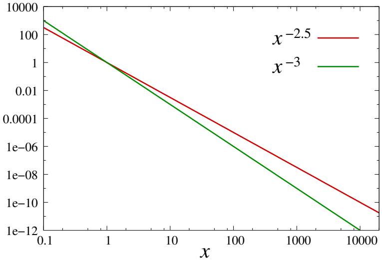
  - Thus, on log-log scale a power-law looks like a linear function, with a slope equal to γ.

Plotting power-laws

Illustration

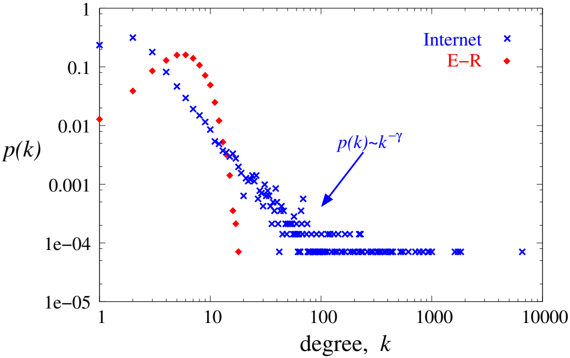






Power-law vs Poisson distribution

The $p(k)$ of the Internet at the level of AS, compared to the $p(k)$ of an Erdős–Rényi graph with the same number of nodes and links:



Scale-free networks

Scale-free networks

- A network is called **scale-free** if the tail of its degree distribution **decays as a power-law**,
$$p(k) \sim k^{-\gamma}.$$
- The exponent γ is often referred to as the node degree exponent or node degree decay exponent.

Degree distribution of real networks

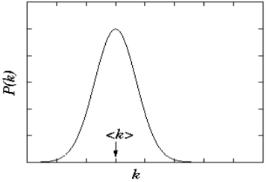
What do you expect?

R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

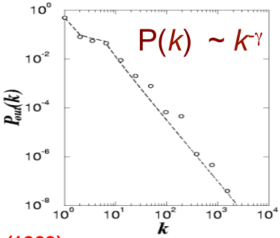
(from the slides of A.-L. Barabási)

Degree distribution of real networks

Exponential Network



Scale-free Network



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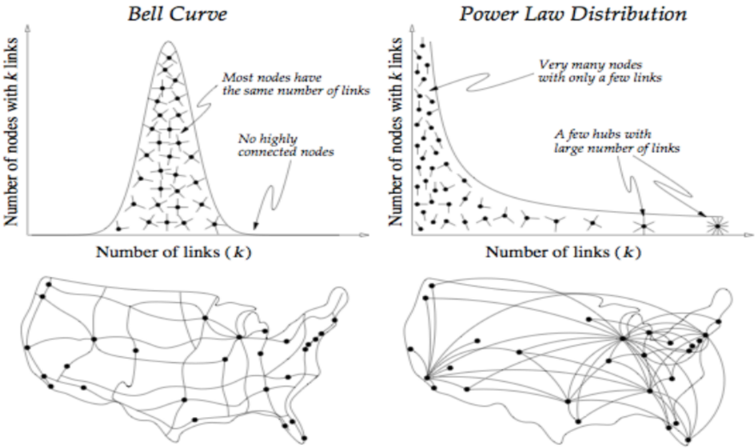
Fundamental difference between scale-free and Poisson-distribution!

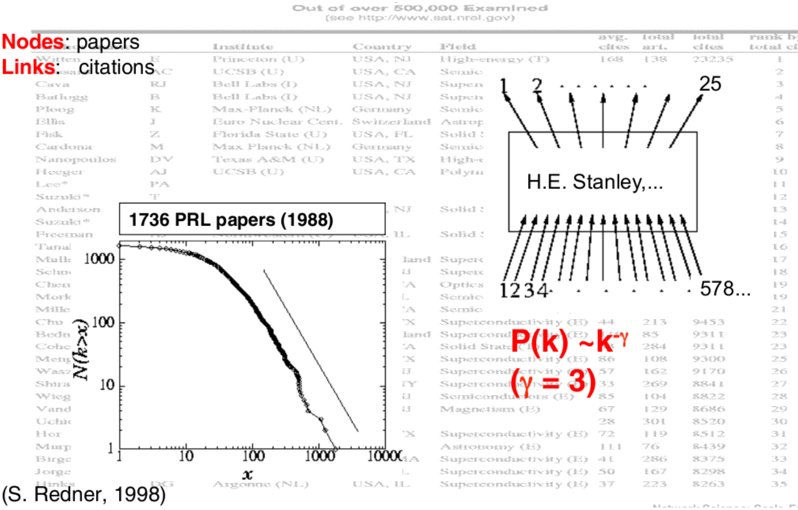


(from the slides of A.-L. Barabási)

Fundamental difference between scale-free and Poisson-distribution!

- The two distributions are strikingly different: **HUBS!**
  - $\langle k \rangle << k$.
  - Thus, no „typical” degree.
  - The degree distribution is very **inhomogeneous** and skewed.

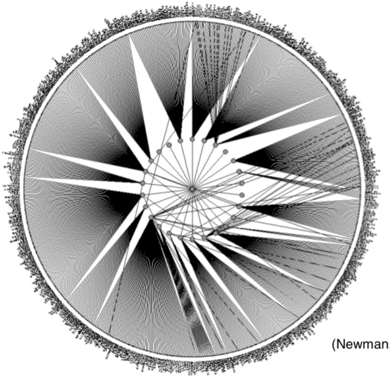
Scale-free $p(k)$ everywhere

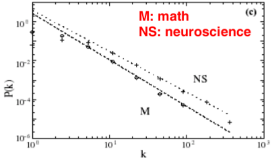


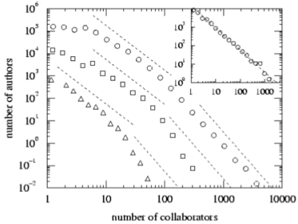
(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

Nodes: scientist (authors)
Links: joint publication







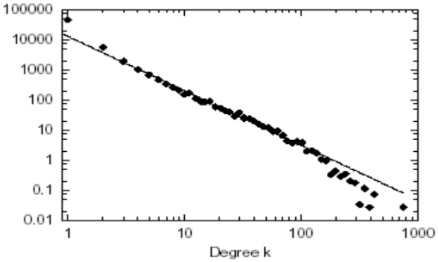
(Newman, 2000, Barabasi et al 2001)

(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

Nodes: online user
Links: email contact

Kiel University log files
112 days, N=59,912 nodes



(from the slides of A.-L. Barabási)

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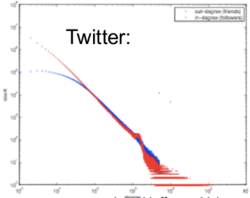
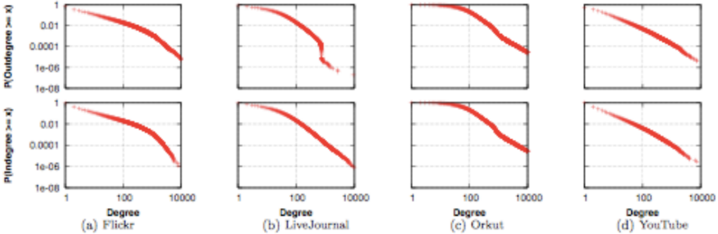


Jake Hoffman, Yahoo,

All distribtions show a fat-tail behavior:
there are orders of magnitude spread in the degrees

Alan Mislove, Measurement and Analysis of Online Social Networks



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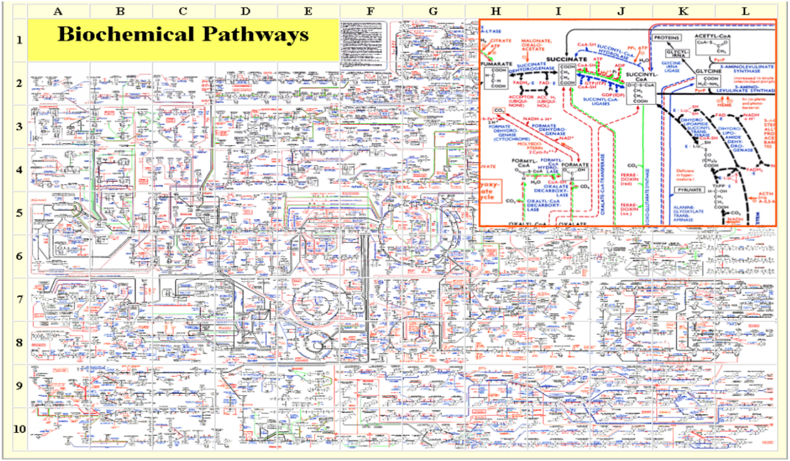

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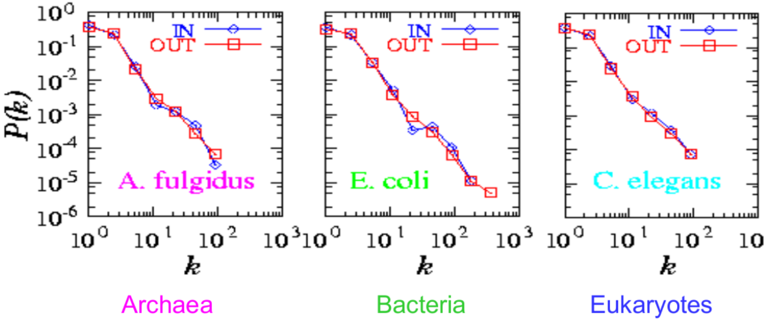


Organisms from all three domains of life are scale-free!

$$P_{in}(k) \approx k^{-2.2}$$

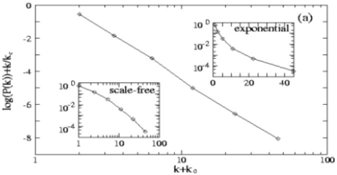
$$P_{out}(k) \approx k^{-2.2}$$

H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, *Nature*, 407 651 (2000)

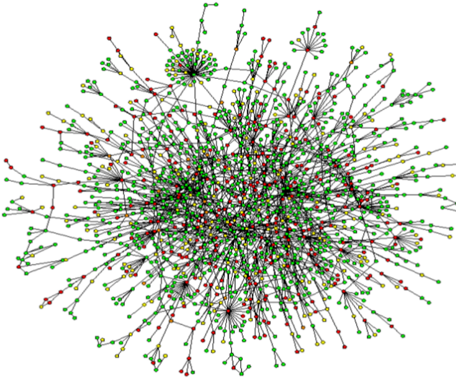
(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

Nodes: proteins
Links: physical interactions-binding



$$P(k) \sim (k + k_0)^{-\gamma} \exp\left(-\frac{k + k_0}{k_\tau}\right)$$

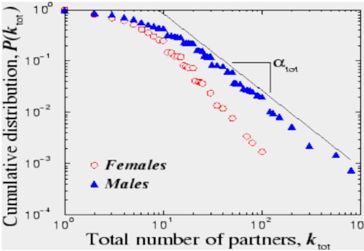


(from the slides of A.-L. Barabási)

Scale-free $p(k)$ everywhere

(from the slides of A.-L. Barabási)

Nodes: people (Females; Males)
Links: sexual relationships



4781 Swedes; 18-74;
59% response rate.

Liljeros et al. Nature 2001

Why „scale-free”?

Why are networks with a power-law degree distribution called **„scale-free”**?

Scaling

- What is **scaling**?

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  Suppose we are interested in two quantities, A and B. The quantity B is scaling

  - linearly with A if $B \propto A$,
  - quadratically with A if $B \propto A^2$,
  - etc.

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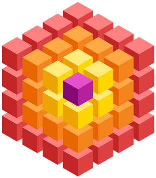
- A trivial example:
  How does the volume of a square scale with the length of its edges?

$$V = l^3$$

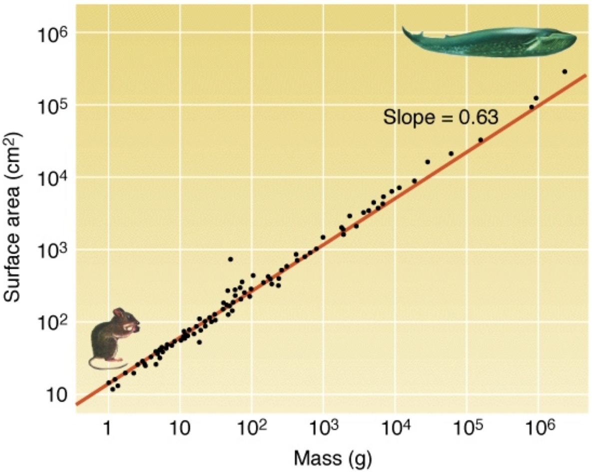


Scaling in nature

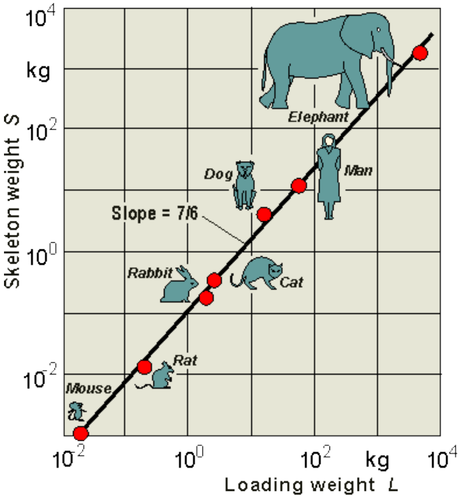
A prominent example of scaling laws in nature is provided by allometric scaling.

Allometric scaling

Body surface



Allometric scaling
Body parts



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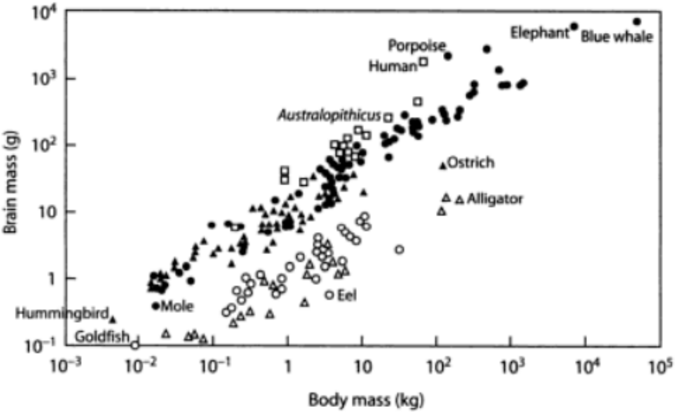
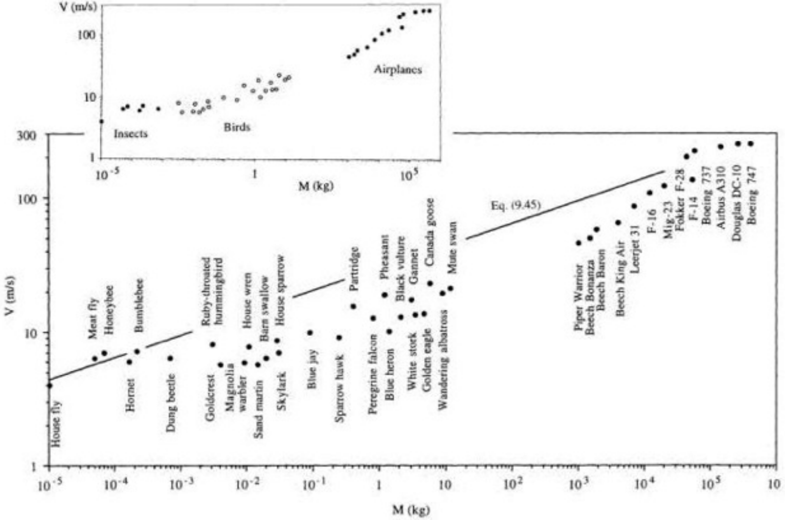
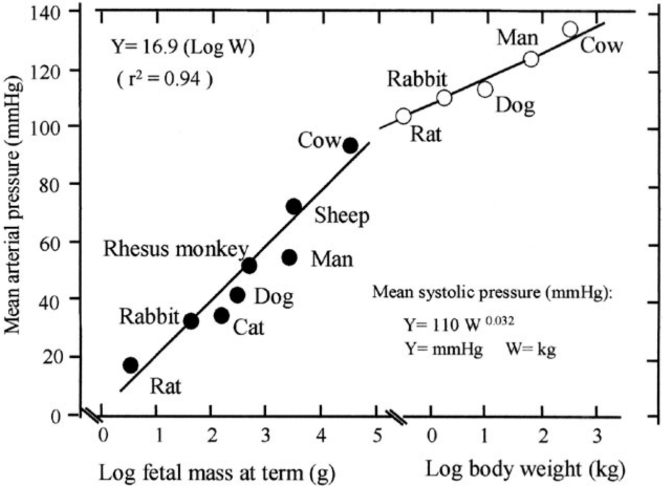


Figure 15. Brain size of 200 species of vertebrates plotted against body size on a log-log graph. Primates are open squares; other mammals are solid dots; birds are solid triangles; bony fishes are open circles; and reptiles are open triangles. (After H. J. Jerison, *The Evolution of the Brain and Intelligence*, 1973)

Allometric scaling
Velocity



Allometric scaling
Vascular system



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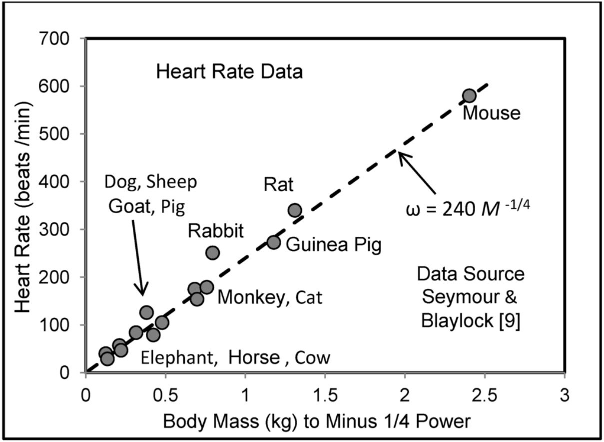
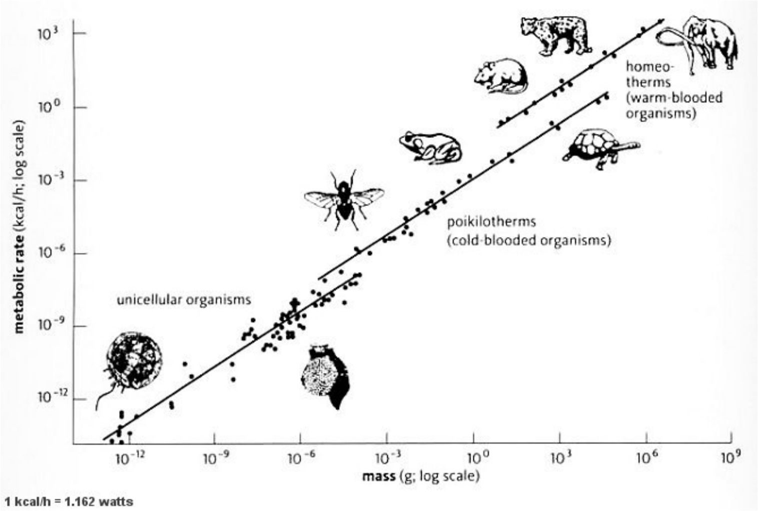


Allometric scaling

Metabolic rate



Scaling function

Scaling function

Definition: a function $F(x)$ is **scaling** if

$$F(a \cdot x) = g(a) \cdot F(x),$$

thus, changing the argument is equivalent to multiplying $F(x)$ with a constant.

Power-laws are scaling

Assume $F(x) = b \cdot x^{\gamma}$.

$$\rightarrow F(a \cdot x) = b \cdot (a \cdot x)^{\gamma} = b \cdot a^{\gamma} \cdot x^{\gamma} = a^{\gamma} \cdot b \cdot x^{\gamma} = a^{\gamma} \cdot F(x).$$

(In addition, it can be proven that all scaling functions are power-laws).

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Scaling distributions

What about the scaling of probability distributions?

Scaling distribution

A probability distribution is **scaling** if the $\rho(x)$ density function behaves as a power-law

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Vilfredo Pareto:

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- The distribution of wealth:

$$\rho(x) = \begin{cases} \frac{\alpha x_{\min}}{x^{\alpha+1}} & x > x_{\min} \\ 0 & x < x_{\min} \end{cases}$$

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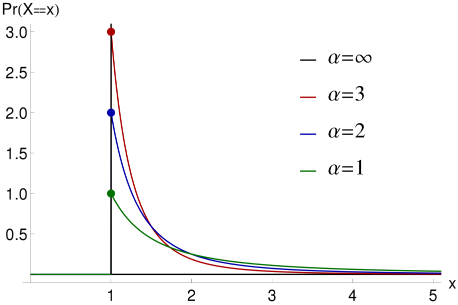

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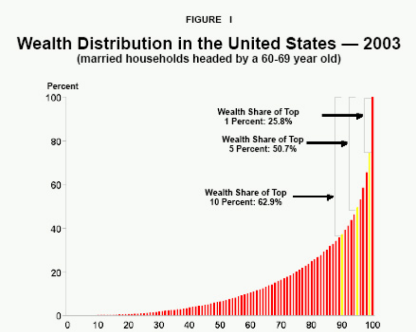

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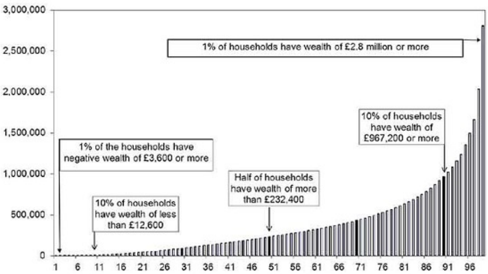
Figure 1: Distribution of total wealth between households, 2008-10, GB



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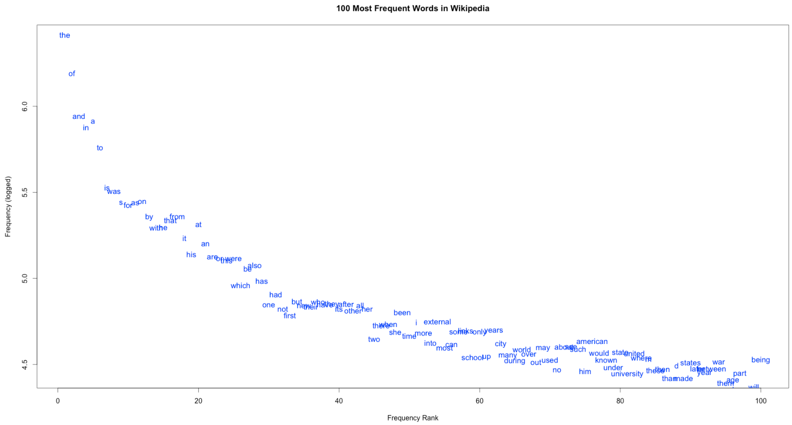
G. K. Zipf

$$p(q) = \frac{C}{q^\gamma}, \qquad C = \sum_{k=1}^{N} \frac{1}{q^\gamma}, \qquad (\gamma = 1)$$

Scaling distributions
Zipf’s law



Scaling distributions

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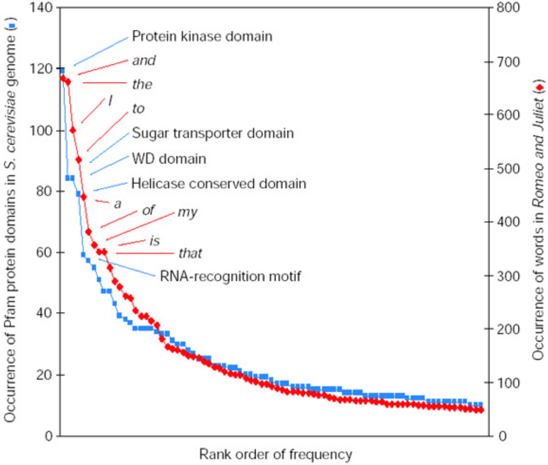

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Why are networks with a power-law degree distribution called **„scale-free”**?

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Why are networks with a power-law degree distribution called **„scale-free”**?

- power-law $p(k)$ → scaling distribution.
- no „typical degree” → no typical scale for the degrees.

Scale-free networks
Measured γ exponents

Scale-free degree distribution: $p(k) \sim k^{-\gamma}$.

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference
WWW	325, 729	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, Barabási 1999
WWW	4×10^7	7		2.38	2.1				Kumar *et al.* 1999
WWW	2×10^8	7.5	4, 000	2.72	2.1	16	8.85	7.61	Broder *et al.* 2000
WWW, site	260, 000				1.94				Huberman, Adamic 2000
Internet, domain$*$	3, 015 - 4, 389	3.42 - 3.76	30 – 40	2.1 - 2.2	2.1 - 2.2	4	6.3	5.2	Faloutsos 1999
Internet, router$*$	3, 888	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos 1999
Internet, router$*$	150, 000	2.66	60	2.4	2.4	11	12.8	7.47	Govindan 2000
Movie actors$*$	212, 250	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási, Albert 1999
Coauthors, SPIRES$*$	56, 627	173	1, 100	1.2	1.2	4	2.12	1.95	Newman 2001b,c
Coauthors, neuro.$*$	209, 293	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.* 2001
Coauthors, math$*$	70, 975	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.* 2001
Sexual contacts$*$	2810			3.4	3.4				Liljeros *et al.* 2001
Metabolic, E. coli	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.* 2000
Protein, S. cerev.$*$	1870	2.39		2.4	2.4				Mason *et al.* 2000
Ythan estuary$*$	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya, Solé 2000
Silwood park$*$	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya, Solé 2000
Citation	783, 339	8.57			3				Redner 1998
Phone-call	53×10^6	3.16		2.1	2.1				Aiello *et al.* 2000
Words, cooccurence$*$	460, 902	70.13		2.7	2.7				Cancho, Solé 2001
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- Thus, the continuous $p(k)$ is given by

$$p(k) = (\gamma - 1)\frac{k^{-\gamma}}{k_{\min}^{1-\gamma}}$$

Divergence...

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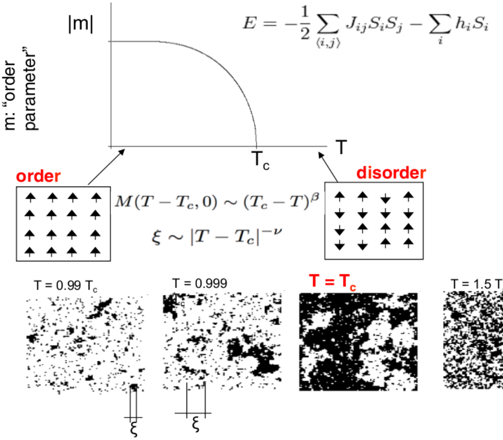
- E.g., ferromagnetic phase transition:



interacting elementary magnets (spins) sitting in a lattice.

Neighboring spins like to be point in the same direction

If the temperature T is high this attraction is not sufficiently strong– there is no net magnetization.

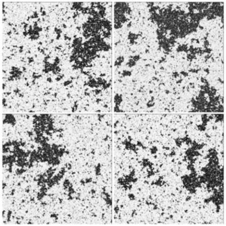
If the temperature is low, the ferromagnetic order sets in. There is a phase transition at T_c.

Divergence...

- At the critical point of the phase transition we observe **divergence** in correlation length, susceptibility, etc.

$$\xi \sim |T - T_c|^{-\nu}$$

At T = Tc: correlation length diverges



Fluctuations emerge at all scales: ***scale-free behavior***

- Phase transition: fluctuations at all scales

$\updownarrow$

- Networks: degrees at all scales.

https://www.youtube.com/watch?v=lQxD1PinDbs

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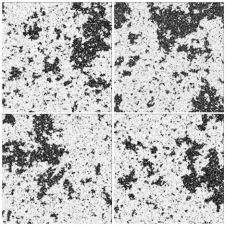

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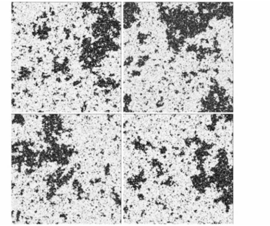

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YES!

Divergence of higher moments

- What is the m^{th} moment of $p(k)$?

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$\rightarrow$ For a fixed γ, all moments with $m > \gamma - 1$ diverge!

Scale-free networks
Measured γ exponents

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference
WWW	$325,729$	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, Barabási 1999
WWW	4×10^7	7		2.38	2.1				Kumar *et al.* 1999
WWW	2×10^8	7.5	$4,000$	2.72	2.1	16	8.85	7.61	Broder *et al.* 2000
WWW, site	$260,000$				1.94				Huberman, Adamic 2000
Internet, domain*	$3,015$ - $4,389$	3.42 - 3.76	$30-40$	2.1 - 2.2	2.1 - 2.2	4	6.3	5.2	Faloutsos 1999
Internet, router*	$3,888$	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos 1999
Internet, router*	$150,000$	2.66	60	2.4	2.4	11	12.8	7.47	Govindan 2000
Movie actors*	$212,250$	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási, Albert 1999
Coauthors, SPIRES*	$56,627$	173	$1,100$	1.2	1.2	4	2.12	1.95	Newman 2001b,c
Coauthors, neuro.*	$209,293$	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.* 2001
Coauthors, math*	$70,975$	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.* 2001
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.* 2001
Metabolic, E. coli	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.* 2000
Protein, S. cerev.*	1870	2.39		2.4	2.4				Mason *et al.* 2000
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya, Solé 2000
Silwood park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya, Solé 2000
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Most measured γ are smaller than 3.

$\rightarrow$ $\langle k^2 \rangle$ **diverges in the $N \rightarrow \infty$ limit!**

Divergence of the variance and σ

- The variance of the degree:

$$\mathrm{Var}(k) \equiv \left\langle (k - \langle k \rangle)^2 \right\rangle = \left\langle k^2 \right\rangle - \langle k \rangle^2 .$$

→ The variance is diverging as well!

- The standard deviation of the degree:

$$\sigma(k) \equiv \sqrt{\mathrm{Var}(k)} = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}.$$

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Divergence of the variance

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WWW: $\langle k \rangle = 7 \pm \infty$
Internet: $\langle k \rangle = 3.5 \pm \infty$
Coauthorship: $\langle k \rangle = 11.5 \pm \infty$
etc.

The $\langle k \rangle$ is not meaningful due to the large fluctuations!

Consequences of the scale-free $p(k)$

Summary of the consequences of the scale-free $p(k)$:

- we plot $p(k)$ on **log-log scale**
- **HUBS!**
- **divergent $\langle k^2 \rangle$!** (for $\gamma < 3$)

  $\rightarrow$ no „typical” degree,
  ($\rightarrow$ anomalous percolation),
  ($\rightarrow$ anomalous spreading)

Average distance in scale-free networks

- Do scale-free networks have the small-world property?

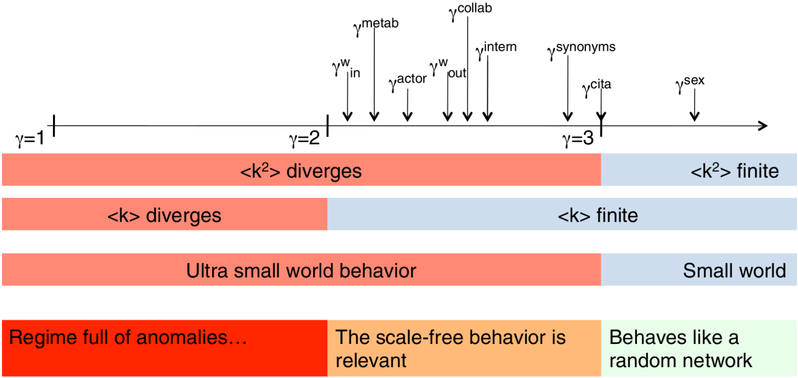
Average distance in scale-free networks

$$\langle l \rangle \sim \begin{cases} \text{const.} & \gamma \leq 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

$\gamma \leq 2$ and $2 < \gamma < 3$: Ultra Small World

$\gamma > 3$: Small World

Summary of the behavior of scale-free networks



(from the slides of A.-L. Barabási)