DMM Summary

Massimo Perego

Contents

1 Decision Problems

1.1 Fundamental definitions

A decision problem is defined by a 6-uple:

$$P = (X, \Omega, F, f, D, \Pi)$$

Where

- X is the feasible region, set of all alternatives
- Ω is the sample space, set of all possible scenarios
- F is the indicator space, set of all possible impacts
- $f : X \times \Omega \to F$ is the impact function, which associates each possible configuration $f(x, \omega)$ of the system to an impact
- D is the set of all decision-makers
- $\Pi : D \to 2^{F \times F}$ is the preference function, which associates to each decision maker a subset of impact pairs

The **alternatives** formally describe the events under the control of the decision-makers. The feasible region X includes all possible choices.

Alternatives are described as vectors of real numbers, each component of the alternative is a **decision variable**, an elementary quantity describing a specific aspect of the alternative. The core of the decision process is choosing an alternative with the best possible outcome (impact), the decision variables create the decision space (all the values that an alternative can assume), and solutions are point within that space that have to be evaluated by the decision process.

The **scenarios** formally describe the events out of the control of the decision-makers. The sample space Ω defines all such events.

The **impacts** model all aspects relevant to the decision, they are described quantitatively as vector of real numbers.

The **impact function** is a vectorial function which associates each configuration to an impact.

A **decision-maker** is whoever takes part in the decision.

The decision-maker takes part directly in the choice of alternative, holding the power and responsibility to decide, while **stakeholders** are all the subjects who don't actively participate in the decision but whose interests are affected by its outcome.

1.2 Preference Relation

For each decision maker, we need a relation between pair of impacts to determine which ones are preferable, $\Pi : D \to 2^{F \times F}$ associates each decision-maker to a subset of impact pairs.

A **weak preference** $f \preceq_d f'$ is when the decision-maker accepts the exchange of f for f'.

Two impacts are **indifferent** if the decision-maker considers both equally as satisfactory, i.e., is willing to exchange one for the other in both directions.

Two impacts are **incomparable** if the decision-maker is unable or unwilling to choose between them, rejecting the exchange in both directions.

In a decision problem, an **outperformer** $x \in X$ for $d \in D$ is an alternative which performs at least as good as any other $x' \in X$ in every scenario $\omega \in \Omega$

$$(f(x,\omega), f(x',\omega)) \in \Pi_d, \quad \forall \omega \in \Omega,\ \forall x' \in X$$

and is strictly better for at least one comparison/scenario

$$\exists x^\circ \in X \quad (f(x,\omega), f(x^\circ,\omega)) \in \Pi_d \wedge (f(x^\circ,\omega), f(x,\omega)) \notin \Pi$$

1.2.1 Properties

Considering impact set F and a binary relation Π, the main properties of said relation can be:

- **Reflexivity:** $\forall a \in F$, $(a,a) \in \Pi$, every impact is in relation with itself; to check for it, on the graph representation every impact has to have a self-loop
- **Transitivity:** $\forall a,b,c \in F$, if $(a,b) \in \Pi$ and $(b,c) \in \Pi$ then $(a,c) \in \Pi$; to check for it, on the graph representation follow every arc and check that every possible "triangle" is complete
- **Antisymmetry:** $\forall a,b \in F$, if $(a,b) \in \Pi$ and $(b,a) \in \Pi$ then $a = b$, two impacts are indifferent only if they're equal; to check for it, on

the graph representation there must be no impacts that point at each other

- **Completeness:** $\forall a, b \in F$, if $(a, b) \notin \Pi$ then $(b, a) \in \Pi$, every impact is in relation with every other, one way or another; to check for it, on the graph representation, from each node there must be an arc (either outgoing or incoming) connecting it to every other

1.2.2 Types of relations (orders)

Combining the properties, we can get different kinds of preferences:

- **Preorder:** reflexivity and transitivity. Guarantees that the set of nondominated alternatives is nonempty, even if not all alternatives are comparable, we can find at least one nondominated solution
- **Partial order:** reflexivity, transitivity and antisymmetry. This limits indifference, but still allows for incomparability, and thus not always leading to a definitive choice
- **Weak order:** reflexivity, transitivity and completeness. Guarantees that nondominated solutions exists and are all mutually indifferent, any of them can be chosen. Such orders admit representation by a value function (with ties), turning the decision problem into an optimization problem
- **Total order:** reflexivity, transitivity, antisymmetry and completeness. Provides a unique linear ranking of alternatives, there is always a unique best alternative

1.2.3 Derived relations

From the weak preference relation, one can derive:

- **Indifference relation:** Ind_Π

  $$(a, b), (b, a) \in \mathrm{Ind}_\Pi \Leftrightarrow (a, b) \in \Pi \wedge (b, a) \in \Pi$$

  To build it from the graph representation, add all self loops and each pair of arcs that point at each other

- **Strict preference relation:** Str_Π

  $$(a, b) \in \mathrm{Str}_\Pi \Leftrightarrow (a, b) \in \Pi \wedge (b, a) \notin \Pi$$

To build it from the graph representation, add all arcs which do not have an equal one in the opposite direction (the set difference of the last one w.r.t. Π)

- **Incomparability relation:** Inc_Π

$$(a, b), (b, a) \in \mathrm{Inc}_\Pi \Leftrightarrow (a, b) \notin \Pi \wedge (b, a) \notin \Pi$$

To build it from the graph representation, add all arcs which are not present in either direction in the graph

2 Basic Decision Models

2.1 Dominance relation

A solution **dominates** another when the impact of the former is preferable to the impact of the latter. When solving a decision problem, it looks natural to choose among the nondominated solutions.

On a strict preference graph, the nondominated solutions correspond to nodes with no ingoing arcs.

2.1.1 Decision making on weak orders

If a preference relation is a weak order

- the induced dominance is a weak order
- and the solution set is finite and nonempty, nondominated solutions exists and are all mutually indifferent

A **value function** $v : F \to \mathbb{R}$ associates each impact to a real value. A value function is **consistent** with preference relation Π when

$$f \preceq f' \Leftrightarrow v(f) \geq v(f'), \quad \forall f, f' \in F$$

If a preference relation Π admits a consistent value function, then Π is a weak order. The opposite is not always true. Consistent value functions are not unique, each one admits infinite equivalent ones that sort impacts in the same way, with different values (composition with any strictly increasing function).

2.1.2 Weak order preference models

Some ways to order preferences starting from a weak order.

Lexicographic order Order the alternatives w.r.t. the value of the first indicator (for some defined ordering of the indicators) and break ties with the subsequent one. This yields a total order.

The variant **with aspiration levels** introduces a "minimum requirement" ϵ_l, rejecting all alternatives with indicator f_l worse than ϵ_l (higher or lower depending on whether it's a benefit or cost). It then applies the lexicographic order on the restricted feasible region.

Utopia point Identify an ideal impact, with the best possible value for each indicator (optimize them independently or estimate such point) and evaluate all alternatives with the distance from the ideal impact.

Different definitions of distance yield different results. Some definitions:

- L_1 Manhattan distance

$$d(f, f') = \sum_{l \in P} |f_l - f'_l|$$

- L_2 Euclidean distance

$$d(f, f') = \sqrt{\sum_{l \in P} \left(f_l - f'_l\right)^2}$$

- L_∞ Chebyshev distance/maximum norm

$$d(f, f') = \max_{l \in P} |f_l - f'_l|$$

Borda count In the case of finite alternatives, they can be sorted by counting how many alternatives are worse than each one

$$B(f) = \left|\left\{f' \in F \mid f \preceq f'\right\}\right|$$

Then

$$f \preceq f' \Leftrightarrow B(f) > B(f')$$

2.2 MAUT

Multi Attribute Utility theory assumes the preference relation of the decision-maker is a weak order, admitting a consistent value function. We need to derive such value function from the preference relation.

2.2.1 Indifference curves

An **indifference curve** is a subset of the impact space $I \subseteq F$ of reciprocally indifferent impacts. By definition:

- The curves cover F
- Any two curves have empty intersection

- Weak order on impacts maps to total order on curves

Usually, **continuity** is assumed (they are mathematical objects and not a general set of points), and each indifference curve is expressed in the implicit form $u(f) = c$, each c identifying a curve.

2.2.2 Determining the utility function

The general process requires sampling and guessing a utility function, with consistency checks. This process can be complex and error prone.

Some properties which allow for easier estimates of $u(f)$:

- **Invertibility:** $u(f)$ can be solved w.r.t. each f_l
- **Monotony:** to compensate the variation of an indicator, the others must vary in a well-defined direction
- **Convexity** or **concavity:** the indifference curves compensate for the increase of an indicator by a certain amount with variations of the other ones that increase (or decrease) with the value of the first indicator

2.2.3 Additive utility function

A utility function is additive when it can be expressed as a sum of functions of the single indicators. If the utility function is additive, the problem to estimate it can be reduced to the estimation of p single-variable functions, simplifying the process.

Preferential independence A subset of indicators L is **preferentially independent** from the complementary subset $P \setminus L$ when given two impacts with identical values of $P \setminus L$ the preference relation does not depend on values in $P \setminus L$.

Mutual preferential independence A problem enjoys **mutual preferential independence** when every proper subset of indicator is independent from its complement.

Mutual preferential independence is *necessary for additivity*. Intuitively if subsets of indicators depend on each other there can't be a way to express the utility function in an additive way by summing such indicators (or value of the subsets).

For a decision problem with $p \geq 3$ indicators mutual preferential independence is a sufficient condition for additivity.

2.2.4 MRS

The **Marginal Rate of Substitution** λ_{12} between two indicators f_1 and f_2 represents how much of f_1 are we willing to *"give up"* for a unit of f_2; e.g., if we're willing to give 4 units of f_1 for a unit of f_2 then $\lambda_{12} = 1/4$.

It's the ratio of the partial derivatives of the utility function with respect to f_1 and f_2

$$\lambda_{12}(f) = \frac{\frac{\partial u}{\partial f_1}}{\frac{\partial u}{\partial f_2}}$$

A uniform MRS corresponds to a linear utility function $u(f) = w_1 f_1 + w_2 f_2$, and as such:

$$\lambda_{12}(f) = \frac{w_1}{w_2}$$

It represents the steepness of the indifference curve (slope).

Corresponding trade-off condition We denote as **corresponding trade-off condition the property**

$$\lambda_{12}(f_1', f_2')\lambda_{12}(f_1'', f_2'') = \lambda_{12}(f_1'', f_2')\lambda_{12}(f_1', f_2'')$$

Multiplying two MRSs on a diagonal equates to multiplying the values on the opposite diagonal.

A preference relation Π admits an additive utility function if and only if it enjoys both mutual preferential independence and the corresponding trade-off condition.

2.2.5 Building an additive utility function

Usually a utility function uses normalized pure values for utilities (pure numbers $\in [0, 1]$) and combines them with nonnegative weights whose total sum is 1.

Then we need to build a function for each one-dimensional impact, for example through the bisection method: get the decision maker to point extreme impacts and points in the middle of each "segment" created, until a function can be determined.

Determining the weights Identify a sufficient number of indifferent impacts and solve the resulting constraints on the weighs (linear equation system).

If indifference is imprecise, this can give wrong weights, so build a complete pairwise comparison for all indicators. The **pairwise comparison matrix** is the matrix reporting all rates of substitution between normalized utilities, expressing relative weight.

A correct pairwise comparison matrix enjoys:

- **Positivity:** each entry is positive, $\tilde{\lambda}_{lm} > 0$, $\forall l, m \in P$
- **Reciprocity:** $\tilde{\lambda}_{lm} = 1/\tilde{\lambda}_{ml}$, $\forall l, m \in P$
- **Consistency:** $\tilde{\lambda}_{ln} = \tilde{\lambda}_{lm} \cdot \tilde{\lambda}_{mn}$, $\forall l, m, n \in P$

To determine the weights from (correct) a matrix:

- Calculate the total of each column
- Divide each entry in the column for the total
- For each row calculate the mean of the resulting values, those means compose the weight vector

2.3 Mathematical Programming: *How to*

Mathematical programming assumes decision problems with a preference relation which admits a known consistent utility function, reducing the problem to classical optimization.

The general process for solving MP problems is:

1. Draw a graphical representation of the feasible region
2. Find nonregular points
3. Write the generalized Lagrangian function
4. Write the KKT conditions
5. Solve the system of conditions to reject candidate points, hoping that few remain (add nonregular points after this)
6. Evaluate the function in all the remaining points, choosing the optimum

Easy enough right? (It's not)

Let's use an example:

$$\begin{aligned} \min f(x) &= (x_1 - 1)^2 + x_2^2 \\ g_1(x) &= -x_1^2 - x_2^2 + 4 \leq 0 \\ g_2(x) &= x_1 - 3/2 \leq 0 \end{aligned}$$

Nonregular points All points in which the gradients of the active constraints are linearly independent are regular. Only active constraints must be considered, the whole feasible region is composed of regular points. A constraint is active when $= 0$.

Why are active constraints zero? The constraints are active only on the borders of the feasible region, since a optimal solution can only be found on that border, "pushing the boundaries" of the problem. Each point strictly inside the feasible region is next to another point, slightly better, slightly more towards the border.

Calculate the gradients of each constraint and check wether they can be zero or not. If the gradient can be zero and in such point:

- The constraint is inactive: business as usual
- The constraint is active: nonregular point, has to be added to the candidate set

In our example, the gradients are:

$$\begin{aligned} \nabla g_1(x) &= [-2x_1 \ -2x_2] \\ \nabla g_2(x) &= [1 \ 0] \end{aligned}$$

And:

- *The first one is 0 only in the origin, point in which the constraint is nonactive ($g_1(0,0) = 4$)*
- *The second one is never zero*

Then check points in which pairs of constraints are active, i.e., make a system in which both are zero (I think equality constraints always have to be added? I'll get back to you on that, not sure).

In our example:

$$\begin{cases} -x_1^2 - x_2^2 + 4 = 0 \\ x_1 - 3/2 = 0 \end{cases} \implies \begin{cases} x_1 = 3/2 \\ x_2^2 = 7/4 \end{cases}$$

From which we get the points

$$A = \left(\frac{3}{2}, \frac{\sqrt{7}}{2}\right), \quad B\left(\frac{3}{2}, -\frac{\sqrt{7}}{2}\right)$$

Check wether the gradients are linearly independent or not in the points found. To verify this the simplest way is to compose a 2×2 matrix with the values of the gradients considered in each point, if the determinant of such matrix is nonzero the gradients are linearly independent in the point considered.

In our example, $\nabla g_1(A) = \begin{bmatrix} -3 & -\sqrt{7} \end{bmatrix}$ *and* $\nabla g_2(A) = [1\ 0]$*, the resulting matrix being*

$$M = \begin{bmatrix} -3 & 1 \\ -\sqrt{7} & 0 \end{bmatrix}$$

Whose determinant is:

$$det(M) = (-3 \cdot 0) - (-\sqrt{7} \cdot 1) = \sqrt{7} \neq 0$$

So the gradients are linearly independent.

Then check points in which triples of constraints are active, similarly to earlier. You can guess how this goes on.

In our example there are no more constraints, but you would simply check that the gradients are linearly independent in the points resulting from the system given by $g_1(x) = 0$*,* $g_2(x) = 0$*,* $g_3(x) = 0$*.*

The aim of this phase is only to find nonregular points. Points with not linearly independent gradients are nonregular and as such have to be added to the candidate set after "sifting" with the KKT conditions.

Generalized Lagrangian function The generalized Lagrangian function is defined as:

$$\ell(x) = f(x) + \sum_{i=1}^{s} \lambda_i h_i(x) + \sum_{j=1}^{m} \mu_j g_j(x)$$

With λ_i free multipliers and $h_i(x)$ equality constraints (always active).

In our example, the function becomes:

$$\begin{aligned}\ell(x) &= f(x) + \mu_1 g_1(x) + \mu_2 g_2(x) \\ &= (x_1 - 1)^2 + x_2^2 + \mu_1 \left(-x_1^2 - x_2^2 + 4\right) + \mu_2(x_1 - 3/2)\end{aligned}$$

KKT Conditions The KKT conditions state that if a point is regular and locally minimal:

1. **Stationarity:** the partial derivatives of the Lagrangian function w.r.t. the x variables are equal to zero ($\partial\ell/\partial x_i = 0$)
2. The partial derivatives of the Lagrangian function w.r.t. the λ multipliers are equal to zero ($\partial\ell/\partial\lambda_i = h_j = 0$), that is, the equality constraints are respected (just add the equality constraints)
3. **Dual feasibility:** the product of the functions expressing the inequality constraints, times the corresponding multipliers are equal to zero ($\mu_k g_k = 0$)
4. **Primal feasibility:** all inequalities constraints are satisfied
5. **Complementary slackness:** all multipliers of the inequality constraints are nonnegative ($\mu_k \geq 0$)

These conditions allow to restrict the number of candidate points from *all regular and nonregular points* to *nonregular and a few regular.*

In our example, the conditions become

$$\begin{aligned}\partial\ell/\partial x_1 &= 2(x_1 - 1) - 2\mu_1 x_1 + \mu_2 = 0 \\ \partial\ell/\partial x_2 &= 2x_2 - 2\mu_1 x_2 = 0 \\ \mu_1 g_1 &= \mu_1(-x_1^2 - x_2^2 + 4) = 0 \\ \mu_2 g_2 &= \mu_2(x_1 - 3/2) = 0 \\ g_1 &= -x_1^2 - x_2^2 + 4 \leq 0 \\ g_2 &= x_1 - 3/2 \leq 0 \\ \mu_1 &\geq 0 \\ \mu_2 &\geq 0\end{aligned}$$

No equalities here, but just add them as constraint and remember to find the value of the free multiplier λ.

Solving conditions To solve the system without exhaustively exploring all possible cases one can use a search tree, whose nodes divide the feasible region in disjoint parts. To do that we build on the stricter conditions, i.e., the complementary slackness conditions (the products $\mu_k g_k = 0$, start with the simplest one). Given a constraint, we can distinguish two cases:

1. $\mu_k = 0$ and $g_k \leq 0$
2. $\mu_k > 0$ and $g_k = 0$

These assumption simplify the system, allowing it to be analyzed in an easier way. The process can be repeated on the resulting sub-problems, if necessary.

The idea is to restrict the possible solutions and find them a little at a time. Divide in "easy" sub-problems and all solutions to such problems are candidate points.

In our example, the constraint chosen is $\mu_2 g_2 = 0$. We now consider the case P^1, with $\mu_2 > 0$ and $g_2(x) = 0$, and thus $x_1 = 3/2$. The constraints now become:

$$\begin{aligned} 1 - 3\mu_1 + \mu_2 &= 0 \\ x_2(1 - \mu_1) &= 0 \\ \mu_1(7/4 - x_2^2) &= 0 \\ 7/4 - x_2^2 &= 0 \\ \mu_1 &\geq 0 \end{aligned}$$

We can now say that $\mu_1 = (\mu_2 + 1)/3 > 0$ and $x_2^2 = 7/4$. This yields candidate points:

$$A = \left(\frac{3}{2}, \frac{\sqrt{7}}{2}\right), \quad B\left(\frac{3}{2}, -\frac{\sqrt{7}}{2}\right)$$

Now consider P^2, with $\mu_2 = 0$ and $g_2 \leq 0$, the constraints become:

$$\begin{aligned}
2(x_1 - 1) - 2\mu_1 x_1 = 0 \quad &\implies x_1(1 - \mu_1) = 1 \implies \mu_1 \neq 1 \\
x_2(1 - \mu_1) = 0 \quad &\implies x_2 = 0 \\
\mu_1(-x_1^2 - x_2^2 + 4) = 0 \quad &\implies \mu_1(4 - x_1^2) = 0 \\
x_1^2 + x_2^2 \geq 4 \quad &\implies x_1^2 \geq 4 \\
x_1 \leq 3/2 & \\
\mu_1 \geq 0 & \\
\implies x_1 = -2 &
\end{aligned}$$

This yields candidate point:

$$C = (-2, 0)$$

Choose optimum We now want to evaluate the function in all the candidate points (remember to consider nonregular points) and choose the best solution.

In our example:

$$\begin{cases} f(A) = 2 \\ f(B) = 2 \\ f(C) = 9 \end{cases}$$

Which implies that both A and B are globally optimal points.

3 Models with complex preferences

The preference relation now is not a weak order.

We denote as **Paretian preference** (in the case of costs) the relation

$$\Pi = \left\{(f, f') \mid f_l \leq f'_l, \text{ for each } l \in \{1, \ldots, p\}\right\}$$

Which is a **partial order**.

We denote as **dominated solution** a solution which is not better than w.r.t. each indicator of another solution, and strictly worse in at least one indicator.

We denote as **Paretian solution** every solution such that no other solution dominates it. We call **Paretian region** the set of all Paretian solutions.

Paretian solutions are not preferable to all other solutions and are not all reciprocally indifferent. Thus we want to identify the whole Paretian region

3.1 Identifying the Paretian region

3.1.1 Applying the definition

In the finite case, the Paretian region can be found by applying the definition through pairwise comparisons.

This is exact but slow.

3.1.2 Inverse transformation method

If the solution can be graphically represented, compute the inverse function $\phi : F \to X$ of $f : X \to F$, build the image of X in F through f, find graphically the nondominated impacts (lower left quadrant empty), find a parametric way to describe such impacts, transform them back using the inverse function.

This is exact, but human intervention is required and is limited to 2 indicators.

The practical steps are:

1. Invert the functions describing the two indicators, i.e., system and describe x_1 and x_2 in terms of f_1 and f_2

2. Turn the constraints in term of fs, i.e., substitute x_1 and x_2 in the constraints with what obtained earlier
3. Draw the graph in terms of f_1 and f_2, with the constraints obtained
4. Find the set of points with an empty lower left quadrant
5. If the segment of points goes from A to B, "invert" the points, i.e., calculate $x(A)$ and $x(B)$, for x_1 and x_2

3.1.3 KKT conditions

KKT conditions can be extended to Paretian preference, by repeating the derivation with minor changes, obtaining a set that is usually larger than the Paretian region (finds an overestimate).

Not usable in discrete problems, as always.

3.1.4 Weighted sum method

Consists in building a linear combination of the indicators and optimizing it. The result is sufficient conditions for a point to be Paretian (underestimate of the region).

3.1.5 ϵ-constraint method

Replace all indicators but one with constraints that require the solution to respect a quality threshold and solve the auxiliary problem. The result is a necessary condition for a point to be Paretian.

It's needed to find parametrically how the Paretian region is described w.r.t. the variation of ϵ. It's just a constraint, so: when it varies, which solutions are feasible? Get the intervals for which the Paretian region doesn't change and optimize the non-replaced indicator for the feasible solutions.

This can be applied to any problem and provides an overestimate of the Paretian region, but it requires to consider all possible values of ϵ and find all globally optimal solutions, increasing the complexity of the problem.

3.2 Weak rationality methods

Decision-makers often can't estimate correctly, so we can fix the matrix or just *embrace* that the pairwise comparison matrix is incorrect and the normalized utilities are incorrect.

3.2.1 Reconstructing consistent matrices

A possible approach force consistency on the matrix is to find the positive, consistent and reciprocal matrix closest to the original one, for some definition of distance (different definitions of distance yield different results). The problem to solve is:

$$\min d\left(W, \tilde{\Lambda}\right)$$

$$W_{lm} = \frac{w_l}{w_m}$$

$$w_l > 0,\ l \in P, \quad \sum_{l \in P} w_l = 1$$

Where $\tilde{\Lambda}$ is the original matrix and W is the consistent one. W is composed by the ratio of the weights, which are the unknown variables (to find).

The minimum distance from a consistent matrix can be considered as a measure of inconsistency of the original one.

If he asks for the eigenvalue method I'm out, I ain't studying that.

3.2.2 AHP

The **Analytic Hierarchy Process** was introduced in 1980 based on the following criticisms:

1. The reconstruction of the single-variable normalized utility functions is subject to strong approximation errors
2. The estimation of the weights is subject to strong approximation errors when the number of attributes p is large
3. The various approximation errors combine in cascade

The method replaces absolute measures with relative ones, and quantitative ratios with qualitative scales, building a hierarchy of indicators to compare only conceptually similar quantities.

The preference among impacts is measured with an arbitrary **qualitative scale**, allowing to compare heterogeneous quantities, translating verbal judgments and building an evaluation matrix.

The decision-maker expresses preference for each impact for each pair of solutions, with a qualitative scale. The utility is on a relative qualitative scale.

Humans find it difficult to compare non-homogeneous things, so build an indicator tree and compare only siblings: leaves include elementary attributes, upper levels summarize them, getting progressively more general. Many small pairwise comparison matrices are built on the different levels.

The weights of the attributes are also determined starting from a qualitative scale, their importance is determined by the decision-maker via pairwise comparisons.

Then the weights are normalized within each group of children nodes and the tree structure allows to build the attribute weight vector level-by-level, from leaves to root. The weights vector must have sum 1, to normalize them is enough to make it so that their sum coincides with the weight of the father node (just multiply).

Rank reversal The main defect of the AHP is that the order of alternatives substantially depends on what alternatives are present, since evaluations are pairwise comparisons. Adding or removing alternatives can change the ranking.

To avoid rank reversal, a proposal is to fix a set of absolute levels for each indicator and make comparisons on levels, instead of alternatives. This allows for a open and long decision process, in which alternatives arrive gradually, but introduces further approximation on the values (very different values can be "flattened" and similar values can be made to look "distant").

3.2.3 ELECTRE methods

The ELECTRE methods start from a criticism of the assumption that the decision-maker is able to compare all pairs of impacts. The idea is to extend the Paretian preference definition with a concept called **outranking**: A could be preferable to B even if it's worse for some attributes, if not by too much.

We say that f **outranks** f' based on the threshold $\epsilon_l \geq 0$ when f is not worse than $f' - \epsilon_l$, $\forall l \in P$. Setting $\epsilon_l = 0$ yields the Paretian preference.

Now the preference means that *exchanging f and f' is not a clear loss.* This relation is reflexive, but is in general nontransitive, noncomplete and nonantisymmetric.

This definition can easily produce too weak or too rich relations (based on the value of ϵ_l), so it can be refined with other conditions: the final relation

will include only pairs that verify all conditions. The conditions take into account additional remarks, such as weights w_l on the relative importance of indicators.

Some conditions:

- **Satisfaction of comparability threshold:** impact f is not worse than f' by more than ϵ_l for all attributes
- **Concordance condition:** a subset of attributes of sufficient weight agree that f is not worse than f'

$$f \preceq_{s_c} f' \Leftrightarrow c_{ff'} = \sum_{l \in P : f_l \geq f'_l} w_l \geq \alpha_c, \quad \alpha_c \in [0,1]$$

- **Discordance condition:** no attributes reject with exceeding strength the statement that f is better than f':

$$f \preceq_{s_d} f' \Leftrightarrow d_{ff'} = \frac{\max_{l \in P} [\max(f'_l - f_l, 0)]}{\max_{l \in P} |f_l - f'_l|} \leq 1 - \alpha_d, \quad \alpha_d \in [0,1]$$

  To calculate it more easily, it's the ratio of the highest positive difference $f'_l - f_l$ (not absolute value, 0 at a minimum) with the highest $|f'_l - f_l|$, which is the biggest difference among any two indicators of the alternative

One can intersect the three definitions to refine the outranking relation. Parameters α_c and α_d are arbitrary, and may need to be tuned.

Concordance/discordance matrix It groups all concordance/discordance coefficients $c_{ff'}/d_{ff'}$. Diagonals are always 0 for discordance and 1 for concordance.

Kernel identification The outranking relation is used to filter solutions out. We denote as **kernel** the subset of alternatives obtained by the procedure:

1. Start with an empty kernel $K := \emptyset$
2. Add to the kernel the subset of all not strictly outranked solutions

$$K := K \cup \left\{ x \in X \mid \nexists x' \in X : x' \prec_S x \right\}$$

3. Remove from X all solutions outranked by a kernel solution

$$X := X \setminus \left\{ x \in X \mid \exists x' \in X : x' \prec_S x \right\}$$

4. Terminate if $X = K$, otherwise go back to 2

If the outranking graph contains circuits the procedure does not terminate.

Creation of a weak ordering A final phase can be added, in which additional criteria are introduced to sort kernel solutions:

- **Topological ordering:** it selects one of the total orders consistent with the original partial one. The forward ordering approach:
  - Start with an empty list
  - Find a solution not strictly outranked, append it to the list and remove it from the graph
  - Repeat the last step if the graph is nonempty, return the list otherwise

  The backward approach proceeds similarly, but with solutions which do not outrank any other and reverses the list at the end. The combined approach
  - Applies the forward and backward approach
  - Computes the Borda count on both lists
  - Sums the two Borda counts to obtain a weak order
- **Ordering with aggregated indices:** build for each impact a concordance and discordance aggregated index:
  - Concordance index: large when many weighty attributes prevail on the other ones

$$C_f = \sum_{g \in F} (c_{fg} - c_{gf}), \quad f \in F$$

  - Discordance index: decreases when the regret for a victory is small

$$D_f = \sum_{g \in F} (d_{fg} - d_{gf}), \quad f \in F$$

4 Models with multiple scenarios

Weak order preference, single decision-maker, but the scenario is not certain. Two possibilities:

- Decisions in *conditions of ignorance*: ω falls in Ω
- Decisions in *conditions of risk*: the probability of each ω is known

An alternative **strongly dominates** another when its impact is at least as good in all scenarios. An alternative **probabilistically dominates** another when for any threshold considered the probability that it has impacts not worse than the threshold is not worse than the other alternative.

Models of uncertainty There are two main ways to describe uncertain situations:

- **Scenario description**: Ω is a finite set, in which the scenarios are explicitly listed
$$\Omega = \left\{\omega^{(1)}, \ldots, \omega^{(|\Omega|)}\right\}$$
- **Interval description**: Ω is the Cartesian product of a finite number of real intervals on the s exogenous variables
$$\Omega = [\omega_1^{\min}, \omega_2^{\max}] \times \cdots \times [\omega_s^{\min}, \omega_s^{\max}]$$

These are not the only possible cases, but they are frequent. The choice depends on the problem

W.r.t. the possible scenarios of a decision problem, the terms

- Conditions of **uncertainty**: describe the whole category of decision problems in which the impact depends on external factor that cannot be predicted (the scenario)
- Conditions of **ignorance**: describe problems in which the only information known about the scenario ω is that it falls inside a set Ω
- Conditions of **risk**: describe problems in which, along with the set of scenarios Ω, a formalization of the probability of each scenario is known (a function $\pi : \Omega \to [0, 1]$, with the sum of all probabilities being 1)

4.1 Conditions of ignorance

We denote as **choice criterium** every definition of $\phi_\Omega(x)$ aimed to replace the impact $f(x, \omega)$.

4.1.1 Worst-case criterium

Assume for each solution the worst scenario, reducing the problem to

$$\min_{x \in X} \phi_{worst}(x) = \min_{x \in X} \max_{\omega \in \Omega} f(x, \omega)$$

It's an approach aimed at avoiding losses, even giving up opportunities.

4.1.2 Best-case criterium

Complementary to the last one, assume for each solution the best scenario possible. The problem becomes:

$$\min_{x \in X} \phi_{best}(x) = \min_{x \in X} \min_{\omega \in \Omega} f(x, \omega)$$

Believe in opportunities, *ignore danger*.

4.1.3 Hurwicz criterium

More balanced, use a convex combination of best and worst-case criterium. The problem becomes

$$\min_{x \in X} \phi_{Hurwicz}(x) = \min_{x \in X} \left[\rho \max_{\omega \in \Omega} f(x, \omega) + (1 - \rho) \min_{\omega \in \Omega} f(x, \omega) \right]$$

Where $\rho \in [0, 1]$ is the pessimism coefficient, as it weighs the worst impact, allowing to tune the weights of the scenarios.

Tuning ρ Get (maybe invent) a pair of reciprocally indifferent alternatives and impose equality of the corresponding $\phi_{Hurwicz}$ values, solving the corresponding equation in ρ.

The simplest way to do that is to get an alternative x and have the decision-maker indicate its certainty equivalent, a (generally fictitious) alternative with uniform impact on all scenarios, indifferent to x.

Sensitivity analysis If ranking is unclear and value of ρ imprecise, find for which values of ρ each solution is optimal.

4.1.4 Laplace criterium

Apply the same weight on all scenarios. This reduces the problem to

$$\min_{x \in X} \phi_{Laplace}(x) = \min_{x \in X} \frac{\sum_{\omega \in \Omega} f(x, \omega)}{|\Omega|}$$

Mean of the impacts on the scenarios, with a balanced approach that takes all scenarios into account.

4.1.5 Regret criterium

Evaluate the regret the decision-maker would feel if the decision taken were wrong, each solution should be compared to other ones, scenario by scenario. How can one measure the regret?

$$\rho(x, \omega) = f(x, \omega) - \min_{x' \in X} f(x', \omega)$$

It's the difference from the solution/scenario considered w.r.t. the best possible solution for that scenario. Then apply the worst-case criterium to this function. This reduces the problem to:

$$\min_{x \in X} \phi_{regret}(x) = \min_{x \in X} \max_{\omega \in \Omega} \rho(x, \omega) = \min_{x \in X} \max_{\omega \in \Omega} \left(f(x, \omega) - \min_{x' \in X} f(x', \omega) \right)$$

In practice:

- Determine the best alternative for each scenario
- For each alternative, evaluate the regret (subtract the value determined earlier) for each scenario
- Pick the maximum regret for each alternative
- Choose the minimum of the values obtained

It's a comparative approach, caring only about unnecessary losses.

4.1.6 Surplus criterium

Complimentary to the regret criterium: consider the minimum guaranteed surplus obtained w.r.t. the worst alternative and maximize it. Surplus is measured as

$$\sigma(x, \omega) = \max_{x' \in X} f(x', \omega) - f(x, \omega)$$

Then apply the worst-case criterium to the surplus function

$$\max_{x \in X} \phi_{surplus}(x) = \max_{x \in X} \min_{\omega \in \Omega} \sigma(x, \omega) = \max_{x \in X} \min_{\omega \in \Omega} \left(\max_{x' \in X} f(x', \omega) - f(x, \omega) \right)$$

In practice:

- Determine the worst alternative for each scenario
- For each alternative, evaluate the surplus (difference with the worst) for each scenario
- Pick the minimum surplus for each alternative
- Choose the maximum of the values obtained

It's a comparative approach, caring only about nonguaranteed gains.

4.1.7 Formal defects of the choice criteria

No criteria satisfies all properties which would be desirable. Let's present such properties.

All criteria presented respect these four properties:

- **Weak order:** the dominance relation is a weak order
- **Labeling independence:** the dominance relation is independent from names and order of alternatives and scenarios
- **Scale invariance:** scalar products of the impacts yield the same dominance relation, the result is independent from unit of measure and offset
- **Strong dominance:** the dominance relation includes the strong dominance relation

$$f(x, \omega) \leq f(x', \omega), \ \forall \omega \in \Omega \implies x \preceq x'$$

However, each criteria breaks at least one of these three:

- **Independence from irrelevant alternatives:** no rank reversal, adding/removing alternatives doesn't modify the other ranks. Regret and surplus violate this property, they are comparative approaches
- **Independence from scenario duplication:** the dominance relation doesn't change by adding scenarios with identical impacts. Laplace violates this property, the weight of the duplicated scenarios increases
- **Uniform variations of a scenario:** the dominance relation does not change if a scenario varies by a uniform amount for all alternatives. Worst-case, best-case and Hurwicz violate the property, changing a scenario can change the max/min across scenarios

It can be proven that these properties are mutually exclusive, no algorithm can satisfy all of them.

4.2 Conditions of risk

With the scenario set Ω, we now have a formalization of their probability. For each solution, the associated impact is a random variable depending on the scenario.

The main approach consists in reducing the problem to the optimization of an auxiliary function which removes dependency on the scenario.

4.2.1 Definitions of probability

Probability is a debated concept, with more than one definition.

Classical definition Probability is the ratio between the number of elementary cases that form a scenario and the total number of possible elementary cases.

$$\pi(\omega) = \frac{n(\omega)}{n(\Omega)}$$

This assumes a finite number of elementary cases with same probability, but what is an elementary case? Why do they have equal probability?

Frequentist definition Probability of a scenario is the limit to which its relative frequency tends to as the number of observation increases

$$\pi(\omega) = \lim_{n \to +\infty} \frac{n(\omega)}{n}$$

This requires empirical information of good quality and a large quantity of observation, also the future behavior must be similar to the past.

Subjective definition Probability is the price one would pay to receive 1 if the scenario happens, 0 if it doesn't. This depends on personal opinion.

Axiomatic definition Probability is any function that respects the axioms:

- Being restricted in $[0, 1]$ for all $\omega \in \Omega$
- Summing to 1 over all $\omega \in \Omega$
- Being additive over sets of disjoint scenarios

The theory that derives is perfectly consistent, but does not indicate how to obtain the value practically.

4.2.2 Expected value criterium

Sums the impacts of a solution over all scenarios with the convex combination of impacts with probabilities

$$\phi_{EV} = E\left[f(x, \omega)\right] = \sum_{\omega \in \Omega} \pi(\omega) f(x, \omega)$$

It's just using the probability as weight for each impact.

Formal defects This criterium can lead to unrealistic consequences. It has some strong defects:

- **Actual preferences inconsistent with expected values:** different combinations of impacts and probabilities with same EV should be reciprocally indifferent, while they often are not

- **Extreme values of probability and impact have paradoxical effects:** combining small probabilities with large impacts is problematic, it can lead to infinite (very large) EVs for very small probabilities, not reflecting reality. This problem can be reduced by considering *utility*, which scales logarithmically with gain, but is solved only when the utility function is upper bounded

4.2.3 Stochastic utility theory

We want to build a choice criterium that satisfies certain desired properties (axioms).

A **finite simple lottery** is a pair of functions $(f(\omega), \pi(\omega))$ where $f(\omega)$ is a random variable on a finite sample space and $\pi(\omega)$ is a probability function on the sample space.

A **degenerate lottery** has a single deterministic scenario (probability 1).

A **compound lottery** is a lottery whose impacts are other lotteries (possibly degenerate). We call $L_{F,\Omega}$ the set of all possible lotteries, simple or compound, on F and Ω.

In a decision problem, each alternative corresponds to a lottery (different results with the associated probability).

A **preference relation between lotteries** is a binary relation on the lottery set: $\Pi \subset 2^{L_{F,\Omega} \times L_{F,\Omega}}$. A preference relation between lotteries Π admits a **consistent stochastic utility function** $u : L_{F,\Omega} \to \mathbb{R}$ when, for every pair of lotteries ℓ and ℓ', the utility of the preferred one exceeds the utility of the other one

$$\ell \preceq \ell' \Leftrightarrow u(\ell) \geq u(\ell')$$

Fundamental axioms The properties required for a rational preference between lotteries, or axioms of stochastic utility, are:

1. **Weak ordering:** the preference relation Π between lotteries is a weak order
2. **Monotony:** lotteries that assign larger probabilities to better impacts/lotteries are preferable
3. **Continuity:** any intermediate impact between two lotteries admits an equivalent compound lottery with the two given ones as outcomes, composing them with a suitable probability value

4. **Independence** (or **substitution**)**:** the preference between two lotteries does not change by combining them with the same lottery with the same probability

5. **Reduction:** any compound lottery is indifferent to the simple lottery with the same final impacts and probabilities given by the laws of conditional and total probabilities; the lottery structure is not relevant, impacts and probabilities are

Von Neumann-Morgenster stochastic utility theorem Given a set of impacts F not all reciprocally indifferent, a sample space Ω and a preference relation Π between lotteries on F and Ω that respects the five axioms, there exists one and only one utility function $u : L_{F,\Omega} \to [0, 1]$ consistent with Π and normalized so as to have value 0 in the worst impact and 1 in the best one.

This can be proved, and means that it's always possible to build a consistent stochastic utility function to compare lotteries, under the five axioms.

4.2.4 Risk aversion and risk propensity

Risk profile is the profile of the stochastic utility function on the degenerate lotteries ℓ_f as impact f varies in F.

The shape of the utility function $u(f)$ for all $f \in F$ determines $u(\ell(f, \pi))$ for all $\ell \in L$, combining $u(f)$ and $\pi(\omega)$ and showing the attitude of the decision-maker towards risk.

Comparing the risk profile with the segment given by connecting the degenerate lotteries for worst and best impacts, there are three relevant cases:

- **Convex case:** risk profile above the segment, lottery preferred to the deterministic impact, **risk-prone** decision-maker
- **Linear case:** risk profile on the segment, lottery and deterministic impact are indifferent, **risk-neutral** decision makers, confirming the EV criterium
- **Concave case:** risk profile below the segments, deterministic impact preferred to the lottery, **risk-averse** decision-maker

Given a lottery ℓ we denote as **certainty equivalent** the deterministic impact f_ℓ equivalent to the lottery, and **risk premium** the difference between

the EV of the lottery and its certainty equivalent.

$$RP(\ell) = \phi_{EV}(\ell) - CE(\ell) = E[f(\ell, \omega)] - f(u(\ell))$$

The risk premium measures the additional utility needed for the decision-maker to accept a lottery instead of the EV, therefore it's positive/zero/negative for risk-averse/neutral/prone decision-makers.

4.3 Decision theory

Decisions are now taken in stages and part of the scenario unravels before part of the decision is taken.

4.3.1 Decision tree

A tree representation is the most common one. It introduces a hierarchical structure on decision and exogenous variables: $2^{\max} + 1$ levels, chronologically ordered: even levels represent the decision-maker's choices, odd levels represent scenario elements fixing, leaves represent final configurations.

The arcs outgoing from a node are possible values of $x^{(t)}$ or $\omega^{(t)}$ (depending on the type of node).

To solve the problem **backward induction** algorithm is used: visit the tree from leaves to root, assigning a value to the current node based on the value of its children:

- Odd levels, below there's a decision node, apply a criterium ϕ and assign to the node the value of ϕ; you want to find out the "value" given by the possible scenarios
- Even levels, below there's a scenario node, choose the best alternative and mark the corresponding arc (a rational decision-maker always takes the best choice)

In the end the marked arcs provide the optimal strategy.

Scenarios conditioned by decisions The state of nature can be influenced by the decision variables, turning probabilities from $\pi(\omega)$ to the conditional value $\pi(\omega|x)$.

4.3.2 Random experiments

Sometimes, the estimate of the probabilities can be refined through a **random experiment**. In general, given an outcome of the experiment, the probabilities change.

They can be incorporated in the decision tree by adding two levels upstream of the basic decision:

- Decide wether to make the experiment or not
- Outcome of the experiment ω'; deterministic if the experiment is not made
- The main decision takes place, solution $x \in X$ for the given problem
- The uncertain scenario $\omega \in \Omega$ has probability $\pi(\omega|\omega')$ conditioned by the outcome of the experiment, and an impact that includes the cost of the experiment

We denote as **information value** the difference between the utility gained performing the experiment and the utility gained not performing it.

Probability computation for the decision tree Usually, we know $\pi(\omega)$ and $\pi(\omega'|\omega)$, but on the tree we need $\pi(\omega')$ and $\pi(\omega|\omega')$. To obtain them from available data

$$\pi(\omega|\omega') = \frac{\pi(\omega'|\omega)\pi(\omega)}{\sum_{\omega\in\Omega}\pi(\omega'|\omega)\pi(\omega)}$$

and the total probabilities can be obtained starting from the conditional probabilities and the probabilities of the scenarios.

$$\pi(\omega') = \sum_{\omega\in\Omega}\pi(\omega',\omega) = \sum_{\omega\in\Omega}\pi(\omega'|\omega)\pi(\omega)$$

Example, if given a decision problem with three alternatives and two scenarios:

$f(x,\omega)$	ω_1	ω_2
x_1	40	40
x_2	20	80
x_3	70	10

$\pi(\omega)$	ω_1	ω_2
	0.6	0.4

$\pi(y\|\omega)$	ω_1	ω_2
y_1	5/6	1/4
y_2	1/6	3/4

The EV without the experiment is:

$$EV(x_1) = f(x_1, \omega_1)\pi(\omega_1) + f(x_1, \omega_2)\pi(\omega_2) = 40, \quad EV(x_2) = 44, \quad EV(x_3) = 46$$

the best being x_3 with 46.

Then find the probability of each outcome of the random experiment:

$$\pi(y_1) = \pi(y_1|\omega_1)\pi(\omega_1) + \pi(y_1|\omega_2)\pi(\omega_2) = 0.6, \quad \pi(y_2) = 0.4$$

Then find the probability of each scenario after each experiment

$$\pi(\omega_1|y_1) = \frac{\pi(\omega_1)\pi(y_1|\omega_1)}{\pi(y_1)} = \frac{5}{6}, \quad \pi(\omega_2|y_1) = \frac{1}{6}$$

$$\pi(\omega_1|y_2) = \frac{1}{4}, \quad \pi(\omega_2|y_2) = \frac{3}{4}$$

The expected values with the experiments:

$$EV(x_1|y_1) = f(x_1, \omega_1)\pi(\omega_1|y_1) + f(x_1, \omega_2)\pi(\omega_2|y_1) = 40,$$

$$EV(x_2|y_1) = 30, \quad EV(x_3|y_1) = 60$$

$$EV(x_1|y_2) = 40, \quad EV(x_2|y_2) = 65, \quad EV(x_3|y_2) = 25$$

For y_1 the best choice is x_3 with 60, for y_2 the best choice is x_2 with 65.

Then the total expected value is:

$$EV_i = EV(x_3|y_1) \cdot \pi(y_1) + EV(x_2|y_2) \cdot \pi(y_2) = 62$$

The information value is then:

$$V = EV_i - EV(x_3) = 16$$

5 Models with multiple decision makers

5.1 Game theory

Game theory studies the situation in which multiple decision-makers have each their own variables, set independently from the others.

Some terminology:

- A decision problem is called a **game**
- A decision-maker is called a **player**
- An impact is called a **payoff**
- A solution is called a **strategy profile**

A **pure strategy** is a deterministic choice a player can make in a game; a basic alternative, setting the subvector of variables for the player considered. A **mixed strategy** is a probability distribution over the set of pure strategies.

Game can be classified from different points of view:

- W.r.t. the **relation between players**
  - Noncooperative: each player is independent
  - Cooperative: players can agree to share payoffs
- W.r.t. the **information on the data**:
  - Complete: all players know the whole of X and f
  - Incomplete: each player d knows only $X^{(d)}$ and $f^{(d)}$
- W.r.t. the **information on the moves**:
  - Perfect: all players know all past moves
  - Imperfect: player d only knows his own past moves

5.1.1 Game representations

There are two main representations of games:

- **Extended form:** the game is represented as a tree, nodes are game states, a turn is $|D|$ consecutive levels, all nodes on a level are associ-

ated with a player, levels are in chronological order, leaves are payoffs and outgoing arcs represent possible moves for the current player

- **Strategic form:** a strategy indicates the move a player should make in each possible state and the game is represented as a matrix in which rows represent strategies for the first player and columns strategies for the second player, the matrix entries are associated with payoffs

5.1.2 Dominance between strategies

Given two strategies, one dominates the other when it yields a better impact for any possible behavior of the other player. Assuming rational players, a dominated strategy will never be chosen and can be removed.

Worst-case strategy If a game requires simultaneous moves (backward induction is ruled out) and it does not reduce to a single nondominated strategy profile, we can calculate the minimum payoff that can be obtained by a player in the worst case by treating other players as scenarios, i.e., the best possible guarantee on the performance of the player, this is called **value of the game for the player**,

5.1.3 Equilibrium

We denote a strategy profile as an **equilibrium point** or **Nash equilibrium** when the strategy profile can't yield a better payoff, i.e., moving away from that point ends up damaging the player.

Determining Nash equilibrium A way to determine equilibria is the **best response method**:

- Scan all players
- For each player d, scan all strategies of the other players and mark the best strategy for player d
- A strategy profile marked for all players is an equilibrium

On a matrix, mark the best row payoff in each column and vice versa.

5.2 Zero-sum games

A zero-sum game is a game in which the overall utility of the game is zero for every single strategy profile. Changing the scale does not modify this

property, so any game with uniform sum can be considered zero-sum.

With two-players, the win of one equals the loss of the other so it's redundant to specify. Definitions of domination and equilibrium need to be adapted to the new representation: row player wants to maximize gain, while column player need to minimize losses.

To find equilibria, look for saddle points in the matrix: mark the maximum in each column and minimum in each row, entries with two marks are equilibria.

Value of the game It's interesting to note that, in a two-player zero-sum game, if $u^{(r)}$ and $u^{(c)}$ are the values of the game for row and column player respectively, then $u^{(r)} \leq u^{(c)}$, and $u^{(r)} = u^{(c)}$ if and only if the game has at least one Nash equilibrium. The guaranteed gain of the row player is bounded by the guaranteed loss of the column player.

5.2.1 Mixed strategies

A mixed strategy $\xi^{(d)}$ for player $d \in D$ is a probability vector defined on $X^{(d)}$ which represents the probability with which d chooses the basic strategy $x^{(d)}$ (alternatively, the frequency with which d chooses $x^{(d)}$ in a repeated game).

Allowing mixed strategies turns the payoffs into random variables, whose EV has to be maximized.

Zero-sum games can remain unsolved in pure strategies, but they **always admit a solution in mixed strategy**.

Minimax theorem In a two-player game, the worst case for any strategy of a player corresponds to one of the pure strategies of the adversary. This allows to assume that the adversary will behave in a deterministic way, using the most damaging pure strategy.

The minimax theorem states that for any two-player zero-sum game there is at least one mixed strategy in which the guaranteed expected gain of the row player coincides with the guaranteed expected loss of the column player.

5.3 Symmetric games

A symmetric game is a game in which all players have the same strategies and, if they exchange them, they correspondingly exchange payoffs. Each player is perfectly interchangeable. This way, the results of the strategies do not depend on the players who apply them.

5.3.1 Taxonomy of two-person two-strategy symmetric games

Classification based on Nash equilibria Considering all payoffs to be different, the general payoff matrix is

	1	2
1	(f_{11}, f_{11})	(f_{12}, f_{21})
2	(f_{21}, f_{12})	(f_{22}, f_{22})

There are only four possible situations:

1. $f_{11} > f_{21}$ and $f_{12} > f_{22}$: strategy 1 dominates for both players, equilibrium in $(1, 1)$
2. $f_{11} > f_{21}$ and $f_{12} < f_{22}$: no dominated strategies, equilibria on $(1, 1)$ and $(2, 2)$
3. $f_{11} < f_{21}$ and $f_{12} > f_{22}$: no dominated strategy, equilibria on $(1, 2)$ and $(2, 1)$
4. $f_{11} < f_{21}$ and $f_{12} < f_{22}$: strategy 2 dominates for both players, equilibrium on $(2, 2)$

Two-person and two-strategy symmetric games have always at least one equilibrium, this is generally not true for asymmetric games, or symmetric ones with more than two strategies.

Classifications based on the order of the payoffs More detailed, classification based on the relative order of the four payoffs. There are as many cases as permutations of the values (12, conventionally $f_{11} > f_{22}$, with no loss of generality).

5.3.2 Games classes

The ideal marriage Corresponds to order

$$f_{11} > f_{12} > f_{21} > f_{22}$$

The strategies are conventionally denoted as Cooperate C and Not cooperate NC, since:

- Mutual cooperation pays more than free-riding
- Free-riding pays more than being exploited
- Being exploited pays more than mutual egoism

Ideal case of cooperation, which dominates NC, and there is one equilibrium in (C, C). The worst-case criterium leads both players to the equilibrium, providing the best payoff.

The stag hunt Corresponds to order

$$f_{11} > f_{21} > f_{22} > f_{12}$$

Two hunters can:

- Cooperate and catch a stag
- One of them can defect and catch a hare, while the other maybe gets the stag
- Both can defect and catch a hare

Under these conditions, no strategy is dominated and there are two equilibria in (C, C) and (NC, NC). The worst-case criterium leads to the NC equilibrium, but the best payoff comes with C, both strategies are rational.

Pure coordination games Corresponds to orders

$$f_{11} > f_{22} > f_{12} > f_{21} \quad \text{and} \quad f_{11} > f_{22} > f_{21} > f_{12}$$

Including the case in which $f_{12} = f_{21}$.

Under these conditions no strategy is dominated, there are two equilibria in $(1, 1)$ and $(2, 2)$ and they are nearly equivalent. This describes situations in which the best results are obtained by using the same strategy, while asymmetric strategies are damaging for both.

The chicken race Corresponds to order

$$f_{21} > f_{11} > f_{12} > f_{22}$$

Two cars drive towards each other: who swerves loses. The largest payoff comes at a risk (of the lowest one) and cannot be obtained by both players.

Under these conditions no strategy is dominated and there are two equilibria in (C, NC) and (NC, C). There is no way to know a priori which one will be chosen.

Battle of the sexes Corresponds to orders

$$f_{12} > f_{21} > f_{11} > f_{22} \quad \text{and} \quad f_{21} > f_{12} > f_{11} > f_{22}$$

Two fiances want to go out but can't communicate: one would like to go to A the other to B, but both would rather be together than alone. Similar to the chicken race, but the payout is good for both.

Under these conditions no strategy is dominated and there are two equilibria in $(1, 2)$ and $(2, 1)$. No way to know a priori which equilibrium will be chosen.

Prisoner's dilemma Corresponds to order

$$f_{21} > f_{11} > f_{12} > f_{22}$$

Two gangsters are arrested: confess for less time? If one snitches, he goes free and the other goes away for a long time, if no one confesses short sentence for both, if both confess intermediate sentence for both.

Under these conditions cooperation is dominated by noncooperation and there is one equilibrium in (NC, NC). The worst-case criterium leads both player to the equilibrium (bad for both).

5.3.3 Finite games and mixed strategies

Nash extended Von Neumann and Morgenstern's mixed strategy theory from zero-sum games to any game with finite sets of players and strategies.

Every finite game admits at least an equilibrium in mixed strategies. The number of equilibria can grow exponentially with players/strategy.

5.4 Group decision-making

Multiple decision-makers, but each one does not fix independently the values of the decision variables, all decision-makers must agree on the decision before taking it.

The problem can be reduced to: *how to aggregate the individual preferences into a group preference?* After that, the problem becomes equivalent to that of a single decision-maker.

Social welfare function The problem is to derive a group preference from a finite set of weak orders, one for each individual. The function that does that is called **social welfare function**. It receives the preferences of all individuals and returns a preference relation for the whole group.

5.4.1 Condorcet method

Also known as simple majority method, is based on the definition:

$$x \preceq_D x' \Leftrightarrow \left|\left\{d \in D \mid x \preceq_d x'\right\}\right| \geq \left|\left\{d \in D \mid x' \preceq_d x\right\}\right|$$

The alternative preferred by more individuals is preferred by the group. Indifferent individuals are counted on both sides.

It's simple, but can lead to a circuit of strict preference: it does not guarantee transitivity. This means that the method can fail to provide a solution preferable to all other ones.

5.4.2 Borda method

Based on the auxiliary definition of Borda count

$$B_d(x) = \left|\left\{x' \in X \mid x \preceq_d x'\right\}\right|$$

Then aggregated

$$B_D(x) = \sum_{d \in D} B_d(x)$$

The group preference is then

$$x \preceq_D x' \Leftrightarrow B_D(x) \geq B_D(x')$$

It's a weak order by construction and always allows to build a welfare function, but it also suffers from rank reversal (X appears in the definition), allowing it to be manipulated.

5.4.3 Plurality system

A function is built based on the number of individuals that prefer each alternative to all other ones

$$V_D(x) = \left|\left\{d \in D \mid x \preceq_d x',\ \forall x' \in X\right\}\right|$$

Then the group preference

$$x \preceq_D x' \Leftrightarrow V_D(x) \geq V_D(x')$$

This guarantees a weak order, but suffers from rank reversal and allows compact minorities to prevail on disjunted majorities, leading to choices abhorred by most; this happens because only the first position in the individual weak orders is considered.

5.4.4 Lexicographic method

A total order on the individual is imposed

$$d_1 \prec \ldots \prec d_{|D|}$$

And to each pair of preference the first strict preference existing is applied

$$x \preceq_D x' \Leftrightarrow \exists d \in D : x \preceq_d x' \text{ and } x \sim_{d'} x',\ \forall d' < d$$

It's a hierarchy of individuals: the first one with a preference rules, it's an absolute monarchy.

It always provides a weak order and doesn't suffer from rank reversal, but it's not democratic, easily unstable and inefficient (people on the lower levels have little incentives to contribute).

5.4.5 Axiomatic approach

The axiomatic approach lists the desired properties and tries to design a function that satisfies them by construction, which has been proved impossible.

A **preference profile** $\Pi(X)$ is any vector of $|D|$ weak orders on X.

Given a solution pair $x, y \in X$ and two profiles $\Pi(X)$ and $\Pi'(X)$, we say that Π' **promotes** x **over** y **more than** Π when

$$x \prec_{\Pi,d} y \implies x \prec_{\Pi',d} y \text{ and } x \sim_{\Pi,d} y \implies x \preceq_{\Pi',d} y$$

We call **dictator** an individual $s \in D$ such that

$$x \prec_{\Pi,s} y \implies x \prec_{\Pi,D} y$$

Every preference of the dictator turns into preference of the group through the social welfare function. The existence of a dictator is property of the social welfare function.

The idea of a dictator can be generalized to a **decisive set** of individuals.

A **minimal decisive set** is a set that cannot lose members without losing the status of decisive.

Arrow's axioms Desirable properties of a social welfare function are:

1. **Nontriviality:** there are at least three alternatives and two individuals
2. **Universality:** $g(\Pi)$ is defined for all profiles Π (solves the problem for all profiles)
3. **Weak order:** $g(\Pi)$ returns a weak order for all profiles
4. **Independence from irrelevant alternatives:** The social welfare function on a restricted alternative set is the restriction of the original group preference
5. **Monotony:** given two alternatives $x, y \in X$ and two preference profiles $\Pi(X)$ and $\Pi'(X)$, if $\Pi'(X)$ promotes x more over y more than $\Pi(X)$ and $x \prec_{\Pi,D} y$, then $x \prec_{\Pi',D} y$, the social welfare function maintains a preference for x over y if x is further promoted

6. **Popular sovereignty:** Every weak order can be obtained using $|D|$ suitable preferences. The social welfare function is surjective
7. **Nondictatorship:** no individual is a dictator

Arrow's theorem states that any social welfare function satisfying axioms 1 through 6 implies a dictator.

5.4.6 Criticism to Arrow's axioms

Criticism arose, trying to undermine the proof suggesting that the axioms are not as obvious as they look.

Nontriviality What if $|X| = 2$? For only two alternatives the Condorcet method satisfies all other axioms. But how to reduce to two the alternatives? It's just another problem.

Personally, I think other criticisms are not as important (can't be bothered to write them).