DMM Summary

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Disclaimer This "summary" it's not supposed to be a standalone tool, it contains more *pragmatic* definitions, aimed more at the exercises/exam itself. It's not complete *and it's not supposed to be.*

Contents

1 Decision Problems

1.1 Preference Relation

1.1.1 Properties

Considering impact set F and a binary relation Π, the main properties of said relation can be:

- **Reflexivity:** $\forall a \in F$, $(a, a) \in \Pi$; to check for it, on the graph representation every impact has to have a self-loop

- **Transitivity:** $\forall a, b, c \in F$, if $(a, b) \in \Pi$ and $(b, c) \in \Pi$ then $(a, c) \in \Pi$; to check for it, on the graph representation follow every arc and check that every possible "triangle" is complete

- **Antisymmetry:** $\forall a, b \in F$, if $(a, b) \in \Pi$ and $(b, a) \in \Pi$ then $a = b$; to check for it, on the graph representation there must be no impacts that point at each other

- **Completeness:** $\forall a, b \in F$, if $(a, b) \notin \Pi$ then $(b, a) \in \Pi$; to check for it, on the graph representation, from each node there must be an arc (either outgoing or incoming) connecting it to every other

1.1.2 Types of relations (orders)

Combining the properties, we can get different kinds of preferences:

- **Preorder:** reflexivity and transitivity. Guarantees that the set of nondominated alternatives is nonempty, even if not all alternatives are comparable, we can find at least one nondominated solution

- **Partial order:** reflexivity, transitivity and antisymmetry. This limits indifference, but still allows for incomparability, and thus not always leading to a definitive choice

- **Weak order:** reflexivity, transitivity and completeness. Guarantees that nondominated solutions exists and are all mutually indifferent, any of them can be chosen. Such orders admit representation by a value function (with ties), turning the decision problem into an optimization problem

- **Total order:** reflexivity, transitivity, antisymmetry and completeness. Provides a unique linear ranking of alternatives, there is always a unique best alternative

1.1.3 Derived relations

From the weak preference relation, one can derive:

- **Indifference relation:** Ind_Π

$$(a, b), (b, a) \in \text{Ind}_\Pi \Leftrightarrow (a, b) \in \Pi \wedge (b, a) \in \Pi$$

  To build it from the graph representation, add all self loops and each pair of arcs that point at each other

- **Strict preference relation:** Str_Π

$$(a, b) \in \text{Str}_\Pi \Leftrightarrow (a, b) \in \Pi \wedge (b, a) \notin \Pi$$

  To build it from the graph representation, add all arcs which do not have an equal one in the opposite direction (the set difference of the last one w.r.t. Π)

- **Incomparability relation:** Inc_Π

$$(a, b), (b, a) \in \text{Inc}_\Pi \Leftrightarrow (a, b) \notin \Pi \wedge (b, a) \notin \Pi$$

  To build it from the graph representation, add all arcs which are not present in either direction in the graph

2   Basic Decision Models

2.1   Structured preferences

2.1.1   Dominance relation

Some ways to sort *the stuff.*

Lexicographic order   Order the alternatives w.r.t. the value of the first indicator (for some ordering of the indicators) and break ties with the subsequent one. This yields a total order.

The variant with aspiration levels introduces a "minimum requirement" ϵ_i, rejecting all alternatives with indicator f_i worse than ϵ_i (higher or lower depending on whether it's a benefit or cost).

Utopia point   Identify an ideal impact with the best possible value for each indicator (optimize them independently) and evaluate all alternatives with the distance from the ideal impact.

Different definitions of distance yield different results. Some definitions:

- L_1 Manhattan distance

$$d(f, f') = \sum_{l \in P} |f_l - f'_l|$$

- L_2 Euclidean distance

$$d(f, f') = \sqrt{\sum_{l \in P} \left(f_l - f'_l\right)^2}$$

- L_∞ Chebyshev distance/maximum norm

$$d(f, f') = \max_{l \in P} |f_l - f'_l|$$

Borda count   In the case of finite alternatives, they can be sorted by counting how many alternatives are worse than each one

$$B(f) = \left|\left\{f' \in F \mid f \preceq f'\right\}\right|$$

2.2 MAUT

2.2.1 Indifference curves

An *indifference curve* is a subset of the impact space $I \subseteq F$ of reciprocally indifferent impacts. By definition:

- The curves cover F

- Any two curves have empty intersection

- Weak order on impacts maps to total order on curves

Usually, continuity is assumed (they are mathematical objects and not a general set of points), and each indifference curve is expressed in the implicit form $u(f) = c$, each c identifies a curve.

2.2.2 MRS

The Marginal Rate of Substitution λ_{12} between two indicators f_1 and f_2 represents how much of f_1 are we willing to "give up" for a unit of f_2; e.g., if we're willing to give 4 units of f_1 for a unit of f_2 then $\lambda_{12} = 1/4$.

It's the ratio of the partial derivatives of the utility function w.r.t. f_1 and f_2

$$\lambda_{12}(f) = \frac{\frac{\partial u}{\partial f_1}}{\frac{\partial u}{\partial f_2}}$$

A uniform MRS corresponds to a linear utility function $u(f) = w_1 f_1 + w_2 f_2$, and as such:

$$\lambda_{12}(f) = \frac{w_1}{w_2}$$

It represents the steepness of the indifference curve (slope).

2.3 Mathematical Programming: *Amateur hour*

The general process for solving MP problems is:

1. Draw a graphical representation of the feasible region

2. Find nonregular points

3. Write the generalized Lagrangian function

4. Write the KKT conditions

5. Solve the system of conditions to reject candidate points, hoping that few remain (add nonregular points after this)

6. Evaluate the function in all the remaining points, choosing the optimum

Easy enough right? (It's not)

Let's use an example:

$$\min f(x) = (x_1 - 1)^2 + x_2^2$$
$$g_1(x) = -x_1^2 - x_2^2 + 4 \leq 0$$
$$g_2(x) = x_1 - 3/2 \leq 0$$

Nonregular points   All points in which the gradients of the active constraints are linearly independent are regular. Only active constraints must be considered, the whole feasible region is composed of regular points. A constraint is active when $= 0$.

Why are active constraints zero? The constraints are active only on the borders of the feasible region, since a optimal solution can only be found on that border, "pushing the boundaries" of the problem. Each point strictly inside the feasible region is next to another point, slightly better, slightly more towards the border.

Calculate the gradients of each constraint and check wether they can be zero or not. If the gradient can be zero and in such point:

- The constraint is inactive: business as usual

- The constraint is active: nonregular point, has to be added to the candidate set

In our example, the gradients are:

$$\nabla g_1(x) = [-2x_1 \quad -2x_2]$$
$$\nabla g_2(x) = [1 \ 0]$$

And:

- *The first one is 0 only in the origin, point in which the constraint is nonactive $(g_1(0,0) = 4)$*

- *The second one is never zero*

Then check points in which pairs of constraints are active, i.e., make a system in which both are zero (I think equality constraints always have to be added? I'll get back to you on that, not sure).

In our example:

$$\begin{cases} -x_1^2 - x_2^2 + 4 = 0 \\ x_1 - 3/2 = 0 \end{cases} \implies \begin{cases} x_1 = 3/2 \\ x_2^2 = 7/4 \end{cases}$$

From which we get the points

$$A = \left(\frac{3}{2}, \frac{\sqrt{7}}{2} \right), \quad B \left(\frac{3}{2}, -\frac{\sqrt{7}}{2} \right)$$

Check wether the gradients are linearly independent or not in the points found. To verify this the simplest way is to compose a 2×2 matrix with the values of the gradients considered in each point, if the determinant of such matrix is nonzero the gradients are linearly independent in the point considered.

In our example, $\nabla g_1(A) = \begin{bmatrix} -3 & -\sqrt{7} \end{bmatrix}$ *and* $\nabla g_2(A) = [1 \ 0]$, *the resulting matrix being*

$$M = \begin{bmatrix} -3 & 1 \\ -\sqrt{7} & 0 \end{bmatrix}$$

Whose determinant is:

$$det(M) = (-3 \cdot 0) - (-\sqrt{7} \cdot 1) = \sqrt{7} \neq 0$$

So the gradients are linearly independent.

Then check points in which triples of constraints are active, similarly to earlier. You can guess how this goes on.

In our example there are no more constraints, but you would simply check that the gradients are linearly independent in the points resulting from the system given by $g_1(x) = 0$, $g_2(x) = 0$, $g_3(x) = 0$.

The aim of this phase is only to find nonregular points. Points with not linearly independent gradients are nonregular and as such have to be added to the candidate set after "sifting" with the KKT conditions.

Generalized Lagrangian function The generalized Lagrangian function is defined as:

$$\ell(x) = f(x) + \sum_{i=1}^{s} \lambda_i h_i(x) + \sum_{j=1}^{m} \mu_j g_j(x)$$

With λ_i free multipliers and $h_i(x)$ equality constraints (always active).

In our example, the function becomes:

$$
\begin{aligned}
\ell(x) &= f(x) + \mu_1 g_1(x) + \mu_2 g_2(x) \\
&= (x_1 - 1)^2 + x_2^2 + \mu_1 \left(-x_1^2 - x_2^2 + 4\right) + \mu_2(x_1 - 3/2)
\end{aligned}
$$

KKT Conditions The KKT conditions state that if a point is regular and locally minimal:

1. The partial derivatives of the Lagrangian function w.r.t. the x variables are equal to zero $(\partial \ell / \partial x_i = 0)$

2. The partial derivatives of the Lagrangian function w.r.t. the λ multipliers are equal to zero $(\partial \ell / \partial \lambda_i = h_j = 0)$, that is, the equality constraints are respected

3. The product of the functions expressing the inequality constraints, times the corresponding multipliers are equal to zero $(\mu_k g_k = 0)$

4. All inequalities constraints are satisfied

5. All multipliers of the inequality constraints are nonnegative $(\mu_k \geq 0)$

These conditions allow to restrict the number of candidate points from *all regular and nonregular points* to *nonregular and a few regular.*

In our example, the conditions become

$$
\begin{aligned}
\partial \ell / \partial x_1 &= 2(x_1 - 1) - 2\mu_1 x_1 + \mu_2 = 0 \\
\partial \ell / \partial x_2 &= 2x_2 - 2\mu_1 x_2 = 0 \\
\mu_1 g_1 &= \mu_1(-x_1^2 - x_2^2 + 4) = 0 \\
\mu_2 g_2 &= \mu_2(x_1 - 3/2) = 0 \\
g_1 &= -x_1^2 - x_2^2 + 4 \leq 0 \\
g_2 &= x_1 - 3/2 \leq 0 \\
\mu_1 &\geq 0 \\
\mu_2 &\geq 0
\end{aligned}
$$

Solving conditions To solve the system without exhaustively exploring all possible cases one can use a search tree, whose nodes divide the feasible region in disjoint parts. To do that we build on the stricter conditions, i.e., the products $\mu_k g_k = 0$ (at first choose the simplest one). Given a constraint, we can distinguish two cases:

1. $\mu_k = 0$ and $g_k \leq 0$

2. $\mu_k > 0$ and $g_k = 0$

These assumption simplify the system, allowing it to be analyzed in an easier way. The process can be repeated on the resulting sub-problems, if necessary.

The idea is to restrict the possible solutions and find them a little at a time. Divide in "easy" sub-problems and all solutions to such problems are candidate points.

In our example, the constraint chosen is $\mu_2 g_2 = 0$. We now consider the case P^1, with $\mu_2 > 0$ and $g_2(x) = 0$, and thus $x_1 = 3/2$. The constraints now become:

$$1 - 3\mu_1 + \mu_2 = 0$$
$$x_2(1 - \mu_1) = 0$$
$$\mu_1(7/4 - x_2^2) = 0$$
$$7/4 - x_2^2 = 0$$
$$\mu_1 \geq 0$$

We can now say that $\mu_1 = (\mu_2 + 1)/3 > 0$ and $x_2^2 = 7/4$. This yields candidate points:

$$A = \left(\frac{3}{2}, \frac{\sqrt{7}}{2}\right), \quad B\left(\frac{3}{2}, -\frac{\sqrt{7}}{2}\right)$$

Now consider P^2, with $\mu_2 = 0$ and $g_2 \leq 0$, the constraints become:

$$2(x_1 - 1) - 2\mu_1 x_1 = 0 \qquad\qquad \implies x_1(1 - \mu_1) = 1 \implies \mu_1 \neq 1$$
$$x_2(1 - \mu_1) = 0 \qquad\qquad \implies x_2 = 0$$
$$\mu_1(-x_1^2 - x_2^2 + 4) = 0 \qquad\qquad \implies \mu_1(4 - x_1^2) = 0$$
$$x_1^2 + x_2^2 \geq 4 \qquad\qquad \implies x_1^2 \geq 4$$
$$x_1 \leq 3/2$$
$$\mu_1 \geq 0$$
$$\implies x_1 = -2$$

This yields candidate point:

$$C = (-2, 0)$$

Choose optimum We now want to evaluate the function in all the candidate points (remember to consider nonregular points) and choose the best solution.

In our example:
$$\begin{cases} f(A) = 2 \\ f(B) = 2 \\ f(C) = 9 \end{cases}$$

Which implies that both A and B are globally optimal points.