Embedding r-Factorizations of Complete Uniform Hypergraphs into s-Factorizations

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Abstract

The problem we study in this thesis asks under which conditions an r-factorization of K_m^h can be embedded into an s-factorization of K_n^h. This problem is a generalization of a problem posed by Peter Cameron which asks under which conditions a 1-factorization of K_m^h can be embedded into a 1-factorization of K_n^h. This was solved by Häggkvist and Hellgren. We study sufficient conditions in the case where $s = h$ and m divides n. To that end, we take inspiration from a paper by Amin Bahmanian and Mike Newman and simplify the problem to the construction of an "acceptable" partition. We introduce the notion of irreducible sums and link them to the main obstacles in constructing acceptable partitions before providing different methods for circumventing these obstacles. Finally, we discuss a series of open problems related to this case.

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Chapter 1

Introduction

1.1 Definitions and Notation

A hypergraph $\mathcal{G}$ consists of a set of vertices $\mathcal{V}$ and a multiset E of multisets of these vertices, which we call edges. We say that a hypergraph $\mathcal{H}$ is a sub-hypergraph of a hypergraph $\mathcal{G}$ if the vertex set of $\mathcal{H}$ is a subset of the vertex set of $\mathcal{G}$ and the edge multiset of $\mathcal{H}$ is a submultiset of the edge multiset of $\mathcal{G}$. The degree of a vertex in a hypergraph is the sum of the multiplicities of that vertex in all the edges. We say that a hypergraph is r-regular if all its vertices have degree r. For a positive integer r, an r-factorization of a hypergraph is a partition of its edges into spanning r-regular sub-hypergraphs. The complete h-uniform hypergraph on n vertices, denoted K_n^h, is the hypergraph on n vertices whose edges are all of the h-subsets of the vertex set. In this case the edge multiset is a set.

For a positive integer n, we denote by $[n]$ the set $\{1, 2, \dots, n\}$.

1.2 Problem Statement and Motivation

The question of interest is the following.

Problem 1.2.1. *Under what conditions can an r-factorization of K_m^h be extended to an s-factorization of K_n^h?*

Problem 1.2.1 is a generalization of a problem posed by Peter Cameron [1] in which he asks under which conditions a 1-factorization, or parallelism, of K_m^h can be embedded into a 1-factorization, or parallelism, of K_n^h. It was conjectured by Baranyai and Brouwer [2] that a 1-factorization of K_m^h could be embedded into a 1-factorization of K_n^h if and only if h divides m and n, and $n \geq 2m$. This conjecture was proved by Häggkvist and Hellgren [3]. Amin Bahmanian and Mike Newman made significant advancements in the generalized case, that is, when considering factorizations of higher degrees. In the case where $r = s$, they proved the following theorem:

Theorem 1.2.2 (Bahmanian and Newman [4], Theorem 1.7)**.** *Let m, n, h, r satisfy the following necessary conditions for some positive integers p, q, c, d.*

$$n \geq 2m \qquad mr = ph \qquad nr = qh \qquad \binom{m-1}{h-1} = cr \qquad \binom{n-1}{h-1} = dr$$

Assume furthermore the following condition.

$$\gcd(m, n, h) = \gcd(n, h)$$

Then there exists an r-factorization of K_n^h containing an embedded r-factorization of K_m^h.

Basic necessary conditions for the existence of an s-factorization of K_n^h that contains an embedded r-factorization of K_m^h are known:

Lemma 1.2.3 (Bahmanian and Newman [5], Lemma 7)**.** *If an r-factorization of K_m^h can be embedded into an s-factorization of K_n^h, then*

(N1) $h \mid rm$, $h \mid sn$;

(N2) $r \mid \binom{m-1}{h-1}$, $s \mid \binom{n-1}{h-1}$;

(N3) $1 \leq s/r \leq \binom{n-1}{h-1}/\binom{m-1}{h-1}$;

(N4) $n \geq \frac{h}{h-1}m$ *if* $1 < s/r < \binom{n-1}{h-1}/\binom{m-1}{h-1}$;

(N5) $n \geq 2m$ *if* $s = r$.

The first two conditions in Lemma 1.2.3 are used to define the notion of "feasible" integers, that is, parameter sets m, n, h, r, s satisfying these divisibility conditions.

Definition 1.2.4. *Positive integers m, n, h, r, s are called* ***feasible*** *if there exist positive integers p, q, c, d such that the following conditions hold, with $n > m \geq h > 1$.*

$$mr = ph \qquad ns = qh \qquad \binom{m-1}{h-1} = cr \qquad \binom{n-1}{h-1} = ds$$

These are necessary conditions for the existence of an s-factorization of K_n^h that contains an embedded r-factorization of K_m^h.

Note that we consider the parameters m, n, h, r, s as an ordered 5-tuple in this definition. The parameters c and d are the numbers of colours in the r-factorization of K_m^h and in the s-factorization of K_n^h, respectively. The parameters p and q are the numbers of edges of each colour in the r-factorization of K_m^h and in the s-factorization of K_n^h, respectively. The first two conditions in Definition 1.2.4 tell us that the number of vertices per colour class is an integer while the last two conditions tell us that the numbers of colours in both factorizations are integers.

The task of solving Problem 1.2.1 has been simplified by Bahmanian and Newman [4] to the creation of a "good collection" with some basic properties.

Definition 1.2.5. *Let m, n, and h be integers such that $m < n$, and assume there exist integers p, q, r, s such that $mr = ph$ and $ns = qh$. A collection $\mathcal{C} = \{\!\{C_1, C_2, \ldots, C_d\}\!\}$ where each $C_j \in \mathcal{C}$ is itself a multiset of h-subsets of $[n]$ is called* **good** *if the following hold:*

C1: $|C_j| = q$ *for each* $C_j \in \mathcal{C}$,

C2: $\sum_{A \in C_j} |A \cap [m]| = ms$ *for each* $C_j \in \mathcal{C}$,

C3: For $0 \leq i \leq h$ *the number of all h-subsets over all $C_j \in \mathcal{C}$ whose intersection with $[m]$ is of size i is exactly* $\binom{m}{i}\binom{n-m}{h-i}$,

C4: Each C_j contains either exactly p sets contained in $[m]$ or none.

Given a good collection $\mathcal{C} = \{\!\{C_1, C_2, \ldots, C_d\}\!\}$, we will refer to the multisets $C_1, C_2, \ldots, C_d$ as colour classes. In 2018, M.A Bahmanian and M. Newman proved the following:

Theorem 1.2.6 (Bahmanian and Newman [4], Theorem 3.2)**.** *Assume we have positive integers m, n, p, q, r, s and a good collection $\mathcal{C}$. Then there exists an s-factorization of K_n^h that contains an embedded r-factorization of K_m^h.*

The authors also make the observation that, given an s-factorization of K_n^h containing an embedded r-factorization of K_m^h, we can construct a good collection by letting $\mathcal{C}$ be the collection of colour classes of edges in the s-factorization of K_n^h. It is easy to check that the four conditions of Definition 1.2.5 are satisfied. Using Theorem 1.2.6, the authors prove the following:

Theorem 1.2.7 (Bahmanian and Newman [4], Theorem 5.10)**.** *Let m, n, h, r, s be feasible integers and let $k = \gcd(n, h)$. Assume furthermore that $n \geq 2m$, $\gcd(n, h) = \gcd(m, n, h)$, and $1 < \frac{s}{r} \leq \frac{m}{k}\left(1 - \binom{m-k}{h}/\binom{m}{h}\right)$. Then there exists an s-factorization of K_n^h that contains an embedded r-factorization of K_m^h.*

Theorem 1.2.7 is proved using a technique based on group actions developed by Bahmanian and Newman in [4]. This group action approach will be used to prove a large portion of the results in this thesis. Suppose we have feasible integers m, n, h, r, s. Let t be any divisor of n and let G_t be the cyclic group of order n/t with generator γ. Considering the integers $1, \ldots, n$ arranged in a circular configuration, we define the action of γ on $[n]$ to be a circular t-shift. This group action extends naturally to a group action on $\mathcal{A} = \binom{[n]}{h}$, the set of all h-subsets of $[n]$. For each $A \in \mathcal{A}$ we denote by $\mathcal{R}_A^t$ its orbit under the action of G_t. The central idea is to colour entire orbits rather than single edges. This has many advantages that will be discussed in greater detail later.

Note that the case where $m = h$ is trivial, as the only factorization of K_m^m is the trivial 1-factorization. From now on we always assume $h < m$.

The remainder of this thesis will have the following structure. In Chapter 2 we introduce the special case of Problem 1.2.1 that is of interest to us and begin the first step of the construction of a good collection. In Chapter 3 we introduce the obstacles that pose the greatest challenge in constructing good collections and we develop methods for circumventing these. In Chapter 4 we introduce an alternative perspective on feasible integers and good collections in order to lighten the task of their construction. We also introduce new methods for constructing these. In Chapter 5 we combine these methods with the methods developed by Bahmanian and Newman to make advancements in the general case. Finally, in Chapter 6 we discuss a few open problems relevant to this problem.

Chapter 2

Embedding factorizations when m divides n and $s = h$

Suppose $n = \alpha m$ for some integer α. Suppose $m, \alpha m, h, r, s$ are feasible and let $k = \gcd(\alpha m, h)$. For a set $A \in \binom{[\alpha m]}{h}$, define $t_A = |\mathcal{R}_A^1|$, that is, t_A is the smallest circular shift that preserves A. From this last observation it is clear that t_A divides αm and that $\frac{\alpha m}{t_A}$ divides h.

For $1 \leq i \leq \alpha m$, define $T_i = \left\{A \in \binom{[\alpha m]}{h} \ : \ t_A = i\right\}$. It follows from the above observation that if $T_i \neq \emptyset$, then $i \mid \alpha m$ and $\frac{\alpha m}{i} \mid h$. Conversely, if $i \mid \alpha m$ and $\frac{\alpha m}{i} \mid h$, we can construct an element of T_i as follows: Divide the αm vertices, still arranged in a circular fashion, into $\frac{\alpha m}{i}$ slices of i consecutive vertices. In each slice, choose the first $\frac{hi}{\alpha m}$ vertices. The result is a set A of $\frac{\alpha m}{i}\frac{hi}{\alpha m} = h$ vertices which, by construction, clearly satisfies $t_A = i$. We define

$$P = \left\{i \in [\alpha m] \ : \ T_i \neq \emptyset\right\}.$$

By the above, we have $P = \left\{i \in [\alpha m] \ : \ i \mid \alpha m \text{ and } \frac{\alpha m}{i} \mid h\right\}$. It is straightforward to see that $P = \left\{\frac{\alpha m}{j} \ : \ j \mid k\right\}$ where $k = \gcd(\alpha m, h)$.

Define

$$D = \bigcup_{A \in \binom{[m]}{h}} \mathcal{R}_A^m.$$

In other words, the elements of D are obtained by taking all h-subsets of $[m]$ and shifting these around by leaps of m. We make a few straightforward observations:

(i) $i \mid |T_i|$ for all $i \in P$, as T_i is closed under 1-shifts. that is, if $A \in T_i$, then $R_A^1 \subseteq T_i$;

(ii) $D \subseteq T_{\alpha m}$;

(iii) $|D| = \binom{m}{h}\alpha$;

(iv) $\binom{[\alpha m]}{h} = \dot{\bigcup\limits_{i \in P}} T_i$.

In the construction of our good collection, we will adhere to the convention of colouring whole orbits by shifts of length 1 or m rather than colouring individual edges. This, as we will see later, will guarantee that condition C2 of Definition 1.2.5 is satisfied. As a first step in the construction, we will assign colours to all orbits of size αm with the goal of fulfilling condition C4.

Case 1: $r = s = h$

In this case, feasibility implies $c = \frac{1}{m}\binom{m}{h}$. Thus, observation (iii) gives us $|D| = c\alpha m$. Furthermore, by observation (ii) we can form c colour classes $C_1, C_2, \ldots, C_c$, each consisting of $p = m$ orbits contained in D (by m-shifts, so that these all have size α) so condition C4 of a good collection is satisfied. Since $\alpha m \mid |T_{\alpha m}|$, the orbits by m-shifts contained in $T_{\alpha m} \setminus D$, each of which has size α, can be bundled m-by-m in any configuration to form colour classes. This process creates colour classes $C_{c+1}, C_{c+2}, \ldots, C_\ell$. We now check that these colour classes satisfy the conditions in Definition 1.2.5.

C1: For all $1 \leq i \leq \ell$, C_i is a union of m orbits by m-shifts. Since each orbit has size α, we have $|C_i| = \alpha m = q$.

C2: For all $1 \leq i \leq \ell$, C_i is of the form $\bigcup\limits_{i=1}^{m} \mathcal{R}_{A_j}^m$ for some edges $A_1, A_2, \ldots, A_m$. For each $1 \leq j \leq m$, we have $\sum\limits_{A \in \mathcal{R}_{A_j}^m} |A \cap [m]| = h$, as each vertex in A_j, when shifted by leaps of m, will appear in $[m]$ exactly once. Thus, we have

$$\begin{aligned} \sum_{A \in C_i} |A \cap [m]| &= \sum_{j=1}^{m} \sum_{A \in \mathcal{R}_{A_j}^m} |A \cap [m]| \\ &= \sum_{j=1}^{m} h \\ &= mh \\ &= ms. \end{aligned}$$

C3: As this condition depends on all of the colour classes, it will be verified in a later section.

C4: For all $1 \leq i \leq c$, C_i, being a union of $m = p$ orbits by m-shifts contained in D, contains exactly $m = p$ edges contained in $[m]$. Since the union $C_1 \cup C_2 \cup \ldots \cup C_c$ contains every orbit by m-shift contained in D, it follows that this union contains every edge contained in $[m]$. Therefore, for all $c + 1 \leq i \leq \ell$, we conclude that C_i contains no edges contained in $[m]$.

It remains to show that the remaining colour classes can be formed from the remaining T_i, $i \in P \setminus \{\alpha m\}$.

Case 2: $r < s = h$

In this case feasibility implies

$$p = \frac{mr}{h} \qquad q = \alpha m \qquad c = \frac{h}{mr}\binom{m}{h} \qquad d = \frac{1}{\alpha m}\binom{\alpha m}{h}$$

We need to form c colour classes, each with $p = \frac{mr}{h}$ edges contained in $[m]$. Unlike in the previous case, we cannot simply take these colour classes to be unions of orbits contained in D, as p such orbits would give $p\alpha = \frac{r}{h}q < q$ edges. To work around this, we will start the construction in the same fashion as the previous case, but we will complete each colour class by adding additional orbits contained in $T_{\alpha m} \setminus D$. First, we need a lemma:

Lemma 2.0.1. *Let m, n, h, r, s be feasible integers with $r \leq s = h$ and $n = \alpha m$. Then*

$$\frac{p}{\alpha}|T_{\alpha m} \setminus D| \geq \binom{m}{h} m\left(1 - \frac{r}{h}\right)$$

Proof. We will prove an equivalent form of this inequality, namely $r \geq \frac{h\alpha}{|T_{\alpha m}|}\binom{m}{h}$. First, we'll show that these two inequalities are indeed equivalent. We have

$$\begin{aligned} p|T_{\alpha m} \setminus D|/\alpha \geq \binom{m}{h} m\left(1 - \frac{r}{h}\right) &\iff \frac{mr}{h}\left(|T_{\alpha m}| - \binom{m}{h}\alpha\right)/\alpha \geq \binom{m}{h} m\left(\frac{h-r}{h}\right) \\ &\iff \frac{r}{\alpha}|T_{\alpha m}| - r\binom{m}{h} \geq \binom{m}{h}(h-r) \\ &\iff r|T_{\alpha m}| \geq h\alpha\binom{m}{h} \\ &\iff r \geq \frac{h\alpha}{|T_{\alpha m}|}\binom{m}{h} \end{aligned}$$

As r must be a positive integer, it suffices to show that $|T_{\alpha m}| \geq h\alpha\binom{m}{h}$. Recall that we are considering the αm vertices to be arranged in a circle, so that we have α slices of m vertices. Take an edge A contained in $[m]$ such that $1 \in A$. The orbit of A by 1-shifts is clearly of length αm. There are $\binom{m-1}{h-1} = \binom{m}{h}\frac{h}{m}$ ways to choose A. Furthermore, since A is chosen to include the first vertex, any pair of orbits created using a different choice of A will be disjoint. So considering every possible A and its orbit, we get $\binom{m}{h}\frac{h}{m}\alpha m = h\alpha\binom{m}{h}$ edges contained in $T_{\alpha m}$, hence the result.

□

Group the orbits contained in D (by m-shifts, noting that these are all of size α) p-by-p along with $m(1 - \frac{r}{h})$ orbits by m-shifts contained in $T_{\alpha m} \setminus D$ to create c colour classes $C_1, C_2, \ldots, C_c$. Note that $m(1 - \frac{r}{h})$ is always an integer, as $m(1 - \frac{r}{h}) = m - \frac{mr}{h} = m - p$. There are enough orbits in $T_{\alpha m} \setminus D$ to do this by Lemma 2.0.1. Any remaining orbits in $T_{\alpha m} \setminus D$ after this process are bundled m-by-m to form colour classes $C_{c+1}, C_{c+2}, \ldots, C_\ell$. This can be done since $\alpha m \mid |T_{\alpha m}|$. We now check that these colour classes satisfy the conditions in Definition 1.2.5.

C1: For all $1 \leq i \leq \ell$, C_i is a union of $m = p + m(1 - \frac{r}{h})$ orbits by m-shifts. Since each orbit has size α, we have $|C_i| = \alpha m = q$.

C2: For all $1 \leq i \leq \ell$, C_i is of the form $\bigcup_{i=1}^{m} \mathcal{R}^m_{A_j}$ for some edges $A_1, A_2, \ldots, A_m$. For each $1 \leq j \leq m$, we have $\sum_{A \in \mathcal{R}^m_{A_j}} |A \cap [m]| = h$, as each vertex in A_j, when shifted by leaps of m, will appear in $[m]$ exactly once. Thus, we have

$$\begin{aligned}
\sum_{A \in C_i} |A \cap [m]| &= \sum_{j=1}^{m} \sum_{A \in \mathcal{R}^m_{A_j}} |A \cap [m]| \\
&= \sum_{j=1}^{m} h \\
&= mh \\
&= ms.
\end{aligned}$$

C3: As this condition depends on all of the colour classes, it will be verified in a later section.

C4: For all $1 \leq i \leq c$, C_i, being a union of p orbits by m-shifts contained in D and $m - p$ orbits by m-shifts not contained in D, contains exactly p edges contained in $[m]$. Since the union $C_1 \cup C_2 \cup \ldots \cup C_c$ contains every orbit by m-shift contained in D, it follows that this union contains every edge contained in $[m]$. Therefore, for all $c + 1 \leq i \leq \ell$, we conclude that C_i contains no edges contained in $[m]$.

It remains to show that the remaining colour classes can be formed from the remaining T_i , $i \in P \setminus \{\alpha m\}$.

2.1 Frobenius promotions

In this section we develop a means by which orbits can be combined or "promoted" to create larger unions of orbits. The end goal, of course, is to perform these promotions until all orbits have been promoted to a union of size αm to which we will assign a colour.

Definition 2.1.1. *Let $a_1, a_2, \ldots, a_n$ be positive integers with $\gcd(a_1, a_2, \ldots, a_n) = 1$. The **Frobenius number** of the set $\{a_1, a_2, \ldots, a_n\}$, denoted by $G(a_1, a_2, \ldots, a_n)$, is the largest integer that cannot be expressed as a linear combination $k_1a_1 + k_2a_2 + \ldots + k_na_n$ where $k_1, k_2, \ldots, k_n$ are non-negative integers.*

In the case where $n = 2$, the Frobenius number is known to be $G(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$ [6]. This gives an upper bound on $G(a_1, a_2, \ldots, a_n)$: Suppose $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ with $n \geq 2$. Then clearly

$$G(a_1, a_2, \ldots a_n) \leq G(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1.$$

This upper bound is satisfactory for our purposes.

Definition 2.1.2. *Let n be a positive integer with at least two prime divisors and let p and q be its two smallest prime divisors. Define*

$$Fr(n) = (p-1)(q-1).$$

$Fr(n)$ provides a lower bound on the successor of the Frobenius number of the set of prime divisors of n, that is, any integer $N \geq Fr(n)$ can be written as a linear combination of the prime divisors of n with non-negative integer coefficients. We formalize this with a lemma:

Lemma 2.1.3. *Let n be a positive integer with at least two prime divisors. Let p and q be the two smallest prime divisors divisors of n and assume $p > q$. Suppose N is a positive integer such that $N \geq Fr(n)$. Then there exist non-negative integers a and b such that $ap+bq = N$.*

Proof. By definition, $Fr(n) = (p-1)(q-1)$, which is in turn equal to $G(p,q)+1$. It therefore follows from the definition of the Frobenius number of $\{p,q\}$ that there exist non-negative integers a and b such that $ap + bq = N$. $\square$

This means that if t is a divisor of k satisfying the condition that k/t has at least two prime divisors and $g(t) \geq Fr(k/t)$ where $g(t)$ is the number of orbits by 1-shifts of size $\frac{\alpha m}{k}t$, then there exist non-negative integers a and b such that the $g(t)$ orbits can be promoted to make a unions of orbits, the size of each union being $\frac{\alpha m}{k}tp$ and b unions of orbits of size $\frac{\alpha m}{k}tq$ where $p > q$ are the two smallest prime divisors of k/t. Note that when we perform a promotion by combining orbits, we do not wish to distinguish the resulting union of orbits from the orbits of the same size. Therefore, for convenience, we use the term "orbit" to refer to unions of orbits as well as orbits.

Given feasible integers m, n, h, r, s with $n = \alpha m$ and $s = h$, the method outlined above will allow us to prove that if $g(t) \geq Fr(k/t)$ for all divisors t of k such that k/t has at least two prime divisors, then there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of K^h_m. Before we formally state and prove this theorem, however, we will need some important results. This theorem is stated and proved in Section 4.1.

Chapter 3

Irreducible and weak irreducible sums

In this chapter we introduce irreducible sums, the main obstacles in creating good collections. We find conditions under which these obstacles are absent, and provide methods of circumventing these obstacles when they are present.

3.1 Irreducible sums

We use the notation $\{\!\{\dots\}\!\}$ for multisets. To describe a multiset whose elements are $d_1, d_2, \dots, d_n$ where each d_i has multiplicity a_i, we will use the form $\{\!\{d_1^{a_1}, d_2^{a_2}, \dots, d_n^{a_n}\}\!\}$.

Definition 3.1.1. *Let k be a positive integer. Denote by $D(k)$ the set of proper divisors of k. For each $d \in D(k)$, let e_d be some non-negative integer such that the e_d are not all zero. The multiset $\{\!\{d^{e_d} : d \in D(k)\}\!\}$ of proper divisors d of k with multiplicity e_d is called* ***irreducible*** *if*

- *$k \mid \sum\limits_{d \in D(k)} e_d d$; and*
- *it admits no submultiset $\{\!\{d^{a_d} : d \in D(k)\}\!\}$ such that $\sum\limits_{d \in D(k)} a_d d = k$*

*For convenience, we adopt the term k-**irreducible sum** (or simply "irreducible sum" in the absence of ambiguity) to mean a linear combination $\sum\limits_{d \in D(k)} e_d d$ of the proper divisors of k with non-negative coefficients such that the sum's total is a multiple of k and for any choice of $0 \le a_d \le e_d$ for all $d \in D(k)$, $\sum\limits_{d \in D(k)} a_d d \neq k$. With this notation, we say that any sum $\sum\limits_{d \in D(k)} a_d d$ with $0 \le a_d \le e_d$ for all $d \in D(k)$ is a **subsum** of the sum $\sum\limits_{d \in D(k)} e_d d$. We also define the **length** of a sum $\sum\limits_{d \in D(k)} e_d d$ to be $\sum\limits_{d \in D(k)} e_d$, that is, the cardinality of its associated irreducible multiset.*

Example 3.1.2. Let $k = 30$. Then $\{\!\{1^1, 2^0, 3^0, 5^0, 6^4, 10^2, 15^1\}\!\}$, or simply $\{\!\{1, 6^4, 10^2, 15\}\!\}$, is an irreducible multiset. Its associated irreducible sum is $1+6+6+6+6+10+10+15 = 60$. No subsum of this sum has a total of 30. In fact, 30 is the smallest integer for which an irreducible sum exists.

Example 3.1.3. Let $k = 84$.
$1 + 4 + 12 + 12 + 12 + 12 + 12 + 12 + 21 + 28 + 42 = 168$ is an irreducible sum. We can combine a 4-term and two 12-terms and promote them to make a 28-term. The resulting sum is $1+12+12+12+12+21+28+28+42 = 168$ and one might observe, as we will soon prove, that it also an irreducible sum. The act of combining terms to create a single larger term is called a **promotion**. It is easy to see that the sum resulting from such promotions on an irreducible sum should always be irreducible. We formalize this observation in a lemma.

Lemma 3.1.4. *Suppose $A = \{\!\{d^{e_d} \ : \ d \in D(k)\}\!\}$ is an irreducible multiset. Suppose $a_1, \dots, a_n$ are integers and $d_1, \dots, d_n, t$ are distinct proper divisors of k for $1 \le i \le n$ such that $t = a_1 d_1 + \dots + a_n d_n$. Suppose also that $1 \le a_i \le e_{d_i}$ for $1 \le i \le n$. Then $A' = A \setminus \{\!\{d_1^{a_1}, \dots, d_n^{a_n}\}\!\} \cup \{\!\{t\}\!\}$ is also irreducible.*

Proof. We have $A' = \{\!\{d^{e'_d} \mid d \in D(k)\}\!\}$ where

$$e'_d = \begin{cases} e_{d_i} - a_i & \text{if } d = d_i \text{ for } 1 \le i \le n \\ e_t + 1 & \text{if } d = t \\ e_d & \text{otherwise} \end{cases}$$

and so

$$\begin{aligned} \sum_{d \in D(k)} e'_d d &= \sum_{i=1}^{n} (e_{d_i} - a_i) d_i + (e_t + 1)t + \sum_{\substack{d \in D(k) \\ d \notin \{d_1, \dots, d_n, t\}}} e_d d \\ &= -t + t + \sum_{d \in D(k)} e_d d \\ &= \sum_{d \in D(k)} e_d d \end{aligned}$$

Therefore this promotion does not change the value of the sum. Thus, k divides the sum. Suppose A' admits a submultiset $B = \{\!\{d^{b_d} \ : \ d \in D(k)\}\!\}$ such that $\sum_{d \in D(k)} b_d d = k$. On one hand, if $b_t \le e_t$, then B is also a submultiset of A, contradicting the assumption that A is irreducible. If, on the other hand, $b_t = e_t + 1$, then $B \setminus \{\!\{t\}\!\} \cup \{\!\{d_1^{a_1}, \dots, d_n^{a_n}\}\!\}$ is a submultiset of A and the sum of its elements (counting multiplicities) is clearly the same as that of B, again contradicting the assumption that A is irreducible. $\square$

When faced with an irreducible sum, it's often convenient to start by applying promotions until no more promotions can be made. Such an irreducible sum has the convenient property that no subsum has a total equal to any divisor of k.

Definition 3.1.5. *Let k be a positive integer. A k-irreducible sum is **minimally k-irreducible** if it admits no sub-sum of length at least two with total equal to any divisor of k.*

A minimally k-irreducible sum is one in which no promotions can be made. In particular, this gives us upper bounds on the coefficients e_d.

Lemma 3.1.6. *Let k be a positive integer. Let $\sum_{d\in D(k)} e_d d$ be a minimally k-irreducible sum. Then, for each $d \in D(k)$, e_d is less than the smallest prime divisor of k/d.*

Proof. Let k be a positive integer and suppose $\sum_{d\in D(k)} e_d d$ is a minimally k-irreducible sum. Let d be some proper divisor of k and let p be the smallest prime divisor of k/d. Suppose toward a contradiction that $e_d \geq p$. Then the sum consisting of p d-terms is a subsum of $\sum_{d\in D(k)} e_d d$ with total pd. However, since p divides k/d, this implies that pd is a divisor of k. This contradicts the assumption that $\sum_{d\in D(k)} e_d d$ is a minimally k-irreducible sum. We can therefore conclude that $e_d < p$. □

It is easy to see that we can obtain a minimally irreducible sum starting with an irreducible sum by performing promotions until no more promotions can be made. We formalize this observation in the following lemma:

Lemma 3.1.7. *Let k be a positive integer. Then there exists a k-irreducible sum if and only if there exists a minimally k-irreducible sum.*

Proof. It follows from the definition that any minimally k-irreducible sum is k-irreducible. Let $\sum_{d\in D(k)} e_d d$ be a k-irreducible sum. If the sum is minimally irreducible, then we are done. If not, then there is a proper divisor t of k and a subsum $\sum_{d\in D(k)} a_d d = t$ where $0 \leq a_d \leq e_d$ for all $d \in D(k)$. After promoting this subsum to a t-term, we obtain a new irreducible sum by Lemma 3.1.4. We repeat this process until no more promotions can be made. Since at every step the length of the sum strictly decreases, we know that this process must terminate. That is, we will eventually obtain a minimally irreducible sum. □

Lemma 3.1.8. *Let k be a prime power. Then there exists no k-irreducible sum.*

Proof. Let $k = p^a$ where p is prime and a is some non-negative integer. Let c be a positive integer and suppose $\sum_{d\in D(k)} e_d d = ck$ is an irreducible sum. By Lemma 3.1.7 we may assume it is minimally irreducible. The proper divisors of k are p^i for $0 \leq i \leq a-1$. Since the sum is minimally irreducible, it follows by Lemma 3.1.6 that $e_{p^i} \leq p-1$ for each $1 \leq i \leq a-1$.

Thus, we have:

$$\begin{aligned} ck &= \sum_{i=0}^{a-1} e_{p^i} p^i \\ &\leq \sum_{i=0}^{a-1} (p-1)p^i \\ &= \sum_{i=0}^{a-1} (p^{i+1} - p^i) \\ &= p^a - 1 \\ &< k \end{aligned}$$

So $c < 1$. Therefore the sum is not irreducible. □

The following lemma is a condition applying to k-irreducible sums where k is a product of exactly two prime powers, if such an irreducible sum exists. It is not known whether there exist values of k with exactly two prime divisors for which a k-irreducible sum exists. However, the next lemma ultimately leads to the construction of an h-factorization of $K^h_{\alpha m}$ containing an embedded r-factorization of K^h_m in the case where $k = \gcd(\alpha m, h)$ is of this form.

Lemma 3.1.9. *Let $k = p^a q^b$ where p and q are distinct primes and a and b are positive integers. Let $\sum_{d \in D(k)} e_d d$ be a k-irreducible sum. Then there exists a proper divisor t of k such that $p^a \nmid t$, $q^b \nmid t$, and $e_t \neq 0$.*

Proof. Suppose $\sum_{d \in D(k)} e_d d$ is a k-irreducible sum such that $e_d = 0$ for all d of the form $p^i q^j$ with $0 \leq i \leq a-1$ and $0 \leq j \leq b-1$. Starting with $i = 0$ and proceeding in increasing order until $i = a-1$, group up the $p^i q^b$-terms p-by-p until fewer than p remain. Starting with $j = 0$ and proceeding in increasing order until $j = b-1$, group up the $p^a q^j$-terms q-by-q until fewer than q remain. We have

$$\begin{aligned} \sum_{d \in D(k)} e_d d &\leq \sum_{i=0}^{a-1} (p-1)p^i q^b + \sum_{j=0}^{b-1} (q-1) p^a q^j \\ &= q^b(p^a - 1) + p^a(q^b - 1) \\ &= 2k - q^b - p^a \\ &< 2k \end{aligned}$$

The total of this sum is either 0 or k, which contradicts the assumption that the sum is irreducible. □

Lemma 3.1.10. *Let k be a positive integer. Let $\sum\limits_{d\in D(k)} e_d d$ be a k-irreducible sum. Then there exists a proper divisor d of k that is not of the form k/p^b where p is a prime and b is a positive integer such that $e_d \neq 0$.*

Proof. Suppose $\sum_{i=1}^{n} a_i \frac{k}{p_i^{b_i}} = ck$ is a k-irreducible sum where the a_i and b_i are positive integers and the p_i are primes. Without loss of generality, we may assume the terms are distinct, that is, that if for some i_1 and i_2 we have $p_{i_1} = p_{i_2}$, then $b_{i_1} \neq b_{i_2}$. For all $1 \le i \le n$, combine the $\frac{k}{p_i^{b_i}}$-terms p_i-by-p_i to create $\frac{k}{p_i^{b_i-1}}$-terms until fewer than p_i remain. Perform these promotions in any order until it is no longer possible for any $1 \le i \le n$. We may now assume that $1 \le a_i \le p_i - 1$ for all $1 \le i \le n$. Let j be the index with $1 \le j \le n$ that maximizes b_i with the property that $p_i = p_1$. Define $\ell = \prod\limits_{\substack{1\le i\le n\\ p_i\neq p_1}} p_i^{b_i}$ Then we have

$$\begin{aligned} \sum_{i=1}^{n} a_i \frac{k}{p_i^{b_i}} = ck &\implies \sum_{i=1}^{n} a_i \frac{\ell p_1^{b_j}}{p_i^{b_i}} = c\ell p_1^{b_j} \\ &\implies p_1^{b_j} \mid \sum_{i=1}^{n} a_i \frac{\ell p_1^{b_j}}{p_i^{b_i}} \end{aligned}$$

Notice that for all $1 \le i \le n$, $a_i \frac{\ell p_1^{b_j}}{p_i^{b_i}}$ is an integer. Furthermore,

- If $p_i \neq p_1$, then $\frac{\ell}{p_i^{b_i}}$ is an integer, so $p_1 \mid a_i \frac{\ell p_1^{b_j}}{p_i^{b_i}}$.
- If $p_i = p_1$ and $i \neq j$, then $\frac{p_1^{b_j}}{p_i^{b_i}} = p_1^{b_j - b_i}$ where $b_j - b_i > 0$, so $p_1 \mid a_i \frac{\ell p_1^{b_j}}{p_i^{b_i}}$.

This means that p_1 must also divide the only remaining term, namely $a_j \frac{\ell p_1^{b_j}}{p_j^{b_j}} = a_j \ell$. Since $p_j = p_1$ does not divide l, it must divide a_j. This is a contradiction as $1 \le a_j \le p_j - 1$. □

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When k has more than two prime divisors

Example 3.1.11. Consider the case $k = 42 = 2 \cdot 3 \cdot 7$. We can show that there exist no 42-irreducible sums. By Lemma 3.1.7, it suffices to check that there are no minimally 42-irreducible sums. The proper divisors of 42 are $1, 2, 3, 6, 7, 14$, and 21. A minimally 42-irreducible sum, if such a sum exists, will be of the form $21a + 14b + 7c + 6d + 3e + 2f + g$ for non-negative integers a, b, c, d, e, f, g that are not all zero. As these coefficients are non-negative, we have a lower bound on all of these, namely 0. We can impose an upper bound too: For any proper divisor t of 42, the coefficient of t in a minimally k-irreducible sum must be less than the smallest prime divisor of k/t, by Lemma 3.1.6. Having bounded all coefficients, we are left with a finite list of potential minimally k-irreducible sums. We have the following bounds on the coefficients:

Divisor	Possible coefficients
21	$a \in \{0,1\}$
14	$b \in \{0,1,2\}$
7	$c \in \{0,1\}$
6	$d \in \{0,1,2,3,4,5,6\}$
3	$e \in \{0,1\}$
2	$f \in \{0,1,2\}$
1	$g \in \{0,1\}$

As there are finitely many ways to assign values to the coefficients a, b, c, d, e, f, and g, there is an upper bound on the possible total of such a sum. We can obtain this upper bound by assigning the maximum value to each coefficient. We get $21 \cdot 1 + 14 \cdot 2 + 7 \cdot 1 + 6 \cdot 6 + 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 100$. But $100 < 3 \cdot 42$, which implies that the total of a 42-irreducible cannot exceed $2 \cdot 42 = 84$. It is also implied by Definition 3.1.1 that the total of a k-irrecucible sum must be at least $2k$, therefore we conclude that any 42-irreducible sum must have 84 as its total. We now look for the coefficients in this range that yield a sum whose total is exactly 84. This is done with the help of a Python script, but can be done manually as well. The Python script in question can be found in Appendix A. The function of interest is `candidate_vectors_with_total_ck`. Inputting $(2, 42)$ into this function will yield the following five possible coefficient assignments:

Solutions	a	b	c	d	e	f	g
(1)	1	2	0	5	0	2	1
(2)	1	1	1	6	1	1	1
(3)	1	2	1	4	1	0	1
(4)	1	2	1	4	0	2	0
(5)	1	2	0	5	1	2	0

Each row corresponds to a unique assignment of values. With the same Python script we can show that all five of these sums contain a subsum with total equal to $k = 42$. However, we will do it manually:

- Solutions (2), (3), and (4) satisfy $a, b, c \geq 1$, so we have $21 + 14 + 7 = 42$ as a subsum for each of these sums.
- Solutions (1) and (5) satisfy $b, d \geq 2, f \geq 1$, so $14 + 14 + 6 + 6 + 2 = 42$ is a subsum for both of these sums.

We conclude that there is no 42-irreducible sum.

One might hope that this same strategy could be applied to all integers but that is not the case:

Example 3.1.12. Let m, n, h, r, s be feasible integers with $n = \alpha m$, $s = h$, and $k = \gcd(\alpha m, h) = 30$. It is conceivable at this point that after carelessly performing promotions, we are left with the following orbits:

- 1 orbit of size $\frac{\alpha m}{30}$,
- 4 orbits of size $6\frac{\alpha m}{30}$,
- 2 orbits of size $10\frac{\alpha m}{30}$, and

- 1 orbit of size $15\frac{\alpha m}{30}$

These account for $2\alpha m$ edges but there is no way to partition these further into two colour classes. This, of course, arises from the 30-irreducible sum from Example 3.1.2. Indeed, if we were able to partition these orbits into two colour classes of size αm, that partition would correspond directly to a partition of the terms in the irreducible sum $1 + 6 + 6 + 6 + 6 + 10 + 10 + 15 = 60$ into two subsums of total 30, which contradicts the fact that the sum is irreducible.

The sum $1 + 6 + 6 + 6 + 6 + 10 + 10 + 15 = 60$ is the only 30-irreducible sum. This can be shown using a method similar to that used in Example 3.1.11.

A minimally 42-irreducible sum, if such a sum exists, will be of the form $21a + 14b + 7c + 6d + 3e + 2f + g$ for non-negative integers a, b, c, d, e, f, g that are not all zero. As these coefficients are non-negative, we have a lower bound on all of these, namely 0. We can impose an upper bound too: For any proper divisor t of 42, the coefficient of t in a minimally k-irreducible sum must be less than the smallest prime divisor of k/t, by Lemma 3.1.6. Having bounded all coefficients, we are left with a finite list of potential minimally k-irreducible sums. We have the following bounds on the coefficients:

This example serves as motivation for the development of strategies for dealing with cases where there exists a k-irreducible sum where $k = \gcd(\alpha m, h)$. Given feasible integers m, n, h, r, s with $n = \alpha m$ and $s = h$, if there is no k-irreducible sum, then we can construct the desired good collection. In fact, we show later that even for $k = 30$, given feasible integers m, n, h, r, s with $n = \alpha m$ and $s = h$, a good collection can always be constructed.

3.2 Weak Irreducible Sums

Definition 3.2.1. *Let k be a positive integer. Denote by $D(k)$ the set of proper divisors of k. For each $d \in D(k)$, let e_d be some non-negative integer. The multiset $\{\!\{d^{e_d} : d \in D(k)\}\!\}$ of proper divisors d of k with multiplicity e_d is called* ***weak k-irreducible*** *if it admits no submultiset $\{\!\{d^{a_d} : d \in D(k)\}\!\}$ such that $\sum_{d \in D(k)} a_d d = k$*

Similarly to k-irreducible multisets, it is convenient to think of weak k-irreducible multisets as sums. To that end, we define a **weak k-irreducible sum** to be a sum of proper divisors of k that admits no subsum with total equal to k, that is, in the definition of a weak k-irreducible sum, the condition that k divides the total is absent. For example, $2 + 2 + 3$ is a weak 6-irreducible sum. A question of interest is: *For a given positive integer k, what is the largest weak k-irreducible sum?* We can also define the notion of a weak minimally k-irreducible sum:

Definition 3.2.2. *Let k be a positive integer. A weak k-irreducible sum is* ***weak minimally k-irreducible*** *if it admits no sub-sum of length at least two with total equal to any divisor of k.*

It is straightforward to see by analogy with Lemma 3.1.7 that for any weak k-irreducible sum, there exists a weak minimally k-irreducible sum with the same total. Therefore the aforementioned question is equivalent to *For a given positive integer k, what is the largest weak minimally k-irreducible sum?* For a positive integer k, define $F(k)$ to be the value of the largest weak minimally k-irreducible sum.

Lemma 3.2.3. *If $k = p^a$ where p is prime and a is a non-negative integer, then $F(k) = p^a - 1$*

Proof. Let $\sum_{i=0}^{a-1} e_{p^i} p^i$ be a weak minimally p^a-irreducible sum. This implies $e_{p^i} < p$ for all $0 \leq i \leq a-1$. Furthermore, it is clear that $\sum_{i=0}^{a-1}(p-1)p^i$ is a weak minimally p^a-irreducible sum, therefore $F(p^a) = \sum_{i=0}^{a-1}(p-1)p^i = \sum_{i=0}^{a-1}(p^{i+1} - p^i) = p^a - 1$. □

Lemma 3.2.4. *If $k = pq$ where p and q are distinct primes, then $F(k) = 2pq - p - q$.*

Proof. Let $p > q$ be prime numbers. Let $e_1 1 + e_p p + e_q q$ be the largest weak minimally pq-irreducible sum. We will show that $e_1 = 0, e_p = q - 1$, and $e_q = p - 1$.

First we will show that $(q-1)p + (p-1)q$ is a weak minimally pq-irreducible sum. Suppose there exist integers a and b with $0 \leq a \leq q-1$ and $0 \leq b \leq p-1$ such that $ap + bq = pq$. Then $p \mid b$ and $q \mid a$. This is a contradiction.

Now suppose $e_1 1 + e_p p + e_q q$ is a larger weak minimally pq-irreducible sum. Then clearly $0 \leq e_p \leq q-1$, $0 \leq e_q \leq p-1$, and $0 \leq e_1 \leq q-1$. We have $pq - e_1 > pq - q > (p-1)(q-1) = Fr(pq)$, so there exist integers a and b such that $pq - e_1 = ap + bq$ where $0 \leq a \leq q-1$ and $0 \leq b \leq p-1$. Thus, $ap + bq + e_1$ is a sub-sum with total pq. This is a contradiction. Therefore $F(pq) = (p-1)q + (q-1)p$, hence the result. □

Proposition 3.2.5. *Let $p > q$ be prime numbers, and a and b be positive integers. Then $F(p^a q^b) < p^a q^b (2 + \frac{1}{p-1})$*

Proof. Let $\sum\limits_{d \in D(p^a q^b)} e_d d$ be a weak minimally $p^a q^b$-irreducible sum. Then:

- For $d = p^i q^j$ with $0 \leq i \leq a$ and $0 \leq j \leq b-1$, we have $e_d \leq q-1$.
- For $d = p^i q^b$ with $0 \leq i \leq a-1$, we have $e_d \leq p-1$.

So,

$$\begin{aligned}
\sum_{d\in D(p^aq^b)} e_d d &\leq \sum_{i=0}^{a}\sum_{j=0}^{b-1} p^i q^j (q-1) + \sum_{i=0}^{a-1} p^i q^b (p-1) \\
&= \sum_{i=0}^{a} p^i \sum_{j=0}^{b-1} (q^{j+1}-q^j) + q^b \sum_{i=0}^{a-1} (p^{i+1}-p^i) \\
&= \frac{1-p^{a+1}}{1-p}(q^b-1) + q^b(p^a-1) \\
&= \frac{(1-p^{a+1})(q^b-1)+q^b(p^a-1)(1-p)}{1-p} \\
&= \frac{-1-2p^{a+1}q^b+p^{a+1}+q^bp^a+q^bp}{1-p} \\
&< \frac{-2p^{a+1}q^b+q^bp^a}{1-p} \\
&= \frac{p^aq^b}{p-1}(2p-1) \\
&= p^aq^b\left(2+\frac{1}{p-1}\right)
\end{aligned}$$

□

It follows from this proposition that if a p^aq^b-irreducible sum exists, it must have $2p^aq^b$ as its total.

Corollary 3.2.6. *Let $k = p^aq^b$ where $p > q$ are primes and a and b are positive integers. Suppose $\sum\limits_{d\in D(k)} e_d d$ is a minimally k-irreducible sum. Then $e_{p^{a-1}q^{b-1}} = 0$.*

Proof. Let $\sum_{d\in D(p^aq^b)} e_d d = 2p^aq^b$ be a minimally p^aq^b-irreducible sum with $e_{p^{a-1}q^{b-1}} > 0$. We will prove that this sum admits a subsum $ep^aq^{b-1} + fp^{a-1}q^b + p^{a-1}q^{b-1} = p^aq^b$, and therefore can't be p^aq^b-irreducible. The right side of the first inequality in the proof of Proposition 3.2.5 assumes the "worst case scenario" where $e_{p^aq^{b-1}} = q-1$. If, however, $e_{p^aq^{b-1}} \leq q-2$, then

$$\begin{aligned}
\sum_{d\in D(p^aq^b)} e_d d &\leq p^aq^b\left(2+\frac{1}{p-1}\right) - p^aq^{b-1} \\
&= p^aq^b\left(2+\frac{1}{p-1}-\frac{1}{q}\right) \\
&< 2k
\end{aligned}$$

This implies that in a minimally p^aq^b-irreducible sum, we must have $e_{p^aq^{b-1}} = q-1$. Similarly, if instead of the "worst case scenario" where $e_{p^{a-1}q^b} = p-1$ we have $e_{p^{a-1}q^b} \leq p-3$, then

$$\sum_{d \in D(p^a q^b)} e_d d \le p^a q^b \Big(2 + \frac{1}{p-1}\Big) - 2p^{a-1}q^b$$
$$= p^a q^b \Big(2 + \frac{1}{p-1} - \frac{2}{p}\Big)$$
$$< 2k$$

Therefore any minimally $p^a q^b$-irreducible sum must satisfy $e_{p^{a-1}q^b} \in \{p-2, p-1\}$.
We have $pq - 1 > (p-1)(q-1)$, so by Lemma 2.1.3 there exist nonnegative integers e and f such that $ep + fq = pq - 1$. We will show that $1 \le e \le q-1$ and $1 \le f \le p-2$.

- Suppose $e = 0$. Then $fq = pq - 1$, which implies that q divides $pq - 1$, which is a contradiction. We conclude that $e \ge 1$.
- Suppose $f = 0$. Then $ep = pq - 1$, which implies that p divides $pq - 1$, which is a contradiction. We conclude that $f \ge 1$.
- Suppose $e \ge q$. Then $pq + fq \le pq - 1$, which implies $fq < -1$. This is a contradiction as neither f nor q can be negative. We conclude that $e \le q - 1$.
- Suppose $f \ge p-1$, then $ep + (p-1)q \le pq - 1$, which implies $ep - q \le -1$. This is a contradiction, as $e \ge 1$ and $p > q$. We conclude that $f \le p - 2$.

Therefore we have $e \le e_{p^a q^{b-1}}$ and $f \le e_{p^{a-1}q^b}$. Thus, from $ep + fq = pq - 1$ we get a subsum $ep^a q^{b-1} + fp^{a-1}q^b + p^{a-1}q^{b-1} = p^a q^b$. This contradicts the assumption that the sum is minimally k-irreducible. □

Theorem 3.2.7. *Let $p > q$ be prime numbers and a and b be positive integers. Then $F(p^a q^b) < p^a q^b(2 + \frac{1}{p-1} - \frac{1}{p} + \frac{1}{pq})$.*

Proof. It follows from Corollary 3.2.6 and Proposition 3.2.5 that

$$F(p^a q^b) < p^a q^b \Big(2 + \frac{1}{p-1}\Big) - (q-1)p^{a-1}q^{b-1}$$
$$= p^a q^b \Big(2 + \frac{1}{p-1} - \frac{1}{p} + \frac{1}{pq}\Big).$$

□

Figure 3.1 is a plot showing the first hundred values of $F(k)$. Using Lemma 3.2.3 and Lemma 3.2.4 we can compute $F(k)$ for k of the form p^a or pq where p and q are prime and a is a positive integer. For other values of k, we proceed by brute force. A Python script has been used to compute $F(k)$ for all k in a finite range. This script, when given a value of k as an input, generates all possible weak minimally k-irreducible sums and outputs the total of its largest. This script can be found in Appendix B.

Several trends are to be seen in Figure 3.1. When k is a prime power, $F(k) = k - 1$ by Lemma 3.2.3. Therefore it is not surprising that the points representing these values form a

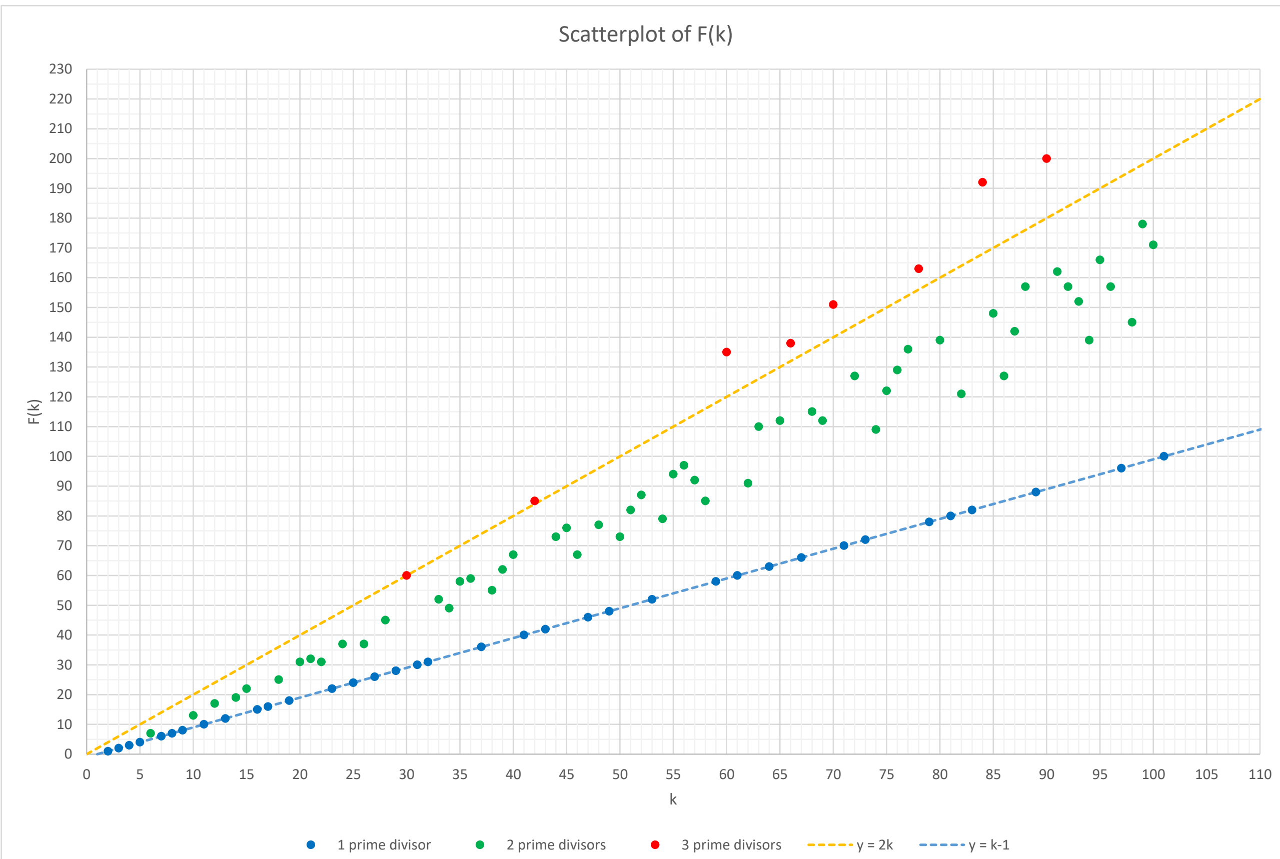


Figure 3.1: $F(k)$ for $2 \leq k \leq 101$

line. The green points, representing values of k with exactly two prime divisors, seem to lie below the line $y = 2k$. If this is shown to be true for all such k (that is, if $F(p^a q^b) < 2p^a q^b$ for all primes p, q and positive integers a, b), then it can be concluded that there exist no k-irreducible sums for values of k with exactly two prime divisors. Theorem 3.2.7 provides a close upper bound on $F(k)$ for these values of k.

Chapter 4

Partitioning the remaining T-sets

4.1 Acceptable Partitions

Given feasible integers m, n, h, r, s with $n = \alpha m$ and $h = s$, the construction from Section 2 leaves us with the task of creating colour classes out of the edges in $\binom{[\alpha m]}{h} \setminus T_{\alpha m}$ where $T_{\alpha m}$ is the set of all edges whose orbits by 1-shifts is of size αm. Recall that $P = \left\{ \frac{\alpha m}{j} \ : \ j \text{ divides } k \right\}$. The question of interest is the following:

For which values of α, m, and h can we guarantee that there exists a way to partition the orbits contained in the sets T_i , $i \in P \setminus \{\alpha m\}$, so that the orbits in each part contain, in total, αm edges?

For such α, m, and h, a good collection can be created. Note that the definition of feasible integers in this case asks that q, the number of edges per colour, be equal to αm. Note also that this question makes no reference to the value of r. Thus, it is more convenient at this point to work with a relaxed version of the definitions of feasible integers and of a good collection.

Definition 4.1.1. *Integers α, m, h with $m \geq h \geq 1$ and $\alpha \geq 2$ are called **acceptable** if $\alpha m \mid \binom{\alpha m}{h}$.*

Note that we consider a set of acceptable integers as an ordered triple. Acceptability is a necessary condition following from feasibility. It says that the total number of colours must be an integer.

Definition 4.1.2. *Let α, m, h be acceptable integers. A partition $\mathcal{C} = \{C_1, C_2, \ldots, C_\ell\}$ of $\binom{[\alpha m]}{h} \setminus T_{\alpha m}$ is called **acceptable** if the following hold:*

A1: $|C_j| = \alpha m$ for all $C_j \in \mathcal{C}$

A2: For all $C_j \in \mathcal{C}$ and h-subset $A \in C_j$, we have $\mathcal{R}_A^1 \subseteq C_j$

Each of the C_j in an acceptable partition can be seen as a colour class. Condition A2 formalizes our convention of colouring edges by the orbit. Acceptable partitions represent the second and final step in the construction of a good collection, the first being the assignment of colours to the orbits of size αm, which was done in Section 2.

Lemma 4.1.3. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let A be an h-subset of $[\alpha m]$. Then* $\sum_{B \in \mathcal{R}_A^1} |B \cap [m]| = |\mathcal{R}_A^1| \frac{h}{\alpha}$.

Proof. Recall that $\mathcal{R}_A^\infty$ is the orbit of A by 1-shifts. We have $|\mathcal{R}_A^1| = \frac{\alpha m}{k/t}$ for some proper divisor t of k. Define $\mathcal{S}_A^1$ to be the multiset of edges obtained by starting with A and performing 1-shifts clockwise αm times. In other words, the elements of $\mathcal{S}_A^1$ are precisely those of $\mathcal{R}_A^1$, and each element has multiplicity k/t.

We will compute the sum $\sum_{B \in \mathcal{S}_A^1} |B \cap [m]|$, keeping in mind that since we are using an index multiset, the number of times each index is used to define a term of the sum is equal to the multiplicity of that index within the index multiset. In other words, we have $\sum_{B \in \mathcal{S}_A^1} |B \cap [m]| = \frac{k}{t} \sum_{B \in \mathcal{R}_A^1} |B \cap [m]|$.

We can compute the sum $\sum_{B \in \mathcal{S}_A^1} |B \cap [m]|$ in a different way: The edge A has h vertices, each of which will, when rotated αm times by a 1-shift, appear exactly once in each of the αm positions. This means that each vertex of the edge will appear exactly m times as an element of $[m]$. Since there are h vertices in the edge, the total number of times a vertex will appear in $[m]$ is mh. We therefore have $\sum_{B \in \mathcal{S}_A^1} |B \cap [m]| = mh = ms$. Paired with the previous computation, this gives us $\sum_{B \in \mathcal{R}_A^1} |B \cap [m]| = \frac{ms}{k/t}$ □

Lemma 4.1.4. *Suppose m, n, h, r, s with $s = h$ and $n = \alpha m$ for some integer α are feasible integers. If there exists an acceptable partition for α, m, h, then there exists a good collection for m, n, h, r, s.*

Proof. Let $k = \gcd(\alpha m, h)$. We start by noticing that α, m, h are acceptable integers. Indeed, feasibility implies that $s \mid \binom{n-1}{h-1}$ which, in our case, is equivalent to $\alpha m \mid \binom{\alpha m}{h}$.

Suppose $\mathcal{C}$ is an acceptable partition for parameters α, m, and h. We know from Section 2 that the set $T_{\alpha m}$ of edges with orbit size αm can be partitioned into colour classes of size αm such that each colour class contains $q = \alpha m$ edges, exactly $p = \frac{mr}{h}$ or 0 of which are contained in $[m]$. Form $\mathcal{C}'$ by taking the union of $\mathcal{C}$ with this partition. We now have a partition of $\binom{[\alpha m]}{h}$ satisfying condition C4 of a good collection. Thus it remains to show that C1-C3 are satisfied.

- C1: Follows from A1.
- C2: It follows from A2 that every $C_j \in \mathcal{C}$ is a union of orbits by 1-shifts. Let ℓ be the number of orbits by 1-shifts that make up C_j. Let $\mathcal{R}_{A_1}^1, \mathcal{R}_{A_2}^1, \ldots \mathcal{R}_{A_\ell}^1$ be the orbits in question and let $\frac{\alpha m}{k/t_1}, \frac{\alpha m}{k/t_2}, \ldots, \frac{\alpha m}{k/t_\ell}$ be their respective sizes, where $t_1, t_2, \ldots, t_\ell$ are

proper divisors of k, noting that these are not necessarily distinct. By A1, we have:

$$\begin{aligned}
&\sum_{i=1}^{\ell} \frac{\alpha m}{k/t_i} = \alpha m \\
\implies &\sum_{i=1}^{\ell} \frac{ms}{k/t_i} = ms \\
\implies &\sum_{i=1}^{\ell} \Big(\sum_{B \in \mathcal{R}^1_{A_i}} |B \cap [m]| \Big) = ms && \text{By Lemma 4.1.3.} \\
\implies &\sum_{A \in C_j} |A \cap [m]| = ms && \text{Since } C_j = \bigcup_{i=1}^{\ell} \mathcal{R}^1_{A_i}.
\end{aligned}$$

On the other hand, any $C_j \in \mathcal{C}' \setminus \mathcal{C}$ consists of a union of m orbits by m-shifts, each of which is of size $h = s$, so it is straightforward to see that the condition is satisfied.

- C3: Follows from the observation that $\mathcal{C}'$ is a partition of the set of h-subsets of $[\alpha m]$.

□

The main implication of this lemma is that we need not consider the value of r after checking the necessary divisibility conditions. In fact, we observe that for any acceptable integers α, m, h, there exists an integer r that makes the parameter set m, n, h, r, s with $n = \alpha m$ and $s = h$ feasible. All that is required of r by the definition of feasibility is that $r \leq h$, $h \mid mr$, and $r \mid \binom{m-1}{h-1}$. The number $r = \frac{h}{gcd(m,h)}$, for example, satisfies all three conditions. In general there are multiple ways to extend acceptable integers to feasible integers.

Lemma 4.1.5. *Let α, m, h be acceptable integers such that $k = \gcd(\alpha m, h)$ has at most one prime divisor. Then there exists an acceptable partition for this parameter set.*

Proof. Suppose $k = 1$. Recall that $P = \{\frac{\alpha m}{k} t \quad : \quad t \text{ divides } k\}$ is the set of possible orbit sizes. In this case, $P = \{\alpha m\}$, that is, every edge has orbit size αm. The task at hand is to find a partition of $\binom{[\alpha m]}{h} \setminus T_{\alpha m} = \emptyset$. Thus, we are done. Recall that colours were already assigned to the edges in $T_{\alpha m}$ in Section 2.

Suppose $k = p^a$ where p is some prime and a is a positive integer. We have $P = \{\frac{\alpha m}{k}, \frac{\alpha m}{k} p, \ldots, \frac{\alpha m}{k} p^a\}$. Start by arbitrarily bundling orbits together to form colour classes in any way. For instance, p orbits of size $\frac{\alpha m}{k} p^{a-1} = \frac{\alpha m}{p}$ may be grouped to make a single colour class. If this process continues until every edge is assigned a colour, then the resulting colouring is an acceptable partition, so we are done. Suppose, on the other hand, that this process terminates with unassigned edges remaining, meaning it is impossible to further promote orbits to colour classes. Since α, m, h are acceptable integers, we know that αm divides $\binom{\alpha m}{h}$, the total number of edges. We also know that αm divides $|T_{\alpha m}|$ since $T_{\alpha m}$ is a union of orbits of size αm, and therefore αm divides $|\binom{[\alpha m]}{h} \setminus T_{\alpha m}|$. Thus, by denoting by a_i the number of unassigned orbits of size $\frac{\alpha m}{k}p^i$, we have

$$\alpha m \mid \sum_{i=0}^{a-1} a_i \frac{\alpha m}{k} p^i$$

Therefore

$$k \mid \sum_{i=0}^{a-1} a_i p^i$$

This is a k-irreducible sum since we assumed it was impossible to further promote orbits to colour classes. However, by Lemma 3.1.8, this is impossible as k is a prime power. We conclude that $a_i = 0$ for all $0 \leq i \leq a-1$, and therefore all orbits have been promoted to colour classes. □

We now have all the pieces necessary to construct an h-factorization of $K^h_{\alpha m}$ containing an embedded r-factorization of K^h_m in the case of feasible integers m, n, h, r, s where $n = \alpha m$, $s = h$, and $k = \gcd(\alpha m, h)$ has at most one prime divisor:

Theorem 4.1.6. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. If $k = \gcd(\alpha m, h)$ has at most one prime divisor, then there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of K^h_m.*

Proof. By Lemma 4.1.5, we can construct an acceptable partition for the acceptable integers α, m, h. By Lemma 4.1.4, from this acceptable partition we can construct a good collection for the feasible integers m, n, h, r, s. Finally, by Theorem 1.2.6, this implies that there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of K^h_m. □

Lemma 4.1.7. *Let α, m, h be acceptable integers. Suppose, after a series of promotions, that the only orbits not promoted to colour classes are of sizes of the form $\frac{\alpha m}{Q}$ for prime powers Q. Then an acceptable partition exists for the parameters α, m, h.*

Proof. Let $p_1^{b_1}, \ldots, p_f^{b_f}$ be the prime powers in question, noting that the p_i are not necessarily distinct. For each $1 \leq i \leq f$, bundle the $\frac{\alpha m}{p_i^{b_i}}$-orbits p_i-by-p_i until fewer than p_i remain. Continue doing this in any order until it is no longer possible for any $1 \leq i \leq f$. We claim that this process terminates only when every orbit has been promoted to an orbit of size αm, that is, a colour class. If it terminated before that, let a_i be the number of $\frac{\alpha m}{p_i^{b_i}}$-orbits remaining for each $1 \leq i \leq f$. The total number of edges in the union of all these orbits must be a multiple of αm, as these edges are the only edges to which a colour has not been assigned. Thus we have $\alpha m \mid \sum_{i=1}^{f} a_i \frac{\alpha m}{p_i^{b_i}}$. By the same argument used in the proof of Lemma 3.1.10, we come to the conclusion that $p_i \mid a_i$ for some $1 \leq i \leq f$ with $a_i > 0$. This contradicts the assumption that the process had terminated, as at least one more promotion could have been made by combining p_i orbits of size $\frac{\alpha m}{p_i^{b_i}}$. □

Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Recall that, for all proper divisors t of $k = \gcd(\alpha m, h)$, we define $g(t)$ to be the number of orbits by 1-shifts of size $\frac{\alpha m}{k} t$.

Theorem 4.1.8. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Suppose that $g(t) \geq Fr(k/t)$ for every proper divisor t of $k = \gcd(\alpha m, h)$ such that k/t has at least two prime divisors. Then there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of K^h_m.*

Proof. Since Theorem 4.1.6 takes care of the cases where k has fewer than two prime divisors, we may assume k has at least two. This guarantees that k will have at least one divisor t where k/t has at least two prime divisors. Define
$H = \{t \in \mathbb{N} \ : \ t \mid k \text{ and } k/t \text{ has at least two prime divisors}\}$. First, label and order the elements of H as $t_1 \preceq t_2 \preceq \ldots \preceq t_j$ in any way respecting the partial order defined by $x \preceq y \iff x$ has at most as many prime divisors as y. Then, starting with $t = t_1$ onward, we do the following:

Let $p > q$ be the smallest prime divisors of k/t. The number of orbits of size $\frac{\alpha m}{k}t$ is at least* $g(t)$, which in turn is at least $Fr(k/t)$. Therefore the number of orbits of size $\frac{\alpha m}{k}t$ can be written as $ap + bq$ for some nonnegative integers a, b. Thus, we can:

- Bundle p orbits of size $\frac{\alpha m}{k}t$ to make a single orbit of size $\frac{\alpha m}{k}tp$. Do this a times.
- Bundle q orbits of size $\frac{\alpha m}{k}t$ to make a single orbit of size $\frac{\alpha m}{k}tq$. Do this b times.

Since at every step orbits are being promoted to strictly larger orbits, this process is guaranteed to terminate, leaving us with no orbits of sizes $\frac{\alpha m}{k}d$ for any $d \in H$. Thus, all that remain are orbits of sizes of the form $\frac{\alpha m}{Q}$ where Q is a prime power. By Lemma 4.1.7, these can be promoted to complete the acceptable partition. By Lemma 4.1.4, there is a good collection for the feasible integers m, n, h, r, s. Finally, by Theorem 1.2.6, there exists an h-factorization of $K^h_{\alpha m}$ containing an embedded r-factorization of K^h_m. □

4.2 Sizes of T-sets

Recall that for $i \in P$, the set T_i is the set of all edges whose orbit under $1-$shifts is of length i. For all positive divisors $t \mid k$, define

- $f(t) = \binom{\frac{\alpha m}{k}t}{\frac{h}{k}t}$ The number of edges with orbit size dividing $\frac{\alpha m}{k}t$;
- $g(t) = |T_{\frac{\alpha m}{k}t}| / (\frac{\alpha m}{k}t)$ The number of orbits of size $\frac{\alpha m}{k}t$.

$f(t)$ is the number of ways to construct an edge which is preserved by an $\frac{\alpha m}{k}t$-shift. These are precisely the edges whose orbit size divides $\frac{\alpha m}{k}t$. It is therefore clear that $f(t) = \sum_{d \mid t} |T_{\frac{\alpha m}{k}d}|$.

Applying Möbius inversion, we get

$$|T_{\frac{\alpha m}{k}t}| = \sum_{d \mid t} \mu(d) f\left(\frac{t}{d}\right)$$

*We say "at least" and not "exactly" since at every step, some orbits are being promoted. These new orbits need to be counted as well.

where μ is the Möbius function defined on positive integers as

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

Lemma 4.2.1. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Let $t = p_1^{a_1} \dots p_\ell^{a_\ell}$ be a proper divisor of k where $p_1, \dots, p_\ell$ are distinct primes and $a_1, \dots, a_\ell$ are positive integers. Define $P(t) = \{p_1, \dots, p_\ell\}$. Then*

$$g(t) = \frac{k}{t\alpha m} \sum_{A \subseteq P(t)} (-1)^{|A|} f\left(\frac{t}{\prod_{p \in A} p}\right).$$

Proof.

$$\begin{aligned}
g(t)\frac{\alpha m}{k} t &= |T_{\frac{\alpha m}{k} t}| \\
&= \sum_{d \mid t} \mu(d) f\left(\frac{t}{d}\right) \\
&= \sum_{\substack{0 \le i_1 \le a_1 \\ \vdots \\ 0 \le i_\ell \le a_\ell}} \mu(p_1^{i_1} \dots p_\ell^{i_\ell}) f(p_1^{a_1 - i_1} \dots p_\ell^{a_\ell - i_\ell}) \\
&= \sum_{\substack{i_1 \in \{0,1\} \\ \vdots \\ i_\ell \in \{0,1\}}} \mu(p_1^{i_1} \dots p_\ell^{i_\ell}) f(p_1^{a_1 - i_1} \dots p_\ell^{a_\ell - i_\ell}) \quad \text{Since } \mu \text{ is non-zero only for square-free inputs.} \\
&= \sum_{A \subseteq \{p_1, \dots, p_\ell\}} \mu\Big(\prod_{p \in A} p\Big) f\left(\frac{p_1^{a_1} \dots p_\ell^{a_\ell}}{\prod_{p \in A} p}\right) \\
&= \sum_{A \subseteq \{p_1, \dots, p_\ell\}} (-1)^{|A|} f\left(\frac{t}{\prod_{p \in A} p}\right),
\end{aligned}$$

hence the result. □

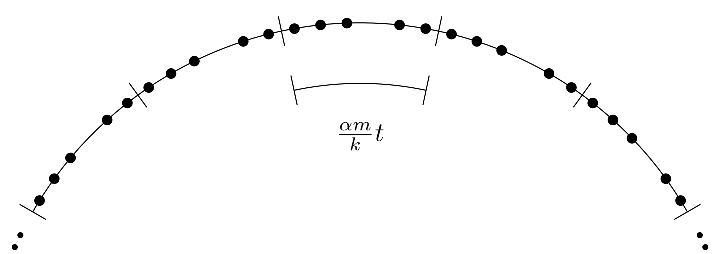
Lemma 4.2.2. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Then $g(1) \geq 1$.*

Proof. The number of orbits of size $\frac{\alpha m}{k}$ is $g(1)$, so it suffices to show that there exists at least one orbit of size $\frac{\alpha m}{k}$. Considering the αm edges arranged in a circular formation as usual, split the circle into k slices of $\frac{\alpha m}{k}$ consecutive vertices. In each slice, choose the first h/k vertices, for a total of h vertices. The edge that consists of these h vertices is of size $\frac{\alpha m}{k}$ because the smallest shift that preserves this edge is clearly $\frac{\alpha m}{k}$. Therefore $g(1) \geq 1$. □

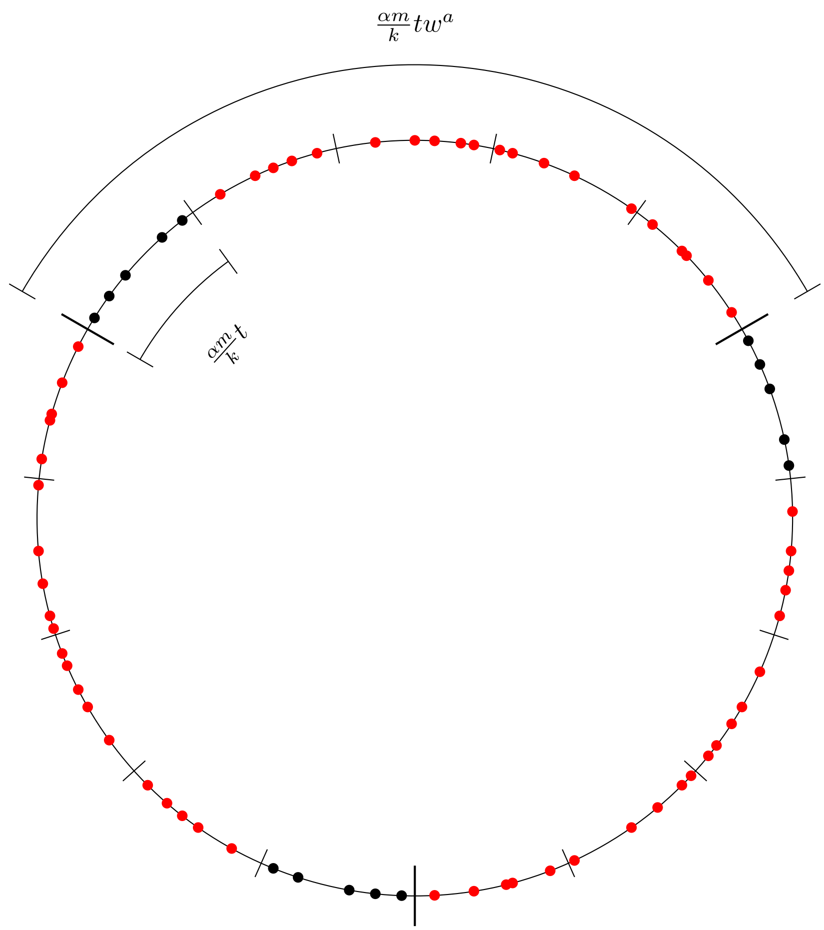
Lemma 4.2.3. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Let w be a prime and let a be a positive integer such that $w^a \mid k$. Suppose t is a divisor of k/w^a. Then*

$$g(tw^a) \geq (f(t) - 1)^{w^a - 1} g(t).$$

Proof. Take an edge in $T_{\frac{\alpha m}{k}t}$. Viewing the vertices as the numbers $1, 2, \ldots, \alpha m$ arranged in a circle, we can divide this circle into k/t slices of length $\frac{\alpha m}{k}t$.



Since the edge's orbit is of size $\frac{\alpha m}{k}t$ the edge in question will be represented identically within each slice. Take any w^a consecutive slices. Notice that if we alter the configuration of the vertices in all but one slice, these w^a consecutive slices uniquely determine an edge in $T_{\frac{\alpha m}{k}tw^a}$ by copying this sequence of slices around the circle:



Thus, we can construct elements of $T_{\frac{\alpha m}{k}tw^a}$ as follows:
Fix one of the slices. There are w^a ways to do this. In the other w^a-1 slices, choose among the $\binom{\frac{\alpha m}{k}t}{\frac{h}{k}t}-1$ other other configurations. There are $\left(\binom{\frac{\alpha m}{k}t}{\frac{h}{k}t}-1\right)^{w^a-1}$ ways to do this. The resulting edge's orbit is of size $\frac{\alpha m}{k}tw^a$. We have

$$|T_{\frac{\alpha m}{k}tw^a}| \geq w^a\left(\binom{\frac{\alpha m}{k}t}{\frac{h}{k}t}-1\right)^{w^a-1}|T_{\frac{\alpha m}{k}t}| = w^a(f(t)-1)^{w^a-1}|T_{\frac{\alpha m}{k}t}|.$$

So

$$g(tw^a) = \frac{|T_{\frac{\alpha m}{k}tw^a}|}{\frac{\alpha m}{k}tw^a} \geq \frac{w^a(f(t)-1)^{w^a-1}|T_{\frac{\alpha m}{k}t}|}{\frac{\alpha m}{k}tw^a} = (f(t)-1)^{w^a-1}g(t).$$

□

Corollary 4.2.4. *Let m,n,h,r,s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Let t be a proper divisor of k. Suppose w is a prime and a is a positive integer such that tw^a divides k. Then $g(tw^a) \geq g(t)$.*

Proof. We have

$$f(t) = \binom{\frac{\alpha m}{k}t}{\frac{h}{k}t} \geq \frac{\alpha m}{k}t \geq \frac{\alpha m}{k} \geq \frac{\alpha m}{h} \geq \alpha \geq 2.$$

So by Lemma 4.2.3,

$$g(tr^a) \geq (f(t)-1)^{r^a-1}g(t) \geq (2-1)g(t) = g(t).$$

□

Lemma 4.2.5. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Suppose d and t are distinct divisors of k such that $d \mid t$. Then $g(t) \geq g(d)$.*

Proof. Write t/d as $p_1^{a_1}p_2^{a_2}\dots p_\ell^{a_\ell}$ where $p_1, p_2, \dots, p_\ell$ are the distinct prime divisors of t/d and $a_1, a_2, \dots, a_\ell$ are positive integers. Then it follows from Corollary 4.2.4 that $g(d) \leq g(dp_1^{a_1})$. Applying Corollary 4.2.4 repeatedly $\ell - 1$ more times, we get $g(d) \leq g(dp_1^{a_1}) \leq g(dp_1^{a_1}p_2^{a_2}) \leq \dots \leq g(dp_1^{a_1}p_2^{a_2}\dots p_\ell^{a_\ell}) = g(t)$, hence the result. □

Theorem 4.2.6. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Suppose t and d are positive divisors of k such that $t > d$. Then $g(t) \geq g(d)$.*

Proof. Let t and d be divisors of k such that $t > d$. If $d \mid t$, then $g(t) \geq g(d)$ by Lemma 4.2.5. We may therefore assume that there exist integers a, b such that $t = ad + b$ with $1 \leq b < d$ and $a \geq 1$. Note that, since $d \nmid t$, we have $d \geq 2$.

Consider an edge $e \in T_{\frac{\alpha m}{k}d}$. It can be seen as $\frac{k}{d}$ identical slices in which $\frac{h}{k}d$ of the $\frac{\alpha m}{k}d$ vertices belong to e. Consider the first $\frac{\alpha m}{k}t$ vertices. This subset of $[\alpha m]$ can be seen as a of the aforementioned identical slices followed clockwise by $\frac{\alpha m}{k}b$ additional vertices. Let $c = \lceil \frac{a}{2} - 1 \rceil$. Let e' be any edge constructed by copying the first $a - c$ slices in e and choosing $\frac{h}{k}(cd + b)$ vertices among the $\frac{\alpha m}{k}(cd + b)$ that follow, and copying this slice of length $\frac{\alpha m}{k}t$ around the circle. Let $\frac{\alpha m}{k}j$ be the size of the orbit of e'. Since e' was constructed by repeating a slice of length $\frac{\alpha m}{k}t$ around the circle, it follows that e' is preserved by an $\frac{\alpha m}{k}t$-shift. Since the size of the orbit of e' is $\frac{\alpha m}{k}j$, it follows that $\frac{\alpha m}{k}j$ is the smallest shift length that preserves e'. Therefore $(\frac{\alpha m}{k}t)/(\frac{\alpha m}{k}j)$ must be an integer, which implies that j divides t. We aim to show that $j = t$.

Suppose $j \neq t$. Since $j \mid t$, we have

$$\begin{aligned} j &\leq \frac{1}{2}t \\ &= \frac{1}{2}(ad + b) \\ &\leq \frac{1}{2}((2c+1)d + b) \\ &= cd + \frac{1}{2}(d + b) \\ &< (c+1)d \\ &\leq (a-c)d \end{aligned}$$

But e' was constructed by copying the first $a-c$ slices in e (that is, e and e' are identical in the first $\frac{\alpha m}{k}(a-c)d$ vertices), so e and e' must be identical in the first $\frac{\alpha m}{k}j$ vertices. Therefore, since e' is preserved by an $\frac{\alpha m}{k}j$-shift, e too must be preserved by an $\frac{\alpha m}{k}j$-shift.

- If $d < j$, we must have $d \mid j$, which in turn implies $d \mid t$. This is a contradiction.
- If $d = j$, we have $d \mid t$. This is again a contradiction.
- Finally, if $d > j$, then $\frac{\alpha m}{k}d$ is not the smallest shift preserving e. This is also a contradiction, as the orbit of e is of size $\frac{\alpha m}{k}d$.

We conclude that $j = t$, so $e' \in T_{\frac{\alpha m}{k}t}$.
For every choice of e there are at least $\binom{\frac{\alpha m}{k}(cd+b)}{\frac{h}{k}(cd+b)}$ ways to choose $e' \in T_{\frac{\alpha m}{k}t}$, as e' is constructed by choosing $\frac{h}{k}(cd+b)$ vertices among $\frac{\alpha m}{k}(cd+b)$. Therefore $|T_{\frac{\alpha m}{k}t}| \geq \binom{\frac{\alpha m}{k}(cd+b)}{\frac{h}{k}(cd+b)}|T_{\frac{\alpha m}{k}d}|$. We have

$$g(t) = \frac{k}{\alpha m t}|T_{\frac{\alpha m}{k}t}| \geq \frac{k}{\alpha m t}\binom{\frac{\alpha m}{k}(cd+b)}{\frac{h}{k}(cd+b)}|T_{\frac{\alpha m}{k}d}| = \binom{\frac{\alpha m}{k}(cd+b)}{\frac{h}{k}(cd+b)}\frac{d}{t}g(d).$$

Recall that, since the case $m = h$ is trivial, we may assume $h < m$. In particular, this implies $\frac{\alpha m}{k} \geq 3$.

- If $a = 1$ or $a = 2$, then

$$\begin{aligned}
\binom{\frac{\alpha m}{k}(cd+b)}{\frac{h}{k}(cd+b)}\frac{d}{t} &= \binom{\frac{\alpha m}{k}b}{\frac{h}{k}b}\frac{d}{t} && \text{Because } c = \lceil a/2 - 1\rceil = 0.\\
&\geq \binom{\frac{\alpha m}{k}b}{\frac{h}{k}b}\frac{d}{2d+b} && \text{Because } t = d+b \text{ or } t = 2d+b.\\
&\geq \frac{\alpha m}{k}b\frac{d}{2d+b} && \\
&\geq 3b\frac{d}{2d+b} && \text{Because } \alpha \geq 2 \text{ and } k \leq h < m.\\
&> 3b\frac{d}{3d} && \text{Because } b < d.\\
&\geq b && \\
&\geq 1 &&
\end{aligned}$$

  Therefore $g(t) \geq \binom{\frac{\alpha m}{k}(cd+b)}{\frac{h}{k}(cd+b)}\frac{d}{t}g(d) > g(d)$.

- If $a \geq 3$, then we have

$$
\begin{aligned}
&(3d-2)a \geq 9d-6 \\
\implies &(3d-2)a \geq 6d && \text{Because } d \geq 2. \\
\implies &3ad-2a \geq 6d \\
\implies &\frac{1}{2}-\frac{1}{3d} \geq \frac{1}{a} \\
\implies &t\Big(\frac{1}{2}-\frac{1}{3d}\Big)+\frac{b}{2} \geq \frac{t}{a} \\
\implies &t\Big(\frac{1}{2}-\frac{1}{3d}\Big)+\frac{b}{2} \geq d \\
\implies &\frac{t}{2}-d+\frac{b}{2} \geq \frac{t}{3d} \\
\implies &\frac{ad+b}{2}-d+\frac{b}{2} \geq \frac{t}{3d}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
g(t) &\geq \frac{d}{t}\frac{\alpha m}{k}(cd+b)g(d) \\
&\geq \frac{d}{t}3\Big(\Big(\frac{a}{2}-1\Big)d+b\Big)\Big)g(d) \\
&= \frac{d}{t}3\Big(\frac{ad+b}{2}-d+\frac{b}{2}\Big)g(d) \\
&\geq \frac{d}{t}3\frac{t}{3d}g(d) && \text{By the above inequality.} \\
&= g(d)
\end{aligned}
$$

□

Lemma 4.2.7. *Let m,n,h,r,s be feasible integers with $n=\alpha m$ and $s=h$. Let $k=\gcd(\alpha m,h)$. Let p be a prime and a be a positive integer such that p^a divides k. Then $g(p^a)\geq(p-1)^2$.*

Proof. Recall that we are assuming $h<m$ to avoid the trivial case $h=m$. Therefore $\frac{\alpha m}{k}\geq 3$. Thus, we have

$$f(1)=\binom{\frac{\alpha m}{k}}{\frac{h}{k}}\geq\frac{\alpha m}{k}\geq 3.$$

By Lemma 4.2.3, we have

$$
\begin{aligned}
g(p^a) &= g(1p^a) \\
&\geq (f(1)-1)^{p^a-1} g(1) \\
&\geq 2^{p^a-1} g(1) && \text{Since } f(1) \geq 3. \\
&\geq 2^{p^a-1} && \text{Since } g(1) \geq 1 \text{, by Lemma 4.2.2.} \\
&\geq 2^{p-1} && \text{Since } a \geq 1.
\end{aligned}
$$

It remains to show that $2^{p-1} \geq (p-1)^2$. For $p = 2$, $p = 3$, and $p = 5$, it is easy to check that this inequality is satisfied:

p	2^{p-1}	$(p-1)^2$
2	2	1
3	4	4
5	16	16

To prove that this inequality holds for all primes larger than 5, it suffices to show that $2^\ell \geq \ell^2$ for all integers $\ell \geq 4$ by induction.

- Base case: For $\ell = 4$, we have $2^\ell = \ell^2 = 16$, so the inequality is verified.
- Induction step: Take an integer $\ell \geq 4$ and suppose $2^\ell \geq \ell^2$. Then we have

$$
\begin{aligned}
2^{\ell+1} &= 2 \cdot 2^\ell \\
&\geq 2\ell^2 && \text{By the induction hypothesis.} \\
&\geq \ell^2 + 4\ell && \text{Because } \ell \geq 4. \\
&\geq \ell^2 + 2\ell + 1 && \text{Because } \ell \geq 4. \\
&= (\ell+1)^2.
\end{aligned}
$$

We conclude that $g(p^a) \geq (p-1)^2$.

□

In general $g(p^a)$ is much larger that $(p-1)^2$. However this inequality suffices for our purposes.

4.3 If k has two prime divisors

We now have all the prerequisites to deal with the case where k has two prime divisors.

Lemma 4.3.1. *Let α, m, h be acceptable integers such that $k = \gcd(\alpha m, h)$ has exactly two prime divisors. Then there exists an acceptable partition for this parameter set.*

Proof. Let $k = p^a q^b$ where $p > q$ are the two prime divisors of k and a and b are positive integers. The possible orbit sizes are of the form $\frac{\alpha m}{k} p^i q^j$ with $0 \leq i \leq a$ and $0 \leq j \leq b$. Omitting the orbits of size αm, as these are irrelevant to the definition of an acceptable partition, the sizes of orbits of interest all belong to one of the following types:

Type 1: $\frac{\alpha m}{k} q^j$ with $0 \leq j \leq b-1$

Type 2: $\frac{\alpha m}{k} p^i q^j$ with $1 \leq i \leq a-1$ and $0 \leq j \leq b-1$

Type 3: $\frac{\alpha m}{k} p^i q^b$ with $0 \leq i \leq a-1$

Type 4: $\frac{\alpha m}{k} p^a q^j$ with $0 \leq j \leq b-1$

For orbits of size of Type 1, starting with $j = 0$ and increasing until $j = b-1$, bundle the orbits of size $\frac{\alpha m}{k} q^j$ q-by-q until fewer than q remain. Note that the result of such a promotion is a union of orbits, but as usual we continue to refer to them as orbits.

Let t be a divisor of k such that the orbit sizes $\frac{\alpha m}{k} t$ are of Type 2. These have the property that $p \mid t$ and $p, q \mid \frac{k}{t}$. By Theorem 4.2.6, we have $g(t) \geq g(p)$. Furthermore, by Lemma 4.2.7, we have $g(p) \geq (p-1)^2$. Since $p > q$, we can combine these two observations to get $g(t) > (p-1)(q-1) = Fr(\frac{k}{t})$. By Lemma 2.1.3, we may now apply Frobenius promotions to the orbits of size of Type 2 in increasing order of orbit size, that is, write the number of orbits of size $\frac{\alpha m}{k} p^i q^j$ with $1 \leq i \leq a-1$ and $0 \leq j \leq b-1$ as $ep + fq$ for nonnegative integers f and q, combine these orbits p-by-p to make e orbits of size $\frac{\alpha m}{k} p^{i+1} q^j$, and combine the rest q-by-q to make f orbits of size $\frac{\alpha m}{k} p^i q^{j+1}$. At each step these promotions may create orbits of size of Type 2, 3, or 4, but never of Type 1. Since these steps are being made in increasing order of orbit sizes, this will promote all orbits of size of Type 2. All that remain now are orbits of size of Type 1, 3, and 4.

For orbits of size of Type 3, starting with $i = 0$ and increasing until $i = a-1$, bundle these p-by-p until fewer than p remain. Such a promotion creates either a colour class (orbit of size αm) or another orbit of size of Type 3.

For orbits of size of Type 4, starting with $j = 0$ and increasing until $j = b-1$, bundle these q-by-q until fewer than q remain. Such a promotion creates either a colour class (orbit of size αm) or another orbit of size of Type 4.

The total number of unassigned edges is bounded above by

$$\begin{aligned}
&\sum_{i=0}^{b-1}(q-1)\frac{\alpha m}{k}q^j + \sum_{i=0}^{a-1}(p-1)\frac{\alpha m}{k}p^i q^b + \sum_{j=0}^{b-1}(q-1)\frac{\alpha m}{k}p^a q^j \\
=&\frac{\alpha m}{k}((q^b-1) + q^b(p^a-1) + p^a(q^b-1)) \\
=&\frac{\alpha m}{k}(2k-1-p^a) \\
<&2\alpha m
\end{aligned}$$

So there are either 0 or αm unassigned edges. In the former case, we are done. In the latter case, these edges form a colour class. $\square$

Theorem 4.3.2. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Assume $k = p^a q^b$ where $p > q$ are primes and a and b are positive integers. Then there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of K^h_m.*

Proof. By Lemma 4.3.1, we can construct an acceptable partition for the acceptable integers α, m, h. By Lemma 4.1.4, from this acceptable partition we can construct a good collection for the feasible integers m, n, h, r, s. Finally, by Theorem 1.2.6, this implies that there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of K^h_m. $\square$

So far, we have shown that if k has at most two prime divisors, then there exists an $h-$factorization of $K^h_{\alpha m}$ that contains an embedded $r-$factorization of k^h_m. We have also introduced new ways to construct embeddings when we don't assume the number of prime divisors of k. This leaves us wondering about cases where k has three or more prime divisors. Particularly, *Are there values of k with more than two prime divisors for which, given feasible integers m, n, h, r, s with $n = \alpha m$, $s = h$, and $k = \gcd(\alpha m, h)$, an h-factorization of $K^h_{\alpha m}$ containing an embedded r-factorization of K^h_m can be constructed?* The smallest positive integer with three prime divisors is 30, so this is a natural place to start.

4.4 If k has three or more prime divisors

Example 4.4.1. We know from Example 3.1.12 that there is exactly one 30-irreducible sum, and we observed in that example that it might be problematic when trying to build an acceptable partition in cases where $k = 30$. The irreducible sum in question is $15 + 2 \cdot 10 + 4 \cdot 6 + 1 = 60$. Therefore, if we had (perhaps after performing promotions) exactly one orbit of size $15\frac{\alpha m}{30}$, two of size $10\frac{\alpha m}{30}$, four of size $6\frac{\alpha m}{30}$, and one of size $\frac{\alpha m}{30}$, then we could not proceed as no further promotions could be made. Therefore we have to carefully build our acceptable partition to avoid this situation. We do this as follows.

Bundle an $\frac{\alpha m}{30}$-orbit with a $2\frac{\alpha m}{30}$-orbit to make a $3\frac{\alpha m}{30}$-orbit. Note that $g(2) \geq g(1)$ (by Theorem 4.2.6, g is a strictly increasing function) so we can repeat this process until we have no more $\frac{\alpha m}{30}$-orbits remaining. The result is a sum

$$\sum_{d \in D(k)} e_d d$$

whose total is a multiple of 30 and such that $e_1 = 0$. This sum has no 1-term. Since there is only one 30-irreducible sum and it has a 1-term, we can conclude that our sum is not 30-irreducible. Therefore we can proceed to promote orbits arbitrarily until every edge has been assigned a colour. The result is an acceptable partition.

One may hope at this point that this technique can be applied to other values of k with at least three prime divisors. There are two details that work in our favour in the case $k = 30$ that do not hold in general. The first is that there is only one 30-irreducible sum.

This is not the case in general, as we see in the next example. The second is that $6 \mid k$, which allows us to combine each $\frac{\alpha m}{30}$-orbit with a $2\frac{\alpha m}{k}$-orbit to make a $3\frac{\alpha m}{k}$-orbit. When applying this method on other values of k, this property is very useful as the smallest orbits tend to be the most difficult to promote. However this technique can be adapted to solve the problem for other values of k with three prime divisors.

Example 4.4.2. For $k = 84$, we can use a Python script to find all irreducible sums. There are exactly three of these. They are

- $42 + 2 \cdot 28 + 5 \cdot 12 + 7 + 3 = 168$,
- $42 + 2 \cdot 28 + 21 + 4 \cdot 12 + 1 = 168$, and
- $42 + 28 + 21 + 6 \cdot 12 + 4 + 1 = 168$.

Two of them have a 1-term and one of them has a 3-term. Group up an $\frac{\alpha m}{84}$-orbit with a $3\frac{\alpha m}{84}$-orbit to make a $4\frac{\alpha m}{84}$-orbit. Since $g(3) \geq g(1)$ by Theorem 4.2.6, we can repeat this until there are no $\frac{\alpha m}{84}$-orbits remaining. Then, pair a $3\frac{\alpha m}{84}$-orbit with a $4\frac{\alpha m}{84}$-orbit to make a $7\frac{\alpha m}{84}$-orbit. Since $g(3) \leq g(4)$ and the previous step increased the number of $4\frac{\alpha m}{84}$-orbits while decreasing the number of $3\frac{\alpha m}{84}$-orbits, this can be repeated until no $3\frac{\alpha m}{84}$-orbits remain. Having eliminated all $\frac{\alpha m}{84}$-orbits and $3\frac{\alpha m}{84}$-orbits, we have effectively avoided all three possible irreducible sums, so an acceptable partition can be made.

4.5 Nested acceptable partitions

Lemma 4.5.1. *Let α, m, h be acceptable integers for which there exists an acceptable partition. Let P be a prime larger than all prime divisors of $k = \gcd(\alpha m, h)$ and a be some positive integer such that $\alpha, P^a m, P^a h$ are acceptable integers. Then there exists an acceptable partition for the latter acceptable integers.*

Proof. Let $\mathcal{C}$ be an acceptable partition for the acceptable integers α, m, h. Let $m' = P^a m$, $h' = P^a h$, and $k' = \gcd(\alpha m', h') = P^a k$. By Lemma 4.1.7 it suffices to form colour classes out of all orbits of size of the form $\frac{\alpha m}{k'} t$ where k'/t has at least two prime divisors.

Let $t_1, t_2, \ldots, t_z$ be the divisors of k' that have at least two prime divisors and are not multiples of P. Note that, since $\frac{\alpha m}{k} t = \frac{\alpha m'}{k'} t$, the number of orbits of size $\frac{\alpha m'}{k'} t$ is equal to the number of orbits of size $\frac{\alpha m}{k} t$ that have been coloured in $\mathcal{C}$. Thus, by copying the colour classes in $\mathcal{C}$, we can promote all orbits of sizes t_1, t_2, …, and t_z to create orbits of size $\alpha m = \frac{\alpha m'}{P^a}$.

Let t be a divisor of k with at least two prime divisors such that $P \mid t$. Then by Theorem 4.2.6 we have $g(t) \geq g(P)$. By Lemma 4.2.7, we have $g(P) \geq (P-1)^2$. Finally, since P is larger than all prime divisors of k, we have $(P-1)^2 \geq Fr(k'/t)$. From these three inequalities we conclude that $g(t) \geq Fr(k'/t)$. Let $t_1, t_2, \ldots, t_\ell$ be every divisor of k' that is a multiple of P and has at least two prime divisors. Without loss of generality we may choose their indices such that, if $i < j$, then t_i has at most as many prime divisors as t_j. For each $1 \le i \le \ell$, let $p_i > q_i$ be the two smallest prime divisors of t_i. Starting with $i = 1$ and proceeding increasingly until $i = \ell$, let c_i be the number of orbits of size $\frac{\alpha m'}{k'}t$ remaining at this step. Note that, since promoting orbits of a certain size never decreases the number of orbits of any larger size, we must have $c_i \ge g(t_i)$, and therefore $c_i \ge Fr(k'/t_i)$. Apply Lemma 2.1.3 to obtain non-negative integers a and b such that $ap_i + bq_i = c_i$. We may now combine the $\frac{\alpha m'}{k'}t_i$-orbits p-by-p to create a orbits of size $\frac{\alpha m'}{k'}pt_i$ and combine the rest of the $\frac{\alpha m'}{k'}t_i$-orbits q-by-q to create b orbits of size $\frac{\alpha m'}{k'}qt_i$, thus leaving us with no orbits of size $\frac{\alpha m'}{k'}t_i$. After this process has been done for $t_1, t_2, \ldots,$ and t_ℓ, the only orbits that have not been assigned a colour are of sizes of the form $\frac{\alpha m}{Q}$ where Q is a prime power. Thus, by Lemma 4.1.7, we have an acceptable partition for the acceptable integers α, m', h'. □

Given the conditions in Lemma 4.5.1, we say that the acceptable partition for the acceptable integers α, m, h is **nested** in the acceptable partition constructed for the integers α, m', h'. We also say, for the sake of consistency, that the acceptable parameter set α, m, h is **nested** in the parameter set α, m', h'.

Theorem 4.5.2. *Let α, m, h be acceptable integers for which there exists an acceptable partition. Let P be a prime larger than all prime divisors of $k = \gcd(\alpha m, h)$, and a and r be positive integers. Let $m' = P^a m$, $n' = \alpha m'$, and $s = h' = P^a h$. If m', n', h', r, s are feasible integers, then there exists an h'-factorization of $K^{h'}_{\alpha m'}$ containing an embedded r-factorization of $K^{h'}_{m'}$.*

Proof. Using Lemma 4.5.1, we can construct an acceptable partition for the acceptable integers α, m', h'. By Lemma 4.1.4, this acceptable partition allows us to construct a good collection for the feasible integers m', n', h', r, s. Finally, by Theorem 1.2.6, there exists an h'-factorization of $K^{h'}_{\alpha m'}$ containing an embedded r-factorization of $K^{h'}_{m'}$. □

This theorem offers us a way to generate feasible integers m', n', h', r, s with $n' = \alpha m'$ and $s = h'$ for which an h'-factorization of $K^{h'}_{\alpha m'}$ containing an embedded r-factorization of $K^{h'}_{m'}$ exists by starting with acceptable integers α, m, h for which an acceptable partition is known to exist and multiplying the parameters m and h by a power of some large prime. Note that while the parameter r appears in the statement of this theorem, the particular choice of its value has no importance beyond feasibility. Note also that since the parameter set α, m, h is acceptable, the parameter set m', n', h', r, s is feasible if and only if $r \le h'$, $P^a\alpha m \mid \binom{P^a \alpha m}{P^a h}$ and $r \mid \binom{m'-1}{h'-1}$. Recall that the condition for acceptability of the parameter set α, m, h is $\alpha m \mid \binom{\alpha m}{h}$. One may hope that the condition $P^a\alpha m \mid \binom{P^a \alpha m}{P^a h}$ follows from this or vice-versa. This is not the case. For example, we have

- $3 \mid \binom{3}{1}$,
- $6 \nmid \binom{6}{3}$, and
- $30 \mid \binom{30}{15}$.

Suppose we have acceptable integers α, m', h' such that $k' = \gcd(\alpha m', h')$ has more than two prime divisors, preventing us from applying Lemma 4.3.1 or Lemma 4.1.5. We would like to create an acceptable partition. We can construct a chain of nested acceptable parameter sets, such that the innermost parameter set, α, m, h, satisfies the condition that

$k = \gcd(\alpha m, h)$ has at most two prime divisors. Then we can apply Lemma 4.3.1 or Lemma 4.1.5 to obtain an acceptable partition for this parameter set. Finally, by repeatedly applying Lemma 4.5.1 starting on the innermost acceptable parameter set and proceeding outward, we obtain an acceptable partition for the outermost parameter set α, m', h'.

The following is an example illustrating this principle.

Example 4.5.3. Let $\alpha = 2, m_1 = 9, h_1 = 7$. One can easily check that $18 \mid \binom{18}{7}$ and therefore the chosen integers are acceptable. We have $k = \gcd(18, 7) = 1$ and can therefore conclude by Lemma 4.1.5 that there exists an acceptable partition for these integers. In fact, we have the following:

- $18 \cdot 2^3 = 144 \mid \binom{144}{56}$, so by Lemma 4.5.1, there exists an acceptable partition for $\alpha = 2, m_2 = 72, h_2 = 56$
- $144 \cdot 5 = 720 \mid \binom{720}{280}$, so by Lemma 4.5.1, there exists an acceptable partition for $\alpha = 2, m_3 = 360, h_3 = 280$
- $720 \cdot 13^2 = 121,680 \mid \binom{121,680}{47,320}$, so by Lemma 4.5.1, there exists an acceptable partition for $\alpha = 2, m_4 = 60,840, h_4 = 47,320$

While the binomial coefficients in this example are relatively large, they can be computed very quickly with a computer. Recall that, given an acceptable partition for acceptable integers α, m, h, it is easy to find all possible values of r for which there exists a good collection in the case where $n = \alpha m$ and $s = h$. By Lemma 4.1.4, it suffices to choose a value of r that makes the parameter set m, n, h, r, s feasible. Given that the parameters α, m, h are acceptable, this condition can be written in terms of r as follows:

- $r \leq h$,
- $h \mid mr$,
- $r \mid \binom{m-1}{h-1}$.

We can write a Python script that takes as input positive integers m and h with $m \geq h$ and outputs a list of all values of r that satisfy those three conditions. Note that the particular choice of α has no relevance here beyond acceptability.

```
import math


# This function takes as input positive integers m and h from an acceptable parameter
# set alpha, m, h and outputs a list of all values of r for which the integers m,
# alpha*m, h, r, s (with s = h) are feasible.
def possible_values_of_r(m, h):
    list = []
    binomial_coefficient = math.comb(m - 1, h - 1)
    for r in range(1, h + 1):
        if m * r % h == 0 and binomial_coefficient % r == 0:
            list.append(r)
    return list
```

```
# Create an array acceptable_integers whose 0 entry is the value of m inputted by the
# user and whose 1 entry is the value of h inputted by the user.
acceptable_integers = input("Enter m and h seperated by a space: ").split()
print("The possible values of r are:")
# Print each element of the list acceptable_integers.
for i in possible_values_of_r(int(acceptable_integers[0]), int(acceptable_integers[1])):
    print(i)
input()
```

For example, in the case of the acceptable integers $\alpha = 2, m_2 = 72, h_2 = 56$, these values of r are $7, 14$, and 28. In the case of the acceptable integers $\alpha = 2, m_3 = 360, h_3 = 280$, the possible values of r are $7, 14, 21, 42, 49, 63, 91, 98, 119, 126, 147, 182, 203, 238$, and 273. Finally, in the case of the acceptable integers $\alpha = 2, m_4 = 60,840, h_4 = 47,320$, there are $4,457$ possible values of r. They are too numerous to list here but they can be found using the above Python script.

Given feasible integers m, n, h, r, s with $n = \alpha m$ and $s = h$, we know from Theorem 4.2.6 that $\frac{\alpha m}{k}$, being the smallest orbit size, is the size of which there are the fewest orbits. This often makes it difficult to promote these orbits. The next lemma presents a condition under which these orbits can be promoted.

Lemma 4.5.4. *Let m, n, h, r, s be feasible integers with $n = \alpha m$ and $s = h$. Let $k = \gcd(\alpha m, h)$. Suppose k has two prime divisors p and q such that $q < p < g(q)$. Then the $\frac{\alpha m}{k}$-orbits can be promoted.*

Proof. If $g(1) \geq q$, bundle the $\frac{\alpha m}{k}$-orbits q-by-q until fewer than q remain. Let a, b, and c be the numbers of $\frac{\alpha m}{k}p$-orbits, $\frac{\alpha m}{k}q$-orbits, and $\frac{\alpha m}{k}$-orbits remaining after these promotions, respectively. Note that if $c = 0$, we are done. Therefore we assume $c \geq 1$. We have

$$a = g(p),\ b \geq g(q),\ \text{and } c \leq q - 1.$$

Furthermore, $pq - c > pq - p - q + 1 = (p-1)(q-1)$, so there exist integers $e \geq 0$ and $f \geq 0$ such that

$$ep + fq = pq - c$$

In fact, we know that $e \geq 1$ and $f \geq 1$ because if, for example, $e = 0$, then $fq = pq - c \implies q \mid c \implies c \geq q$, which is a contradiction. Thus, we have

- $e \leq q - 1$ because if $e \geq q$, then $pq - c \geq pq + fq$, implying $-c \geq fq$, which is a contradiction. Furthermore, since $q < p < g(q)$, we have $e < g(q)$. By Theorem 4.2.6, we have $g(q) \leq g(p)$. Finally, since $g(p) = a$, we conclude that $e \leq a$.
- $f \leq p - 1$ because if $f \geq p$, then $pq - c \geq ep + pq$, implying $-c \geq ep$, which is a contradiction. Furthermore, since $p < g(q)$ and $b \geq g(q)$, we conclude that $f \leq b$.

Therefore we can combine

- f $\frac{\alpha m}{k}q$-orbits,
- e $\frac{\alpha m}{k}p$-orbits, and
- c $\frac{\alpha m}{k}$-orbits

to make one orbit of size $f\frac{\alpha m}{k}q + e\frac{\alpha m}{k}p + c\frac{\alpha m}{k} = \frac{\alpha m}{k}(fq + ep + c) = \frac{\alpha m}{k}pq$. This promotion uses every remaining orbit of size $\frac{\alpha m}{k}$. ∎

4.6 Divisibility of binomial coefficients

Many questions related to the divisibility of binomial coefficients arise when working on this problem. For instance in Section 4.5 we would like conditions on prime powers P^a with $P \nmid \alpha m$ under which, when $\alpha m \mid \binom{\alpha m}{h}$, we have $P^a \alpha m \mid \binom{P^a \alpha m}{P^a h}$.

Definition 4.6.1. *For a prime power p and a positive integer n, the p-adic valuation of n, denoted $\nu_p(n)$, is the exponent of the largest power of p that divides n.*

Let p be a prime and n and m be integers. It is clear from Definition 4.6.1 that $\nu_p(n \cdot m) = \nu_p(n) + \nu_p(m)$. It is also well-known that

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \text{ [7]}$$

This is known as Legendre's formula.

Lemma 4.6.2. *Let p be a prime and a, h, and n be positive integers with $h \leq n$. Then*

$$\nu_p\left(\binom{np^a}{hp^a}\right) = \nu_p\left(\binom{n}{h}\right)$$

Proof. We have

$$\begin{aligned}
\nu_p((np^a)!) &= \sum_{i=1}^{\infty} \left\lfloor \frac{np^a}{p^i} \right\rfloor \\
&= \sum_{i=1}^{a} \left\lfloor np^{a-i} \right\rfloor + \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \\
&= n \sum_{i=1}^{a} p^{a-i} + \nu_p(n!) \\
&= n \sum_{i=0}^{a-1} p^{i} + \nu_p(n!) \\
&= n \frac{1-p^a}{1-p} + \nu_p(n!).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu_p((hp^a)!) &= h \frac{1-p^a}{1-p} + \nu_p(h!) \text{ and} \\
\nu_p(((n-h)p^a)!) &= (n-h) \frac{1-p^a}{1-p} + \nu_p((n-h)!).
\end{aligned}$$

So,

$$\begin{aligned}
\nu_p\left(\binom{np^a}{hp^a}\right) &= \nu_p((np^a)) - [\nu_p((hp^a)!) + \nu_p(((n-h)p^a)!)] \\
&= n\frac{1-p^a}{1-p} + \nu_p(n!) - \left[h\frac{1-p^a}{1-p} + \nu_p(h!) + (n-h)\frac{1-p^a}{1-p} + \nu_p((n-h)!)\right] \\
&= \nu_p(n!) - [\nu_p(h!) + \nu_p((n-h)!)] \\
&= \nu_p\left(\binom{n}{h}\right).
\end{aligned}$$

□

It is a straightforward consequence of this lemma that if p is prime and a, h, and n are positive integers with $h \leq n$, then $p^a \mid \binom{np^a}{hp^a} \iff p^a \mid \binom{n}{h}$. The case that pertains to us in the context of nested acceptable partitions is that in which $p \nmid n$. In this case we can say more:

Lemma 4.6.3. *Let a, h, and n be positive integers with $h \leq n$. Let p be a prime not dividing n. Then*

$$np^a \left| \binom{np^a}{hp^a} \iff n \right| \binom{np^a}{hp^a} \textit{ and } p^a \left| \binom{n}{h} \right.$$

Proof. We have

$$np^a \left| \binom{np^a}{hp^a} \iff n \right| \binom{np^a}{hp^a} \text{ and } p^a \left| \binom{np^a}{hp^a} \right.$$

since $\gcd(n, p^a) = 1$. Hence

$$np^a \left| \binom{np^a}{hp^a} \iff n \right| \binom{np^a}{hp^a} \text{ and } p^a \left| \binom{n}{h} \right.$$

By Lemma 4.6.2. □

Chapter 5

Two-step embeddings

We use the notation $(r, K_m^h) \hookrightarrow (s, K_n^h)$ to represent an embedding of an r-factorization of K_m^h in an s-factorization of K_n^h. The results in previous chapters give us embeddings of the form $(r, K_m^h) \hookrightarrow (h, K_{\alpha m}^h)$, so it would be useful to create embeddings as follows:

$$(r, K_m^h) \hookrightarrow (h, K_{\alpha m}^h) \hookrightarrow (s, K_n^h)$$

In principle the first embedding in this chain would be of the type discussed in the case where $m \mid n$ and $r \leq h = s$ while the second is of the type obtained by M.A. Bahmanian and Mike Newman [4] in Theorem 1.2.7.

Theorem 5.0.1. *Let m, n, h, r, s be feasible integers, and let $k = \gcd(n, h)$. Suppose there exists an integer $\alpha \geq 2$ such that the following conditions are satisfied.*

- *There exists an h-factorization of $K_{\alpha m}^h$ containing an embedded r-factorization of K_m^h;*
- *$n \geq 2\alpha m$;*
- *$\gcd(n, h) \mid \alpha m$; and*
- *$1 < \frac{s}{h} \leq \frac{\alpha m}{k}\left[1 - \binom{\alpha m - k}{h} / \binom{\alpha m}{h}\right]$.*

Then there exists an s-factorization of K_n^h containing an embedded r-factorization of K_m^h.

Proof. The existence of the embedding $(r, K_m^h) \hookrightarrow (h, K_{\alpha m}^h)$ guarantees, by feasibility, that h divides $\binom{\alpha m - 1}{h-1}$, which is equivalent to the condition that αm divides $\binom{\alpha m}{h}$. $\alpha m \mid \binom{\alpha m}{h}$. This condition, paired with the condition that m, n, h, r, s are feasible, implies that the integers $\alpha m, n, h, h, s$ are feasible. This is necessary for the existence of an embedding $(h, K_{\alpha m}^h) \hookrightarrow (s, K_n^h)$. The condition $\gcd(n, h) \mid \alpha m$ is equivalent to the condition $\gcd(\alpha m, n, h) = \gcd(n, h)$. We are now in the position to apply Theorem 1.2.7 to ensure the existence of the latter embedding. $\square$

Example 5.0.2. Let $m = 26, n = 1285, h = 20, r = 10$, and $s = 28$. It is straightforward to verify that the chosen integers are feasible.

Let $\alpha = 10$. For the embedding $(10, K_{26}^{20}) \hookrightarrow (20, K_{260}^{20})$, it is straightforward to check that the integers $26, 260, 20, 10, 20$ are feasible. Furthermore, $\gcd(\alpha m, h) = 20$ has exactly two prime divisors, so there exists a 20-factorization of K_{260}^{20} that contains an embedded 10-factorization of K_{26}^{20}, by Theorem 4.3.2.

The remaining three conditions of Theorem 5.0.1 are straightforward to verify. We can therefore apply Theorem 5.0.1 and conclude that there exists a 28-factorization of K_{1285}^{20} that contains an embedded 10-factorization of K_{26}^{20}.

It should be noted that Theorem 1.2.7 could not have been applied directly to obtain the embedding in a single step as the condition $\gcd(m, n, h) = \gcd(n, h)$ is not satisfied.

Chapter 6

Open problems

We conclude with a collections of open problems.

Problem 6.0.1. *Do there exist arbitrarily long chains of nested acceptable partitions?*

The task of creating a chain of nested acceptable partitions of length t is equivalent to the task of finding acceptable integers α, m_1, h_1 for which there exists an acceptable partition and creating a sequence of pairs of positive integers $(\alpha m_1, h_1), (\alpha m_2, h_2), ..., (\alpha m_t, h_t)$ such that for all $1 \leq i \leq t$, $\alpha m_i \mid \binom{\alpha m_i}{h_i}$ and for all $1 \leq i \leq t-1$, $(\alpha m_{i+1}, h_{i+1}) = (\alpha m_i p_i^{a_i}, h_i p_i^{a_i})$ for some prime p_i larger than all prime divisors of $k_i = \gcd(\alpha m_i, h_i)$ and some positive integer a_i.

One method used in constructing long sequences of this form was to set $\alpha = 2$ and $m_1 = h_1$. Bertrand's postulate states that for any integer $h > 1$, there is always at least one prime p such that $h < p < 2h$. Thus, for every $i \geq 1$ we can choose a prime p_i such that $h_i < p_i < 2h_i$ and set $(\alpha m_{i+1}, h_{i+1}) = (2h_i p_i, h_i p_i)$. At every step, it is clear that $p_i \mid \binom{2h_i}{h_i}$. In order to conclude that $2h_i p_i \mid \binom{2h_i p_i}{h_i p_i}$, it is necessary and sufficient to have $2h_i \mid \binom{2h_i p_i}{h_i p_i}$, by Lemma 4.6.3. This condition is not always satisfied. However, it has been verified with a Python script that for any $2 \leq h \leq 950$, there exists a prime p with $h < p < 2h$ such that $2hp \mid \binom{2hp}{hp}$.

Problem 6.0.2. *Does there exist a positive integer k with exactly two prime divisors such that there exists a k-irreducible sum?*

For an integer k of this form, I have verified that no k-irreducible sum exists if:

- k is not a multiple of 6 and $k \leq 4,000,000$, or
- $k = 2^i 3^j$ with $1 \leq i \leq 4$ and $1 \leq j \leq 4$, or
- $k = p^a q^b$ where $p > q$ are primes, a and b are positive integers, and $p^a(p - q^b) + pq^b > 1$.

The first two cases were verified using a lightly modified version the Python script in Appendix A. The third case is proved as follows:

Let $\sum_{d \in D(p^a q^b)} e_d d$ be a weak minimally $p^a q^b$-irreducible sum and suppose $p^a(p - q^b) + pq^b \geq 1$. Then:

- For $d = p^i q^j$ with $0 \leq i \leq a$ and $0 \leq j \leq b-1$, we have $e_d \leq q - 1$.
- For $d = p^i q^b$ with $0 \leq i \leq a-1$, we have $e_d \leq p - 1$.

Just like in the proof of Proposition 3.2.5, we have

$$\sum_{d \in D(p^a q^b)} e_d d \leq \frac{-1 - 2p^{a+1}q^b + p^{a+1} + q^b p^a + q^b p}{1 - p}$$

It is easy to check that, the condition $p^a(p - q^b) + pq^b > 1$ is equivalent to the condition $\frac{-1-2p^{a+1}q^b+p^{a+1}+q^b p^a+q^b p}{1-p} < 2p^a q^b$. Therefore $F(p^a q^b) < 2p^a q^b$, so there exist no $p^a q^b$-irreducible sums.

In the case where k is of the form $2^a 3^b$ for positive integers a and b, Theorem 3.2.7 guarantees $F(k) < k(2 + \frac{1}{3})$. However, by using a method similar to the one used in the proof of Corollary 3.2.6, it can be shown that $F(k) < k(2 + \frac{1}{6})$. This upper bound is relatively close to $2k$. If it can be shown that $F(k) < 2k$ for all values of k of the form $2^a 3^b$ where a and b are positive integers, then it can be concluded that there exist no k-irreducible sums for values of k of this form.

Problem 6.0.3. *Is it the case that, for any integer $k > 1$, we have that $F(k) \leq \ell k$ where ℓ is the number of prime divisors of k?*

This is certainly the case when k has one prime divisor. When k is of the form $p^a q^b$ for primes $p > q$ and positive integers a and b, we have $F(k) < k(2 + \frac{1}{p-1} - \frac{1}{p} + \frac{1}{pq})$ by Theorem 3.2.7. This upper bound is fairly close to $2k$. If it can be shown that $F(k) < 2k$ for all values of k with exactly two prime divisors, then it can be concluded that there exist no k-irreducible sums for values of k of this form.

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Appendix A

Python code for finding all minimally k-irreducible sums

```
import math
import itertools


# This function returns True if x divides y and False otherwise.
def divides(x, y):
    return y % x == 0


# This function returns True if the integer x is prime, and False otherwise.
def is_prime(x):
    if x < 2:
        return False
    for i in range(2, int(math.sqrt(x)) + 1):
        if divides(i, x):
            return False
    return True


# This function takes as input a positive integer k and returns a list of its proper
# divisors, ordered increasingly.
def proper_divisors(k):
    # Define D to be an empty list. It will represent the list of all proper divisors
    # of k.
    D = []
    for i in range(1, k):
        if divides(i, k):
            D.append(i)
    return D


# This function takes as input a positive integer x and returns the smallest prime
# divisor of x.
def smallest_prime_divisor(x):
    for i in range(2, x + 1):
        if divides(i, x) and is_prime(i):
```

```
            return i


# This function takes as input a positive integer and returns a two-dimensional array.
# The firs row is
# a list of k's prime divisors, ordered increasingly, and the second row is a list of
# the exponents of
# prime divisor.
def prime_divisors(k):
    divisors = []
    exponents = []
    divisor = 2
    while divisor ** 2 <= k:
        exponent = 0
        while divides(divisor, k):
            exponent += 1
            k //= divisor

        if exponent > 0:
            divisors.append(divisor)
            exponents.append(exponent)

        divisor += 1

    if k > 1:
        divisors.append(k)
        exponents.append(1)

    return [divisors, exponents]


# The following function returns a broad list of candidate minimally k-irreducible
# sums. For any minimally k-irreducible sum a_1d_1 + ,,, + a_nd_n, the corresponding
# vector [a_1, ..., a_n] is guaranteed to belong to the list candidate_vectors(k).
def candidate_vectors(k):
    D = proper_divisors(k)
    n = len(D)  # D is the list [d_1, d_2, ..., d_n] of proper divisors of k.
    I = []
    for i in range(0, n):
        # Define I[i+1] to be the interval of integers j with 0 <= j <= p_i - 1 where
        # p_i is the smallest prime divisor of k/d_i
        I.append([*range(0, smallest_prime_divisor(k // D[i]))])
    return list(itertools.product(*I))


# Given a positive integer k with proper divisors d_1, ..., d_n and a vector x = [x_1,
# .., x_n], the following function outputs the total of the
# sum x_1d_1 + ... + x_nd_n.
def corresponding_sum_total(x, k):
    D = proper_divisors(k)
    n = len(D)
    sum = 0
    for i in range(0, n):
        sum += x[i] * D[i]
    return sum
```

```
# Given a positive integer k with proper divisors d_1, ..., d_n and a positive integer
# c>=2, the following function outputs the set V of all vectors [x_1, ..., x_n] satisfying
# x_1d_1 + ... + x_nd_n = ck.
def candidate_vectors_with_total_ck(c, k):
    U = candidate_vectors(k)
    V = []
    for x in U:
        if corresponding_sum_total(x, k) == c * k:
            V.append(x)
    return V


print(candidate_vectors_with_total_ck(2, 42))


# Given a positive integer k with proper divisors d_1, ..., d_n, the following function
# takes as input a vector x = [x_1, ..., x_n], makes a list of all vectors y = [y_1,
# ..., y_n] satisfying
# 0<=y_i<=x_i for all 1<=i<=n and y_1 + ... + y_n >= 2, and outputs True if one such y
# has a corresponding total dividing k, and outputs False otherwise.
def admits_subsum_with_total_dividing_k(x, k):
    D = proper_divisors(k)
    n = len(D)
    I = []  # The entries of the list I will be the closed intervals [1, x_i] for 0<=i<=n.
    for i in range(0, n):
        I.append([*range(0, x[i] + 1)])
    X = list(itertools.product(
        *I))  # A list representing the cartesian product of every interval in I.
    # Define Y to be the sublist of X obtained by removing the vector [0,...,0] as well
    # as any vector of the form [0, ..., 0, 1, 0, ..., 0] (i.e. any vector with a
    # 1-norm less than 2) as we are only interested in non-trivial subsums.
    Y = []
    for x in X:
        if sum(x) > 1:
            Y.append(x)
    # We check the total of the sums corresponding to each vector in X. If one of the
    # totals is a divisor of k, then the function returns True. Otherwise it returns
    # False.
    for y in Y:
        if divides(corresponding_sum_total(y, k), k):
            return True
    return False


# The following function takes as input positive integers c and k and outputs a list of
# all minimally k-irreducible sums with total ck.
def all_minimally_k_irreducible_sums_with_total_ck(c, k):
    V = candidate_vectors_with_total_ck(c, k)
    L = []  # List of all vectors corresponding to minimally k-irreducible sums with
    # total ck.
    for x in V:
        if not admits_subsum_with_total_dividing_k(x, k):
            L.append(x)
    return L


# The following function takes as input a positive integer k and a vector x = [x_1,
```

```
# ..., x_n] and returns the string 'x_1d_1 + ... + x_nd_n', the sum corresponding to x.
def vector_as_sum(k, x):
    D = proper_divisors(k)
    n = len(D)
    s = ""
    # Append 'x_id_i + ' to s for all 0 <= i <= n-1.
    for i in range(0, n):
        if x[i] != 0:
            s = s + str(x[i]) + "*" + str(D[i]) + " + "
    # Remove the superfluous ' + ' at the end of s
    s = s[0:len(s) - 3]
    return s


# Prompt user to input a value of c and k.
integers = input("input positive integers c > 1 and k > 1 separated by a space: ").split()
# Display the list of all minimally k-irreducible sums with total ck.
list = all_minimally_k_irreducible_sums_with_total_ck(int(integers[0]), int(integers[1]))
if len(list) == 0:
    print("There are no k-irreducible sums with total ck for the chosen values.")
else:
    print("List of all minimally k-irreducible sums with total ck:")
    for x in list:
        print(vector_as_sum(int(integers[1]), x))
input()
```

Appendix B

Python code for computing $F(k)$

The following is a Python script that prompts the user to input a value of k and proceeds to compute and display the value of $F(k)$.

```
import itertools
import math


# This function returns True if x divides y and False otherwise.
def divides(x, y):
    return y % x == 0


# This function returns True if the positive integer x is prime, and False otherwise.
def is_prime(x):
    if x < 2:
        return False
    for i in range(2, int(math.sqrt(x)) + 1):
        if divides(i, x):
            return False
    return True


# This function takes as input a positive integer k and returns a list of its proper
# divisors, ordered increasingly.
def proper_divisors(k):
    # Define D to be an empty list. It will represent the list of all proper divisors
    # of k.
    D = []
    for i in range(1, k):
        if divides(i, k):
            D.append(i)
    return D


# This function takes as input a positive integer x and returns the smallest prime
# divisor of x.
def smallest_prime_divisor(x):
    for i in range(2, x + 1):
        if divides(i, x) and is_prime(i):
```

```
            return i


# The following function returns a broad list of candidate minimally k-irreducible
# sums. It is the cartesian product of all the closed intervals [0, p_d - 1] (for
# proper divisors d of k) where p_d is the smallest prime divisor of k/d. For any
# minimally k-irreducible sum a_1d_1 + ,,, + a_nd_n, the corresponding vector [a_1,
# ..., a_n] is guaranteed to belong to the list candidate_vectors(k).
def candidate_vectors(k):
    D = proper_divisors(k)
    n = len(D)  # D is the list [d_1, d_2, ..., d_n] of proper divisors of k.
    I = []
    for i in range(0, n):
        # Define I[i+1] to be the interval of integers j with 0 <= j <= p_i - 1 where
        # p_i is the smallest prime divisor of k/d_i
        I.append([*range(0, smallest_prime_divisor(k // D[i]))])
    return list(itertools.product(*I))


# Given a positive integer k with proper divisors d_1, ..., d_n and a vector x = [x_1,
# .., x_n], the following function outputs the total of the sum x_1d_1 + ... + x_nd_n.
def corresponding_sum_total(x, k):
    D = proper_divisors(k)
    n = len(D)
    sum = 0
    for i in range(0, n):
        sum += x[i] * D[i]
    return sum


# Given a positive integer k with proper divisors d_1, ..., d_n, the following function
# takes as input a vector x = [ x_1, ..., x_n], makes a list of all vectors y = [y_1,
# ..., y_n] satisfying 0<=y_i<=x_i for all 1<=i<=n and y_1 + ... + y_n >= 2,
# and outputs True if one such y has a corresponding total dividing k, and outputs
# False otherwise.
def admits_subsum_with_total_dividing_k(x, k):
    D = proper_divisors(k)
    n = len(D)
    I = []  # The entries of the list I will be the closed intervals [1, x_i] for 0<=i<=n.
    for i in range(0, n):
        I.append([*range(0, x[i] + 1)])
    X = list(itertools.product(*I))  # A list representing the cartesian product of
    # every interval in I.

    # Define Y to be the sublist of X obtained by removing the vector [0,...,0] as well
    # as any vector of the form [0, ..., 0, 1, 0, ..., 0] (i.e. any vector with a
    # 1-norm less than 2) as we are only interested in non-trivial subsums.
    Y = []
    for x in X:
        if sum(x) > 1:
            Y.append(x)
    # We check the total of the sums corresponding to each vector in X. If one of the
    # totals is a divisor of k, then the function returns True. Otherwise it returns
    # False.
    for y in Y:
        if divides(corresponding_sum_total(y, k), k):
            return True
```

```
    return False


# The following function takes as input a positive integer k and outputs a list of all
# vectors corresponding to weak minimally k-irreducible sums.
def all_weak_minimally_k_irreducible_sums(k):
    X = candidate_vectors(k)
    list = []  # Define list to be an empty list. Append to it every vector in X whose
    # corresponding sum is weak
    # minimally k-irreducible.
    for x in X:
        if not admits_subsum_with_total_dividing_k(x, k):
            list.append(x)
    return list


# This function takes a positive integer n as input and outputs the number of prime
# divisors of n.
def number_of_prime_divisors(n):
    count = 0
    for i in range(2, int(n + 1)):
        if divides(i, n) and is_prime(i):
            count += 1
    return count


# This function takes as input a positive integer n and outputs True if n is a product
# of two primes and False otherwise.
def is_product_of_two_primes(n):
    if number_of_prime_divisors(n) == 2 and is_prime(int(n / smallest_prime_divisor(n))):
        return True
    else:
        return False


# The following function takes as input a positive integer k and outputs F(k),
# the maximum total achievable by a weak k-irreducible sum.
def max_weak_k_irreducible_sum_total(k):
    # In the case where k is a prime power or a product of two primes, we apply a known
    # formula to obtain the maximum total.
    if number_of_prime_divisors(k) == 1:
        return k - 1
    if is_product_of_two_primes(k):
        p = smallest_prime_divisor(k)
        q = int(k / p)
        return (p - 1) * q + (q - 1) * p
    # In the remaining cases, we proceed by brute force:
    X = all_weak_minimally_k_irreducible_sums(k)
    totals = []  # Define totals to be the empty list. For every vector x in X,
    # append the total of the sum corresponding to x.
    for x in X:
        totals.append(corresponding_sum_total(x, k))
    return max(totals)


# Prompt the user to input an integer k > 1 and print the value of F(k)
print("F(k) = ", max_weak_k_irreducible_sum_total(int(input("Input an integer k > 1: "))))
```

```
input()
```