A General Understanding of Incidence Rate and Recovery Rate in SIR Model

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1 Introduction

The compartmental models in epidemiology, which was firstly introduced by Anderson and May (1992) [1], has been considered as one of the most powerful ways to analyze and predict the spread and control of a directly transmitted infectious disease. The two essential terms in this model, the incidence term, which is defined to be the number of new infections caused by infected individuals per unit time, and the recovery term, which is defined to be the number of infected individuals moving to the recovery group per unit time, can distinguishably affect the dynamics of the diseases. Therefore, the studies on the incidence and the recovery become very meaningful.

On the contrary to previous models with concrete incidence and recovery terms, we analyzed and gave a general understanding of the epidemiologic models with incidence and recovery rates of abstract functions, examined their dynamics, and tried to determine the incidence and recovery rates from the behavior of an epidemic to provide guidelines for the public and policy makers.

2 Original Model

We present the original SIR model as follows:

$$\begin{cases} \dot{S} = -\beta S I \\ \dot{I} = \beta S I - r I \\ \dot{R} = r I \end{cases} \tag{1}$$

where S, I, and R denote the size of population of the susceptible, the infected, and the removed respectively. We require that the disease transmission rate $\beta > 0$ and that the recovery rate $r > 0$.

We claim without proofs that SIR model has the following properties (Weiss, 2013) [2]

- Long term limits exist.
- The disease always dies out.
- The epidemic threshold theorem.
- The total population is always N.

We define the *effective reproductive number* $R_e = \dfrac{S(0)\beta}{r} = \tau(K\dfrac{S(0)}{N})\dfrac{1}{r}$.

We now show that R_e is the threshold value or tipping point that determines whether an infectious disease will quickly die out or whether it will invade the population and cause an epidemic.

Theorem 1. The epidemic threshold theorem

1. *If $R_e \leq 1$, then $I(t)$ decreases monotonically to zero as $t \to \infty$.*

2. *If $R_e > 1$, then $I(t)$ starts increasing, reaches its maximum, and then decreases to zero as $t \to \infty$. We call this scenario of increasing numbers of infected individuals an epidemic.*

The epidemic threshold theorem provides strategies for public health experts to prevent an epidemic by reducing R_e to less than one. For example, for the flu:

1. Increase the recovery rate with antivirals;

2. Reduce the transmission rate by self-isolation of susceptible individuals (request that they stay at home and skip school or work);

3. Reduce the population of susceptible individuals by offering flu vaccines;

4. Reduce the transmissibility τ by encouraging frequent hand washing and, in some cultures, distributing face masks.

These strategies provide a theoretical underpinning for public health interventions.

3 The Modified Model

The incidence and recovery terms can be represented in various ways. Firstly, Anderson and May (1992) [1] used the bilinear incidence term (βSI, where β is the transmitting rate, S denotes the susceptible, and I denotes the infectious) and the linear recovery term (rI, where r is the recovery rate) and concluded that the disease always dies out in different behaviors regarding to different effective reproductive number. However, the assumption that the transmitting and the recovery rates are constant is not realistic when isolation of infectious individuals from susceptible individuals applies or the medical resources are limited. Several concrete types of nonlinear incidence term and recovery term have been studied. Dubey, Dubey, and Dubey (2016) [3] took the incidence term as Crowley-Martin type and recovery term as Holling type III (saturated treatment function), which considered the intervention to control the infective, and concluded that the infection persists along with the low availability of treatment when basic reproduction number is greater than one. Li and Zhang (2017) [4] used the $\beta(I) = \frac{\beta I}{aI^2+cI+1}$ and $\mu = \mu(b, I) = \mu_0 + (\mu_1 - \mu_0)\frac{b}{I+b}$ to be incidence and recovery rates to understand the influence by government intervention and hospitalization condition variation in the spread of diseases, and found that the basic reproduction number R_0 is not a threshold parameter, and that the modified model undergoes backward bifurcation when there is limited number of medical resources.

Nevertheless, the concrete terms above may not be practical in ever-changing situations and conditions, so researches, predictions, and policy-decisions based on these assumptions can be inaccurate and may lead to a violent disease outbreak. Therefore, we analyzed the epidemiologic models with incidence and recovery rates of abstract functions [5], examined their dynamics, and tried to determine the incidence rate and recovery rate from the behavior of an epidemic to provide guidelines for the public and policy makers.

To obtain a general understanding, we investigated three different types of modifications of the original SIR model.

The first one is:

$$\begin{cases} \dot{S} = dN - dS - \dfrac{\beta SI}{\phi(I)} \\ \dot{I} = \dfrac{\beta SI}{\phi(I)} - rI - dI \\ \dot{R} = rI - dR \end{cases} \tag{2}$$

where $\phi(I)$ is a function of I representing how population size of infected individual could impact the incidence rate. Without loss of generality, we can assume that $\phi(0) = 1$. Some examples of $\phi(I)$ are:

$$\phi_1(I) = 1 + I,$$
$$\phi_2(I) = 1 + 0.001I + 0.001I^2,$$
$$\phi_3(I) = 0.5 + 0.5e^{-I}.$$

The second one is:

$$\begin{cases} \dot{S} = dN - dS - \beta SI \\ \dot{I} = \beta SI - dI - \dfrac{rI}{\Omega(I)} \\ \dot{R} = \dfrac{rI}{\Omega(I)} - dR \end{cases} \tag{3}$$

where $\Omega(I)$ is also a function of I representing how population size of infected individual could impact the recovery rate. Typically, $\Omega(I)$ is an increasing function due to the limitation on medical resources. Without loss of generality, we can assume that $\Omega(0) = 1$. An example of $\Omega(I)$ is:

$$\frac{r}{\Omega(I)} = \mu_0 + (\mu_1 - \mu_0)\frac{b}{I+b}.$$

The third one is:

$$\begin{cases} \dot{S} = dN - dS - \dfrac{\beta SI}{\phi(I)} \\ \dot{I} = \dfrac{\beta SI}{\phi(I)} - dI - r(I)I \\ \dot{R} = r(I)I - dR \end{cases} \tag{4}$$

where $\phi(I)$ is a function of I representing how population size of infected individual could impact the incidence rate, $r(I)$ is a function of I representing how population size of infected individual could impact the recovery rate. Without loss of generality, we can assume that $\phi(0) = 1$. Some examples of $\phi(I), r(I)$ are as above.

4 Analysis of the Modified Model

4.1 Change of Incidence Rate

First we rewrite the modified model:

$$\begin{cases} \dot{S} = dN - dS - \dfrac{\beta SI}{\phi(I)} \\ \dot{I} = \dfrac{\beta SI}{\phi(I)} - rI - dI \\ \dot{R} = rI - dR \end{cases}, \tag{5}$$

4.1.1 Equilibrium Points

Let

$$\begin{cases} dN - dS - \dfrac{\beta SI}{\phi(I)} = 0 \\ \dfrac{\beta SI}{\phi(I)} - rI - dI = 0 \\ rI - dR = 0 \end{cases}, \tag{6}$$

there can be possibly 2 equilibrium points:

$$\begin{cases} S^* = N \\ I^* = 0 \\ R^* = 0 \end{cases} or \begin{cases} S^* = \dfrac{d+r}{\beta}\phi(I^*) \\ I^* = I^* \\ R^* = \dfrac{r}{d}I^* \end{cases}, \tag{7}$$

where the second equilibrium point requires:

$$I^* < \frac{dN}{r+d},\ \phi(I^*) < \frac{\beta N}{r+d}.$$

4.1.2 Linear Stability Analysis

We can calculate the Jacobi for stability (LSA):

$$J(S,I,R) = \begin{pmatrix} -d - \dfrac{\beta I}{\phi(I)} & -\dfrac{\beta S\phi(I) - \beta SI\phi'(I)}{\phi^2(I)} & 0 \\ \dfrac{\beta I}{\phi(I)} & \dfrac{\beta S\phi(I) - \beta SI\phi'(I)}{\phi^2(I)} - d - r & 0 \\ 0 & r & -d \end{pmatrix}.$$

For the equilibrium point (define it as EP) $(N, 0, 0)$, we have:

$$J(N,0,0) = \begin{pmatrix} -d & -\beta N & 0 \\ 0 & \beta N - d - r & 0 \\ 0 & r & -d \end{pmatrix}.$$

The characteristic equation is

$$(\lambda + d)^2(\lambda + d + r - \beta N) = 0,$$

which gives eigenvalues:

$$\lambda_1 = \lambda_2 = -d, \lambda_3 = -d - r + \beta N.$$

Since d, r, β, $N > 0$, we have $\lambda_1 = \lambda_2 = -d < 0$ and

$$\begin{aligned} \lambda_3 < 0 \Leftrightarrow -d - r + \frac{\beta N}{\phi(0)} < 0 \\ \Leftrightarrow \phi(0) > \frac{\beta N}{r+d}, \end{aligned}$$

so the EP $(N, 0, 0)$ is attracting if

$$\phi(0) > \frac{\beta N}{r+d};$$

is unstable if

$$\phi(0) < \frac{\beta N}{r+d}.$$

For the EP $(\frac{d+r}{\beta}\phi(I^*), I^*, \frac{r}{d}I^*)$, we have:

$$J(\frac{d+r}{\beta}\phi(I^*), I^*, \frac{r}{d}I^*) = \begin{pmatrix} -d - \dfrac{\beta I^*}{\phi(I^*)} & -\dfrac{\beta S^*\phi(I^*) - \beta S^* I^* \phi'(I^*)}{\phi^2(I^*)} & 0 \\ \dfrac{\beta I^*}{\phi(I^*)} & \dfrac{\beta S^*\phi(I^*) - \beta S^* I^* \phi'(I^*)}{\phi^2(I^*)} - d - r & 0 \\ 0 & r & -d \end{pmatrix}.$$

The characteristic equation is

$$(\lambda + d)(\lambda^2 + a\lambda + b) = 0,$$

where

$$\begin{aligned} a &= d + \frac{\beta I^*}{\phi(I^*)} + \frac{(d+r)I^*\phi'(I^*)}{\phi(I^*)} \\ b &= \frac{(d+r)I^*}{\phi(I^*)}(\beta + d\phi'(I^*)), \end{aligned}$$

which gives:

$$\begin{aligned} &\lambda_1 = -d < 0, \\ &\lambda_2 + \lambda_3 = -a, \\ &\lambda_2\lambda_3 = b. \end{aligned}$$

To guarantee that the EP is attracting, we require: $a > 0$ and $b > 0$.

So $(\frac{d+r}{\beta}\phi(I^*), I^*, \frac{r}{d}I^*)$ is attracting if

$$a > 0 \quad \text{and} \quad b > 0.$$

4.2 Change of Recovery Rate

First we rewrite the modified model:

$$\begin{cases} \dot{S} = dN - dS - \beta SI \\ \dot{I} = \beta SI - dI - \dfrac{rI}{\Omega(I)} \\ \dot{R} = \dfrac{rI}{\Omega(I)} - dR \end{cases} . \tag{8}$$

For convenience, we can remove without loss of generality the third equation (due to a conserved quantity $S(t) + I(t) + R(t)$), and the ODEs become:

$$\begin{cases} \dot{S} = dN - dS - \beta SI \\ \dot{I} = \beta SI - dI - \dfrac{rI}{\Omega(I)} \end{cases} . \tag{9}$$

4.2.1 Equilibrium Points

Let

$$\begin{cases} dN - dS - \beta SI = 0 \\ \beta SI - dI - \dfrac{rI}{\Omega(I)} = 0 \end{cases} , \tag{10}$$

there can be possibly 2 equilibrium points:

$$\begin{cases} S^* = N \\ I^* = 0 \end{cases} or \begin{cases} S^* = S^* \\ I^* = I^* \end{cases} , \tag{11}$$

where

$$\begin{cases} dN - dS^* - \beta S^* I^* = 0 \\ \beta S^* - d - \dfrac{r}{\Omega(I^*)} = 0 \end{cases} if\ exists.$$

4.2.2 Linear Stability Analysis

We can calculate the Jacobi for stability (LSA):

$$J(S, I) = \begin{pmatrix} -d - \beta I & -\beta S \\ \beta I & \beta S - d - r\frac{\Omega(I) - \Omega'(I)I}{\Omega^2(I)} \end{pmatrix} .$$

For the EP $(N, 0)$, we have:

$$J(N, 0) = \begin{pmatrix} -d & -\beta N \\ 0 & \beta N - d - r \end{pmatrix} .$$

The characteristic equation is

$$(\lambda + d)(\lambda + d + r - \beta N) = 0,$$

which gives eigenvalues:

$$\lambda_1 = -d, \lambda_2 = -d - r + \beta N.$$

Since d, r, β, $N > 0$, then $\lambda_1 = -d < 0$ and

$$\begin{aligned} \lambda_2 < 0 &\Leftrightarrow -d - r + \beta N < 0 \\ &\Leftrightarrow \frac{\beta N}{d + r} < 1, \end{aligned}$$

so the EP $(N, 0)$ is attracting if

$$\frac{\beta N}{d + r} < 1;$$

is unstable if

$$\frac{\beta N}{d + r} > 1.$$

For the EP (S^*, I^*), we have:

$$J(S^*, I^*) = \begin{pmatrix} -d - \beta I^* & -\beta S^* \\ \beta I^* & \frac{r\Omega'(I^*)I^*}{\Omega^2(I^*)} \end{pmatrix}.$$

The characteristic equation is

$$(\lambda^2 + A\lambda + B) = 0,$$

where

$$A = d + \beta I^* - \frac{r\Omega'(I^*)I^*}{\Omega^2(I^*)}$$
$$b = \beta^2 S^* I^* - (d + \beta I^*)\frac{r\Omega'(I^*)I^*}{\Omega^2(I^*)},$$

which gives:

$$Real(\lambda_1) < 0, Real(\lambda_2) < 0) \Leftrightarrow A > 0, B > 0.$$

So the EP (S^*, I^*) is attracting if and only if

$$A > 0 \quad \text{and} \quad B > 0.$$

4.3 Change of Incidence Rate and Recovery Rate

First we rewrite the modified model:

$$\begin{cases} \dot{S} = dN - dS - \dfrac{\beta SI}{\phi(I)} \\ \dot{I} = \dfrac{\beta SI}{\phi(I)} - dI - r(I)I \\ \dot{R} = r(I)I - dR \end{cases} , \tag{12}$$

4.3.1 Equilibrium Points

Let

$$\begin{cases} dN - dS - \dfrac{\beta SI}{\phi(I)} = 0 \\ \dfrac{\beta SI}{\phi(I)} - dI - r(I)I = 0 \\ r(I)I - dR = 0 \end{cases}, \tag{13}$$

there can be possibly 2 equilibrium points:

$$\begin{cases} S^* = N \\ I^* = 0 \\ R^* = 0 \end{cases} \quad or \quad \begin{cases} S^* = \dfrac{d + r(I^*)}{\beta}\phi(I^*) \\ I^* = I^* \\ R^* = \dfrac{r(I^*)}{d} I^* \end{cases}, \tag{14}$$

where the second EP requires:

$$I^* < \frac{dN}{r(I^*) + d}, \ \phi(I^*) < \frac{\beta N}{r(I^*) + d}.$$

4.3.2 Linear Stability Analysis

We can calculate the Jacobi for stability (LSA):

$$J(S, I, R) = \begin{pmatrix} -d - \dfrac{\beta I}{\phi(I)} & -\dfrac{\beta S\phi(I) - \beta SI\phi'(I)}{\phi^2(I)} & 0 \\ \dfrac{\beta I}{\phi(I)} & \dfrac{\beta S\phi(I) - \beta SI\phi'(I)}{\phi^2(I)} - d - r'(I) - r(I) & 0 \\ 0 & r'(I)I + r(I) & -d \end{pmatrix}.$$

For the EP $(N, 0, 0)$, we have:

$$J(N, 0, 0) = \begin{pmatrix} -d & -\beta N & 0 \\ 0 & \beta N - d - r(0) & 0 \\ 0 & r(0) & -d \end{pmatrix}.$$

The characteristic equation is

$$(\lambda + d)^2(\lambda + d + r(0) - \beta N) = 0,$$

which gives eigenvalues:

$$\lambda_1 = \lambda_2 = -d, \lambda_3 = -d - r(0) + \beta N.$$

Since d, β, N, $r(0) > 0$, we have $\lambda_1 = \lambda_2 = -d < 0$ and

$$\begin{aligned} \lambda_3 < 0 &\Leftrightarrow -d - r(0) + \frac{\beta N}{\phi(0)} < 0 \\ &\Leftrightarrow \phi(0) > \frac{\beta N}{r(0)+d} \end{aligned},$$

so the EP $(N, 0, 0)$ is attracting if

$$\phi(0) > \frac{\beta N}{r(0)+d};$$

is unstable if

$$\phi(0) < \frac{\beta N}{r(0)+d}.$$

For the EP $(\frac{d+r(I^*)}{\beta}\phi(I^*), I^*, \frac{r(I^*)}{d}I^*)$, we have:

$$J(\frac{d+r(I^*)}{\beta}\phi(I^*), I^*, \frac{r(I^*)}{d}I^*) = \begin{pmatrix} -d - \dfrac{\beta I^*}{\phi(I^*)} & -\dfrac{\beta S^*\phi(I^*) - \beta S^* I^* \phi'(I^*)}{\phi^2(I^*)} & 0 \\ \dfrac{\beta I^*}{\phi(I^*)} & \dfrac{\beta S^*\phi(I^*) - \beta S^* I^* \phi'(I^*)}{\phi^2(I^*)} - d - r'(I^*)I^* - r(I^*) & 0 \\ 0 & r'(I^*)I^* + r(I^*) & -d \end{pmatrix}.$$

The characteristic equation is

$$(\lambda + d)(\lambda^2 + C\lambda + D) = 0,$$

where

$$\begin{aligned} C &= d + \frac{\beta I^*}{\phi(I^*)} + \frac{(d + r(I^*))I^*\phi'(I^*)}{\phi(I^*)} + r'(I^*)I^* \\ D &= (d + \frac{\beta I^*}{\phi(I^*)})(\frac{(d + r(I^*))I^*\phi'(I^*)}{\phi(I^*)} + r'(I^*)I^*) + \frac{\beta I^*}{\phi(I^*)}(d + r(I^*))(1 - \frac{I^*\phi'(I^*)}{\phi(I^*)}) \end{aligned},$$

which gives:

$$\begin{aligned} \lambda_1 &= -d, \\ \lambda_2 + \lambda_3 &= -C, \\ \lambda_2\lambda_3 &= D. \end{aligned}$$

To guarantee that the equilibrium point is attracting, we require: $C > 0$ and $D > 0$.

So the EP $(\frac{d+r(I^*)}{\beta}\phi(I^*), I^*, \frac{r(I^*)}{d}I^*)$ is attracting if

$$C > 0 \quad \text{and} \quad D > 0.$$

5 Numerical Simulations and Applications

We operated several simulations in Matlab with different combinations of incidence and recovery rates.

Notice that all the following parameters selected satisfies the stability condition mentioned before.

5.1 Change of Incidence Rate

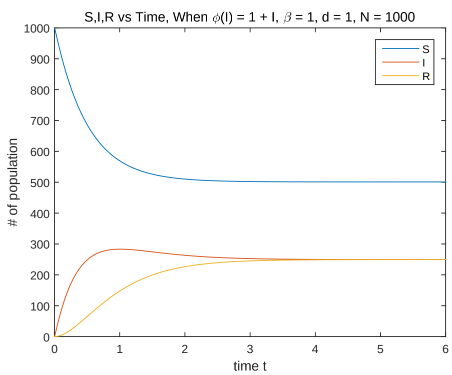
We choose the set of parameters as follows:

$$\beta = 1, d = 1, N = 1000 \ and \ r = 1.$$

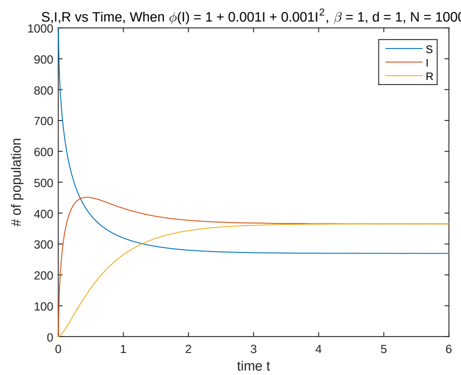
Letting

$$\phi_1(I) = 1 + I,$$
$$\phi_2(I) = 1 + 0.001I + 0.001I^2,$$

we can get Figure 1 and Figure 2.



1: S, I, R vs Time, when $\phi(I) = 1 + I$



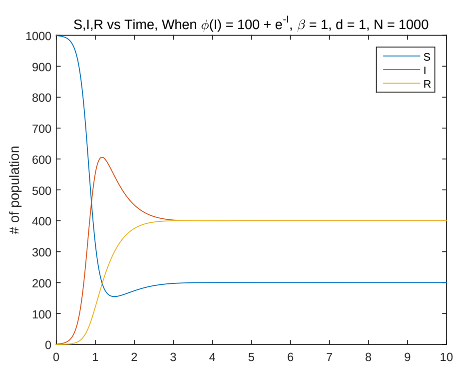
2: S, I, R vs Time, when $\phi(I) = 1 + 0.001I + 0.001I^2$

From the two graphs we know that from a certain initial state, the solution of ODEs will go to a stable state as $t \to \infty$, which is consistent with our expectation.

Now let $\phi(I) = a + be^{-I}$. Considering the visualization beauty of our simulation results, we assume that

$$\phi_3(I) = 100 + e^{-I},$$

which still satisfies the condition where the equilibrium point exists and is stable. Then we can get Figure 3.



3: S, I, R vs Time, when $\phi(I) = 100 + e^{-I}$

Similarly we know that the solution can show a property of stability. Thus, in the following simulations we won't repeat such analysis any more and just show the results.

5.2 Change of Recovery Rate

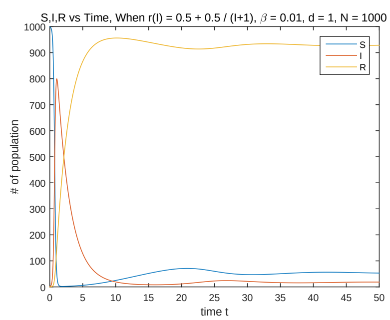
We choose the set of parameters as follows:

$$\beta = 0.01, b = 1, N = 1000 \ and \ r = 1.$$

Letting

$$r(I) = 0.5 + \frac{1}{2(1+I)},$$

we can get Figure 4



4: S, I, R vs Time, when $r(I) = 0.5 + \frac{1}{2(1+I)}$

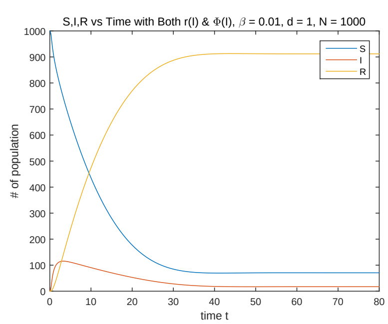
5.3 Change of Both Incidence Rate and Recovery Rate

We choose the set of parameters same as the previous subsection.

Considering both $\phi(I)$ and $r(I)$, let

$$\phi(I) = 1 + 0.001I + 0.001I^2,$$
$$r(I) = 0.5 + \frac{1}{2(1+I)},$$

then we can get Figure 5.



5: S, I, R vs Time, when considering both $\phi(I)$ and $r(I)$

6 A Brief Conclusion

Compared with the starting SIR model, where the dynamics of the system is completely determined by the basic reproduction number R_0, our modified epidemiologic models show different dynamics highly depending on the non-trivial incidence and recovery rates. The results here have several important biological implications.

The question of how an epidemiologic model can be determined by a given parametrization of function has been discussed for many because, in a practical and complicated real-life circumstance, this model is often used for numerical simulations to estimate the parameters and hence help make decisions on how to control the disease or prevent future outbreaks, which naturally implies various incidence and recovery functions. Our models present a general understanding of the functions of incidence and recovery, including a general analysis of stability at equilibrium and simulations of diverse incidence and recovery rates. Our results may be helpful for those who would like to identify the specific incidence and recovery rate functions from population data and those who would like to see how government control or medicine limitation would influence on an epidemic outbreak.

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