Basics
Maxwell's Mixture Model
(Under Untruthful Responding)

Direct Question $\rightarrow$ Did you cheat on the exam?
Indirect Question $\rightarrow$ Were you honest on the exam?
Unrelated Question $\rightarrow$ Were you born in April?

You choose p,q,a

Untruthfulness

1. Direct Question:
   Yes $\rightarrow$ No
   No $\rightarrow$ No

2. Indirect Question:
   Yes $\rightarrow$ Yes
   No $\rightarrow$ Yes

3. Unrelated Question:
   Yes $\rightarrow$ Yes
   No $\rightarrow$ No

Calculator of Estimator

$$P(yes) = p\pi_x a + q(1-\pi_x) + q\pi_x(1-a) + (1-p-q)\pi_y$$
$$P(yes) = \pi_x[pa - q + q(1-a)] + q + (1-p-q)\pi_y$$
$$P(yes) = \pi_x(pa - qa) + q + (1-p-q)\pi_y$$
$$P(yes) = \pi_x a(p-q) + q + (1-p-q)\pi_y \tag{1}$$
$$\hat{\pi}_x = \frac{\hat{P}_y - q - (1-p-q)\pi_y}{a(p-q)} \tag{2}$$

If you are aware that untruthfulness was going on, then

$$E(\hat{\pi}_x) = \frac{E(\hat{P}_y) - q - (1-p-q)\pi_y}{a(p-q)}$$
$$E(\hat{\pi}_x) = \frac{[\pi_x a(p-q) + q + (1-p-q)\pi_y] - q - (1-p-q)}{a(p-q)}$$
$$E(\hat{\pi}_x) = \pi_x$$
$$Var(\hat{\pi}_x) = \frac{P_y^*(1-P_y^*)}{a^2(n-1)(p-q)^2}$$
$$Bias(\hat{\pi}_x) = 0$$

But if you do not know what untruthfulness is occuring

$$\hat{\pi}_x = \frac{\hat{P}_y - q - (1-p-q)\pi_y}{p-q} \tag{3}$$

With $E(\hat{P}_y)$ given by (1)

$$E(\hat{\pi}_x) = \frac{\pi_x a(p-q)}{p-q}$$

$$Var(\hat{\pi}_x) = \frac{P_y^*(1-P_y^*)}{(n-1)(p-q)^2}$$
$$Bias(\hat{\pi}_x) = (\pi_x a) - \pi_x = \pi_x(a-1)$$

Correct Estimator is given by (2) with unknown untruthfulness
Incorrect Estimator is given by (3)

To find MSE Theoretically, we proceed as:

$$MSE(\hat{\pi}_x) = Var(\hat{\pi}_x) + (Bias(\hat{\pi}_x))^2$$

To find MSE Empirically, we proceed as:

$$Bias(\hat{\pi}_x) = \frac{\sum_{i=1}^{n}(\hat{\pi}_i)}{n} - \pi_x$$
$$MSE(\hat{\pi}_x) = \frac{\sum_{i=1}^{n}(\hat{\pi}_i - \pi_x)^2}{n}$$

Using n simulations for a given p, q, π_x, π_y, a. With $\hat{\pi}_i = \frac{\hat{P}_y - q - (1-p-q)\pi_y}{(p-q)}$