Nonlinear Hyperbolic Conservation Laws - A Brief Informal Introduction

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ABSTRACT: *Conservation* is a fundamental principle of the physical world. Matter may move around and redistribute but it does not appear or disappear. *Hyperbolicity* means that news that happen at a given point A, take time before they may affect affairs at another point B. *Nonlinear* means that the manner in which news propagate depends on what type of news it is. This makes the subject of nonlinear hyperbolic conservation laws fascinating, rich and challenging to study. These notes form a brief and informal introduction to the mathematical theory of nonlinear hyperbolic conservation laws. They are intended to give good intuitive foundation for those contemplating to become practitioners in the field. Key concepts are emphasized over mathematical rigor, with the aim of developing good understanding of the mathematical structure of the equations and their solutions. Inevitably, these notes are incomplete, and should serve primarily as a prelude to a good text.

Keywords: Conservation, Characteristics, Weak solution, Shock wave, Rarefaction, the Riemann problem

1 INTRODUCTION

A hyperbolic system of conservation laws is a time dependent set of coupled partial differential equations (PDEs) which in one space dimension takes the form

$$W_t + F(W)_x = 0 \,, \qquad W(x,0) = W_0(x) \tag{1}$$

here $W = (w_1, w_2, \cdots, w_n)$ is the vector of conserved variables, and $F(W) = (f_1, f_2, \cdots, f_n)$ the vector of flux functions, both of length n. The system may be written in *quasi-linear* form

$$W_t + A(W)W_x = 0 \tag{2}$$

where $A(W)$ is the $n \times n$ *Jacobian Matrix*, $A(W) = \frac{\partial F(W)}{\partial W}$ with entries $A_{i,j} = \frac{\partial f_i}{\partial w_j}$. The system is said to be *hyperbolic* if $A(W)$ has *real* eigenvalues, and a *complete set* of linearly independent eigenvectors.

These equations are wave-like, they support the propagation of disturbances, the eigenvalues are associated with the (finite) speed of propagation of disturbances, and the eigenvectors carry information about their structure. The flux vector $F(W)$ is often a nonlinear function of W, admitting a rich spectrum of wave phenomena. A flow that starts off being smooth may remain smooth, or it may steepen up and develop discontinuous fronts, known as shock waves. On the other hand, a flow that starts off discontinuous may propagate as discontinuous fronts, or it may break up instantaneously into smooth flow. These notes are a very brief and informal introduction to nonlinear hyperbolic conservation laws. We emphasize working concepts over mathematical rigor. Our aim is to develop the language and basic tools needed to provide good understanding of the mathematical structure of the equations and their solutions. We hope that these notes give a taste of the beauty and richness of this fascinating subject, as well as of its importance. Obviously, these notes are incomplete, and should serve primarily as a prelude to a good text. For comprehensive discussions, there are numerous excellent text books, some are mentioned in the bibliography.

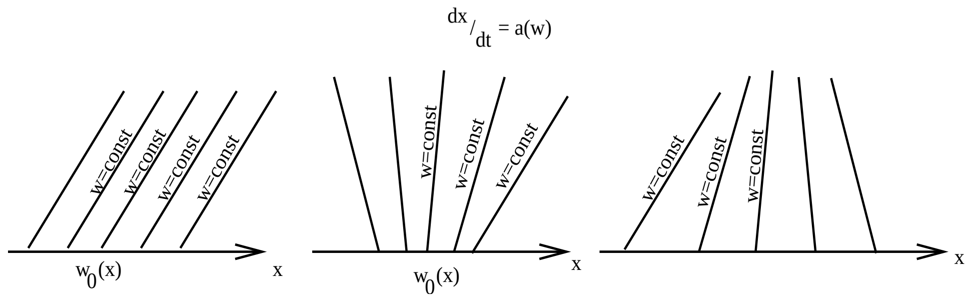

Figure 1. Initial data propagates along characterititistic: linear (left) and nonlinear (center, right) advection

Derivation

To see how conservation considerations lead to equations of the form (1), we consider the classical example of mass conservation in gas dynamics. We will use the gas dynamics equations to motivate discussion throughout these notes. Consider a one dimensional tube filled with gas of density $\rho(x,t)$ moving with velocity $u(x,t)$. The flow rate, or mass flux, is given by $\rho(x,t)u(x,t)$. The total gas mass in a tube segment $[x_1, x_2]$ at time t is

$$\int_{x_1}^{x_2} \rho(x,t) \text{ dx}$$

and the total mass crossing at point x during the time period $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} \rho(x,t)u(x,t) \text{ dt} .$$

Consider a tube segment $[x_1, x_2]$ over a time period $[t_1, t_2]$. Mass conservation implies that the total mass in the segment can only change over a period of time due to net mass flow through the segment boundaries. This can be expressed in the *integral form* of the conservation law

$$\int_{x_1}^{x_2} \rho(x,t_2) \text{ dx} = \int_{x_1}^{x_2} \rho(x,t_1) \text{ dx} + \int_{t_1}^{t_2} \rho(x_1,t)u(x_1,t) \text{ dt} - \int_{t_1}^{t_2} \rho(x_2,t)u(x_2,t) \text{ dt} . \qquad (3)$$

If the functions $\rho(x,t)$ and $u(x,t)$ are differentiable, the Fundamental Theorem of Calculus gives $\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \rho_t \text{dt}$, $\rho(x_2,t)u(x_2,t) - \rho(x_1,t)u(x_1,t) = \int_{x_1}^{x_2} (\rho u)_x \text{dx}$ and equation (3) can be written as

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \left(\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}(\rho(x,t)u(x,t) \right) \text{dxdt} = 0 .$$

Since the above is true for any $[x_1, x_2] \times [t_1, t_2]$, we conclude that the integrand itself must vanish, and obtain the *differential form* of the conservation law

$$\rho_t + (\rho u)_x = 0 \qquad (4)$$

We make the following remarks:

- The derivation of the integral form (3) of the conservation law does not assume that the flow is smooth, and is therefore a fundamental conservation statement. For a general conservation law on domain $I = [x_1, x_2]$ and time interval $T = [t_1, t_2]$, the *integral form* of the equation is given by

$$\int_I W(x,t_2) \text{ dx} = \int_I W(x,t_1) \text{ dx} - \left[\int_T F(W(x_2,t)) \text{ dt} - \int_T F(W(x_1,t)) \text{ dt} \right] \qquad (5)$$

- To be able to solve (4), we need to specify the velocity. If the velocity $u = a$ is a prescribed constant, as might describe the passive advection of a contaminant in a uniform flow, the flux function $f = a\rho$, and equation (4) is *linear*. If u is a function of ρ, as might be the case in traffic flow, for example $u = V(1-\rho) = u(\rho)$, then $f(\rho) = V\rho(1-\rho)$ is a quadratic (more generally, nonlinear) function of ρ. Nonlinearity gives rise to tremendous richness in solution phenomena, which we will explore first in the scalar case, then in the case of coupled systems.

- In a similar way, one may derive equations expressing the conservation of momentum and energy. Changes in total momentum may result from net momentum transport and net pressure forces. Changes in total energy may result from net energy transport and net work done by the pressure forces. This leads to the Euler equations of gas dynamics (37).
- Processes such as diffusion due to viscous effects or heat conduction may also be incorporated. Viscous effects may become important if solutions develop sharp fronts, but in many flow situations they are considered small enough to be neglected. As we will see later, it is sometimes necessary to appeal to those neglected terms in order to resolve ambiguities in the solution.

2 SCALAR CONSERVATION LAWS

Consider the scalar conservation law

$$w_t + f(w)_x = 0 \,, \qquad w(x,0) = w_0(x) \,, \tag{6}$$

which we also write in *quasi-linear form*

$$w_t + a(w)w_x = 0 \,, \qquad w(x,0) = w_0(x) \,, \tag{7}$$

where $a(w) = f'(w)$. On dimensional grounds, we observe that $a(w)$ has physical units of a velocity. As we will see below, it corresponds to the propagation speed of disturbances.

Consider the solution of (6) along special curves in the $x-t$ plane satisfying $\dfrac{dx}{dt} = a(w)$. Along such curves, $w = w(x(t),t) = \tilde{w}(t)$ satisfies

$$\frac{d\tilde{w}}{dt} = \frac{d}{dt}w(x(t),t) = w_t + \frac{dx}{dt}w_x = w_t + a(w)w_x = 0 \,. \tag{8}$$

that is

$$w = \text{const} \qquad \text{along} \qquad \frac{dx}{dt} = a(w)$$

The special curves $dx/dt = a(w)$ are called *characteristics*. Along these special curves, the PDE (6) reduces to an ODE (8), in this case a very simple one, $\tilde{w}'(t) = 0$, that can be easily integrated to establish that w is constant along characteristics. Furthermore, since the slope of the characteristics is $a(w)$ and since w is constant along them, so is the slope. It follows that the characteristics are *straight lines*. This gives a simple geometric way to construct the solution for $t > 0$, by propagating initial values of w along straight lines with slope $dx/dt = a(w)$ (see Figure 1).

2.1 *Linear Advection*

The simplest case corresponds to $a(w) = a$ is constant, giving the *linear advection* equation

$$w_t + aw_x = 0 \,, \qquad w(x,0) = w_0(x) \tag{9}$$

The characteristics here are straight lines parallel to each other with slopes $\frac{dx}{dt} = a$. The solution $w(x,t)$ remains constant along characteristics and may be traced back from (x,t) along the characteristic to the initial line to give

$$w(x,t) = w_0(x - at) \tag{10}$$

describing the simple translation without distortion of the initial condition $w_0(x)$ at speed a in the x direction.

The solution at (x,t) depends on initial data at a single point $x_0 = x - at$ along the initial line. More generally, the solution at (x,t) may depend on several data points along the initial line, or on an entire

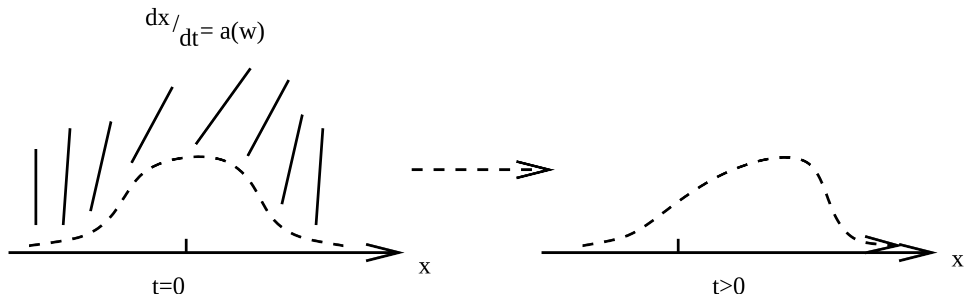

Figure 2. Solution steepens up as initial data propagate along characteristics.

interval, called the *domain of dependence*. Since disturbances propagate at a finite speed, the domain of dependence is a finite interval.

The Riemann Problem

The Riemann problem is an important building block in constructing solutions for conservation laws,

$$w_t + f(w)_x = 0\,, \qquad w_0(x) = \begin{cases} w_L & x < 0 \\ w_R & x > 0 \end{cases}\,, \tag{11}$$

with $w_{L,R}$ constants. In the linear case, the solution of (11) is given by

$$w(x,t) = \begin{cases} w_L & x/t < a \\ w_R & x/t > a \end{cases}$$

and describes a wave, a discontinuity of strength $[w] = w_R - w_L$, which starts off at $x = 0$ and propagates downstream (with the flow) at the characteristic speed a. The wave front $x/t = a$ separates the $x - t$ plane into two regions where the solution is constant, w_L on its left and w_R on its right (see Figure 8).

2.2 *Nonlinear Advection*

The general nonlinear scalar equation takes the form $w_t + f(w)_x = 0$ where the flux function f is a nonlinear function of w. We will assume that f is a convex function of w, $f''(w) > 0$ (more precisely that f'(w)=a(w) is monotone in w, either increasing or decreasing). This condition simplifies the solution structure, and is satisfied, for example, by various gas dynamics models. The solution may still be constructed in the nonconvex case, but its structure is more complex.

Burger's Equation

The simplest nonlinear example is that of Burger's equation

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0 \tag{12}$$

with $f(w) = w^2/2$, and characteristic speed $a(w) = f'(w) = w$. Here, w is constant along characteristics, and characteristics are straight lines, but they are no longer parallel to each other. They have slopes $a(w) = w$. Higher values of w propagate faster than lower values, possibly overtaking them at some finite time t_c. When characteristics carrying different solution values intersect, the solution becomes multivalued, the PDE ceases to be valid and the notion of solution needs to be generalized.

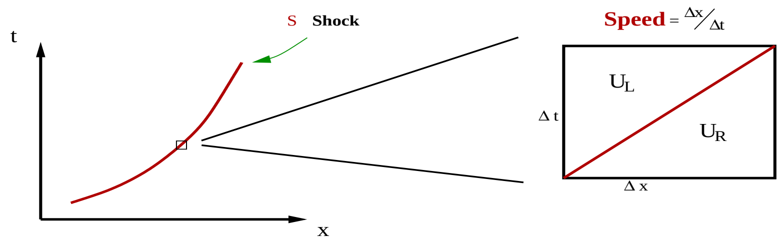

Figure 3. Shock relations arise naturally by integrating (5) over a small control volume.

If initial data $w_0(x)$ is smooth, the solution will start off smooth, and propagate along characteristics. The solution at (x, t) may be expressed implicitly by tracing it back to the initial line along the characteristics

$$w(x,t) = w_0(x - a(w)t) = w_0(x - w(x,t)t) = w_0(x_0)$$

Implicit differentiation gives

$$w_x = w_0'(x_0)\,(1 - w_x t)\ , \qquad w_t = w_0'(x_0)\,(-w_0(x_0) - w_t t)$$

yielding after rearrangement

$$w_x = \frac{w_0'(x_0)}{1 + w_0'(x_0)t}\ , \qquad w_t = -\frac{w_0'(x_0)w_0(x_0)}{1 + w_0'(x_0)t}\ . \tag{13}$$

We distinguish two different scenarios: (i) If the initial data $w_0(x)$ is an increasing function of x, $w_0'(x_0) > 0$, the denominator in (13) is always positive, w_t and w_x remain bounded for all $t > 0$; (ii) If $w_0'(x_0) < 0$ for some x_0, the denominator will vanish at some finite time t_c, and w_t and w_x will become unbounded. This will happen first at

$$t_c = -\frac{1}{\min w_0'(x_0)}$$

The former scenario corresponds to characteristics opening up in a fan across which wave speeds increase monotonically and never intersect, the latter corresponds to fast characteristics overtaking slower ones, with solution becoming multivalued at t_c, w_t and w_x blow up and no longer satisfy the PDE.

To extend the solution beyond t_c, its notion needs to be generalized. This is done by allowing $w(x, t)$ to become discontinuous. A curve of discontinuity is inserted, separating two regions in the $x - t$ plane where the solution is smooth. These discontinuities are moving internal boundaries. The basic rule governing their propagation may be obtained in an informal way, by appealing to the fundamental principle of conservation (see Figure 3). We draw a small box of dimension Δx by Δt in the $x - t$ plane, small enough so that the solution to either side of the discontinuity can be thought of as approximately constant, denoted by w_L, w_R, and the speed of the discontinuity is approximately constant, denoted by s, with the box dimensions chosen so that $s = \Delta x/\Delta t$. Requiring that the total 'mass' at the new time equals the old 'mass plus net mass flow through the boundaries gives

$$\Delta x w_L \approx \Delta x w_R + \Delta t f(w_L) - \Delta t f(w_R)\ , \qquad \Longrightarrow \qquad \frac{\Delta x}{\Delta t}\,(w_R - w_L) \approx f(w_R) - f(w_L)\ .$$

As $\Delta x, \Delta t \to 0$ with $s = \Delta x/\Delta t$, we obtain the *Rankine Hugoniot* jump conditions

$$s\,[w] = [f] \tag{14}$$

A discontinuity that satisfies (14) is called a *shock wave*. The formation of shock waves in a finite time is generic for nonlinear hyperbolic equations, even when initial data is smooth. Beyond the time of shock formation, the differential form of the equation (6) ceases to be valid, but the integral form, which is the more fundamental form of the equation, continues to hold. Solutions are extended as *weak solutions* (see

Figure 4. Shock waves as the limit of traveling wave solutions, as viscosity coefficient $\epsilon \to 0$.

below). These are piecewise smooth solutions, separated by curves of discontinuity, they satisfy the PDE (6) in smooth parts and the shock jump conditions (14) across discontinuities.

Weak Solutions
A convenient way to define weak solutions makes use of test functions. A test function $\phi(x,t)$ is a smooth function with compact support, that is it vanishes outside a finite box $[-R,R]\times[0,T]$, for some R and T. Multiply (1) by a test function ϕ, and integrate over the top half $x-t$ plane

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}(w_t+f(w)_x)\,\phi\;\mathrm{dx}\;\mathrm{dt}=0$$

Integration by parts gives

$$\int_{-\infty}^{+\infty}\left(w\phi\big|_{-\infty}^{+\infty}-w\phi_t\right)\mathrm{dx}+\int_0^{\infty}\left(f(w)\phi\big|_0^{+\infty}-f(w)\phi_x\right)\mathrm{dt}=0$$

Due to compactness of ϕ, all boundary terms vanish except the contribution along the initial line, which gives after rearrangement

$$\int_{-\infty}^{+\infty}w\phi_t\;\mathrm{dx}+\int_0^{\infty}f(w)\phi_x\;\mathrm{dt}=\int_{-\infty}^{+\infty}w_0(x)\phi(x,0)\mathrm{dx}\tag{15}$$

w is said to be a *weak solution* of the conservation law (6) if w satisfies (15) for all smooth compactly supported test functions ϕ. It can be shown that weak solutions satisfy the PDE (6) in smooth regions, and satisfy the Rankine Hugoniot jump conditions (14) across discontinuities.

Traveling Waves and Vanishing Viscosity Solutions
Real physical flows may develop sharp fronts but not discontinuities. In these flow situations, it may no longer be justified to neglect the viscous effect. Indeed it is precisely these viscous effects that counter balance the tendency of the solution to steepen up, and lead to a rapidly varying but continuous front. When viscous effects are included, the equation takes the form

$$w_t^\epsilon+f(w^\epsilon)_x=\epsilon w_{xx}^\epsilon\tag{16}$$

with ϵ the (small) viscosity coefficient. *Traveling wave* solutions are solutions of (16) satisfying

$$w^\epsilon(x,t)=\tilde{w}\left(\frac{x-st}{\epsilon}\right)=\tilde{w}(\xi)\;,\qquad\lim_{x\to\mp\infty}\tilde{w}(\xi)=w_{L,R}\;,\qquad\lim_{x\to\mp\infty}\tilde{w}'(\xi)=0\tag{17}$$

for some constant propagation speed s. Substituting (17) into (16) gives

$$-\frac{s}{\epsilon}\tilde{w}'+\frac{1}{\epsilon}f'(\tilde{w})=\epsilon\frac{1}{\epsilon^2}\tilde{w}''$$

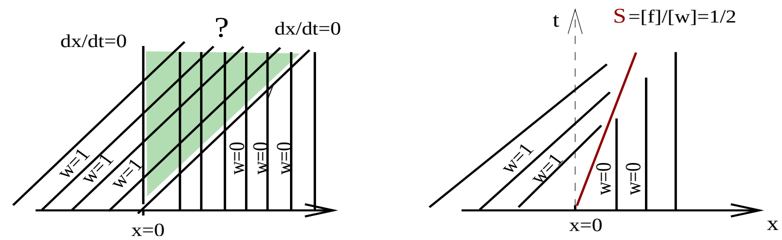

Figure 5. Multivalued solution due to overlapping characteristics, and formation of a shock wave.

which can be integrated once to give

$$-s(w_R - w_L) + f(w_R) - f(w_L) = \tilde{w}'\big|_{-\infty}^{\infty} = 0\,, \qquad \Longrightarrow \qquad s[w] = [f]\,,$$

that is, traveling wave solutions are admissible provided $W_{L,R}$ at $\pm\infty$ and s satisfy the shock jump conditions (14). Shock waves may be regarded as travelling wave solutions in the limit as $\epsilon \to 0$, see Figure 4. Indeed, the solutions for (6) that are of physical relevance and mathematical interest are those that can be obtained as limits of viscous solutions $w^\epsilon(x,t)$ as $\epsilon \to 0$. The concept of *vanishing viscosity solutions* is important in selecting the physically relevant weak solution of (6) in cases of ambiguity.

Example 1
Consider the Riemann problem

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0\,, \qquad w_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \tag{18}$$

We observe that $a(w_L) = w_L = 1$ is greater than $a(w_R) = w_R = 0$, a situation that leads to the formation of shock wave. Geometric construction of the solution using the characteristics establishes (see Figure 5). that for $x < 0$, equivalently $x/t < 0$, the initial value $w_L = 1$ propagates along $dx/dt = 1$ and $w(x,t) = 1$. Similarly, for $x/t > 1$, $w_R = 0$ propagates along vertical characteristics to give $w(x,t) = 0$. But the region $0 < x/t < 1$ is covered by two sets of characteristics and the solution there is multivalued. To resolve the ambiguity, we insert a shock wave. The jump conditions (14) give us the speed of the shock as

$$s = \frac{[f]}{[w]} = \frac{\frac{1}{2}w_R^2 - \frac{1}{2}w_L^2}{w_R - w_L} = \frac{w_L + w_R}{2} = \frac{1+0}{2} = \frac{1}{2}\,,$$

and the complete solution is

$$w(x,t) = \begin{cases} 1 & x/t < 1/2 \\ 0 & x/t > 1/2 \end{cases}$$

Note that the solution $w(x,t)$ does not depend on x and t independently, but on the ratio $\xi = x/t$. This property is also true for solutions of the Riemann problem in the case of systems.

Example 2
Consider now the Riemann problem

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0\,, \qquad w_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{19}$$

Extending the solution along characteristics from the initial line, we obtain $w(x,t) = 0$ for $x/t < 0$, and $w(x,t) = 1$ for $x/t > 1$. But the region $0 < x/t < 1$ is not covered by characteristics, and it is not clear

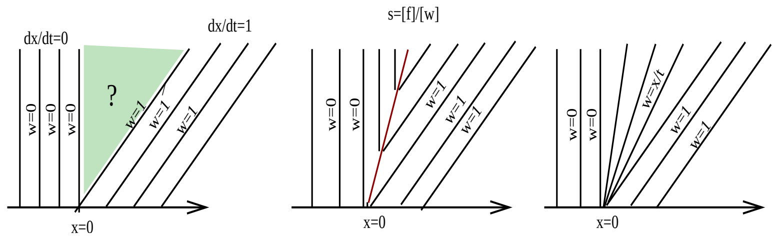

Figure 6. Weak solutions are not unique

how the solution there is determined. One possibility is to insert a shock front, with $s = [f]/[u] = 1/2$

$$w(x,t) = \begin{cases} 0 & x/t < 1/2 \\ 1 & x/t > 1/2 \end{cases}$$

But there are other options. It is easy to check that for $t > 0$ the function $w(x,t) = x/t$ is smooth and satisfies

$$w_t + (\frac{1}{2}w^2)_x = w_t + ww_x = -\frac{x}{t^2} + \frac{x}{t}\frac{1}{t} = 0$$

and therefore satisfies the PDE (19). This offers another possible way to complete the characteristics in the missing region in a smooth manner

$$w(x,t) = \begin{cases} 0 & x/t < 0 \\ x/t & 0 < x/t < 1 \\ 1 & x/t > 1 \end{cases}$$

This fan-like solution is called a *rarefaction wave*. The name is borrowed from gas dynamics, where the gas rarefies as it propagates across such fans, as we will see later. Both solutions are illustrated in Figure 6. And, in fact, there are many other possible weak solutions. That is, weak solutions are *not* unique. Which solution is the right one?

We appeal to the principle of *stability*: solutions in the physical world are often stable: if we change the problem by a small amount, we expect that the solution changes by a small amount. We can apply this principle here to help us determine which of the above solutions is stable. We replace the discontinuous initial data in (19) by a rapidly varying but continuous function over $[-\epsilon, \epsilon]$

$$w_0(x) = \begin{cases} 0 & x < -\epsilon \\ (x+\epsilon)/(2\epsilon) & -\epsilon \le x \le \epsilon \\ 1 & \epsilon < x \end{cases}$$

The initial condition is now smooth, the characteristics open up to a fan that covers the entire $x - t$ plane, and the solution can be easily constructed for all times $t > 0$. In the regularized region, we obtain

$$w(x,t) = w_0(x - wt) = \frac{1}{2\epsilon}(x - wt + \epsilon) \qquad \Longrightarrow \qquad w(x,t) = \frac{x+\epsilon}{t+2\epsilon}$$

As $\epsilon \to 0$, the solution clearly converges to the smooth rarefaction fan (see Figure 7).

Alternatively, we can appeal to the principle of vanishing viscosity and consider solution $w^\epsilon(x,t)$ of

$$w_t^\epsilon + \left(\frac{1}{2}(w^\epsilon)^2\right)_x = \epsilon w_{xx}^\epsilon \ ,$$

with $w_L = 0$, $w_R = 1$, in the limit of vanishing viscosity $\epsilon \to 0$. This process can also be shown to select the rarefaction fan as the vanishing viscosity solution.

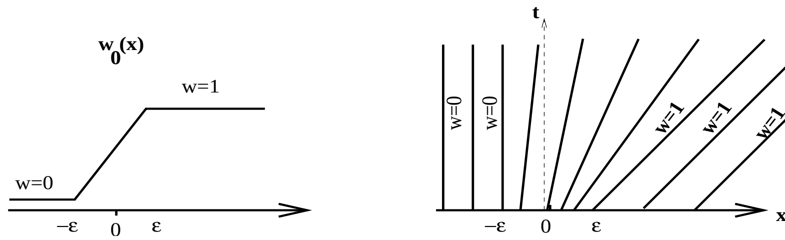

Figure 7. The initial data (left) and characteristics (right) for the regularized problem (19).

The Entropy Condition

Since weak solutions are not unique, an aditional condition is needed to select the physically relevant solution. This condition is known as the *Entropy Condition*, and can be formulated in a variety of ways. Weak solutions that satisfy the entropy condition are called *entropy solutions*. In the scalar case, a simple geometric condition, due to Lax, says that a shock is admissible provided the characteristics converge into it. In other words, a shock wave satisfying the Rankine-Hugoniot jump conditions (14) is allowed only if

$$f'(w_L) > s > f'(w_R)\ . \tag{20}$$

Intuitively, the solution in Figure 6 (center) doesn't seem right, the characteristics emanate from the shock, and the solution in $0 < x/t < 1$ does not depend on the initial data but rather on some information that is generated at the shock. It is also unstable to small changes either in the initial data or in the equation. This solution also violates the entropy condition (20). A more general condition that holds also for the nonconvex case (due to Oleinik) is

$$\frac{f(w) - f(w_L)}{w - w_L} \geq s \geq \frac{f(w) - f(w_R)}{w - w_R}\ , \tag{21}$$

for all w between w_L and w_R.

Another formulation of the entropy condition uses entropy functions. We define $\eta(w)$ and $\psi(w)$ with $\psi'(w) = \eta'(w)f'(w)$ to be an *entropy function* and *entropy flux* associated with the conservation law (6). Then it is easy to verify that for smooth solutions, η satisfies the conservation law $\eta(w)_t + \psi(w)_x = 0$. Using the vanishing viscosity limit of the viscous equation (16), it can be shown that in the nonsmooth case $\eta_t + \psi(w)_x < 0$ in a weak sense. Putting the two together we get that w is the entropy solution of (6) if for all entropy-entropy flux pairs

$$\eta_t + \psi(w)_x \leq 0\ . \tag{22}$$

Finally, it is sometimes convenient to express the entropy inequality using test functions: $w(x,t)$ is the entropy solution of (6) if it satisfies

$$\int_0^\infty \int_{-\infty}^\infty \phi_t \eta(w) + \phi_x \psi(w)\ \mathrm{dx\ dt} \leq \int_{-\infty}^\infty \phi(x,0)\eta(w_0(x))\ \mathrm{dx} \tag{23}$$

for all (positive) test functions $\phi(x,t)$, smooth with compact support.

Nonlinear Transformations

Nonlinear transformation have a surprising effect on solutions of nonlinear hyperbolic conservation laws. Consider Burger's equation

$$w_t + \left(\frac{1}{2}w^2\right)_x = 0\ . \tag{24}$$

We write equation (24) in quasilinear form $w_t + ww_x = 0$, and multiplying by $2w$ gives $2ww_t + 2w^2 w_x = 0$ which can be recast as a conservation law

$$\left(w^2\right)_t + \left(\frac{2}{3}w^3\right)_x = 0 \tag{25}$$

This amounts to a nonlinear transformation of w into w^2. It can be shown that both equations (24) and (25) admit the same *smooth* solutions. As soon as shock waves are formed, however, this is no longer true. Shock waves will propagate with speed $S_1 = [\frac{1}{2}w^2]/[w] = (w_L + w_R)/2$ under (24) and with speed $S_2 = [\frac{2}{3}w^3]/[w^2] = \frac{2}{3}(w_L^2 + w_L w_R + w_R^2)$ under (25). For weak shocks, $S_1 \approx S_2$ but they become very different as the shock gets stronger. To determine which shock speed is the correct one, we need to know which equation expresses conservation correctly: if w is the conserved quantity, (24) is the correct equation and S_1 is correct shock speed; if w^2 is the quantity which is conserved, then (25) is the correct equation and S_2 is the correct speed. We note that the scalar case is special in that the transformed equation can be recast in a new conservation form. In the systems case, this is generally not true: Nonlinear transformations generally transform conservation laws into hyperbolic systems that cannot be recast in conservation form. It is important to note that even if they do, this may not represent correct physical conservation, and the transformed equations are only valid as long as the solution is smooth.

3 HYPERBOLIC SYSTEMS

Consider the hyperbolic system of conservation laws

$$W_t + F(W)_x = 0\ , \qquad W(x,0) = W_0(x) \tag{26}$$

where $W = (w_1, w_2, ..., w_n)^T$ and $F(W) = (f_1(W), f_2(W), ..., f_n(W))^T$ are vectors of length n. The system may be written in quasi-linear form $w_t + A(w)w_x = 0$ where $A(w)$ is the *Jacobian Matrix*

$$A(W) = \frac{\partial F(W)}{\partial W} = \begin{pmatrix} \partial f_1/\partial w_1 & \partial f_1/\partial w_2 & \cdots & \partial f_1/\partial w_n \\ \partial f_2/\partial w_1 & \partial f_2/\partial w_2 & \cdots & \partial f_2/\partial w_n \\ \vdots & \vdots & & \vdots \\ \partial f_n/\partial w_1 & \partial f_n/\partial w_2 & \cdots & \partial f_n/\partial w_n \end{pmatrix} .$$

The system is hyperbolic with real eigenvalues λ_k and a full set of eigenvectors r_k. We assume the eigenvalues are ordered

$$\lambda_1(W) \le \lambda_2(W) \le \cdots \le \lambda_n(W)\ .$$

The term *strict hyperbolicity* is sometimes used if the eigenvalues are distinct, in which case the eigenvectors are linearly independent, hence form a complete set. We denote by $R = (r_1, r_2, \cdots, r_n)$ the matrix of right eigenvectors, and by Λ the diagonal matrix of eigenvalues. We have $Ar_k = \lambda_k r_k\ , k = 1, \cdots, n$ which in matrix form reads $AR = R\Lambda$, or equivalently

$$A = R\Lambda R^{-1}\ .$$

We denote by (w) the set of left eigenvectors satisfying $l_k A = \lambda_k l_k$ and the matrix of left eigenvectors L. In general, the set of left and right eigenvectors are different from each other and satisfy the *bi-orthogonality* condition $l_i \cdot r_j = \delta_{i,j}$ or equivalently $LR = I$. If $A(W)$ is symmetric, the two sets of eigenvectors are identical, that is $L = R^{-1} = R^T$. In the general case $A = A(W)$, and λ_k, r_k and l_k all depend on W. We will discuss the linear case before moving to the nonlinear case.

3.1 *Linear Systems*

If $A(w) = A$ is constant, $F(W) = AW$ and the system of equations is linear

$$W_t + AW_x = 0 \tag{27}$$

Given initial data $W_0(x)$ at $t = 0$, the solution may be constructed using characteristics. We multiply (27) by R^{-1} and denote by $V = R^{-1}W$ then

$$R^{-1}W_t + R^{-1}AW_x = R^{-1}W_t + R^{-1}ARR^{-1}W_x = V_t + \Lambda V_x = 0$$

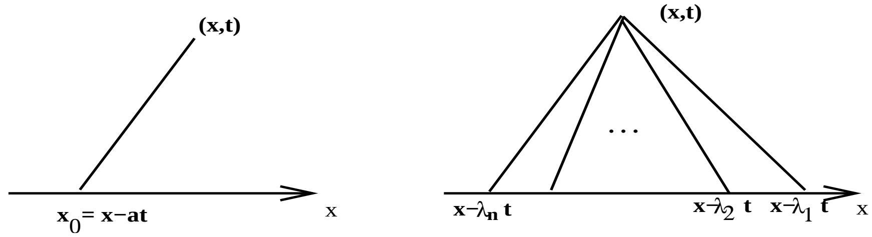

Figure 8. Solution domain of Dependence: Scalar (left) and systems (right).

and the system decouples into n scalar advection equations in the components of the vector $V = (v_1, v_2, \cdots, v_n)$

$$\begin{array}{cc} (v_1)_t + \lambda_1 (v_1)_x = 0 \,, & v_1(x,0) = (R^{-1}W_0(x))_1 = l_1 W_0(x) \\ (v_1)_2 + \lambda_2 (v_2)_x = 0 \,, & v_2(x,0) = (R^{-1}W_0(x))_2 = l_2 W_0(x) \\ \vdots & \vdots \\ (v_n)_t + \lambda_n (v_n)_x = 0 \,, & v_n(x,0) = (R^{-1}W_0(x))_n = l_n W_0(x) \end{array}$$

where we have also translated the initial data in terms of V. The components of V, given by $v_k = l_k W$, are called the *characteristic variables*, the eigenvalues λ_k are the *characteristic speeds*, and can be either positive or negative. Lines in the $x - t$ plane satisfying $dx/dt = \lambda_k$ are the *characteristics* associated with the k^{th} characteristic field. Characteristic decomposition is the main vehicle to generalize tools and concepts from the scalar case to systems. The problem takes a simpler form when formulated in terms of the characteristic variables, it decouples into n scalar pieces. Each scalar piece may be solved independently, as we have found in (10), $v_k(x,t) = v_k(x - \lambda_k t, 0)$ and the pieces may be transformed back in terms of $W = RV$

$$W(x,t) = RV(x,t) = \sum_{k=1}^{n} v_k(x,t) r_k = \sum_{k=1}^{n} v_k(x - \lambda_k t, 0) r_k$$

The above formula expresses the geometric construction of the solution W at (x,t) by tracing back the characteristics to the initial line. The domain of dependence now consists of n points along the initial line $x_k = x - \lambda_k t$. The solution is a superposition of waves proportional to eigenvectors r_k propagating at speeds λ_k (see Figure 8).

The Riemann Problem

Consider the Riemann problem

$$W_t + AW_x = 0 \,, \qquad W_0(x) = \begin{cases} W_L = \sum_{k=1}^{n} a_k r_k & x < 0 \\ W_R = \sum_{k=1}^{n} b_k r_k & x > 0 \end{cases} \tag{28}$$

where we have expanded the initial states $W_{L,R}$ in terms of the eigenvectors r_k. We also write the initial jump as

$$[W] = W_R - W_L = \sum_{k=1}^{n} (b_k - a_k) r_k = \sum_{k=1}^{n} \alpha_k r_k$$

We also have

$$W(x,0) = Rv(x,0) = \sum_{k=1}^{n} v_k(x,0) r_k = \begin{cases} W_L & x < 0 \\ W_R & x > 0 \end{cases} \tag{29}$$

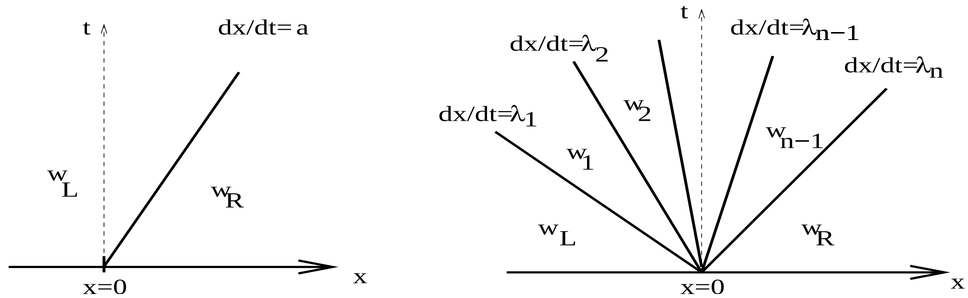

Figure 9. Solution of the Riemann problem for linear equations: scalar (left) and systems (right).

and by equating the coefficients of r_k in (28) and (29) we obtain

$$v_k(x,0) = \begin{cases} a_k & x<0 \\ b_k & x>0 \end{cases}, \qquad \Longrightarrow \qquad v_k(x,t) = v_k(x-\lambda_k t, 0) = \begin{cases} a_k & x/t<\lambda_k \\ b_k & x/t>\lambda_k \end{cases}.$$

Consider the solution $W(x,t)$ in the wedge $\lambda_p < x/t < \lambda_{p+1}$, so that $x - \lambda_k t > 0$, for $k = 1, ..., p$ and $x - \lambda_k t < 0$, for $k = p+1, ..., n$. We obtain

$$\begin{aligned} W(x,t) &= \sum_{k=1}^{n} v_k(x,t) r_k = \sum_{k=1}^{n} v_k(x-\lambda_k t, 0) r_k \\ &= \sum_{k=1}^{p} b_k r_k + \sum_{k=p+1}^{n} a_k r_k = \sum_{k=1}^{n} a_k r_k + \sum_{k=1}^{p} (b_k - a_k) r_k \\ &= W_L + \sum_{k=1}^{p} \alpha_k r_k = W_R - \sum_{k=p+1}^{n} \alpha_k r_k \end{aligned}$$

Put differently

$$W(x,t) = \begin{cases} W_L & x/t < \lambda_1 \\ W_L + \alpha_1 r_1 & \lambda_1 < x/t < \lambda_2 \\ W_L + \alpha_1 r_1 + \alpha_2 r_2 & \lambda_2 < x/t < \lambda_3 \\ \quad\vdots & \vdots \\ W_L + \sum_{k=1}^{p} \alpha_k r_k & \lambda_p < x/t < \lambda_{p+1} \\ \quad\vdots & \vdots \\ W_L + \sum_{k=1}^{n} \alpha_k r_k = W_R & \lambda_n < x/t \end{cases} \tag{30}$$

Equation (30) provides an extremely useful picture of the wave structure of the solution for the Riemann problem. As in the scalar case, the solution depends on the ratio $\xi = x/t$, it is piecewise constant in wedges $\lambda_p < \xi < \lambda_{p+1}$, separated by discontinuous wave fronts of strengths $[w_k] = \alpha_k r_k$ propagating at speeds λ_k. The initial jump in the Riemann data is projected onto the characteristic fields, $W_R - W_L = \sum_{k=1}^{n} \alpha_k r_k$, and each piece $\alpha_k r_k$ propagates at its corresponding speed λ_k (see Figure 9). Solutions for which all components but one are zero are called *simple waves*, they constitute important building blocks in the construction of solutions.

Riemann Solution in Phase Space

For simplicity, we consider a 2 by 2 system with $W = (w_1, w_2)$, and r_k, λ_k, $k = 1, 2$. We observe that if the initial states satisfy $[W] = W_R - W_L \propto r_k$, then $[W]$ propagates as a single discontinuity (simple wave) with speed λ_k. We change the question around and ask: Given a state $\widehat{W}$, what states W can be connected

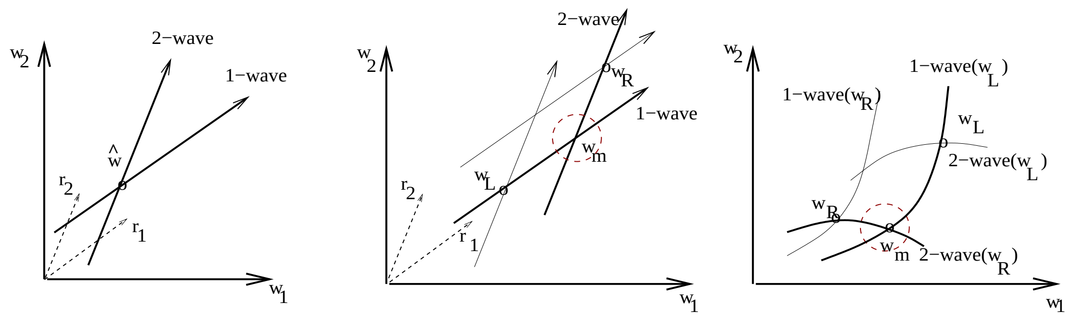

Figure 10. The Riemann problem in phase space: Simple wave through $\widehat{W}$ (left), intersection of simple wave curves through $\widehat{W}_{L,R}$ (middle) and simple wave curves in the nonlinear case (right)

to $\widehat{W}$ via a 1-wave, that is via a single wave in the 1-family. This is best answered geometrically in phase plane. The states that are connectable to a given $\widehat{W}$ via a 1-wave are a one parameter family of states along the line parallel to r_1 through $\widehat{W} = (\hat{w}_1, \hat{w}_2)$ (see Figure 10). Similarly, the states that can be connected to a given $\widehat{W}$ via a 2-wave are represented by a line parallel to r_2 through $\widehat{W}$. The complete solution of the Riemann problem in this case reduces to the following question: Given left/right states $W_{L,R}$, find a middle state W_m that can be connected to W_L via a 1-wave, and to W_R via a 2-wave. This is easily done geometrically, by drawing all the states that are connectable to W_L via a 1-wave, and all the states that are connectable to W_R via a 2-wave, and look for their point of intersection (see Figure 10). There are two points of intersection of the simple wave curves, but only one of them respects the order of the waves $\lambda_1 < \lambda_2$, connecting W_L to W_m via a 1-wave, followed by a 2-wave that connects W_m to W_R. The linear independence of r_1 and r_2 guarantees that the point of intersection that we are seeking exists and is unique, which in turns guarantees the existence of a unique solution to the Riemann problem.

This approach generalizes to the nonlinear case. The solution for the Riemann problem is given by intersection of elementary wave curves in phase space. In the nonlinear case, the elementary waves must allow for shock waves and rarefaction fans, and the curves they produce are not generally straight lines through $W_{L,R}$, as is illustrated schematically in Figure 10. The intersection point is generally sought by nonlinear rootfinding and is not guaranteed to exist or to be unique. This will be pursued in the next section.

3.2 *Nonlinear Systems*

In the nonlinear case, $A = A(W)$, the eigenvalues $\lambda_k(W)$ and corresponding right/left eigenvectors $r_k(W)$ and $l_k(W)$, now all depend on W. As in the scalar case, elementary waves may propagate as shock waves, or as rarefaction fans, except now they may occur in each of the n characteristic fields. Another type of wave is of a linear flavor and is called a *contact discontinuity*. The solution for the Riemann problem, as we will see below, consists of a combination of such waves. As in the scalar case, the solution structure is simplified if the flux is convex. For scalar equation, we have $f''(w) = a'(w) > 0$ for all$w$, or $f''(w) = a'(w) < 0$ for all$w$ that is the propagation speed is a monotone function of w. This condition generalizes to systems through the quantity

$$\nabla\lambda(W) \cdot r(W) \; .$$

Here $\nabla\lambda(W) = (\partial\lambda/\partial w_1, \partial\lambda/\partial w_2, \cdots, \partial\lambda/\partial w_n)$. We distinguish between two possibilities. We say that the k^{th} characteristic field is *genuinely nonlinear* if $\nabla\lambda_k(W) \cdot r_k(W) \neq 0$ for all W, and we say that the k^{th} characteristic field is *linearly degenerate* if $\nabla\lambda_k(W) \cdot r_k(W) = 0$. This distinction arises naturally in the construction of simple wave solutions of (26), as we will see below. For now, we make the following comments: (i) this condition generalizes the scalar case (where we identify λ_k with $a(w)$ and take r_k to be 1): in the genuinely nonlinear case, it reduces to the scalar convexity condition $a'(w) \neq 0$for all$w$; in the linear case, a is constant and $\nabla a = a' = 0$; (ii) a genuinely nonlinear field admits elementary waves that are either shock waves (converging characteristics) or expansion fans (diverging characteristics), while a linearly degenerate field admits simple waves called *contact discontinuities*, across which the characteristics remain parallel as in the linear case.

As a first step, we pursue characteristic decomposition, as in the linear case, and multiply (26) by $l_k(W)$

$$l_k W_t + l_k A(W) W_x = l_k W_t + \lambda_k l_k W_x = 0$$

which reduces the PDE (26) to an ODE along the *characteristics*

$$l_k(W) \cdot dW = 0 \ , \qquad \text{along } dx/dt = \lambda_k(W) \tag{31}$$

This differential relation along the characteristic is often not simple enough to be integrated, and the Riemann Invariants along characteristics may not exist. It is nonetheless useful, for example in imposing boundary conditions on truncated infinite domains. What is available are the so-called *Generalized Riemann Invariants*, these are solution invariants *across* characteristics, as we will see below.

The Riemann Problem

To solve the Riemann problem

$$W_t + F(W)x = 0 \ , \qquad W(x,0) = \begin{cases} W_L & x < 0 \\ W_R & x > 0 \end{cases} \ , \tag{32}$$

we consider the solution building blockes, namely rarefaction fans (denoted by $\mathcal{R}$), shock waves (denoted by $\mathcal{S}$) and contact discontinuities (denoted by $\mathcal{C}$).

Rarefaction Waves

We take lead from the scalar case, and seek solutions $W(x,t) = V(x/t) = V(\xi)$ of the form

$$W(x,t) = \begin{cases} W_L & x/t < \xi_1 \\ V(\xi) & \xi_1 \le x/t \le \xi_2 \\ W_R & \xi_2 < x/t \end{cases} \ . \tag{33}$$

For $W(x,t)$ of the form (33)

$$W_t = -\frac{x}{t^2} V'(x/t) \ , \quad W_x = \frac{1}{t} V'(x/t)$$

and substituting into (32) gives

$$W_t + A(W) W_x = -\frac{x}{t^2} v'(x/t) + A(V) \frac{1}{t} V'(x/t) = 0$$

or

$$A(V(\xi)) V'(\xi) = \xi V'(\xi) \ .$$

It follows that for $W(x,t)$ of the form (33) to be a solution, ξ must be an eigenvalue, and $V'(\xi)$ must be the corresponding eigenvector,

$$V'(\xi) = \alpha(\xi) r_k(V(\xi)) \ , \qquad \xi = \lambda_k(V(\xi)) \tag{34}$$

$V(\xi)$ defines a curve whose tangent, $V'(\xi)$, points in the direction of the eigenvector $r_k(V(\xi))$. Such curves are called *integral curves*, and Riemann problems for which solutions of this form exist have the property that W_L and W_R lie on the same integral curve. We further observe that since $\lambda_k(V(\xi)) = \xi$, the eigenvalue λ_k increases monotonically as ξ changes from ξ_1 to ξ_2 along the integral curve. This is precisely the condition of genuine nonlinearity. It arises naturally if one tries to solve the above system of ODEs:

Differentiating (34) with respect to ξ gives

$$1 = \nabla\lambda_k(V(\xi)) \cdot V'(\xi) = \nabla\lambda_k(V(\xi))) \cdot (\alpha(\xi) r_k(V(\xi)) = \alpha(\xi)\nabla\lambda_k(V(\xi)) \cdot r_k(V(\xi))$$

and the proportionality scalar $\alpha(\xi)$ is given by $\alpha(\xi) = 1/\nabla\lambda_k(V(\xi)) \cdot r_k(V(\xi))$ provided $\nabla\lambda_k(V(\xi)) \cdot r_k(V(\xi)) \neq 0$, that is provided that the field is genuinely nonlinear. Rarefaction fans are curves satisfying the system of ODE's

$$V'(\xi) = \frac{1}{\nabla\lambda_k(V(\xi)) \cdot r_k(V(\xi))} r_k(V(\xi)) \ , \ \ V(\xi_1) = W_L \ , \ \ V(\xi_2) = W_R \ .$$

The parameter $\xi = x/t$ may be eliminated to give differential relations between the solution components that hold along integral curves. The relation $V'(\xi) \propto r_k(\xi)$ may be expressed as $dV \propto r_k(V)$ or

$$\frac{dv_1}{(r_k)_1} = \frac{dv_2}{(r_k)_2} = \cdots = \frac{dv_n}{(r_k)_n}$$

which may be integrated in phase space to give $n-1$ solution invariants *across* simple waves called the *Generalized Riemann Invariants*. Given a state $\widehat{W}$, the set of all states that can be connected to $\widehat{W}$ via a simple wave in the k^{th} characteristic field form a one parameter family states, characterized by $n-1$ Riemann Invariants. We will illustrate these curves by considering specific examples.

Shock Waves
Given a state $\widehat{W}$, we would like to determine all states W that can be connected to $\widehat{W}$ via a single shock in the k-th characteristic field. The shock relations (14) give

$$s(W - \widehat{W}) = F(W) - F(\widehat{W}) \tag{35}$$

They form a nonlinear system of n equations in $n+1$ unknowns, $w = (w_1, w_2, \cdots, w_n)$ and the speed of the shock s, and defines a one-parameter family of states (provided a solution exists). We denote those states by $W^k(\alpha)$. In the linear case,

$$W^k(\alpha) = \widehat{W} + \alpha r_k \ , \qquad s_k(\alpha) = \lambda_k$$

the parameter α measures the streangth of the wave, and $W^k(\alpha)$ describes a straight line through $\hat{W}$ parallel to r_k. In the nonlinear case,

$$F(W^k(\alpha)) - F(\widehat{W}) = s^k(\alpha) \left(W^k(\alpha) - \widehat{W} \right) \tag{36}$$

and the family of states $W^k(\alpha)$ generally defines a curve in phase space. Again, we will illustrate these curves by considering specific example. For now, we consider the shock curve in the limit of weak shock, $\alpha \to 0$, $W^k(\alpha) \to \hat{W}$. We differentiate (36) with respect to α

$$F'(W^k(\alpha))(W^k)'(\alpha) = (s^k)(\alpha)(W^k(\alpha) - \widehat{W}) + s^k(\alpha)(W^k)'(\alpha)$$

and evaluate at $\alpha = 0$,

$$F'(\widehat{W})(W^k)'(0) = s^k(0)(W^k)'(0) \ .$$

It follows that $(W^k)'(0)$ is an eigenvector of $F'(\widehat{W}) = A(\widehat{W})$ and $s^k(0) = \lambda_k(\widehat{W})$ is the corresponding eigenvalue, which reiterates the fact that weak shock waves resembles linear waves, they propagate at speed λ_k and their strength is proportional to the corresponding eigenvector r_k. We note further that the tangent to the shock curve at $\widehat{W}$, $\hat{(}W^k)'(0)$, points in the direction of $r_k(\widehat{W})$, just like the integral curve describing rarefaction fans. Both these curves pass through $\widehat{W}$ and have the same slope. It can be shown that they also have the same curvature (!). One curve represent smooth solutions, across which the entropy remains unchanged, the other represent discontinuous shock waves, across which the entropy changes. The fact that these two curves match to second order in α at $\widehat{W}$ is also reflected in the fact that entropy changes across shock fronts are third order in the shock strength.

The entropy condition generalizes to systems in two parts. A shock in the k^{th} characteristic field is admissible provided

$$\lambda_k(W_L) > s > \lambda_k(W_R)\ , \qquad \lambda_{k-1}(W_L) < s < \lambda_{k+1}(W_R)$$

The first condition implies that the characteristics in the k^{th} field converge into the shock, and the second condition implies that those characteristics are the only ones that run into the shock, characteristics in all other fields run 'through' the shock.

Examples

The Euler Equations

The Euler equations model the dynamics of compressible, inviscid non heat conducting gases, and express the conservation of mass, momentum and energy. In one space dimension, they take the form

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix}_x = 0\ . \tag{37}$$

Here, ρ, u and p are the density, velocity and pressure, and E total energy of the gas which is the sum of the kinetic plus internal energy $E = \frac{1}{2}\rho u^2 + \rho e$. To close the system, thermodynamic considerations determine how the pressure depends on the flow variables. This relation is called The *Equation of State* (EOS). For ideal gases, the specific internal energy e is assumed to depend linearly on the temperature T, and the EOS becomes

$$p = p(\rho, e) = (\gamma - 1)\rho e \tag{38}$$

where γ is a gas constant called the specific heat ratio, $1 < \gamma < 3$. For air, $\gamma = 1.4$. We further use $\eta = p/\rho^\gamma$ to denote the entropy, c to denote the speed of sound, with $c^2 = \frac{dp}{d\rho}\big|_\eta$, and h the (specific) enthalpy $h = (E+p)/\rho = \frac{1}{2}u^2 + \frac{c^2}{\gamma-1}$. For ideal gas, $c^2 = \gamma p/\rho$. The system is hyperbolic, with eigenstructure

$$R = \begin{pmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ h-uc & \frac{1}{2}u^2 & h+uc \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} u-c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+c \end{pmatrix}.$$

Nonlinear Transformations and Primitive Formulations

Certain calculations are simplified by writing the equations in terms of other sets of variables, for example the physical variables $W^p = (\rho, u, p)$. There is generally no physical conservation associated with these variables, so the resulting system of equations are in quasi-linear form, not in conservation form. These equations are valid only as long as the flow is smooth. One place where nonconservative formulations simplify things is in figuring out the structure of simple wave solutions.

Consider a (nonlinear) transformation from the conserved variables W^c to some primitive set of variables W^p, and denote by T the Jacobian of the transformation, $T = \frac{\partial W^p}{\partial W^c}$. On physical grounds, we expect the characteristic speeds of the transformed system in *any* set of variables to be unchanged. We write the conservative Euler system (37) as

$$W_t^c + F(W^c)_x = W_t^c + A(W^c)W_x^c = 0$$

and pre-multiply by T to give

$$TW_t^c + TA(W^c)W_x^c = 0 \qquad \Longrightarrow \qquad W_t^p + TA(W^c)T^{-1}TW_x^c = 0$$

or

$$W_t^p + \tilde{A}(W^p)W_x^p = 0$$

where $\tilde{A}(W^p) = TA(W^c)T^{-1}$. The transformation is a similarity transformation, which preserves the eigenvalues as expected. The eigenvectors are related through the transformation matrix $r^p = Tr^c$, with obvious notation.

The Euler system in terms of the primitive set of variables $W^p = (\rho, u, p)$ and the corresponding eigenvectors are given by

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_x = 0 \ , \qquad R^{(\rho,u,p)} = \begin{pmatrix} 1 & 1 & 1 \\ -c/\rho & 0 & c/\rho \\ c^2 & 0 & c^2 \end{pmatrix} \ , \tag{39}$$

We will use reduced 2×2 models of gas dynamics to illustrate the Riemann Invariants, shock relations, integral and shock curves. We will then return to (37) and complete the discussion for the full Euler system.

Reduced Gas Dynamics Models

If the pressure p is assumed to depend on the density ρ alone, the first two equations of the Euler system (37) provide a complete flow description, and can be solved as a 2×2 system. The total energy in such models is not guaranteed to be conserved.

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \end{pmatrix}_x = 0 \ .$$

It is easy to verify that the system is hyperbolic provided $p'(\rho) > 0$. The positive quantity $p'(\rho)$ is denoted by c^2, with c the *speed of sound*. The coefficient matrix $A(W)$ and eigenstructure are given by

$$A(W) = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix} \ , \quad \Lambda = \begin{pmatrix} u - c & 0 \\ 0 & u + c \end{pmatrix} \ , \quad R = \begin{pmatrix} 1 & 1 \\ u - c & u + c \end{pmatrix} \ . \tag{40}$$

Within this simplifying assumption, we present two examples: (i) The Isothermal Gas Dynamics and (ii) Isentropic Gas Dynamics.

Isothermal Gas Dynamics

For ideal gas, the equation of state is $p/\rho = RT$ with R the gas constant. If the flow is assumed isothermal (constant temperature flow), it follows that $p/\rho = RT = const$ and the speed of sound $c^2 = p'(\rho) = RT$ is constant.

It is easy to check that both characteristic fields are genuinely nonlinear

$$\left(\nabla_{(\rho,\rho u)}\, \lambda_{1,2}\right) \cdot r_{1,2} = \nabla_{(\rho,\rho u)} \, \left(\frac{\rho u}{\rho} \mp c\right) \cdot (1, u \mp c)^T = \left(-\frac{\rho u}{\rho^2}, \frac{1}{\rho}\right) \cdot (1, u \mp c)^T = \mp \frac{c}{\rho} \neq 0 \ ,$$

and support the propagation of elementary waves that are either shock waves or rarefaction fans.

While gas particles move *with* the flow velocity u, these simple waves move at speeds $u \mp c$, thus they move *through* the flow, with relative velocity $\mp c$. These waves are *sound waves* or *acoustic waves*, and they represent a genuine nonlinear phenomenon. A gas particle that starts off ahead of the wave, will find itself behind the wave, once the wave has passed. These waves have a 'head' and a 'tail', and they are being crossed by gas particles from front to back.

In terms of the primitive variables $W = (\rho, u)$ the system becomes

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0$$

The eigenvalues are unchanged, the eigenvectors are $r_1 = (1, -c/\rho)^T$, $r_2 = (1, c/\rho)^T$. For smooth flows, the primitive formulation can be used to simplify some of the calculations. For example, genuine nonlinearity is easily verifiable

$$\nabla\lambda_k \cdot r_k = \nabla_{(\rho,u)}(u \mp c) \cdot (1, \mp c/\rho)^T = (0, 1) \cdot (1, \mp c/\rho) = \mp c/\rho \neq 0 \, .$$

Rarefaction Waves
Integral curves satisfy the differential relations

$$\frac{d\rho}{1} = \frac{du}{\mp c/\rho} \qquad \Longrightarrow \qquad \frac{du}{d\rho} = \mp\frac{c}{\rho}$$

which can be integrated to give the Riemann Invariant across expansion fans

$$\begin{aligned} I^- &= u + \int \frac{c}{\rho} = u + c \log \rho = const \qquad \text{across } \mathcal{R}_1 \\ I^+ &= u - \int \frac{c}{\rho} = u - c \log \rho = const \qquad \text{across } \mathcal{R}_2 \end{aligned} \tag{41}$$

Finally, consider two states $W_{L,R}$ that connect through a rarefaction $\mathcal{R}_1$. We note that W_L is ahead of the wave, and W_R is behind it. From the monotonicity of $\lambda_1 = u - c$ across the wave, we have $u_L - c < u_R - c$, implying $u_L < u_R$. Since $du/d\rho = -c/\rho < 0$, density and velocity change in opposite directions. It follows that $\rho_R < \rho_L$, implying that the density ahead (left) of the $\mathcal{R}_1$ is higher than the density behind (right) the fan, that is the gas *rarefies* as it flows across the $\mathcal{R}_1$ fan. The same is easily verified for a $\mathcal{R}_2$ fan. Hence the name *rarefaction fan*.

We summarize by giving the integral curves through a given state $\hat{w} = (\hat{\rho}, \hat{u})$

$$\begin{array}{llll} u = \hat{u} - c \log(\rho/\hat{\rho}) \, , & \text{and} & \rho < \hat{\rho} & \text{behind } \mathcal{R}_1 \\ u = \hat{u} + c \log(\rho/\hat{\rho}) \, , & \text{and} & \rho < \hat{\rho} & \text{behind } \mathcal{R}_2 \end{array} \, .$$

Shock Curves
The shock relations are

$$\begin{aligned} \rho u - \hat{\rho}\hat{u} &= s(\rho - \hat{\rho}) \\ \rho u^2 + c^2\rho - (\hat{\rho}\hat{u}^2 + c^2\hat{\rho}) &= s(\rho u - \hat{\rho}\hat{u}) \end{aligned}$$

which is a nonlinear system of 2 equations in 3 unknowns, ρ, u and s. We write

$$\begin{aligned} \rho(u - s) &= \hat{\rho}(\hat{u} - s) = M \\ M(u - \hat{u}) &= -c^2(\rho - \hat{\rho}) \end{aligned}$$

and rearrange

$$M(u - \hat{u}) = M((u - s) - (\hat{u} - s)) = M^2(\frac{1}{\rho} - \frac{1}{\hat{\rho}}) = -M^2(\frac{\rho - \hat{\rho}}{\rho\hat{\rho}})$$

to give

$$M^2(\frac{\rho - \hat{\rho}}{\rho\hat{\rho}}) = c^2(\rho - \hat{\rho}) = -M(u - \hat{u}) \, .$$

This can be solved to give $M = \pm c\sqrt{\rho\hat{\rho}}$ and the one-parameter family of states that can be connected to a given $\widehat{W} = (\hat{\rho}, \hat{u})$ is given by

$$u = \hat{u} \pm c\frac{\rho - \hat{\rho}}{\sqrt{\rho\hat{\rho}}} \tag{42}$$

We get both $\pm$ signs because the shock relations that we have derived apply to either characteristic fields. To determine which sign corresponds to which field, we use the shock speed and recall that in the weak shock limit $s \to \lambda(\widehat{W})$. We substitute u from (42) into

$$s = \frac{\rho u - \hat{\rho}\hat{u}}{\rho - \hat{\rho}} = \hat{u} \mp \sqrt{\rho/\hat{\rho}}$$

in the limit of weak shock $\rho \to \hat{\rho}$, $u \to \hat{u}$. We observe that

$$s^- = \hat{u} - c\sqrt{\rho/\hat{\rho}} \to \hat{u} - c = \lambda_1(\widehat{W})$$

$$s^+ = \hat{u} + c\sqrt{\rho/\hat{\rho}} \to \hat{u} - c = \lambda_2(\widehat{W})$$

which establishes that the minus sign corresponds to $\mathcal{S}_1$, and the plus sign to $\mathcal{S}_2$. The weak shock limit also implies that the eigenvectors $r_{1,2}(\hat{W})$ are tangent to the respective curves $\mathcal{S}_{1,2}$ at $\widehat{W}$.

Finally, the entropy condition needs to be taken into consideration, namely that for a shock to be admissible, the characteristics must converge into the shock, see (20). Consider a $\widehat{W} = W_R$ ahead (to the right) of a shock $\mathcal{S}_2$, and W behind it. From the entropy condition, we have

$$\lambda_2(W) = u + c > s = \hat{u} + c\sqrt{\rho/\hat{\rho}} > \hat{u} + c = \lambda_2(\widehat{W})$$

and we conclude that for $\mathcal{S}_2$ propagating into a right state $\widehat{W} = W_R$, $u > \hat{u}$. Using the shock relation with the appropriate (plus) sign, gives $\rho > \hat{\rho}$, that is the density behind $\mathcal{S}_2$ is *higher* than ahead of it. A similar calculation establishes that the density behind an $\mathcal{S}_1$ shock is also higher then ahead of it. That is, shocks that are admissible by the entropy condition have higher density behind them than ahead of them, in other words they are *compressive*.

We now have the four curves of interest. We denote by $\widehat{W}$ the state *ahead* of the wave and summerize the results we have obtained

$$u = \hat{u} - c\frac{\rho - \hat{\rho}}{\sqrt{\rho\hat{\rho}}}\,, \qquad \rho > \hat{\rho}\,, \qquad \text{behind } \mathcal{S}_1$$

$$u = \hat{u} + c\log\left(\frac{\rho}{\hat{\rho}}\right), \qquad \rho < \hat{\rho}\,, \qquad \text{behind } \mathcal{R}_1$$

$$u = \hat{u} + c\frac{\rho - \hat{\rho}}{\sqrt{\rho\hat{\rho}}}\,, \qquad \rho > \hat{\rho}\,, \qquad \text{behind } \mathcal{S}_2$$

$$u = \hat{u} - c\log\left(\frac{\rho}{\hat{\rho}}\right), \qquad \rho < \hat{\rho}\,, \qquad \text{behind } \mathcal{R}_2$$

The Riemann Problem

We now have all we need to solve the Riemann problem. Assume a left state W_L and a right state W_R can be connected via a 1-shock followed by a 2-shock, and denote that middle state by W_m (see Figure **??**). To find W_m we note that W_m is connectable to W_L via a 1-shock, and to W_R via a 2-shock. It therefore

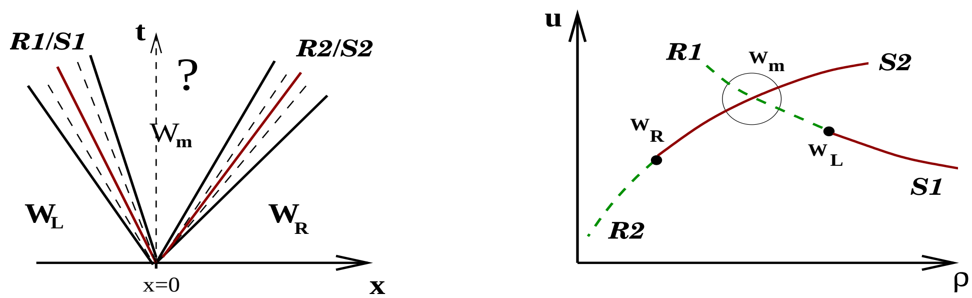

Figure 11. Solution of the Riemann problem for the isothermal model: characteristics in $x-t$ plane (left) and intersection of elementary curves in phase space (right).

satisfies

$$u_m = u_L - c\frac{\rho_m - \rho_L}{\sqrt{\rho_m \rho_L}} \ , \qquad u_m = u_R + c\frac{\rho_m - \rho_R}{\sqrt{\rho_m \rho_R}} \ .$$

Eliminating u_m between these two equations gives a quadratic equation in $\sqrt{\rho_m}$, with the physically relevant solution given by the positive root.

Of course not every two states may be connected by a sequence of entropy satisfying shocks. For a general $n \times n$ system, we make the following remarks: If W_R is close to W_L, a unique solution exists, connecting W_L to W_R through a sequence of shock waves

$$\begin{array}{ll} W_L & \\ W^1(\alpha_1) \ , & \text{1-parameter family of states connectable to } u_L \text{ via 1-shock} \\ W^2(\alpha_1,\alpha_2) \ , & \text{1-parameter family states connectable to } u_1(\alpha_1) \text{ via 2-shock} \\ W^3(\alpha_1,\alpha_2,\alpha_3) \ , & \\ \quad \vdots & \vdots \\ W^n(\alpha_1,\alpha_2,\cdots,\alpha_n) = W_R & \end{array}$$

but these shocks may not be entropy satisfying. If $W_R - W_L$ is not small, the shock curves may fail to intersect altogether.

If on the other hand, $W_{L,R}$ can be connected by 2 rarefaction fans, the the middle state W_m satisfies

$$u = u_L - c\log\rho/\rho_L \ , \qquad u = u_R + c\log\rho/\rho_R$$

and eliminating u between these two equations gives an equation for ρ

$$\rho = \sqrt{\rho_L \rho_R}\exp\frac{1}{2c}(u_L - u_R) \ .$$

More generally, given a left and right states $W_{L,R}$, we seek a middle state W_m that can be connected to W_L via a 1-wave (admissible shock or rarefaction) and to W_R via a 2-wave (admissible shock or rarefaction). Geometrically, this amounts to the following construction. Draw the 1-shock curve and 1-fan curve through W_L, those represent all the admissible states that can be connected to W_L via a 1-wave of any kind. Similarly, draw the 2-shock curve and 2-fan through W_R. The middle state W_m is on both sets of curves and solving the Riemann problem amounts to finding the intersection point of the relevant curves. This reduces to a nonlinear root finding problem in ρ. This is illustrated in Figure 11.

We note that the curves $\mathcal{S}_1$ and $\mathcal{R}_1$ through $\widehat{W}$ ahead of a wave are only meaningfully defined for $\rho > \hat{\rho}$ on $\mathcal{S}_1$ and $\rho < \hat{\rho}$ on $\mathcal{R}_1$, in agreement with the shock being compressive, and the rarefaction fan being expansive. We also note that at $\widehat{W}$ itself, both curves are tangent to $r_1(\widehat{W})$, that is they connect smoothly. It is not difficult to show that they also have the same curvature, that is they match up to and including 2nd derivatives. The same is true for $\mathcal{R}_2$ and $\mathcal{S}_2$. This is related to the entropy jump across a shock being 3^{rd} order in the shock strength.

Isentropic Gas Dynamics
If the flow is isentropic, $p/\rho^\gamma = A(\eta) = const$, again $p = p(\rho)$, and system (37) reduces to

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + A(\eta)\rho^\gamma \end{pmatrix}_x = 0\,. \tag{43}$$

The eigenstructure of this system is essentially the same as that of (40), with $c^2 = p'(\rho)\big|_\eta = \gamma A(\eta)\rho^{\gamma-1}$.
Using (41), the Generalized Riemann Invariants are

$$\begin{aligned} I^- &= u + \int \frac{c}{\rho}\, d\rho = u + \int \sqrt{\gamma A(\eta)}\rho^{\frac{\gamma-3}{2}}\, d\rho = u + \frac{2}{\gamma-1}c = const \qquad \text{across } \mathcal{R}_1 \\ I^+ &= u - \int \frac{c}{\rho}\, d\rho = u - \int \sqrt{\gamma A(\eta)}\rho^{\frac{\gamma-3}{2}}\, d\rho = u - \frac{2}{\gamma-1}c = const \qquad \text{across } \mathcal{R}_2 \end{aligned}$$

Equivalently, the integral curves through $\widehat{W} = (\hat{\rho}, \hat{u})$ are

$$u = \hat{u} - \frac{2}{\gamma-1}(c - \hat{c})\ , \qquad \text{behind } \mathcal{R}_1$$

$$u = \hat{u} + \frac{2}{\gamma-1}(c - \hat{c})\ , \qquad \text{behind } \mathcal{R}_2$$

The Hugoniot curve can be obtained as follows.

$$\left.\begin{aligned} s[\rho] &= [\rho u] \\ s[\rho u] &= [\rho u^2 + p] \end{aligned}\right\} \implies \left.\begin{aligned} \rho(u-s) &= \hat{\rho}(\hat{u}-s) = M \\ \rho u(u-s) + p &= \hat{\rho}\hat{u}(\hat{u}-s) + \hat{p} \end{aligned}\right\}$$

yielding

$$M[u] + [p] = M^2[\tau] + [p] = 0$$

here $\tau = \rho^{-1}$ is the specific volume. Eliminating M gives $[u]^2 = -[\tau][p]$ and the Hugoniot curve in the $\rho - u$ plane is given by

$$u = \hat{u} \pm \sqrt{-[\tau][p]}\,,$$

with the $-$ sign corresponding to $\mathcal{S}_1$, and the $+$ sign to $\mathcal{S}_2$. As in the isothermal flow example, the entropy condition implies that shocks are compressive, that is the gas density behind the shock is greater than ahead of it. Similarly, expansion fans are rarefactions, that is the gas density behind the fan is lower than ahead of it. The Riemann problem with given $W_{L,R}$ can now be solved by drawing the $(\mathcal{R},\mathcal{S})_1$ through W_L, and $(\mathcal{R},\mathcal{S})_2$ through W_R and seek a middle state W_m that is the intersection point of those curves.

Back to the Euler Equations
We now complete the picture for the Euler system (37). While the derivations are a little more involved, we should not loose sight of the similarities with the reduced systems. The derivation follows exactly the same steps, some of the details will be skipped.

Premultiplying (37) by the respective left eigenvectors leads to the following differential relations along characteristics

$$\begin{array}{lll} dp - \rho c du = 0 & \text{along} & dx/dt = u - c \\ dp - c^2 d\rho = 0 & \text{along} & dx/dt = u \\ dp + \rho c du = 0 & \text{along} & dx/dt = u + c \end{array} . \tag{44}$$

It is not difficult to verify that the first/third characteristic fields are genuinely nonlinear

$$\nabla\lambda_1(W) \cdot r_1(W) = -\frac{\gamma+1}{2}\frac{c}{\rho} < 0 \, , \qquad \nabla\lambda_3(W) \cdot r_3(W) = \frac{\gamma+1}{2}\frac{c}{\rho} > 0 \, , \tag{45}$$

admitting shock waves and rarefaction fans. As in the isothermal/isentropic systems, admissible shocks are compressive and rarefaction fans are expansive. There are *two* Generalized Riemann Invariants across each simple wave. They can be found most easily from the physical variables $W = (\rho, u, p)$ and associated eigenvectors. Across $\mathcal{R}_1$, we have

$$\frac{d\rho}{\rho} = \frac{du}{-c} = \frac{dp}{\gamma p}$$

where we have used $\rho c^2 = \gamma p$ for ideal gas. We integrate and get

$$\frac{d\rho}{\rho} = \frac{dp}{\gamma p} \qquad \Longrightarrow \qquad \gamma \log \rho = \log p - \log A \qquad \Longrightarrow \qquad \frac{p}{\rho^\gamma} = Const$$

that is the flow is isentropic. The other Riemann Invariant can be found as in the isentropic flow case to give

$$u + \frac{2}{\gamma - 1} c = const$$

The two Generalized Riemann Invariants are

$$\begin{array}{llll} I_1 = u + \dfrac{2c}{\gamma-1} \, , & \text{and} \quad I_2 = \dfrac{p}{\rho^\gamma} & & \text{across} \ \mathcal{R}_1 \\ I_1 = u - \dfrac{2c}{\gamma-1} \, , & \text{and} \quad I_2 = \dfrac{p}{\rho^\gamma} & & \text{across} \ \mathcal{R}_3 \end{array} .$$

Substituting I_2 into I_1 yields useful expressions for the one parameter family of states connectable to a given $W_{L,R}$ via a rarefaction

$$u = u_L - \frac{2c_L}{\gamma-1}\left[\left(\frac{p}{p_L}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] \, , \qquad \text{and} \quad p < p_L \qquad \text{across} \ \mathcal{R}_1$$

$$u = u_R + \frac{2c_R}{\gamma-1}\left[\left(\frac{p}{p_R}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] \, , \qquad \text{and} \quad p < p_R \qquad \text{across} \ \mathcal{R}_3$$

Shock relations are

$$\begin{array}{rl} [\rho u] & = s\,[\rho] \\ [\rho u^2 + p] & = s\,[\rho u] \\ [u(E+p)] & = s\,[E] \end{array}$$

For a given state ahead of the shock $\widehat{W} = (\hat{\rho}, \hat{u}, \hat{p})$, they form a nonlinear system of three equations in four unknowns: the state variables behind the shock, $W = (\rho, u, p)$, and the speed of the shock s, and define a

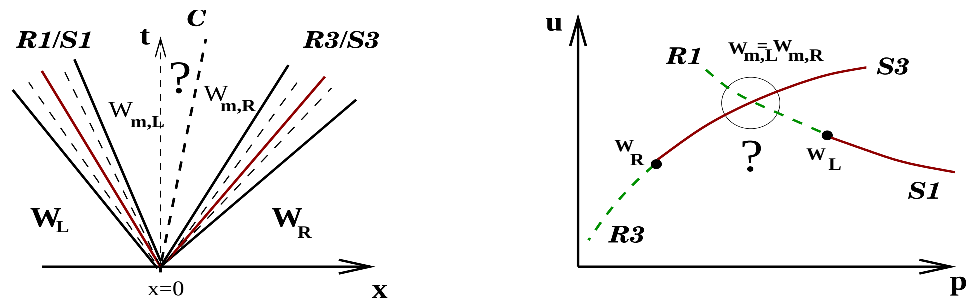

Figure 12. Solution of the Riemann problem for the Euler equations: characteristics in $x - t$ plane (left) and intersection of elementary curves in phase space (right).

one parameter family of states. Elimination of ρ and s yields a useful expression for this family. We use τ to denote the specific volume $\tau = \rho^{-1}$, and skip the details,

$$u = u_L - (p - p_L)\left[\frac{2\tau_L}{(\gamma_1)p + (\gamma - 1)p_L}\right]^{1/2}, \qquad \text{and} \quad p > p_L \qquad \text{across} \quad \mathcal{S}_1$$

$$u = u_R + (p - p_R)\left[\frac{2\tau_R}{(\gamma_1)p + (\gamma - 1)p_R}\right]^{1/2}, \qquad \text{and} \quad p > p_R \qquad \text{across} \quad \mathcal{S}_3$$

Where we get something new is in the particle wave, associated with speed $\lambda_2 = u$. Here, we can use $W^c = (\rho, \rho u, E)$ to obtain

$$\nabla\lambda_2(W) \cdot r_2(W) = \left(-\frac{\rho u}{\rho^2}, \frac{1}{\rho}, 0\right) \cdot \left(1, u, u^2/2\right) = 0$$

or more easily use the primitive variables $W^p = (\rho, u, p)$ and associated eigenvector

$$\nabla\lambda_2(W) \cdot r_2(W) = (0, 1, 0) \cdot (1, 0, 0) = 0\,,$$

establishing either way that the particle field $\lambda_2 = u$ is *linearly degenerate*.

The Riemann invariants across this wave are found most easily from the primitive formulation $W = (\rho, u, p)$

$$\frac{d\rho}{1} = \frac{du}{0} = \frac{dp}{0}$$

giving the two Generalized Riemann Invariants

$$I_1 = u\,, \quad \text{and} \qquad I_2 = p \qquad \text{across} \;\; \mathcal{C}_2.$$

We make the following comments:

- The integral curve is tangent everywhere to $r_2 = (1, u, u^2/2)$, and since the eigenvalue $\lambda_2 = u$ is constant along the curve, so is r_2. It follows that the integral curve is a straight line, parallel to $r_2 = (1, u, u^2/2)$.
- It is easy to verify that the same data, $\rho_L \neq \rho_R, \;\; u_L = u_R \equiv u, \;\; p_L = p_R \equiv p$ and $\lambda = s = u$ also satisfy the shock jump conditions

$$s[W] = u\begin{pmatrix}[\rho]\\ [\rho u]\\ [E]\end{pmatrix} = \begin{pmatrix}[\rho u]\\ [\rho u^2 + p]\\ [u(E+p]\end{pmatrix}, \quad \text{and} \quad [W] = \begin{pmatrix}[\rho]\\ [\rho u]\\ [E]\end{pmatrix} = [\rho]\begin{pmatrix}1\\ u\\ \frac{1}{2}u^2\end{pmatrix} = [\rho]\, r_2(W)$$

that is the shock curve and the integral curve coincide in this degenerate case.

- This type of wave is called a *contact discontinuity*. The characteristics speed is constant across the wave, that is the characteristics are *parallel* to the discontinuity, as they do in linear systems.

The general solution for the Riemann problem now contains 3 waves, two nonlinear (acoustic) waves associated with $\lambda_{1,3}$ which are either shocks or rarefaction fans, and a contact (particle) wave, associated with λ_2, with two middle states, $W_{m,L}$ and $W_{m,R}$ (see Figure). To solve the Riemann problem amounts to describing the family of states $W_{m,L}$ using $(\mathcal{R}, \mathcal{S})_1$ and the family of states $W_{m,R}$ using $(\mathcal{R}, \mathcal{S})_2$, and use the fact that $u_{m,L} = u_{m,R}$ and $p_{m,L} = p_{m,R}$ to eliminate u between the 1-curve and the 3-curve, and solve for the common pressure p by nonlinear rootfinding. This process in illustrated geometrically in Figure 12.

4 SUMMARY

In this brief set of notes, we have identified the key ideas in the field of nonlinear hyperbolic conservation laws. We tried to give a taste of how they interact, and of the surprising richness of phenomena that they admit. As stated at the outset, these notes are informal and somewhat cursory. They are aimed at giving a good intuitive foundation for those intending to become practitioners in the field. They are by no means complete, and should primarily serve as a prelude to a good text book. Fortunately, there are numerous excellent text books, some of which you can find in the bibliography section.

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