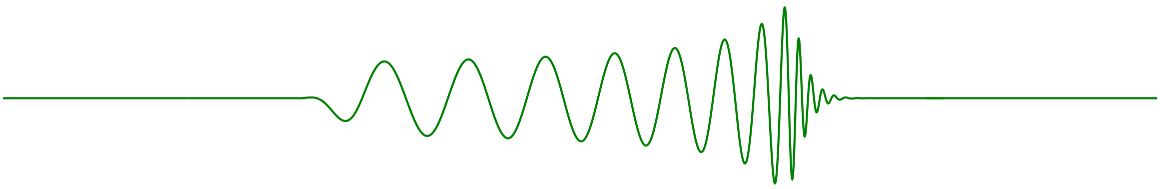
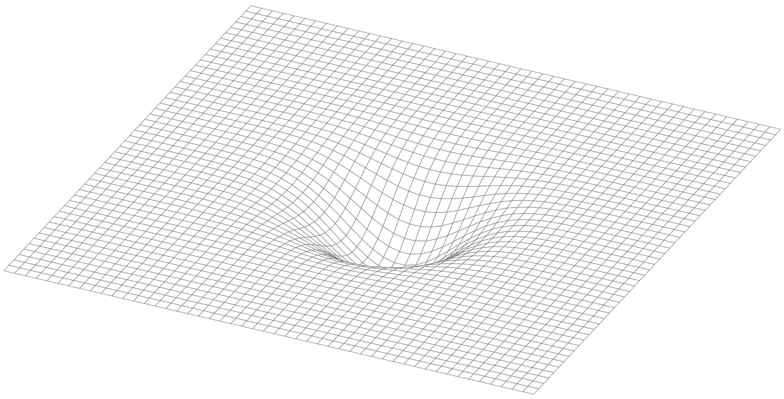


Summary

Relativity



Max Melching

1 Mathematics

1.1 Gravitational Physics Summary – Math Part

1.1.1 Basics of Manifolds

to describe curved spacetime, we need a coordinate-independent notion of spaces; this is given by manifolds, which are described using charts=coordinates but have an independent, invariant meaning; similarly, they can often be pictured to be embedded in some higher-dimensional Euclidian space, but that need not be the case

therefore, physics happens on manifolds, so events are points on it and more

defining vectors on manifolds is a non-trivial topic, they are now completely distinct notion from points and cannot be visualized as pointing from some origin to this point (problem: thinking of an embedded manifold for now, the vectors would point out of the manifold); instead, we can define vectors locally (infinitesimally) via derivatives of curves (i.e. as *tangent vectors*; the corresponding set of all tangent vectors is called *tangent space* V); for a function $f(x^\alpha)$ on the manifold, we can calculate the derivative along a curve $\gamma = \gamma(\sigma)$ ($\sigma \in \mathbb{R}$ parametrizes γ):

$$\frac{df}{d\sigma} = \frac{df(x^\alpha(\sigma))}{d\sigma} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\sigma} \tag{1.1}$$

this is a simply application of the chain rule and it yields the tangent vector components

$$t^\alpha = \frac{dx^\alpha}{d\sigma} = \underline{t} \cdot x^\alpha \tag{1.2}$$

because the vector is supposed to act as

$$\underline{t} \cdot f = \left(t^\alpha \frac{\partial}{\partial x^\alpha}\right) \cdot f = \frac{\partial f}{\partial x^\alpha} t^\alpha \tag{1.3}$$

(remember: we identify it with a derivative, which can in turn be expressed using partial derivatives with respect to the coordinates)

formally, we can express this as

$$\underline{t} = t^\alpha \underline{e_\alpha} = t^\alpha \frac{\partial}{\partial x^\alpha} = \frac{d}{d\sigma} \tag{1.4}$$

(note: $\frac{\partial}{\partial x^\alpha}$ is often abbreviated as ∂_α)

tangent vectors are invariant quantities, they do not (and should not!) depend on the coordinates we use to express them; their components, on the other hand, are *not* invariant; they obey the following transformation rule:

$$\underline{t} = t^\alpha \frac{\partial}{\partial x^\alpha} = t^\alpha \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta} = t'^\beta \frac{\partial}{\partial x'^\beta} \qquad \Leftrightarrow \qquad \boxed{t'^\beta = t^\alpha \frac{\partial x'^\beta}{\partial x^\alpha}} \tag{1.5}$$

once again, this is basically just an application of the chain rule

next natural step: linear maps on tangent space V (= set/space of tangent vectors); these are called *covectors* or *one forms* (elements of the dual space or *cotangent space* V^*) and it turns out that we can identify them with differentials/gradients of functions

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha \tag{1.6}$$

where we chose a convenient basis $\{\underline{e}^\alpha\}_\alpha = \{dx^\alpha\}_\alpha$ of the dual vector space; these satisfy

$$dx^\alpha \left(\frac{\partial}{\partial x^\beta} \right) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \tag{1.7}$$

more generally, such covectors $w : V \to \mathbb{R}$ obey

$$w\left(\alpha \underline{a} + \beta \underline{b}\right) = \alpha w\left(\underline{a}\right) + \beta w\left(\underline{b}\right), \qquad \forall a, b \in \mathbb{R},\ \underline{a}, \underline{b} \in V \tag{1.8}$$

we can also characterize covectors via tuples of components

$$w_\alpha = w\left(\underline{e_\alpha}\right) = w(\partial_\alpha) = \partial_\alpha w \tag{1.9}$$

in general, we can also write

$$w\left(\underline{a}\right) = w_\alpha \underline{e}^\alpha \left(a^\beta \underline{e_\beta}\right) = w_\alpha a^\alpha \tag{1.10}$$

to see how covector components in different coordinates are related, we look at the following inner product (which is also invariant)

$$w\left(\underline{t}\right) = w_\alpha t^\alpha \stackrel{!}{=} w'_\beta t'^\beta = w'_\beta \frac{\partial x'^\beta}{\partial x^\alpha} t^\alpha \qquad \Leftrightarrow \qquad \boxed{w'_\beta = w_\alpha \frac{\partial x^\alpha}{\partial x'^\beta}} \tag{1.11}$$

1.1.2 Tensors

We have seen how covectors are maps from V to the real numbers. Similarly, one can show that there is a unique identification between vectors from V and maps from the dual space V^* to the real numbers – vectors are also maps. It is possible to generalize this concept to coordinate-independent entities which map multiple vectors, covectors or mixes of them to the real numbers. Linear maps

$$T : V^n \times (V^*)^m = V \times \dots \times V \times V^* \times \dots V^* \to \mathbb{R} \tag{1.12}$$

are called *tensors* of rank $m+n$. Due to their invariance under coordinate-transformations, every physical quantity has to be expressed as a tensor.

Just like vectors can be collected in components $t^\alpha = \underline{t} \cdot x^\alpha = \partial_\sigma x^\alpha$ and covectors in components $w_\alpha = w(\partial_\alpha)$, we can characterize a tensor of rank $m+n$ using components

$$\boxed{T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = T\left(\underline{e_1}, \dots, \underline{e_n}, \underline{e^1}, \dots, \underline{e^m}\right)} \, . \tag{1.13}$$

Remark: it is no typo that there are m upper and n lower indices. This reflects the fact that a tensor of rank $m+n$ can map m covectors with its m "vectorial" indices and n vectors with its n "covectorial" indices.

These components do change under coordinate transformations. The corresponding behaviour can be derived from the ones for vectors (1.5) and covectors (1.11),

$$\boxed{T'^{\alpha\beta\dots}{}_{\gamma\delta\dots} = T^{\mu\nu\dots}{}_{\lambda\sigma\dots} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \dots \frac{\partial x^\lambda}{\partial x'^\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \dots} \, . \tag{1.14}$$

This is the important *tensor transformation law*.

The rank of a tensor can be reduced if we insert a fixed object into one of the "slots", i.e. in the example of a rank-4-tensor

$$T(\cdot,\cdot,\cdot,\cdot) \to T'(\cdot,\cdot,\cdot) = T(\underline{t},\cdot,\cdot,\cdot) \qquad T^{\alpha\beta}{}_{\gamma\delta} T'^{\alpha\beta}{}_{\delta} = T^{\alpha\beta}{}_{\gamma\delta} t^\gamma \tag{1.15}$$

or

$$T(\cdot,\cdot,\cdot,\cdot) \to T'(\cdot,\cdot,\cdot) = T(\cdot,\cdot,w,\cdot) \qquad T^{\alpha\beta}{}_{\gamma\delta} \to T'^{\beta}{}_{\gamma\delta} = T^{\alpha\beta}{}_{\gamma\delta} w_\alpha \tag{1.16}$$

Remark: might be inconsistent to write components like this because vectorial indices come first but the first arguments in T are also vectorial (which they connect to a covectorial index).

Example 1.1: Known Tensors

We have already encountered several examples of tensors. Vectors and covectors are rank-1-tensors, which should not be surprising because we used them to derive general tensors. However, scalars are also tensors, namely of rank 0 – they can be thought of as mapping the real numbers to themselves without taking any further arguments.

Another example of a tensor, which plays a great role in geometry on manifolds and thus – as we will see later – also in physics, is the *metric*. The following properties can be used as a definition for this 2-tensor:

(1.) The metric is symmetric.

(2.) The metric is non-degenerate.

Together with the usual properties of a tensor, like linearity, this defines a (pseudo)metric. In case of spacetime, this is characterized by the fact that the metric has three positive and one negative eigenvalue.

Just like any other tensor, the metric can be characterized by their components. These we can also be read off from the line element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu := g_{\mu\nu}\, dx^\mu \otimes dx^\nu \tag{1.17}$$

which is a frequently used tool in geometry (for example to measure lengths). Often, one thinks of the covectors dx^μ in this expression as infinitesimal changes in the coordinate x^μ and of the corresponding component a in adx^μ as the effect of this change. This is justified by the fact that $adx^\mu(\partial_\nu) = a\delta^\mu_\nu$, the coefficient of dx^μ indeed contains all information about the direction ∂_μ which is present in the whole object.

Metrics can be used to define inner products, which are not natively present on manifolds, in the following manner:

$$\underline{A} \cdot \underline{B} := g\left(\underline{A}, \underline{B}\right) = g_{\mu\nu} A^\mu B^\nu \,. \tag{1.18}$$

Inner products shall be symmetric, i.e. $\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}$, which is why we demanded symmetry of g. That means

$$g\left(\underline{A}, \underline{B}\right) = g\left(\underline{A}, \underline{B}\right) \qquad\qquad g_{\mu\nu} A^\mu B^\nu = g_{\nu\mu} A^\nu B^\mu \,. \tag{1.19}$$

The metric provides us with a natural identification between vectors and covectors because $g\left(\cdot, \underline{A}\right)$ is nothing but a map which takes a vector and maps it to a real number – which is the definition of a covector. Similarly, we can identify covectors w with the unique vector $\underline{A}$ that fulfils $w\left(\underline{B}\right) = g\left(\underline{A}, \underline{B}\right)$. In components, these requirements read

$$\boxed{A_\mu = g_{\mu\nu} A^\nu} \qquad\qquad \boxed{A^\mu = g^{\mu\nu} A_\nu} \tag{1.20}$$

where $g^{\mu\nu}$ denote the components of the inverse metric, which is defined by

$$\boxed{g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu} \,. \tag{1.21}$$

Apparently, it is almost trivial to change from vectors to covectors and vice versa in this component notation. For this reason, the strict distinction between A^μ and A_μ is often dropped (at least for interpretation purposes).

1.1.3 Covariant Derivative

1.2 Notes & Thoughts

to be able to develop an appropriate/meaningful notion of parallelism, we need a "better" derivative. This will be provided by a connection

1.2.1 Giulini GR lectures May 19 and 26

in Riemann normal coordinates, all Christoffel symbols vanish; but they only exist in neighbourhoods around points p (are Riemann normal coordinates *at* p); they are *very* helpful in calculations because tensor equations only have to be proven in a single coordinate system, which we can choose to be Riemann normal coordinates because we have just shown that they do exist

uhh, Christoffel symbols satisfy an affine transformation law, not linear (indicator of not tensorial) because there is term without $\Gamma^{\sigma}_{\mu\nu}$

formel of Koszul only holds like this for torsion-free and metric connections; and by substituting $X = e_\alpha, Y = e_\beta, Z = e_\gamma$ into it and then contracting with certain component of inverse metric gives rise to formula for connection coefficients (commonly called Christoffel symbols for Levi-Civita connection) and these uniquely determine the connection (which is proof for uniqueness); note that connection coefficients (= covariant derivative with only basis vectors) already determines the connection because connection is $\mathbb{R}$-linear, tensorial (C^∞-linear) and obeys the Leibniz rule

$\nabla_X Y = X^\alpha (\nabla_\alpha Y^\beta) \frac{\partial}{\partial x^\beta}$, so components of $\nabla_X Y$ are partial derivatives of components plus extra term, i.e. $\nabla_{\frac{\partial}{\partial x^\alpha}} Y =: (\nabla_\alpha Y^\beta) \frac{\partial}{\partial x^\alpha}$; however, they to read this is *not* covariant derivative of Y^β since this would be the covariant derivative of a function, but instead as the β-component of the covariant derivative $\nabla_\alpha Y$

here it is, reason why covariant derivative of covector (with respect to some vector field X) looks the way it looks:

$$\begin{aligned}
\nabla_X \omega &= \nabla_{X^\alpha \frac{\partial}{\partial x^\alpha}} \left(\omega_\beta dx^\beta \right) \\
&= X^\alpha \left(dx^\beta \nabla_{\frac{\partial}{\partial x^\alpha}} \omega_\beta + \omega_\beta \nabla_{\frac{\partial}{\partial x^\alpha}} dx^\beta \right) \\
&= X^\alpha \frac{\partial \omega_\beta}{\partial x^\alpha} dx^\beta
\end{aligned}$$

but $dx^\beta \left(\frac{\partial}{\partial x^\alpha} \right) = \delta^\beta_\alpha$, so by taking the derivative of this equation we see that ... $\left(\nabla_{\frac{\partial}{\partial x^\gamma}} dx^\beta \right) \left(\frac{\partial}{\partial x^\alpha} \right) = \left(\nabla_{\frac{\partial}{\partial x^\gamma}} dx^\beta \right)^\alpha = -\Gamma^\beta_{\gamma\alpha}$

from that we get general formula for tensors of arbitrary rank because of Leibniz rule;

therefore, we get the "master formula"

$$\begin{aligned}\nabla_X T &= T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}\frac{\partial}{\partial x^{\alpha_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\alpha_k}}\otimes dx^{\beta_1}\otimes\cdots\otimes dx^{\beta_m}\\ &= X^\gamma\left(\nabla_\gamma T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}\right)\frac{\partial}{\partial x^{\alpha_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\alpha_k}}\otimes dx^{\beta_1}\otimes\cdots\otimes dx^{\beta_m}\\ \nabla_\gamma T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} &= \left(\frac{\partial T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}}{\partial x^\gamma}+\sum_{i=1}^k\Gamma^{\alpha_i}_{\gamma\lambda}T^{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\alpha_k}_{\beta_1\ldots\beta_m}-\sum_{j=1}^m\Gamma^\lambda_{\gamma\beta_j}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{j-1}\lambda\beta_{j+1}\ldots\beta_m}\right)\end{aligned}\tag{1.22}$$

there is also another notion of derivative on manifolds, the Lie derivative; to define it, we do not need any additional structure (unlike for connection, connection coefficients need metric); the Lie derivative can often be used to express symmetries; components of it are

$$(\mathcal{L}_X T)^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} = X^\alpha\frac{\partial}{\partial x^\alpha}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}-\sum_{i=1}^k\frac{\partial X^{\alpha_i}}{\partial x^\lambda}T^{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_k}_{\beta_1\ldots\beta_m}+\sum_{j=1}^m\frac{\partial X^\lambda}{\partial x^{\beta_i}}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{i-1}\lambda\beta_{i+1}\ldots\beta_m}\tag{1.23}$$

we notice: for upper index we now have minus, lower index has plus (reversed compared to connection); interesting property: all partial derivatives could be replaced by covariant derivatives without changing the formula (despite them being defined independently of each other!); if a certain vector field defines a symmetry, i.e. the metric does not change under the flow of that vector field (stays constant along it/integral curves defined by it), then we can express that as the vanishing of the Lie-derivative of this symmetry-generating vector field; these vector fields are called Killing fields; note that $(\mathcal{L}_X g)_{\alpha\beta} = (\nabla_\alpha X^\gamma)g_{\gamma\beta} + (\nabla_\beta X^\gamma)g_{\alpha\gamma} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha$ (we just use writing in terms of covariant derivative for first equality) -¿ shouldn't that be equal to $\nabla_{[\alpha}X_{\beta]}$; would also explain $\mathcal{L} = \mathrm{Alt}(\nabla)$ statement I heard; ah no, this is the *symmetrized* part... But maybe that supports view, symmetric part is zero for Killing field (but this has vanishing Lie derivative, so antisymmetric part also zero, right?)

very interesting: Einstein tensor is divergence-free, i.e. $\nabla_\alpha G^{\alpha\beta} = 0$

we know that, in general, $[\nabla_\mu, \nabla_\nu]T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} \neq 0$; expanding this quantity for an arbitrary vector field X^α, $[\nabla_\mu, \nabla_\nu]X^\alpha = \cdots = \left(\frac{\partial}{\partial x^\mu}\Gamma^\alpha_{\nu\beta} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\beta}\right)X^\beta$ + terms proportional to $\Gamma = R^\alpha_{\beta\mu\nu}X^\beta$ + terms proportional to Γ; in Riemann-normal coordinates, $\Gamma = 0$ and $[\nabla_\mu, \nabla_\nu]X^\alpha = R^\alpha_{\beta\mu\nu}X^\beta$ and since both sides are tensors, this equation holds in general; curvature is related to (Giulini said "obstruction") commutivity of second derivatives; furthermore, pulling down the index α yields $[\nabla_\mu, \nabla_\nu]X_\alpha = -R^\beta_{\alpha\mu\nu}X_\beta$, which tells us how this quantity acts on a covector (again, has on other sign and acts on other index, like it was for covariant derivative itself); thus, we get the general formula $[\nabla_\mu, \nabla_\nu]T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} = \sum_{i=1}^k R^{\alpha_i}_{\lambda\mu\nu}T^{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_k}_{\beta_1\ldots\beta_m} - \sum_{j=1}^m R^\lambda_{\beta_j\mu\nu}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{i-1}\lambda\beta_{i+1}\ldots\beta_m}$

application of that formula: $\nabla_\mu\nabla_\nu X_\beta = R^\alpha_{\mu\nu\beta}X_\alpha$ holds for any Killing vector field X; second derivatives are determined by vector field itself; similarly, $\frac{\partial}{\partial x^\alpha}X_\beta + \frac{\partial}{\partial x^\beta}X_\alpha = 2\Gamma^\gamma_{\alpha\beta}X_\gamma$, the symmetric part of first derivative of Killing vector field is determined by field itself as well; the only free parameters are value of the field itself and anti-symmetric part $\frac{\partial}{\partial x^\alpha}X_\beta - \frac{\partial}{\partial x^\beta}X_\alpha$ of first derivative (all derivatives of higher order are determined by relation to curvature tensor); solutions of linear differential equations (no matter of partial or not) constitute a vector space because we can add them together and multiply with numbers and maximum number of dimensions is given by number of freely specifiable initial conditions; here, these are values $X^\alpha\big|_p$ of Killing field at a specific point, i.e. $n = \dim(M)$, and $\frac{\partial}{\partial x^\alpha} X_\beta - \frac{\partial}{\partial x^\beta} X_\alpha\Big|_p$, i.e. $\frac{1}{2}n(n+1)$; in total, that means there are at most $n + \frac{1}{2}n(n+1)$ independent solutions to the Killing equation; in Minkowski space there are indeed 10, in that sense it is maximally symmetric (these generate symmetries of the space, which are given by Poincare group)

Riemann tensor has 20 components and there are 10 different traces (because of antisymmetry in first two indices, which means trace is always zero); our goal is now decomposing it into trace and traceless parts, both contain information about Riemann tensor; the trace part is nothing but the Ricci tensor $R_{\alpha\beta} = R^\lambda_{\alpha\lambda\beta}$, the "rest" (trace-free part) is the Weyl tensor (which he writes down in terms of some weird product); this product has the same symmetries as the Riemann tensor, so Weyl tensor also has them and it is trace free in addition, i.e. $W^\lambda_{\alpha\lambda\beta} = 0$ (taking these conditions into account, the Weyl tensor has 10 independent components in 4 dimensions; very interesting property is that *only* in 4 dimensions, the amount of information in Weyl, Ricci Tensor is the same; in 3 dimensions, Weyl tensor has no information and in higher ones much more than Ricci); in index-form it is given by $W^\alpha_{\beta\mu\nu}$

oof:

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= \frac{1}{2}(g \cdot \mathrm{Ric})_{\alpha\beta\gamma\delta} - \frac{1}{12}R(g \cdot g)_{\alpha\beta\gamma\delta} + W_{\alpha\beta\gamma\delta} \\
&= \frac{1}{2}\big(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\big) \\
&\quad - \frac{1}{12}R\big(g_{\alpha\gamma}g_{\beta\delta} + g_{\beta\delta}g_{\alpha\gamma} - g_{\alpha\delta}g_{\beta\gamma} - g_{\beta\gamma}g_{\alpha\delta}\big) + W_{\alpha\beta\gamma\delta} \\
&= \frac{1}{2}\big(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\big) - \frac{1}{6}R\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big) + W_{\alpha\beta\gamma\delta}
\end{aligned}
$$

long talk about constant curvature; interesting statement (Schur's theorem): constant curvature $\Leftrightarrow$ Gaussian/sectional curvature of each point does not depend on the choice of the 2-tangent-plane through the point

the Weyl curvature $W^\alpha_{\beta\mu\nu}$ (which is a function of the metric g) has the important property of being conformally invariant, i.e. $W^\alpha_{\beta\mu\nu}(\Omega^2 g) = W^\alpha_{\beta\mu\nu}(g)$ for some function Ω and even the reverse statement is true: if $W^\alpha_{\beta\mu\nu}(g_1) = W^\alpha_{\beta\mu\nu}(g_2)$, then $\exists$ locally a function $\Omega \in C^\infty(M;\mathbb{R})$ without zeros such that $g_1 = \Omega^2 g_2$; for example, a vanishing Weyl tensor means that the space is locally, conformally flat -¿ all of these statements are valid only for $n \geq 3$

1.2.2 Order

basically take order from Penrose?; other way to put it: from summary H_Analysis, but with less math; also Carroll?

first: do manifolds; then go to tangent space (we want vectors); then go to bundles (first tangent bundle, then more general vector bundles); then define tensors and tensor bundle; then go to differential geometry

1.2.3 General Thoughts

Schwarzschild metric contains information on many effects of BHs in its components! coefficient $1 - \frac{2M}{r}$ in front of dt^2 tells us about time dilation close to BH (more t goes by the closer you get) and $\frac{1}{1-\frac{2M}{r}}$ in front of dr^2 tells us about curvature of space (increases as r decreases)

1.2.4 Math Stuff

regarding tensor product: throughout the discussions, linearity of objects was very important (we have used it for the differential, many mappings, etc.); however, a very important notion that is not linear is the underlying spaces we have looked at; take for example $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$: here, we do not have linearity, which would require $(2, 1) = 2 \cdot (1, 1)$ and clearly, this is not true; however, we might be interested in such a space and this is what is called a tensor product space; there are more tensor products, we also have to make sense of the one of objects in this space, but this is a good motivation

goal of derivatives is approximation to first order, which is expressed in demanding linearity of operators

differentiation is linear and has Leibniz rule, so it already fulfils requirements for tensor derivative (thus it makes sense to demand ∇, d acting like D on functions); multiple generalizations of Df to something like "Ds" for sections s exist, which is fine because from ∇ we can easily get many others e.g. by $d = \mathrm{Alt}(\nabla)$ (not sure if equality is true, but from Carroll eq 1.82 it looks like this), that is by suitable mappings

1.2.5 From Wald

the notion of curvature, intuitively, corresponds to the one of a 2-sphere in 3D space; however, this is extrinsic curvature which is only visible in embeddings, but what we are interested in is something like intrinsic curvature; how can we detect that?

1.2.6 From Penrose

tangent space in point $p \in M$ is immediate/infinitesimal vicinity of M "stretched out"; more formally, a linearisation of the manifold

to do physics, we cannot just work with vector spaces or affine spaces like the Euclidian space (basically $\mathbb{R}^n$, but no need to fix origin), but we need manifolds; however, manifolds do not have enough natural structure to build up the theory that is needed to describe physics, so we need some additional (local) structure (e.g. enabling us to measure infinitesimal distances in case of a metric structure); this structure is often encoded to/using the tangent spaces (which are present naturally for manifolds), which are vector spaces again

problem of abstract notion of "no structure" is for example: no general, meaningful (well-defined) notion of differentiation (does exist for functions, but not for vector fields, 1-forms or other tensors); exterior derivative is something like that, but it does not really give information about varying of the forms (nice is that it maps p-forms to $p + 1$-forms)

some structure can also be provided by connection; although not every structure can reproduced, metrics uniquely determine a connection (Levi-Civita connection)

goal of derivative operators: measure constancy and deviations from it; in case of vectors, this is equivalent to a notion of parallelism; note: we will go reverse route, define derivative and get parallelism from that; this notion will have the unusual feature of path-dependence, where unusual is meant with respect to what we know from Euclidian space; while it is possible to do this (see Wald), but this is mainly by making the "right" guess and thus not really helpful (idea is to say we want something where change of v is proportional to difference Δx and then we say: this works; welp)

which requirements make sense? since tangent space is linearisation of manifold, there should also be linear dependence on direction that we differentiate along; more generally, pointwise linearity means that functions can be dragged across the operator; when acting on tensors however, a product rule has to be specified: $\nabla_X(fs) = (\nabla_X f)s + f\nabla_X s$ makes sense (without argument X, this becomes $\nabla(fs) = (\nabla f) \otimes s + f\nabla s$) (?)

ideas come from the fact that our goal is to generalize action of derivative D; therefore, demanding $\nabla f = df$ also makes a lot of sense

extension to more than one tensor field is possible by demanding additivity $\nabla(s+t) = \nabla s + \nabla t$ and by specifying product/Leibniz rule $\nabla(s \otimes t) = (\nabla s) \otimes t + s \otimes (\nabla t)$; to uniquely determine this generalization, it is also necessary to demand compatibility with trace/contraction (which also helps with defining these things in the first place)

interesting: local connection can be defined uniquely from Gaussian basis vectors

2 Special Relativity

In modern day physics, there are often two competing viewpoints. One is very much based on intuition and the other is based solely on a mathematical description. This shows especially in the theory of special relativity, where one can deal (i) in much detail with groups and transformations or (ii) with a much more pictorial version of the theory, mostly utilizing very basic geometry in so-called spacetime diagrams.

Both approaches can lead to a rich and of course equivalent understanding, but it is often tempting to focus on only one of them. In my personal experience, this is often the mathematical description because students are often more familiar with the required math, so teaching the alternative and rather new intuitive-based approach would actually more complicated. This, however, can often lead to a lack of intuition, which is still fundamental to fully understand relativity as a whole. For this reason, our goal is to learn about both approaches.

2.1 Newtonian Physics

2.1.1 Space & Time

maybe start from argument for speed of light is limit and then say that close to the speed of light, Newtonian description fails; this means we cannot look at time and space as separate concepts anymore, instead look at things like spacetime diagrams; from that, we can also derive something like notion of distance and then we can formalize this using notion of metric (another way to see this: we want invariant notion of distance in Minkowski space and we have already seen one; mathematically, that means we work with pseudo-metric)

surely state that Newton is conform with our intuition and holds at small distances, speeds; for larger ones, however, inconsistencies show up

Maxwell's equation tells us that the speed of light is a constant, $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$; or Michelson-Morley experiment, shows that speed of light is the same in all inertial frames (i.e. that uniform motion does not influence it); therefore, the distance light travels is related to this constant velocity via the well-known formula for uniform motion $c^2(\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$; any observer in a different inertial frame also has to measure the same velocity, despite having different coordinates, i.e. $c^2(\Delta t')^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$; this seemingly simple idea will turn out to be very powerful and fundamental in building the theory of relativity

argument for maximal speed? Ah, we do not even need maximum speed for showing inconsistency: speed of light being constant contradicts Newtonian addition of velocities; maybe this just magically comes out because otherwise, we would violate causality

as end of this section: however, the Newtonian physics is not complete; as the Michelson-Morley experiment showed, the speed of light is constant (or rather say that Maxwell equations tell us this, M-M was after SR has been developed); in Newtonian physics, however, a photon emitted by an observer moving with velocity v will have velocity $c + v$ in the frame of an observer at rest (because of Galilei transform); to solve this inconsistency, we will have to rethink the Newtonian concepts of time and space

2.2 Relativity

2.2.1 What is wrong with Newton?

make it subsection here? Or mention in Newton? -¿ nope, here is good

2.2.2 Einstein Postulates

we only demand that physics must not depend on observer (inertial system) and that speed of light is constant

Definition 2.1: Inertial Frame

An *inertial frame of reference* is a frame where $F^k = ma^k = m\ddot{r}^k$ holds.

Postulate 2.2: Principle of Relativity

All physical observations must hold independently of the inertial frame that is chosen.

Einstein was not the first one to notice this, nothing really revolutionary or new compared to Newton

this means we have to abandon notion of universal space, there is no preferred frame of reference!

This means that if you are in a closed room without any windows, you cannot perform any experiment to determine if you are at rest or moving at a constant velocity. In other words, there is no absolute notion of rest; it's all relative (hence the name “relativity principle”).

second thing we demand (very reasonable, but this is assumption we build theory upon):

Postulate 2.3: Relativity Principle

The speed of light c is constant in all inertial frames.

this constant speed of light makes it very special because it is independent of the observer/inertial system and thus allows to transform statements between them (make invariant statements?)

Property 2.4: Universal Speed Limit

The speed of light c is the maximum velocity for all interactions.

no information can be transmitted at higher speed, in particular no particle can travel faster! Follows from relativity principle; in principle, it is still possible to construct theories where speeds greater than light are allowed, but we will use this property as an assumption; there is also experimental evidence which supports this (or at least does not falsify/contradict it): Neutrinos also travel with speed of light (measured to high accuracy) -¿ thus no real, independent postulate (rather consequence from relativity principle)

to end subsection: we want equivalent results in all frames; if we try to measure a distance, this requires a notion of "at the same time"; harder than it sounds (communication cannot happen instantaneously, as we have just learned), but this is why we will deal with measuring time now, i.e. clocks

2.2.3 Spacetime Diagrams

natural idea for visualization: time evolution of events and world lines; thus we have one spatial and one temporal axis (which is rescaled for convenience purposes; after all, we are often interested in velocities close to speed of light, thus slopes would be huge die to c being on the order of 10^8)

-¿ hmmm, maybe even do spacetime diagrams here? Just basic idea of visualizing time and spatial axis (does not have to be something with multiple frames, can be added to picture later)

-¿ y-axis of spacetime diagram is time, but in units of space (rescaled to make visualizations more convenient)

-¿ even do light cones here?

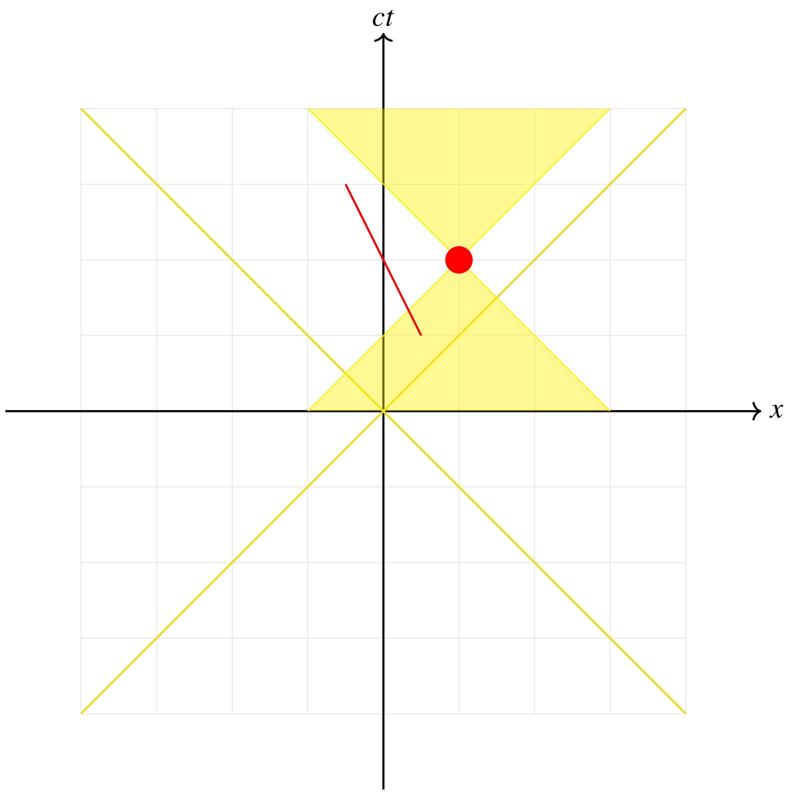


Figure 2.1

2.3 Clocks

Until now, we have not really touched the notion of time. Partly, this is because we natively have a very clear, intuitive understanding of time: we look at clocks to measure it and this notion can be employed anywhere in space – just take a look at equal clocks in different points and compare their readings. This definition is employed in Newtonian physics, without much more attention being needed.

However, as it is the story for much of relativity theory, this notion essentially breaks down once we go to more extreme situations like distances on cosmic scales or clocks moving with respect to each other with high velocities. In both of those cases, timing the reading of a clock and hence comparing if they show the same time is difficult. For large distances, this is rather easy to see because information is transmitted at a finite speed $\leq c$, so when receiving information about the measurement result t of a far-away clock, we have to take into account the time it travelled to us in order to find out which event happened simultaneously to t.

This is problematic since many notions implicitly rely on the fact that we can measure quantities at the same time, i.e. on a notion of simultaneity. A very important example are lengths, which are defined as the separation of points – at a fixed time. Therefore, a well-defined notion of "at a fixed time" is required for us to be able to measure lengths and until now, we have no such notion. In everyday life, it is easy to avoid such difficulties: after all, we can look at clocks side-by-side, make sure they show the same time and then move one of them away to the desired position. This procedure ensures the clocks are synchronized, so we can simply take the desired measurements and compare the times later on. However, this is not really feasible to do that for measurements between planets or galaxies and clearly, an alternative, perhaps more general, way of communicating time measurements is needed.

All of that motivates the need for a synchronization procedure of clocks. We will here present the one proposed by Einstein, starting with its definition for resting observers and then look at it for the case of moving observers. Throughout this section, we will adopt visualizations from [**?**], while many of the definitions follow [**?**] more closely.

2.3.1 Synchrony of Clocks

Resting Observers Our setting is identical copies of an ideal clock being attached event/point in space. For a consistent, well-defined notion of "time", however, we now have to make sure these clocks show equivalent times. To do that, we will synchronize them by adopting the following definition, originally proposed by Einstein.

Definition 2.5: Einstein Synchronization

Two clocks C, C' with times t, t' attached to observers O, O' at rest are *synchronized*, i.e. $t = t'$, if light signals sent out from them meet exactly in the midpoint of $\overline{OO'}$.

In principle, there is no unique procedure to synchronize clocks. However, in accordance with the equivalence principle, it would be desirable to for the procedure to work independently of the chosen inertial frame. A very straightforward idea is to exploit the relativity principle and use light to communicate times between different observers and their respective clocks. That is what lead Einstein to this definition of synchrony.[1] It gives rise to the following notion of simultaneity.

Definition 2.6: Simultaneity

Two events E at t and E' at t' are called *simultaneous* if the locally simultaneous clock readings of synchronized clocks at these events are identical.

Essentially by definition, the following properties hold.

Property 2.7: Simultaneity as an Equivalence Relation

Simultaneity defines an equivalence relation on the set of all clocks in an inertial frame, i.e. the following properties hold:

1. Every event is simultaneous to itself.
2. If E is simultaneous to E', then E' is simultaneous to E.
3. If E is simultaneous to E' and E' is simultaneous to E'', then E is simultaneous to E''.

Moreover, a notion of simultaneity attached to some clock C partitions the set of all events $\{E\}$ into several, mutually disjoint subsets (equivalence classes), each containing events which are simultaneous to each other. A representative of the former is a family of synchronized clocks which show equivalent times.

Our main takeaway is the following: events that simultaneous for one observer O are simultaneous for every other observer O' at rest with respect to O.

This also justifies the following interpretation: two events E at t and E' at t' are simultaneous if light signals sent out by them reach the *referee* $\mathcal{R}$ at the same time. Here, the referee is a third observer, which is defined by the property that he always has an equal distance to the two other observers O, O'. This is meant in the sense that light signals sent out from $\mathcal{R}$ take the same time to go to O and come back as they do to go to O' and come back. Therefore, we reduce simultaneity for the spatially separated observers O, O' to simultaneity for a single observer $\mathcal{R}$, which is precisely the requirement stated in 2.6. More precisely, events E at t (measured by clock C) and E' at t' (measured by clock C') are simultaneous, i.e. $t = t'$, if they show the same time to the referee.

Besides governing which events happen at the time, the synchronization procedure further paves way to possibilities to determine times t' shown on a clock C' from times measured by another clock C. As figure 2.2 (a) shows, the travel time of a light signal from O to O' is

[1] As Giulini elaborates on in 2.1 of [**?**], this freedom in defining synchronization indeed exists.

the same as the travel time from O' to O. Using t' to denote the time light from O sees on C' when intersecting O' and using that C' is synchronized with C (which implies $t' = t$), equality of light travel times implies:

$$t - t_- = t_+ - t \qquad \Leftrightarrow \qquad \boxed{t = \frac{t_+ + t_-}{2}}. \tag{2.1}$$

Here, t_+ is the time on C when the first signal is sent out and t_+ the time on C when the second signal arrives. In just the same manner,

$$\boxed{t' = \frac{t'_+ + t'_-}{2}} \tag{2.2}$$

which should also be clear since $t'_- = t_-, t'_+ = t_+$.

In the definition of synchrony, we have assumed to be the observers O, O' to be at rest, which is also what 2.2 (a) represents. However, one requirement was that synchronization should also work in other inertial systems. This situation where O, O' are at rest with respect to each other, but move uniformly with respect to another observer, is shown in 2.2 (b).[2] Indeed, by constructing the referee $\mathcal{R}'$ and drawing the light signals sent to and from each observer (which form a rectangle, sometimes named *lightangle*), one obtains the times $t''_-, t''_+, t'''_-, t'''_+$ by prolonging the edges of the lightangle. This is equivalent to what has been done for resting observers in 2.2 (a) and likewise,

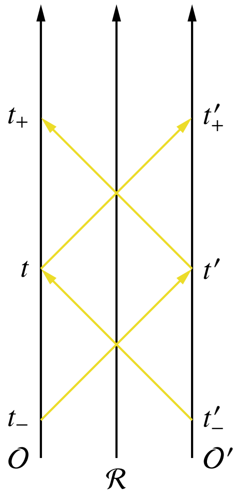
$$t'' = \frac{t''_+ + t''_2}{2} \qquad\qquad t''' = \frac{t'''_+ + t'''_-}{2}, \tag{2.3}$$

just as before, so the observers agree on $t'' = t'''$ (to verify that, we can also look at the distance of t_-, t, t_+ on the vertical axis we use to depict time). But nonetheless, something seems off. Figure 2.2 (c), where for simplicity we assumed that $t = t''$ in the event where O, O'' intersect, shows this more clearly. While the synchronization process still works, its induced notion of simultaneity for C'', C''' is *not* the same as the one for C, C'. Geometrically speaking, the "lines of simultaneity" (diagonal from left to right in lightangle) change from being horizontal in figure 2.2 (a) to being tilted in (b).

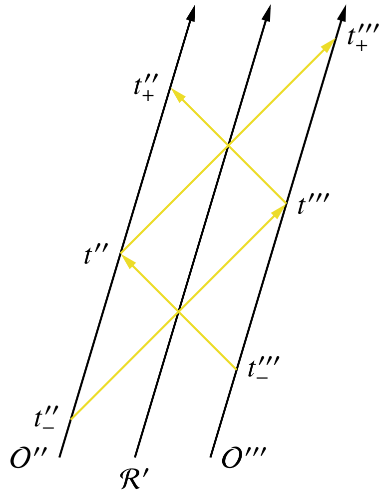
The origin of this difference is that the time of flight for light on $\overline{OO'}$ is different from the one on $\overline{O'O}$ (equivalent: time on $\overline{O\mathcal{R}}$ differs from the one on $\overline{\mathcal{R}O'}$) because the observers are moving uniformly in the same direction. Does that point to an inconsistency and thus error in the synchronization procedure? The answer is no because the roundtrip time is equal for $\overline{\mathcal{R}O\mathcal{R}}$ and $\overline{\mathcal{R}O'\mathcal{R}}$ – and this is the only distance/time that can be quantified using a single clock like the one at $\mathcal{R}$.[3] The events at t'' and t''' are said to be simultaneous because they do have the same roundtrip time of light signals, as one can verify in figure 2.2 (b).

[2] For a better distinction between the situations, the observers are named O'', O''' instead of O, O'. This does not change any interpretations that were mentioned.

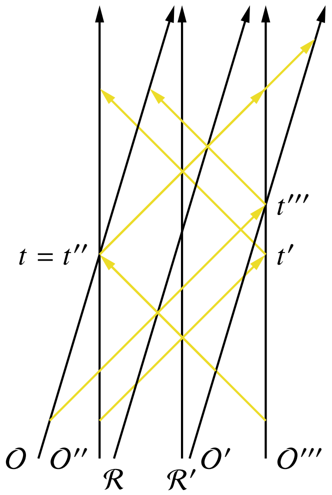
[3] To do that, synchronized clocks would be required. However, we wish to accomplish synchronization using the referee and the roundtrip time for him, so reasoning in this way does not work.



(**a**) Observers at rest



(**b**) Moving observers a rest with respect to each other



(**c**) Comparison of all observers from (a) and (b). For simplicity, we assume that the clock C and C'' show the same time $t = t''$ on intersection. Despite this fact, they cannot agree on notion of simultaneity.

Figure 2.2: Spacetime diagrams for different sets of observers. We draw the observers $\mathcal{O}, \mathcal{O}'$ whose clocks we wish to synchronize, the referee $\mathcal{R}$ for them and the light pulses they exchange in order to achieve the synchronization.
Clocks carried by $\mathcal{O}$ and $\mathcal{O}'$ measure the events at times $t = \frac{t_+ + t_-}{2}$ and $t' = \frac{t'_+ + t'_-}{2}$ to be simultaneous, i.e. $t = t'$. This is because light (represented as yellow lines) sent out simultaneously by the referee $\mathcal{R}$ to $\mathcal{O}$ and $\mathcal{O}'$ "sees" times t and t' on the clocks before returning simultaneously to $\mathcal{R}$, where the results are "reported".

Moving Observers We have already seen how uniform movement changes the notion of simultaneity, even if the corresponding observers remain at rest with respect to each other (which means it is a relative notion, dependent on the motional state of the observer). Now, we will deal with the case where O and O' move with respect to each other with a relative velocity v (see figure 2.3 for examples of that).

This relative velocity results in a change of the distance $\overline{OO'}$ over time, making comparisons of clock readings much harder than before. One can prove the following theorem.

Theorem 2.8: Minkowski's Theorem

For two observers O, O' that move relative to each other with velocity v and an event E' occurring on the world line of O' at time τ',

$$\tau' = \sqrt{t_+ t_-} = \sqrt{1 - v^2/c^2}\, t \,. \tag{2.4}$$

Here, t_-, t_+ are the times measured by a synchronized clock on O where light signals to E' have been sent out and received back.

We assume that the synchronization when world lines of O, O' intersected and that clocks have been set to $t = 0 = t'$ there (although it is not necessary that this happened at $t = 0$, just at any time smaller than τ'). Just like before, we can visualize the synchronization process in terms of a referee $\mathcal{R}$ (figure 2.3). For him, the time passing between emitting light and receiving it back is equal for O and O', i.e. $\tau = \tau'$ for synchronized clocks and the corresponding events are simultaneous.

However, we have already seen that observers in other inertial frames can perceive simultaneity differently. Indeed, the travel time seen by O is $t_+ - t_-$. By the synchronization procedure we have employed, he assumes an equal light travel time and hence that the time τ light signals sent out by him "saw" is the same as the time t defined by

$$t - t_- = t_+ - t \qquad \Leftrightarrow \qquad t = \frac{t_+ + t_-}{2} \,,$$

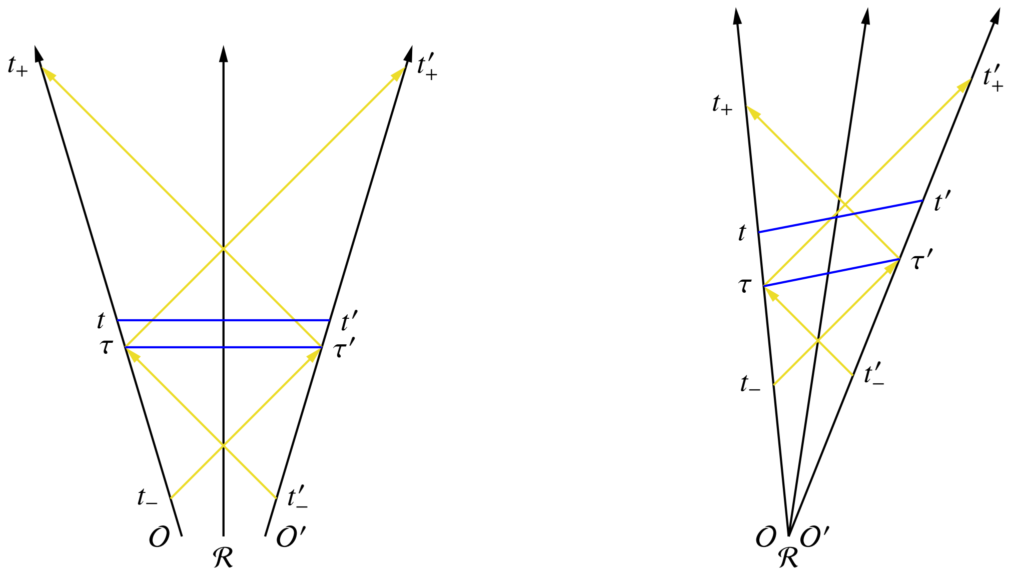
just like (2.1). However, theorem 2.8 shows that

$$\boxed{\tau' = \sqrt{t_+ t_-} = \sqrt{1 - v^2/c^2}\, t \leq t = \frac{t_+ + t_-}{2}} \,. \tag{2.5}$$

For the observer O, a time t has passed since the synchronization at $t = 0 = t'$, while for O' the *smaller* time τ' has passed – moving clocks tick slower.

This seems very puzzling. How does the situation look from the inertial frame where O' is at rest? For a clock attached to O' the time $t' = \frac{t'_+ + t'_-}{2}$ has passed, while for one attached to O it is $\tau = \sqrt{t'_+ t'_-} = \sqrt{1 - v^2/c^2}\, t' \leq t'$ – the effect is mutual. That makes things even more

[4] Beware that the diagrams are not perfect, e.g. the referee should be exactly where the light beams cross again. This is most likely due to an error in my code, which I was unable to locate. Nonetheless, the most important ideas should still be conveyed, which is why I decided to keep the graphics.



(**a**) Observers moving with relative velocity $v = 0.6c$

(**b**) Observers moving with relative velocity $v = 0.5c$

Figure 2.3: [4]

puzzling and in fact seems paradoxical: how can both claim that less time went by for the other observer? The answer lies in the fact that in (special) relativity, there is no absolute, universal time anymore. Time standard become relative and so do notions like simultaneity. As a consequence, both observers have the right to claim less time went by for the other and both are right in doing so. This is a key fact to understand in relativity and no different from the uniform motion of observers relative to each other. Both can claim that they are at rest, while the other one moves because it does not change the physics exhibited by the situation.

The same observation can explain why the synchronization procedure for the referee, who does indeed see E' at τ' simultaneous to E at τ (which are the events and times light signals see which are sent out by him), does not yield the same notions of simultaneity for O, O': both of them are not at rest to $\mathcal{R}$ anymore and this differing motional state leads to the effects observed now. If this was the case, the relative velocity v would be $v = 0$ and the geometric mean $\sqrt{t_+ t_-}$ would reduce back to the arithmetic mean $\frac{t_+ + t_-}{2}$, yielding the familiar result

$$t = \tau = \tau' = t' \,. \tag{2.6}$$

Apparently, Minkowski's theorem is a more general version of the results (2.1), (2.2) and it reduces back to them in case of $v = 0$.

2.3.2 Time Dilation

The behaviour of time for moving observers seems puzzling at first, one may ask: is this a bug or a feature? In other words, is this effect that less time seems to pass on moving clocks a real physical effect or is it caused by our choice of synchronization? It is especially the mutuality of effect that may add to these doubts. However, as it was also argued in the last subsection, this phenomenon is fundamental *feature* of time. Resting observers agree on times elapsed along clocks carried by them, moving observers do not. Abandoning this idea of universal time may seem very unintuitive compared to what we experience in everyday life, but it turns out to hold true. Just like the answer to "where does event E happen?" depends on the coordinates we choose, the answer to "at which time did event E happen?" depends on the (world line of the) clock we use in our measurements.

There are multiple ways to see that this is not a consequence of our notion of simultaneity being flawed, causing the inconsistent result. A very striking one, at least in my personal opinion, is that what we have shown here as a synchronization procedure is equivalent to setting the same time on clocks and then moving them to different points in space (in the slow transport limit, such that no time dilation occurs -¿ mention time dilation previously maybe? Fits very well here, but mentioning for the first time is not good). This is what we would do intuitively to synchronize clocks on Earth, which shows us that our unfamiliarity with these effects comes from our lack of experience with velocities $v \approx c$ or cosmic distances rather than pointing to unphysical effects.

-¿ Maybe nice transition: both resting and moving observer claim less time passes for other, who is right? To appropriately answer the question, we need them to synchronize, get one moving and then get them back together, so we need acceleration (should lead to infinitesimal proper time interval; here we see that less time passed for moving observer) -¿ but: we can see that it is in fact not needed because we can also accelerate resting observer for the time (so whatever effect that has cancels out and we can really attribute it to uniform motion)

-¿ we have seen how simultaneity depends on the observer; however, time itself fundamentally does as well! This is shown very instructively by an effect called *time dilation*. -¿ let's take two observers starting in origin; one moves uniformly to some other point in space and comes back; let's say they meet again in some event for which resting observer/clock measures time t; clearly, they meet simultaneous in this event (by definition, otherwise they would not meet); however, by our notion of simultaneity, we now know that the clock of the moving observer shows the time $\tau = \sqrt{1 - v^2}\, t$!!!

ahhh, direction in which observer moves does not matter for $\tau = \sqrt{1 - v^2}\, t$ because we can always go in inertial frame where one of them is at rest to see that only the relative velocity of the observers matters (this is equally true in all other frames, where both are moving, but more clear in this example)

now, what in case an observer that moves with the speed of light? No other light pulse sent from some distance can ever reach him, so how can they agree on time measurements? Well, they cannot because speed of light is universal limit (no signal can reach light), from which we conclude that no time passes for this observer. This is confirmed by $\tau = \sqrt{1 - c^2/c^2}\, t = 0$

note for the end: all of these effects only become relevant for $v \approx c$; otherwise, $t = \tau \approx t'$ for all observers (they measure the same time, just like our experience and intuition tells us; this is something like universal time then once we synchronize them)

2.3.3 Doppler Shift

do it? Or maybe do in time dilation section? But I do think it is not 100% necessary, so should not be priority

2.3.4 Length Contraction

there is an induced effect on lengths, which are defined as distances measured *at the same time*; here we see how observers may measure different lengths, they have potentially different notions of simultaneity

2.4 Lorentz Transformation

basically follow structure Einstein paper, derive coordinate transformation first; then we can see some of the effects that follow from it, like time dilation for example (is just logical consequence; we demand something in the postulates, then see what implications that has for coordinates and physics) -¿ better structure: do time measurement first, which shows that time and space are not independent anymore, then talk about how we need transformation between equivalent frames (mention at the end that one can also impose this straight from relativity principle, derive Lorentz and then examine effects like time dilation)

mention that, as we have seen, time is now a notion which depends on the observer, so it is just natural that we also have to transform it

another motivation: speed of light constant, so adding velocities like in Galilei transform cannot be right anymore; thus, we need new way to transform between frames of reference, one that respects relativity principle

-¿ much of clock section was arguing and sort of awkward pictures because they are often hard to interpret (not meant to be 100% rigorous); now, however, we will come to validation of all the results -¿ idea: show how transformation for spatial coordinates works, then compute $\Delta t = (x_1 - x_2)/c$ in primed coordinates, i.e. how $\Delta t' = (x_1' - x_2')/c$ is related to Δt (no change, again, comes from the fact that c is constant for every observer); will turn out that this is in very similar manner to spatial ones; implies that time transforms between observers as well, i.e. that clock results are correct! -¿ confirms time dilation as "real", physical effect arising from relativity principle rather than induced side-effect from our chosen definition of synchrony/simultaneity (although this derivation is not as rigorous as the one in clock section, where we set up consistent procedure for time measurement)

definitely include plot of γ-factor to show that it only becomes significant for speeds close to c, otherwise 1 to good approximation

2.4.1 Light Cones

we can visualize Lorentz transformation very nicely in geometric manner/fashion

mention future and past light cone

uhhh for spacelike events (equivalent: they do not lie in light cone of other event), their order depends on inertial frame we choose! This is what is meant with causality problems for spacelike events and world lines, cause and effect depend on observer!

talk about spacetime diagrams here -¿ after all, we will use them in section on clocks (not really anymore) -¿ better motivation: can visualize effects of Lorentz transformation very nicely

spacetime diagram is basically intersection in light cone, right? Where we only show one spatial dimension simultaneous events in Minkowski space are one a line parallel to spatial axis; gleichortige (-¿ equilocal?) events are on a line parallel to time axis

explanation why primed axes are shifted inwards in Minkowski diagrams: we wish to express these axes in the unprimed coordinate system, i.e. we do not apply the Lorentz transform from unprimed (at rest) to primed (moving with v), but rather the inverse one (thus we have to use $-v$ in transformation)

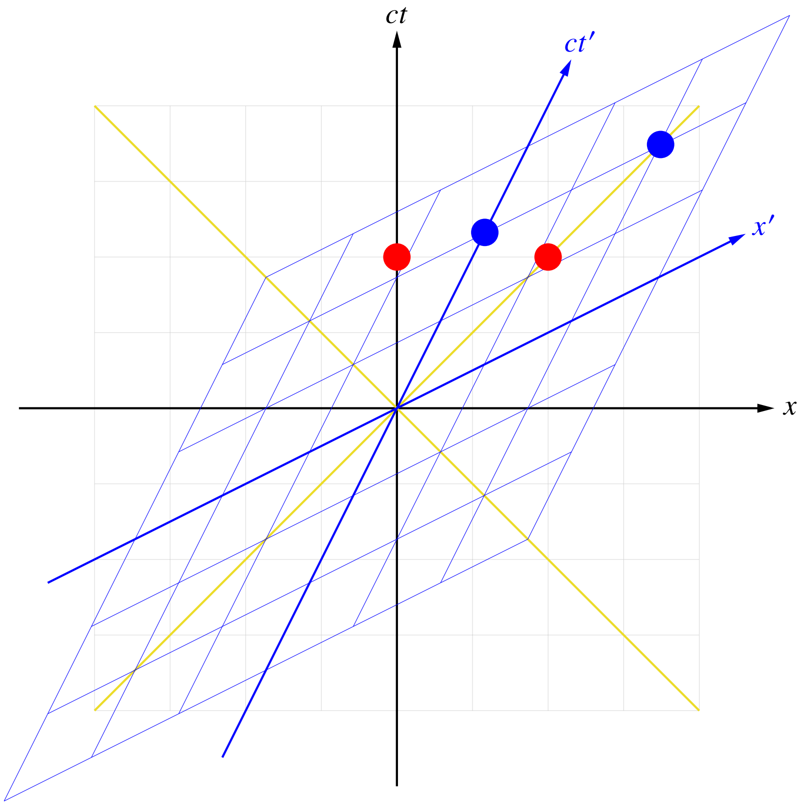


Figure 2.4: Two events at spacetime points $(0, 2), (2, 2)$. Red dots show the points with these coordinates in the (x, ct)-frame and blue dots show the same coordinates in the (ct', x')-frame moving with $v = 0.5c$.
We can see very nicely how each observer perceives time differently. Events happening simultaneously to both red dots (i.e. which lie on the line between them at $t = 2$), do *not* happen at $t' = t$, but at $t' = \tau = \sqrt{1 - v^2}\, t$ (which is evident from the fact that the blue dots are at $t' = t$). The same can be said for the moving observer in blue, which sees events at $t = \sqrt{1 - v^2}\, t'$ simultaneous to the blue dots at $t' = 2$.

2.5 Minkowski Space

over the course of the last sections, we saw more and more how space time work in relativity and how they are related; Einstein's insights and Lorentz transform show how clocks work differently than their Newtonian pendants once relativity principle is incorporated, they are not independent anymore; but it is only with the Minkowskian viewpoint that this connection is fully unveiled/completed by uniting them into one spacetime (which goes along with unified mathematical formulation on sound/fond footing)

do mathematical stuff; also make interpretation of Lorentz transformation as symmetry group stuff (?)

events are now spacetime points

2.5.1 Notes

turns out there is a very convenient way to formulate what we just learned, in terms of manifolds (where we use coordinates only as a tool, similar to what we require for different inertial frames); will also enable to formulate some properties in a very nice manner

ok, this might be nice: in spacetime diagrams, we see the idea how space and time (which are, as we know now, no separate notions) can be combined into one entity – describe them as coordinates in one space; this space is called Minkowski space and spacetime diagrams can be seen as a visualization of it; SR is basically which geometry Minkowski space possesses and mathematically, we can describe that in metric; this metric we get from invariant notion we have already seen

---

-¿ the geometric structure of Minkowski space can be described using the relation $c^2(\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ because this naturally gives us a line element ds^2 and corresponding metric (these are mathematical tools needed in order to do geometry; we choose this specific metric because it makes sense from a physics point of view, we can make statements about causality etc.); $ds^2 = 0$ means light, similarly > 0 and < 0 characterize certain things (here we need argument for c as maximum speed, right? Because for other particles, $v^2t^2 < c^2t^2$ for sure, so $ds^2 < 0$ -¿ is this a valid argument (I think so, this is how timelike worldlines are defined)? Because the Δx_i will be smaller as well, right?)

-¿ two observers can only communicate if their distance does not increase with a certain critical velocity (something like this), which is motivation for taking this particular metric and inner product (we can check causality)

-¿ observer-independent statements mean that coordinates have no physical meaning; advantage of metric: statements using that (for example distances and angles) are coordinate-independent, so the metric allows us to make physically meaningful statements

-¿ just state spacetime diagrams as convenient way to visualize Minkowski space

---

more thoughts inspired by Dragon script: we have seen how to measure time τ that elapses for observer; we can formalize this into scalar product which takes two tangent vectors as input; the value of this inner product tells us about the properties of the world line we compute the tangent vectors for, if it is timelike/lightlike/spacelike (right?); in particular, by taking the difference between events as an input here, we see if events have a causal connection (which is the case if difference is timelike); hmmm, so basically: the time that an observer measures tells us something about the observer, for example its speed (for light, no time elapses! For fast observers, a smaller amount of time elapses than for a resting one, etc.)

basic question is what the distance between events in spacetime is; this is a non-trivial question because observers might be moving, which means the spatial distance between them is not constant and that makes measuring them more complicated; in fact, it requires to also take time into account (time becomes coordinate time and is no invariant quantity anymore); our way to measure time will exploit that the speed of light is constant and independent of the observer (big advantage!); distance measure in new $4D$ spacetime will be defined based on how long light is travelling between observer and event, that leads to an invariant notion of time which naturally takes velocity of observers into account, the proper time τ; this is then defined into an inner product to give us a systematic way of arguing about causality in relativity because based on the value of this inner product for the difference vector between two events, we can learn something about if they can potentially influence each other

4-momentum is related to velocity, that is tangent vector; therefore, we can also compute inner product for it

physically way to say things: we want observer-independent statements, for example on spatial distances and time; mathematical: coordinate-independent, invariant notions like distance and metric

---

solution: end separate description of space and time, go to 4D-Minkowski space; events can then be described using 4-vectors $\xi = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ (called *event* from now on); we choose to scale time such that all components have the same units; factors of c are kind of nasty in relativity, which is why very often c is set to 1 (this way, first coordinate is simply time but scaling and computations are still convenient)

to be more general than moving with constant speed, let's use the very general definition of how to measure lengths of curves $\gamma(t)$ defined on some interval I: $L(\gamma) = \int_I \sqrt{g_{\mu\nu} v^\mu v^\nu}\, dt = \int_I \sqrt{g(v,v)}\, dt$, $v = \gamma'(t)$ (t-dependence omitted); this is done using the metric g and generalizing from the example where we subtract spatial coordinates from time yields

$$g = \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.7}$$

in basis $\xi = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$; this already determines the (pseudo) metric as a tensor because of invariance when changing the basis

the corresponding line element is

$$ds^2 := (ds)^2 = g_{\mu\nu} v^\mu v^\nu = c^2 dt^2 - \|d\vec{x}\|^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{2.8}$$

to see equivalence to what was deduced for non-constant speed, we rewrite condition from example **??**:

$$c^2 t^2 = c^2 t'^2 - v^2 t'^2 = c^2 t'^2 - \|\vec{x}\|^2$$

times t, distances $\|x\|$ are nothing but differences from 0 and for infinitesimal differences, this reads

$$c^2 dt^2 = c^2 dt'^2 - \|d\vec{x}\|^2$$

which is nothing but ds^2 (is invariant, just like τ); this shows us very cool feature of formalized description, effects like time dilation simply follow from demanding that ds^2 or $d\tau^2$ (equivalent) do not depend on the inertial/coordinate system they are measured in, i.e. $ds^2 = ds'^2$

therefore, we can compute the time elapsed along a world line Γ (which is nothing but a curve in Minkowski space) according to

$$\tau = \int d\tau = \frac{1}{c} \int ds \tag{2.9}$$

if the time t in some frame of reference can be measured, the proper time element can be rewritten as:

$$d\tau = \frac{ds}{c} = \frac{\sqrt{ds^2}}{c} = \frac{1}{c}\sqrt{c^2 dt^2 - d\vec{x}^2} = \frac{dt}{c}\sqrt{c^2 - \frac{d\vec{x}}{dt}\frac{d\vec{x}}{dt}} = dt\sqrt{1 - \frac{v^2}{c^2}} =: \gamma dt \tag{2.10}$$

where γ is the *Lorentz-factor*

if we have some world line $\gamma(\sigma) : \sigma \mapsto (t, x, y, z) = (t(\sigma), x(\sigma), y(\sigma), z(\sigma))$ parametrized by a parameter $\sigma \in [a, b]$, we can also compute the proper time using this parametrization:

$$\tau = \int_\Gamma d\tau = \int_a^b \frac{dt}{d\sigma} \gamma \, d\sigma = \frac{1}{c} \int_a^b \sqrt{\left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{d\vec{x}}{d\sigma}\right)^2} \, d\sigma \tag{2.11}$$

should be correct, is just transformation law for integrals... last equality from Dragon, should be equivalent to previous ones (guess it just depends on situation which one is used); this formula is valid for accelerated clocks as well (reduces to the previous one for time difference in case of uniformly moving clock/observer) we can see that for constant velocity, $\tau = \gamma t$ as well (where we now talk about differences which are not infinitesimal)

2.5.2 Lorentz Transformations – V2

now that we know how times from different frames of reference are related, we may ask for more general relations between them

when interpreting Minkowski space as a manifold and working with coordinates/charts ξ, we know the results should be independent of ξ; in particular, that means they hold in other charts as well and changing coordinates is an important part; the basis changes even have a distinct name, *Lorentz transformation*; this is basically group theory due to the symmetries that Minkowski space possesses (known from logic and experiments) ?right?

2.5.3 Metric

We have now seen how physics can be conveniently described using a 4D manifold, which we called spacetime. Points on this manifold are events and we can change coordinates or inertial frames using Lorentz-transformations. Moreover, there are several quantities that can be defined naturally on manifolds, for example curves, vectors, and covectors (maps that take vectors as input). While manifolds do have an additional natural structure, this is given by topology. In physics, however, we are also interested in statements concerned with distances between events and to measure them we need additional structure. More specifically, we have to specify a metric that will allow measuring distances, as well as norms of vectors via the induced inner product.

Mathematically, metrics are objects called tensors and they have the convenient property that they are invariant under coordinate changes. Therefore, distances are physical statements because they do not depend on the inertial frame we compute them on. In the tradition of invariant quantities that have been encountered so far, we may guess that the metric will be related to light in some way. From the universality of the speed if light c, distances s are equivalent to times t for light, $s = ct$. Because of that, a natural measure for distances is the time elapsed on a clock, i.e. the geometric structure of Minkowski space is determined by Minkowski's theorem 2.8. Instead of denoting time with the usual variable t, we will now switch to the *proper time* τ since the time elapsed a clock between events $(0, 0, 0, 0)$ and (ct, x, y, z) is

$$\boxed{\tau = \sqrt{1 - \frac{v^2}{c^2}}\, t = \sqrt{1 - \left(\frac{x}{ct}\right)^2 - \left(\frac{y}{ct}\right)^2 - \left(\frac{z}{ct}\right)^2}\, t = \sqrt{t^2 - (x^2 + y^2 + z^2)/c^2}} \tag{2.12}$$

This depends on the trajectory taken by the clock/corresponding observer (more specifically, on the uniform velocity v), but will in general not be equal to t. What was denoted with τ here, as well as the coordinates (ct, x, y, z), in reality is a difference $\Delta\tau$ of proper times and coordinates and if we make these differences infinitesimally small, i.e. $\Delta \to d$, we obtain the infinitesimal distance or *proper time element* (strictly mathematically: *line element*)[5]

$$\boxed{d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}}\,. \tag{2.13}$$

This implies the following *proper line element*

$$\boxed{ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2} \tag{2.14}$$

and corresponding *proper distance* $s = c\tau$ between events. This specific form of the line element immediately gives us the components of the *Minkowski metric* η, which can be conveniently arranged in a matrix

$$\boxed{\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}\,. \tag{2.15}$$

At this point, we shall make a remark regarding the signs: in principle, we could have chosen η to have only one minus (in the time component) since only the proper time is constrained by physical observations, while the proper distance could equally well be defined as $ds^2 = -c^2 d\tau^2$. There are good arguments for both conventions, but I personally find it more natural to keep the signs when going from times to distances, which is why the $(+,-,-,-)$-convention is adopted here.

From the metric, we can infer statements about the geometry of relativity. In Euclidian space, points of constant distance s lie on a circle around the origin, which is determined by the equation $s^2 = x^2 + y^2 + z^2$. In Minkowski space, events of equal distance lie on a hyperboloid of constant proper times τ and are determined by

$$\boxed{c^2\tau^2 = c^2 t^2 - x^2 - y^2 - z^2}\,. \tag{2.16}$$

Setting $\tau = 1$ yields a hyperbola, whose intersection with time axes in spacetime diagrams determines the "length of one time unit" (figure 2.5).

One final note concerns an alternative derivation of the metric. It is also based on the relativity principle, but instead of constructing clocks etc. explicitly, it uses that light propagates as a spherical wave with velocity c. Writing this equation in multiple inertial frames yields

$$c^2t^2 = x^2+y^2+z^2 \;\text{ and }\; c^2t'^2 = x'^2+y'^2+z'^2 \qquad \Leftrightarrow \qquad c^2t^2-x^2-y^2-z^2 \stackrel{!}{=} c^2t'^2-x'^2-y'^2-z'^2\,. \tag{2.17}$$

[5]We adopt the common notation $dx^2 := (dx)^2$.

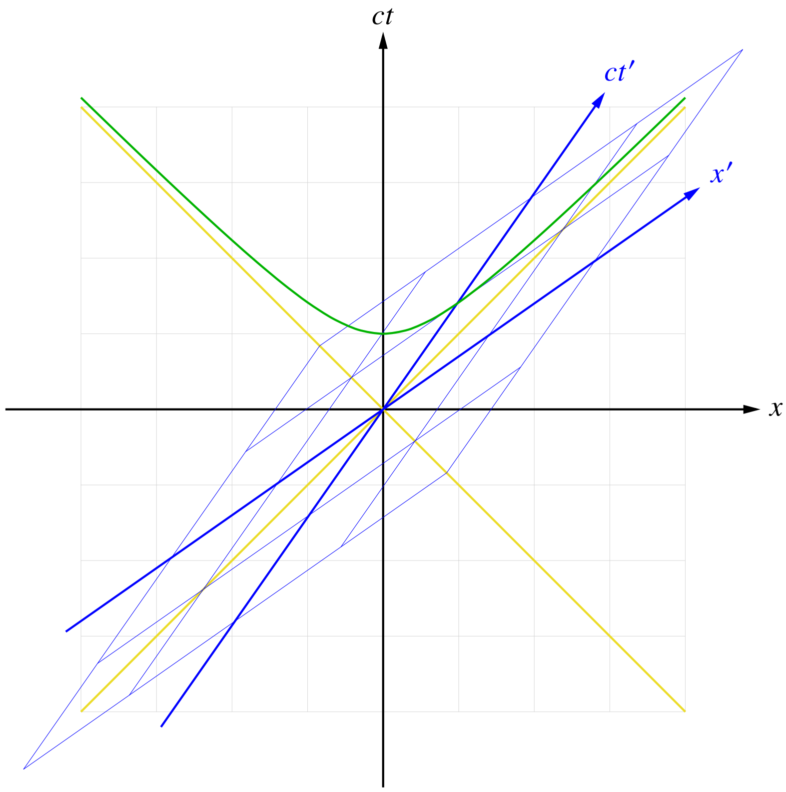


Figure 2.5: As we can see, this yields same time steps as Lorentz transform, hyperbola intersects ct'-axis perfectly at $ct' = 1$ (as it should).

by the equivalence principle. This points to the invariant proper distance we have called s (which vanishes for light) and it also shows where the ambiguity in overall sign comes from.

"derivation": $c^2t^2 = x^2 + y^2 + z^2$ holds for light independently of frame we choose, so it is an invariant object and it also represents what we do with clocks (bad phrasing, but idea good), so rearranging it is good choice for metric; however, that leaves some freedom in sign, physics is of course the same no matter which one we choose, but still we have to select one of them

how Dragon phrases it: Minkowski theorem gives us a geometrical structure of spacetime, tells us about distances in them (2.5 Uhrzeit); in particular, time elapsed on uniformly moving clock between $(0, 0, 0, 0)$ and $(ct, x, y, z) = t(c, v_x, v_y, v_z)$ is $\tau = \sqrt{1 - v^2/c^2}\, t = t^2 - (x^2 + y^2 + z^2)/c^2$; for example, a clock at rest will measure the difference $\tau = t$ (which makes sense, after all the difference is measured between $(0, 0, 0, 0)$ and $(ct, 0, 0, 0)$, only temporal distance); these results are all coordinate-independent, i.e. a clock moving with velocity v will measure $\tau = \tau' = \sqrt{1 - v^2}\, t'$ for the time elapsed on the resting clock (i.e. will see it simultaneous to the event $(ct', x', y', z') = t'(c, v_x, v_y, v_z)$, where it is simultaneous to $(ct, 0, 0, 0)$; in the language of clock section, $t' = \frac{t'_+ + t'_-}{2}$ with light travel times t'_+, t'_-; $t' \geq \tau$, which once again shows that time dilation is mutual effect); this shows how $c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$, the time τ is an invariant quantity that measures the distance between events (by measuring time elapsed on clock that moves uniformly from origin to this event); only possible by use of Einstein synchronization, properly synchronized clocks will measure the same time τ for events

geometric interpretation: equidistant events in Minkowski space lie on hyperboloid of constant $\tau^2 = c^2t^2 - x^2 - y^2 - z^2$ -¿ can show that as Eichhyperbel (gauge hyperbola?) in Minkowski diagram; of course, choosing the origin was only conventional (we have this freedom in choice of coordinates), so more generally we would write $(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$

-¿ shows even more clearly how time and space are intertwined in special relativity, there is no distinction anymore, just one 4D-spacetime

therefore, it makes sense to choose this (or rather the infinitesimal version; equivalent for uniform movement, but more rigorous and as it turns out, more general) as a line element: $d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ (where we abbreviate $(dx)^2 =: dx^2$, is customary); we note that only opposite sign in front of time and spatial coordinates is fixed, so we have freedom in choosing the overall sign); we note that $ds^2 = c^2d\tau^2$ is also a line element (dimensions of time are not really what we would associate with metric and spatial distances); allows to compute proper times and proper distances, respectively

metric also allows very compact formulations of causality (light cone structure): world lines of light (and in general particles moving at light speed) are characterized by $ds^2 = 0$ (these trajectories are called *null* or *lightlike*), the ones for massive particles by $ds^2 > 0$ ($<$ for convention with other sign; also called *timelike*), while $ds^2 < 0 \quad \Leftrightarrow \quad c < v$ and this is a violation of principles we used to build the theory (*spacelike*); timelike means trajectories are always in light/null cone (Penrose continues to make interesting point on that on page 407: light cones more fundamental than metric)

2.5.4 Relativistic Dynamics

we have dependence on world line of our distance measure; this is nothing unusual in mathematical theory of metric spaces, but it raises an important physical question: what is the preferred trajectory of particles, i.e. what is the time that usually elapses for them? Turns out that it is extremal proper time (minimal for us, depends on sign convention of metric, right?), which yields straight lines.

idea: new transformation law means we have new dynamics; these can be formulated conveniently using points and vectors in Minkowski space; talk about four-momentum and forces etc.

four-velocity: using chain rule, we can write $U^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dt}{d\tau}\frac{dx^\alpha}{dt} =: \gamma(c, \vec{v})$ in general; this $\gamma = \frac{dt}{d\tau} = \frac{\partial t}{\partial \tau}$ then depends on the metric, it is the (negative of, depends on convention that is chosen) component g_{tt}; for the Minkowski metric in SR (where only uniform velocities occur), we can further simplify this by using $d\tau^2 = ds^2/c^2 = (c^2dt^2 - dx^2 - dy^2 - dz^2)/c^2 = dt^2(1 - \frac{dx^\alpha}{dt}\frac{dx^\alpha}{dt}/c^2) =: dt^2/\gamma^2$; this γ us negligible for many problems, only has significant effects if $v \approx c$ (which is why it is does not show up in daily life)

redshift leads to higher energy, but frequency also changes such that all in all speed of light remains constant (might be wrong this way, but heard something like that)

2.6 Acceleration

using the generalization we just made, we can also deal with accelerations, i.e. non-uniform movement; this is easier (or at least more straightforward) to do than intuitive version, at least for now

we have seen how to compute proper times in special relativity, the formula could be broken down to

$$\tau = \int d\tau = \frac{1}{c} \int ds = \frac{1}{c} \int \sqrt{g_{\mu\nu} v^\mu v^\nu} \, dt \tag{2.18}$$

in fact, this formula also holds for $v = v(t)$, i.e. when a time-dependent velocity and thus acceleration is present (we can compute dynamics); this is because it does not involve absolute differences like $x - x'$, but infinitesimal ones dx along the whole path, so changes in v are incorporated automatically; however, we have to use other metrics in this case and general relativity presents a general way to compute the metric

for non-uniform velocity, connection procedure does not work between macroscopic events anymore (because getting distance from velocity is not multiplication, now integration), but only infinitesimally (equivalent: just like we have to integrate velocity for distance, we have to integrate for travel time of light; since time = distance for light -¿ nice visualization: rectification of curve, works perfect for straight line, but if changing derivative comes into play, not perfect anymore, have to go to infinitesimal distances for that); motivation for $d\tau$

3 General Relativity

SR dealt with uniformly moving frames, now we want to use the insights gained there to generalize discussions to accelerated frames – this is what general relativity does (as it turns out, acceleration is very closely related to gravity, so GR is a theory of gravity as well)

-¿ wrong, SR can handle acceleration (contrary to popular belief I feel)! GR is really about incorporating gravity

3.1 Generalizing Relativity

3.1.1 What is wrong with Newton (and SR)?

gravitational redshift and instantaneous effect of gravity

3.1.2 Einstein Postulates

do postulates by Einstein again as start, but now the ones for GR; weak equivalence principle + Einstein equivalence principle

-¿ what about Mach principle?

3.1.3 Notes

Penrose has incredibly well written section 17.9 on intuition about metric and light cone structure in GR

3.2 Giulini Lectures

from 10 on (until 16) he deals with GWs, noice

3.3 Gravitational Physics Summary – Physics Part

Remark: we use units of $c = G = 1$ (*geometric units*).

Remark: we adopt the Einstein summation convention where repeated combinations of upper and lower indices are summed over, that is we abbreviate $\sum_\mu x^\mu y_\mu = x^\mu y_\mu$.

3.3.1 Newtonian Gravity

Newtonian gravity can be captured by his famous formula

$$\boxed{F_g = -\frac{m_1 m_2}{r^2}} \tag{3.1}$$

which describes the gravitational force that an object with mass m_1 exerts onto another object with mass m_2. From Newton's second law, we know that the same force is exerted from the second object onto the first.

This force can also be brought into the form

$$F_g = m_2 \frac{d}{dr}\left(\frac{m_1}{r}\right) = -m_2 \frac{d\Phi_g}{dr} \tag{3.2}$$

which tells us that gravitation is a conservative force with potential

$$\boxed{\Phi_g = -\frac{m_1}{r}} \tag{3.3}$$

produced by some object with mass m_1. We get the conservative property from equation (3.2) alone because gravitational force only has a radial and no angular component (thinking in polar/spherical coordinates) such that any derivative with respect to angular coordinates vanishes. Thus, (3.2) is equivalent to the more general condition for conservative forces,

$$\boxed{\vec{F} = -\vec{\nabla}\Phi} \qquad F^k = -\delta^{kl}\frac{\partial \Phi}{\partial x^l}\,. \tag{3.4}$$

As a consequence, knowing the potential is sufficient to know how gravity acts. Thus, we are interested in how to determine Φ and this can be done using the Poisson equation. For a point particle, it takes the form

$$\boxed{\Delta\Phi = \nabla^2\Phi = 0} \tag{3.5}$$

and for a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\Delta\Phi(\vec{x}) = 4\pi\rho(\vec{x})} \qquad \delta^{ij}\frac{\partial^2\Phi(\vec{x})}{\partial x^i \partial x^j} = 4\phi\rho(\vec{x})\,. \tag{3.6}$$

Another perspective is not to look at forces $\vec{F}$, but at associated accelerations which comes from Newton's second law

$$\boxed{\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}} \qquad\qquad F^k = ma^k = m\ddot{r}^k \tag{3.7}$$

or at momenta $\vec{p}$ which are defined by

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} \quad \Leftrightarrow \quad \vec{p} = m\vec{v}} \qquad\qquad p^k = mv^k\,. \tag{3.8}$$

Example 3.1: Gravity on Earth

The gravity exerted by Earth on objects with mass m (assuming they stand on Earth's surface for now) is

$$F_g = -m\frac{m_e}{r_e^2} = -mg \tag{3.9}$$

Comparing that with Newton's second formula, $F = ma$, we see that such an object experiences an acceleration

$$a = -g = -9.81\frac{\mathrm{m}}{\mathrm{s}^2} = -1.1 \cdot 10^{-16}\frac{1}{\mathrm{m}}\,. \tag{3.10}$$

Remark: note that we implicitly assume that gravitational mass and inertial mass are equal here. This is a non-trivial statement, which has been experimentally verified with high accuracy.

To see how much potential energy is needed to lift objects of mass m to a height $h \ll r_e$ above Earth's surface, we can do a Taylor expansion around $h = 0$:

$$\begin{aligned}
\Phi_g = -\frac{m_e}{r_e + h} &\simeq -\left.\frac{m_e}{r_e + h}\right|_{h=0} + h\left.\frac{d}{dh}\left(-\frac{m_e}{r_e + h}\right)\right|_{h=0} + O(h^2) \\
&= -\frac{m_e}{r_e} + h\left.\frac{m_e}{(r_e + h)^2}\right|_{h=0} + O(h^2) \\
&= -\frac{m_e}{r_e} + h\frac{m_e}{r_e^2} + O(h^2) = -\frac{m_e}{r_e} + hg + O(h^2)
\end{aligned}$$

However, the first contribution is nothing but the energy at Earth's surface. The potential energy that at h and thus the energy which is needed to lift an object of mass m to this height h (which is what one is interested in most of the time) is given to first order by the difference

$$\Phi_g = -\frac{m_e}{r_e} + gh - -\frac{m_e}{r_e} = gh\,. \tag{3.11}$$

This corresponds to gauging our measurements such that Earth's surface is the value with zero potential energy.

We see that gravity is related to a potential and thus to potential energy. Hence, we expect an objects energy to change if it moves in a gravitational field (in radial direction). This has interesting consequences, for example because light will also be affected by this.

Example 3.2: Gravitational Redshift

somehow we could build perpetual motion machine is the argument, don't get it

I rather think about it like this (should be equivalent): SR tells us that photons have a certain mass $m = \frac{E}{c^2}$; therefore, it is also affected by a gravitational potential and to move against gravity, some of its energy has to be converted; that corresponds to a change in frequency, (since $f = \frac{E}{h}$):

$$\frac{f_{\text{top}}}{f_{\text{bottom}}} = \frac{E_{\text{top}}}{E_{\text{bottom}}} = \frac{m - mgh}{m} = 1 - gh \tag{3.12}$$

Remark: in script, this is only true to first order, so derivation might be wrong... Result there reads $\frac{1}{m+mgh}$. Ahhh, because there things are defined differently: photon starts from top, thus it has more energy at ground

A natural consequence because time is inversely proportional to frequency (time differences are) is that clocks tick faster at higher altitude, i.e. for a stronger gravitational potential. It is also possible to derive it in reverse order, that is by showing that clocks tick slower in a stronger gravitational field. This causes a change in frequency and thus also a redshift.

3.3.2 Special Relativity

The theory of relativity is about how physical laws depend on the observer. We will begin with the theory of *special relativity* (SR), which generalizes Newtonian dynamics.

Definition 3.3: Inertial Frame

An *intertial frame of reference* is a coordinate frame where $\vec{F} = m\vec{a}$ holds. In particular, that means objects move with constant speed when no force is acting on them.

From the definition we can immediately see that there is no unique inertial frame because we can always get other inertial frames from existing ones by looking at frames which move with constant speed with respect to them.

We also see that all laws of physics hold equally in all inertial frames because

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}(t)}{dt} = m\frac{d\vec{v}'(t)}{dt} \tag{3.13}$$

as long as $\vec{v} - \vec{v}' = \text{const}$. That also means only laws and quantities which are invariant under transformations between inertial frames have physical meaning. In particular, that means coordinates of events have no physical meanings. As an alternative, we can look at distances between events which turn out to be invariant because they are related to a metric:

$$\begin{aligned}(\Delta s)^2 &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= \underline{\Delta x} \cdot \eta \cdot \underline{\Delta x} = \begin{pmatrix}\Delta t & \Delta x & \Delta y & \Delta z\end{pmatrix} \cdot \begin{pmatrix}-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}\Delta t \\ \Delta x \\ \Delta y \\ \Delta z\end{pmatrix} \\ &= \eta_{\mu\nu}(\Delta x)^\mu (\Delta x)^\nu = (\Delta x)^\mu (\Delta x)_\mu \,.\end{aligned}$$

Making these differences Δ infinitesimally small gives the line element of the metric η

$$ds^2 := (ds)^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{3.14}$$

Let us now think about gravity in special relativity. In principle, the Newtonian description is kept, but some effects can be examined in different manner now, e.g. due to the new notion/tool of different inertial frames. However, a frame where gravity acts is *not* inertial (because we have external force in gravity, right?)! Thus, to do physics on Earth, we have to find a reference frame in which the effect of gravity is cancelled out. Obviously, earths surface is not sufficient and neither is a uniformly moving one. In free fall, however, we experience no gravity, that is a freely falling frame cancels out the effect of gravity. This can be stated more formally:

Property 3.4: Weak Equivalence Principle

The effects of a gravitational field are indistinguishable from an accelerated frame of reference.

Basically, that means only a freely falling frame can serve as an inertial frame on Earth. That raises the question what happens to the laws of physics in such a freely falling frame.

Property 3.5: Einstein Equivalence Principle

The laws of physics in a freely falling frame are locally described by SR without gravity. For this reason, such a frame is also called *local inertial frame (LIF)*.

Speaking strictly mathematically, "locally" means in an infinitesimally small neighbourhood of points. The degree to which this can be extended in practice depends on the physical effects of interest.

Since gravity acts radially, its direction changes on different places around Earth. That implies there is no uniform direction of acceleration, so there can be no global freely falling frame/LIF. Other properties of gravity which are known from experience are the following:

(a) All bodies which start with the same initial velocity move through a gravitational field along the same curve

(b) Bodies which move initially parallel to each other in a freely falling frame do not necessarily move parallel at all times if an external gravitational field is present (this effect is due to *tidal forces* acting on them)

Property (b) can be further examined and quantified. To do that, we note that for a particle with world line $x^k(\tau)$ we have

$$\frac{d^2x^k}{d\tau^2} = -\delta^{kl}\frac{\partial\Phi}{\partial x^l} = -\delta^{kl}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}}$$

because of (3.4) and Newton's second law $F^k = m\frac{d^2x^k}{d\tau^2}$. Similarly, for another particle starting close to the first one (i.e. with world line $x^k + \xi^k$, where $\left|\xi^k\xi_k\right| \ll 1$)

$$\begin{aligned}\frac{d^2\left(x^k+\xi^k\right)}{d\tau^2} &= -\delta^{kl}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}+\vec{\xi}} \\ &\simeq -\delta^{kl}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}} - \delta^{kl}\xi^m\frac{\partial}{\partial x^m}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}}\end{aligned}$$

where we used a Taylor approximation to first order. The linearity of derivatives yields

$$\boxed{\frac{d^2\xi}{d\tau^2} = \frac{d^2\left(x^k+\xi^k\right)}{d\tau^2} - \frac{d^2x^k}{d\tau^2} = -\delta^{kl}\frac{\partial^2\Phi}{\partial x^m\partial x^l}\xi^m} \tag{3.15}$$

Remark: note that the evaluation is still at the point $\vec{x}$, not at something related to ξ^k!

This is the *Newtonian deviation equation*. We see that tidal forces are governed by the tidal acceleration tensor $\frac{\partial^2\Phi}{\partial x^m\partial x^l}$. Tidal forces are a way to detect gravity as opposed to constant acceleration (which would affect the world lines x^k and $x^k + \xi^k$ equally)

3.3.3 Curved Spacetime

One problem in SR is that the Newtonian description of gravity is still taken to be valid. That, however, is a problem because there are many inconsistencies between them, for example the instantaneous effect of gravity (gravitational redshift is also puzzling). However, we can come up with generalized description: for anybody familiar with differential/Riemannian geometry, the effects (a), (b) of gravity stated above sound very much like the ones associated with a curved space. This motivates the (mathematical) description of gravity as a geometrical effect in Minkowskian spacetime (which will become a curved space in this process). Many relations known from SR will remain, but with different quantities and most prominently, a different metric other than η. The basic goal of *general relativity* (GR) will be to find ways to derive the metric which contains information about spacetime curvature and thus gravity.

The approach in this subsection will always be to look how generalizations can be made using the metric and other tools of geometry, while recovering SR in a LIF. Such a check, however, has not been made for the metric itself yet! Thus, we will now look at how the mathematical term "locally" is to be thought of. Taking an arbitrary metric with components $g_{\mu\nu}$ in some basis, we can always transform to other coordinates using the tensor transformation law. This

say something about how LIF can be characterized using metric; is important we always want to recover results in there, for example statements on timelike etc. (around 2.36)

The most basic thing we need to know is how test particles move in curved spaces. We will start with the case where no force is present, i.e. free movement. The Newtonian theory/SR gives us $\ddot{\vec{a}} = 0$ and thus a movement on straight lines. This has to be reproduced locally (that is in a LIF), but originating from a more general concept. Finding this generalization is based on the observation that tangent vectors remain constant along straight lines. That leads to the *geodesic equation*

$$\boxed{\nabla_{\underline{t}} t^\beta = t^\alpha \nabla_\alpha t^\beta = 0}\,, \tag{3.16}$$

which just expresses that t^β remains constant as long as we take the derivative along the curve that it is tangent to (which explains the $t^\alpha \nabla_\alpha$ part). Consequently, the world lines test particles are *geodesics*.

One can obtain the same statement from a completely different approach: by demanding that world lines are the curves in Minkowski space which extremize the proper time/distance

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{-g_{\alpha\beta} dx^\alpha dx^\beta} = \int_0^1 \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}}\, d\sigma \tag{3.17}$$

between two events A, B.

In both cases, we obtain the following coordinate version of the geodesic equation:

$$\boxed{\frac{d^2 x^\beta}{d\sigma^2} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma}} \tag{3.18}$$

Proof. In the first approach, we use

$$\begin{aligned} t^\alpha \nabla_\alpha t^\beta &= \frac{dx^\alpha}{d\sigma} \nabla_\alpha \frac{dx^\beta}{d\sigma} \\ &= \frac{dx^\alpha}{d\sigma} \left(\frac{\partial}{\partial x^\alpha} \frac{dx^\beta}{d\sigma} + \Gamma^\beta_{\alpha\delta} \frac{dx^\delta}{d\sigma} \right) \\ &= \frac{dx^\alpha}{d\sigma} \frac{\partial}{\partial x^\alpha} \frac{dx^\beta}{d\sigma} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma} \\ &\underset{\text{chain rule}}{=} \frac{d}{d\sigma} \frac{dx^\beta}{d\sigma} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma} \\ &= \frac{d^2 x^\beta}{d\sigma^2} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma} \,. \end{aligned}$$

In the second approach, we derive the Euler-Lagrange equations by varying the proper time integral:

$$\boxed{\frac{\partial L}{\partial x^\alpha} = \frac{d}{d\sigma} \frac{\partial L}{\partial dx^\alpha / d\sigma}} \tag{3.19}$$

where we introduced the Lagrangian

$$\boxed{L = L\left(x^\alpha, \frac{dx^\alpha}{d\sigma}\right) = \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}}} \,. \tag{3.20}$$

Notable and useful properties in this context are that L is constant along the geodesic because

$$\begin{aligned} \frac{d}{d\sigma} \left(g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right) &= t^\gamma \partial_\gamma \left(g_{\alpha\beta} t^\alpha t^\beta \right) = t^\gamma \nabla_\gamma \left(g_{\alpha\beta} t^\alpha t^\beta \right) \\ &= t^\alpha t^\beta \nabla_\gamma g_{\alpha\beta} + g_{\alpha\beta} t^\beta t^\gamma \nabla_\gamma t^\alpha + g_{\alpha\beta} t^\alpha t^\gamma \nabla_\gamma t^\beta = 0 \,. \end{aligned}$$

Note that this does *not* imply $\partial_\alpha L = 0$ (so the Euler-Lagrange equations still make sense). It does, however, mean that we are free to change the parametrization from σ to any *affine parameter* $\sigma' = a\sigma + b,\ a, b \in \mathbb{R}$ while only picking up a factor $\frac{1}{a}$. The proper time τ is defined as the parameter σ' with

$$L = 1 \qquad \Leftrightarrow \qquad g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = -1 \tag{3.21}$$

which comes from the known normalization of the four-velocity $U^\alpha = \frac{dx^\alpha}{d\tau}$. Therefore, we can always replace $d\sigma \to d\tau = L d\sigma$ or vice versa. □

The advantage of having two approaches is that, in some cases, calculating the Christoffel symbols might be easier or might have already been done, while in others the Euler-Lagrange equations reveal very useful properties of the problem/system (for example in case one coordinate does not appear explicitly in L; then, we have immediately found a quantity which is conserved by the system, i.e. it does not change as time evolves/as we vary σ when going along the geodesic, this quantity being $\frac{\partial L}{\partial dx^\alpha / d\sigma}$).

A physical consequence from $L = \text{const}$ is that geodesics/world lines which are timelike/null/spacelike somewhere have this property everywhere! Test particles (with mass $m > 0$) always move along timelike geodesics while massless particles like photons move along null geodesics (spacelike ones violate causality).

3.3.4 Weakly Curved Spacetimes

basic idea here: re-derive first subsection from GR (confirm that it reproduces Newtonian results), that is using the spacetime metric

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\left(dx^2 + dy^2 + dz^2\right) \tag{3.22}$$

which is the first order approximation in case the Newtonian potential $\Phi = -\frac{m}{r}$ fulfils $|\Phi| \ll 1$; we will see where it comes from later on, will be used as motivation for now

we can formulate the geodesic equation for test particles in many ways, for example

$$U^\alpha \nabla_\alpha U^\beta = 0 \qquad\qquad p^\alpha \nabla_\alpha p^\beta \tag{3.23}$$

where $U^\alpha = \frac{dx^\alpha}{d\tau}$ is the four-velocity of a particle with world line x^α and $p^\alpha = mU^\alpha$ is the four-momentum (sometimes preferred quantity because it is well-defined also for photons)

for the first/time component $p^0 = E \approx m$ ($c^2 = 1$, this is an approximation to lowest order) we obtain

$$0 = m\frac{dp^0}{d\tau} + \Gamma^0_{\alpha\beta} p^\alpha p^\beta \approx m\frac{dp^0}{d\tau} + \Gamma^0_{00}\left(p^0\right)^2 \qquad \Leftrightarrow \qquad \frac{dp^0}{d\tau} \approx -m\frac{\partial \Phi}{\partial t} \tag{3.24}$$

which matches the Newtonian result that if the gravitational field does not change over time, then the energy p^0 will be conserved over time

similarly, the equations for the spatial components

$$0 = m\frac{dp^k}{d\tau} + \Gamma^k_{\alpha\beta} p^\alpha p^\beta \approx m\frac{dp^k}{d\tau} + \Gamma^k_{00} m^2 \qquad \Leftrightarrow \qquad \frac{dp^k}{d\tau} \approx -m\delta^{kl}\frac{\partial \Phi}{\partial x^l} \tag{3.25}$$

match the Newtonian result (3.4) that gravity acts as a conservative force

In the Newtonian case, tidal forces (3.15) could be used to detect gravity as opposed to constant acceleration; something similar should be possible for curvature (which is how we described gravity now) and indeed, we obtain the *geodesic deviation equation*

$$\boxed{\nabla_{\underline{U}} \nabla_{\underline{U}} \xi^\beta = U^\sigma \nabla_\sigma U^\alpha \nabla_\alpha \xi^\beta = -R^\beta{}_{\gamma\delta\epsilon} U^\gamma \xi^\delta U^\epsilon} \,. \tag{3.26}$$

Therefore, gravity/curvature is determined and measured by the *Riemann curvature tensor*

$$\boxed{R^{\beta}{}_{\gamma\delta\epsilon} = \frac{\partial \Gamma^{\beta}{}_{\gamma\epsilon}}{\partial x^{\delta}} - \frac{\partial \Gamma^{\beta}{}_{\gamma\delta}}{\partial x^{\epsilon}} + \Gamma^{\beta}{}_{\delta\mu}\Gamma^{\mu}{}_{\gamma\epsilon} - \Gamma^{\beta}{}_{\epsilon\mu}\Gamma^{\mu}{}_{\gamma\delta}}\,. \tag{3.27}$$

A helpful way to think about the Riemann tensor is as the "difference"/commutator of subsequent derivatives, which is motivated by the following equation:

$$\boxed{\left[\nabla_{\alpha}, \nabla_{\beta}\right] V^{\mu} = R^{\mu}{}_{\gamma\alpha\beta} V^{\gamma}}\,. \tag{3.28}$$

We can simplify this equation by choosing a LIF with $\underline{e}'_0 = \underline{e}_{\tau}$ (one basis vector along proper time). Since this coordinate system is parallel transported along the geodesic,

$$U^{\gamma}\nabla_{\gamma}\frac{dx'^{\beta}}{dx^{\alpha}} = U^{\gamma}\nabla_{\gamma}\left(\underline{e}'^{\beta}\right)_{\alpha} = 0$$

and we obtain

$$\frac{d^2\xi^{\beta}}{d\tau^2} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} U^{\sigma}\nabla_{\sigma}U^{\delta}\nabla_{\delta}\xi^{\alpha} = -\frac{\partial x'^{\beta}}{\partial x^{\alpha}} R^{\alpha}{}_{\gamma\delta\epsilon}\frac{\partial x^{\gamma}}{\partial \tau}\xi^{\delta}\frac{\partial x^{\epsilon}}{\partial \tau} = -R'^{\beta}{}_{\tau\delta\tau}\xi'^{\delta} \tag{3.29}$$

Remark: note that the evaluation is still at the point $\vec{x}$, not at something related to ξ^k! due to the transformation law for vectors and tensors (?). This equation looks very much like the Newtonian expression (3.15). Moreover, we see that the Riemann tensor might play a role which is similar to the tidal acceleration tensor $\frac{\partial^2\Phi}{\partial x^m \partial x^l}$. Hence, it might also play a similar role in causing gravity...

3.3.5 Einstein Equation

We have seen how Newtonian theory can be generalized, for example by going from a potential Φ for the gravity field to the metric $g_{\mu\nu}$. Many effects could be derived from that, e.g. equations of motion and the effect of gravity in tidal forces – but we have no way of determining the metric yet! Thus, we will now look for a field equation like (3.6). As it turns out, there is no way of truly deriving the result. Instead, we can only motivate it sufficiently.

First of all, we need an analogue to the mass distribution $\rho(\vec{x})$ to describe the effect of matter. This should be a tensorial quantity in accordance with previous generalizations.

This definition also allows to summarize several conservation formulas into one, coordinate-independent equation:

$$\nabla_{\alpha}T^{\alpha\beta} = 0\,. \tag{3.30}$$

This gives us

$$\partial_t\rho + \partial_i\pi^i = 0 \qquad\qquad \partial_t\pi^i + \partial_j T^{ij} = 0 \tag{3.31}$$

which corresponds to energy and momentum conservation (which is the continuum version of $F^k = ma^k$ since spatial components of the stress-energy tensor describe forces).

Now that we have something which causes gravity, we need to equate it with something that describes gravity – a natural quantity would be the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$. However, the stress-energy tensor only has two indices, while the Riemann tensor has four. Thus, we have to get rid of two indices, which can be done by contraction. Since $T = T^{\alpha}{}_{\alpha} \neq 0$, we are only interested a 2-tensor trace component, which turns out to be the *Ricci tensor*

$$\boxed{R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}} = R_{\beta\alpha}\,. \tag{3.32}$$

However, the Ricci tensor is not divergent free, so a field equation of the kind $R^{\alpha\beta} = bT^{\alpha\beta}$ cannot hold. Luckily, we can construct divergent-free tensor from it by rearranging

$$\nabla_{\alpha} R^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \nabla_{\alpha} R$$

which can be obtained using the Bianchi identity. Since $\nabla_{\alpha} g^{\alpha\beta} = 0$,

$$\nabla_{\alpha}\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right) = 0 \tag{3.33}$$

and we see that the *Einstein tensor*

$$\boxed{G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = G^{\beta\alpha}} \tag{3.34}$$

fulfils the required conservation law. Here, R is the *Ricci scalar* (or *scalar curvature*)

$$\boxed{R = R^{\alpha}_{\alpha}}\,, \tag{3.35}$$

part of another trace component of the Riemann tensor. Thus, the field equation of interest (also called *Einstein equation*) is

$$\boxed{G^{\alpha\beta} = 8\pi T^{\alpha\beta}} \tag{3.36}$$

where the constant of proportionality 8π is determined by requiring that they reduce to the Newtonian weak-field limit. Now, remembering that $\nabla_{\gamma} g^{\alpha\beta} = 0$ tells us we are free to add a term of the kind $ag^{\alpha\beta}$ to the field equation. Therefore,

$$\boxed{G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta}} \tag{3.37}$$

is an equally valid version of the Einstein equation. Λ is the (in)famous *cosmological constant* and its contribution can be thought of as "stress-energy of empty space". However, this can be absorbed in $T^{\alpha\beta}$, so we will mostly stick to using (3.36). The Einstein equation can be interpreted following this famous quote by John Archibald Wheeler:

"Spacetime tells matter how to move; matter tells spacetime how to curve."

To simplify things one can look empty space first, where $T^{\alpha\beta} = T = 0$. For the Einstein equation, that means

$$G^{\alpha\beta} = 0\,.$$

But since trace of the Einstein equation yields $R = -8\pi T$ (= 0 in vacuum), we obtain the following *vacuum Einstein equation*:

$$\boxed{R^{\alpha\beta} = 0}\,. \tag{3.38}$$

Now we will come to few examples of solutions of the Einstein equation.

Example 3.6: Solutions of the Einstein equation

(a) Minkowski space with $g_{\alpha\beta} = \eta_{\alpha\beta}$ and

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.39}$$

(b) Schwarzschild black hole: unique (non-trivial) solution for a static, spherically symmetric spacetime containing a mass M (a *black hole*)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2\left(d\theta^2 + \sin(\theta)^2 d\phi^2\right) \tag{3.40}$$

(c) weak (Newtonian) gravitational potential: Schwarzschild solution far away from the source (such that it can be approximated as point mass)

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\left(dR^2 + R^2 d\theta^2 + R^2\sin(\theta)^2 d\phi^2\right) \tag{3.41}$$

$$= -(1+2\Phi)dt^2 + (1-2\Phi)\left(dx^2 + dy^2 + dz^2\right) \tag{3.42}$$

Is obtained from defining the potential $\Phi = -\frac{r}{r-M}$, Taylor-expanding to first order in Φ and then making the gauge transformation $r \to R = r - M$.

(d) Kerr solution of a rotating black hole

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin(\theta)^2}{\rho^2}dt^2 - 2a\frac{2Mr\sin(\theta)^2}{\rho^2}dt\,d\phi + \frac{\rho^2}{\Delta}dr^2 \\ &+ \frac{\left(r^2+a^2\right)^2 - a^2\Delta\sin(\theta)^2}{\rho^2}\sin(\theta)^2 d\phi^2 + \rho^2 d\theta^2 \end{aligned} \tag{3.43}$$

where $\Delta = r^2 - 2Mr + a^2, \rho = r^2 + a^2\cos(\theta)^2$ and a parametrizes the rotation speed

(e) Friedmann-Lemaitre-Robertson-Walker (FLWR) metric of an isotropic, homogenous, expanding universe

$$ds^2 = -dt^2 - a(t)^2\left[\frac{1}{1-kr^2}dr^2 + r^2\left(d\theta^2 + \sin(\theta)^2 d\phi^2\right)\right] \tag{3.44}$$

As we can see, there are not too many analytical solutions. This is because the Einstein equation forms a coupled system of 10 non-linear, second-order partial differential equations, which makes them very hard to solve in general (even numerically).

3.3.6 Existence of Gravitational Waves

gravitational wave = GW

3.3.7 Effect of Gravitational Waves

Gravitational waves are small perturbations of spacetime, so they should have a measurable effect. To find this effect, we will now look at a particles in the TT gauge. We assume it to be at some position x^α and at rest at $t = 0$, that is

$$U^\alpha\bigg|_{t=0} = \frac{dx^\alpha}{d\tau}\bigg|_{t=0} = (1, 0, 0, 0)\,.$$

Evaluating the geodesic equation at $t = 0$ yields

$$\begin{aligned}\frac{dU^\alpha}{d\tau} = \frac{d^2x^\alpha}{d\tau^2} &= -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\Gamma^\alpha{}_{tt} \\ &= -\frac{1}{2}g^{\alpha\mu}\left(2\partial_t g_{t\mu} - \partial_\mu g_{tt}\right) = 0\,.\end{aligned} \tag{3.45}$$

The four-velocity remains constant, so the coordinates of the particle do not change! That, however, does not mean GWs have no effect – after all, coordinates have no invariant meaning anyway. Instead, we have to look at physically meaningful quantities like the proper distance between particles which is related to the metric and thus may be affected by a GW.

To simplify calculations, we will now assume a GW propagating in z-direction, with linear polarization and $h_+ = h_+(t-z), h_\times = 0$. The proper distance between two particles which are initially separated by L in x-direction then becomes

$$L_x(t) = \int_0^L \sqrt{g_{xx}dxdx} = \int_0^L \sqrt{1+h_+}\,dx \approx \int_0^L \left(1 + \frac{h_+}{2}\right)dx = L\left(1 + \frac{h_+(t)}{2}\right) \tag{3.46}$$

assuming that the wavelength is much longer than L (whence the amplitude h_+ does not change much during the propagation from $x = 0$ to $x = L$). Therefore, a GW causes a (time-dependent) relative length change

$$\frac{\Delta L_x}{L} = \frac{L_x - L}{L} \approx \frac{h_+(t)}{2} \tag{3.47}$$

which is often called *strain*.

Another way to quantify the effect of a GW is to look at geodesic deviation. In a LIF, we recall that the corresponding equation (3.26) reads

$$\frac{d^2\xi^\alpha}{d\tau^2} = -R^\alpha{}_{\tau\mu\tau}\xi^\mu \, .$$

It is possible to calculate the relevant components of the Riemann tensor in TT gauge:

$$R^x{}_{\tau x\tau} = -\frac{1}{2}\frac{\partial^2 h_+}{\partial\tau^2} \qquad R^y{}_{\tau x\tau} = -\frac{1}{2}\frac{\partial^2 h_\times}{\partial\tau^2} \qquad R^y{}_{\tau y\tau} = \frac{1}{2}\frac{\partial^2 h_+}{\partial\tau^2} \, . \tag{3.48}$$

Assuming an initial separation in x-direction again, i.e. $\underline{\xi} = (0, \xi, 0, 0)$, we obtain

$$\frac{d^2\xi^x}{d\tau^2} = \frac{\xi}{2}\frac{\partial^2 h_+}{\partial\tau^2} \qquad \frac{d^2\xi^y}{d\tau^2} = \frac{\xi}{2}\frac{\partial^2 h_\times}{\partial\tau^2} \, . \tag{3.49}$$

For an initial separation in y-direction, $\underline{\xi} = (0, 0, \xi, 0)$, the roles are reversed:

$$\frac{d^2\xi^x}{d\tau^2} = \frac{\xi}{2}\frac{\partial^2 h_\times}{\partial\tau^2} \qquad \frac{d^2\xi^y}{d\tau^2} = -\frac{\xi}{2}\frac{\partial^2 h_+}{\partial\tau^2} \, . \tag{3.50}$$

Consequently, GWs exert a force which changes the proper distance between particles, i.e. they stretch and squeeze spacetime between particles (not only in a LIF). Moreover, we see that the two polarizations have an equivalent effect, they are just rotated against each other.