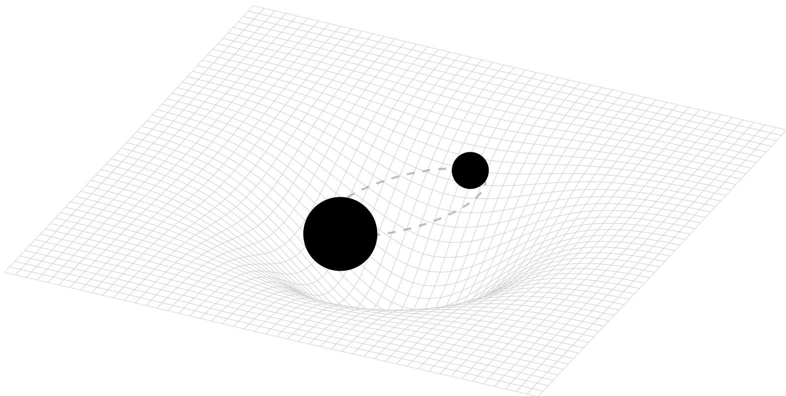
Relativity

A Summary Of What I Have Learned About It



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Code for spacetime diagrams is inspired by the one presented in https://de.overleaf.com/latex/templates/minkowski-spacetime-diagram-generator/kqskfzgkjrvq

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Lecture information by Giulini at https://www.itp.uni-hannover.de/de/ag/giulini/lehre/ws2122-foundations-and-applications-of-special-relativity/, where script is linked as well (although link to script did not work last time I checked, unfortunately). Also link to Dragon script there, which is pretty much the same as book (just formatting different)

Lecture by Frank Ohme with accompanying notes

I can really recommend reading what Einstein himself wrote about his theories; in particular, the original paper "On the electrodynamics of moving bodies" and his Nobel lecture are definitely worth the read (or rather multiple reads)

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Acronyms

SR special relativity

GR general relativity

ICCs inertial Cartesian coordinates

MCIF momentarily comoving inertial frame

LIF local inertial frame

Mathematical Notation

a thin letters (lower and upper case) are used to denote scalar values and variables with scalar values

$\vec{a}$ thin letters with arrows are used to denote three-vectors. Their components have latin indices, e.g. a^k

$\underline{a}$ thin, underlined letters are used to denote four-vectors. Their components have greek indices, e.g. a^μ

$\mathbf{A}$ bold, capital letters are used to denote more general tensors. Their components have greek indices, e.g. A^μ

$a^\mu b_\mu$ the Einstein summation convention is adopted, where repeated indices denote sums:

$$a^\mu b_\mu = \sum_\mu a^\mu b_\mu$$

$\equiv$ this symbol means something like "corresponds to" or "is equivalent"

$:=$ this symbol is used to define new symbols and names

$\simeq$ this symbol is used to indicate Taylor expansions

1 Mathematics

Mathematically speaking, the theory of relativity is a geometrical one. While most of the more involved mathematics can be avoided when dealing with special relativity (in the sense that they are not strictly necessary to understand it), general relativity is founded on ideas from (Riemannian) geometry. For this reason, a certain amount of knowledge in this area is required in order to understand it. However, even for special relativity, learning how some of the physical concepts are realized in mathematical terms can be enlightening.

Consequently, dealing with mathematics is inevitable. We will do it as a preface before the physics chapters. It has to be noted that the topics covered here can easily fill two full courses and perhaps, this amount of time is also needed to be able to fully comprehend all the definitions and concepts. It is virtually impossible to provide an adequate, thorough treatment here since this a physics-oriented summary. Therefore, not all things can be treated in depth and proofs that show certain properties (for example that notions are well-defined) are left out – some corners have to be cut, unfortunately.

1.1 Basics of Manifolds

remark: in principle, one has to distinguish between maps from the manifold to some other space and maps from Euclidean space to some other space, which would use inverse charts first. For us, however, these distinctions are not too important because points in spacetime come from $\mathbb{R}^4$, the main differences occur for other structures like the metric. At least in special relativity. -¿ but wait, isn't that true for all manifolds? Perhaps not, but why not? For a manifold of dimension n, we can always take all points from $\mathbb{R}^n$... Ahhh, maybe difference is global chart for SR? Identity even

to describe curved spacetime, we need a coordinate-independent notion of spaces; this is given by manifolds, which are described using coordinates but have an independent, invariant meaning (specific choice of coordinates does not matter for its properties); similarly, they can often be pictured as being embedded in some higher-dimensional Euclidian space, but that need not be the case

therefore, physics happens on *manifolds* M, so events are points on them and so on; a manifold is a collection of points which locally looks like $\mathbb{R}^n$ (in so-called *charts* which map from parts of the manifold to coordinates in Euclidian space)

remark: We will restrict ourselves to smooth manifolds, where charts are smooth.

1.1.1 Tangent Vectors

defining vectors on manifolds is a non-trivial topic, they are now completely distinct notion from points and cannot be visualized as pointing from some origin to this point (problem: thinking of an embedded manifold for now, the vectors would point out of the manifold); instead, we can define vectors locally (!) via derivatives of curves (i.e. as *tangent vectors*; the corresponding set of all tangent vectors is called *tangent space* V)

to describe tangent vectors $\underline{t}$ more explicitly, we have to expand them with respect to some basis and a convenient choice for this is the set of partial derivatives along the coordinate axes of a chart x^α, $\{\frac{\partial}{\partial x^\alpha}\}_\alpha$ (note: $\frac{\partial}{\partial x^\alpha}$ is often abbreviated as ∂_α); this choice yields

$$\underline{t} = t^\alpha \underline{e_\alpha} = \frac{dx^\alpha}{d\sigma}\frac{\partial}{\partial x^\alpha} = \frac{d}{d\sigma} \tag{1.1}$$

where $\sigma \in \mathbb{R}$ is the parameter of the curve that $\underline{t}$ is tangent to; that means we can also think of $\underline{t}$ as a tuple of components t^α

in this representation vector $\leftrightarrow$ derivative, the derivative of a function $f(x^\alpha)$ along γ is (note that we consider a function and curve on the coordinate space, not the manifold itself):

$$\frac{df}{d\sigma} = \frac{df(x^\alpha(\sigma))}{d\sigma} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\sigma} =: \underline{t} \cdot f\,, \tag{1.2}$$

which is an application of the chain rule and justifies the particular form of vector components in (1.1) (because we want to identify directional derivatives like the one in (1.2) with our new vector notion, the tangent vector)

tangent vectors $\underline{t}$ are invariant quantities, they do not (and should not!) depend on the coordinates we use to express them; their components, on the other hand, are *not* invariant (because the basis vectors change, so must the components for the whole object $\underline{t}$ to stay the same); they obey the following transformation law:

$$\underline{t} = t^\alpha \frac{\partial}{\partial x^\alpha} = t^\alpha \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial}{\partial x'^\beta} \stackrel{!}{=} t'^\beta \frac{\partial}{\partial x'^\beta} \qquad \Leftrightarrow \qquad \boxed{t'^\beta = t^\alpha \frac{\partial x'^\beta}{\partial x^\alpha}} \tag{1.3}$$

once again, this is basically just an application of the chain rule

1.1.2 Differential Equations

state it with vector field stuff -¿ or make separate bundle section?

1.1.3 One Forms

next natural step: linear maps on tangent space V; these are called *covectors* or *one forms* (elements of the dual space or *cotangent space* V^*) and it turns out that we can identify them with differentials/gradients of functions

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha \tag{1.4}$$

where we chose a convenient basis $\{\underline{e}^\alpha\}_\alpha = \{dx^\alpha\}_\alpha$ of the dual vector space, satisfying

$$\boxed{dx^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta} \,. \tag{1.5}$$

More generally, covectors $w : V \to \mathbb{R}$ obey

$$w\left(\alpha\underline{a} + \beta\underline{b}\right) = \alpha w\left(\underline{a}\right) + \beta w\left(\underline{b}\right), \qquad \forall a, b \in \mathbb{R},\; \underline{a}, \underline{b} \in V\,. \tag{1.6}$$

We can also characterize them via tuples of components

$$w_\alpha = w\left(\underline{e_\alpha}\right) = w(\partial_\alpha) = \partial_\alpha w\,. \tag{1.7}$$

In general, that implies we can write

$$w\left(\underline{a}\right) = w_\alpha \underline{e^\alpha}\left(a^\beta \underline{e_\beta}\right) = w_\alpha a^\alpha \,. \tag{1.8}$$

To see how covector components in different coordinates are related, we look at the following inner product (which is also invariant):

$$w\left(\underline{t}\right) = w_\alpha t^\alpha \stackrel{!}{=} w'_\beta t'^\beta = w'_\beta \frac{\partial x'^\beta}{\partial x^\alpha} t^\alpha \qquad \Leftrightarrow \qquad \boxed{w'_\beta = w_\alpha \frac{\partial x^\alpha}{\partial x'^\beta}} \,. \tag{1.9}$$

1.1.4 Tensors

We have seen how covectors are maps from V to the real numbers. Similarly, one can show that there is a unique identification between vectors from V and maps from the dual space V^* to the real numbers – vectors are also maps. It is possible to generalize this concept to coordinate-independent entities which map multiple vectors, covectors or mixes of them to the real numbers. Linear maps

$$T : V^n \times (V^*)^m = V \times \ldots \times V \times V^* \times \ldots V^* \to \mathbb{R} \tag{1.10}$$

are called *tensors* of rank $n+m$. Due to their invariance under coordinate-transformations, every physical quantity has to be expressed as a tensor (this follows from the requirements of special relativity).

Just like vectors can be collected in components $t^\alpha = \underline{t} \cdot x^\alpha = \partial_\sigma x^\alpha$ and covectors in components $w_\alpha = w(\partial_\alpha)$, we can characterize a tensor of rank $n+m$ using components

$$\boxed{T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} = T\left(\underline{e_{\beta_1}}, \ldots, \underline{e_{\beta_n}}, \underline{e^{\alpha_1}}, \ldots, \underline{e^{\alpha_m}}\right)} \,, \tag{1.11}$$

remark: it is no typo that there are m upper and n lower indices. This reflects the fact that a tensor of rank $m+n$ can map m covectors with its m "vectorial" indices and n vectors with its n "covectorial" indices. This is just like $w_\mu = w(\partial_\mu)$, a vector is needed to get the covector component (and vice versa).

which also means that tensors are linear combinations of products of vectors and covectors,

$$\boxed{T = T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} \, \underline{e_{\alpha_1}} \ldots \underline{e_{\alpha_m}} \underline{e^{\beta_1}} \ldots \underline{e^{\beta_n}}} \,. \tag{1.12}$$

remark: perhaps, it is a bad idea, but tensor products $\otimes$ between the $\underline{e}_\alpha, \underline{e}^\beta$ are omitted.

These components do change under coordinate transformations. The corresponding behaviour can be derived from the ones for vectors (1.3) and covectors (1.9),

$$\boxed{T'^{\alpha\beta\dots}{}_{\gamma\delta\dots} = T^{\mu\nu\dots}{}_{\lambda\sigma\dots} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \dots \frac{\partial x^\lambda}{\partial x'^\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \dots} \,. \tag{1.13}$$

This is the important *tensor transformation law*.

The rank of a tensor can be reduced if we insert a fixed object into one of the "slots", i.e. in the example of a rank-4-tensor

$$T(\cdot,\cdot,\cdot,\cdot) \to T'(\cdot,\cdot,\cdot) = T(\underline{t},\cdot,\cdot,\cdot) \qquad T^{\alpha\beta}{}_{\gamma\delta} \to T'^{\alpha\beta}{}_{\delta} = T^{\alpha\beta}{}_{\gamma\delta}\, t^\gamma \tag{1.14}$$

or

$$T(\cdot,\cdot,\cdot,\cdot) \to T'(\cdot,\cdot,\cdot) = T(\cdot,\cdot,w,\cdot) \qquad T^{\alpha\beta}{}_{\gamma\delta} \to T'^{\beta}{}_{\gamma\delta} = T^{\alpha\beta}{}_{\gamma\delta}\, w_\alpha \,. \tag{1.15}$$

These operations are called *contraction*.

remark: might be inconsistent to write components like this because vectorial indices come first but the first arguments in T are vectorial too (which connect to covectorial index).

Example 1.1: Known Tensors

We have already encountered several examples of tensors. Vectors and covectors are rank-1-tensors, which should not be surprising because we used them to derive general tensors. However, scalars are also tensors, namely of rank 0 – they can be thought of as mapping the real numbers to themselves without taking any further arguments.

remark: here, we do not distinguish explicitly between tensors and tensor fields, which are maps from the manifold to the space of tensors of a certain rank (they assign a tensor to each point). For example, we change fluently between vectors and vector fields or scalars and (scalar) functions.

Another example of a tensor, which plays a great role in geometry on manifolds and thus – as we will see later – also in physics, is the *metric*. The following properties can be used as a definition for this 2-tensor:

(1.) The metric is symmetric. (2.) The metric is non-degenerate.

Together with the usual requirements for a tensor, like linearity, these properties are sufficient to define a (pseudo)metric. In case of spacetime, this "pseudo" property is characterized by the fact that the metric has three positive eigenvalues and one negative.

Just like any other tensor, metrics can be characterized by their components. These we can also read off from the *line element* ($\equiv$ infinitesimal distance between points)

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu := g_{\mu\nu}\, dx^\mu \otimes dx^\nu} \tag{1.16}$$

which is a frequently used tool in geometry (for example to measure lengths). Often, one thinks of the covectors dx^μ in this expression as infinitesimal changes in the coordinate x^μ and of the corresponding component a in adx^μ as the effect of this change. This is justified by the fact that $adx^\mu(\partial_\nu) = a\delta^\mu_\nu$, the coefficient a of dx^μ indeed already contains all information from ds^2 about the behaviour/change in direction ∂_μ.

Metrics can be used to define inner products, which are not natively present on manifolds,[1] in the following manner:

$$\underline{A} \cdot \underline{B} := g\left(\underline{A}, \underline{B}\right) = g_{\mu\nu} A^\mu B^\nu \,. \tag{1.17}$$

Just like before, the components are obtained via evaluation in basis vectors,

$$\boxed{g_{\mu\nu} = g(\underline{e_\mu}, \underline{e_\nu})}\,. \tag{1.18}$$

Inner products shall be symmetric, i.e. $\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}$, so

$$g_{\mu\nu} = g_{\nu\mu} \qquad\qquad g\left(\underline{A}, \underline{B}\right) = g_{\mu\nu} A^\mu B^\nu = g_{\nu\mu} A^\nu B^\mu = g\left(\underline{A}, \underline{B}\right). \tag{1.19}$$

The metric provides us with a natural identification between vectors and covectors because $g\left(\cdot, \underline{A}\right)$ is nothing but a map which takes a vector and maps it to a real number – which is the definition of a covector. Similarly, we can identify covectors w with the unique vector $\underline{A}$ that fulfils $w\left(\underline{B}\right) = g\left(\underline{A}, \underline{B}\right),\ \forall \underline{B} \in V$. In components, these requirements read

$$\boxed{A_\mu = g_{\mu\nu} A^\nu} \qquad\qquad \boxed{A^\mu = g^{\mu\nu} A_\nu}\,. \tag{1.20}$$

Here, $g^{\mu\nu}$ denote the components of the inverse metric, which is defined by

$$\boxed{g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu}\,. \tag{1.21}$$

Apparently, it is almost trivial to change from vectors to covectors and vice versa in this component notation. For this reason, the strict distinction between A^μ and A_μ is often dropped

[1] Non-tensorial quantities might be not invariant and thus ill-defined. At the same time, the properties do not fix a tensor uniquely, so one has to be specified (no native metric exists).

(at least for interpretation purposes). This transfers to tensors of arbitrary rank. Additionally, we can write

$$\boxed{\underline{A} \cdot \underline{B} = A^\mu B_\mu = A_\mu B^\mu} \,. \tag{1.22}$$

1.2 Riemannian Geometry

1.2.1 Metric

do here and not in tensors section?

1.2.2 Covariant Derivative

It is a natural question – especially in physics – to ask for the change of a quantity. The tool we know until now to answer this question is the differential df of a function. In coordinates x^α, it can be expressed as

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha \equiv \{\partial_\alpha f\}_\alpha \, ,$$

i.e. essentially using partial derivatives ∂_α. Similarly, derivatives of vectors are $\partial_\alpha V^\beta$. That, however, is a problem because the components are not tensors, so the result of this quantity is not invariant. The reason is simple: the basis vectors also change, but $\partial_\alpha V^\beta$ does not account for that. Hence, we need a new operator to take derivatives of tensors like $\underline{V}$.

This new operator will be called *covariant derivative* ∇ and we want it to be a map from tensors (tensor fields) to tensors (tensor fields). We can define it by demanding

(1.) $\nabla f = df$ for functions f

(2.) ∇ fulfils a product rule for tensors: $\nabla(ST) = S\nabla T + (\nabla S)T$

(3.) ∇ is compatible with contraction, that is we can interchange these operations

Combining these properties with the usual requirement

$$\boxed{\nabla_{\underline{V}} = \nabla_{V^\alpha \partial_\alpha} \overset{!}{=} V^\alpha \nabla_{\partial_\alpha} =: V^\alpha \nabla_\alpha} \tag{1.23}$$

for tensors[2] is already sufficient for a well-defined operator. From that, we can derive what the components of the result are. We will start by looking at the action of ∇ on a vector:

$$\nabla_\alpha \underline{V} = \nabla_\alpha V^\beta \underline{e_\beta} = \left(\nabla_\alpha V^\beta\right) \underline{e_\beta} + V^\beta \nabla_\alpha \underline{e_\beta} = \left(\partial_\alpha V^\gamma + V^\beta \Gamma^\gamma{}_{\alpha\beta}\right) \underline{e_\gamma} =: \left(\nabla_\alpha \underline{V}\right)^\gamma \underline{e_\gamma} \, . \tag{1.24}$$

remark: sometimes, $(\nabla_\alpha \underline{V})^\gamma$ is written as $\nabla_\alpha V^\gamma$. In my opinion, this is misleading since the components V^γ are functions, so ∇_α would act like d and thus essentially ∂_α (it is the whole point of a covariant derivative, that its β-component is *not* the derivative of the β-component

[2] This is not linearity in the "classical" sense (which follows from the property (2.), interpreting scalars as 0-tensors and using $\nabla a = 0$), but rather C^∞-linearity for functions in the lower argument of ∇.

of the vector). Nonetheless, I still wanted to mention this.

In the third step, we have simply used that ∇ maps vectors to vectors and expressed the result $\nabla_\alpha \underline{e_\beta}$ in terms of the corresponding coefficients

$$\boxed{\Gamma^\gamma{}_{\alpha\beta} := \left(\nabla_\alpha \underline{e_\beta}\right)^\gamma} \tag{1.25}$$

(where indices α, β have been added to them to clarify what these components belong to). These coefficients are called *Christoffel symbols* and they do *not* constitute a tensor. Based on these considerations, we can also give components for the object $\nabla_{\underline{W}} \underline{V}$:

$$\nabla_{\underline{W}} \underline{V} = W^\alpha \nabla_\alpha \underline{V} = W^\alpha \left(\partial_\alpha V^\gamma + V^\beta \Gamma^\gamma{}_{\alpha\beta}\right) \underline{e_\gamma} = W^\alpha (\nabla_\alpha \underline{V})^\gamma \underline{e_\gamma} = W^\alpha \left(\nabla_{\underline{W}} \underline{V}\right)^\gamma_\alpha \underline{e_\gamma} \,. \tag{1.26}$$

To see how the covariant derivative of a covector looks like, we use a trick similar to the one used when the transformation laws were derived:

$$\begin{aligned}
\nabla_\alpha \left(V^\beta w_\beta\right) &= \partial_\alpha \left(V^\beta w_\beta\right) = V^\beta \partial_\alpha w_\beta + w_\beta \partial_\alpha V^\beta \\
&= V^\beta \partial_\alpha w_\beta - w_\beta V^\gamma \Gamma^\beta{}_{\alpha\gamma} + w_\beta \partial_\alpha V^\beta + w_\beta V^\gamma \Gamma^\beta{}_{\alpha\gamma} \\
&= V^\beta \left(\partial_\alpha w_\beta - w_\gamma \Gamma^\gamma{}_{\alpha\beta}\right) + w_\beta \nabla_\alpha V^\beta \\
&\overset{!}{=} \nabla_\alpha (w(\underline{V})) = (\nabla_\alpha w)(\underline{V}) + w(\nabla_\alpha \underline{V}) \\
&= V^\beta (\nabla_\alpha w)_\beta + (\nabla_\alpha V)^\beta w_\beta \,.
\end{aligned}$$

Therefore, the action of ∇_α on vectors and covectors is:

$$\boxed{(\nabla_\alpha V)^\beta = \partial_\alpha V^\beta + V^\gamma \Gamma^\beta{}_{\alpha\gamma}} \qquad \boxed{(\nabla_\alpha w)_\beta = \partial_\alpha w_\beta - w_\gamma \Gamma^\gamma{}_{\alpha\beta}} \,. \tag{1.27}$$

From that, we also get the action on general tensors because according to (1.12) they are nothing but products of vectors and covectors. Thus, applying the product rule yields:

$$\boxed{\begin{aligned}(\nabla_\sigma T)^{\alpha\beta\dots}{}_{\gamma\delta\dots} = \partial_\sigma T^{\alpha\beta\dots}{}_{\gamma\delta\dots} &+ \Gamma^\alpha{}_{\sigma\nu} T^{\nu\beta\dots}{}_{\beta\delta\dots} + \Gamma^\beta{}_{\sigma\nu} T^{\alpha\nu\dots}{}_{\gamma\delta\dots} \\ &- \Gamma^\nu{}_{\sigma\gamma} T^{\alpha\beta\dots}{}_{\nu\delta\dots} - \Gamma^\nu{}_{\sigma\delta} T^{\alpha\beta\dots}{}_{\gamma\nu\dots} \pm \dots\end{aligned}} \,. \tag{1.28}$$

While it is possible to calculate the Christoffel symbols according to (1.25), this only works if we know how to calculate changes of the basis vectors $\underline{e_\alpha}$ using ∇. That is not very convenient, after all we might wish to define ∇ using them, so we will now show an alternative way to do it. The whole idea is that coordinate axes in flat space do not change direction such that applying ∇_α reduces to application of ∂_α. However, we can relax the requirements from flat space to "almost flat"/"locally flat" space in certain charts. These charts are called *Riemannian normal coordinates* or, more common in physics, *local inertial frames* (LIFs) and their characteristic properties are

$$\Gamma^{\gamma}{}_{\alpha\beta} = 0 \qquad\qquad \nabla_\alpha = \partial_\alpha \,. \tag{1.29}$$

Moreover, flat ↔ constant metric, so metric and covariant derivative are compatible,

$$\partial_\gamma g_{\alpha\beta} = 0 = \nabla_\gamma g_{\alpha\beta} \,, \tag{1.30}$$

in a LIF. However, we have written it as a tensorial equation, so it is valid in any frame. In a similar manner,

$$\Gamma^{\gamma}{}_{\alpha\beta} = \Gamma^{\gamma}{}_{\beta\alpha} \qquad \Leftrightarrow \qquad \Gamma^{\gamma}{}_{\alpha\beta}\partial_\gamma = \Gamma^{\gamma}{}_{\beta\alpha}\partial_\gamma \qquad \Leftrightarrow \qquad \nabla_\alpha \underline{e_\beta} = \nabla_\beta \underline{e_\alpha} \,, \tag{1.31}$$

which clearly holds in a LIF due to $\Gamma^{\gamma}{}_{\alpha\beta} = 0$, can be generalized to arbitrary coordinate systems. These two properties can be combined to show that

$$\boxed{\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}\right)} \,. \tag{1.32}$$

To end this introduction to covariant derivatives, it is very important to mention that we deal with a special type of them, the *Levi-Civita connection* (connection is another name for covariant derivative). This is because a physical requirement[3] is $\nabla_\gamma g_{\alpha\beta} = 0$ and $\Gamma^{\gamma}{}_{\alpha\beta} = \Gamma^{\gamma}{}_{\beta\alpha}$, which also characterizes a Levi-Civita connection. This very special type of connection can be uniquely derived from the metric $g_{\alpha\beta}$ because it is essentially partial derivative plus Christoffel symbols and those are determined by the metric, as (1.32) shows. More arbitrary connections are *not* unique for a given manifold, so this is indeed a special property of the Levi-Civita connection.

1.2.3 Curvature

Here, we will collect properties of the Riemann tensor as defined in (**??**) via

$$R^{\beta}{}_{\gamma\delta\epsilon} = \frac{\partial \Gamma^{\beta}{}_{\gamma\epsilon}}{\partial x^\delta} - \frac{\partial \Gamma^{\beta}{}_{\gamma\delta}}{\partial x^\epsilon} + \Gamma^{\beta}{}_{\delta\mu}\Gamma^{\mu}{}_{\gamma\epsilon} - \Gamma^{\beta}{}_{\epsilon\mu}\Gamma^{\mu}{}_{\gamma\delta} \,.$$

Besides (**??**), which is helpful to interpret the action of the Riemann tensor, it has several helpful properties to use in calculations and some of those we will list here. In a LIF

$$\boxed{R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\gamma\partial_\beta g_{\alpha\delta} - \partial_\gamma\partial_\alpha g_{\beta\delta} - \partial_\delta\partial_\beta g_{\alpha\gamma} + \partial_\delta\partial_\alpha g_{\beta\gamma}\right)} \,. \tag{1.33}$$

[3] In LIFs and thus globally. Comes from the Einstein Equivalence Principle 4.4.

Using this formula rather than the general one simplifies proving the following properties:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \qquad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{1.34}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \qquad R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0\,. \tag{1.35}$$

These properties reduce the number of independent components in case of a 4-dimensional manifold like spacetime to 20 (= number of non-zero second derivatives of $g_{\alpha\beta}$, which makes sense because R is a linear combination of them). The *Bianchi identity* also plays a very important role:

$$\boxed{\nabla_\mu R_{\alpha\beta\gamma\delta} + \nabla_\gamma R_{\alpha\beta\delta\mu} + \nabla_\delta R_{\alpha\beta\mu\gamma} = 0}\,. \tag{1.36}$$

Local Inertial Frames To end this subsection, we will make some remarks connecting curvature to LIFs as defined by (4.14). Due to the vanishing of the first derivatives of the metric,

$$\boxed{\Gamma^\gamma{}_{\alpha\beta}(p) = 0} \tag{1.37}$$

in a LIF. However, that does not imply vanishing of the Riemann tensor because the second derivatives $\partial_\alpha \partial_\beta g_{\gamma\delta}$ do *not* necessarily vanish. Hence, the Riemann tensor is not zero in general (same derivatives of the Christoffel symbols). This also marks a difference between LIFs and cartesian coordinates (= flat space), where the coordinate axes are constant and Christoffel symbols as well as Riemann tensor are identically zero.

1.3 Notes & Thoughts

to be able to develop an appropriate/meaningful notion of parallelism, we need a "better" derivative. This will be provided by a connection

1.3.1 Giulini GR lectures May 19 and 26

in Riemann normal coordinates, all Christoffel symbols vanish; but they only exist in neighbourhoods around points p (are Riemann normal coordinates *at* p); they are *very* helpful in calculations because tensor equations only have to be proven in a single coordinate system, which we can choose to be Riemann normal coordinates because we have just shown that they do exist

uhh, Christoffel symbols satisfy an affine transformation law, not linear (indicator of not tensorial) because there is term without $\Gamma^{\sigma}_{\mu\nu}$

formel of Koszul only holds like this for torsion-free and metric connections; and by substituting $X = e_\alpha, Y = e_\beta, Z = e_\gamma$ into it and then contracting with certain component of inverse metric gives rise to formula for connection coefficients (commonly called Christoffel symbols for Levi-Civita connection) and these uniquely determine the connection (which is proof for uniqueness); note that connection coefficients (= covariant derivative with only basis vectors) already determines the connection because connection is $\mathbb{R}$-linear, tensorial (C^∞-linear) and obeys the Leibniz rule

$\nabla_X Y = X^\alpha (\nabla_\alpha Y^\beta) \frac{\partial}{\partial x^\beta}$, so components of $\nabla_X Y$ are partial derivatives of components plus extra term, i.e. $\nabla_{\frac{\partial}{\partial x^\alpha}} Y =: \left(\nabla_\alpha Y^\beta\right) \frac{\partial}{\partial x^\alpha}$; however, they to read this is *not* covariant derivative of Y^β since this would be the covariant derivative of a function, but instead as the β-component of the covariant derivative $\nabla_\alpha Y$

here it is, reason why covariant derivative of covector (with respect to some vector field X) looks the way it looks:

$$\begin{aligned}
\nabla_X \omega &= \nabla_{X^\alpha \frac{\partial}{\partial x^\alpha}} \left(\omega_\beta dx^\beta\right) \\
&= X^\alpha \left(dx^\beta \nabla_{\frac{\partial}{\partial x^\alpha}} \omega_\beta + \omega_\beta \nabla_{\frac{\partial}{\partial x^\alpha}} dx^\beta\right) \\
&= X^\alpha \frac{\partial \omega_\beta}{\partial x^\alpha} dx^\beta
\end{aligned}$$

but $dx^\beta\left(\frac{\partial}{\partial x^\alpha}\right) = \delta^\beta_\alpha$, so by taking the derivative of this equation we see that ... $\left(\nabla_{\frac{\partial}{\partial x^\gamma}} dx^\beta\right)\left(\frac{\partial}{\partial x^\alpha}\right) = \left(\nabla_{\frac{\partial}{\partial x^\gamma}} dx^\beta\right)^\alpha = -\Gamma^\beta_{\gamma\alpha}$

from that we get general formula for tensors of arbitrary rank because of Leibniz rule; therefore, we get the "master formula"

$$\begin{aligned}\nabla_X T &= T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}\frac{\partial}{\partial x^{\alpha_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\alpha_k}}\otimes dx^{\beta_1}\otimes\cdots\otimes dx^{\beta_m}\\ &= X^\gamma\left(\nabla_\gamma T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}\right)\frac{\partial}{\partial x^{\alpha_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\alpha_k}}\otimes dx^{\beta_1}\otimes\cdots\otimes dx^{\beta_m}\\ \nabla_\gamma T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} &= \left(\frac{\partial T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}}{\partial x^\gamma}+\sum_{i=1}^k\Gamma^{\alpha_i}_{\gamma\lambda}T^{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\alpha_k}_{\beta_1\ldots\beta_m}-\sum_{j=1}^m\Gamma^\lambda_{\gamma\beta_j}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{j-1}\lambda\beta_{j+1}\ldots\beta_m}\right)\end{aligned}\tag{1.38}$$

there is also another notion of derivative on manifolds, the Lie derivative; to define it, we do not need any additional structure (unlike for connection, connection coefficients need metric); the Lie derivative can often be used to express symmetries; components of it are

$$(\mathcal{L}_X T)^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} = X^\alpha\frac{\partial}{\partial x^\alpha}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}-\sum_{i=1}^k\frac{\partial X^{\alpha_i}}{\partial x^\lambda}T^{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_k}_{\beta_1\ldots\beta_m}+\sum_{j=1}^m\frac{\partial X^\lambda}{\partial x^{\beta_i}}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{i-1}\lambda\beta_{i+1}\ldots\beta_m}\tag{1.39}$$

we notice: for upper index we now have minus, lower index has plus (reversed compared to connection); interesting property: all partial derivatives could be replaced by covariant derivatives without changing the formula (despite them being defined independently of each other!); if a certain vector field defines a symmetry, i.e. the metric does not change under the flow of that vector field (stays constant along it/integral curves defined by it), then we can express that as the vanishing of the Lie-derivative of this symmetry-generating vector field; these vector fields are called Killing fields; note that $(\mathcal{L}_X g)_{\alpha\beta} = (\nabla_\alpha X^\gamma)g_{\gamma\beta}+\left(\nabla_\beta X^\gamma\right)g_{\alpha\gamma} = \nabla_\alpha X_\beta+\nabla_\beta X_\alpha$ (we just use writing in terms of covariant derivative for first equality) -¿ shouldn't that be equal to $\nabla_{[\alpha}X_{\beta]}$; would also explain $\mathcal{L} = \mathrm{Alt}(\nabla)$ statement I heard; ah no, this is the *symmetrized* part... But maybe that supports view, symmetric part is zero for Killing field (but this has vanishing Lie derivative, so antisymmetric part also zero, right?)

very interesting: Einstein tensor is divergence-free, i.e. $\nabla_\alpha G^{\alpha\beta} = 0$

we know that, in general, $\left[\nabla_\mu,\nabla_\nu\right]T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m}\neq 0$; expanding this quantity for an arbitrary vector field X^α, $\left[\nabla_\mu,\nabla_\nu\right]X^\alpha = \cdots = \left(\frac{\partial}{\partial x^\mu}\Gamma^\alpha_{\nu\beta}-\frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\beta}\right)X^\beta +$ terms proportional to $\Gamma = R^\alpha_{\beta\mu\nu}X^\beta +$ terms proportional to Γ; in Riemann-normal coordinates, $\Gamma = 0$ and $\left[\nabla_\mu,\nabla_\nu\right]X^\alpha = R^\alpha_{\beta\mu\nu}X^\beta$ and since both sides are tensors, this equation holds in general; curvature is related to (Giulini said "obstruction") commutivity of second derivatives; furthermore, pulling down the index α yields $\left[\nabla_\mu,\nabla_\nu\right]X_\alpha = -R^\beta_{\alpha\mu\nu}X_\beta$, which tells us how this quantity acts on a covector (again, has on other sign and acts on other index, like it was for covariant derivative itself); thus, we get the general formula $\left[\nabla_\mu,\nabla_\nu\right]T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_m} = \sum_{i=1}^k R^{\alpha_i}_{\lambda\mu\nu}T^{\alpha_1\ldots\alpha_{i-1}\lambda\alpha_{i+1}\ldots\alpha_k}_{\beta_1\ldots\beta_m}-\sum_{j=1}^m R^\lambda_{\beta_j\mu\nu}T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{i-1}\lambda\beta_{i+1}\ldots\beta_m}$

application of that formula: $\nabla_\mu\nabla_\nu X_\beta = R^\alpha_{\mu\nu\beta}X_\alpha$ holds for any Killing vector field X; second derivatives are determined by vector field itself; similarly, $\frac{\partial}{\partial x^\alpha}X_\beta+\frac{\partial}{\partial x^\beta}X_\alpha = 2\Gamma^\gamma_{\alpha\beta}X_\gamma$, the symmetric part of first derivative of Killing vector field is determined by field itself as well; the only free parameters are value of the field itself and anti-symmetric part $\frac{\partial}{\partial x^\alpha} X_\beta - \frac{\partial}{\partial x^\beta} X_\alpha$ of first derivative (all derivatives of higher order are determined by relation to curvature tensor); solutions of linear differential equations (no matter of partial or not) constitute a vector space because we can add them together and multiply with numbers and maximum number of dimensions is given by number of freely specifiable initial conditions; here, these are values $X^\alpha\big|_p$ of Killing field at a specific point, i.e. $n = \dim(M)$, and $\frac{\partial}{\partial x^\alpha} X_\beta - \frac{\partial}{\partial x^\beta} X_\alpha\Big|_p$, i.e. $\frac{1}{2}n(n+1)$; in total, that means there are at most $n + \frac{1}{2}n(n+1)$ independent solutions to the Killing equation; in Minkowski space there are indeed 10, in that sense it is maximally symmetric (these generate symmetries of the space, which are given by Poincare group)

Riemann tensor has 20 components and there are 10 different traces (because of antisymmetry in first two indices, which means trace is always zero); our goal is now decomposing it into trace and traceless parts, both contain information about Riemann tensor; the trace part is nothing but the Ricci tensor $R_{\alpha\beta} = R^\lambda_{\ \alpha\lambda\beta}$, the "rest" (trace-free part) is the Weyl tensor (which he writes down in terms of some weird product); this product has the same symmetries as the Riemann tensor, so Weyl tensor also has them and it is trace free in addition, i.e. $W^\lambda_{\ \alpha\lambda\beta} = 0$ (taking these conditions into account, the Weyl tensor has 10 independent components in 4 dimensions; very interesting property is that *only* in 4 dimensions, the amount of information in Weyl, Ricci Tensor is the same; in 3 dimensions, Weyl tensor has no information and in higher ones much more than Ricci); in index-form it is given by $W^\alpha_{\ \beta\mu\nu}$

oof:

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= \frac{1}{2}(g \cdot \mathrm{Ric})_{\alpha\beta\gamma\delta} - \frac{1}{12}R(g \cdot g)_{\alpha\beta\gamma\delta} + W_{\alpha\beta\gamma\delta} \\
&= \frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\right) \\
&\quad - \frac{1}{12}R\left(g_{\alpha\gamma}g_{\beta\delta} + g_{\beta\delta}g_{\alpha\gamma} - g_{\alpha\delta}g_{\beta\gamma} - g_{\beta\gamma}g_{\alpha\delta}\right) + W_{\alpha\beta\gamma\delta} \\
&= \frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\right) - \frac{1}{6}R\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) + W_{\alpha\beta\gamma\delta}
\end{aligned}
$$

long talk about constant curvature; interesting statement (Schur's theorem): constant curvature $\Leftrightarrow$ Gaussian/sectional curvature of each point does not depend on the choice of the 2-tangent-plane through the point

the Weyl curvature $W^\alpha_{\ \beta\mu\nu}$ (which is a function of the metric g) has the important property of being conformally invariant, i.e. $W^\alpha_{\ \beta\mu\nu}(\Omega^2 g) = W^\alpha_{\ \beta\mu\nu}(g)$ for some function Ω and even the reverse statement is true: if $W^\alpha_{\ \beta\mu\nu}(g_1) = W^\alpha_{\ \beta\mu\nu}(g_2)$, then $\exists$ locally a function $\Omega \in C^\infty(M;\mathbb{R})$ without zeros such that $g_1 = \Omega^2 g_2$; for example, a vanishing Weyl tensor means that the space is locally, conformally flat -¿ all of these statements are valid only for $n \geq 3$

1.3.2 Order

basically take order from Penrose?; other way to put it: from summary H_Analysis, but with less math; also Carroll?

first: do manifolds; then go to tangent space (we want vectors); then go to bundles (first tangent bundle, then more general vector bundles); then define tensors and tensor bundle; then go to differential geometry

1.3.3 General Thoughts

Schwarzschild metric contains information on many effects of BHs in its components! coefficient $1 - \frac{2M}{r}$ in front of dt^2 tells us about time dilation close to BH (more t goes by the closer you get) and $\frac{1}{1-\frac{2M}{r}}$ in front of dr^2 tells us about curvature of space (increases as r decreases)

-¿ this is general job of metric components, tell us how distances are affected (which can also be temporal ones in 4D spacetime)

1.3.4 Math Stuff

regarding tensor product: throughout the discussions, linearity of objects was very important (we have used it for the differential, many mappings, etc.); however, a very important notion that is not linear is the underlying spaces we have looked at; take for example $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$: here, we do not have linearity, which would require $(2, 1) = 2 \cdot (1, 1)$ and clearly, this is not true; however, we might be interested in such a space and this is what is called a tensor product space; there are more tensor products, we also have to make sense of the one of objects in this space, but this is a good motivation

goal of derivatives is approximation to first order, which is expressed in demanding linearity of operators

differentiation is linear and has Leibniz rule, so it already fulfils requirements for tensor derivative (thus it makes sense to demand ∇, d acting like D on functions); multiple generalizations of Df to something like "Ds" for sections s exist, which is fine because from ∇ we can easily get many others e.g. by $d = \mathrm{Alt}(\nabla)$ (not sure if equality is true, but from Carroll eq 1.82 it looks like this), that is by suitable mappings

1.3.5 From Wald

the notion of curvature, intuitively, corresponds to the one of a 2-sphere in 3D space; however, this is extrinsic curvature which is only visible in embeddings, but what we are interested in is something like intrinsic curvature; how can we detect that?

1.3.6 From Penrose

tangent space in point $p \in M$ is immediate/infinitesimal vicinity of M "stretched out"; more formally, a linearisation of the manifold

to do physics, we cannot just work with vector spaces or affine spaces like the Euclidian space (basically $\mathbb{R}^n$, but no need to fix origin), but we need manifolds; however, manifolds do not have enough natural structure to build up the theory that is needed to describe physics, so we need some additional (local) structure (e.g. enabling us to measure infinitesimal distances in case of a metric structure); this structure is often encoded to/using the tangent spaces (which are present naturally for manifolds), which are vector spaces again

problem of abstract notion of "no structure" is for example: no general, meaningful (well-defined) notion of differentiation (does exist for functions, but not for vector fields, 1-forms or other tensors); exterior derivative is something like that, but it does not really give information about varying of the forms (nice is that it maps p-forms to $p+1$-forms)

some structure can also be provided by connection; although not every structure can reproduced, metrics uniquely determine a connection (Levi-Civita connection)

goal of derivative operators: measure constancy and deviations from it; in case of vectors, this is equivalent to a notion of parallelism; note: we will go reverse route, define derivative and get parallelism from that; this notion will have the unusual feature of path-dependence, where unusual is meant with respect to what we know from Euclidian space; while it is possible to do this (see Wald), but this is mainly by making the "right" guess and thus not really helpful (idea is to say we want something where change of v is proportional to difference Δx and then we say: this works; welp)

which requirements make sense? since tangent space is linearisation of manifold, there should also be linear dependence on direction that we differentiate along; more generally, pointwise linearity means that functions can be dragged across the operator; when acting on tensors however, a product rule has to be specified: $\nabla_X(fs) = (\nabla_X f)s + f\nabla_X s$ makes sense (without argument X, this becomes $\nabla(fs) = (\nabla f) \otimes s + f\nabla s$) (?)

ideas come from the fact that our goal is to generalize action of derivative D; therefore, demanding $\nabla f = df$ also makes a lot of sense

extension to more than one tensor field is possible by demanding additivity $\nabla(s+t) = \nabla s + \nabla t$ and by specifying product/Leibniz rule $\nabla(s \otimes t) = (\nabla s) \otimes t + s \otimes (\nabla t)$; to uniquely determine this generalization, it is also necessary to demand compatibility with trace/contraction (which also helps with defining these things in the first place)

interesting: local connection can be defined uniquely from Gaussian basis vectors

2 Special Relativity

In modern day (theoretical) physics, I personally feel like there are two competing viewpoints. One is very much based on intuition and the other is based almost solely on a mathematical description. This shows especially in the theory of special relativity, where one can deal (i) in much detail with groups and transformations or (ii) with a much more pictorial version of the theory, mostly utilizing very basic geometry in so-called spacetime diagrams.

Both approaches can lead to a rich and of course equivalent understanding, but it is often tempting to focus on only one of them. In my own experience, this is mostly the mathematical description because students are often more familiar with the required math, so teaching the alternative and rather new intuitive-based approach would actually be more complicated. This, however, can often lead to a lack of intuition, which is still fundamental to fully understand and embrace relativity as a whole. For this reason, the (admittedly, quite ambitious) goal of this summary is to treat both approaches. In this chapter, we will start by dealing with the intuitive approach.

2.1 Newtonian Physics

Sir Isaac Newton shaped the way physics was made for decades with his work "Philosophiae Naturalis Principia Mathematica" from 1687, which essentially founded the theory of classical mechanics. We will go briefly over the most important insights and notions needed to understand relativity.

It is based on the following principles.

Postulate 2.1: Newton Axioms

1. Every body remains at rest or in a uniform motion, unless a force acts upon it.
2. If a force $\vec{F}$ acts upon an object of mass m, it causes an acceleration

$$\boxed{\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}} \qquad \boxed{F^k = ma^k = m\ddot{r}^k}\,. \tag{2.1}$$

3. If two bodies exert forces on each other, these forces have the same magnitude but opposite directions (actio = reactio),

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}} \qquad \boxed{F^k_{12} = -F^k_{21}}\,. \tag{2.2}$$

These axioms (sometimes also called laws) define how objects (which are sometimes given the abstract name *observer* O; these give us viewpoints to describe physics from) experience physics. In order to describe this action mathematically, we also need explicit ways to assign the position of observers to points $\vec{r} \in \mathbb{R}^3$ in the Euclidean space we live in. Such an assignment is what physicists call *coordinates* or *frames*.

Newton's laws have rich implications, some of which we discuss now.

First Law The first law tells us something about which special class of frames is suitable to describe physics.

Definition 2.2: Inertial Frame

An *inertial frame (of reference)* is a frame where $F^k = ma^k$ holds.

Basically, if you put a ball in front of you and let it go, then you are in an inertial frame if the ball just remains where it is.

For this to be valid, either $\vec{r}$ or $\vec{v}$ have to be constant (both direction and magnitude) in this inertial frame if $\vec{F} = 0$, which is exactly what Newton's first law imposes. From the definition we can also see that there is no unique inertial frame because we can always get other inertial frames from existing ones by looking at frames which move with constant speed with respect to them.

A very important realization is that all inertial frames are suited equally well to describe physics because the laws of physics do not depend on the frame we choose:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}_1(t)}{dt} = m\frac{d\vec{v}_2(t)}{dt} \tag{2.3}$$

as long as $\vec{v}_1 - \vec{v}_2 = \text{const}$. Therefore, while the values of $\vec{v}_1$ and $\vec{v}_2$ might differ and thus depend on the inertial frame we choose, the physics inferred from them does not (forces is what we observe). This explains the preferred role of inertial frames when describing physics. Laws stated in inertial frames hold in all of them, while laws in non-inertial frames do not, whence it is often non-trivial to find out whether effects can be attributed to some physical process or to the frame/coordinates used to describe the situation (for example, the Coriolis force is needed to explain effects on Earth's surface, which is a rotating and thus non-inertial frame of reference).

If all inertial frames are suited equally well to describe physics, one might ask if a relation between them exists, i.e. if there are ways to transform between inertial frames. Besides coordinate transformations (for example from Cartesian to spherical coordinates), which we will not discuss further here, the other possible transformation is between uniformly moving frames. These changes are admitted by the *Galilei transform* and assuming the velocity v to be in x-direction, this maps coordinates (x, y, z) according to

$$\boxed{x \longrightarrow x' = x + vt} \qquad \boxed{y \longrightarrow y' = y} \qquad \boxed{z \longrightarrow z' = z}\,. \tag{2.4}$$

Second Law The second law is probably the second most famous formula in physics. It is a special case of constant mass $m = \text{const.}$ of the formula

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \qquad \boxed{F^k = \frac{dp^k}{dt}} \tag{2.5}$$

that relates forces to *momenta* $\vec{p} = m\vec{v}$. Because of their relation to forces, some fundamental principles involve momenta. For example, the collision of particles with no external forces being present can be examined using momenta due to the property of *momentum conservation*,

$$0 = F^k_{\text{total}} = \sum_k p^k_j \tag{2.6}$$

remark: I do not understand this anymore. Does this belong here? See https://en.wikipedia.org/wiki/Momentum#Conservation. Probably better after third law. And second equality is definitely wrong (though I can see equivalence) -¿ make as example after third law? Logic is: for closed system (no other, external forces), third law tells us sum over forces they exert on each other is zero, so that the sum of their momenta is conserved; this is true at all time, which means the total momentum is conserved (and in particular, the one before and after the collision must be equal)

which has to hold before and after the collision, yielding

$$\sum_j p_j^k = \sum_j \tilde{p}_j^k \,. \tag{2.7}$$

Here, p_j^k denotes the momentum of the j-th object before the collision and $\tilde{p}_j^k$ after.

In either form, the second law determines dynamics in the Newtonian physics.

Third Law The third law is probably the hardest to grasp, but it has profound implications on how one should think about forces etc. intuitively.

-¿ when I push table, I feel the force it exerts on me (not the one I exert)

Example 2.3

astronaut example who pushes stones away in space

mathematically speaking, it tells us something about symmetries of Newtonian dynamics (don't know how to elaborate further on that...)

2.2 Relativity

2.2.1 What Is Wrong with Newton?

Newtonian theory works beautifully for many applications, even today where the theories of relativity and quantum mechanics are available. However, it does not describe the entirety of physics. This is also what physicists realized in the late 19th/early 20th century. At least to some degree, if not fully, every contradiction that surfaced at this time can be traced back to the nature of light.

In the 1860s, James Clerk Maxwell derived the *Maxwell equations* describing electromagnetic fields. They predicted electromagnetic waves, whose existence was confirmed in 1886 in experiments conducted by Heinrich Hertz. These waves propagate with the speed of light,

$$\boxed{c = \frac{1}{\sqrt{\epsilon_0\,\mu_0}} = 299\,792\,458\,\frac{\mathrm{m}}{\mathrm{s}}}, \tag{2.8}$$

so it was (and is) natural to identify them with light.

Several conflicts with Newtonian physics arise from this:

1. c is constant for all observers measuring it. The Galilei transform describing transformations in Newtonian theory, however, predicts different speeds. More specifically, light emitted by a moving observer is measured to have a speed of $c + v$ by a resting observer.
2. Maxwell's equations are not invariant under the Galilei transform. Instead, the corresponding symmetry transformations are modified versions of them (confer section 2.4 on those Lorentz transformations).

On the other hand, Newton's theory had tremendous success itself over many decades, so why would one conclude that this theory is at fault rather than the new Maxwell theory? Because (a) not all of Newtonian physics is to be replaced and (b) because of overwhelming experimental evidence. In addition to Hertz's experiments, the Michelson-Morley experiment in 1887 confirmed that c is constant for all observers to high precision.

For this reason, some new concepts are needed. While one can handle the subject strictly mathematically and e.g. simply derive the "correct" coordinate transformations that leave the Maxwell equations invariant, this tells us only little about the new physics that may arise. As we will learn, quite some rethinking of the concepts of space and time is needed to obtain equivalent results from a more intuitive approach and this work was mainly done by Einstein, e.g. in his famous paper "On the electrodynamics of moving bodies" [Ein05].

2.2.2 Einstein Postulates

Einstein's theory is based on two fundamental ideas, which are formulated in postulates.

The first idea is to make the invariance under chosen reference frame, which has been known a long time beforehand, a building block of the theory rather than a consequence.

Postulate 2.4: Principle of Relativity

All physical observations must hold independently of the inertial frame that is chosen.

This principle can be traced back to Galileo Galilei, who formulated it in 1632. It means if you are in a closed room without any windows, you cannot perform any experiment to determine if you are at rest or moving at a constant velocity. In other words, there is no absolute notion of rest; it is all relative (hence the name "relativity principle").

The second postulate concerns the speed of light and is definitely something new.

Postulate 2.5: Universality of Speed of Light

The vacuum speed of light c is constant for observers in all inertial frames.

Note that we do not demand "equivalent" statements like the relativity principle did – c has *the exact same* value for all observers. This postulate was based on the insights mentioned in the previous subsection, i.e. based on theory *and* experiment, and even more experiment evidence has been collected since.

However, it has profound implications. One of them concerns the very notion of space itself. In Newtonian physics, the notion of space is an absolute one. This means there is one reference frame that describes "real" space and while frames moving with respect to this one exhibit the same physical observations, this frame always has a special role. One way to characterize this observer is by measuring the speed of light sent out by him. If it is c and not differing by some amount, then the frame is at rest (otherwise it would be $c + v$).[1] However, since all observers measure c now, there is no way to determine this preferred observer at rest – the notion of rest/motion and therefore the connected notion of space itself becomes completely relative. Albeit the relativity principle was known before him, Einstein was the first one to recognize all implications it has and that it ultimately leads to the abandonment of absolute space when combined with c being constant.

At the same time, this constance makes c very special because statements related to it are independent of the observer/inertial system and thus allow to make invariant statements. This

[1] Note that we can assume the value of c to be known because it can be measured e.g. using a mirror by measuring the round-trip time. Strictly speaking, this only yields the two-way speed of light, but details on inferring one-way speeds require theory we are yet to built up.

property will be used routinely when properties in relativity are derived.

A direct corollary of postulate 2.5 is that c is the maximum speed at which *any* signal, not just light, can be transmitted.

Property 2.6: Universal Speed Limit

The speed of light c is the maximum velocity for all interactions.

The proof of this is very similar to the argument provided beforehand.[2]

Proof. If there was a speed c' higher than c, then for moving observers the speeds would be dependent on their velocity v and observed to be $c' + v$. Consequently, one could single out an observer sending signals at c' and this observer would be at rest – a violation of the relativity principle. □

This implication is a reason why experimental tests of this speed limit are important, they are confirmations of postulate 2.5. Such tests have been conducted successfully for neutrinos and gravitational waves, both of which do indeed propagate at the speed of light (to high experimental accuracy).[3] Interestingly, both the constancy of c and it being the maximum speed attainable by any signal/information implies that relativistic addition $+_R$ must yield

$$\boxed{c +_R v = c} \tag{2.9}$$

not matter the value of v. As we will see later, this statement is indeed true.

2.2.3 Light Cones & Spacetime Diagrams

Before we start to evaluate implications of these two postulates, it is customary to introduce a technique to visualize the time evolution of physical systems. The most straightforward choice is to map position x on the x-axis and time t on the y-axis, i.e. a space-time diagram (which we will call *spacetime diagram* instead, for reasons that will become clear later). Since we are often interested in velocities close to c, it is customary to rescale the time-axis and display ct-values instead of t-values on it (otherwise slopes would be very small, small time step would correspond to large change in position).

In doing that, we discard two of the three spatial dimensions of Euclidean space. Nonetheless, it is sufficient to visualize important ideas (see figure 2.1 for a simple example).

[2] The idea was taken from Dragon [Dra12], but I tried to provide more details. However, I am not 100% sure about them being correct, so if you find an error to it, you might be right.

[3] This also means confirmations of other theories, which introduce new fields to circumvent the relativity principle and still allow speeds $> c$, cannot be confirmed.

- A single point in such a diagram gives position and time of an object (e.g. particle, person or rocket). We will refer to these points as *events* and use the symbol E.
- The trajectory of an object as a function of time (essentially a collection of events) is called *world line* Γ. An example for $v = 0.5c$ is shown as a red line in figure 2.1.
- Light is described by the relation $x = ct$, which means it is always a diagonal at $\pm 45°$ (corresponds to slope ± 1), irrespective of the observer for which the diagram is drawn. This is why we often depict light sent out from and received by the origin (in both spatial directions), see the yellow lines in figure 2.1.

For the first time, we encounter what is often called *light cone structure* of relativity, which is essentially a corollary of c being a universal speed limit. This fact shapes the causal structure in relativity and for spacetime diagrams, it tells us that only events connected by straight world lines with slope $|v| \leq c$ can causally influence each other. This defines two cones above and below the event, the possible future and past of this event, each defined by events that the event can have interacted with or can potentially interact with. Together, these cones are called *light cone* (see figure 2.1 for an example). In terms of light cones, the structure of relativity can be states as follows.

Definition 2.7: Timelike, Lightlike, Spacelike

A straight world line Γ connecting events E_1, E_2 is called

- *timelike*, if it lies inside of E_1's and E_2's light cone ($|v| < c$)
- *null/lightlike*, if it lies on the edge of E_1's and E_2's light cone ($|v| = c$)
- *spacelike*, if it lies outside of E_1's and E_2's light cone ($|v| > c$)

Accordingly, the events E_1, E_2 are called timelike-separated, null-/lightlike-separated and spacelike-separated.

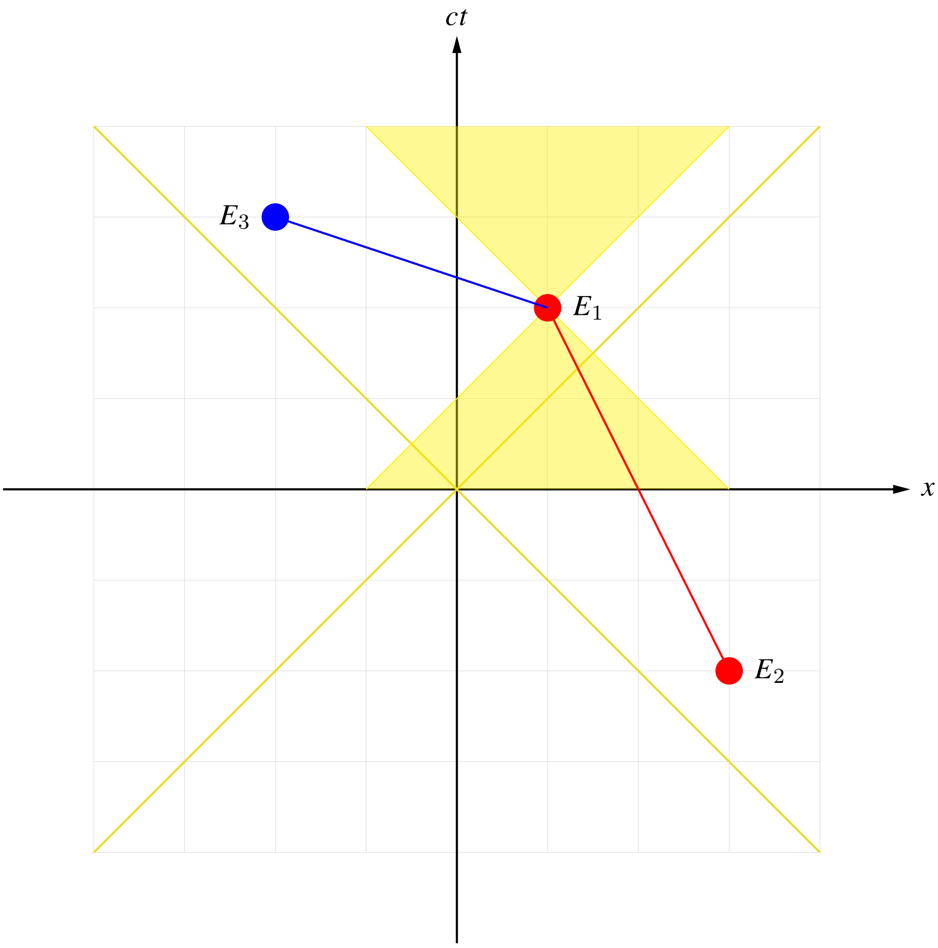


Figure 2.1: A simple spacetime diagram.
Red dots visualize three events E_1, E_2, E_3 at $(x, ct) = (1, 2), (3, -2), (-2, 3)$. For E_1, the light cone is drawn as well. Additionally, the world lines connecting E_1, E_2 (red; timelike) and E_1, E_3 (blue; spacelike) are shown.

2.3 Clocks

Until now, we have not really discussed the notion of time. Partly, this is because we natively have a very clear, intuitive understanding of time: we look at clocks to measure it and this notion can be employed anywhere in space – just take a look at equal clocks in different points and compare their readings. This definition is employed in Newtonian physics, without much more attention being needed.

-¿ what requirements do we even have on a notion of time for it to be considered "good"? As Misner-Thorne-Wheeler state, it has to make the trajectory of free particles look simple. To see how these trajectories might change depending on this choice (that we have always assumed to be given until now), let us take a look at (2.5), which is the equation of motion. For a free particle, $v = \text{const.}$ and thus $F = 0$ – free particles move in a straight line. However, the denominator contains t, just like $v^k = \frac{dx^k}{dt}$. For this reason, our notion of "constant" itself depends on the time standard we have chosen! As MTW show very illustratively, if we were to measure time in solar days, this would mean that the distance zurückgelegt in order to keep the velocity constant would change – which makes it look like a force is at work. This is a very undesirable behaviour -¿ could even show with chain rule how things look like written out in terms of the usual notion of time, showing that we get something of the form $\cdots = \frac{dp^k}{dt}$

However, as it is the story for much of relativity theory, this notion essentially breaks down once we go to more extreme situations like distances on cosmic scales or clocks moving relative to each other with high velocities. In both of those cases, timing the reading of a clock and hence comparing if they show the same time is difficult. For large distances, this is rather easy to see because information is transmitted at a finite speed $\leq c$, so when receiving information about the measurement result t of a far-away clock, we have to take into account the time it travelled to us in order to find out which event happened simultaneously to t.

This is problematic since many notions implicitly rely on the fact that we can measure quantities at the same time, i.e. on a notion of simultaneity. A very important example are lengths, which are defined as the separation of points – at a fixed time. Therefore, a well-defined notion of "at a fixed time" is required for us to be able to measure lengths and until now, we have no such notion. In everyday life, it is easy to avoid such difficulties: after all, we can look at clocks side-by-side, make sure they show the same time and then move one of them away to the desired position. This procedure ensures the clocks are synchronized, so we can simply take the desired measurements and compare the times later on. However, this is not really feasible to do that for measurements between planets or galaxies and clearly, an alternative, perhaps more general, way of communicating time measurements is needed.

All of that motivates the need for a synchronization procedure of clocks. We will here present the one proposed by Einstein, starting with its definition for resting observers and then look at it for the case of moving observers. Throughout this section, we will adopt visualizations from

[Dra12], while many of the definitions follow [Giu21] more closely.

2.3.1 Synchrony of Clocks

Resting Observers Our setting is identical copies of an ideal clock being attached to each point in space (fig. 2.2). For a consistent, well-defined notion of "global time", however, we have to make sure these clocks show equivalent times. To do that, we will synchronize them by adopting the following definition, originally proposed by Einstein in [Ein05].

Definition 2.8: Einstein Synchronization

Two clocks C, C' with readings t, t' attached to observers O, O' at rest (relative to each other) are *synchronized* if light signals sent out from them travel the same time on $\overline{OO'}$ as they do on $\overline{O'O}$.

More explicitly, consider the following scenario: O sends out light with his clock C showing the time t_+, O' reads the time t' off of his clock C' upon arrival of this light, immediately reflecting it back to O who receives it with his clock C showing t_-. Then C and C' are said to synchronize if

$$t' - t_- = t_+ - t' \quad \Leftrightarrow \quad \boxed{t' = \frac{t_+ + t_-}{2}}\,. \tag{2.10}$$

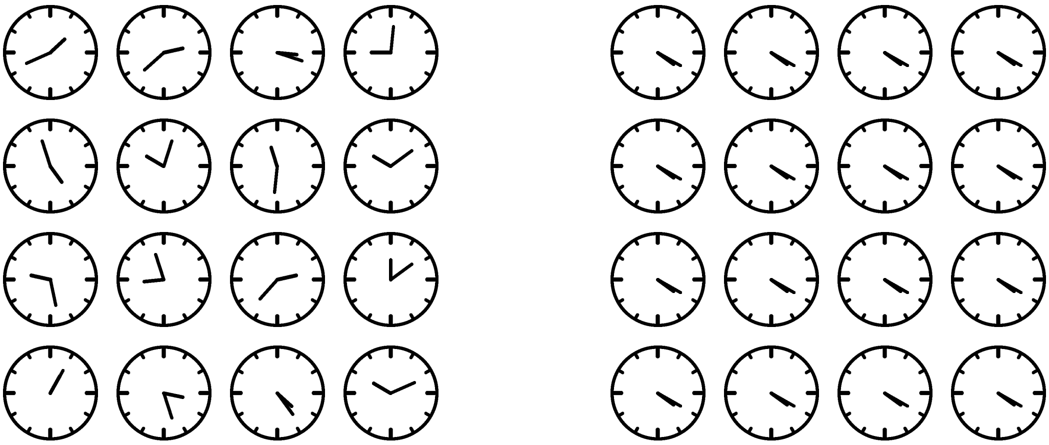
If C' were to show a different time, then in order to synchronize it with C we would *set*it so that (2.10) holds. An eventual asynchrony would be measurable since an offset $t' \to t' + \Delta t$ of C' compared to C would lead to

$$(t' + \Delta t) - t_- = t_+ - (t' + \Delta t) \quad \Leftrightarrow \quad t' = \frac{t_+ + t_-}{2} - \Delta t\,, \tag{2.11}$$

i.e. by comparing t' (which we assume to be reported to O, without bothering about the details of a practical implementation of this report) to $\frac{t_+ + t_-}{2}$, we can check if C and C' are synchronized. In principle, this confirmation of synchrony has to be done just once because synchronized clocks of observers which are at rest relative to each other remain synchronized for all times (assuming ideal, identical clocks). Visually speaking (thus referring to figure 2.3 (a), (b), for example), that is because changing the times involved only shift the whole setup of light signals parallel to the observers world lines, but do not change their relations (length, intersections, etc.). In particular, the clocks C, C attached to O, O' remain synchronized.

An important aspect of the scheme we present here is to realize that the freedom to *set* t' according to (2.10) does actually exist. The reason is that using a single clock C showing time t, only the two-way speed of light (i.e. c) as an average speed of the roundtrip $\overline{OO'O}$ can be measured. Therefore,

$$\boxed{c(t_+ - t_-) = \overline{OO'O} = 2\,\overline{OO'}}\,. \tag{2.12}$$



(a) Unsynchronized Clocks (b) Synchronized Clocks

Figure 2.2: (Desired) Effect of Synchronization for clocks in different positions. Note that both axes are spatial ones here, no temporal.
While all clocks are equal in (a), i.e. they tick at the same speed, they do not show the same time. In contrast, all clocks in (b) show the same time 4:20.

The Einstein synchronization of C then utilizes a second clock with time t' to makes sure that $t' - t_- = t_+ - t'$, which implies

$$c(t_+ - t_-) = c(t_+ - t') + c(t_- - t') = 2\,\overline{OO'} \quad \Leftrightarrow \quad \boxed{t' - t_- = \frac{\overline{OO'}}{c} = t_+ - t'}\,, \tag{2.13}$$

which means the one-way speeds on both paths are also c.[4]

This notion of synchrony gives rise to the following notion of simultaneity.

Definition 2.9: Simultaneity

Two events E at t and E' at t' are called *simultaneous* if the locally simultaneous clock readings of synchronized clocks C, C' at these events are identical, i.e. $t = t'$.

Now we are able to answer the question which time C shows when the light signal sent out from O at t_- is reflected by O' (at t'). It is precisely

$$\boxed{t = t' = \frac{t_+ + t_-}{2}}\,. \tag{2.14}$$

Therefore, O can infer the time t' at which the event E' occurred using only readings t_-, t_+ of C (which reflects the "locally simultaneous" part), a crucial property to have, remembering the

[4]See [MM03] for more details. [Giu21] also elaborates on this topic in section 2.1.

introduction of this section. After all, we know that for synchronized clocks C' showed t', so the event E where C shows t must occur at the same time as (i.e. be simultaneous to) E'. Of course, the synchronization is mutual, i.e. the time t as seen by light sent out and received from O' is

$$\boxed{t = \frac{t'_+ + t'_-}{2}} \tag{2.15}$$

holds as well (another way to see this is that $t = t' \Rightarrow t_+ = t'_+$ and $t_- = t'_-$).

In addition to the mutuality ≡ symmetry, synchronization (or, equivalently, the induced notion of simultaneity) has some other properties, which can be proven by logic.

Property 2.10: Simultaneity as an Equivalence Relation

Simultaneity defines an equivalence relation on the set of all events in an inertial frame, i.e. the following properties hold:

1. Reflexivity: every event is simultaneous to itself.
2. Symmetry: if E is simultaneous to E', then E' is simultaneous to E.
3. Transitivity: if E is simultaneous to E' and E' is simultaneous to E'', then E is simultaneous to E''.

Moreover, a family of synchronized clocks partitions the set of all events $\{E\}$ into several, mutually disjoint subsets (equivalence classes) containing events which are simultaneous to each other.

A key takeaway is the following: events that simultaneous for one observer O are simultaneous for every other observer O' at rest relative to O. It is very important to stress the "at rest" part here, synchrony of clocks and thus simultaneity of events are tied to specific inertial systems. In particular, other inertial systems may and will define synchrony and simultaneity differently (for the same set of events). We will look at the consequences of that later in this section. For now, though, we focus on an implication of the transitivity. This property allows the following, equivalent definition of clock synchronization (in some sense another characterization of Einstein synchronization).

Definition 2.11: Einstein Synchronization 2

Two clocks C, C' with readings t, t' attached to observers O, O' at rest (relative to each other) are *synchronized* if light signals sent out from them at the same reading of their respective clocks meet exactly in the midpoint of $\overline{OO'}$.

This is clearly equivalent to equality of total travel time $\overline{OO'O} = 2\,\overline{OO'}$ as demanded in 2.8, but stating it in this way has one advantage (which is where the transitivity from property 2.10 is utilized): two clocks C, C' attached to observers O, O' are synchronous if they show equal times to the observer that is exactly between them. This is meant in the sense that light signals sent out from $\mathcal{R}$ take the same time to go to O and come back as they do to go to O' and come back. In accordance with [Dra12], we will refer to this third observer as *referee* $\mathcal{R}$. Just like synchrony, simultaneity can be defined in terms of $\mathcal{R}$ as well.

Definition 2.12: Simultaneity 2

Two events E at t and E' at t' are called *simultaneous* if light sent out at the same time from the referee $\mathcal{R}$ arrives O at E and O' at t'.

Hence, instead of tuning t' such that no Δt occurs (as it was used in (2.11)), we now argue via light signals sent out from $\mathcal{R}$ to O, O' at time t_-^R. t^R is time measured by a clock C^R attached to $\mathcal{R}$. By definition, the reflected signals come back to $\mathcal{R}$ at the same time t_+^R. If the times t, t' at which the light has been reflected by O, O' fulfil

$$t = t' = t^R = \frac{t_+^R + t_-^R}{2}\,, \tag{2.16}$$

then the clocks C^R, C, C' are synchronized. By prolonging the beams sent out from and received by $\mathcal{R}$ (which form what is sometimes called a *lightangle*), one can then obtain the times t_-, t_+, t'_-, t'_+ and also the same light signal paths that the previous definition 2.8 yielded. This situation is illustrated in 2.3 (a).

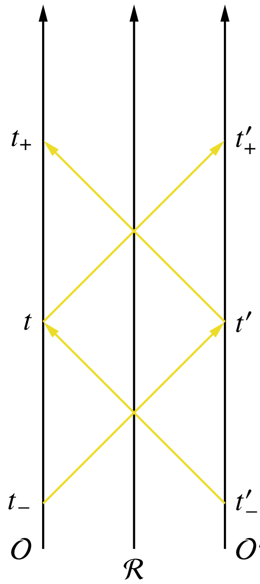
Resting Observers from a Different Frame We have already mentioned how the Einstein synchronization is not unique. However, this scheme has one big advantage: it respects the relativity principle, as figure 2.3 (b) illustrates. The idea is to analyze the situation from figure 2.3 (a) in a different inertial frame that is moving relative to both observers (which stay at rest relative to each other).[5] Indeed, by constructing the referee $\mathcal{R}'$ and drawing the light signals sent to and from each observer (which form a lightangle again), one obtains the times $t''_-, t''_+, t'''_-, t'''_+$ by prolonging the edges of the lightangle. This is equivalent to what has been done for resting observers in 2.3 (a) and likewise,

$$t'' = \frac{t''_+ + t''_2}{2} \qquad\qquad t''' = \frac{t'''_+ + t'''_-}{2}\,. \tag{2.17}$$

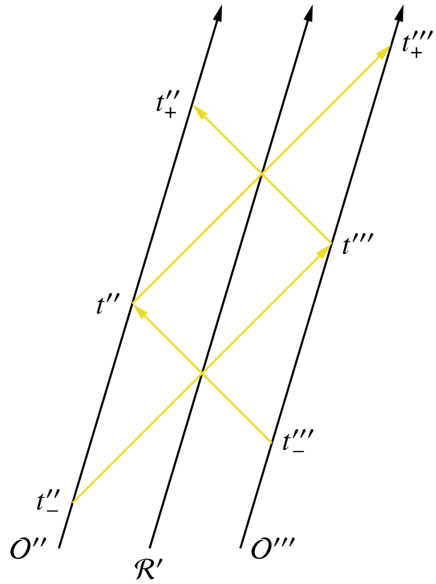
Therefore, the observers O'', O''' agree on $t'' = t'''$ (to verify that, we can also look at the distance of t_-, t, t_+ on the vertical axis we use to depict time or the distance along the world lines of the observers).

But nonetheless, something seems to be different. We get an idea of what it might be from a comparison of the pairs of synchronized clocks in figure 2.3 (c), where $t = t''$ has been assumed

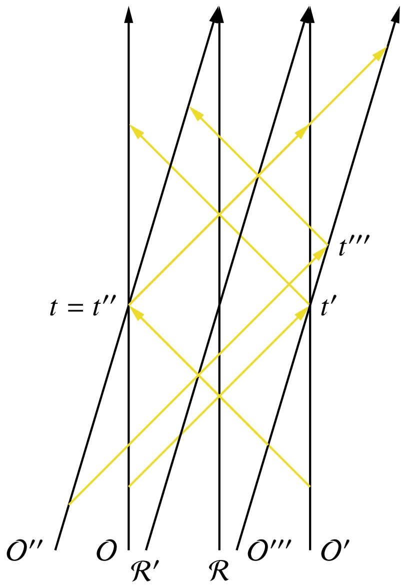
[5] For a better distinction between these situations, the observers are named O'', O''' instead of O, O'. This does not change any interpretations that were mentioned.



(a) Observers at rest relative to each other in their rest frame



(b) Observers at rest relative to each other in frame moving relative to them



(c) Comparison of all observers from (a) and (b). For simplicity, we assume that the clock $\mathcal{C}$ and $\mathcal{C}''$ show the same time $t = t''$ on intersection of $\mathcal{O}, \mathcal{O}''$. Despite that, they cannot agree on the time of intersection of $\mathcal{O}', \mathcal{O}''', t' \neq t'''$.

Figure 2.3: Synchronization procedure in spacetime diagrams for different sets of observers. We draw the observers $\mathcal{O}, \mathcal{O}'$ whose clocks we wish to synchronize, the referee $\mathcal{R}$ for them and the light pulses they exchange in order to achieve the synchronization. Clocks carried by $\mathcal{O}$ and $\mathcal{O}'$ measure the events at times $t = \frac{t_+ + t_-}{2}$ and $t' = \frac{t'_+ + t'_-}{2}$ to be simultaneous, i.e. $t = t'$. This is because light (represented as yellow lines) sent out simultaneously by the referee $\mathcal{R}$ to $\mathcal{O}$ and $\mathcal{O}'$ "sees" times t and t' on the clocks before returning simultaneously to $\mathcal{R}$, where the results are "reported".

in the event where O, O'' intersect for a simpler discussion. While both pairs of clocks are synchronized among themselves, the induced notion of simultaneity for C'', C''' is *not* the same as the one for C, C'. We can see that from t''' being higher than t' on the time axes of both the resting and moving observers (which are parallel to their world lines). Geometrically speaking, the "lines of simultaneity" (diagonal from left to right corner in lightangle) change from being horizontal in figure 2.3 (a) to being tilted in (b).

The origin of these differences lies in the change of inertial frame. While light has the same world line as before due to the constancy of c, the world lines of both clocks, O'' and O''', are tilted now (which causes the observed change in the lightangle). Therefore, the time of flight for light on $\overline{O''O'''}$ is different from the one on $\overline{O'''O''}$ (equivalent: time needed for $\overline{O''\mathcal{R}'}$ differs from the one for $\overline{O'''\mathcal{R}'}$) because the observers are moving uniformly in the same direction. More explicitly, while the one-way speeds of light are still c for both paths, the uniform motion of both clocks leads to differences $v(t''' - t''_-), -v(t''_+ - t''')$ in separation $\overline{O''O'''}$ and thus (for the resting observer)

$$\boxed{c(t''' - t''_-) = \overline{O''O'''} + v(t''' - t''_-)} \quad \text{and} \quad \boxed{c(t''_+ - t''') = \overline{O''O'''} - v(t''_+ - t''')} \,. \tag{2.18}$$

Consequently,

$$\boxed{t''' - t''_- = \frac{\overline{O''O'''}}{c - v} \neq \frac{\overline{O'''O''}}{c + v} = t''_+ - t'''} \,, \; \boxed{t'' - t'''_- = \frac{\overline{O'''O''}}{c + v} \neq \frac{\overline{O''O'''}}{c - v} = t'''_+ - t''} \,, \tag{2.19}$$

in contrast to (2.13) – the one-way speeds as seen from the stationary observer is *not* c, the clocks are not seen to by synchronized ("t''' is higher on the time axis as t'").

Moving Observers As we have already mentioned and just demonstrated, the Einstein synchronization does respect the relativity principle, which means it can be applied to clocks resting relative to each other in arbitrary inertial frames. However, just like different inertial observers can disagree on the speed of an object, that does not mean synchronized clocks in different inertial frames make equivalent statements, e.g. regarding the simultaneity of events. Using figure 2.3 (c), we made a first try to visualize this and also started discussing it at the end of the last paragraph. Now, we shall complete this discussion by examining the relation of clocks moving relative to one another in more detail. For clocks which have been synchronized at some point in time, which we choose to be $t = 0 = t'$ (any time smaller than τ' would work equally well), the following theorem holds.

Theorem 2.13: Minkowski's Theorem

For two observers O, O' that move relative to each other with velocity v and an event E'

occurring on the world line of O' at time τ',

$$\boxed{\tau' = \sqrt{t_+ t_-} = \sqrt{1 - v^2/c^2}\, t = t - \left(1 - \sqrt{1 - v^2/c^2}\right) t}\ . \tag{2.20}$$

Here, t_-, t_+ are the times measured by a clock C on O when light signals to E' have been sent out and received back.

A geometrical proof can be found in [Dra12]. Einstein's original derivation was different and an analogous version is presented in subsection 2.4.1.

This result is probably very puzzling at first, so let us elaborate on it, starting with the viewpoint of O (for the accompanying illustrations, see figure 2.4). Because the clocks have been synchronized in the beginning, O expects that the light signals sent out and received at t_-, t_+ saw O' when C' showed

$$t = \frac{t_+ + t_-}{2}$$

the time a clock synchronized with C, i.e. at rest relative to it, would show. However, the time that C' really showed is given by Minkowski's theorem as

$$\tau' = \sqrt{1 - v^2/c^2}\, t = t - (1 - \sqrt{1 - v^2/c^2}) t\,.$$

This is the same time τ that C has shown a bit earlier, $\tau = \tau' < t$ (for $v > 0$), the moving clock is slow by $1 - \sqrt{1 - v^2/c^2}$ seconds per second. Accordingly, C' expects C to show that

$$t' = \frac{t'_+ + t'_-}{2}$$

has passed when light sent out from O' at t'_- arrives there. Instead though, C shows

$$\tau = \sqrt{1 - v^2/c^2}\, t'\,,$$

which is equal to the time $\tau' < t'$ that C' has shown before. At first sight, this makes the result even more puzzling and in fact seems paradoxical – how can both observers claim less time has passed for the other one?

To resolve the confusion, let us first summarize the results. All in all, it is safe to say O and O' de-synchronize if they are moving relative to each other, causing

$$\boxed{t \neq \tau'} \qquad \text{and} \qquad \boxed{t' \neq \tau}\,. \tag{2.21}$$

While they do not agree on times shown on each others clock and not even on which clock lags behind, both observers do agree that (i) they cannot agree on a consistent notion of simultaneity, which causes (ii) less time to go by on moving clocks.

On the other hand, thinking back to the definition of our synchronization procedure, this effect should not be too surprising anymore. Just like before, it is convenient visualize the synchronization process in terms of a referee $\mathcal{R}$ (fig. 2.4). For him, the time passing between emitting light and receiving it back is equal for O and O', i.e.

$$\tau = \tau' , \tag{2.22}$$

which is what we expect. However, O and O' are not in the same inertial system as $\mathcal{R}$, they move relative to him. Consequently, their notions of simultaneity are different, which is just an equivalent way of phrasing de-synchronization. Thinking about previous results of relativity, this should not surprise us at all: both observers can claim the other one moves while they are stationary (a manifestation of the relativity principle). Therefore, they can also both claim that the other clock moves and thus slows down, a stationary observer has no business telling moving observers something about simultaneity (since there is not absolute notion of being stationary). Nonetheless, all observers will agree on physically relevant statements like the order of events ($t_1 < t_2 \;\Leftrightarrow\; \tau_1 < \tau_2$).

One can take the viewpoint that Minkowski's theorem gives us the general way of comparing times between clocks. After all, for resting observers with relative velocity $v = 0$, the geometric mean $\sqrt{t_+ t_-}$ reduces back to the arithmetic mean $\frac{t_+ + t_-}{2}$, yielding the familiar result

$$t = \frac{t_+ + t_-}{2} = \tau = \tau' = \frac{t'_+ + t'_-}{2} = t' . \tag{2.23}$$

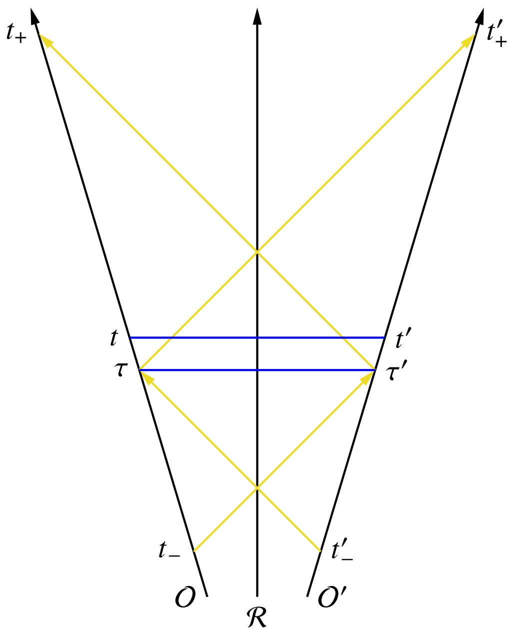
One final note concerns some related vocabulary. It is commonly said that "moving clocks run slowly", perhaps also in this summary. However, one has to be careful taking this too literally: we are looking at *identical* clocks here, i.e. if we slowed down the moving ones, the rate of *all* clocks we have dealt with would be perfectly equal. It is just that their readings are different for the same event if measured from different inertial frames (which is, admittedly, a very subtle difference). But again, this effect *not* caused by one of the clocks ticking slower, they are merely a tool *revealing* a property of time.[7]

2.3.2 Time Dilation

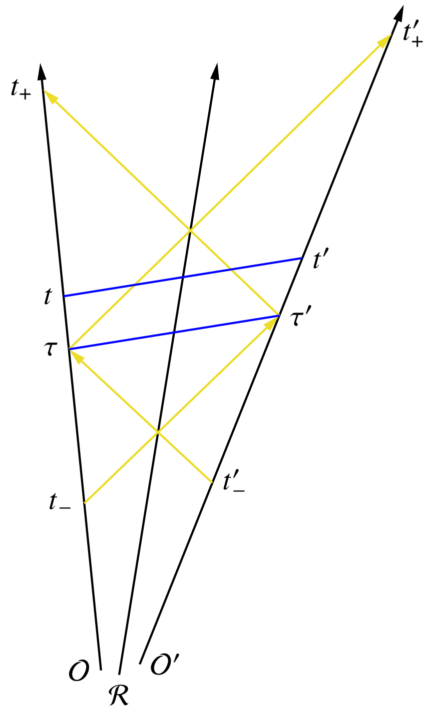
In principle, there is no unique procedure to synchronize clocks. Admittedly, the Einstein synchronization does have a few very convenient properties, like respecting the relativity principle (which is achieved by exploiting the constancy of c again). However, it produces strange results for observers moving relative to each other, namely that less time passes on moving

[6] Beware that the diagrams are not perfect, e.g. the referee should be exactly where the light beams cross again. This is most likely due to an error in my code, which I was unable to locate. Nonetheless, the most important ideas should still be conveyed (for an analogous graphic, see figure 2.8 of [Dra12]).

[7] Supported by the fact that more than two clocks are needed to see mutuality of effect, at least one resting and one moving pair (so that synchronization is possible). See [Giu21] for an detailed discussion.



(a) Observers moving with relative velocity $v = 0.6c$



(b) Observers moving with relative velocity $v = 0.5c$

Figure 2.4: Synchronization procedure in spacetime diagrams for moving observers.[6]

We also note that it is really the relative velocity that matters because we could always go in the rest frame of one of the observers and examine the situation from there. Also, it does not matter which rest frame because the effect is mutual, i.e. it does not change if we replace $v \to -v$.

clocks (something we do not know from our everyday life). Therefore, it is a valid question to ask whether or not this is an effect induced by our specific choice of synchrony, i.e. if other procedures could build consistent notions of global time free of these effects. In other words: is this a bug or a feature of time?

Not surprisingly, though, the answer is that we have discovered a fundamental *feature* of time, which uncovered by and not caused by the Einstein synchronization (otherwise, we would not have chosen this scheme). Time does indeed slows down for moving observer, an effect called *time dilation*. The implications of this are profound: in relativity, the notion of an absolute, universal time, that all observers agree on, has to be abandoned. Instead, it becomes relative and depends on the motional state of the observer that measures it. Of course, if the time standard becomes relative, so do notions like simultaneity. As a consequence, both observers have the right to claim less time went by for the other and both are right in doing so. This is a key fact to understand relativity and no different from the abandonment of absolute space, which had analogous implications on observers moving uniformly relative to each other. Here, both can claim to be at rest, while the other one moves because it does not change the physics exhibited by the situation. The same is true for times (although it might not be clear yet). Put differently, just like the answer to "where does event E happen?" depends on the coordinates we choose, the answer to "at which time did event E happen?" depends on the (world line of the) clock we use in our measurements.

An argument that even further supports choosing the Einstein synchronization is equivalent to the synchronization via clock transport. Here, one sets the same time on clocks which are right next to each other and then moves one of these clocks (sometimes called master clock) to different points in space so that other the readings of other clocks can be compared to this master clock (in a slow transport limit, such that negligible time dilation occurs on the way; Giulini discusses this explicitly in 5.1 of [Giu21]). This is what we would do intuitively to synchronize clocks on Earth, which shows us that our unfamiliarity with time dilation comes from our lack of experience with velocities $v \approx c$ or cosmic distances rather than pointing to unphysical effects.

Our next goal is to resolve the seemingly paradoxical mutuality of time dilation. One might suspect that the de-synchronization of clocks is caused by changes in the travel time as O and O' move farther away from each other, i.e. by the increasing spatial separation leading to difficulties in communication between them and thus confusing the synchronization. Albeit it is true that the separation does increase, the de-synchronization arising from this is not a technical issue of the procedure. We can convince ourselves of this by looking at a scenario where the clock readings are compared in immediate spatial vicinity. It is commonly stated in the following form.

Example 2.14: Twin Paradox

It deals with two observers (commonly taken to be twins, i.e. of the same age), one of them at rest and the other one moving uniformly to some destination, for example Mars M. Both start at the same time and at the same point in space and meet again in this point at some later time. The interesting question is how much time has passed for both of them, after all we have just learned that uniform motion affects clocks. In reality, this experiment would involve some kind of acceleration (turning around), but as Dragon points out in [Dra12] this would not change the results because one can accelerate the resting observer in this time as well. Hence, any result we infer from instantaneous accelerations (and uniform movement in between) already contains the relevant ingredients and can be attributed to uniform motion. It should also be noted that certainly no general relativity is needed to explain the effects (as long as any effect of gravity is neglected), contrary to what is claimed in many popular-science explanations. This statement is correct no matter if acceleration is taken into account or not.

From now on, O_R will be used to denote the resting observer and O_M to denote the second observer moving at v relative to O_R. If O_R resting in Earth's orbit measures O_M to reach M at time t, then $\tau = \sqrt{1 - v^2/c^2}\, t$ has elapsed on a clock carried by O_M. Immediately turning around at Mars and travelling back with velocity v' (could be $v' = -v$, but no need for that), then times t' and $\tau' = \sqrt{1 - v'^2/c^2}\, t'$ elapse for O_R and O_M, respectively. All in all, for the roundtrip times we have

$$t + t' = \frac{\tau}{\sqrt{1 - v^2/c^2}} + \frac{\tau'}{\sqrt{1 - v'^2/c^2}} \geq \tau + \tau' \,.$$

Furthermore, there is no ambiguity here in the perception of O_M because they end up witnessing the same event, $E = E'$, so there can be no ambiguity in simultaneity – the twin paradox is not paradoxical after all. One can attribute this to the fact that on his journey, the observer O_M has to turn around in order to come back, so he is changing reference frames on the journey. Hence, three inertial frames are involved in total and there is no frame where the moving observer is stationary over the whole journey, which eliminates a mutual description from moving to resting observer (from all three frames one can prove that less time goes by for observer on the journey, which is the striking part here). This is demonstrated explicitly in the second treatment of this example using spacetime diagrams (example 2.15).

Therefore, it is indeed true that less time goes by on moving clocks. This effect has also been verified experimentally. It should be noted, though, that it only becomes relevant for velocities $v \approx c$, otherwise $\sqrt{1 - v^2/c^2} \approx 1$ and $\tau \approx t$. Once again, we emphasize that this is the reason why the Newtonian way of describing time as an absolute, universal notion has worked for so long and is perhaps also more intuitive.

An interesting consequence of time dilation arises for observers moving with the speed of light, like e.g. photons. No other light pulse sent from some distance can ever reach him, so how can this observer agree on time measurements with others? Well, they cannot agree because the speed of light is a universal limit, no signal can reach the moving observer, from which we conclude that no time passes for this observer. This result is in accordance with

$$\tau = \sqrt{1 - c^2/c^2}\, t = 0\,. \tag{2.24}$$

2.3.3 Length Contraction

Time works different in relativity, we must learn to accept that. Part of that is examining induced effects, for example on lengths. After all, when thinking about it a bit more, lengths are defined as readings of a measuring rod/ruler *at the same time*. Since observers have different notions of simultaneity, the very definition of lengths will also be affected by this.

It is natural to say that two rods are of the same length if they have the same length for the referee $\mathcal{R}$ we have already used to define simultaneity. As figure 2.11 of [Dra12] illustrates very well (so well that I was not able to reproduce it), the observers do not see both rods to have equal length. Because of them measuring with different notions of simultaneity, they do not agree on lengths. An observer O who carries a ruler of length l measures the length l' of a ruler carried by an observer O' moving with velocity v relative to O to have length

$$\boxed{l' = \sqrt{1 - v^2/c^2}\, l}\,. \tag{2.25}$$

Again, this effect is mutual. O' measures a length of

$$l = \sqrt{1 - v^2/c^2}\, l'$$

for a ruler of length l carried by O.

2.3.4 Doppler Shift & Aberration

Time being different for moving observers has implications on frequencies, which are nothing but inverse period lengths T,

$$f = \frac{1}{T}\,. \tag{2.26}$$

ah, while $\kappa(-v) = 1/\kappa(v)$, this is not true for times: $\tau(-v) = \tau(v)$; thus it is not needed to explain time shifts, but instead it is useful for frequencies

Doppler factors could be helpful because frequencies are inverse time intervals, i.e. through arriving of light signals (thus blue-, redshift), one can argue for time intervals that the observers see as well! Due to longer travel time of light, light signals reach at different times, leading to different perceptions of simultaneity (Wikipedia on twin paradox is nice for that)

also helpful because from them, we can deduce how addition of velocities works; so maybe might be worthwhile to treat them after all... -¿ however, doing that in Lorentz transformation section might also be fine...

2.4 Lorentz Transformation

The effects of time dilation and length contraction point to new effects that occur upon changing inertial frames to look at problems. These effects are not predicted by the Galilei transform, which maps spatial coordinates between two inertial frames (one unprimed, one primed, moving relative to each other with velocity v) according to

$$x' = x - vt \qquad \Leftrightarrow \qquad x = x' + vt$$

and does not change time at all, i.e.

$$t' = t\,.$$

Over the course of the last sections, however, we have seen that time is a notion which depends on the observer measuring it as well. Therefore, it is natural that it has to be transformed as well. Additionally, observers measure different speeds of light for this Galilei transform, a violation of the relativity principle since observers have to agree on c by postulate 2.5. The goal of this section is to find a new, corrected transformation between the unprimed and primed coordinates. Note that we will restrict ourselves to one spatial dimension x instead of three for now (a generalized treatment will be done in subsection 3.1.2).

Since we now that the Galilei transformation does work for velocities $v \ll c$, there is no need for a completely new expression. Our ansatz can simply be

$$x' = \gamma(x - vt) \qquad \text{and} \qquad x = \gamma(x' + vt') \tag{2.27}$$

for some factor $\gamma = \gamma(v)$, which fulfils $\gamma \approx 1$ for $v \ll c$. Note that we have used $\gamma(v) = \gamma(-v)$ here, which is the reason why $x \rightarrow x'$ has the same structure as $x' \rightarrow x$. This is a necessary condition on the transformation (and thus γ) since otherwise, there would be a preferred direction, which is forbidden by the relativity principle.

To obtain the way $t \rightarrow t'$ transforms, we use postulate 2.5 again, which tells us that $x = ct$ and $x' = ct'$ for light. We can rearrange this to read

$$ct' = x' = \gamma(x - vt) = \gamma(ct - \frac{v}{c}x) \qquad \text{and} \qquad ct = x = \gamma(x' + vt') = \gamma(ct' + \frac{v}{c}x')\,. \tag{2.28}$$

We are free to switch the expressions here since we are looking at light and it is necessary because $t' \overset{!}{=} t$ for $v = 0$.

The last step is determining the factor γ, which can be done by inserting (2.28) into (2.27)

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(\gamma(x' + vt') - v\gamma/c(ct' + \frac{v}{c}x')) \\ &= \gamma^2(x' + vt' - vt' - \frac{v^2}{c^2}x') = \gamma^2\left(1 - \frac{v^2}{c^2}\right)x' \end{aligned}$$

from which we obtain the *Lorentz-factor*

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}} \tag{2.29}$$

Inserting this result into (2.28) and (2.27) yields the *Lorentz transformation*

$$\boxed{x' = \gamma(x - vt) = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad \boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right) = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}} \tag{2.30a}$$

$$\boxed{x = \gamma(x' + vt') = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad \boxed{t = \gamma\left(t' + \frac{v}{c^2}x'\right) = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}}\,. \tag{2.30b}$$

Often, one abbreviates $\beta = \frac{v}{c}$, which is the velocity given in multiples of c (equal to v in units where $c = 1$). This set of equations shows how different inertial frames are related, respecting the the constancy of c.[8]

From its definition and figure 2.5, it is clear that γ diverges as v approaches c. The Lorentz transformation, on the other hand, does not diverge because of $x = ct$, which implies

$$x' = \left.\frac{c - v}{\frac{1}{c^2}\sqrt{c^2 - v^2}}\right|_{v=c} t = 0 \qquad\qquad t' = \left.\frac{c - v}{\frac{1}{c^2}\sqrt{c^2 - v^2}}\right|_{v=c} \frac{t}{c} = 0\,.$$

Hence, $x' = ct'$ still holds, as required. Them being equal to zero, as measured from the unprimed coordinates, simply comes from the fact that exchanging signals with light is impossible, so one cannot receive any information regarding times t' or positions x'.

The other limit we have to check is $v \ll c$, where we should re-obtain the Galilei transform. As figure 2.5 shows, γ exhibits the desired behaviour of $\gamma(v) \approx 1,\ v \ll c$. More quantitatively, a

[8] The inertial part is crucial here. It implies that the transformation have to be linear in x, ct since otherwise, $\frac{d^2x'}{dt^2} \neq \frac{d^2x}{dt^2}$ and the frames would not be inertial anymore.

Taylor expansion in v/c shows (for one more term, see (3.34))

$$\boxed{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}} \simeq \left.\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right|_{v=0} + \left.\frac{\frac{-1}{2}\cdot\frac{-2v}{c}}{(1-\frac{v^2}{c^2})^{3/2}}\right|_{v/c=0} \frac{v}{c} + \ldots \boxed{= 1 + O\left(\frac{v^2}{c^2}\right)} . \tag{2.31}$$

Combining that with $v \ll c \ll c^2$, we obtain the desired result to zero-th order in v:

$$x' \simeq x - vt \qquad\qquad t' \simeq t .$$

One final note: the derivation presented here is perhaps not be the most rigorous one. Alternatively, one can argue using a diagram like 2.6, which is the spacetime diagram of a moving rod. Using a bit of geometry, one can show that the angle α between x, x' and the one between ct, ct' are equal. Moreover ct' is the line of equilocality of $x' = 0$, while in unprimed coordinates it is the line $x = vt$. [Giu21] argues in detail how $\tan(\alpha) = \frac{v}{c}$ and more relations, which also leads to (2.30). Another approach is mentioned in subsection 3.1.2.

2.4.1 Time Dilation & Length Contraction 2

In principle, it is not surprising that a new transformation has to be used. After all, we have demanded some new postulates to be true and there is no reason for the "old" transformation to fulfil it. On the other hand, we have already examined quite a few predictions of these postulates in the clock section 2.3 and likewise, it is not guaranteed that our new Lorentz transformation reproduces the results which have been found there.

Most prominently, time dilation and length contraction showed up as new effects. We will now check if they can be reproduced in coordinates related via the Lorentz transformation.

Measuring in Unprimed Coordinates Consider a rod with ends at positions x_1, x_2. Their spatial distance and thus the length of the rod in unprimed coordinates is

$$L = x_2 - x_1 .$$

From the primed coordinates, however, one would measure

$$L' = x_2' - x_1' = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) = \gamma(x_2 - x_1) = \gamma L = \frac{L}{\sqrt{1-\frac{v^2}{x^2}}} . \tag{2.32}$$

Here we assume $t_1 = t_2$ because by definition, distances are measured at the same time (simul-

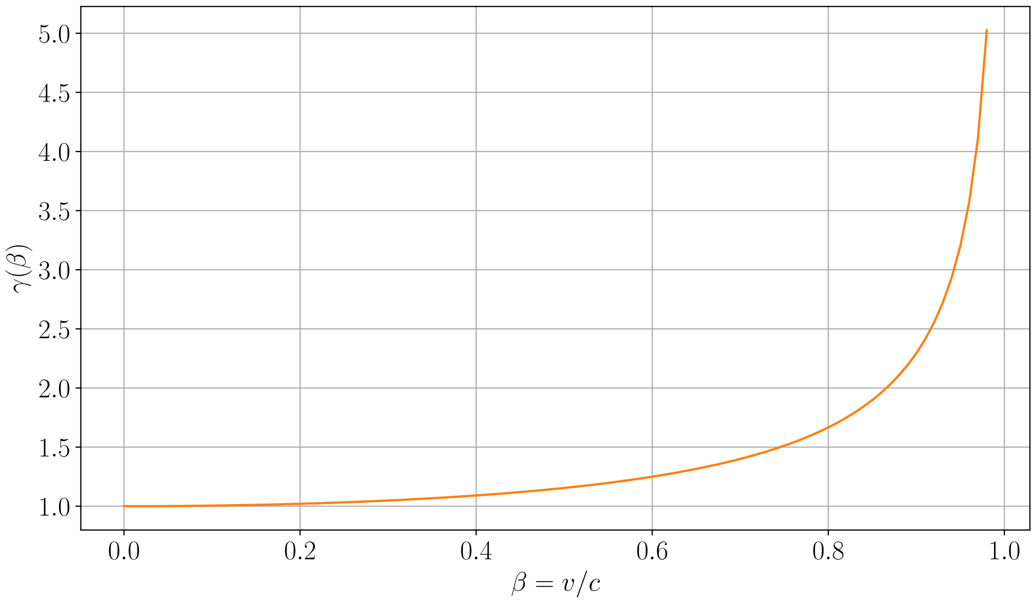


Figure 2.5: Plot of the Lorentz-factor $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. Clearly, its effect only becomes significant for speeds which are significant fractions of c.

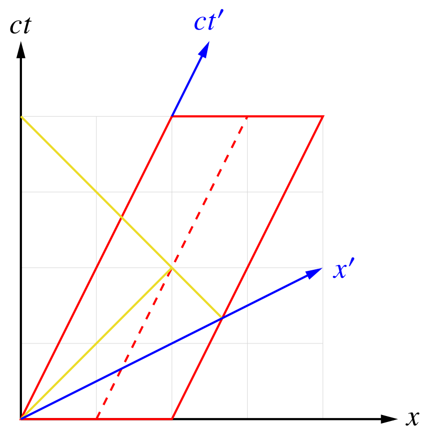


Figure 2.6: Spacetime diagram for a rod moving with velocity v, whose ends are depicted as red lines (midpoint using dashed line).
By definition, the axis ct' is the left end of the rod at $x' = 0$. In the (ct, x)-frame, it is given by the line $x = vt$. For the x'-axis, one can construct the point on the world line of the right end of the rod that is simultaneous to $t' = 0$ as measured by the left end of the rod. In accordance with the Einstein synchronization, this can be done by looking at light sent out from the origin and its intersection with the midpoint (dashed line). Tracing back this intersection point using light that is sent out by the right end of the rod in negative x-direction, one finds the point simultaneous to $t = 0 = t'$ in the origin (i.e. the line of simultaneity, which is nothing but x').

taneously). This result is not what we expect from the previous discussions. Is the Lorentz transformation at fault? Luckily, no. We have emphasized how simultaneity is important for the length to be meaningful. However,

$$t_1' = \gamma(t_1 - \frac{v}{c^2}x_1) \neq \gamma(t_2 - \frac{v}{c^2}x_2) = t_2' \quad \Leftrightarrow \quad \Delta t' = t_2' - t_1' = \gamma \frac{v}{c^2}(x_1 - x_2) \neq 0\,, \tag{2.33}$$

the measurements in primed coordinates are not simultaneous, so this L' is *not* what a ruler in primed coordinates measures. In order to simulate this result, we have to tune the times t_1, t_2 to achieve $t_1' = t_2'$. Choosing

$$\Delta t = t_2 - t_1 = -\Delta t' = -\gamma \frac{v}{c^2}(x_1 - x_2) = \frac{v}{c^2}\gamma L \tag{2.34}$$

leads to a cancellation of the time difference in (2.33). The "true" length measured in unprimed coordinates then becomes

$$\begin{aligned} \boxed{L'} &= x_2' - x_1' = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) = \gamma L - v\Delta t \\ &= \gamma L - \frac{v^2}{c^2}\gamma L = \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} L = \sqrt{1 - \frac{v^2}{c^2}} L = \boxed{\frac{L}{\gamma}}\,. \end{aligned} \tag{2.35}$$

Moving rulers measure smaller distances, physics is still intact. Effectively, this corresponds to waiting a longer time until the second measurement in unprimed coordinates, which in turn causes the transformed times t_1', t_2' to be equal (note that the length L does not change because it is at rest in unprimed coordinates, i.e. x_1, x_2 are constant, independent of time). An intuitive explanation for this is that the moving observer claims the system at rest is moving, i.e. the end of the rod moves away from him, shortening it. In order to compensate for that, we have to wait a bit longer until taking the second measurement at t_2' and correspondingly, increase t_2.

Consider a temporal distances $T = t_2 - t_1$ measured by a clock resting in unprimed coordinates (i.e. $x_1 = x_2$). This time difference measured from primed coordinates is

$$T' = t_2' - t_1' = \gamma(t_2 - \frac{v}{c^2}x_2) - \gamma(t_1 - \frac{v}{c^2}x_1) = \gamma(t_2 - t_1) = \gamma T = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}\,.$$

The primed observer measures a time T' between two events on his clock, while on the clock resting in unprimed coordinates,

$$\boxed{T = \sqrt{1 - v^2/c^2}\, T'} \tag{2.36}$$

go by. Since the primed observer sees the unprimed one moving, that confirms what we already knew: moving clocks tick slower. This effect can be attributed to the synchronization of clocks. Despite $x_1 = x_2$, $x_1' \neq x_2'$, so the measurements in primed coordinates are taken by two different clocks (which are properly synchronized, but in the primed frame).

Measuring in Primed Coordinates Now consider a rod with end points at x'_1, x'_2. Its length measured from the primed coordinates is

$$L' = x'_2 - x'_1\,.$$

This measurement is conducted simultaneously in primed coordinates, so $t'_1 = t'_2$. From unprimed coordinates, however, we measure at $t_1 = t_2$. For this reason, L' becomes

$$L' = x'_2 - x'_1 = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) = \gamma(x_2 - x_1) = \gamma L\,.$$

Consequently, the length L of the rod measured in unprimed coordinates is

$$\boxed{L = \frac{L'}{\gamma}}\,, \tag{2.37}$$

which confirms the mutuality of length contraction.

Next, consider a temporal distance $T' = t'_2 - t'_1$ measured by a clock resting in primed coordinates (i.e. $x'_1 = x'_2$). Measuring in this difference in unprimed coordinates, where the clock has moved from x_1 to $x_2 = x_1 + vT,\ T = t_2 - t_1$ during this time, yields

$$\boxed{T'} = t'_2 - t'_1 = \gamma(t_2 - \frac{v}{c^2}x_2) - \gamma(t_1 - \frac{v}{c^2}x_1) = \gamma(t_2 - t_1 - \frac{v^2}{c^2}T)\boxed{= \frac{T}{\gamma}}\,. \tag{2.38}$$

Hence, we have also obtained the mutuality of time dilation.

Albeit the calculations are largely equivalent, we have argued slightly differently here, which was done deliberately to show there is not a single correct way of calculating the corresponding effects. In the end, we can can confirm that both length contraction and time dilation are reproduced by the Lorentz transformation. Furthermore, the calculations are much more straightforward compared to what had to be done previously (setting up synchronization, drawing complicated diagrams etc.). For this reason, it is customary to introduce the Lorentz transformation without covering any details on clocks. However, in my personal experience, that often results in a lack of intuition on these topics – one is bound to the understanding in terms of Lorentz transformations, although they are *not* necessary to understand what is going on. Since special relativity is a confusing topic in itself due to several frames and new concepts, intuition is an important part and helps tremendously with interpreting calculations.

2.4.2 Addition of Velocities

Let us now turn to velocities in relativity. Intuitively, we expect the velocity of points with fixed coordinates in the primed frame to move with velocity v in the unprimed frame, i.e. we expect them to fulfil an equation of the form

$$x = vt + b$$

for some offset/shift b. However, looking at the Lorentz transformation (2.30), we may get confused by all the γ's and ask ourselves: is this really the case? Rearranging yields

$$x = vt + x'/\gamma \tag{2.39}$$

as the world line of a fixed point x' in unprimed coordinates (x, ct). Hence, the velocity of this world line (again, in unprimed coordinates) is

$$\frac{dx}{dt} = \frac{d}{dt}(vt + x'/\gamma) = v\,. \tag{2.40}$$

Velocities work as expected between two uniformly moving frames.

As more frames get involved, something has to be different though. This is because of c being a universal speed limit, so adding velocities in relativity cannot work as before, which would allow velocities $> c$. To find out how it works, let us look at a third frame (x'', ct'') moving with velocity w relative to (x', ct'). Inserting the Lorentz transformation twice yields

$$\begin{aligned} x'' &= \gamma_w(x' - wt') = \gamma_w\left(\gamma_v(x - vt) - w\gamma_v\left(t - \frac{v}{c^2}x\right)\right) \\ &= \gamma_v\gamma_w\left(\left(1 - \frac{vw}{c^2}\right)x - (v + w)t\right) \end{aligned}$$

Thus, the world line of $x'' = 0$ (convenient choice, no offset, so that velocity can be read off immediately) is given in the unprimed (x, ct)-coordinates by

$$x'' = 0 \qquad \Leftrightarrow \qquad x = \frac{v + w}{1 + vw/c^2}\,t \qquad \Leftrightarrow \qquad \boxed{v +_R w = \frac{v + w}{1 + vw/c^2}}\,. \tag{2.41}$$

Relativistic addition of velocities is much more complicated than non-relativistic. But setting $v = c$ we see that

$$\boxed{c +_R w = \frac{c + w}{1 + cw/c^2} = c\,\frac{1 + w/c}{1 + w/c} = c}\,, \tag{2.42}$$

the $+_R$ we have derived does admit the correct behaviour. Moreover, for $v, w \ll c$

$$v +_R w = \frac{v + w}{1 + vw/c^2} \approx v + w\,, \tag{2.43}$$

it reduces to the usual addition in the Newtonian limit. Thus, we have to accept that addition now works differently, $+_R$ conforms everything we demand from it.

Until now, we have assumed v, w to be parallel because we have only taken into account a single spatial dimension. In the more general case with three spatial dimensions, however,

$$\boxed{\vec{v} +_R \vec{w} \neq \vec{w} +_R \vec{v}} \quad \text{and} \quad \boxed{\vec{v}_1 +_R (\vec{v}_2 +_R \vec{v}_3) \neq (\vec{v}_1 +_R \vec{v}_2) +_R \vec{v}_3}, \tag{2.44}$$

adding velocities relativistically is neither commutative nor associative.

2.4.3 Spacetime Diagrams 2

Even more help with understanding relativity is provided by visualizations like spacetime diagrams, which are especially helpful because one can visualize multiple observers in a single diagram (see e.g. fig. 2.6 or fig. 2.7, where grid lines are shown as well). Essentially, this is a geometric way of visualizing what Lorentz transformations do. By marking the coordinates (x, ct) of an event in a single, resting frame we can immediately read off the coordinates in other frames as well by showing the axes as it is done in 2.7. In principle, a Lorentz transformation shifts the points $(x, ct) = (0, ct)$ by $-vt$ to the left. This means the ct'-axis is rotated to the left compared to the ct-axis, so events would have different positions in diagrams. For this reason, the ct'-axis is rotated to the right, such that the position of events stays the same (an analogous argument can be made for the x'-axis). Reading off coordinates then works by looking at the transformed lines of simultaneity (parallel to x'-axis) and equilocality (parallel to ct'-axis), i.e. projecting the event parallel to these lines until we find the intersection with the ct'- and x-axis, respectively.

We have already discussed the causal structure of relativity, timelike trajectories are always in the light cone and spacelike ones are outside of it. For this reason, the events marked by red dots in figure 2.8 are clearly spacelike. Now we get a a visual explanation of why spacelike events are problematic: while the unprimed, resting observer sees the left event E_1 happening before the right one E_2, the primed observer sees E_2 happening before E_1. Thus, if c was not the speed limit and spacelike events were able to communicate with each other, observers could disagree on cause and effect. Luckily, no evidence for such a transmission with $v > c$ has been found (yet), so we do not have to rebuild our understanding of causality.

The new addition of being able to depict multiple observers to spacetime diagrams makes them a powerful tool suitable to explain many effects of relativity. For example, the fairly complicated twin paradox can be explained and – perhaps, even more important – visualized conveniently.

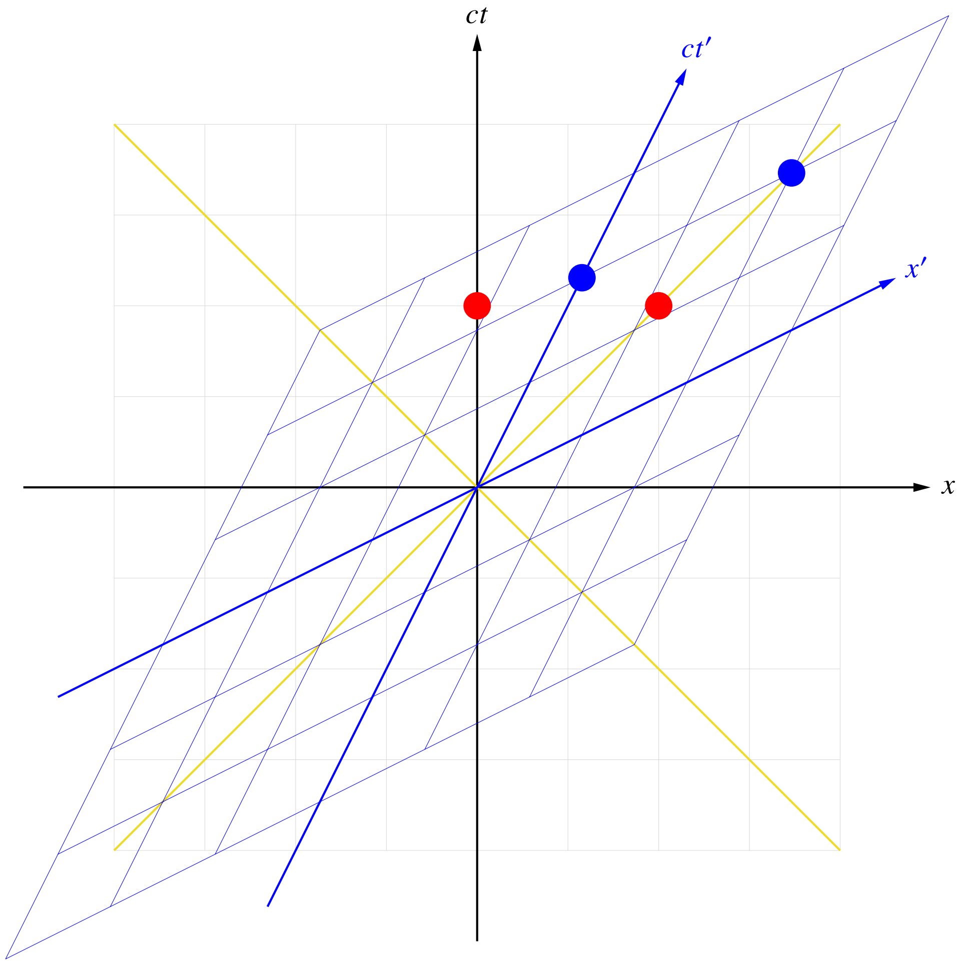


Figure 2.7: Events in frames moving with $v = 0.5c$ relative to each other. Red dots show two events at spacetime points $(x, ct) = (0, 2), (2, 2)$ (the corresponding primed coordinates are $(x', ct') = (-1.15, 2.3), (1.15, 1.15)$). Blue dots show the same coordinates in the (x', ct')-frame (corresponding unprimed coordinates: $(x, ct) = (1.15, 2.3), (3.46, 3.46)$).
We can see very nicely how each observer perceives time differently. Events happening simultaneously to both red dots (i.e. which lie on the line between them at $t = 2$), do *not* happen at $t' = t$, but at $t' = \tau = \sqrt{1 - v^2}\, t$ (which is evident from the fact that the blue dots are at $t' = t$). The same can be said for the moving observer in blue, which sees events at $t = \sqrt{1 - v^2}\, t'$ simultaneous to the blue dots at $t' = 2$. Analogous arguments can be made for positions and equilocality.

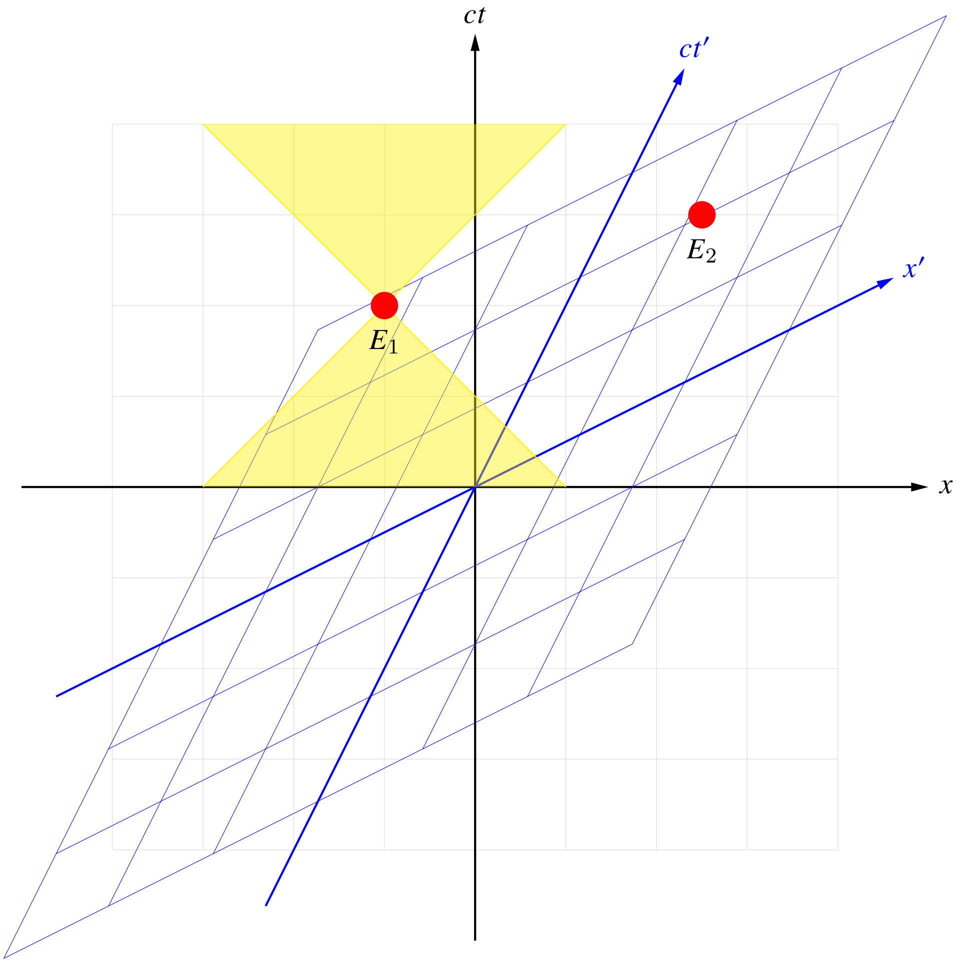


Figure 2.8: Spacelike events.

It is very obvious that E_2 does not lie in the light cone of E_1, which has been visualized to make that clear (this implies E_1 does not lie in the light cone of E_2). The unprimed, black observer sees E_1 happening before E_2, while the blue observer moving at $v = 0.5c$ sees it the other way around.

Example 2.15: Twin Paradox 2

As promised, here comes the detailed demonstration of the twin paradox, which has been started in example 2.14. We will discuss the setup shown in figure 2.9, i.e. treat one observer at rest (will be commonly referred to as "unprimed" one) and two observers moving with velocities $v = \pm 0.5c$ relative to the unprimed observer (these will be called "primed" and "double-primed", in accordance with their axis labels in 2.9).

Our approach will be to compute the roundtrip time needed to go from S to E (i.e. the time passing on the world line on ct axis) and the time needed to go from S to T to E (i.e. the time passing on the other world line shown in 2.9). Each of these quantities will be computed from clocks resting in all three of the inertial frames shown in figure 2.9 (reminder: three are involved due to turning around, rest frame of moving observer changes there).

During the process, we have to distinguish between four times: (i) the time t_{ST} passing on the resting clock between S and T, (ii) the time t_{TE} passing on the resting clock between T and E, (iii) the time τ_{ST} passing on the moving clock between S and T and (iv) the time τ_{TE} passing on the moving clock between T and E. From that we get the total times for resting and moving clock,

$$t_{SE} = t_{ST} + t_{TE} \qquad\qquad \tau_{SE} = \tau_{ST} + \tau_{TE} = t'_{ST} + t''_{TE} \, .$$

One final note concerns the velocities involved: the world line is drawn for $v = 0.5c$ on the way from S to T and $v = -0.5c$ on the way from T to E (same velocity, different direction), where v is the velocity of the respective moving frame compared to the unprimed, resting one. This implies a relative velocity of

$$v_2 = \frac{0.5c + 0.5c}{1 + 0.5^2} = 0.8c$$

from double-primed to primed frame.

- **Measuring from unprimed coordinates**

  Clearly,

  $$t_{ST} = 2 = t_{TE} \qquad \Leftrightarrow \qquad t_{SE} = t_{ST} + t_{TE} = 4$$

  in arbitrary time units (where one time unit goes by between two grid lines). From that, Minkowski's theorem tells us

  $$t'_{SE} = \sqrt{1 - v^2}\, t_{SE} = 3.464 \, .$$

Simultaneously, by looking at how much time elapses between the intersection of gray grid lines and the blue ct'-axis, one can see that as a rough estimate $t'_{ST} \lesssim 2$. For the exact result, we apply Minkowski's theorem:

$$t'_{ST} = \tau_{ST} = \sqrt{1 - v^2/c^2}\, t_{ST} = \sqrt{1 - 0.5^2} \cdot 2 = 1.732\,.$$

Applying the same procedure to the double-primed coordinates yields

$$t''_{TE} = \tau_{TE} = \sqrt{1 - v^2/c^2}\, t_{TE} = \sqrt{1 - (-0.5)^2} \cdot 2 = 1.732 = \tau_{ST}\,.$$

This is because time dilation does not depend on the direction, only on the absolute velocity. Therefore, we can confirm that

$$t_{SE} = t_{ST} + t_{TE} = 4 > \tau_{SE} = \tau_{ST} + \tau_{TE} = 3.464\,,$$

less time goes by on the moving clock.

- **Measuring from primed coordinates**

While from these coordinates one still sees four time units going by on the roundtrip for the resting observer, i.e. $t_{SE} = 4$, a rough estimate for the roundtrip time measured by a clock resting in the primed coordinates ct' is $t'_{SE} \gtrsim 4$. More precisely, Minkowski's theorem yields

$$t_{SE} = \sqrt{1 - v^2/c^2}\, t'_{SE} \qquad \Leftrightarrow \qquad t'_{SE} = \frac{t_{SE}}{\sqrt{1 - v^2/c^2}} = \frac{4}{\sqrt{1 - (-0.5)^2}} = 4.619\,.$$

This is a consequence from the mutuality of time dilation, an observer resting in primed coordinates sees the unprimed observer moving at $v = -0.5c$ and therefore measures more time passing in his own frame.

However, t'_{SE} is not what a clock in the primed coordinates sees. Instead,

$$\tau_{SE} = \tau_{ST} + \tau_{TE} = t'_{ST} + t''_{TE}$$

as stated before. For t'_{ST}, however, we cannot simply use Minkowski's theorem and thus the time

$$\frac{t_{ST}}{\sqrt{1 - v^2/c^2}} = 2.309\,,$$

which cannot be quite correct since from the diagram we get the estimate $t'_{ST} \lesssim 3$. This is because the world line is moving with respect to the rest frame of ct, so we would only get statements about what is the time as seen from this frame. However, we wish to measure the primed time from the primed coordinates. This requires a Lorentz transformation of the events S, T, E:

$$t'_{ST} = t'_T - t'_S = \frac{t_T - \frac{v}{c}x_T}{\sqrt{1 - v^2/c^2}} - \frac{t_S - \frac{v}{c}x_S}{\sqrt{1 - v^2/c^2}} = \frac{t_{ST} - \frac{v}{c}x_T}{\sqrt{1 - v^2/c^2}} = \frac{2 - 0.5 \cdot 1}{\sqrt{1 - 0.5^2}} = 1.732\,.$$

In the same manner, we obtain

$$t'_{TE} = t'_E - t'_T = \frac{t_E - \frac{v}{c}x_E}{\sqrt{1 - v^2/c^2}} - \frac{t_T - \frac{v}{c}x_T}{\sqrt{1 - v^2/c^2}} = \frac{t_{TE} + \frac{v}{c}x_T}{\sqrt{1 - v^2/c^2}} = \frac{2 + 0.5 \cdot 1}{\sqrt{1 - (-0.5)^2}} = 2.887\,.$$

To get t''_{TE}, we have to prolong the axes of simultaneity of the primed coordinates and count the number of time units which go by between the intersections of them with ct'' (i.e. the number of intersections with green lines on the way). This yields roughly $t''_{TE} \gtrsim 1.5$ again. For the time passing simultaneously on a clock in primed coordinates, we can read off roughly $t'_{TE} \approx 4$. The correct numbers can be obtained from Minkowski's theorem again:

$$t''_{TE} = \sqrt{1 - (-v_2)^2/c^2}\, t'_{TE} = \sqrt{1 - (-0.8)^2}\, 2.887 = 1.732\,.$$

All together, the primed observer measures a roundtrip time

$$t'_{SE} = t'_{ST} + t'_{TE} = 4.619 > \tau_{SE} = \tau_{ST} + \tau_{TE} = t'_{ST} + t''_{TE} = 3.464\,.$$

We find the same result that less time has passed on the moving clock. Furthermore, we see that the absolute value for this $\tau_{SE} = t'_{SE}$ and the one computed from Minkowski's theorem in the first calculation agree (because corresponding world line is parallel to ct). On the other hand, it should not be surprising that the absolute values measured for τ_{ST} and τ_{TE} do not agree. After all, they are still measured from different reference frames and with respect to frames moving with different relative velocities (± 0.5 for primed; $0.5, 0.8$ for unprimed).

- **Measuring from double-primed coordinates**

Just as before, $t_{SE} = 4$ is measured, while roughly $t''_{SE} \gtrsim 4$ and precisely

$$t''_{SE} = \frac{t_{SE}}{\sqrt{1 - v^2/c^2}} = \frac{4}{\sqrt{1 - (-0.5)^2}} = 4.619$$

are measured by a clock resting in the double-primed coordinates. As one can confirm by looking at the result above, this is the same time a clock resting in primed coordinates measures. We expect more of these equal results since the double-primed coordinate system is moves with the same relative velocity as the primed one, just in the other direction (sign of velocity is different).

A rough estimate for t''_{TE} is $t''_{TE} \gtrsim 2$ and the exact result is

$$t''_{TE} = t''_E - t''_T = \frac{t_E - \frac{v}{c}x_E}{\sqrt{1-v^2/c^2}} - \frac{t_T - \frac{v}{c}x_T}{\sqrt{1-v^2/c^2}} = \frac{t_{TE} + \frac{v}{c}x_T}{\sqrt{1-v^2/c^2}} = \frac{2 - 0.5 \cdot 1}{\sqrt{1-(-0.5)^2}} = 1.732\,.$$

An analogous calculation yields

$$t''_{ST} = t''_T - t''_S = \frac{t_T - \frac{v}{c}x_T}{\sqrt{1-v^2/c^2}} - \frac{t_S - \frac{v}{c}x_S}{\sqrt{1-v^2/c^2}} = \frac{t_{ST} - \frac{v}{c}x_T}{\sqrt{1-v^2/c^2}} = \frac{2 + 0.5 \cdot 1}{\sqrt{1-0.5^2}} = 2.887\,.$$

As promised, we get more results that we have already seen in calculations in the primed coordinates, but now they are switched (due to the transition $v \to -v$).

Now we wish to compute t'_{ST} and expect roughly $t'_{ST} \gtrsim 1.5$. Indeed, we obtain

$$t'_{ST} = \sqrt{1 - v_2^2/c^2}\, t'_{TE} = \sqrt{1-0.8^2}\, 2.887 = 1.732\,.$$

All in all, the double-primed observer measures a roundtrip time

$$t''_{SE} = t''_{ST} + t''_{TE} = 4.619 > \tau_{SE} = \tau_{ST} + \tau_{TE} = t'_{ST} + t''_{TE} = 3.464\,.$$

Once again, these results look very familiar.

A conclusion of these extensive calculations is that physics is not broken, despite relativity sometimes being unintuitive at first glance. All inertial frames play equal roles, which shows in the mutual slowing of moving clocks. In frames where both clocks seem to be moving, we can further confirm that the effect increases with the velocity v, as implied by (2.20).

It should also be noted that the agreement of all three observers regarding τ_{SE} is really a coincidence due to the symmetric setup we have chosen – for other scenarios, e.g. with unequal velocities on the first and second part of the journey or even if we assume that ct is not at rest after all (but keeping the relative velocities of primed and unprimed system, i.e. rotating the whole setup), this will not be the case anymore. In the same manner,

$$t'_{ST} \neq t''_{TE}$$

in general. However, if one of the clocks moves on a world line parallel to the ct-axis of one of the observers, *all* of them will agree on the time elapsed along this clock (so t_{SE} being equal for all observers is really not a coincidence; the same goes for t'_{ST} and t''_{TE} in this setup). This is despite observers measuring different times on their own clocks and is due to Minkowski's theorem, which tells us how much time has passed simultaneously on a clock in another frame.

-¿ note: although there is no preferred system in the sense that it is "more correct" than others (no absolute space), one can argue it numbers computed in stationary system are the ones that make most sense because assuming the moving observer stops again and rests after his journey, this rest frame is where they compare their clocks in and thus also where an experiment would be conducted (i.e. these times would be the one which are measured)

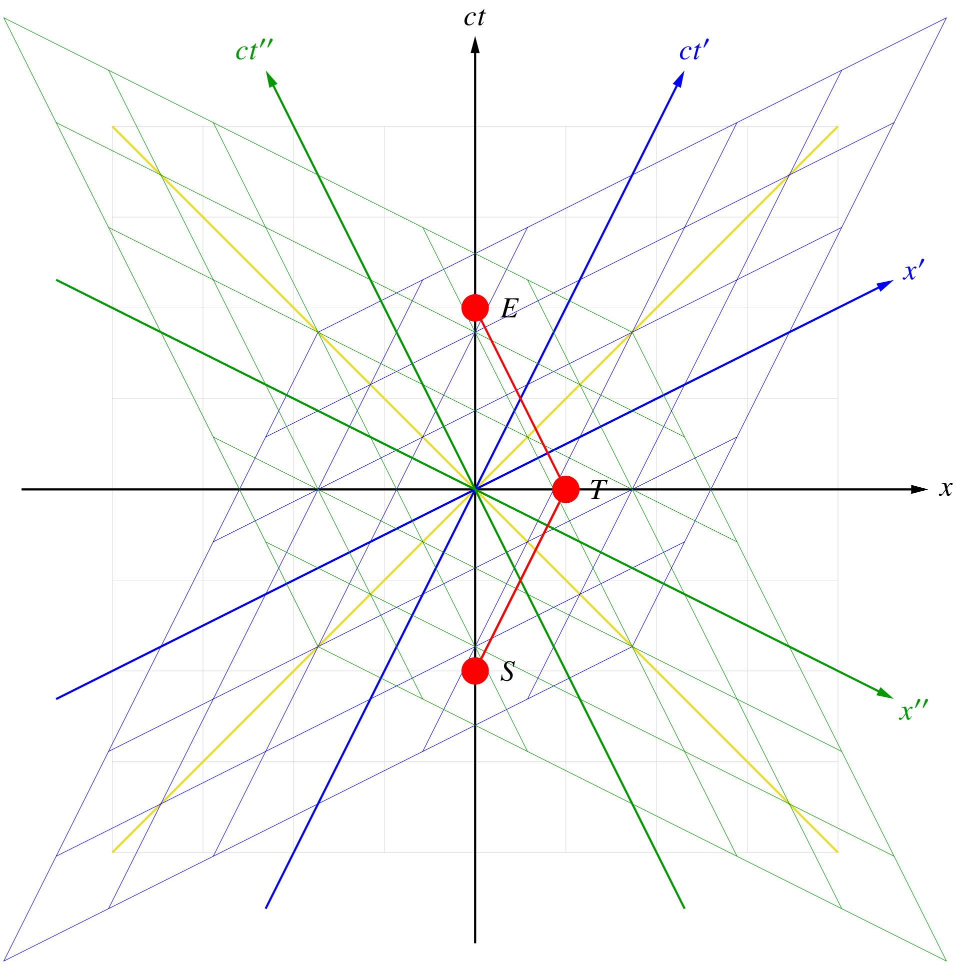


Figure 2.9: Visual explanation of twin paradox. Event S is the starting point, T turning point and E end point. The primed observer has a relative velocity of $v = 0.5v$ with respect to the unprimed one, while the double-primed observer has $v = -0.5c$.
We can measure times passing between events in a certain frame O by prolonging the corresponding lines of simultaneity (parallel to spatial axis of O) from event to the time axis of O and then count the number of time units passing on the time axis between the intersection points. Similarly, we could also prolong them until they intersect the time axis of any other frame O'. In this case, counting the number of time steps passing between the intersection points on the time axis of O' would yield the time passing for O', measured from O.

3 Bridging The Gap Between Special And General Relativity

The last chapter was dedicated to learning the effects of special relativity in an intuitive-based manner. It has already been teased that a mathematical discussion of the same subject can be equally as fruitful, but we have not really started it yet. Not only that, the formalization of relativistic physics leads naturally to the theory of general relativity, where such a description is required.

Therefore, this chapter in some ways builds a bridge between special and general relativity because we start to treat everything on the same footing.

3.1 Minkowski Space

Throughout the last sections, we discovered more and more how space and time work in relativity and how they are related. Important contributions to that picture were made by the insights of Einstein regarding synchronization of clocks as well as Lorentz and Poincaré, who developed a corrected version of the Galilei transform.

It might be clear to the reader that this implies space and time are not independent anymore, but instead have to be treated on the same footing. Historically speaking, however, this final step was not made until Minkowski proposed his viewpoint that physics should take place in a four-dimensional *spacetime*. This unification is an essential part of how the theory of relativity is described in modern literature, in particular because it allows a description in terms of a well-developed mathematical theory – the theory of manifolds. We will also adopt the usage from now on.

First, it is necessary to state what spacetime is in a formal, mathematical way.

Definition 3.1: Minkowski Space

In special relativity, *Spacetime* is described as a 4-manifold M with one time and three spatial coordinates. Another common name for M is *Minkowski space* with corresponding symbol $\mathbb{M}$ (i.e. we use $M = \mathbb{M}$).

Events E can be identified with points in M and thus specified using *inertial Cartesian coordinates* (ICCs)

$$\boxed{x^\mu = (ct, x, y, z) = (ct, \vec{x})} \tag{3.1}$$

where (x, y, z) are the Cartesian coordinates of Euclidean space and t the coordinate time measured by (synchronized) clocks in this space.

World lines are curves $\Gamma = \Gamma(\sigma) : I = [a, b] \to M$.

We adopt the convention to rescale the time-component to have the same units as the spatial components. To avoid these conversion factors, we could choose units where $c = 1$, as it is common practice when dealing with relativity. However, we have elected not to do this here. Another note on the coordinates is that choosing inertial Cartesian coordinates is by no means necessary[1], just like spherical coordinates are equally suited to describe physics in Euclidean space as Cartesian ones. It is simply more convenient for now and we maintain generality of our discussion because the theory of manifolds provides us with natural ways of changing coordinates. This mathematical way of demanding coordinate-independent statements and using invariant notions like tensors is completely equivalent to what relativity demanded: physics has to be invariant of the observer, so only special quantities like proper times have an invariant and thus physical meaning (coordinate times or positions do not). This correspondence

[1] In fact, coordinates themselves have *no* physical meaning.

of underlying ideas and concepts is the reason why describing relativity in the language of manifolds is a very natural idea.

Having defined spacetime, we can see why space-time diagrams have been called space-time diagrams: they visualize the new-defined entity M, just like coordinate planes with x-, y-axis visualize Euclidean space. This also explains why they are often called *Minkowski diagrams*.

While points and vectors do not necessarily have a direct identification with each other, elements of the vector space $\mathbb{M}$ are also points in M because both are elements of $\mathbb{R}^4$ (which further motivates $M = \mathbb{R}^4$).[2] For this reason, points $x^\mu = (ct, x, y, z)$ in coordinates are often also interpreted as (the components of) a four-vector $\underline{x}$ (so that $\mathbb{M}$ really forms a vector space, justifying the name Minkowski *space*). Over the course of the next sections, we will develop the mathematical tools to describe spacetime. As we will learn throughout this section, it has more structure than what is natively given by "pure" manifolds, namely a metric. Therefore, $\mathbb{M}$ will turn out to be a *pseudo-Riemannian manifold* or *Lorentz-manifold*, $\mathbb{M} = (\mathbb{R}^4, \eta)$. At this point, the departure from Euclidean space with metric g and line element $ds^2 = dt^2 + dx^2 + dy^2 + dz^2$ begins.

3.1.1 Metric & Inner Product

We have now seen how physics can be conveniently described using a 4D manifold, which we called spacetime. Points on this manifold are events and we can change coordinates or inertial frames using Lorentz-transformations. Moreover, there are several quantities that can be defined naturally on manifolds, for example curves, vectors, and covectors (maps that take vectors as input). While manifolds do have an additional natural structure, this is given by topology. In physics, however, we are also interested in statements concerned with distances between events and to measure them we need additional structure. More specifically, we have to specify a metric that will allow measuring distances, as well as norms of vectors via the induced inner product.

Mathematically, metrics are objects called tensors and they have the convenient property that they are invariant under coordinate changes. Therefore, distances are physically meaningful statements because they do not depend on the inertial frame we compute them on. In the tradition of invariant quantities that have been encountered so far, we may guess that the metric will be related to light in some way. From the universality of the speed if light c, distances s are equivalent to times t for light, $s = ct$. Because of that, a natural measure for distances is the time elapsed on a clock, i.e. the geometric structure of Minkowski space is determined by Minkowski's theorem 2.13. Instead of denoting time with the usual variable t, we will now switch to the *proper time* τ since the time elapsed a clock between events $(0, 0, 0, 0)$ and (ct, x, y, z)

[2] To be more precise, $\mathbb{M}$ is the affine space of $\mathbb{R}^4$, i.e. we keep the space itself, but not the attached notions of distances and angles. In particular, there is no origin (in accordance with relativity principle).

is

$$\boxed{\tau = \sqrt{1 - \frac{v^2}{c^2}}\, t = \sqrt{1 - \left(\frac{x}{ct}\right)^2 - \left(\frac{y}{ct}\right)^2 - \left(\frac{z}{ct}\right)^2}\, t = \sqrt{t^2 - (x^2 + y^2 + z^2)/c^2}} \tag{3.2}$$

This distance notion depends on the trajectory taken by the clock/corresponding observer (more specifically, on the uniform velocity v), but will in general not be equal to t, which is the time measured simultaneously by a clock resting in the corresponding frame (but we can still compute τ from this t because it directly accounts for time dilation). This does *not* mean τ depends on the coordinates we use to compute it, i.e. the specific inertial frame chosen, the striking factor is the path taken by the clock we measure time for.

Example 3.2: Proper Time vs. Coordinate Time

We will use an example to elaborate a bit more on the meaning of all the symbols in (3.2). Say we are in an inertial frame with coordinates (ct, x, y, z).

The time elapsed between two events A clock resting in this frame will measure the proper time between events $E_1 = (ct_1, 0, 0, 0)$, $E_2 = (ct_2, 0, 0, 0)$, i.e. it will show

$$\tau_{E_2,E_1} = t_2 - t_1$$

to be elapsed between them. If we look at events $E_3 = (ct_1, x_1, y_1, z_1)$, $E_4 = (ct_2, x_2, y_2, z_2)$, however, it will still measure $t_2 - t_1$. This is not equal to the proper time

$$\tau_{E_3,E_4} = \sqrt{(t_2 - t_1)^2 - ((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)/c}$$

between these events, which would be measured by a clock moving on the straight world line that connects them.

Going to a frame with coordinates (ct', x', y', z'), which moves uniformly between E_3 and E_4, the situation is different. A clock resting in this frame moves on a trajectory between E_3 and E_4, which means the time t' measured by it now coincides with the proper time between these two events,

$$\tau_{E_3,E_4} = t_2' - t_1' \,.$$

This is because the spatial coordinates of E_3 and E_4 are equal in the primed frame, the Lorentz transformation automatically incorporates all spatial movement happening in unprimed coordinates into t'. However,

$$t_2' - t_1' = \tau_{E_1,E_2} = \sqrt{(t_2' - t_1')^2 - ((x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2)/c}$$ ***TODO:*** shouldn't this be $t_2 - t_1$ on the

because the trajectory connecting E_1 and E_2 is not parallel to ct'. This is the mutuality of time dilation. Nonetheless, it shows a general way how different observers can agree on times: by using the proper time between events. Not only do the numbers agree in this case, conceptually it also makes a lot of sense to look at times which are measured by clocks which actually "see" both events, i.e. move on a world line connecting them, instead of using clocks far away from the event.

This whole behaviour might seem familiar, we have already encountered it in example 2.15, where the twin paradox has been discussed in detail. Here it already showed how different observers do not necessarily agree on times t their own clocks measure between two events, but they do agree on times they infer to be measured by a clock moving on a trajectory connecting these two events – the proper time. Moreover, since all observers agree on the proper time, one can immediately infer effects like time dilation, which becomes as easy as

$$t_2' - t_1' = \tau'_{E_3,E_4} = \tau_{E_3,E_4} < t_2 - t_1 \,,$$

the primed, moving observers measures smaller times than the unprimed, resting one.

As we have seen in this example, in some coordinates it is very easy to compute proper times because the object or particle we look at is at rest in this frame, which implies that the coordinate time t is already the proper time, $\tau = t$. These coordinates are often given a special name.

Definition 3.3: Instantaneous Rest Frame

A frame (t, x, y, z) where

$$\boxed{x = y = z = 0 \qquad \Rightarrow \qquad \tau = t} \tag{3.3}$$

along the world line of a particle is called *instantaneous rest frame* or *comoving frame*.

This definition can also be extended to non-uniform velocities $v = v(t)$ because instantaneous rest frames taken at different times are related by Lorentz transforms. We will now see how the general definition of the proper time may be extended to this case.

Generalized Proper Time What was denoted with τ in equation (3.2), in reality is a difference $\Delta\tau$ of proper times (just measured with respect to $\tau = 0$), and the same is true for the coordinates (ct, x, y, z). Making these differences infinitesimally small, i.e. $\Delta \longrightarrow d$, we obtain the infinitesimal distance or *proper time element*[3]

$$\boxed{d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} = (1 - \frac{v^2}{c^2})\, dt^2 = dt^2/\gamma^2} \,. \tag{3.4}$$

[3]We adopt the common notation $dx^2 := (dx)^2 := dx \otimes dx$ ($\otimes$ is the tensor product, usually omitted here).

From the general form $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ of line elements, one can immediately read off the components of the *Minkowski metric* η, which can be conveniently arranged in a matrix

$$\boxed{\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}} . \tag{3.5}$$

Potentially contrary to popular belief, the scope special relativity is not restricted to observers with uniform velocities. Using some of the quantities and tools we have just derived, it is possible to extend the description of certain dynamics to observers moving with non-uniform velocities, i.e. accelerating ones. For distances, this is possible using integration on manifolds, where the length of a curve $\Gamma(\sigma)$ defined on an interval $I \subset \mathbb{R}$ is

$$L(\Gamma) = \int_\Gamma ds = \int_\Gamma \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_I d\Gamma := \int_I \sqrt{g_{\mu\nu} t^\mu t^\nu} \, d\sigma = \int_I \sqrt{g(\underline{t}, \underline{t})} \, d\sigma . \tag{3.6}$$

$\underline{t} = \frac{d\Gamma(\sigma)}{d\sigma}$ is the tangent vector (field) along Γ. Because we are equipped with a metric $g_{\mu\nu} = \eta_{\mu\nu}$ and corresponding line element $d\tau$, we can compute proper times for arbitrary kinds of movements.

Postulate 3.4: Clock Postulate

Given a world line Γ parametrized by $\sigma \in I = [a, b]$, i.e.

$$\Gamma(\sigma) : \sigma \mapsto (t, x, y, z) = (t(\sigma), x(\sigma), y(\sigma), z(\sigma)) ,$$

the proper time elapsed along Γ is

$$\boxed{\begin{aligned} \tau &= \int_\Gamma d\tau = \int_\Gamma dt/\gamma(t) = \int_\Gamma \sqrt{1 - \frac{v(t)^2}{c^2}} \, dt \\ &= \int_a^b \frac{d\tau}{d\sigma} d\sigma = \int_a^b \sqrt{\left(\frac{dt}{d\sigma}\right)^2 - \frac{1}{c^2} \frac{dx^\alpha}{d\sigma} \frac{dx^\alpha}{d\sigma}} \, d\sigma \end{aligned}} . \tag{3.7}$$

Note that while this "derivation" we provided here does make a lot sense, there is no guarantee for it to be correct – accelerating clocks could be vastly different from uniformly moving and resting ones. For this reason, we call (3.7) a postulate rather than a property. Once again, experiments have tested this postulate to very high accelerations of $\approx 10^{16} g$ ($g = 9.81 \frac{\mathrm{m}}{\mathrm{s}^2}$), which showed no dependence on it (only on the speed v).

Intuitively, we can see that (3.7) works because it utilizes infinitesimal steps $d\tau$ where $v(t)$ does not change, so we can apply what we know at this point and integrate up the results from all points (idea is similar to the rectification of curves). It includes the case of uniform movement $v = \text{const.}$, whence the integrand γ is constant and evaluation of the integral simply yields $\tau = \gamma(t_b - t_a)$. In case $v = v(t)$, the most convenient formula to use from (3.7) really depends on what is given – it may be the parametrization that is explicitly known or the velocity as a function of time.

Instead of σ, one could also choose arbitrary linear combinations $\sigma' = e\sigma + f$ $(e, f \in \mathbb{R})$ to parametrize Γ and then use $d\sigma' = e\, d\sigma$, which means that the integral in (3.7) is invariant under changes of this *affine parameter*. A very common choice is $\sigma = \tau$ and we will later see how this specific parametrization can be characterized.

Another remark on this definition is related to the metric. In coordinates x'^μ which are not inertial Cartesian ones (nothing prevents us from using spherical coordinates for the spatial part, as an example), the metric is very likely to have different components $\eta'_{\mu\nu} \neq \eta_{\mu\nu}$. In this case, the last equality would not hold true anymore, although analogous formulas can be obtained from (again, assuming a world line Γ with tangent vector $\underline{t}$)

$$\boxed{\tau = \int_\Gamma d\tau = \frac{1}{c}\int_\Gamma \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{c}\int_I \sqrt{\eta_{\mu\nu} t^\mu t^\nu}\, d\sigma = \frac{1}{c}\int_I \sqrt{\eta'_{\mu\nu} t'^\mu t'^\nu}\, d\sigma}\,, \tag{3.8}$$

which holds due to the transformation rule for integrals (note that the vector components have to be transformed as well, $t^\mu \to t'^\mu$). This rule is also what implies the invariance of τ under the coordinates/frame it is computed in, which is a mathematical way of stating that all observers agree on the proper time.

remark: the factor $1/c$ appears due to the coordinates being (ct, x, y, z), which implies $\eta_{\mu\nu} dx^\mu dx^\nu = d(ct)^2 - dx^2 - dy^2 - dz^2 = \frac{1}{c^2} d\tau^2$.

Proper Distance While it is an important task to measure times, an invariant notion of distances is also important. Distances are commonly associated with metrics and line elements as well, for example

$$dx^2 + dy^2 + dz^2$$

that measures lengths in the three-dimensional Euclidean space we live in. But what is an analogous quantity for four-dimensional Minkowski space?

We have already derived a line element suitable for relativistic analyses of spacetime, $d\tau$ as given by (3.4), and using the work put in there to define distances is a straightforward idea. Let us start by looking at the conceptually simplest case of measuring distance between two events E_1 at (t_1, x_1, y_1, z_1) and E_2 at (t_2, x_2, y_2, z_2), where the rod placed between (x_1, y_1, z_1) and

(x_2, y_2, z_2) is at rest. In this case, the distance measured by this rod is

$$s_{E_1,E_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)} \tag{3.9}$$

independently of the times t_1, t_2. This quantity is called the *proper length* of the measuring rod. However, for a general and thus observer-independent quantity, we have to remember that space and time are not separated anymore. This is because for an observer moving relative to the rod, the time at which he takes the measurements have a clear impact on the length that he measures the rod to have (since the endpoints of the rod are moving in this inertial frame). For invariant statements, we will now define s in terms of the proper time τ. Clearly, if we choose $t_1 = t_2$ in the rest frame of the rod, its proper length is

$$s_{E_1,E_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = c\sqrt{-\tau^2_{E_1,E_2}}\,,$$

confer (3.2). Different frames will not perceive E_2 and E_1 as simultaneous, which means the general definition will necessarily include a difference of times. Since τ does this naturally, a "generalized proper length" is

$$\boxed{s_{E_1,E_2} = c\sqrt{-\tau^2_{E_1,E_2}} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2}}\,. \tag{3.10}$$

This quantity is called *proper distance* and allows for an invariant notion of spatial distance between events.[4] Just like the proper time automatically incorporated a correction for time dilation so that all inertial frames get the same results, the proper distance corrects for length contraction occurring if the length of a rod is measured by an observer moving relative to it. This is a non-trivial statement, so it deserves to be verified explicitly.

TODO: we should note here that inspite of their formal, mathematical relation, proper times and prop

Proof. We have discussed this scenario for one spatial dimension in subsection 2.4.1. One of the results was that we have to adjust the transformed time t'_2 in order to achieve $t'_2 = t'_1$ (cf. (2.35)). As (2.33) shows, the adjustment was $\Delta t' = v/c^2(x'_2 - x'_1)$. For s, only the transformed and *not* the corrected coordinates are used. This, however, turns out to be the crucial part in getting an invariant result, i.e. why no length contraction occurs in s. We can verify this by inserting all the quantities we have mentioned into it:

$$\begin{aligned}(x_2 - x_1)^2 &\overset{!}{=} (x'_2 - x'_1)^2 - c^2(t'_2 - t'_1)^2 = (x'_2 - x'_1)^2 - c^2(\Delta t)^2\\ &= (x'_2 - x'_1)^2 - c^2\frac{v^2}{c^4}(x'_2 - x'_1)^2 = (x'_2 - x'_1)^2\left(1 - v^2/c^2\right) = (x_2 - x_1)^2\end{aligned}$$

where (2.32) was used, which shows that $(x'_1 - x'_2) = \gamma(x_2 - x_1)$. This calculation proves that

[4] Of course, one can compute it for events that are not simultaneous in the current frame as well. The result will be the proper length in some inertial frame, namely the one where the events are simultaneous.

observers from all inertial frames agree on proper distances. □

Just like the proper time, proper distances between events are also path-dependent. The proper time element (3.4) can be used to define the analogous *(proper) line element*

$$\boxed{ds^2 = -c^2 d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2} \tag{3.11}$$

which is suitable to define the proper distance covered when following an arbitrary, not necessarily straight world line Γ (with parameter $\sigma \in I \subset \mathbb{R}$ and tangent vector $\underline{t}$),

$$\boxed{s = \int_\Gamma ds = \int_\Gamma \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \int_I \sqrt{-\eta_{\mu\nu} t^\mu t^\nu}\, d\sigma} \,. \tag{3.12}$$

At this point, we shall make a remark regarding the signs: in principle, one can choose η to have only one minus (in the time component) without changing physics, if signs in equations where the metric occurs are changed accordingly. For example, the proper time and line elements would read

$$d\tau^2 = -\frac{1}{c^2}\eta_{\mu\nu} dx^\mu dx^\nu \qquad\qquad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{3.13}$$

instead. Good arguments exist for both conventions, but in the way metric and line element have been introduced here (via the proper time), it was more natural to adopt the convention with signature $(+, -, -, -)$.

The fact that this overall sign does not matter is supported by the following alternative derivation of the metric. It is also based c being constant, but instead of constructing clocks etc. explicitly, it uses that light propagates as a spherical wave with velocity c. Writing this equation in multiple inertial frames yields

$$c^2 t^2 = x^2 + y^2 + z^2 \,,\; c^2 t'^2 = x'^2 + y'^2 + z'^2 \quad \Leftrightarrow \quad c^2 t^2 - x^2 - y^2 - z^2 \overset{!}{=} c^2 t'^2 - x'^2 - y'^2 - z'^2 \,. \tag{3.14}$$

by the relativity principle. This points to the invariant proper distance we have called s or equivalently (upon multiplication of both sides with $1/c^2$) the proper time τ.

remark: this is how we can argue metric should be symmetric, right? Natural notion of distance is defined via proper time. In Minkowski, this is symmetric (as we have seen) and such a property is invariant for a given vector space/metric. Thus all metric in relativity must be symmetric

Proper Time & Distance in Spacetime Diagrams It is the invariance of proper time, distance that allows us to infer statements about the geometry of Minkowski space from them (where we now assume $v = \text{const.}$ again and thus use (3.2), (3.10)). In Euclidean space, points of constant distance s lie on a circle around the origin, which is determined by the equation $s^2 = x^2 + y^2 + z^2$.

In Minkowski space, events of equal (temporal) distance lie on a hyperboloid of constant proper times τ and are determined by

$$c^2\tau^2 = c^2t^2 - x^2 - y^2 - z^2 \,.$$

Setting $c\tau = 1$ to find a unit time length and neglecting two spatial dimensions yields

$$\boxed{c^2t^2 = 1 + x^2} \,. \tag{3.15}$$

This equation defines a hyperbola, whose intersection with time axes in spacetime diagrams determines the "length of one time unit", a very convenient way to see how moving clocks are perceived to be slower from a resting observers point of view (fig. 3.1).

In the same manner, one can visualize events of equal spatial distance by looking at

$$s^2 = x^2 + y^2 + z^2 = c^2t^2 - c^2\tau^2 = x'^2 + y'^2 + z'^2 + c^2t^2 - c^2t'^2 \,.$$

In spacetime diagrams, two of the three spatial dimensions are neglected and the unit length as a distance from the origin can be found by setting $t' = 0, x' = 1$. Therefore, we obtain

$$\boxed{x^2 = 1 + c^2t^2} \,, \tag{3.16}$$

another hyperbolic equation. The corresponding curve is visualized in figure 3.1. Similarly to what was said about time before, this curve determines the "length of one spatial unit" in spacetime diagrams by its intersection with the x-axis of any observer that we choose to put into the diagram.

Inner Product The notion of a metric allows for the construction of a rich theory. We have seen how it can be used to define distances and now we will deal with another important structure on manifolds, which has already been used in (3.8).

Definition 3.5: Minkowski Inner Product

The *Minkowski inner product* of two vectors $\underline{v}, \underline{w}$ is

$$\boxed{\underline{v} \cdot \underline{w} := \eta(\underline{v}, \underline{w}) = \eta_{\mu\nu} dx^\mu(\underline{v}) dx^\nu(\underline{w}) = \eta_{\mu\nu} v^\mu w^\nu} \,. \tag{3.17}$$

The induced norm of a vector is

$$\boxed{\|\underline{v}\|^2 = \eta(\underline{v}, \underline{v}) = \eta_{\mu\nu} v^\mu v^\nu} \,. \tag{3.18}$$

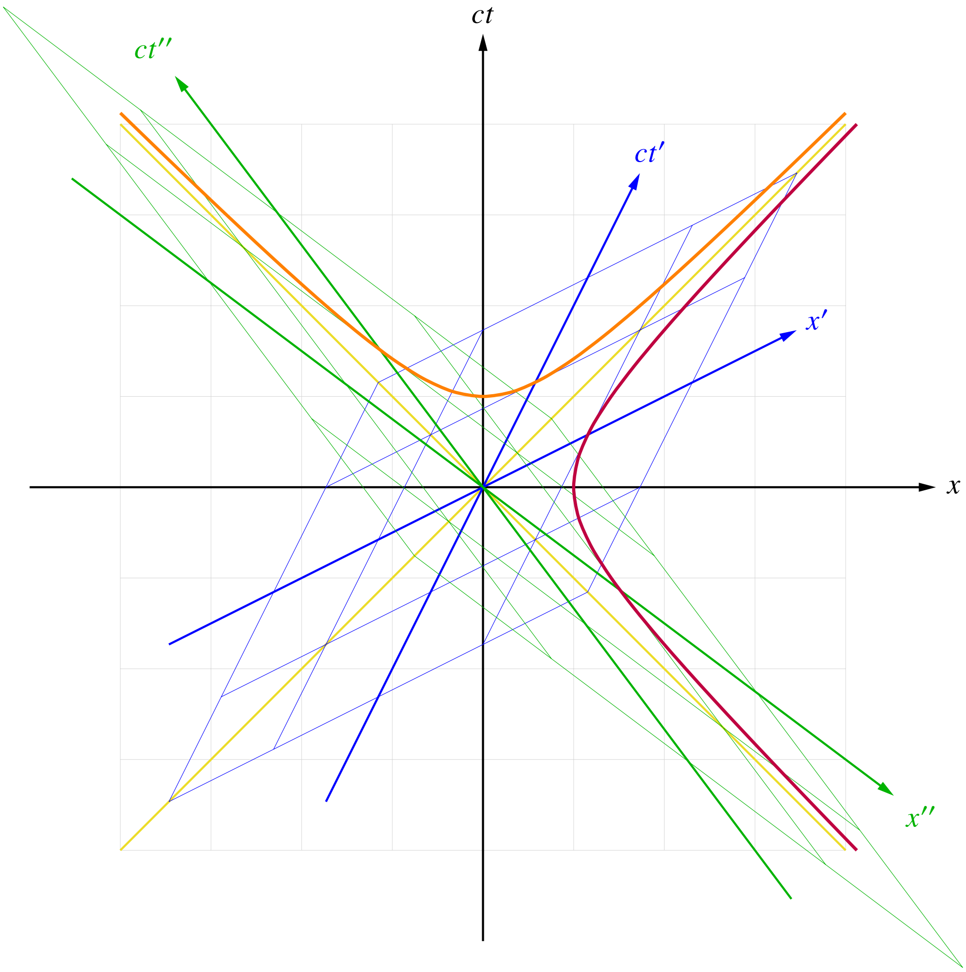


Figure 3.1: Plot of the curves fulfilling $\pm 1 = c^2t^2 - x^2$ (orange for +, purple for −). Only the positive solutions are shown to make diagram less confusing. In addition, three three inertial frames are drawn: one rest frame in black and frames moving with $v = 0.5c, -0.75$ relative to resting one in blue, green, respectively.
As we can see, this yields same time steps as Lorentz transform, the orange hyperbola intersects the time-axes of all observers perfectly at one time unit on their respective axis (as it should). Similarly, the brown hyperbola intersects the space-axes of all observers at one length unit on their respective axis.

This is also the line element of the curve that $\underline{t}$ is tangent to.

Strictly mathematically speaking, this does not define an inner product because it is not positive-definite, "only" non-degenerate. For this reason, η is also called *pseudo-Riemannian metric*. For simplicity (in typical physicist-manner) it is commonly called Minkowski inner product despite that. Due to the non-degeneracy of η, vectors can have negative or even vanishing norm. Other consequences include unit vectors now being defined by $\left\|\underline{t}\right\|^2 = \pm 1$ instead of just $+1$ in case of Euclidean geometry.

Based on (3.17), we naturally get an equivalent, more formal way of quantifying causality.

Definition 3.6: Timelike, Lightlike, Spacelike

A world line Γ with tangent vector $\underline{t}$ is called

- *timelike*, if $|v| < c \;\;\Leftrightarrow\;\; c^2 d\tau^2 = -ds^2 = \eta(\underline{t}, \underline{t}) > 0$ along Γ
- *null/lightlike*, if $|v| = c \;\;\Leftrightarrow\;\; c^2 d\tau^2 = -ds^2 = \eta(\underline{t}, \underline{t}) = 0$ along Γ
- *spacelike*, if $|v| > c \;\;\Leftrightarrow\;\; c^2 d\tau^2 = -ds^2 = \eta(\underline{t}, \underline{t}) < 0$ along Γ

This very compact characterization translates to events connected by world lines and adds a mathematical counterpart to the intuitive definition 2.7 from before. In particular, they show that due to them being related to the metric η, the causal relation of two events does not depend on the coordinates we use to denote/visualize them.

Using this definition, one can also interpret causality in terms of proper times and distances along the world line that connect them – and vice versa. First of all, while one can compute these quantities for arbitrary events, we immediately see that proper times make "more sense" for timelike- or null-separated events than they do for spacelike-separated ones, while proper distances make "more sense" for spacelike- or null-separated events than they do for timelike-separated events. "Making sense" here refers to the value of $d\tau$ and ds, respectively, which can be real or, in the cases that make less sense, be complex. Complex quantities, however, are unphysical, at least in the scope of relativity. This is also related to the interpretation of each quantity: proper time can be thought of as the time measured by a clock which moves between the events and this clock can only move with $|v| \leq c$ (i.e. not on spacelike world lines); distances, on the other hand, often make most sense when measured simultaneously (i.e. as a proper length in a certain frame), whence the corresponding events are naturally spacelike-separated (due to their spatial distance being greater than 0, which is their temporal separation). Consequently, proper distances are not to be thought of as the distance covered by a clock moving along the world line (in fact, this interpretation is more appropriate for the proper time).

remark: this is related to the fact that for events separated by a timelike interval, there exists an inertial frame where they happen at the same place, while for spacelike separation, there exists one where they happen at the same time.

-¿ Penrose continues to make interesting point on that on page 407: light cones more fundamental than metric

3.1.2 Lorentz Transformations 2

Lorentz transformations have been introduced in section 2.4, from which we know how they look like, that they correctly reproduce effects of relativity and how they can be visualized in spacetime diagrams. Nonetheless, it is worthwhile to revisit them now because of the knowledge we have gained since then, in particular regarding the mathematical interpretation of spacetime as a 4-manifold. We have already seen how this formulation comes with a natural structure, like vectors. A crucial part of the theory of manifolds is yet to be discussed. After all, one of the key motivations to use manifolds was that while they are described in terms of coordinates, but their properties exist independently of them, i.e. they do not depend on the specific choice of coordinates. Consequently, changing coordinates is an important part of the whole theory and the coordinate transformations of spacetime are those between different inertial frames – Lorentz transformations. This interpretation is the first time we encounter the more general role they play since there is a rich mathematical structure related to them.

-¿ for more on group-theoretic nature (and mathematical treatment of SR in general), I can recommend [Giu06]

-¿ after metric is nice, then we can present mathematical viewpoint; do not change norm of four-vector (this is well-known property of rotations, but there is also a second class of Lorentz transformations, which are called boosts; they represent what we have derived in previous section, change to other inertial frame that moves uniformly with respect to first one); also note that they admit group structure, i.e. note properties here; maybe even introduce rapidity and note connection of Lie group and Lie algebra? But not elaborate on this

maybe just note that Lorentz transformation = change of coordinates/charts (which is corresponding term in language of manifolds); one condition to obtain them is by demanding norm of four-vector does not change (ahh, might be confusing because I think Nolting refers to this kind of in Euclidean space; what he calls norm is really proper time passing on clock which moves from origin to point, isn't it? Would make sense to demand this better stay the same after transformation after we have put so much effort into invariant definition) -¿ I like this, but does not fit before introduction of norm; so maybe make subsection on transformations after metric?

Lorentz-scalar or Lorentz-invariant is scalar quantity, which does not change under Lorentz-transformation; example is proper time, but also mass etc. are of this nature

when interpreting Minkowski space as a manifold and working with coordinates/charts ξ, we know the results should be independent of ξ; in particular, that means they hold in other charts as well and changing coordinates is an important part; the basis changes even have a distinct name, *Lorentz transformation*; this is basically group theory due to the symmetries that Minkowski space possesses (known from logic and experiments) ?right?

hmmm, parallel transformations form a group (as Einstein states); commutativity is trivial, but associativity not:

$$\begin{aligned}(v_1 +_R v_2) +_R v_3 &= \frac{v_1 + v_2}{1 + v_1 v_2/c^2} +_R v_3 = \frac{\frac{v_1+v_2}{1+v_1 v_2/c^2} + v_3}{1 + \frac{v_1+v_2}{1+v_1 v_2/c^2} v_3/c^2} \\ &= \frac{\frac{v_1+v_2+v_3+v_1 v_2 v_3/c^2}{1+v_1 v_2/c^2}}{1 + \frac{v_1 v_3+v_2 v_3}{1+v_1 v_2/c^2}/c^2} = \frac{v_1 + v_2 + v_3 + v_1 v_2 v_3/c^2}{1 + v_1 v_2/c^2 + (v_1 v_3 + v_2 v_3)/c^2} \\ &= \frac{v_1 + v_2 + v_3 + v_1 v_2 v_3/c^2}{1 + (v_1 v_2 + v_1 v_3 + v_2 v_3)/c^2} \end{aligned} \tag{3.19}$$

this last expression is very symmetric, so we might suspect already that it is associative; essentially by repeating these calculations, we obtain that $v_1 +_R (v_2 +_R v_3)$ evaluates to the same expression, so parallel transformations (boosts) admit group structure with group operation being $+_R$

3.2 Relativistic Kinematics

Up until this point, we have mostly dealt with uniform motion. To be able to make realistic physics, however, this is not sufficient because rarely is a motion ever fully uniform. Thus, in order to deal with relativistic dynamics we have to find ways of treating non-uniform, accelerated motion as well.

It seems to be a common misconception that such topics cannot be treated using the tools of special relativity.[5] It should be stressed that, no matter what may be claimed otherwise, special relativity *can* certainly handle acceleration. Without any problems, the tools developed over the course of the last sections is able to visualize world lines of accelerating observers and calculate proper times/lengths for them using (3.7). However, all of these calculations are still made in inertial frames (which plays an important role in why the equations are valid in the first place), what about accelerating reference frames? As it turns out, this is a topic that cannot be generalized so easily from the theory dealing with uniform velocities – but nonetheless, it is possible and we will talk about it briefly.

Acceleration is not the only topic that has to be developed in a relativistic manner, though. Many other notions will have to be looked at and reformulated using four-vectors in Minkowski space, for example momentum, energy, force. To start this development, we have to think about velocities at first. Excellent sources for this section are [Fle19; Far13].

remark: We elect to formulate everything as general as possible, which means we go from using just ICCs (which are nothing but coordinates for inertial frames) to arbitrary frames of reference. This will require e.g. using covariant derivates ∇ because the Christoffel symbols do not generally vanish, unlike in inertial frames.

3.2.1 Four-Velocity

Trajectories or world lines are nothing but curves on manifolds, as we have already seen. A natural question, however, was left unanswered until now: what is the velocity of such a trajectory? Since its role is to quantify how fast and in which direction the trajectory is changing, velocity now becomes a tangent vector. In analogy to the previous Newtonian description, we may try

$$\underline{u} = \frac{d}{dt} \qquad \Leftrightarrow \qquad u^\alpha = \frac{dx^\alpha}{dt} = (c, \vec{v})\,. \tag{3.20}$$

Thinking back to the previous sections on clocks and proper time, though, we can immediately see how this definition is flawed: coordinate times t are not invariant, i.e. the velocity of a world

[5] Many authors claim that many people claim this. I have no idea how frequently this actually happens, but it one reason is probably that during the beginnings of relativity, the line between special and general relativity was not so clear (and the nomenclature was not, i.e. what was meant by the “special” and “general” parts). Nowadays, though, the consensus among scientists is that the line is gravity – acceleration falls into the scope of special relativity.

line would change depending on the observer. At the same time, we can immediately come up with a solution: replacing t with the proper time τ a clock would measure along the world line. This leads to the *four-velocity*

$$\boxed{\underline{u} = \frac{d}{d\tau} = u^\alpha \frac{d}{dx^\alpha}} \tag{3.21}$$

with components (in arbitrary coordinates x^α)

$$\boxed{u^\alpha = \frac{dx^\alpha}{d\tau} = (c\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}) = \frac{dt}{d\tau}\,(c, \frac{d\vec{x}}{dt}) = \gamma(c, \vec{v})}\,, \tag{3.22}$$

$\vec{v}$ is the "regular" three-velocity, which finds its way into the equation by a simple application of the chain rule. Clearly, although u^α is a four-vector, it has only three independent components because the first one is constant. An interesting corollary from this constancy is that the temporal component u^0 never vanishes, so time never stops (even if there is no spatial motion, $v = 0 \;\Leftrightarrow\; u^1 = u^2 = u^3 = 0$).

An important thing to note is that $\underline{u}$ is only defined for timelike world lines because $d\tau = 0$ for lightlike ones (dividing by zero is not defined), while $d\tau$ is complex for spacelike ones (regardless of whether one can make sense of this derivative mathematically, complex velocities do not make sense physically).

Information we get from $\vec{v}$ is the direction of an object (through the unit vector $\vec{v}/\|v\|$) and how fast (through $v = \|v\|$). As a tangent vector to certain points $x^\alpha(\tau)$ on the corresponding world line Γ, $\underline{u}$ naturally contains information about the direction, so what about $\|\underline{u}\|$?

$$\boxed{\|\underline{u}\|^2 = \eta_{\mu\nu}u^\mu u^\nu = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{\eta_{\mu\nu}dx^\mu dx^\nu}{d\tau^2} = c^2\frac{d\tau^2}{d\tau^2} = c^2}\,. \tag{3.23}$$

The four-velocity possesses a "built-in" normalization, it has a constant magnitude, the speed of light – no matter at which actual speed v the particle moves (and also, independent of the time τ we evaluate it at, i.e. independent of the point in manifold)! Not only does that imply all observers trivially agree on its value, it also points to the special role of τ as affine parameter σ of the world line. Note that the calculation here only works because the Minkowski metric has constant components. Equivalently, we could calculate

$$\|\underline{u}\|^2 = \eta_{\mu\nu}u^\mu u^\nu = \gamma^2(c^2 - v^2) = c^2\frac{1 - v^2/c^2}{1 - v^2/c^2} = c^2\,.$$

3.2.2 Four-Acceleration

Now we can come to acceleration. In principle, treating it does not seem to hard, it should just be a matter of taking one more derivative of $\underline{x}$. Indeed, one can treat acceleration this way – but we have to realize that we still work in inertial, i.e. non-accelerated, frames when doing that (which is completely valid, just important to note). To start this subsection, we will also go this route of taking a derivative of $\underline{u}$. For tangent vectors, however, this is not as straightforward as taking derivative of x^α, instead one has to use a covariant derivative. If we to that, things work out beautifully and we obtain

$$\boxed{\underline{a} = \nabla_{\underline{u}}\underline{u}} \qquad \boxed{a^\alpha = (\nabla_{\underline{u}}\underline{u})^\alpha = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\delta}u^\beta u^\delta} \,. \tag{3.24}$$

In inertial frames (i.e. in ICCs), $\nabla_{\underline{u}} = \frac{d}{d\tau}$ and we obtain more explicit versions:

$$\boxed{\underline{a} = \frac{d\underline{u}}{d\tau} = a^\alpha \frac{d}{dx^\alpha}} \qquad \boxed{a^\alpha = \frac{d^2x^\alpha}{d\tau^2} = \frac{du^\alpha}{d\tau} = \gamma\frac{du^\alpha}{dt} = \gamma(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\vec{v} + \gamma\vec{a})} \tag{3.25}$$

where $\vec{a} = \frac{dv}{dt}$ is the familiar three-acceleration (which is sometimes also called *proper acceleration*). One can even further massage this expression:

$$\begin{aligned}
\boxed{a^\alpha} &= \gamma(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\vec{v} + \gamma\vec{a}) = \gamma\frac{d\gamma}{dt}\,(c, \frac{d\gamma}{dt}\vec{v}) + \gamma^2(0, \vec{a}) \\
&= \gamma\frac{d}{dt}\left(\frac{1}{\sqrt{1 - \vec{v}\cdot\vec{v}/c^2}}\right)\left(c, \frac{d\gamma}{dt}\vec{v}\right) + \gamma^2(0, \vec{a}) \\
&= \gamma\frac{d}{dt}\left(1 - \vec{v}\cdot\vec{v}/c^2\right)\frac{-1}{2}\frac{1}{(1 - \vec{v}\cdot\vec{v}/c^2)^{3/2}}\left(c, \frac{d\gamma}{dt}\vec{v}\right) + \gamma^2(0, \vec{a}) \\
&\boxed{= \frac{\gamma^4}{c^2}\,\vec{v}\cdot\vec{a}\,(c, \vec{v}) + \gamma^2(0, \vec{a})}
\end{aligned} \tag{3.26}$$

Note that $\vec{v}\cdot\vec{a}$ is a scalar product of three-vectors, which means it refers to the one from 3D Euclidean space here, $x^k y_k$.

Essentially, the four-acceleration consists of two terms, the first one being proportional to the four-velocity and the second one being the four-acceleration measured in the rest frame of of the object whose world line we are looking at (whence $\frac{d\gamma}{dt} = 0$, constant velocity). The second term could have been expected, it is just the regular three-acceleration multiplied with $(dt/d\tau)^2$, which appears due to the use of τ instead of t (and becomes 1 in the rest frame of the world line). The first term, however, is new and interesting: it depends on the relative orientation of $\vec{v}, \vec{a}$; it scales with $\gamma^4 = (\gamma^2)^2$ and thus becomes dominant for high velocities (while both are on same scale for small v); lastly, it influences the time component and it determines how much the three-velocity $\vec{v}$ adds to the "pure" three-acceleration $\vec{a}$.

The norm of $\underline{u}$ was a very interesting result, how about the one of $\underline{a}$? For simplicity, we will compute it in ICCs (which is fine because norm is invariant scalar):

$$\begin{aligned}
\boxed{\|\underline{a}\|^2} &= a^\mu a_\mu = \frac{\gamma^8}{c^4}(\vec{v}\cdot\vec{a})^2 c^2 - \left(\frac{\gamma^4}{c^2}(\vec{v}\cdot\vec{a})\vec{v} + \gamma^2\vec{a}\right)^2 \\
&= \frac{\gamma^8}{c^4}(\vec{v}\cdot\vec{a})^2 c^2 - \frac{\gamma^8}{c^4}(\vec{v}\cdot\vec{a})^2 v^2 - 2\frac{\gamma^6}{c^2}(\vec{v}\cdot\vec{a})^2 - \gamma^4 a^2 \\
&= \frac{\gamma^8}{c^4}(\vec{v}\cdot\vec{a})^2\left(c^2 - v^2\right) - 2\frac{\gamma^6}{c^2}(\vec{v}\cdot\vec{a})^2 - \gamma^4 a^2 \\
&= \frac{\gamma^8}{c^4}(\vec{v}\cdot\vec{a})^2\frac{c^2}{\gamma^2} - 2\frac{\gamma^6}{c^2}(\vec{v}\cdot\vec{a})^2 - \gamma^4 a^2 \\
&\boxed{= -\frac{\gamma^6}{c^2}(\vec{v}\cdot\vec{a})^2 - \gamma^4 a^2}
\end{aligned} \tag{3.27}$$

Two notable special cases exist:

$$\vec{v} \parallel \vec{a} \quad\Rightarrow\quad \vec{v}\cdot\vec{a} = va \quad\Rightarrow\quad \boxed{\|a\|^2 = \gamma^6 a^2} \tag{3.28}$$

$$\vec{v} \perp \vec{a} \quad\Rightarrow\quad \vec{v}\cdot\vec{a} = 0 \quad\Rightarrow\quad \boxed{\|a\|^2 = \gamma^4 a^2} \tag{3.29}$$

This amounts to the fact that accelerations of world lines measured in different coordinates are not necessarily equal (just like four-velocities can differ between inertial frames).

Another notable property of $\underline{a}$ is its orientation relative to $\underline{u}$ as measured by the Minkowski inner product $\underline{a}\cdot\underline{u} = a^\mu u_\mu$:

$$\boxed{0 = \frac{dc^2}{d\tau} = \frac{d}{d\tau}u^\mu u_\mu = 2a^\mu u_\mu}\,, \tag{3.30}$$

the four-acceleration is orthogonal to the four-velocity (note: this does not mean $\vec{a}$ is orthogonal to $\vec{v}$, orthogonality is meant strictly in a Minkowskian way).

Example 3.7: Uniformly Accelerated World Lines

maybe treat it here? Can be done analytically

https://de.wikipedia.org/wiki/Zeitdilatation#Bewegung_mit_konstanter_Beschleunigung

see fig. 3.2

Accelerated Frames It has been mentioned a number of times until now that accelerated frames of reference are harder to deal with than inertial ones. Why is that? To see it, let us think back to how the Lorentz transformation has been derived. One cornerstone was the constancy of c in vacuum, which had strong implications on how inertial frames are to be converted into each other.

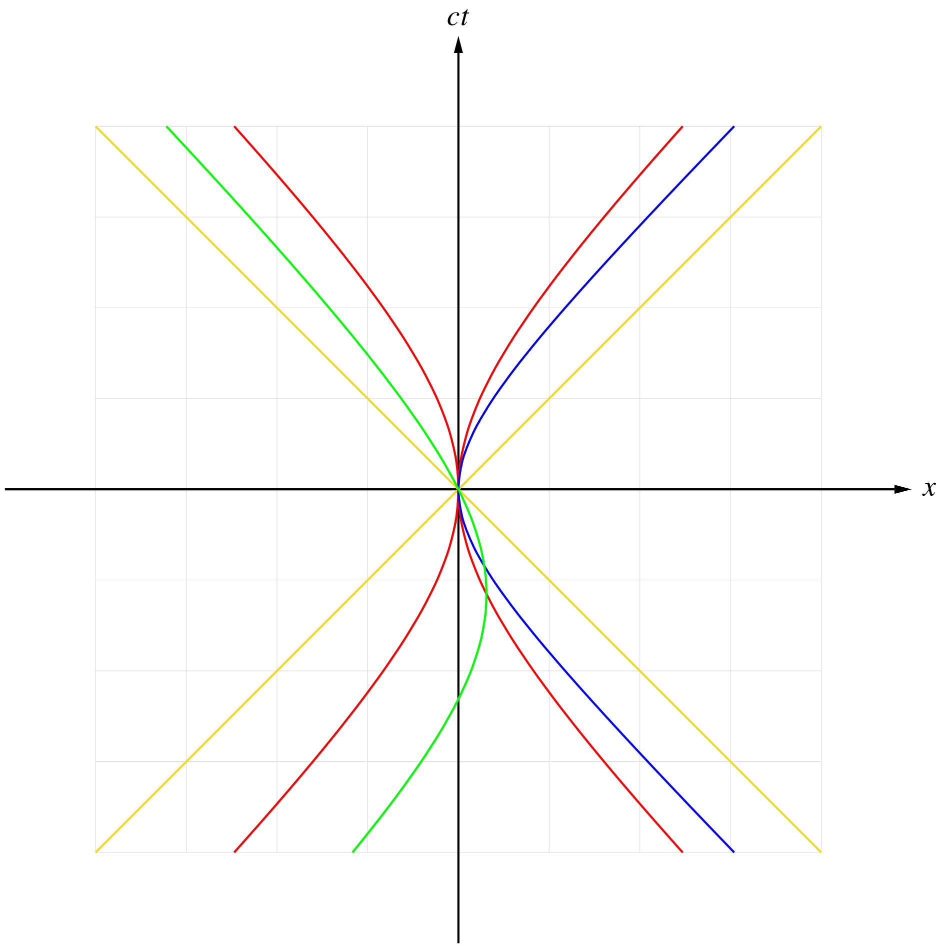


Figure 3.2: Uniformly accelerated world lines.
The (v_0, a_0)-values are $(0, \pm 0.5c/\mathrm{s})$ in red, $(0, 0.9c/\mathrm{s})$ in blue and $(-0.5c, -0.5c/\mathrm{s})$, choosing one time unit in the diagram to be a second. We can clearly see the hyperbolic shapes of all world lines, which are characteristic for uniform acceleration.

Looking back at postulate 2.5, this constancy is only postulated for inertial, i.e. non-accelerating, frames. Therefore, the Lorentz transformation cannot be used anymore and deriving general transformations is much harder because we have lost this property of c being constant that was so helpful when deriving relativistic quantities. Instead, several alternatives exist.

In case of uniform acceleration, one can utilize *Rindler coordinates* as the coordinates of the corresponding accelerated frame. We will focus on a different idea, though. The line of thought leading to this idea is as follows (where we assume to look at a world line in some inertial, but otherwise arbitrary, frame of reference): at each point p on an accelerated world line Γ, the object moves with a certain velocity (which does not change in p, of course, because it is a point). Therefore, by going into the inertial frame that moves with this velocity relative to the initial one, we find the rest frame of the object at this specific point p, the so-called *momentarily comoving inertial frame* (MCIF).[6] The same procedure can be applied to find the rest frame at every point on the Γ, which means we can make sense of the notion of a rest frame along Γ. Moreover, the MCIFs at different points are just inertial frames with different relative velocities, which means we can transform between them using Lorentz transformations. Note the resemblance of this idea to what the clock postulate 3.4 stated for the computation of proper times: we replace $v \rightarrow v(t)$, so that what we have learned about uniform motion can be applied. This, however, only works for infinitesimal intervals, which means the replacement only works in a specific point. For the general result, some more work is needed, namely transforming between the points (which is now possible using Lorentz transformations).[7]

In practice, this whole procedure is still complicated, but at least the idea should now be clear. Furthermore, this procedure being complicated does not completely prevent us from studying accelerated world lines, though. After all, one can still examine their behaviour from inertial frames. Only for some phenomena, it is more convenient to actually analyze the situation from within an accelerated frame.

-¿ on transforming into MCIF: https://math.ucr.edu/home/baez/physics/Relativity/SR/clock.html

A final note concerns acceleration in such a MCIF. While one works in rest frames here, this rest is only momentarily. Hence, there is still an infinitesimal change in velocity (affecting the neighbouring points), i.e. acceleration. More precisely,

$$\boxed{a^\alpha = (0, \vec{a})} \qquad \Leftrightarrow \boxed{\|\underline{a}\|^2 = \vec{a} \cdot \vec{a} = a^2} \,. \tag{3.31}$$

The four-acceleration has only spatial parts here and consequently, its magnitude is equal to

[6] Other names are I have seen which should refer to the same thing are instantaneously comoving frame or momentarily comoving reference frame (MCRF).

[7] Two very good visualizations of this are provided here: https://en.wikipedia.org/wiki/Spacetime_diagram#Accelerating_observers.

the magnitude of the three-acceleration. However, this norm being constant does not mean the components a^α are equal in different MCIFs. Moreover, even if this was the case, coordinate axes of MCIFs at different points will, in general, point into different directions (as seen from another, fixed inertial frame), so constant spatial components in MCIFs would not be equivalent to the three-acceleration pointing into the same direction at all time (as seen from the fixed frame again). Yet another addition to this is that constancy of $\|\underline{a}\|$ does not necessarily imply uniform acceleration because in a general, perhaps even inertial, frame the magnitude of each component a^α need not be constant (for example, $\underline{a}$ having only spatial components in MCIFs tells us nothing about components in other inertial frames).

-¿ do we even have constancy of a at different points? I mean at a fixed point it has to be constant in all frames, clearly, but at different points? For non-uniform acceleration, this would not be true, would it?

Accelerated Clocks Having gained more knowledge as to how acceleration works in relativity, it is appropriate to provide some thoughts on the clock postulate 3.4 now.[8]

The main statement/takeaway was that even for non-uniform motion $v = v(t)$, the infinitesimal proper time interval still has the form

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt$$

and involves no additional terms proportional to acceleration (which also implies the speed of light is c, at least locally, i.e. in the corresponding MCIFs). That is not to say acceleration has no effect on a clock's rate at all – the corresponding changing speeds do change $d\tau$, but it is really only through v in $\gamma(v)$.[9] To incorporate that, integrals over world lines have been introduced, which essentially sum up all the infinitesimal time intervals $d\tau$. An equivalent interpretation using accelerated frames is that one sums up the times going by in each MCIF along the world line, where the speed is momentarily constant such that the formula for uniform motion can be used – the clock rate on accelerating world line is at all points equal to rate of clock in the corresponding MCIF at each point. Since these frames have to be changed in every point, these times are also the infinitesimal ones $d\tau$ and summing them all up results in the same formula.

The clock postulate applies to more than just the rate of clocks. Proper distances can be measured using integrals over the infinitesimal results obtained for uniform motion as well (which should be clear since ds can be expressed using $d\tau$) and as it turns out, many other quantities involving the Lorentz-factor γ can be generalized to $v = v(t)$ by simply using this non-uniform velocity in γ, for example $\underline{u}$.

[8] Many of these thoughts are inspired by articles in the special relativity section of [].

[9] This is despite (3.27) having a non-zero time component, which means that the rate at which the clock rate itself changes is not constant. But still, the clock postulate (which is the only relevant statement in this regard) tells us that this does still not lead to an additional contribution from a in $d\tau$.

However, it must always be remembered that the clock postulate cannot be proven rigorously (and its statement is by no means obvious, everything could very well depend change once $a \neq 0$). One can only *argue* that it makes sense the way it is (like we just did) and there is also experimental evidence supporting it.

Example 3.8: Rotating Disk

cool: https://en.wikipedia.org/wiki/Proper_time#Example_2:_The_rotating_disk

here, $\vec{v} \perp \vec{a}$ and therefore, no induced effect of acceleration at all (since $v =$ const.)

Example 3.9: Twin Paradox 3

acceleration *does* effect the rate of clocks!!! The claim we made during twin paradox was just that acceleration is not the reason for different ageing, uniform motion is sufficient for that; but rapid braking in rocket for example will definitely affect time on clock in this rocket – and analyzing situation from within the rocket is really hard (and thus statements like it runs slower or faster; is much easier to analyze this from inertial frames)

straightforward (not necessarily easy) way is find parametrization of world line, then take derivative with respect to coordinate time t, then do integral (potentially numerically)

3.2.3 Four-Momentum & Energy

Newtonian dynamics in its general form was formulated in terms of momentum, the product of mass and velocity. Continuing the analogous formulation, the *four-momentum* is

$$\boxed{\underline{p} = m\underline{u}} \qquad\qquad \boxed{p^\alpha = mu^\alpha = \gamma(mc, m\vec{v}) = (\gamma mc, \vec{p}_r)}\,. \tag{3.32}$$

The spatial components can be understood as a *relativistic momentum*

$$\boxed{\vec{p}_r = \gamma\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}} \tag{3.33}$$

and are clearly just a generalization of the “usual” Newtonian momentum $\vec{p}$, picking up a factor of γ in comparison to the Newtonian definition. As (2.31) tells us, this factor is ≈ 1 for $v \ll c$, so we retrieve $\vec{p}_r = \vec{p} = m\vec{v}$ in this case.

Conversely, the time component is harder to interpret. Since whatever it represents is very likely to have a Newtonian counterpart, let us look at p^0 in case $v \ll c$. Continuing (2.31) shows

$$\boxed{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}} \simeq \left.\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right|_{v=0} + \left.\frac{\frac{1}{2}\cdot\frac{-2v}{c}}{(1-\frac{v^2}{c^2})^{3/2}}\right|_{v/c=0} \frac{v}{c} + \frac{1}{2}\left.\frac{d}{dv/c}\left(\frac{\frac{1}{2}\cdot\frac{-2v}{c}}{(1-\frac{v^2}{c^2})^{3/2}}\right)\right|_{v/c=0} \frac{v^2}{c^2} + \dots$$
$$= \left.\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right|_{v=0} + \left.\frac{\frac{1}{2}\cdot\frac{-2v}{c}}{(1-\frac{v^2}{c^2})^{3/2}}\right|_{v/c=0} \frac{v}{c} + \frac{1}{2}\left.\left(\frac{\left(\frac{-1}{2}\cdot\frac{-2v}{c}\right)^2}{(1-\frac{v^2}{c^2})^{5/2}} + \frac{\frac{-1}{2}\cdot(-2)}{(1-\frac{v^2}{c^2})^{3/2}}\right)\right|_{v/c=0} \frac{v^2}{c^2} + \dots$$
$$\boxed{= 1 + \frac{v^2}{2c^2} + O\left(\frac{v^3}{c^3}\right)}\,. \tag{3.34}$$

and thus

$$cp^0 = \gamma mc^2 \approx mc^2 + \frac{mv^2}{2}\,.$$

The first term is a constant with dimension of an energy that we do not know yet, but the second term looks familiar: it is kinetic energy. For this reason, we identify

$$\boxed{p^0 = \frac{E}{c}} \qquad\qquad \boxed{E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}}\,, \tag{3.35}$$

E being the *relativistic energy* of a free particle. Apparently, the kinetic energy (which is the energy of a free particle) has more contributions than just the familiar one $\sim v^2$. Since the terms of higher order in v only become relevant for velocities comparable to c, however, it is no surprise that they are discovered by special relativity and not known from Newtonian mechanics. A rather interesting consequence arises for the other velocity limit $v = 0$ whence

$$\boxed{E = mc^2}\,. \tag{3.36}$$

Here we have the most famous formula of physics (remember I mentioned second most before; we have now finally completed quest for most famous one). Just like many insights from relativity, this formula has profound implications: it tells us that mass and energy are equivalent (strictly speaking, we can see that from (3.35) already, but there we might still suspect it comes from kinetic part). Even particles at rest still have energy, namely an amount proportional to what is often called *rest mass* m in this context.[10] This *rest energy* scales with c^2 and therefore, even very small masses contain huge amounts of energy (which shows e.g. in the fact that we gain energy from nuclei nowadays). This motivates defining the *relativistic kinetic energy* as

$$\boxed{T = E - mc^2 = (\gamma - 1)\, mc^2}\,. \tag{3.37}$$

[10]Because sometimes, a *relativistic mass* $m(v) = \gamma m = \frac{m}{\sqrt{1-v^2/c^2}}$ is defined. We will not use this definition.

Equivalently, since

$$\boxed{m^2c^2 = m^2u^\mu u_\mu = p^\mu p_\mu = (p^0)^2 - \vec{p}^2 = E^2/c^2 - \vec{p}_r^2} \tag{3.38}$$

we can also write

$$\boxed{E = \sqrt{(mc^2)^2 + c^2\vec{p}_r^2}}\,. \tag{3.39}$$

One major advantage of using momenta as tangent vectors instead of velocities is that p^α can be defined for timelike *and* null vectors. A priori,

$$p^\alpha = mu^\alpha = m\frac{dx^\alpha}{d\tau} = \frac{0}{0}$$

since $d\tau = 0$ for photons and they have no mass, i.e. $m = 0$. Apparently, we gain no insight from this definition. However, we do know that photons have energy

$$\boxed{E = hf}\,, \tag{3.40}$$

so

$$0 = p^\mu p_\mu = \frac{h^2f^2}{c^2} - p_r^2\,.$$

Therefore, we can associate the spatial components of photon's momentum with a three-vector of magnitude

$$\boxed{p_r = \hbar k = \frac{h}{\lambda} = \frac{hf}{c} = \frac{E}{c}} \tag{3.41}$$

pointing into the direction of some unit vector $\vec{e}_k$, which is the direction that the photon propagates into. This vector $\vec{k} = k\vec{e}_k$ is called *wave vector*.

3.2.4 Four-Force & Equation of Motion

Our goal in this section will be to find a way to treat dynamics in a (special-)relativistic way, which is equivalent to finding the equation of motion. As it turns out, there is not much work to do. Despite some aspects of Newton's theory being limited in their scope, it is not entirely wrong and nothing we have found out until now speaks against continuing to use Newton's second law (in an appropriate four-vector version).

One difference in notation will be the use of coordinate-independent language, i.e. covariant derivative over "regular", just like it has been used to define the four-acceleration, but the interpretation and idea remains the same. Equating this derivative of $\underline{P}$ with the *four-force* $\underline{F}$

yields the following equation of motion:

$$\boxed{\underline{F} = \nabla_{\frac{d}{d\tau}\underline{p}} = \nabla_{\underline{u}}\underline{p} = \frac{1}{m}\nabla_{\underline{p}}\underline{p}} \qquad \boxed{F^\alpha = (\nabla_{\underline{u}}\underline{p})^\alpha = \frac{dp^\alpha}{d\tau} + \Gamma^\alpha_{\beta\delta}u^\beta p^\delta} \tag{3.42}$$

Using the product rule for ∇, we can write that out a little further to obtain

$$\boxed{\underline{F} = \frac{dm}{d\tau}\underline{u} + m\nabla_{\underline{u}}\underline{u}} \qquad \boxed{F^\alpha = \frac{dm}{d\tau}u^\alpha + m\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\delta}u^\beta p^\delta} \tag{3.43}$$

where $\nabla_{\underline{u}}m = \frac{dm}{d\tau}$ since the mass is a scalar function. Therefore, we can write the components of $\underline{F}$ as (dropping the assumption m = const.)

$$\boxed{\underline{F} = (\frac{1}{c}\frac{dE}{d\tau}, \frac{d\vec{p}_r}{d\tau}) = \gamma(\frac{1}{c}\frac{dE}{dt}, \frac{d\gamma}{dt}\vec{p} + \gamma\vec{F}) = (F^0, \gamma\vec{F}_r)} \tag{3.44}$$

with

$$\boxed{\vec{F}_r = \frac{d\vec{p}_r}{dt}} \qquad \boxed{\vec{F} = \frac{d\vec{p}}{dt}} \tag{3.45}$$

being the relativistic and non-relativistic three-forces, respectively and F^0 being the a "force in time" or, perhaps more illustrative, a change of energy P^0 over time, i.e. the *power* of the force. This power is not independent of the other, spatial force components as a quick calculation in shows (assuming ICCs for simplicity):

$$\begin{aligned} F^\mu u_\mu &= \frac{dm}{d\tau}u^\mu u_\mu + m\,a^\mu u_\mu \underset{(3.30)}{=} \frac{dm}{d\tau}c^2 = \gamma c^2\frac{dm}{dt} \\ &= F^0 u_0 - \gamma\frac{d\vec{p}_r}{d\tau} = \gamma^2\frac{dE}{dt} - \gamma^2\frac{d\vec{p}_r}{dt} \\ \Leftrightarrow \qquad & \boxed{F^0 = \frac{\gamma}{c}\frac{dE}{dt} = \gamma c\frac{dm}{dt} + \frac{\gamma}{c}\frac{d\vec{p}_r}{dt}\cdot\vec{v}} \end{aligned} \tag{3.46}$$

Therefore, any change in energy corresponds either to a change in mass or to the (Euclidean) scalar product between what is often interpreted as a relativistic three-force in $\frac{d(\gamma\vec{p})}{dt}$ and the velocity $\vec{v}$, i.e. work done on the system.

For constant mass m, (3.43) simplifies to

$$\boxed{\underline{F} = m\nabla_{\underline{u}}\underline{u} = m\underline{a}} \qquad \boxed{F^\alpha = m\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\delta}u^\beta p^\delta = m\,a^\alpha}, \tag{3.47}$$

which clearly resembles Newton's second law (2.5). In fact, it reduces to this exact equation if we assume ICCs, where $\nabla_{\frac{d}{d\tau}} = \frac{d}{d\tau}$ and thus

$$\boxed{\underline{F} = m\frac{d\underline{u}}{d\tau}} \qquad \boxed{F^\alpha = m\frac{du^\alpha}{d\tau}} \,.$$

Split up into components, that means

$$\boxed{\frac{1}{c}\frac{dE}{dt} = \frac{d\vec{p}_r}{dt}\cdot\vec{v}} \qquad \boxed{\vec{F} = m\,\vec{a}} \,. \tag{3.48}$$

Both of these equations are known from Newtonian mechanics as well.

In the force-free case, i.e. for a free particle, (3.42) becomes

$$\boxed{\nabla_{\underline{p}}\underline{p} = 0} \qquad \boxed{u^\alpha\nabla_\alpha p^\beta = u^\alpha(\nabla_\alpha\underline{p})^\beta = 0} \,. \tag{3.49}$$

This is the *geodesic equation*. It expresses that U^β remains constant as long as we take the derivative along the curve that it is tangent to (which explains the $u^\alpha\nabla_\alpha$ part). Consequently, the world lines of freely moving test particles are *geodesics*.

For a free particle with constant mass m, we can further simplify

$$\boxed{\nabla_{\underline{u}}\underline{u} = 0} \qquad \boxed{u^\alpha\nabla_\alpha u^\beta = u^\alpha(\nabla_\alpha\underline{u})^\beta = 0} \,. \tag{3.50}$$

Coordinate versions of equations (3.42), (3.47) are

$$\boxed{F^\alpha = \frac{dm}{d\tau}\frac{dx^\alpha}{d\tau} + m\frac{d^2x^\alpha}{d\tau^2} + m\,\Gamma^\alpha_{\beta\delta}\frac{dx^\beta}{d\tau}\frac{dx^\delta}{d\tau}} \qquad \boxed{F^\alpha = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\delta}\frac{dx^\beta}{d\tau}\frac{dx^\delta}{d\tau}} \tag{3.51}$$

from which the one for the geodesic equation follow by setting $F^\alpha = 0$.

Proof. We can get to (3.51) via a straightforward calculation to express $\nabla_{\underline{u}}\underline{u}$ in coordinates. In fact, one can prove this for an arbitrary tangent vector $\underline{t}$ along Γ, which is parametrized by an arbitrary parameter σ and not necessarily the proper time τ.

$$\begin{aligned}
t^\alpha \nabla_\alpha t^\beta &= \frac{dx^\alpha}{d\sigma} \nabla_\alpha \frac{dx^\beta}{d\sigma} \\
&= \frac{dx^\alpha}{d\sigma} \left(\frac{\partial}{\partial x^\alpha} \frac{dx^\beta}{d\sigma} + \Gamma^\beta_{\alpha\delta} \frac{dx^\delta}{d\sigma} \right) \\
&= \frac{dx^\alpha}{d\sigma} \frac{\partial}{\partial x^\alpha} \frac{dx^\beta}{d\sigma} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma} \\
&\underset{\text{chain rule}}{=} \frac{d}{d\sigma} \frac{dx^\beta}{d\sigma} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma} \\
&= \frac{d^2 x^\beta}{d\sigma^2} + \Gamma^\beta_{\alpha\delta} \frac{dx^\alpha}{d\sigma} \frac{dx^\delta}{d\sigma} \qquad \square
\end{aligned}$$

Faraoni has interesting stuff regarding momentum conservation, relates $F^\alpha u_\alpha$ to $\frac{dm}{d\tau}$

interesting take, “forces are not convenient”: https://phys.libretexts.org/Bookshelves/University_Physics/Book%3A_Mechanics_and_Relativity_(Idema)/15%3A_Relativistic_Forces_and_Waves/15.01%3A_The_Force_Four-Vector; is this because of way it transforms?

maybe something similar discussed here: https://de.wikipedia.org/wiki/Beschleunigung_(spezielle_Relativit%C3%A4tstheorie)#Beschleunigung_und_Kraft

Example 3.10: Rotating Frame 2

Christoffel symbols are zero in ICCs (more generally: inertial frames -¿ right?), but not necessarily in other coordinates; in non-inertial frames, they lead to extra terms, the fictitious forces; for example, Coriolis and centripetal come out of there very naturally

3.2.5 Lagrangian Approach

review Lagrangian approach here

say that geodesic equation (which is equation of motion for free particle) can be derived by extremizing proper time elapsed along world line as well (minimizing or maximizing depends on sign of metric we choose, right? Ah no, should always be minimizing proper time, but how this translates into requirement for metric is different)

we have dependence on world line of our distance measure; this is nothing unusual in mathematical theory of metric spaces, but it raises an important physical question: what is the preferred trajectory of particles, i.e. what is the time that usually elapses for them? Turns out that it is extremal proper time (minimal for us, depends on sign convention of metric, right?), which yields straight lines relativistic Lagrangian is

$$\boxed{L = -(mc^2 + U)\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.52}$$

with potential energy U; we get that from (see Fleury 3.66) action, which is naturally defined as

$$\boxed{S = \int L\,d\tau = -\int mc^2 + U\,d\tau = -\int (mc^2 + U)\sqrt{1 - \frac{v^2}{c^2}}\,dt} \tag{3.53}$$

3.3 Relativistic Electrodynamics

now we come to the key reason why relativity as formulated by Einstein exists in the first place, electrodynamics

4 General Relativity

despite its great success, Einstein soon realized that SR is incomplete

SR dealt with uniformly moving frames, now we want to use the insights gained there to generalize discussions to accelerated frames – this is what general relativity does (as it turns out, acceleration is very closely related to gravity, so GR is a theory of gravity as well)

-¿ wrong, SR can handle acceleration (contrary to popular belief I feel)! GR is really about incorporating gravity

4.1 Generalizing Relativity

4.1.1 Newtonian Gravity

Newtonian gravity can be captured in Newton's famous formula

$$\boxed{F_g = -\frac{m_1 m_2}{r^2}} \tag{4.1}$$

which describes the gravitational force that an object with mass m_1 exerts onto another object with mass m_2 that is at a distance r. From Newton's second law, we know that the same force is exerted from the second object onto the first.

This force can also be brought into the form

$$F_g = m_2 \frac{d}{dr}\left(\frac{m_1}{r}\right) = -m_2 \frac{d\Phi_g}{dr} \tag{4.2}$$

which tells us that gravitation is a conservative force with potential

$$\boxed{\Phi_g = -\frac{m_1}{r}} \tag{4.3}$$

produced by some object with mass m_1. We get the conservative property from equation (4.2) alone because gravitational force only has a radial and no angular component (thinking in polar/spherical coordinates). At the same time, Φ does not depend on angular coordinates, so any derivative with respect to them vanishes. Thus, (4.2) is equivalent to the more general condition for conservative forces,

$$\boxed{\vec{F} = -\vec{\nabla}\Phi} \qquad F^k = -\delta^{kl}\frac{\partial\Phi}{\partial x^l}\,. \tag{4.4}$$

As a consequence, knowing the potential is sufficient for knowing how gravity acts. Thus, we are interested in how to determine Φ and this can be done using the Poisson equation. For a point-like particle, it takes the form

$$\boxed{\Delta\Phi = \nabla^2\Phi = 0} \tag{4.5}$$

while for a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\Delta\Phi(\vec{x}) = 4\pi\rho(\vec{x})} \qquad \delta^{ij}\frac{\partial^2\Phi(\vec{x})}{\partial x^i\partial x^j} = 4\phi\rho(\vec{x})\,. \tag{4.6}$$

Another perspective is not to look at forces $\vec{F}$, but at associated accelerations, which comes from Newton's second law

$$\boxed{\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}} \qquad F^k = ma^k = m\ddot{r}^k \tag{4.7}$$

or at momenta $\vec{p}$, which are defined by

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} \quad\Leftrightarrow\quad \vec{p} = m\vec{v}} \qquad p^k = mv^k\,. \tag{4.8}$$

Example 4.1: Gravity on Earth

The gravity exerted by Earth on objects with mass m (assuming they are on Earth's surface for now) is

$$F_g = -m\frac{m_e}{r_e^2} = \boxed{-mg} \tag{4.9}$$

Comparing that to Newton's second formula, $F = ma$, we see that such an object experiences an acceleration

$$a = -g = -9.81\frac{\mathrm{m}}{\mathrm{s}^2} = -1.1\cdot 10^{-16}\frac{1}{\mathrm{m}}\,. \tag{4.10}$$

Example 4.1: Gravity on Earth

remark: note that we implicitly assume that gravitational mass and inertial mass are equal here. This is a non-trivial statement, which has been experimentally tested and verified to high precision.

To see how much potential energy is needed to lift objects of mass m to a height $h \ll r_e$ above Earth's surface, we can do a Taylor expansion around $h = 0$:

$$\begin{aligned}\Phi_g = -\frac{m_e}{r_e + h} &\simeq -\left.\frac{m_e}{r_e + h}\right|_{h=0} + h \left.\frac{d}{dh}\left(-\frac{m_e}{r_e + h}\right)\right|_{h=0} + O(h^2)\\ &= -\frac{m_e}{r_e} + h \left.\frac{m_e}{(r_e + h)^2}\right|_{h=0} + O(h^2)\\ &= -\frac{m_e}{r_e} + h\frac{m_e}{r_e^2} + O(h^2) = -\frac{m_e}{r_e} + hg + O(h^2)\end{aligned}$$

However, the first contribution is nothing but the energy at Earth's surface. The energy that is needed to lift an object of mass m to this height h (which is what one is interested in most of the time; corresponds to gauging our measurements such that Earth's surface is the value with zero potential energy) is given to first order by the difference

$$\Phi_g = -\frac{m_e}{r_e} + gh - \left(-\frac{m_e}{r_e}\right) = \boxed{gh} \,. \tag{4.11}$$

This is a well-known formula from classical mechanics.

We see that gravity is related to a potential and thus to potential energy. Hence, we expect an objects energy to change if it moves in a gravitational field (in radial direction). This has interesting consequences, for example because light will also be affected by this.

Example 4.2: Gravitational Redshift

we could build perpetual motion machine if redshift does not occur, idea: send particle with certain energy from top, convert it into photon at bottom and send photon back to top, where it gets converted into particle again; since particle picks up potential energy when falling, while photon is massless and does not need energy to go up, we would gain energy with each iteration; therefore, frequency and thus energy of photon must change on way up, that is gravity has effect on photons; how does that make sense, they have no mass?!

I rather think about it like this (should be equivalent -¿ probably does not make too much sense, should gravity act on rest mass? Perhaps not, then this following does not make sense): SR tells us that photons have a certain mass $m = \frac{E}{c^2}$; therefore, it is also affected by a gravitational potential and to move against gravity, some of its energy has to be converted;

that corresponds to a change in frequency when going from bottom to top, (since $f = \frac{E}{h}$; here, h is the Planck constant, not height!):

$$\frac{f_{\text{top}}}{f_{\text{bottom}}} = \frac{E_{\text{top}}}{E_{\text{bottom}}} = \frac{m - mgh}{m} = 1 - gh \tag{4.12}$$

Example 4.2: Gravitational Redshift

remark: in script, this is only true to first order, so derivation might be wrong... Result there reads $\frac{1}{1+gh} \approx 1 - gh$ after Taylor around $gh = 0$. Ahhh, because of different setting: experiment starts from top, thus there is more energy at ground

A natural consequence because time/time differences are inversely proportional to frequency is that clocks tick faster at higher altitude, i.e. for a weaker gravitational potential (more time passes compared to bottom, although we look at same object). It is also possible to derive this in reverse order, that is by showing that clocks tick slower in a stronger gravitational field. This causes a change in frequency and thus also a redshift.

4.1.2 What is wrong with Newton (and SR)?

why is there even a need for generalizing relativity

gravitational redshift and instantaneous effect of gravity

special relativity came with the abandonment of absolute space and time – so radical changes are to be expected if we want to incorporate gravity now... indeed, it will turn out that gravity is *not* a force, but a fundamental geometrical property/feature of spacetime

now, spacetime $\mathbb{M}$ is not Minkowski space anymore!!!

geometric description we have begun (manifolds with metric) will be continued

coordinate-related statements have no physical meaning due to relativity principle! Only covariant quantities have, in particular tensors (which is why we like them so much)

4.1.3 Einstein Postulates

note: we built on SR and all its postulates, i.e. we assume relativity principle, constancy of c and clock postulate

nice transition: do gravity on Earth example at the end of Newtonian gravity section, then state that we have used equivalence principle there already; different formulation is acceleration = gravity and Einstein recognized that instead of being some strange coincidence, this points to a fundamental feature in the nature of gravity

do postulates by Einstein again as start, but now the ones for GR; weak equivalence principle + Einstein equivalence principle

-¿ what about Mach principle? Ah, indeed needed (see https://de.wikipedia.org/wiki/Relativit%C3%A4tsprinzip)

we need inertial mass = gravitational mass for interpretation acceleration = gravity, right?

question: As in, if I'm accelerating away from the Earth, then does the Earth also appear to be accelerating away from me at the same rate? Or is there something to "break" this type of symmetry?; answers: 1. 9

Kinematically, yes. In terms of describing the positions of objects, it is equivalent to say "A is accelerating away from B" and "B is accelerating away from A". However, it is an observed fact that the universe treats these two situations differently. A and B can check whether they feel artificial gravity in their reference frame. If so, it's accelerating. As far as I know, the "way the universe decides" to break this symmetry is a topic of continuing speculation. 2. in GR, a frame is inertial if it's defined by a free-falling particle (A person in a rocket ship accelerating away from the earth is not free-falling)

In an accelerating frame, the equivalence principle tells us that measurements will come out the same as if there were a gravitational field. But if the spacetime is flat, describing it in an accelerating frame doesn't make it curved. (Curvature is invariant under any smooth coordinate transformation.) Thus relativity allows us to have gravitational fields in flat space — but only for certain special configurations like uniform fields. SR is capable of operating just fine in this context.

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thoughts on equivalence principle:

acceleration is the same as effect of gravity; that means the fictitious forces caused there are analogous to gravity – except that gravity occurs in inertial frames and is *no* coordinate effect; nonetheless, that means they have same source – mathematically speaking, this is Christoffel symbols (which imply non-zero curvature); therefore, since these features are tied to metric (in case of Levi-Civita connection, which is natural choice in physics) we see that cause of gravity is now geometry of spacetime itself – gravity is *not* a force (although its effect is similar to acceleration)

4.1.4 Gravity in SR

Let us now think about gravity in special relativity. In principle, the Newtonian description is kept, but some effects can be examined in different manner now, e.g. due to the new notion/tool of different inertial frames. However, a frame where gravity acts is *not* inertial (objects are accelerated due to the external force in gravity)! Thus, to do physics on Earth, we have to find a reference frame in which the effect of gravity is cancelled out. Obviously, Earths surface is not sufficient and neither is a frame that is uniformly moving with respect to it. In free fall, however, we experience no gravity, that is a freely falling frame cancels out the effect of gravity. This can be stated more formally:

Property 4.3: Weak Equivalence Principle

The effects of a gravitational field are indistinguishable from an accelerated frame of reference.

Basically, that means only a freely falling frame can serve as an inertial frame on Earth. That raises the question what happens to the laws of physics in such a freely falling frame.

Property 4.4: Einstein Equivalence Principle

The laws of physics in a freely falling frame are locally described by SR without gravity. For this reason, such a frame is also called *local inertial frame (LIF)*.

Strictly mathematically speaking, "locally" refers to an infinitesimally small neighbourhood around points. The degree to which this locality can be extended (in practice, e.g. in calculations) depends on the physical effects of interest.

Since gravity acts radially, its direction changes in different places around Earth. That implies there is no uniform direction of acceleration, so there can be no global freely falling frame/LIF. Other properties of gravity which are known from experience are the following:

(a) All bodies (independent of structure and mass), which start with the same initial velocity, move through a gravitational field along the same curve

(b) Bodies, which move initially parallel to each other in a freely falling frame, do not necessarily move parallel at all times if an external gravitational field is present (this effect is due to *tidal forces* acting on them)

Property (b) can be further quantified. For a particle with world line $x^k(\tau)$

$$\frac{d^2x^k}{d\tau^2} = -\delta^{kl}\frac{\partial\Phi}{\partial x^l} = -\delta^{kl}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}}$$

because of (4.4) and Newton's second law $F^k = m\frac{d^2x^k}{d\tau^2}$. Similarly, for another particle starting close to the first one (i.e. with world line $x^k + \xi^k$ with $\left|\xi^k\xi_k\right| \ll 1$)

$$\frac{d^2\left(x^k+\xi^k\right)}{d\tau^2} = -\delta^{kl}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}+\vec{\xi}} \simeq -\delta^{kl}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}} - \delta^{kl}\xi^m\frac{\partial}{\partial x^m}\left.\frac{\partial\Phi}{\partial x^l}\right|_{\vec{x}}$$

where we used a Taylor expansion to first order in ξ^k. The linearity of derivatives yields

$$\boxed{\frac{d^2\xi}{d\tau^2} = \frac{d^2\left(x^k + \xi^k\right)}{d\tau^2} - \frac{d^2 x^k}{d\tau^2} = -\delta^{kl}\,\frac{\partial^2\Phi}{\partial x^m \partial x^l}\,\xi^m} \,. \tag{4.13}$$

remark: note that the evaluation is still at the point $\vec{x}$, not at something related to ξ^k!

This is the *Newtonian deviation equation*. We see that tidal forces are governed by the *tidal acceleration tensor* $\frac{\partial^2\Phi}{\partial x^m \partial x^l}$. Tidal forces are a way to detect gravity as opposed to constant acceleration (which would affect the world lines x^k and $x^k + \xi^k$ equally).

4.1.5 Curved Spacetime

In SR, the Newtonian description of gravity is still taken to be valid. That, however, is a problem because there are many inconsistencies between them, for example the instantaneous effect of gravity (gravitational redshift is also puzzling). However, we can come up with a generalized description: for anybody familiar with differential/Riemannian geometry, the effects (a), (b) of gravity stated above sound very much like the ones associated with a curved manifold. This motivates the (mathematical) description of gravity as a geometrical effect in Minkowskian spacetime (which will become a curved space in this process). Many relations known from SR will remain, but with different quantities and most prominently, a metric other than η. The basic goal of *general relativity* (GR) will be to find ways to derive the metric, which contains information about spacetime curvature and thus gravity.

The approach in this subsection will always be to look how generalizations can be made using the metric and other tools of geometry, while recovering SR in a LIF. The first quantity to perform this check for will be the metric itself. Thus, we will now look at how the mathematical term "locally" is to be thought of. Taking an arbitrary metric with components $g_{\mu\nu}$ in some basis, we can always transform to other coordinates using the tensor transformation law (1.13). This gives us a certain freedom in choosing suitable coordinates and since the number of components in the transformation is $4 \cdot 4 = 16$, while the metric only has 10 independent components (due to its symmetry), we can always achieve

$$g'_{\mu\nu}(p) = \eta_{\mu\nu}$$

at a given point p (not globally!). Going one step further, we can even achieve an equality in the first derivatives using the same idea of gauge freedom. This procedure stops at the second order, though. All in all, we can always find a frame where

$$\boxed{g_{\mu\nu}(p) = \eta_{\mu\nu}} \qquad \boxed{\left.\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right|_p = 0} \qquad \boxed{\left.\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right|_p \neq 0} \tag{4.14}$$

and this frame is a LIF. Here, the metric is flat in p, but deviates more and more from that as we go farther away from p (it is "flat to first order").

Another note on the use of metrics concerns causality. In SR (and thus in a LIF), the value of inner products $v^\alpha v_\alpha$ determined if v^α/the corresponding trajectory is timelike/null/spacelike. Since $v^\alpha v_\alpha$ is a function/number and thus a 0-tensor, this statement is coordinate-independent. Therefore, every other frame inherits the lightcone structure from SR.

4.1.6 Notes

Penrose has incredibly well written section 17.9 on intuition about metric and light cone structure in GR

-¿ interesting, acceleration is absolute in SR, but relative in GR (because spacetime curved there)

we have seen how to compute proper times in special relativity, the formula could be broken down to

$$\tau = \int d\tau = \frac{1}{c} \int ds = \frac{1}{c} \int \sqrt{g_{\mu\nu} v^\mu v^\nu}\, dt \tag{4.15}$$

in fact, this formula also holds for $v = v(t)$, i.e. when a time-dependent velocity and thus acceleration is present (we can compute dynamics); this is because it does not involve absolute differences like $x - x'$, but infinitesimal ones dx along the whole path, so changes in v are incorporated automatically; however, we have to use other metrics in this case and general relativity presents a general way to compute the metric

four-velocity: using chain rule, we can write $U^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dt}{d\tau}\frac{dx^\alpha}{dt} =: \gamma(c, \vec{v})$ in general; this $\gamma = \frac{dt}{d\tau} = \frac{\partial t}{\partial \tau}$ then depends on the metric, it is the square root of the (negative of, depends convention that is chosen regarding metric) component g_{tt}

4.2 Giulini Lectures

from 10 on (until 16) he deals with GWs, noice

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