TOPOLOGY

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Fourth Semester

Contents

Chapter 1

METRIC AND TOPOLOGICAL SPACES

1.1 Metric Spaces and Examples

January 6th.

A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function, called a *metric* on X, satisfying the following properties for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$ (*positive definiteness*).

(ii) $d(x, y) = d(y, x)$ (*symmetry*).

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Let us look at some examples of metric spaces.

Example 1.1. Any set X can be made into a metric space by defining the *discrete metric* d as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \tag{1.1}$$

It is easy to verify that d satisfies all the properties of a metric.

Example 1.2. Recall that a normed space $(V, \|\cdot\|)$ was a vector space V equipped with a *norm* $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and $\alpha \in \mathbb{F}$:

(i) $\|v\| = 0$ if and only if $v = 0$ (positive definiteness).

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ (*absolute homogeneity*).

(iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).

Given a normed space $(V, \|\cdot\|)$, we can define a metric d on V as follows:

$$d(u, v) = \|u - v\| \quad \forall u, v \in V. \tag{1.2}$$

Yet again, it is straightforward to verify that d satisfies all the properties of a metric. Given a vector space V, we can have multiple norms on it, and hence multiple metrics. For example, consider the vector space $\mathbb{R}^n$. We have the following norms on $\mathbb{R}^n$:

- The ℓ^1 *norm*: $\|x\|_1 = \sum_{i=1}^n |x_i|$,
- the *Euclidean norm*: $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$,

- the *supremum norm*: $\|x\|_\infty = \max_{1\le i\le n} |x_i|$,

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Each of these norms induces a different metric on $\mathbb{R}^n$.

The notion of open and closed balls is also abstracted to metric spaces as follows.

Definition 1.3. Let (X, d) be a metric space. The *open ball* of radius $r > 0$ centered at a point $x \in X$ is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \tag{1.3}$$

and the *closed ball* of radius $r > 0$ centered at x is the set

$$B[x, r] = \{y \in X \mid d(x, y) \le r\}. \tag{1.4}$$

Note that in the discrete metric space, $B(x, 1) = \{x\} = B(x, \frac{1}{2})$, and $B(x, 2) = X = B(y, 2)$ for any $x, y \in X$. Thus, $B(x, r) = B(y, \rho)$ does not imply that $x = y$ or $r = \rho$ in general.

Example 1.4. Let p be a prime, say $p = 3$. Define a function $|\cdot|_3 : \mathbb{Z} \to \mathbb{R}_{\ge 0}$ as follows: for any non-zero integer m, write $m = 3^k m'$ where m' is not divisible by 3, and set $|m|_3 = 3^{-k}$. Also, set $|0|_3 = 0$. This function $|\cdot|_3$ is called the 3-adic absolute value on $\mathbb{Z}$. In general, for any prime p, the *p-adic absolute value* is defined similarly.
This 3-adic absolute value induces a norm d_3 on $\mathbb{Q}$ as follows:

$$|q|_3 = \begin{cases} 0 & \text{if } q = 0, \\ |m|_3 \,/\, |n|_3 & \text{if } q = m/n \text{ in lowest terms.} \end{cases} \tag{1.5}$$

This induces a metric on $\mathbb{Q}$ defined by $d_3(x, y) = |x - y|_3$ for all $x, y \in \mathbb{Q}$. This metric space $(\mathbb{Q}, d_3)$ is called the 3-adic metric space, and in general $(\mathbb{Q}, d_p)$ is called the *p-adic metric space*. The completion of $(\mathbb{Q}, d_p)$ gives us the *field of p-adic numbers*, denoted by $\mathbb{Q}_p$. This metric space has some interesting properties; for instance, the triangle inequality is strengthened to the *ultrametric inequality*:

$$d_p(x, z) \le \max\{d_p(x, y), d_p(y, z)\} \quad \forall x, y, z \in \mathbb{Q}. \tag{1.6}$$

Lemma 1.5 (*Hausdorff property*)**.** *Let (X, d) be a metric space. For any distinct $x, y \in X$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$.*

Proof. Verify that choosing any $r \le \frac{1}{2} d(x, y)$ works. ■

Let (X, d) be a metric space. Then a subset $A \subseteq X$ can also be made into a metric space by restricting the metric d to $A \times A$. In the metric space $(A, d|_{A\times A})$, the open balls are given by $B_A(x, r) = B_X(x, r) \cap A$ for all $x \in A$ and $r > 0$, where $B_X(x, r)$ is the open ball in (X, d).
Again, as before, the notion of open sets is abstracted to metric spaces as follows.

Definition 1.6. Let (X, d) be a metric space. A subset $U \subseteq X$ is said to be an *open set* if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

As a small lemma, one can show that every open ball in a metric space is an open set. As an exercise, show that the complement of the closed ball $B[x, r]^c = \{y \mid d(x, y) > r\}$ is also an open set.

Proposition 1.7. *Let (X, d) be a metric space. Let $\tau = \{U \subseteq X \mid U \text{ is open}\}$, that is, the collection of all open sets in X. Then the following hold true.*

(i) $\emptyset, X \in \tau$.

(ii) *For $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \tau$, we have $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$. That is, an arbitrary union of open sets is open.*

(iii) For $U_1, U_2, \ldots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of open sets is open.

Proof. The proof of the first property is trivial. For the second property, let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ such that $B(x,r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$. Thus, $\bigcup_{\alpha\in\Lambda} U_\alpha$ is open.

For the third property, let $x \in \bigcap_{i=1}^n U_i$. Then $x \in U_i$ for all $1 \leq i \leq n$. Since each U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for all $1 \leq i \leq n$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then we have

$$B(x,r) \subseteq B(x,r_i) \subseteq U_i \quad \forall 1 \leq i \leq n, \tag{1.7}$$

which implies that $B(x,r) \subseteq \bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ■

1.2 Topological Spaces and Examples

A *topological space* is a pair (X,τ) where X is a set and τ is a collection of subsets of X satisfying the following properties:

(i) $\emptyset, X \in \tau$.

(ii) For $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \tau$, we have $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$. That is, an arbitrary union of sets in τ is in τ.

(iii) For $U_1, U_2, \ldots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of sets in τ is in τ.

These are the exact same properties that the collection of open sets in a metric space satisfy. Hence, every metric space (X,d) gives rise to a topological space (X,τ_d) where τ_d is the collection of all open sets in (X,d). Such a topology τ_d is called the topology induced by the metric d.

As a smaller example, let $X = \{0,1,2,3,4\}$ and consider the collection $\tau = \{\emptyset, X, \{0\}, \{0,1\}, \{2,4\}\}$. Then the pair (X,τ) is *not* a topological space since $\{0,1\} \cup \{2,4\} = \{0,1,2,4\} \notin \tau$. However, the pair (X,τ') where $\tau' = \{\emptyset, X, \{0\}, \{0,1\}, \{2,4\}, \{0,1,2,4\}\}$ is a topological space.

Description of open sets in $\mathbb{R}$

Theorem 1.8. *A non-empty open set in $\mathbb{R}$ is a countable union of pairwise disjoint open intervals.*

Proof. Let $U \subseteq \mathbb{R}$ be a non-empty open set. For each $x \in U$, define

$$I_x = \bigcup \{(a,b) \mid x \in (a,b) \subseteq U\}. \tag{1.8}$$

Note that $x \in I_x \subseteq U$. Let $a_x = \inf I_x$ and $b_x = \sup I_x$. We claim that $I_x = (a_x, b_x)$. For $a_x < z < b_x$, there exists $a, b \in I_x$ such that $a_x < a < z < b < b_x$. Since $z \in (a,b) \subseteq I_x$, we have $z \in I_x$. Thus, $(a_x, b_x) \subseteq I_x$. The other inclusion is trivial. Hence, $I_x = (a_x, b_x)$ is an open interval.

We now claim that if $x \neq y$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose that $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is an interval containing both x and y and contained in U. By the definition of I_x and I_y, we have $I_x \cup I_y \subseteq I_x$ and $I_x \cup I_y \subseteq I_y$. Thus, $I_x = I_y$.

Finally, let $U = \bigcup_{x\in U} I_x$. By the above claim, the collection $\{I_x \mid x \in U\}$ consists of pairwise disjoint open intervals. Since each I_x contains a rational number (by the density of $\mathbb{Q}$ in $\mathbb{R}$), for each I_x, we can choose a distinct rational number $q_x \in I_x$. This gives $I_x = I_{q_x}$. Thus, we have

$$U = \bigcup_{x\in U} I_x = \bigcup_{q\in\mathbb{Q}\cap U} I_q, \tag{1.9}$$

which is a countable union of pairwise disjoint open intervals. ■

Definition 1.9. Let (X,d_1) and (X,d_2) be two metric spaces on the same set X. The metrics d_1 and d_2 are said to be *equivalent metrics*, $d_1 \sim d_2$, if open sets in (X,d_1) are exactly the open sets in (X,d_2).

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Definition 1.10. Let $A \subseteq X$ be a subset of a metric space (X, d). The *interior* of A, denoted by $\mathrm{Int}(A)$, is defined as

$$\mathrm{Int}(A) := \{x \in A \mid \exists r > 0 \text{ such that } B(x, r) \subseteq A\}. \tag{1.10}$$

For a general topological space, the interior of a set A is defined as the largest open set contained in A. For another definition, we have

$$\mathrm{Int}(A) := \{x \in A \mid \exists U \in \tau \text{ such that } x \in U \subseteq A\}. \tag{1.11}$$

Lemma 1.11. *The above definitions of the interior of a set in a topological space are equivalent.*

Proof. Suppose we affirm the second definition. Let $x \in \mathrm{Int}(A)$. Then there exists an open set $U_x \in \tau$ such that $x \in U_x \subseteq A$. Thus, $\bigcup_{x\in\mathrm{Int}(A)} U_x \subseteq A$ is contained in A and is open. Moreover, if $z \in \mathrm{Int}(A) \setminus \bigcup_{x\in\mathrm{Int}(A)} U_x$, then there exists an open set $V \in \tau$ such that $z \in V \subseteq A$. But then $z \in U_z \subseteq \bigcup_{x\in\mathrm{Int}(A)} U_x$, a contradiction. Thus, $\mathrm{Int}(A) = \bigcup_{x\in\mathrm{Int}(A)} U_x$ is the largest open set contained in A. If V is any open set contained in A, then for any $y \in V$, there exists an open set $V_y \in \tau$ such that $y \in V_y \subseteq A$. Thus, $y \in \mathrm{Int}(A)$, which implies that $V \subseteq \mathrm{Int}(A)$. Hence, the first definition holds. ■

As an example, with the standard topology on $\mathbb{R}$, we have $\mathrm{Int}([0, 1]) = (0, 1)$, $\mathrm{Int}((0, 1) \cup \{2\}) = (0, 1)$, and $\mathrm{Int}(\mathbb{Q}) = \emptyset$. Note that the interior of an open set is the set itself; $\mathrm{Int}(U) = U$ for any open set U.

In the spirit of an induced metric, the subspace topology is defined as follows.

Definition 1.12. Let (X, τ) be a topological space and let $Y \subseteq X$. The *subspace topology* on Y is defined as

$$\tau_Y = \{U \cap Y \mid U \in \tau\}. \tag{1.12}$$

One can verify that τ_Y is a topology on Y. Let us look at some examples of topological spaces.

Example 1.13. The *discrete topology* on a set X is the topology $\tau = \mathcal{P}(X)$, the power set of X. Every subset of X is open in this topology. In contrast, the *indiscrete topology* (trivial topology) on X is the topology $\tau = \{\emptyset, X\}$. Only the empty set and the whole set are open in this topology.

Example 1.14. Let X be any set. The *cofinite topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is finite}\}. \tag{1.13}$$

One can verify that τ is a topology on X; both $\emptyset$ and X are in τ. For an arbitrary union of sets in τ, if any one of them is X, then the union is X. Otherwise, the complement of the union is the intersection of finite sets, which is finite. For a finite intersection of sets in τ, the complement of the intersection is the finite union of finite sets, which is finite. Thus, τ is a topology on X.

Example 1.15. Let X be any set. The *cocountable topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is countable}\}. \tag{1.14}$$

One can verify that τ is a topology on X; both $\emptyset$ and X are in τ. For an arbitrary union of sets in τ, if any one of them is X, then the union is X. Otherwise, the complement of the union is the intersection of countable sets, which is countable. For a finite intersection of sets in τ, the complement of the intersection is the finite union of countable sets, which is countable. Thus, τ is a topology on X.

1.2.1 Basis

Definition 1.16. Let (X, τ) be a topological space. A collection $\mathcal{B} \subseteq \tau$ is said to be a *basis* for τ if

(i) For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

(ii) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

For any set X, if we have a collection $\mathcal{B}$ of subsets of X satisfying the above two properties, then the collection

$$\tau = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \tag{1.15}$$

is a topology on X, and $\mathcal{B}$ is a basis for this topology.

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For example, choosing $\mathcal{B} = \mathcal{P}(X)$ gives us the discrete topology on X, and choosing $\mathcal{B} = \{X\}$ gives us the indiscrete topology on X.

Lemma 1.17. *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ. Then τ is the collection of all possible unions of elements in $\mathcal{B}$.*

Proof. Note that every union of elements in $\mathcal{B}$ is in τ by the definition of a topology; we need to show the other inclusion, that is, every set in τ can be expressed as a union of elements in $\mathcal{B}$. Let $U \in \tau$. For each $x \in U$, by the definition of a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Thus, we have

$$U = \bigcup_{x \in U} B_x, \tag{1.16}$$

which is a union of elements in $\mathcal{B}$. ■

Definition 1.18. Let (X, τ_1) and (X, τ_2) be two topological spaces on the same set X. We say $\tau_1 \supseteq \tau_2$, or that τ_1 is (strictly) *finer* than τ_2, if every open set in τ_2 is also an open set in τ_1. Conversely, we say τ_2 is (strictly) *coarser* than τ_1 if every open set in τ_1 is also an open set in τ_2, that is, $\tau_2 \subseteq \tau_1$.

Lemma 1.19. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies τ and τ' on the same set X. Then the following are equivalent:*

(i) $\tau' \supseteq \tau$.

(ii) *For all $x \in X$ and all $B \in \mathcal{B}$ such that $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. For the reverse implication, let $U \in \tau$. Then U can be written as $U = \bigcup_{\alpha\in\Lambda} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for all $\alpha \in \Lambda$. For each $\alpha \in \Lambda$ and each $x \in B_\alpha$, by the second property, there exists $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subseteq B_\alpha$. Thus, we have

$$U = \bigcup_{\alpha\in\Lambda} B_\alpha = \bigcup_{\alpha\in\Lambda} \bigcup_{x \in B_\alpha} B'_x, \tag{1.17}$$

which is a union of elements in $\mathcal{B}'$. Thus, $U \in \tau'$, and hence $\tau' \supseteq \tau$. For the forward implication, let $B \in \mathcal{B}$ and $x \in B$. Since $B \in \tau \subseteq \tau'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. ■

Let us look at some topologies generated by bases.

Example 1.20. Let $X = \mathbb{R}$ and $\mathcal{B} = \{[a, b) \mid a < b,\, a, b \in \mathbb{R}\}$. We claim that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Clearly, for every $x \in \mathbb{R}$, there exists $[x, x+1) \in \mathcal{B}$ such that $x \in [x, x+1)$. Now, let $B_1 = [a_1, b_1), B_2 = [a_2, b_2) \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Then we have three cases:

- If $a_1 \le a_2 \le x \le b_1 \le b_2$, then choose $B_3 = [a_2, b_1)$.

- If $a_2 \leq a_1 \leq x \leq b_2 \leq b_1$, then choose $B_3 = [a_1, b_2)$.
- If $a_1 \leq x \leq b_2 \leq b_1$ (the case $a_2 \leq x \leq b_1 \leq b_2$ is similar), then choose $B_3 = [x, b_2)$.

In all cases, we have $x \in B_3 \subseteq B_1 \cap B_2$. Thus, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$, called the *lower limit topology* denoted by $\mathbb{R}_l$. Similarly, the collection $\mathcal{B}' = \{(a, b] \mid a < b,\ a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$, called the *upper limit topology* denoted by $\mathbb{R}_u$.

Example 1.21. Ket $K = \{1/n \mid n \in \mathbb{N}\}$. Consider the collection

$$\mathcal{B} = \{(a, b), (a, b) \setminus K \mid a < b,\ a, b \in \mathbb{R}\}. \tag{1.18}$$

Verify that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$, called the *K-topology* on $\mathbb{R}$, denoted by $\mathbb{R}_K$.

Lemma 1.22. *The standard topology on $\mathbb{R}$ is strictly finer than $\mathbb{R}_l$, and also strictly finer than $\mathbb{R}_K$. However, $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.*

Proof. For $\mathbb{R}_l$: Let $x \in \mathbb{R}$ and let $B = (a, b) \in \mathcal{B}$ such that $x \in B$. Then choose $B' = [x, b) \in \mathcal{B}$ such that $x \in B' \subseteq B$. Thus, by the previous lemma, the standard topology on $\mathbb{R}$ is finer than $\mathbb{R}_l$. To see that the inclusion is strict, note that $[a, b)$ is open in $\mathbb{R}_l$ but not in the standard topology on $\mathbb{R}$.

For $\mathbb{R}_K$: Note that inclusion is trivial since every basis element of $\mathbb{R}_K$ is also a basis element of the standard topology on $\mathbb{R}$. To see that the inclusion is strict, note that $(-1, 1) \setminus K$ is open in $\mathbb{R}_K$ but not in the standard topology on $\mathbb{R}$.

For non-comparability of $\mathbb{R}_l$ and $\mathbb{R}_K$: Note that $[5, 6)$ is open in $\mathbb{R}_l$ but not in $\mathbb{R}_K$ since it cannot be expressed as a union of basis elements of $\mathbb{R}_K$. Also, note that $(-1, 1) \setminus K$ is open in $\mathbb{R}_K$ but not in $\mathbb{R}_l$ since it cannot be expressed as a union of basis elements of $\mathbb{R}_l$. ■

1.3 Closed Sets

Definition 1.23. Let (X, τ) be a topological space. A subset $A \subseteq X$ is said to be a *closed set* if its complement $A^c = X \setminus A$ is an open set.

Using De Morgan's laws, one can easily verify the following properties of closed sets in a topological space.

Proposition 1.24. *Let (X, τ) be a topological space. Then the following hold true.*

(i) $\emptyset$ and X are closed sets.

(ii) For any collection $\{A_\alpha\}_{\alpha \in \Lambda}$ of closed sets, we have $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a closed set. That is, an arbitrary intersection of closed sets is closed.

(iii) For closed sets $A_1, A_2, \ldots, A_n$, we have $\bigcup_{i=1}^n A_i$ is a closed set. That is, a finite union of closed sets is closed.

Lemma 1.25. *Any finite set of a metric space (X, d) is closed.*

Proof. Note that it is enough to show that a singleton set $\{x\}$ is closed for any $x \in X$. Let $y \in \{x\}^c$. Then $y \neq x$, and by the Hausdorff property, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$. Thus, $B(y, r) \subseteq \{x\}^c$, which implies that $\{x\}^c$ is open. Hence, $\{x\}$ is closed. ■

Similar to the open sets, we can define closed sets in a subspace as follows.

Theorem 1.26. *Let (X, τ) be a topological space and let $Y \subseteq X$ be a subspace with the subspace topology τ_Y. Then a set $C \subseteq Y$ is closed in (Y, τ_Y) if and only if there exists a closed set D in (X, τ) such that $C = D \cap Y$.*

Proof. Let C be closed in (Y, τ_Y). Then $C^c = Y \setminus C$ is open in (Y, τ_Y). Thus, there exists an open set $U \in \tau$ such that $C^c = U \cap Y$. Let $D = U^c$, which is closed in (X, τ). Then we have

$$C = Y \setminus C^c = Y \setminus (U \cap Y) = Y \cap U^c = Y \cap D. \tag{1.19}$$

Conversely, let D be closed in (X, τ) and let $C = D \cap Y$. Then D^c is open in (X, τ). Thus, we have

$$C^c = Y \setminus C = Y \setminus (D \cap Y) = Y \cap D^c, \tag{1.20}$$

which is open in (Y, τ_Y). Hence, C is closed in (Y, τ_Y). ■

Analogous to the interior of a set, we have the notion of the closure of a set.

Definition 1.27. Let (X, τ) be a topological space and let $A \subseteq X$. The *closure* of A, denoted by $\overline{A}$, is defined as the smallest closed set containing A. That is,

$$\overline{A} := \bigcap \{C \subseteq X \mid C \text{ is closed and } A \subseteq C\}. \tag{1.21}$$

Theorem 1.28. *The closure of a set A in a topological space (X, τ) is equivalent to saying*

$$\overline{A} = \{x \in X \mid \forall U \in \tau \text{ with } x \in U,\, U \cap A \neq \emptyset\}. \tag{1.22}$$

Proof. Let $x \in \overline{A}$ with $x \in U \in \tau$. Suppose for the sake of contradiction that $U \cap A = \emptyset$. Then we have $A \subseteq U^c$, where U^c is closed. This must then imply that $\overline{A} \subseteq U^c$, which is a contradiction since $x \in U$. Thus, we have $U \cap A \neq \emptyset$.

For the converse inclusion, let $x \notin \overline{A}$. Then there exists a closed set C such that $A \subseteq C$ but $x \notin C$. Thus, $x \in C^c$, where C^c is open. Since $A \subseteq C$, we have $C^c \cap A = \emptyset$. Hence, there exists an open set C^c containing x such that $C^c \cap A = \emptyset$. ■

Instead of using "$\forall U \in \tau$" in the above theorem, one can equivalently use "$\forall B \in \mathcal{B}$" where $\mathcal{B}$ is a basis for the topology τ.

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Lemma 1.29. *A is closed if and only if $A = \overline{A}$. Moreover, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.*

Proof. The first assertion is left as an exercise. For the second assertion, let $x \in \overline{A}$. Then for every open set U containing x, we have $U \cap A \neq \emptyset$. Since $A \subseteq B$, we have $U \cap B \neq \emptyset$. Thus, $x \in \overline{B}$. ■

Theorem 1.30. *Let $Y \subseteq X$ be a subspace of a topological space (X, τ). For any $A \subseteq Y$, we have*

$$\overline{A}^Y = \overline{A} \cap Y, \tag{1.23}$$

where $\overline{A}^Y$ is the closure of A in the subspace (Y, τ_Y) and $\overline{A}$ is the closure of A in (X, τ).

Proof. We have

$$\overline{A}^Y = \{y \in Y \mid y \in U \cap Y, U \in \tau, U \cap Y \cap A \neq \emptyset\} \subseteq \overline{A} \cap Y. \tag{1.24}$$

Now let $y_0 \in \overline{A} \cap Y$. Then for every $U \in \tau$ such that $y_0 \in U$, we have $U \cap A \neq \emptyset$. Thus, taking intersection over all closed sets containing A in (X, τ) and then intersecting with Y, we have

$$\bigcap_{\substack{C \subseteq X \\ C \text{ closed} \\ A \subseteq C}} (C \cap Y) = \overline{A} \cap Y. \tag{1.25}$$

Thus, $y_0 \in \overline{A}^Y$. Hence, we have $\overline{A}^Y = \overline{A} \cap Y$. ■

1.4 Convergence and Hausdorff Spaces

Definition 1.31. Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a *limit point* of A if for every open set $U \in \tau$ containing x, we have $(U \setminus \{x\}) \cap A \neq \emptyset$.

Let us denote the set of all limit points of A by A'. Then $A' \subseteq \overline{A}$ since for every open set U containing a limit point x, we have $U \cap A \neq \emptyset$. Note that nothing can be said about the relation between A and A'.

For example, if $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ with the standard topology, then $A' = \{0\}$ and $\overline{A} = A \cup \{0\}$. If $A = (0, 1) \cup \{2\}$, then $A' = [0, 1]$ and $\overline{A} = [0, 1] \cup \{2\}$. If $A = \mathbb{Q} \subseteq \mathbb{R}$ with the standard topology, then $A' = \mathbb{R}$ and $\overline{A} = \mathbb{R}$. There is a nice characterization of the closure of a set using limit points.

Theorem 1.32. *Let (X, τ) be a topological space and let $A \subseteq X$. Then*

$$\overline{A} = A \cup A'. \tag{1.26}$$

Proof. One direction is easy since we have already seen that $A, A' \subseteq \overline{A}$. For the other direction, let $x \in \overline{A}$. If $x \in A$, then we are done. So suppose that $x \notin A$. Then for every open set U containing x, we have $U \cap A \neq \emptyset$. Since $x \notin A$, we have $(U \setminus \{x\}) \cap A \neq \emptyset$. Thus, x is a limit point of A, and hence $x \in A'$. Therefore, we have $\overline{A} \subseteq A \cup A'$, and hence the result follows. ■

Corollary 1.33. *A set A is closed if and only if $A' \subseteq A$.*

We are now ready to define convergence in a (metric) topological space.

Definition 1.34. Let (X, d) be a metric space. A sequence $(x_n)_{n \geq 1} \subseteq X$ is said to *converge* to a point $x \in X$ if for every $\varepsilon > 0$, there exists a natural $N(\varepsilon) \equiv N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \forall n \geq N. \tag{1.27}$$

Equivalently, the sequence $(d(x_n, x))_{n \geq 1} \subseteq \mathbb{R}$ converges to 0 in the usual sense.

In the topological sense, we say $(x_n)_{n \geq 1}$ converges to x if for every open set U containing x, there exists a natural $N(U) \equiv N \in \mathbb{N}$ such that

$$x_n \in U \quad \forall n \geq N. \tag{1.28}$$

Note that the limit of a sequence in a metric space is unique. However, in a general topological space, the limit of a sequence need not be unique. For example, consider the set X with the indiscrete topology. Then every sequence in X converges to every point in X.

Theorem 1.35. *Let (X, d) be a metric space and let $(x_n)_{n \geq 1} \subseteq X$ be a sequence converging to both x and y in X. Then $x = y$.*

Proof. For every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon/2 \quad \forall n \geq N_1. \tag{1.29}$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, y) < \varepsilon/2 \quad \forall n \geq N_2. \tag{1.30}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{1.31}$$

Since $\varepsilon > 0$ is arbitrary, we have $d(x, y) = 0$, which implies that $x = y$. ■

The concept of Hausdorff property can be generalized to topological spaces as follows: for every $x, y \in X$ with $x \neq y$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Theorem 1.36. *Let (X, τ) be a Hausdorff topological space and let $(x_n)_{n\geq 1} \subseteq X$ be a sequence converging to both x and y in X. Then $x = y$.*

Proof. The proof is similar to that in the metric space case. For $x \neq y$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since $(x_n)_{n\geq 1}$ converges to x, there exists $N_1 \in \mathbb{N}$ such that

$$x_n \in U \quad \forall n \geq N_1. \tag{1.32}$$

Similarly, since $(x_n)_{n\geq 1}$ converges to y, there exists $N_2 \in \mathbb{N}$ such that

$$x_n \in V \quad \forall n \geq N_2. \tag{1.33}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have $x_n \in U$ and $x_n \in V$, which implies that $x_n \in U \cap V$. This is a contradiction since $U \cap V = \emptyset$. Hence, we must have $x = y$. ■

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Lemma 1.37. *Let (X, τ) be a Hausdorff topological space. Then every finite subset of X is closed.*

Proof. It is enough to show that a singleton set $\{x\}$ is closed for any $x \in X$. Let $y \in \{x\}^c$. Then $y \neq x$, and by the Hausdorff property, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, $V \subseteq \{x\}^c$, which implies that $\{x\}^c$ is open. Hence, $\{x\}$ is closed. ■

We ask this: is it true that if every finite subset of a topological space (X, τ) is closed, then (X, τ) is Hausdorff? The answer is no, since a counterexample is provided by the cofinite topology on an infinite set X. In this topology, every finite subset is closed, but the space is not Hausdorff.

Definition 1.38. We say that a topological space (X, τ) is *T1* if finite subsets of X are closed.

Theorem 1.39. *Suppose (X, τ) is a T1 topological space. Let $A \subseteq X$. Then $x \in A'$ if and only if for every open set $U \in \tau$ containing x, we have $U \cap A$ is an infinite set.*

Proof. The reverse implication is obvious by the very definition of A'. For the forward implication, let $x \in A'$ and let $U \in \tau$ be any open set containing x. Suppose for the sake of contradiction that $U \cap A$ is a finite set. Since (X, τ) is T1, finite subsets of X are closed. Thus, $(U \cap A)^c$ is open. Moreover, we have

$$x \in U \cap (U \cap A)^c, \quad (U \cap (U \cap A)^c) \cap A = \emptyset, \tag{1.34}$$

which is a contradiction since $x \in A'$. Hence, $U \cap A$ is an infinite set. ■

Let us look at some examples of Hausdorff spaces.

Example 1.40. If X is Hausdorff, then the subspace $Y \subseteq X$ with the subspace topology is also Hausdorff. This is because for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there exist open sets $U, V \in \tau$ such that $y_1 \in U$, $y_2 \in V$, and $U \cap V = \emptyset$. Thus, $y_1 \in U \cap Y$, $y_2 \in V \cap Y$, and $(U \cap Y) \cap (V \cap Y) = \emptyset$.

Example 1.41. For a space (X, τ), and a subset $A \subseteq X$, the *boundary* of A is defined as $\mathrm{Bd}(A) = \overline{A} \cap \overline{X \setminus A}$. Then $\mathrm{Bd}(A)$ and $\mathrm{Int}(A)$ are disoint; indeed, let $x \in \mathrm{Bd}(A) \cap \mathrm{Int}(A)$. Then there exists an open set $U \in \tau$ such that $x \in U \subseteq A$. Since $x \in \mathrm{Bd}(A)$, we have $x \in \overline{X \setminus A}$. Thus, for every open set $V \in \tau$ containing x, we have $V \cap (X \setminus A) \neq \emptyset$. In particular, taking $V = U$, we have $U \cap (X \setminus A) \neq \emptyset$, which is a contradiction since $U \subseteq A$. Hence, we must have $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$.
Moreover, we have $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$. The reverse inclusion is trivial since $\mathrm{Bd}(A) \subseteq \overline{A}$ and $\mathrm{Int}(A) \subseteq A \subseteq \overline{A}$. For the forward inclusion, let $x \in \overline{A}$. If $x \in \mathrm{Bd}(A)$, then we are done. So suppose that $x \notin \mathrm{Bd}(A)$. Then $x \notin \overline{X \setminus A}$, which implies that there exists an open set $U \in \tau$ such that $x \in U$ and $U \cap (X \setminus A) = \emptyset$. Thus, we have $U \subseteq A$, which implies that $x \in \mathrm{Int}(A)$. Therefore, we have $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.
Finally, if $\mathrm{Bd}(A) = \emptyset$, then A is both open and closed. Indeed, since $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$, we have $\overline{A} = \mathrm{Int}(A)$. Thus, A is closed. Also, since $\mathrm{Bd}(A) = \emptyset$, we have $\overline{X \setminus A} = \mathrm{Int}(X \setminus A)$. Thus, $X \setminus A$ is closed, which implies that A is open.

1.5 Product Spaces

For two metric spaces (X, d_X) and (Y, d_Y), we can define a metric on the product set $X \times Y$ as follows:

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2).\} \tag{1.35}$$

A basis of $X \times Y$ with the product metric is given by

$$\mathcal{B} = \{B_X(x, r) \times B_Y(y, r) \mid x \in X, y \in Y, r > 0\}. \tag{1.36}$$

In particular, if (X, τ_X) and (Y, τ_Y) are two topological spaces, then we can define a topology on $X \times Y$, called the *product topology*, by taking the basis

$$\mathcal{B} = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}. \tag{1.37}$$

Then, the topology $\tau_{X\times Y}$ generated by the basis $\mathcal{B}$ is called the product topology on $X \times Y$. Verify that $\tau_{X\times Y}$ is indeed a topology on $X \times Y$. Note that if $\tau_{X\times Y}$ was taken to be $\{U \times V \mid U \in \tau_X, V \in \tau_Y\}$, then it would not be a topology since it is not closed under arbitrary unions. Also worth noting is that if X and Y are both Hausdorff, then so is $X \times Y$ with the product topology.

Example 1.42. X is Hausdorff if and only if the diagonal set $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed in $X \times X$ with the product topology; suppose X is Hausdorff. Let $(x, y) \in (X \times X) \setminus \Delta$. Then $x \neq y$, and by the Hausdorff property, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, we have

$$(U \times V) \cap \Delta = \emptyset, \quad (x, y) \in U \times V \subseteq (X \times X) \setminus \Delta, \tag{1.38}$$

which implies that $(X \times X) \setminus \Delta$ is open. Hence, Δ is closed. Conversely, suppose Δ is closed. Let $x, y \in X$ with $x \neq y$. Then $(x, y) \in (X \times X) \setminus \Delta$, which is open. Thus, there exists a basis element $U \times V$ such that

$$(x, y) \in U \times V \subseteq (X \times X) \setminus \Delta. \tag{1.39}$$

This implies that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Hence, X is Hausdorff.

Chapter 2

MAPPINGS

2.1 Continuous Functions

Recall the definition of a continuous function on $\mathbb{R}$; a function $f : \mathbb{R} \to \mathbb{R}$ was said to be continuous at a point $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta. \tag{2.1}$$

We can abstract this definition to metric spaces as follows.

Definition 2.1. A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be *continuous* at a point $x \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever } d_X(x, y) < \delta. \tag{2.2}$$

It is continuous at every point of X if it is continuous at each $x \in X$.

Also recall that a function $f : \mathbb{R} \to \mathbb{R}$ was continuous at $x \in \mathbb{R}$ if and only if for every open set $V \subseteq \mathbb{R}$ containing $f(x)$, the preimage $f^{-1}(V)$ is an open set in $\mathbb{R}$ containing x. We can use this characterization to define continuity in topological spaces.

Definition 2.2. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ between two topological spaces is said to be *continuous* at a point $x \in X$ if for every open set $V \in \tau_Y$ containing $f(x)$, the preimage $f^{-1}(V)$ is an open set in τ_X containing x. That is,

$$\forall V \in \tau_Y, f(x) \in V \implies x \in f^{-1}(V) \in \tau_X. \tag{2.3}$$

It is continuous at every point of X if it is continuous at each $x \in X$. That is,

$$\forall V \in \tau_Y \implies f^{-1}(V) \in \tau_X. \tag{2.4}$$

The above statement can be made more tight via bases: f is continuous if and only if the preimage of every basis element of Y is an open set in X.

Lemma 2.3. *A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is continuous if and only if $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, where τ_X and τ_Y are the topologies induced by the metrics d_X and d_Y respectively.*

Proof. Suppose $f : (X, d_X) \to (Y, d_Y)$ is continuous. Let $V \in \tau_Y$ be an open set in Y. Then, for every $y \in V$, there exists an $\varepsilon_y > 0$ such that $B_Y(y, \varepsilon_y) \subseteq V$. Since f is continuous, for every $x \in f^{-1}(V)$, there exists a $\delta_x > 0$ such that

$$f(B_X(x, \delta_x)) \subseteq B_Y(f(x), \varepsilon_{f(x)}) \subseteq V. \tag{2.5}$$

Thus, $B_X(x, \delta_x) \subseteq f^{-1}(V)$. Hence, $f^{-1}(V)$ is open in X and so $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous.

For the converse, suppose $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous. That is, for every open set $V \in \tau_Y$, the preimage $f^{-1}(V)$ is an open set in τ_X. Let $\varepsilon > 0$ be given. Consider the open ball $B_Y(f(x), \varepsilon) \in \tau_Y$. Since f is continuous, $f^{-1}(B_Y(f(x), \varepsilon))$ is open in X and contains x. Thus, there exists a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon))$. Hence, for every $y \in B_X(x, \delta)$, we have

$$f(y) \in B_Y(f(x), \varepsilon), \tag{2.6}$$

proving that $f : (X, d_X) \to (Y, d_Y)$ is continuous. ■

Lemma 2.4. *Let (X, τ) be a topological space and let $A \subseteq X$. Then the following hold.*

(i) For every sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, if $a_n \to x$ in X, then $x \in \overline{A}$. That is, if $x \in U$ for some open set $U \in \tau$, there exists a $N(U) \equiv N \in \mathbb{N}$ such that $a_n \in A \cap U$ for all $n \geq N$.

(ii) The converse holds if X is a metric space.

(iii) In general, the converse need not hold.

Proof. We prove the second statement only. Suppose $x \in \overline{A}$. Then $B(x, \frac{1}{n}) \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Thus, we can choose $a_n \in B(x, \frac{1}{n}) \cap A$ for each $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there exists a $N \equiv N(\varepsilon) \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Thus, for every $n \geq N$, we have

$$d(a_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon, \tag{2.7}$$

proving that $a_n \to x$ in X. ■

There is a third definition for metric spaces. A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be *continuous* at a point $x \in X$ if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y. In fact, the open-set definition implies the sequential definition.

Proof. Assume the open-set definition of continuity. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence such that $x_n \to x$ in X. Let $\varepsilon > 0$ be given. Consider the open ball $B_Y(f(x), \varepsilon) \in \tau_Y$. Since f is continuous, $f^{-1}(B_Y(f(x), \varepsilon))$ is open in X and contains x. Thus, there exists a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon))$. Since $x_n \to x$ in X, there exists a $N \equiv N(\delta) \in \mathbb{N}$ such that for every $n \geq N$, we have

$$x_n \in B_X(x, \delta) \implies f(x_n) \in B_Y(f(x), \varepsilon), \tag{2.8}$$

proving that $f(x_n) \to f(x)$ in Y. ■

For metric spaces, the sequential definition also implies the open-set definition.

Example 2.5. Recall $\mathbb{R}_l$, which was generated from the basis $\mathcal{B} = \{[a, b) : a < b, a, b \in \mathbb{R}\}$. Consider the identity function $f : \mathbb{R} \to \mathbb{R}_l$, with $x \mapsto x$. Then $[0, 1)$ is open in $\mathbb{R}_l$, but its preimage under f is $[0, 1)$, which is not open in $\mathbb{R}$.

Theorem 2.6. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function between two topological spaces. Then the following are equivalent.*

(i) f is (open-set) continuous.

(ii) For every closed $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

(iii) For every $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

(iv) For all $f(x) \in V$ with $V \in \tau_Y$, there exists $U \in \tau_X$ such that $x \in U$ and $f(U) \subseteq V$.

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Proof. We first show that (i) and (ii) are equivalent. Suppose C is closed in Y. Then, $Y \setminus C$ is open in Y. Since f is continuous, the preimage $f^{-1}(Y \setminus C)$ is open in X. But,

$$f^{-1}(Y \setminus C) = X \setminus f^{-1}(C). \tag{2.9}$$

Thus, $f^{-1}(C)$ is closed in X. One can follow the converse argument to show that (ii) implies (i).

For (i) implies (iii), let $A \subseteq X$ and let $x \in \overline{A}$. Let V be an open neighbourhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(V)$ is an open neighbourhood of x in X. Thus, $f^{-1}(V) \cap A \neq \emptyset$. Let $a \in f^{-1}(V) \cap A$. Then, $f(a) \in V$ and $f(a) \in f(A)$. Since V was arbitrary, we must have $V \cap f(A) \neq \emptyset$. Thus, $f(x) \in \overline{f(A)}$, proving that $f(\overline{A}) \subseteq \overline{f(A)}$. For (iii) implies (ii), let V be a closed set in Y. Then

$$f^{-1}(V) \subseteq X \implies f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} = V \tag{2.10}$$

which tells us $\overline{f^{-1}(V)} \subseteq f^{-1}(V)$, proving that $f^{-1}(V)$ is closed in X.

For (i) implies (iv), let $f(x) \in V$ with $V \in \tau_Y$. Since f is continuous, $f^{-1}(V)$ is open in X and contains x. Thus, we can take $U = f^{-1}(V)$, proving (iv). For (iv) implies (i), let $V \in \tau_Y$. For every $x \in f^{-1}(V)$, we have $f(x) \in V$. By (iv), there exists an open set $U_x \in \tau_X$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Thus, taking the union over all $x \in f^{-1}(V)$ gives $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$, proving that $f^{-1}(V)$ is open in X. ■

2.1.1 Rules of Continuous Functions

Note that the constant function $f : X \to Y$ defined by $f(x) = y_0$ for some fixed $y_0 \in Y$ is continuous. Also, the identity function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ is continuous.

Let X be a topological space, and A be a subset of X with the subspace topology. Then the inclusion map $i : A \hookrightarrow X$ defined by $i(a) = a$ is continuous. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions between topological spaces, then the composition $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ is also continuous.

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