

TOPOLOGY

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Fourth Semester

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Chapter 1

METRIC AND TOPOLOGICAL SPACES

1.1 Metric Spaces and Examples

January 6th.

A *metric space* is a pair (X, d) where X is a set and $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is a function, called a *metric* on X , satisfying the following properties for all $x, y, z \in X$:

- (i) $d(x, y) = 0$ if and only if $x = y$ (*positive definiteness*).
- (ii) $d(x, y) = d(y, x)$ (*symmetry*).
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Let us look at some examples of metric spaces.

Example 1.1. Any set X can be made into a metric space by defining the *discrete metric* d as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \quad (1.1)$$

It is easy to verify that d satisfies all the properties of a metric.

Example 1.2. Recall that a normed space $(V, \|\cdot\|)$ was a vector space V equipped with a *norm* $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and $\alpha \in \mathbb{F}$:

- (i) $\|v\| = 0$ if and only if $v = 0$ (*positive definiteness*).
- (ii) $\|\alpha v\| = |\alpha| \|v\|$ (*absolute homogeneity*).
- (iii) $\|u + v\| \leq \|u\| + \|v\|$ (*triangle inequality*).

Given a normed space $(V, \|\cdot\|)$, we can define a metric d on V as follows:

$$d(u, v) = \|u - v\| \quad \forall u, v \in V. \quad (1.2)$$

Yet again, it is straightforward to verify that d satisfies all the properties of a metric. Given a vector space V , we can have multiple norms on it, and hence multiple metrics. For example, consider the vector space \mathbb{R}^n . We have the following norms on \mathbb{R}^n :

- The ℓ^1 norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$,
- the *Euclidean norm*: $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$,

- the *supremum norm*: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$,

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Each of these norms induces a different metric on \mathbb{R}^n .

The notion of open and closed balls is also abstracted to metric spaces as follows.

Definition 1.3. Let (X, d) be a metric space. The *open ball* of radius $r > 0$ centered at a point $x \in X$ is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \quad (1.3)$$

and the *closed ball* of radius $r > 0$ centered at x is the set

$$B[x, r] = \{y \in X \mid d(x, y) \leq r\}. \quad (1.4)$$

Note that in the discrete metric space, $B(x, 1) = \{x\} = B(x, \frac{1}{2})$, and $B(x, 2) = X = B(y, 2)$ for any $x, y \in X$. Thus, $B(x, r) = B(y, \rho)$ does not imply that $x = y$ or $r = \rho$ in general.

Example 1.4. Let p be a prime, say $p = 3$. Define a function $|\cdot|_3 : \mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ as follows: for any non-zero integer m , write $m = 3^k m'$ where m' is not divisible by 3, and set $|m|_3 = 3^{-k}$. Also, set $|0|_3 = 0$. This function $|\cdot|_3$ is called the 3-adic absolute value on \mathbb{Z} . In general, for any prime p , the *p-adic absolute value* is defined similarly.

This 3-adic absolute value induces a norm d_3 on \mathbb{Q} as follows:

$$|q|_3 = \begin{cases} 0 & \text{if } q = 0, \\ |m|_3 / |n|_3 & \text{if } q = m/n \text{ in lowest terms.} \end{cases} \quad (1.5)$$

This induces a metric on \mathbb{Q} defined by $d_3(x, y) = |x - y|_3$ for all $x, y \in \mathbb{Q}$. This metric space (\mathbb{Q}, d_3) is called the 3-adic metric space, and in general (\mathbb{Q}, d_p) is called the *p-adic metric space*. The completion of (\mathbb{Q}, d_p) gives us the *field of p-adic numbers*, denoted by \mathbb{Q}_p . This metric space has some interesting properties; for instance, the triangle inequality is strengthened to the *ultrametric inequality*:

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\} \quad \forall x, y, z \in \mathbb{Q}. \quad (1.6)$$

Lemma 1.5 (Hausdorff property). Let (X, d) be a metric space. For any distinct $x, y \in X$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$.

Proof. Verify that choosing any $r \leq \frac{1}{2}d(x, y)$ works. ■

Let (X, d) be a metric space. Then a subset $A \subseteq X$ can also be made into a metric space by restricting the metric d to $A \times A$. In the metric space $(A, d|_{A \times A})$, the open balls are given by $B_A(x, r) = B_X(x, r) \cap A$ for all $x \in A$ and $r > 0$, where $B_X(x, r)$ is the open ball in (X, d) .

Again, as before, the notion of open sets is abstracted to metric spaces as follows.

Definition 1.6. Let (X, d) be a metric space. A subset $U \subseteq X$ is said to be an *open set* if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

As a small lemma, one can show that every open ball in a metric space is an open set. As an exercise, show that the complement of the closed ball $B[x, r]^c = \{y \mid d(x, y) > r\}$ is also an open set.

Proposition 1.7. Let (X, d) be a metric space. Let $\tau = \{U \subseteq X \mid U \text{ is open}\}$, that is, the collection of all open sets in X . Then the following hold true.

- (i) $\emptyset, X \in \tau$.
- (ii) For $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$, we have $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$. That is, an arbitrary union of open sets is open.

(iii) For $U_1, U_2, \dots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of open sets is open.

Proof. The proof of the first property is trivial. For the second property, let $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$. Thus, $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open.

For the third property, let $x \in \bigcap_{i=1}^n U_i$. Then $x \in U_i$ for all $1 \leq i \leq n$. Since each U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for all $1 \leq i \leq n$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then we have

$$B(x, r) \subseteq B(x, r_i) \subseteq U_i \quad \forall 1 \leq i \leq n, \quad (1.7)$$

which implies that $B(x, r) \subseteq \bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ■

1.2 Topological Spaces and Examples

A *topological space* is a pair (X, τ) where X is a set and τ is a collection of subsets of X satisfying the following properties:

- (i) $\emptyset, X \in \tau$.
- (ii) For $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$, we have $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$. That is, an arbitrary union of sets in τ is in τ .
- (iii) For $U_1, U_2, \dots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of sets in τ is in τ .

These are the exact same properties that the collection of open sets in a metric space satisfy. Hence, every metric space (X, d) gives rise to a topological space (X, τ_d) where τ_d is the collection of all open sets in (X, d) . Such a topology τ_d is called the topology induced by the metric d .

As a smaller example, let $X = \{0, 1, 2, 3, 4\}$ and consider the collection $\tau = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}\}$. Then the pair (X, τ) is *not* a topological space since $\{0, 1\} \cup \{2, 4\} = \{0, 1, 2, 4\} \notin \tau$. However, the pair (X, τ') where $\tau' = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}, \{0, 1, 2, 4\}\}$ is a topological space.

Description of open sets in \mathbb{R}

Theorem 1.8. A non-empty open set in \mathbb{R} is a countable union of pairwise disjoint open intervals.

Proof. Let $U \subseteq \mathbb{R}$ be a non-empty open set. For each $x \in U$, define

$$I_x = \bigcup \{(a, b) \mid x \in (a, b) \subseteq U\}. \quad (1.8)$$

Note that $x \in I_x \subseteq U$. Let $a_x = \inf I_x$ and $b_x = \sup I_x$. We claim that $I_x = (a_x, b_x)$. For $a_x < z < b_x$, there exists $a, b \in I_x$ such that $a_x < a < z < b < b_x$. Since $z \in (a, b) \subseteq I_x$, we have $z \in I_x$. Thus, $(a_x, b_x) \subseteq I_x$. The other inclusion is trivial. Hence, $I_x = (a_x, b_x)$ is an open interval.

We now claim that if $x \neq y$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose that $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is an interval containing both x and y and contained in U . By the definition of I_x and I_y , we have $I_x \cup I_y \subseteq I_x$ and $I_x \cup I_y \subseteq I_y$. Thus, $I_x = I_y$.

Finally, let $U = \bigcup_{x \in U} I_x$. By the above claim, the collection $\{I_x \mid x \in U\}$ consists of pairwise disjoint open intervals. Since each I_x contains a rational number (by the density of \mathbb{Q} in \mathbb{R}), for each I_x , we can choose a distinct rational number $q_x \in I_x$. This gives $I_x = I_{q_x}$. Thus, we have

$$U = \bigcup_{x \in U} I_x = \bigcup_{q \in \mathbb{Q} \cap U} I_q, \quad (1.9)$$

which is a countable union of pairwise disjoint open intervals. ■

Definition 1.9. Let (X, d_1) and (X, d_2) be two metric spaces on the same set X . The metrics d_1 and d_2 are said to be *equivalent metrics*, $d_1 \sim d_2$, if open sets in (X, d_1) are exactly the open sets in (X, d_2) .

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Definition 1.10. Let $A \subseteq X$ be a subset of a metric space (X, d) . The *interior* of A , denoted by $\text{Int}(A)$, is defined as

$$\text{Int}(A) := \{x \in A \mid \exists r > 0 \text{ such that } B(x, r) \subseteq A\}. \quad (1.10)$$

For a general topological space, the interior of a set A is defined as the largest open set contained in A . For another definition, we have

$$\text{Int}(A) := \{x \in A \mid \exists U \in \tau \text{ such that } x \in U \subseteq A\}. \quad (1.11)$$

Lemma 1.11. *The above definitions of the interior of a set in a topological space are equivalent.*

Proof. Suppose we affirm the second definition. Let $x \in \text{Int}(A)$. Then there exists an open set $U_x \in \tau$ such that $x \in U_x \subseteq A$. Thus, $\bigcup_{x \in \text{Int}(A)} U_x \subseteq A$ is contained in A and is open. Moreover, if $z \in \text{Int}(A) \setminus \bigcup_{x \in \text{Int}(A)} U_x$, then there exists an open set $V \in \tau$ such that $z \in V \subseteq A$. But then $z \in U_z \subseteq \bigcup_{x \in \text{Int}(A)} U_x$, a contradiction. Thus, $\text{Int}(A) = \bigcup_{x \in \text{Int}(A)} U_x$ is the largest open set contained in A . If V is any open set contained in A , then for any $y \in V$, there exists an open set $V_y \in \tau$ such that $y \in V_y \subseteq A$. Thus, $y \in \text{Int}(A)$, which implies that $V \subseteq \text{Int}(A)$. Hence, the first definition holds. ■

As an example, with the standard topology on \mathbb{R} , we have $\text{Int}([0, 1]) = (0, 1)$, $\text{Int}((0, 1) \cup \{2\}) = (0, 1)$, and $\text{Int}(\mathbb{Q}) = \emptyset$. Note that the interior of an open set is the set itself; $\text{Int}(U) = U$ for any open set U .

In the spirit of an induced metric, the subspace topology is defined as follows.

Definition 1.12. Let (X, τ) be a topological space and let $Y \subseteq X$. The *subspace topology* on Y is defined as

$$\tau_Y = \{U \cap Y \mid U \in \tau\}. \quad (1.12)$$

One can verify that τ_Y is a topology on Y . Let us look at some examples of topological spaces.

Example 1.13. The *discrete topology* on a set X is the topology $\tau = \mathcal{P}(X)$, the power set of X . Every subset of X is open in this topology. In contrast, the *indiscrete topology* (trivial topology) on X is the topology $\tau = \{\emptyset, X\}$. Only the empty set and the whole set are open in this topology.

Example 1.14. Let X be any set. The *cofinite topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is finite}\}. \quad (1.13)$$

One can verify that τ is a topology on X ; both \emptyset and X are in τ . For an arbitrary union of sets in τ , if any one of them is X , then the union is X . Otherwise, the complement of the union is the intersection of finite sets, which is finite. For a finite intersection of sets in τ , the complement of the intersection is the finite union of finite sets, which is finite. Thus, τ is a topology on X .

Example 1.15. Let X be any set. The *cocountable topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is countable}\}. \quad (1.14)$$

One can verify that τ is a topology on X ; both \emptyset and X are in τ . For an arbitrary union of sets in τ , if any one of them is X , then the union is X . Otherwise, the complement of the union is the intersection of countable sets, which is countable. For a finite intersection of sets in τ , the complement of the intersection is the finite union of countable sets, which is countable. Thus, τ is a topology on X .

1.2.1 Basis

Definition 1.16. Let (X, τ) be a topological space. A collection $\mathcal{B} \subseteq \tau$ is said to be a *basis* for τ if

- (i) For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
- (ii) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

For any set X , if we have a collection \mathcal{B} of subsets of X satisfying the above two properties, then the collection

$$\tau = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \quad (1.15)$$

is a topology on X , and \mathcal{B} is a basis for this topology.

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For example, choosing $\mathcal{B} = \mathcal{P}(X)$ gives us the discrete topology on X , and choosing $\mathcal{B} = \{X\}$ gives us the indiscrete topology on X .

Lemma 1.17. Let (X, τ) be a topological space and let \mathcal{B} be a basis for τ . Then τ is the collection of all possible unions of elements in \mathcal{B} .

Proof. Note that every union of elements in \mathcal{B} is in τ by the definition of a topology; we need to show the other inclusion, that is, every set in τ can be expressed as a union of elements in \mathcal{B} . Let $U \in \tau$. For each $x \in U$, by the definition of a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Thus, we have

$$U = \bigcup_{x \in U} B_x, \quad (1.16)$$

which is a union of elements in \mathcal{B} . ■

Definition 1.18. Let (X, τ_1) and (X, τ_2) be two topological spaces on the same set X . We say $\tau_1 \supseteq \tau_2$, or that τ_1 is (strictly) *finer* than τ_2 , if every open set in τ_2 is also an open set in τ_1 . Conversely, we say τ_2 is (strictly) *coarser* than τ_1 if every open set in τ_1 is also an open set in τ_2 , that is, $\tau_2 \subseteq \tau_1$.

Lemma 1.19. Let \mathcal{B} and \mathcal{B}' be bases for topologies τ and τ' on the same set X . Then the following are equivalent:

- (i) $\tau' \supseteq \tau$.
- (ii) For all $x \in X$ and all $B \in \mathcal{B}$ such that $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. For the reverse implication, let $U \in \tau$. Then U can be written as $U = \bigcup_{\alpha \in \Lambda} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for all $\alpha \in \Lambda$. For each $\alpha \in \Lambda$ and each $x \in B_\alpha$, by the second property, there exists $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subseteq B_\alpha$. Thus, we have

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{x \in B_\alpha} B'_x, \quad (1.17)$$

which is a union of elements in \mathcal{B}' . Thus, $U \in \tau'$, and hence $\tau' \supseteq \tau$. For the forward implication, let $B \in \mathcal{B}$ and $x \in B$. Since $B \in \tau \subseteq \tau'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. ■

Let us look at some topologies generated by bases.

Example 1.20. Let $X = \mathbb{R}$ and $\mathcal{B} = \{[a, b) \mid a < b, a, b \in \mathbb{R}\}$. We claim that \mathcal{B} is a basis for a topology on \mathbb{R} . Clearly, for every $x \in \mathbb{R}$, there exists $[x, x+1) \in \mathcal{B}$ such that $x \in [x, x+1)$. Now, let $B_1 = [a_1, b_1)$, $B_2 = [a_2, b_2) \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Then we have three cases:

- If $a_1 \leq a_2 \leq x \leq b_1 \leq b_2$, then choose $B_3 = [a_2, b_1)$.

- If $a_2 \leq a_1 \leq x \leq b_2 \leq b_1$, then choose $B_3 = [a_1, b_2]$.
- If $a_1 \leq x \leq b_2 \leq b_1$ (the case $a_2 \leq x \leq b_1 \leq b_2$ is similar), then choose $B_3 = [x, b_2]$.

In all cases, we have $x \in B_3 \subseteq B_1 \cap B_2$. Thus, \mathcal{B} is a basis for a topology on \mathbb{R} , called the *lower limit topology* denoted by \mathbb{R}_l . Similarly, the collection $\mathcal{B}' = \{(a, b] \mid a < b, a, b \in \mathbb{R}\}$ is a basis for a topology on \mathbb{R} , called the *upper limit topology* denoted by \mathbb{R}_u .

Example 1.21. Let $K = \{1/n \mid n \in \mathbb{N}\}$. Consider the collection

$$\mathcal{B} = \{(a, b), (a, b) \setminus K \mid a < b, a, b \in \mathbb{R}\}. \quad (1.18)$$

Verify that \mathcal{B} is a basis for a topology on \mathbb{R} , called the *K-topology* on \mathbb{R} , denoted by \mathbb{R}_K .

Lemma 1.22. *The standard topology on \mathbb{R} is strictly finer than \mathbb{R}_l , and also strictly finer than \mathbb{R}_K . However, \mathbb{R}_l and \mathbb{R}_K are not comparable.*

Proof. For \mathbb{R}_l : Let $x \in \mathbb{R}$ and let $B = (a, b) \in \mathcal{B}$ such that $x \in B$. Then choose $B' = [x, b) \in \mathcal{B}$ such that $x \in B' \subseteq B$. Thus, by the previous lemma, the standard topology on \mathbb{R} is finer than \mathbb{R}_l . To see that the inclusion is strict, note that $[a, b)$ is open in \mathbb{R}_l but not in the standard topology on \mathbb{R} .

For \mathbb{R}_K : Note that inclusion is trivial since every basis element of \mathbb{R}_K is also a basis element of the standard topology on \mathbb{R} . To see that the inclusion is strict, note that $(-1, 1) \setminus K$ is open in \mathbb{R}_K but not in the standard topology on \mathbb{R} .

For non-comparability of \mathbb{R}_l and \mathbb{R}_K : Note that $[5, 6)$ is open in \mathbb{R}_l but not in \mathbb{R}_K since it cannot be expressed as a union of basis elements of \mathbb{R}_K . Also, note that $(-1, 1) \setminus K$ is open in \mathbb{R}_K but not in \mathbb{R}_l since it cannot be expressed as a union of basis elements of \mathbb{R}_l . ■

1.3 Closed Sets

Definition 1.23. Let (X, τ) be a topological space. A subset $A \subseteq X$ is said to be a *closed set* if its complement $A^c = X \setminus A$ is an open set.

Using De Morgan's laws, one can easily verify the following properties of closed sets in a topological space.

Proposition 1.24. *Let (X, τ) be a topological space. Then the following hold true.*

- \emptyset and X are closed sets.
- For any collection $\{A_\alpha\}_{\alpha \in \Lambda}$ of closed sets, we have $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a closed set. That is, an arbitrary intersection of closed sets is closed.
- For closed sets A_1, A_2, \dots, A_n , we have $\bigcup_{i=1}^n A_i$ is a closed set. That is, a finite union of closed sets is closed.

Lemma 1.25. *Any finite set of a metric space (X, d) is closed.*

Proof. Note that it is enough to show that a singleton set $\{x\}$ is closed for any $x \in X$. Let $y \in \{x\}^c$. Then $y \neq x$, and by the Hausdorff property, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$. Thus, $B(y, r) \subseteq \{x\}^c$, which implies that $\{x\}^c$ is open. Hence, $\{x\}$ is closed. ■

Similar to the open sets, we can define closed sets in a subspace as follows.

Theorem 1.26. Let (X, τ) be a topological space and let $Y \subseteq X$ be a subspace with the subspace topology τ_Y . Then a set $C \subseteq Y$ is closed in (Y, τ_Y) if and only if there exists a closed set D in (X, τ) such that $C = D \cap Y$.

Proof. Let C be closed in (Y, τ_Y) . Then $C^c = Y \setminus C$ is open in (Y, τ_Y) . Thus, there exists an open set $U \in \tau$ such that $C^c = U \cap Y$. Let $D = U^c$, which is closed in (X, τ) . Then we have

$$C = Y \setminus C^c = Y \setminus (U \cap Y) = Y \cap U^c = Y \cap D. \quad (1.19)$$

Conversely, let D be closed in (X, τ) and let $C = D \cap Y$. Then D^c is open in (X, τ) . Thus, we have

$$C^c = Y \setminus C = Y \setminus (D \cap Y) = Y \cap D^c, \quad (1.20)$$

which is open in (Y, τ_Y) . Hence, C is closed in (Y, τ_Y) . ■

Analogous to the interior of a set, we have the notion of the closure of a set.

Definition 1.27. Let (X, τ) be a topological space and let $A \subseteq X$. The *closure* of A , denoted by \overline{A} , is defined as the smallest closed set containing A . That is,

$$\overline{A} := \bigcap \{C \subseteq X \mid C \text{ is closed and } A \subseteq C\}. \quad (1.21)$$

Theorem 1.28. The closure of a set A in a topological space (X, τ) is equivalent to saying

$$\overline{A} = \{x \in X \mid \forall U \in \tau \text{ with } x \in U, U \cap A \neq \emptyset\}. \quad (1.22)$$

Proof. Let $x \in \overline{A}$ with $x \in U \in \tau$. Suppose for the sake of contradiction that $U \cap A = \emptyset$. Then we have $A \subseteq U^c$, where U^c is closed. This must then imply that $\overline{A} \subseteq U^c$, which is a contradiction since $x \in U$. Thus, we have $U \cap A \neq \emptyset$.

For the converse inclusion, let $x \notin \overline{A}$. Then there exists a closed set C such that $A \subseteq C$ but $x \notin C$. Thus, $x \in C^c$, where C^c is open. Since $A \subseteq C$, we have $C^c \cap A = \emptyset$. Hence, there exists an open set C^c containing x such that $C^c \cap A = \emptyset$. ■

Instead of using “ $\forall U \in \tau$ ” in the above theorem, one can equivalently use “ $\forall B \in \mathcal{B}$ ” where \mathcal{B} is a basis for the topology τ .

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Lemma 1.29. A is closed if and only if $A = \overline{A}$. Moreover, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

Proof. The first assertion is left as an exercise. For the second assertion, let $x \in \overline{A}$. Then for every open set U containing x , we have $U \cap A \neq \emptyset$. Since $A \subseteq B$, we have $U \cap B \neq \emptyset$. Thus, $x \in \overline{B}$. ■

Theorem 1.30. Let $Y \subseteq X$ be a subspace of a topological space (X, τ) . For any $A \subseteq Y$, we have

$$\overline{A}^Y = \overline{A} \cap Y, \quad (1.23)$$

where \overline{A}^Y is the closure of A in the subspace (Y, τ_Y) and \overline{A} is the closure of A in (X, τ) .

Proof. We have

$$\overline{A}^Y = \{y \in Y \mid y \in U \cap Y, U \in \tau, U \cap Y \cap A \neq \emptyset\} \subseteq \overline{A} \cap Y. \quad (1.24)$$

Now let $y_0 \in \overline{A} \cap Y$. Then for every $U \in \tau$ such that $y_0 \in U$, we have $U \cap A \neq \emptyset$. Thus, taking intersection over all closed sets containing A in (X, τ) and then intersecting with Y , we have

$$\bigcap_{\substack{C \subseteq X \\ C \text{ closed} \\ A \subseteq C}} (C \cap Y) = \overline{A} \cap Y. \quad (1.25)$$

Thus, $y_0 \in \overline{A}^Y$. Hence, we have $\overline{A}^Y = \overline{A} \cap Y$. ■

1.4 Convergence

Definition 1.31. Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a *limit point* of A if for every open set $U \in \tau$ containing x , we have $(U \setminus \{x\}) \cap A \neq \emptyset$.

Let us denote the set of all limit points of A by A' . Then $A' \subseteq \overline{A}$ since for every open set U containing a limit point x , we have $U \cap A \neq \emptyset$. Note that nothing can be said about the relation between A and A' .

For example, if $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ with the standard topology, then $A' = \{0\}$ and $\overline{A} = A \cup \{0\}$. If $A = (0, 1) \cup \{2\}$, then $A' = [0, 1]$ and $\overline{A} = [0, 1] \cup \{2\}$. If $A = \mathbb{Q} \subseteq \mathbb{R}$ with the standard topology, then $A' = \mathbb{R}$ and $\overline{A} = \mathbb{R}$. There is a nice characterization of the closure of a set using limit points.

Theorem 1.32. Let (X, τ) be a topological space and let $A \subseteq X$. Then

$$\overline{A} = A \cup A'. \quad (1.26)$$

Proof. One direction is easy since we have already seen that $A, A' \subseteq \overline{A}$. For the other direction, let $x \in \overline{A}$. If $x \in A$, then we are done. So suppose that $x \notin A$. Then for every open set U containing x , we have $U \cap A \neq \emptyset$. Since $x \notin A$, we have $(U \setminus \{x\}) \cap A \neq \emptyset$. Thus, x is a limit point of A , and hence $x \in A'$. Therefore, we have $\overline{A} \subseteq A \cup A'$, and hence the result follows. ■

Corollary 1.33. A set A is closed if and only if $A' \subseteq A$.

We are now ready to define convergence in a (metric) topological space.

Definition 1.34. Let (X, d) be a metric space. A sequence $(x_n)_{n \geq 1} \subseteq X$ is said to *converge* to a point $x \in X$ if for every $\varepsilon > 0$, there exists a natural $N(\varepsilon) \equiv N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \forall n \geq N. \quad (1.27)$$

Equivalently, the sequence $(d(x_n, x))_{n \geq 1} \subseteq \mathbb{R}$ converges to 0 in the usual sense.

In the topological sense, we say $(x_n)_{n \geq 1}$ converges to x if for every open set U containing x , there exists a natural $N(U) \equiv N \in \mathbb{N}$ such that

$$x_n \in U \quad \forall n \geq N. \quad (1.28)$$

Note that the limit of a sequence in a metric space is unique. However, in a general topological space, the limit of a sequence need not be unique. For example, consider the set X with the indiscrete topology. Then every sequence in X converges to every point in X .

Theorem 1.35. Let (X, d) be a metric space and let $(x_n)_{n \geq 1} \subseteq X$ be a sequence converging to both x and y in X . Then $x = y$.

Proof. For every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon/2 \quad \forall n \geq N_1. \quad (1.29)$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, y) < \varepsilon/2 \quad \forall n \geq N_2. \quad (1.30)$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (1.31)$$

Since $\varepsilon > 0$ is arbitrary, we have $d(x, y) = 0$, which implies that $x = y$. ■

The concept of Hausdorff property can be generalized to topological spaces as follows: for every $x, y \in X$ with $x \neq y$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Theorem 1.36. *Let (X, τ) be a Hausdorff topological space and let $(x_n)_{n \geq 1} \subseteq X$ be a sequence converging to both x and y in X . Then $x = y$.*

Proof. The proof is similar to that in the metric space case. For $x \neq y$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since $(x_n)_{n \geq 1}$ converges to x , there exists $N_1 \in \mathbb{N}$ such that

$$x_n \in U \quad \forall n \geq N_1. \quad (1.32)$$

Similarly, since $(x_n)_{n \geq 1}$ converges to y , there exists $N_2 \in \mathbb{N}$ such that

$$x_n \in V \quad \forall n \geq N_2. \quad (1.33)$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have $x_n \in U$ and $x_n \in V$, which implies that $x_n \in U \cap V$. This is a contradiction since $U \cap V = \emptyset$. Hence, we must have $x = y$. ■

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