# TOPOLOGY

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Fourth Semester

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Chapter 1

# METRIC AND TOPOLOGICAL SPACES

## 1.1 Metric Spaces and Examples

*January 6th.*

A *metric space* is a pair $(X, d)$ where $X$ is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function, called a *metric* on $X$, satisfying the following properties for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$ (*positive definiteness*).

(ii) $d(x, y) = d(y, x)$ (*symmetry*).

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Let us look at some examples of metric spaces.

**Example 1.1.** Any set $X$ can be made into a metric space by defining the *discrete metric* $d$ as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \tag{1.1}$$

It is easy to verify that $d$ satisfies all the properties of a metric.

**Example 1.2.** Recall that a normed space $(V, \|\cdot\|)$ was a vector space $V$ equipped with a *norm* $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and $\alpha \in \mathbb{F}$:

(i) $\|v\| = 0$ if and only if $v = 0$ (positive definiteness).

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ (*absolute homogeneity*).

(iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).

Given a normed space $(V, \|\cdot\|)$, we can define a metric $d$ on $V$ as follows:

$$d(u, v) = \|u - v\| \quad \forall u, v \in V. \tag{1.2}$$

Yet again, it is straightforward to verify that $d$ satisfies all the properties of a metric. Given a vector space $V$, we can have multiple norms on it, and hence multiple metrics. For example, consider the vector space $\mathbb{R}^n$. We have the following norms on $\mathbb{R}^n$:

- The $\ell^1$ *norm*: $\|x\|_1 = \sum_{i=1}^n |x_i|$,
- the *Euclidean norm*: $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$,

- the *supremum norm*: $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$,

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Each of these norms induces a different metric on $\mathbb{R}^n$.

The notion of open and closed balls is also abstracted to metric spaces as follows.

**Definition 1.3.** Let $(X, d)$ be a metric space. The *open ball* of radius $r > 0$ centered at a point $x \in X$ is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \tag{1.3}$$

and the *closed ball* of radius $r > 0$ centered at $x$ is the set

$$B[x, r] = \{y \in X \mid d(x, y) \le r\}. \tag{1.4}$$

Note that in the discrete metric space, $B(x, 1) = \{x\} = B(x, \frac{1}{2})$, and $B(x, 2) = X = B(y, 2)$ for any $x, y \in X$. Thus, $B(x, r) = B(y, \rho)$ does not imply that $x = y$ or $r = \rho$ in general.

**Example 1.4.** Let $p$ be a prime, say $p = 3$. Define a function $|\cdot|_3 : \mathbb{Z} \to \mathbb{R}_{\ge 0}$ as follows: for any non-zero integer $m$, write $m = 3^k m'$ where $m'$ is not divisible by 3, and set $|m|_3 = 3^{-k}$. Also, set $|0|_3 = 0$. This function $|\cdot|_3$ is called the 3-adic absolute value on $\mathbb{Z}$. In general, for any prime $p$, the *p-adic absolute value* is defined similarly.
This 3-adic absolute value induces a norm $d_3$ on $\mathbb{Q}$ as follows:

$$|q|_3 = \begin{cases} 0 & \text{if } q = 0, \\ |m|_3 \,/\, |n|_3 & \text{if } q = m/n \text{ in lowest terms.} \end{cases} \tag{1.5}$$

This induces a metric on $\mathbb{Q}$ defined by $d_3(x, y) = |x - y|_3$ for all $x, y \in \mathbb{Q}$. This metric space $(\mathbb{Q}, d_3)$ is called the 3-adic metric space, and in general $(\mathbb{Q}, d_p)$ is called the *p-adic metric space*. The completion of $(\mathbb{Q}, d_p)$ gives us the *field of p-adic numbers*, denoted by $\mathbb{Q}_p$. This metric space has some interesting properties; for instance, the triangle inequality is strengthened to the *ultrametric inequality*:

$$d_p(x, z) \le \max\{d_p(x, y), d_p(y, z)\} \quad \forall x, y, z \in \mathbb{Q}. \tag{1.6}$$

**Lemma 1.5** (*Hausdorff property*)**.** *Let $(X, d)$ be a metric space. For any distinct $x, y \in X$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$.*

*Proof.* Verify that choosing any $r \le \frac{1}{2} d(x, y)$ works. ■

Let $(X, d)$ be a metric space. Then a subset $A \subseteq X$ can also be made into a metric space by restricting the metric $d$ to $A \times A$. In the metric space $(A, d|_{A \times A})$, the open balls are given by $B_A(x, r) = B_X(x, r) \cap A$ for all $x \in A$ and $r > 0$, where $B_X(x, r)$ is the open ball in $(X, d)$.
Again, as before, the notion of open sets is abstracted to metric spaces as follows.

**Definition 1.6.** Let $(X, d)$ be a metric space. A subset $U \subseteq X$ is said to be an *open set* if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

As a small lemma, one can show that every open ball in a metric space is an open set. As an exercise, show that the complement of the closed ball $B[x, r]^c = \{y \mid d(x, y) > r\}$ is also an open set.

**Proposition 1.7.** *Let $(X, d)$ be a metric space. Let $\tau = \{U \subseteq X \mid U \text{ is open}\}$, that is, the collection of all open sets in $X$. Then the following hold true.*

*(i) $\emptyset, X \in \tau$.*

*(ii) For $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$, we have $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$. That is, an arbitrary union of open sets is open.*

*(iii) For $U_1, U_2, \ldots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of open sets is open.*

*Proof.* The proof of the first property is trivial. For the second property, let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0}$ is open, there exists $r > 0$ such that $B(x,r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$. Thus, $\bigcup_{\alpha\in\Lambda} U_\alpha$ is open.

For the third property, let $x \in \bigcap_{i=1}^n U_i$. Then $x \in U_i$ for all $1 \leq i \leq n$. Since each $U_i$ is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for all $1 \leq i \leq n$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then we have

$$B(x,r) \subseteq B(x,r_i) \subseteq U_i \quad \forall 1 \leq i \leq n, \tag{1.7}$$

which implies that $B(x,r) \subseteq \bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ∎

## 1.2 Topological Spaces and Examples

A *topological space* is a pair $(X, \tau)$ where $X$ is a set and $\tau$ is a collection of subsets of $X$ satisfying the following properties:

(i) $\emptyset, X \in \tau$.

(ii) For $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \tau$, we have $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$. That is, an arbitrary union of sets in $\tau$ is in $\tau$.

(iii) For $U_1, U_2, \ldots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of sets in $\tau$ is in $\tau$.

These are the exact same properties that the collection of open sets in a metric space satisfy. Hence, every metric space $(X, d)$ gives rise to a topological space $(X, \tau_d)$ where $\tau_d$ is the collection of all open sets in $(X, d)$. Such a topology $\tau_d$ is called the topology induced by the metric $d$.

As a smaller example, let $X = \{0, 1, 2, 3, 4\}$ and consider the collection $\tau = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}\}$. Then the pair $(X, \tau)$ is *not* a topological space since $\{0, 1\} \cup \{2, 4\} = \{0, 1, 2, 4\} \notin \tau$. However, the pair $(X, \tau')$ where $\tau' = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}, \{0, 1, 2, 4\}\}$ is a topological space.

#### Description of open sets in $\mathbb{R}$

**Theorem 1.8.** *A non-empty open set in $\mathbb{R}$ is a countable union of pairwise disjoint open intervals.*

*Proof.* Let $U \subseteq \mathbb{R}$ be a non-empty open set. For each $x \in U$, define

$$I_x = \bigcup \{(a, b) \mid x \in (a, b) \subseteq U\}. \tag{1.8}$$

Note that $x \in I_x \subseteq U$. Let $a_x = \inf I_x$ and $b_x = \sup I_x$. We claim that $I_x = (a_x, b_x)$. For $a_x < z < b_x$, there exists $a, b \in I_x$ such that $a_x < a < z < b < b_x$. Since $z \in (a, b) \subseteq I_x$, we have $z \in I_x$. Thus, $(a_x, b_x) \subseteq I_x$. The other inclusion is trivial. Hence, $I_x = (a_x, b_x)$ is an open interval.

We now claim that if $x \neq y$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose that $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is an interval containing both $x$ and $y$ and contained in $U$. By the definition of $I_x$ and $I_y$, we have $I_x \cup I_y \subseteq I_x$ and $I_x \cup I_y \subseteq I_y$. Thus, $I_x = I_y$.

Finally, let $U = \bigcup_{x\in U} I_x$. By the above claim, the collection $\{I_x \mid x \in U\}$ consists of pairwise disjoint open intervals. Since each $I_x$ contains a rational number (by the density of $\mathbb{Q}$ in $\mathbb{R}$), for each $I_x$, we can choose a distinct rational number $q_x \in I_x$. This gives $I_x = I_{q_x}$. Thus, we have

$$U = \bigcup_{x\in U} I_x = \bigcup_{q\in\mathbb{Q}\cap U} I_q, \tag{1.9}$$

which is a countable union of pairwise disjoint open intervals. ∎

**Definition 1.9.** Let $(X, d_1)$ and $(X, d_2)$ be two metric spaces on the same set $X$. The metrics $d_1$ and $d_2$ are said to be *equivalent metrics*, $d_1 \sim d_2$, if open sets in $(X, d_1)$ are exactly the open sets in $(X, d_2)$.

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