

# **TOPOLOGY**

Shreyasi Datta, notes by Ramdas Singh

Fourth Semester

# Contents

1	METRIC AND TOPOLOGICAL SPACES	1
1.1	Metric Spaces and Examples	1
1.2	Topological Spaces and Examples	3
1.2.1	Basis	5
1.3	Closed Sets	6
1.4	Convergence and Hausdorff Spaces	8
1.5	Product Spaces	10
2	MAPPINGS	11
2.1	Continuous Functions	11
2.1.1	Rules of Continuous Functions	13
2.2	Product Topology	14
2.3	Connectedness	17
2.3.1	Components	19
2.4	Compactness	20
	Index	21

## Chapter 1

# METRIC AND TOPOLOGICAL SPACES

### 1.1 Metric Spaces and Examples

January 6th.

A *metric space* is a pair  $(X, d)$  where  $X$  is a set and  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$  is a function, called a *metric* on  $X$ , satisfying the following properties for all  $x, y, z \in X$ :

- (i)  $d(x, y) = 0$  if and only if  $x = y$  (*positive definiteness*).
- (ii)  $d(x, y) = d(y, x)$  (*symmetry*).
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$  (*triangle inequality*).

Let us look at some examples of metric spaces.

**Example 1.1.** Any set  $X$  can be made into a metric space by defining the *discrete metric*  $d$  as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \quad (1.1)$$

It is easy to verify that  $d$  satisfies all the properties of a metric.

**Example 1.2.** Recall that a normed space  $(V, \|\cdot\|)$  was a vector space  $V$  equipped with a *norm*  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties for all  $u, v \in V$  and  $\alpha \in \mathbb{F}$ :

- (i)  $\|v\| = 0$  if and only if  $v = 0$  (*positive definiteness*).
- (ii)  $\|\alpha v\| = |\alpha| \|v\|$  (*absolute homogeneity*).
- (iii)  $\|u + v\| \leq \|u\| + \|v\|$  (*triangle inequality*).

Given a normed space  $(V, \|\cdot\|)$ , we can define a metric  $d$  on  $V$  as follows:

$$d(u, v) = \|u - v\| \quad \forall u, v \in V. \quad (1.2)$$

Yet again, it is straightforward to verify that  $d$  satisfies all the properties of a metric. Given a vector space  $V$ , we can have multiple norms on it, and hence multiple metrics. For example, consider the vector space  $\mathbb{R}^n$ . We have the following norms on  $\mathbb{R}^n$ :

- The  $\ell^1$  norm:  $\|x\|_1 = \sum_{i=1}^n |x_i|$ ,
- the *Euclidean norm*:  $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$ ,

- the supremum norm:  $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ ,

for all  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . Each of these norms induces a different metric on  $\mathbb{R}^n$ .

The notion of open and closed balls is also abstracted to metric spaces as follows.

**Definition 1.3.** Let  $(X, d)$  be a metric space. The *open ball* of radius  $r > 0$  centered at a point  $x \in X$  is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \quad (1.3)$$

and the *closed ball* of radius  $r > 0$  centered at  $x$  is the set

$$B[x, r] = \{y \in X \mid d(x, y) \leq r\}. \quad (1.4)$$

Note that in the discrete metric space,  $B(x, 1) = \{x\} = B(x, \frac{1}{2})$ , and  $B(x, 2) = X = B(y, 2)$  for any  $x, y \in X$ . Thus,  $B(x, r) = B(y, \rho)$  does not imply that  $x = y$  or  $r = \rho$  in general.

**Example 1.4.** Let  $p$  be a prime, say  $p = 3$ . Define a function  $|\cdot|_3 : \mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$  as follows: for any non-zero integer  $m$ , write  $m = 3^k m'$  where  $m'$  is not divisible by 3, and set  $|m|_3 = 3^{-k}$ . Also, set  $|0|_3 = 0$ . This function  $|\cdot|_3$  is called the  $3$ -adic absolute value on  $\mathbb{Z}$ . In general, for any prime  $p$ , the  $p$ -adic absolute value is defined similarly.

This 3-adic absolute value induces a norm  $d_3$  on  $\mathbb{Q}$  as follows:

$$|q|_3 = \begin{cases} 0 & \text{if } q = 0, \\ |m|_3 / |n|_3 & \text{if } q = m/n \text{ in lowest terms.} \end{cases} \quad (1.5)$$

This induces a metric on  $\mathbb{Q}$  defined by  $d_3(x, y) = |x - y|_3$  for all  $x, y \in \mathbb{Q}$ . This metric space  $(\mathbb{Q}, d_3)$  is called the 3-adic metric space, and in general  $(\mathbb{Q}, d_p)$  is called the  $p$ -adic metric space. The completion of  $(\mathbb{Q}, d_p)$  gives us the field of  $p$ -adic numbers, denoted by  $\mathbb{Q}_p$ . This metric space has some interesting properties; for instance, the triangle inequality is strengthened to the *ultrametric inequality*:

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\} \quad \forall x, y, z \in \mathbb{Q}. \quad (1.6)$$

**Lemma 1.5 (Hausdorff property).** Let  $(X, d)$  be a metric space. For any distinct  $x, y \in X$ , there exists  $r > 0$  such that  $B(x, r) \cap B(y, r) = \emptyset$ .

*Proof.* Verify that choosing any  $r \leq \frac{1}{2}d(x, y)$  works. ■

Let  $(X, d)$  be a metric space. Then a subset  $A \subseteq X$  can also be made into a metric space by restricting the metric  $d$  to  $A \times A$ . In the metric space  $(A, d|_{A \times A})$ , the open balls are given by  $B_A(x, r) = B_X(x, r) \cap A$  for all  $x \in A$  and  $r > 0$ , where  $B_X(x, r)$  is the open ball in  $(X, d)$ .

Again, as before, the notion of open sets is abstracted to metric spaces as follows.

**Definition 1.6.** Let  $(X, d)$  be a metric space. A subset  $U \subseteq X$  is said to be an *open set* if for every  $x \in U$ , there exists  $r > 0$  such that  $B(x, r) \subseteq U$ .

As a small lemma, one can show that every open ball in a metric space is an open set. As an exercise, show that the complement of the closed ball  $B[x, r]^c = \{y \mid d(x, y) > r\}$  is also an open set.

**Proposition 1.7.** Let  $(X, d)$  be a metric space. Let  $\tau = \{U \subseteq X \mid U \text{ is open}\}$ , that is, the collection of all open sets in  $X$ . Then the following hold true.

(i)  $\emptyset, X \in \tau$ .

(ii) For  $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$ , we have  $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$ . That is, an arbitrary union of open sets is open.

(iii) For  $U_1, U_2, \dots, U_n \in \tau$ , we have  $\bigcap_{i=1}^n U_i \in \tau$ . That is, a finite intersection of open sets is open.

*Proof.* The proof of the first property is trivial. For the second property, let  $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$ . Then there exists some  $\alpha_0 \in \Lambda$  such that  $x \in U_{\alpha_0}$ . Since  $U_{\alpha_0}$  is open, there exists  $r > 0$  such that  $B(x, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$ . Thus,  $\bigcup_{\alpha \in \Lambda} U_\alpha$  is open.

For the third property, let  $x \in \bigcap_{i=1}^n U_i$ . Then  $x \in U_i$  for all  $1 \leq i \leq n$ . Since each  $U_i$  is open, there exists  $r_i > 0$  such that  $B(x, r_i) \subseteq U_i$  for all  $1 \leq i \leq n$ . Let  $r = \min\{r_1, r_2, \dots, r_n\}$ . Then we have

$$B(x, r) \subseteq B(x, r_i) \subseteq U_i \quad \forall 1 \leq i \leq n, \quad (1.7)$$

which implies that  $B(x, r) \subseteq \bigcap_{i=1}^n U_i$ . Thus,  $\bigcap_{i=1}^n U_i$  is open. ■

## 1.2 Topological Spaces and Examples

A *topological space* is a pair  $(X, \tau)$  where  $X$  is a set and  $\tau$  is a collection of subsets of  $X$  satisfying the following properties:

- (i)  $\emptyset, X \in \tau$ .
- (ii) For  $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$ , we have  $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$ . That is, an arbitrary union of sets in  $\tau$  is in  $\tau$ .
- (iii) For  $U_1, U_2, \dots, U_n \in \tau$ , we have  $\bigcap_{i=1}^n U_i \in \tau$ . That is, a finite intersection of sets in  $\tau$  is in  $\tau$ .

These are the exact same properties that the collection of open sets in a metric space satisfy. Hence, every metric space  $(X, d)$  gives rise to a topological space  $(X, \tau_d)$  where  $\tau_d$  is the collection of all open sets in  $(X, d)$ . Such a topology  $\tau_d$  is called the topology induced by the metric  $d$ .

As a smaller example, let  $X = \{0, 1, 2, 3, 4\}$  and consider the collection  $\tau = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}\}$ . Then the pair  $(X, \tau)$  is *not* a topological space since  $\{0, 1\} \cup \{2, 4\} = \{0, 1, 2, 4\} \notin \tau$ . However, the pair  $(X, \tau')$  where  $\tau' = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}, \{0, 1, 2, 4\}\}$  is a topological space.

### Description of open sets in $\mathbb{R}$

**Theorem 1.8.** A non-empty open set in  $\mathbb{R}$  is a countable union of pairwise disjoint open intervals.

*Proof.* Let  $U \subseteq \mathbb{R}$  be a non-empty open set. For each  $x \in U$ , define

$$I_x = \bigcup \{(a, b) \mid x \in (a, b) \subseteq U\}. \quad (1.8)$$

Note that  $x \in I_x \subseteq U$ . Let  $a_x = \inf I_x$  and  $b_x = \sup I_x$ . We claim that  $I_x = (a_x, b_x)$ . For  $a_x < z < b_x$ , there exists  $a, b \in I_x$  such that  $a_x < a < z < b < b_x$ . Since  $z \in (a, b) \subseteq I_x$ , we have  $z \in I_x$ . Thus,  $(a_x, b_x) \subseteq I_x$ . The other inclusion is trivial. Hence,  $I_x = (a_x, b_x)$  is an open interval.

We now claim that if  $x \neq y$ , then either  $I_x = I_y$  or  $I_x \cap I_y = \emptyset$ . Suppose that  $I_x \cap I_y \neq \emptyset$ . Then  $I_x \cup I_y$  is an interval containing both  $x$  and  $y$  and contained in  $U$ . By the definition of  $I_x$  and  $I_y$ , we have  $I_x \cup I_y \subseteq I_x$  and  $I_x \cup I_y \subseteq I_y$ . Thus,  $I_x = I_y$ .

Finally, let  $U = \bigcup_{x \in U} I_x$ . By the above claim, the collection  $\{I_x \mid x \in U\}$  consists of pairwise disjoint open intervals. Since each  $I_x$  contains a rational number (by the density of  $\mathbb{Q}$  in  $\mathbb{R}$ ), for each  $I_x$ , we can choose a distinct rational number  $q_x \in I_x$ . This gives  $I_x = I_{q_x}$ . Thus, we have

$$U = \bigcup_{x \in U} I_x = \bigcup_{q \in \mathbb{Q} \cap U} I_q, \quad (1.9)$$

which is a countable union of pairwise disjoint open intervals. ■

**Definition 1.9.** Let  $(X, d_1)$  and  $(X, d_2)$  be two metric spaces on the same set  $X$ . The metrics  $d_1$  and  $d_2$  are said to be *equivalent metrics*,  $d_1 \sim d_2$ , if open sets in  $(X, d_1)$  are exactly the open sets in  $(X, d_2)$ .

January 8th.

**Definition 1.10.** Let  $A \subseteq X$  be a subset of a metric space  $(X, d)$ . The *interior* of  $A$ , denoted by  $\text{Int}(A)$ , is defined as

$$\text{Int}(A) := \{x \in A \mid \exists r > 0 \text{ such that } B(x, r) \subseteq A\}. \quad (1.10)$$

For a general topological space, the interior of a set  $A$  is defined as the largest open set contained in  $A$ . For another definition, we have

$$\text{Int}(A) := \{x \in A \mid \exists U \in \tau \text{ such that } x \in U \subseteq A\}. \quad (1.11)$$

**Lemma 1.11.** *The above definitions of the interior of a set in a topological space are equivalent.*

*Proof.* Suppose we affirm the second definition. Let  $x \in \text{Int}(A)$ . Then there exists an open set  $U_x \in \tau$  such that  $x \in U_x \subseteq A$ . Thus,  $\bigcup_{x \in \text{Int}(A)} U_x \subseteq A$  is contained in  $A$  and is open. Moreover, if  $z \in \text{Int}(A) \setminus \bigcup_{x \in \text{Int}(A)} U_x$ , then there exists an open set  $V \in \tau$  such that  $z \in V \subseteq A$ . But then  $z \in U_z \subseteq \bigcup_{x \in \text{Int}(A)} U_x$ , a contradiction. Thus,  $\text{Int}(A) = \bigcup_{x \in \text{Int}(A)} U_x$  is the largest open set contained in  $A$ . If  $V$  is any open set contained in  $A$ , then for any  $y \in V$ , there exists an open set  $V_y \in \tau$  such that  $y \in V_y \subseteq A$ . Thus,  $y \in \text{Int}(A)$ , which implies that  $V \subseteq \text{Int}(A)$ . Hence, the first definition holds. ■

As an example, with the standard topology on  $\mathbb{R}$ , we have  $\text{Int}([0, 1]) = (0, 1)$ ,  $\text{Int}((0, 1) \cup \{2\}) = (0, 1)$ , and  $\text{Int}(\mathbb{Q}) = \emptyset$ . Note that the interior of an open set is the set itself;  $\text{Int}(U) = U$  for any open set  $U$ .

In the spirit of an induced metric, the subspace topology is defined as follows.

**Definition 1.12.** Let  $(X, \tau)$  be a topological space and let  $Y \subseteq X$ . The *subspace topology* on  $Y$  is defined as

$$\tau_Y = \{U \cap Y \mid U \in \tau\}. \quad (1.12)$$

One can verify that  $\tau_Y$  is a topology on  $Y$ . Let us look at some examples of topological spaces.

**Example 1.13.** The *discrete topology* on a set  $X$  is the topology  $\tau = \mathcal{P}(X)$ , the power set of  $X$ . Every subset of  $X$  is open in this topology. In contrast, the *indiscrete topology* (trivial topology) on  $X$  is the topology  $\tau = \{\emptyset, X\}$ . Only the empty set and the whole set are open in this topology.

**Example 1.14.** Let  $X$  be any set. The *cofinite topology* on  $X$  is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is finite}\}. \quad (1.13)$$

One can verify that  $\tau$  is a topology on  $X$ ; both  $\emptyset$  and  $X$  are in  $\tau$ . For an arbitrary union of sets in  $\tau$ , if any one of them is  $X$ , then the union is  $X$ . Otherwise, the complement of the union is the intersection of finite sets, which is finite. For a finite intersection of sets in  $\tau$ , the complement of the intersection is the finite union of finite sets, which is finite. Thus,  $\tau$  is a topology on  $X$ .

**Example 1.15.** Let  $X$  be any set. The *cocountable topology* on  $X$  is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is countable}\}. \quad (1.14)$$

One can verify that  $\tau$  is a topology on  $X$ ; both  $\emptyset$  and  $X$  are in  $\tau$ . For an arbitrary union of sets in  $\tau$ , if any one of them is  $X$ , then the union is  $X$ . Otherwise, the complement of the union is the intersection of countable sets, which is countable. For a finite intersection of sets in  $\tau$ , the complement of the intersection is the finite union of countable sets, which is countable. Thus,  $\tau$  is a topology on  $X$ .

### 1.2.1 Basis

**Definition 1.16.** Let  $(X, \tau)$  be a topological space. A collection  $\mathcal{B} \subseteq \tau$  is said to be a *basis* for  $\tau$  if

- (i) For every  $x \in X$ , there exists  $B \in \mathcal{B}$  such that  $x \in B$ .
- (ii) For any  $B_1, B_2 \in \mathcal{B}$  and any  $x \in B_1 \cap B_2$ , there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

For any set  $X$ , if we have a collection  $\mathcal{B}$  of subsets of  $X$  satisfying the above two properties, then the collection

$$\tau = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \quad (1.15)$$

is a topology on  $X$ , and  $\mathcal{B}$  is a basis for this topology.

*January 9th.*

For example, choosing  $\mathcal{B} = \mathcal{P}(X)$  gives us the discrete topology on  $X$ , and choosing  $\mathcal{B} = \{X\}$  gives us the indiscrete topology on  $X$ .

**Lemma 1.17.** Let  $(X, \tau)$  be a topological space and let  $\mathcal{B}$  be a basis for  $\tau$ . Then  $\tau$  is the collection of all possible unions of elements in  $\mathcal{B}$ .

*Proof.* Note that every union of elements in  $\mathcal{B}$  is in  $\tau$  by the definition of a topology; we need to show the other inclusion, that is, every set in  $\tau$  can be expressed as a union of elements in  $\mathcal{B}$ . Let  $U \in \tau$ . For each  $x \in U$ , by the definition of a basis, there exists  $B_x \in \mathcal{B}$  such that  $x \in B_x \subseteq U$ . Thus, we have

$$U = \bigcup_{x \in U} B_x, \quad (1.16)$$

which is a union of elements in  $\mathcal{B}$ . ■

**Definition 1.18.** Let  $(X, \tau_1)$  and  $(X, \tau_2)$  be two topological spaces on the same set  $X$ . We say  $\tau_1 \supseteq \tau_2$ , or that  $\tau_1$  is (strictly) *finer* than  $\tau_2$ , if every open set in  $\tau_2$  is also an open set in  $\tau_1$ . Conversely, we say  $\tau_2$  is (strictly) *coarser* than  $\tau_1$  if every open set in  $\tau_1$  is also an open set in  $\tau_2$ , that is,  $\tau_2 \subseteq \tau_1$ .

**Lemma 1.19.** Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for topologies  $\tau$  and  $\tau'$  on the same set  $X$ . Then the following are equivalent:

- (i)  $\tau' \supseteq \tau$ .
- (ii) For all  $x \in X$  and all  $B \in \mathcal{B}$  such that  $x \in B$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

*Proof.* For the reverse implication, let  $U \in \tau$ . Then  $U$  can be written as  $U = \bigcup_{\alpha \in \Lambda} B_\alpha$  where  $B_\alpha \in \mathcal{B}$  for all  $\alpha \in \Lambda$ . For each  $\alpha \in \Lambda$  and each  $x \in B_\alpha$ , by the second property, there exists  $B'_x \in \mathcal{B}'$  such that  $x \in B'_x \subseteq B_\alpha$ . Thus, we have

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{x \in B_\alpha} B'_x, \quad (1.17)$$

which is a union of elements in  $\mathcal{B}'$ . Thus,  $U \in \tau'$ , and hence  $\tau' \supseteq \tau$ . For the forward implication, let  $B \in \mathcal{B}$  and  $x \in B$ . Since  $B \in \tau \subseteq \tau'$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ . ■

Let us look at some topologies generated by bases.

**Example 1.20.** Let  $X = \mathbb{R}$  and  $\mathcal{B} = \{[a, b) \mid a < b, a, b \in \mathbb{R}\}$ . We claim that  $\mathcal{B}$  is a basis for a topology on  $\mathbb{R}$ . Clearly, for every  $x \in \mathbb{R}$ , there exists  $[x, x+1) \in \mathcal{B}$  such that  $x \in [x, x+1)$ . Now, let  $B_1 = [a_1, b_1), B_2 = [a_2, b_2) \in \mathcal{B}$  and let  $x \in B_1 \cap B_2$ . Then we have three cases:

- If  $a_1 \leq a_2 \leq x \leq b_1 \leq b_2$ , then choose  $B_3 = [a_2, b_1)$ .

- If  $a_2 \leq a_1 \leq x \leq b_2 \leq b_1$ , then choose  $B_3 = [a_1, b_2]$ .
- If  $a_1 \leq x \leq b_2 \leq b_1$  (the case  $a_2 \leq x \leq b_1 \leq b_2$  is similar), then choose  $B_3 = [x, b_2]$ .

In all cases, we have  $x \in B_3 \subseteq B_1 \cap B_2$ . Thus,  $\mathcal{B}$  is a basis for a topology on  $\mathbb{R}$ , called the *lower limit topology* denoted by  $\mathbb{R}_l$ . Similarly, the collection  $\mathcal{B}' = \{(a, b] \mid a < b, a, b \in \mathbb{R}\}$  is a basis for a topology on  $\mathbb{R}$ , called the *upper limit topology* denoted by  $\mathbb{R}_u$ .

**Example 1.21.** Let  $K = \{1/n \mid n \in \mathbb{N}\}$ . Consider the collection

$$\mathcal{B} = \{(a, b), (a, b) \setminus K \mid a < b, a, b \in \mathbb{R}\}. \quad (1.18)$$

Verify that  $\mathcal{B}$  is a basis for a topology on  $\mathbb{R}$ , called the *K-topology* on  $\mathbb{R}$ , denoted by  $\mathbb{R}_K$ .

**Lemma 1.22.** *The standard topology on  $\mathbb{R}$  is strictly finer than  $\mathbb{R}_l$ , and also strictly finer than  $\mathbb{R}_K$ . However,  $\mathbb{R}_l$  and  $\mathbb{R}_K$  are not comparable.*

*Proof.* For  $\mathbb{R}_l$ : Let  $x \in \mathbb{R}$  and let  $B = (a, b) \in \mathcal{B}$  such that  $x \in B$ . Then choose  $B' = [x, b) \in \mathcal{B}$  such that  $x \in B' \subseteq B$ . Thus, by the previous lemma, the standard topology on  $\mathbb{R}$  is finer than  $\mathbb{R}_l$ . To see that the inclusion is strict, note that  $[a, b)$  is open in  $\mathbb{R}_l$  but not in the standard topology on  $\mathbb{R}$ .

For  $\mathbb{R}_K$ : Note that inclusion is trivial since every basis element of  $\mathbb{R}_K$  is also a basis element of the standard topology on  $\mathbb{R}$ . To see that the inclusion is strict, note that  $(-1, 1) \setminus K$  is open in  $\mathbb{R}_K$  but not in the standard topology on  $\mathbb{R}$ .

For non-comparability of  $\mathbb{R}_l$  and  $\mathbb{R}_K$ : Note that  $[5, 6)$  is open in  $\mathbb{R}_l$  but not in  $\mathbb{R}_K$  since it cannot be expressed as a union of basis elements of  $\mathbb{R}_K$ . Also, note that  $(-1, 1) \setminus K$  is open in  $\mathbb{R}_K$  but not in  $\mathbb{R}_l$  since it cannot be expressed as a union of basis elements of  $\mathbb{R}_l$ . ■

### 1.3 Closed Sets

**Definition 1.23.** Let  $(X, \tau)$  be a topological space. A subset  $A \subseteq X$  is said to be a *closed set* if its complement  $A^c = X \setminus A$  is an open set.

Using De Morgan's laws, one can easily verify the following properties of closed sets in a topological space.

**Proposition 1.24.** *Let  $(X, \tau)$  be a topological space. Then the following hold true.*

- $\emptyset$  and  $X$  are closed sets.
- For any collection  $\{A_\alpha\}_{\alpha \in \Lambda}$  of closed sets, we have  $\bigcap_{\alpha \in \Lambda} A_\alpha$  is a closed set. That is, an arbitrary intersection of closed sets is closed.
- For closed sets  $A_1, A_2, \dots, A_n$ , we have  $\bigcup_{i=1}^n A_i$  is a closed set. That is, a finite union of closed sets is closed.

**Lemma 1.25.** *Any finite set of a metric space  $(X, d)$  is closed.*

*Proof.* Note that it is enough to show that a singleton set  $\{x\}$  is closed for any  $x \in X$ . Let  $y \in \{x\}^c$ . Then  $y \neq x$ , and by the Hausdorff property, there exists  $r > 0$  such that  $B(x, r) \cap B(y, r) = \emptyset$ . Thus,  $B(y, r) \subseteq \{x\}^c$ , which implies that  $\{x\}^c$  is open. Hence,  $\{x\}$  is closed. ■

Similar to the open sets, we can define closed sets in a subspace as follows.

**Theorem 1.26.** Let  $(X, \tau)$  be a topological space and let  $Y \subseteq X$  be a subspace with the subspace topology  $\tau_Y$ . Then a set  $C \subseteq Y$  is closed in  $(Y, \tau_Y)$  if and only if there exists a closed set  $D$  in  $(X, \tau)$  such that  $C = D \cap Y$ .

*Proof.* Let  $C$  be closed in  $(Y, \tau_Y)$ . Then  $C^c = Y \setminus C$  is open in  $(Y, \tau_Y)$ . Thus, there exists an open set  $U \in \tau$  such that  $C^c = U \cap Y$ . Let  $D = U^c$ , which is closed in  $(X, \tau)$ . Then we have

$$C = Y \setminus C^c = Y \setminus (U \cap Y) = Y \cap U^c = Y \cap D. \quad (1.19)$$

Conversely, let  $D$  be closed in  $(X, \tau)$  and let  $C = D \cap Y$ . Then  $D^c$  is open in  $(X, \tau)$ . Thus, we have

$$C^c = Y \setminus C = Y \setminus (D \cap Y) = Y \cap D^c, \quad (1.20)$$

which is open in  $(Y, \tau_Y)$ . Hence,  $C$  is closed in  $(Y, \tau_Y)$ . ■

Analogous to the interior of a set, we have the notion of the closure of a set.

**Definition 1.27.** Let  $(X, \tau)$  be a topological space and let  $A \subseteq X$ . The *closure* of  $A$ , denoted by  $\bar{A}$ , is defined as the smallest closed set containing  $A$ . That is,

$$\bar{A} := \bigcap \{C \subseteq X \mid C \text{ is closed and } A \subseteq C\}. \quad (1.21)$$

**Theorem 1.28.** The closure of a set  $A$  in a topological space  $(X, \tau)$  is equivalent to saying

$$\bar{A} = \{x \in X \mid \forall U \in \tau \text{ with } x \in U, U \cap A \neq \emptyset\}. \quad (1.22)$$

*Proof.* Let  $x \in \bar{A}$  with  $x \in U \in \tau$ . Suppose for the sake of contradiction that  $U \cap A = \emptyset$ . Then we have  $A \subseteq U^c$ , where  $U^c$  is closed. This must then imply that  $\bar{A} \subseteq U^c$ , which is a contradiction since  $x \in U$ . Thus, we have  $U \cap A \neq \emptyset$ .

For the converse inclusion, let  $x \notin \bar{A}$ . Then there exists a closed set  $C$  such that  $A \subseteq C$  but  $x \notin C$ . Thus,  $x \in C^c$ , where  $C^c$  is open. Since  $A \subseteq C$ , we have  $C^c \cap A = \emptyset$ . Hence, there exists an open set  $C^c$  containing  $x$  such that  $C^c \cap A = \emptyset$ . ■

Instead of using “ $\forall U \in \tau$ ” in the above theorem, one can equivalently use “ $\forall B \in \mathcal{B}$ ” where  $\mathcal{B}$  is a basis for the topology  $\tau$ .

January 12th.

**Lemma 1.29.**  $A$  is closed if and only if  $A = \bar{A}$ . Moreover,  $A \subseteq B$  implies  $\bar{A} \subseteq \bar{B}$ .

*Proof.* The first assertion is left as an exercise. For the second assertion, let  $x \in \bar{A}$ . Then for every open set  $U$  containing  $x$ , we have  $U \cap A \neq \emptyset$ . Since  $A \subseteq B$ , we have  $U \cap B \neq \emptyset$ . Thus,  $x \in \bar{B}$ . ■

**Theorem 1.30.** Let  $Y \subseteq X$  be a subspace of a topological space  $(X, \tau)$ . For any  $A \subseteq Y$ , we have

$$\bar{A}^Y = \bar{A} \cap Y, \quad (1.23)$$

where  $\bar{A}^Y$  is the closure of  $A$  in the subspace  $(Y, \tau_Y)$  and  $\bar{A}$  is the closure of  $A$  in  $(X, \tau)$ .

*Proof.* We have

$$\bar{A}^Y = \{y \in Y \mid y \in U \cap Y, U \in \tau, U \cap Y \cap A \neq \emptyset\} \subseteq \bar{A} \cap Y. \quad (1.24)$$

Now let  $y_0 \in \overline{A} \cap Y$ . Then for every  $U \in \tau$  such that  $y_0 \in U$ , we have  $U \cap A \neq \emptyset$ . Thus, taking intersection over all closed sets containing  $A$  in  $(X, \tau)$  and then intersecting with  $Y$ , we have

$$\bigcap_{\substack{C \subseteq X \\ C \text{ closed} \\ A \subseteq C}} (C \cap Y) = \overline{A} \cap Y. \quad (1.25)$$

Thus,  $y_0 \in \overline{A}^Y$ . Hence, we have  $\overline{A}^Y = \overline{A} \cap Y$ . ■

## 1.4 Convergence and Hausdorff Spaces

**Definition 1.31.** Let  $(X, \tau)$  be a topological space and let  $A \subseteq X$ . A point  $x \in X$  is said to be a *limit point* of  $A$  if for every open set  $U \in \tau$  containing  $x$ , we have  $(U \setminus \{x\}) \cap A \neq \emptyset$ .

Let us denote the set of all limit points of  $A$  by  $A'$ . Then  $A' \subseteq \overline{A}$  since for every open set  $U$  containing a limit point  $x$ , we have  $U \cap A \neq \emptyset$ . Note that nothing can be said about the relation between  $A$  and  $A'$ .

For example, if  $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$  with the standard topology, then  $A' = \{0\}$  and  $\overline{A} = A \cup \{0\}$ . If  $A = (0, 1) \cup \{2\}$ , then  $A' = [0, 1]$  and  $\overline{A} = [0, 1] \cup \{2\}$ . If  $A = \mathbb{Q} \subseteq \mathbb{R}$  with the standard topology, then  $A' = \mathbb{R}$  and  $\overline{A} = \mathbb{R}$ . There is a nice characterization of the closure of a set using limit points.

**Theorem 1.32.** Let  $(X, \tau)$  be a topological space and let  $A \subseteq X$ . Then

$$\overline{A} = A \cup A'. \quad (1.26)$$

*Proof.* One direction is easy since we have already seen that  $A, A' \subseteq \overline{A}$ . For the other direction, let  $x \in \overline{A}$ . If  $x \in A$ , then we are done. So suppose that  $x \notin A$ . Then for every open set  $U$  containing  $x$ , we have  $U \cap A \neq \emptyset$ . Since  $x \notin A$ , we have  $(U \setminus \{x\}) \cap A \neq \emptyset$ . Thus,  $x$  is a limit point of  $A$ , and hence  $x \in A'$ . Therefore, we have  $\overline{A} \subseteq A \cup A'$ , and hence the result follows. ■

**Corollary 1.33.** A set  $A$  is closed if and only if  $A' \subseteq A$ .

We are now ready to define convergence in a (metric) topological space.

**Definition 1.34.** Let  $(X, d)$  be a metric space. A sequence  $(x_n)_{n \geq 1} \subseteq X$  is said to *converge* to a point  $x \in X$  if for every  $\varepsilon > 0$ , there exists a natural  $N(\varepsilon) \equiv N \in \mathbb{N}$  such that

$$d(x_n, x) < \varepsilon \quad \forall n \geq N. \quad (1.27)$$

Equivalently, the sequence  $(d(x_n, x))_{n \geq 1} \subseteq \mathbb{R}$  converges to 0 in the usual sense.

In the topological sense, we say  $(x_n)_{n \geq 1}$  converges to  $x$  if for every open set  $U$  containing  $x$ , there exists a natural  $N(U) \equiv N \in \mathbb{N}$  such that

$$x_n \in U \quad \forall n \geq N. \quad (1.28)$$

Note that the limit of a sequence in a metric space is unique. However, in a general topological space, the limit of a sequence need not be unique. For example, consider the set  $X$  with the indiscrete topology. Then every sequence in  $X$  converges to every point in  $X$ .

**Theorem 1.35.** Let  $(X, d)$  be a metric space and let  $(x_n)_{n \geq 1} \subseteq X$  be a sequence converging to both  $x$  and  $y$  in  $X$ . Then  $x = y$ .

*Proof.* For every  $\varepsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that

$$d(x_n, x) < \varepsilon/2 \quad \forall n \geq N_1. \quad (1.29)$$

Similarly, there exists  $N_2 \in \mathbb{N}$  such that

$$d(x_n, y) < \varepsilon/2 \quad \forall n \geq N_2. \quad (1.30)$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (1.31)$$

Since  $\varepsilon > 0$  is arbitrary, we have  $d(x, y) = 0$ , which implies that  $x = y$ .  $\blacksquare$

The concept of Hausdorff property can be generalized to topological spaces as follows: for every  $x, y \in X$  with  $x \neq y$ , there exist open sets  $U, V \in \tau$  such that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ .

**Theorem 1.36.** *Let  $(X, \tau)$  be a Hausdorff topological space and let  $(x_n)_{n \geq 1} \subseteq X$  be a sequence converging to both  $x$  and  $y$  in  $X$ . Then  $x = y$ .*

*Proof.* The proof is similar to that in the metric space case. For  $x \neq y$ , there exist open sets  $U, V \in \tau$  such that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ . Since  $(x_n)_{n \geq 1}$  converges to  $x$ , there exists  $N_1 \in \mathbb{N}$  such that

$$x_n \in U \quad \forall n \geq N_1. \quad (1.32)$$

Similarly, since  $(x_n)_{n \geq 1}$  converges to  $y$ , there exists  $N_2 \in \mathbb{N}$  such that

$$x_n \in V \quad \forall n \geq N_2. \quad (1.33)$$

Let  $N = \max\{N_1, N_2\}$ . Then for all  $n \geq N$ , we have  $x_n \in U$  and  $x_n \in V$ , which implies that  $x_n \in U \cap V$ . This is a contradiction since  $U \cap V = \emptyset$ . Hence, we must have  $x = y$ .  $\blacksquare$

January 13th.

**Lemma 1.37.** *Let  $(X, \tau)$  be a Hausdorff topological space. Then every finite subset of  $X$  is closed.*

*Proof.* It is enough to show that a singleton set  $\{x\}$  is closed for any  $x \in X$ . Let  $y \in \{x\}^c$ . Then  $y \neq x$ , and by the Hausdorff property, there exist open sets  $U, V \in \tau$  such that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ . Thus,  $V \subseteq \{x\}^c$ , which implies that  $\{x\}^c$  is open. Hence,  $\{x\}$  is closed.  $\blacksquare$

We ask this: is it true that if every finite subset of a topological space  $(X, \tau)$  is closed, then  $(X, \tau)$  is Hausdorff? The answer is no, since a counterexample is provided by the cofinite topology on an infinite set  $X$ . In this topology, every finite subset is closed, but the space is not Hausdorff.

**Definition 1.38.** We say that a topological space  $(X, \tau)$  is *T1* if finite subsets of  $X$  are closed.

**Theorem 1.39.** *Suppose  $(X, \tau)$  is a T1 topological space. Let  $A \subseteq X$ . Then  $x \in A'$  if and only if for every open set  $U \in \tau$  containing  $x$ , we have  $U \cap A$  is an infinite set.*

*Proof.* The reverse implication is obvious by the very definition of  $A'$ . For the forward implication, let  $x \in A'$  and let  $U \in \tau$  be any open set containing  $x$ . Suppose for the sake of contradiction that  $U \cap A$  is a finite set. Since  $(X, \tau)$  is T1, finite subsets of  $X$  are closed. Thus,  $(U \cap A)^c$  is open. Moreover, we have

$$x \in U \cap (U \cap A)^c, \quad (U \cap (U \cap A)^c) \cap A = \emptyset, \quad (1.34)$$

which is a contradiction since  $x \in A'$ . Hence,  $U \cap A$  is an infinite set.  $\blacksquare$

Let us look at some examples of Hausdorff spaces.

**Example 1.40.** If  $X$  is Hausdorff, then the subspace  $Y \subseteq X$  with the subspace topology is also Hausdorff. This is because for any  $y_1, y_2 \in Y$  with  $y_1 \neq y_2$ , there exist open sets  $U, V \in \tau$  such that  $y_1 \in U$ ,  $y_2 \in V$ , and  $U \cap V = \emptyset$ . Thus,  $y_1 \in U \cap Y$ ,  $y_2 \in V \cap Y$ , and  $(U \cap Y) \cap (V \cap Y) = \emptyset$ .

**Example 1.41.** For a space  $(X, \tau)$ , and a subset  $A \subseteq X$ , the *boundary* of  $A$  is defined as  $\text{Bd}(A) = \overline{A} \cap \overline{X \setminus A}$ . Then  $\text{Bd}(A)$  and  $\text{Int}(A)$  are disjoint; indeed, let  $x \in \text{Bd}(A) \cap \text{Int}(A)$ . Then there exists an open set  $U \in \tau$  such that  $x \in U \subseteq A$ . Since  $x \in \text{Bd}(A)$ , we have  $x \in \overline{X \setminus A}$ . Thus, for every open set  $V \in \tau$  containing  $x$ , we have  $V \cap (X \setminus A) \neq \emptyset$ . In particular, taking  $V = U$ , we have  $U \cap (X \setminus A) \neq \emptyset$ , which is a contradiction since  $U \subseteq A$ . Hence, we must have  $\text{Bd}(A) \cap \text{Int}(A) = \emptyset$ .

Moreover, we have  $\overline{A} = \text{Int}(A) \cup \text{Bd}(A)$ . The reverse inclusion is trivial since  $\text{Bd}(A) \subseteq \overline{A}$  and  $\text{Int}(A) \subseteq A \subseteq \overline{A}$ . For the forward inclusion, let  $x \in \overline{A}$ . If  $x \in \text{Bd}(A)$ , then we are done. So suppose that  $x \notin \text{Bd}(A)$ . Then  $x \notin \overline{X \setminus A}$ , which implies that there exists an open set  $U \in \tau$  such that  $x \in U$  and  $U \cap (X \setminus A) = \emptyset$ . Thus, we have  $U \subseteq A$ , which implies that  $x \in \text{Int}(A)$ . Therefore, we have  $\overline{A} = \text{Int}(A) \cup \text{Bd}(A)$ .

Finally, if  $\text{Bd}(A) = \emptyset$ , then  $A$  is both open and closed. Indeed, since  $\overline{A} = \text{Int}(A) \cup \text{Bd}(A)$ , we have  $\overline{A} = \text{Int}(A)$ . Thus,  $A$  is closed. Also, since  $\text{Bd}(A) = \emptyset$ , we have  $\overline{X \setminus A} = \text{Int}(X \setminus A)$ . Thus,  $X \setminus A$  is closed, which implies that  $A$  is open.

## 1.5 Product Spaces

For two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , we can define a metric on the product set  $X \times Y$  as follows:

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}. \quad (1.35)$$

A basis of  $X \times Y$  with the product metric is given by

$$\mathcal{B} = \{B_X(x, r) \times B_Y(y, r) \mid x \in X, y \in Y, r > 0\}. \quad (1.36)$$

In particular, if  $(X, \tau_X)$  and  $(Y, \tau_Y)$  are two topological spaces, then we can define a topology on  $X \times Y$ , called the *product topology*, by taking the basis

$$\mathcal{B} = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}. \quad (1.37)$$

Then, the topology  $\tau_{X \times Y}$  generated by the basis  $\mathcal{B}$  is called the product topology on  $X \times Y$ . Verify that  $\tau_{X \times Y}$  is indeed a topology on  $X \times Y$ . Note that if  $\tau_{X \times Y}$  was taken to be  $\{U \times V \mid U \in \tau_X, V \in \tau_Y\}$ , then it would not be a topology since it is not closed under arbitrary unions. Also worth noting is that if  $X$  and  $Y$  are both Hausdorff, then so is  $X \times Y$  with the product topology.

**Example 1.42.**  $X$  is Hausdorff if and only if the diagonal set  $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$  is closed in  $X \times X$  with the product topology; suppose  $X$  is Hausdorff. Let  $(x, y) \in (X \times X) \setminus \Delta$ . Then  $x \neq y$ , and by the Hausdorff property, there exist open sets  $U, V \in \tau$  such that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ . Thus, we have

$$(U \times V) \cap \Delta = \emptyset, \quad (x, y) \in U \times V \subseteq (X \times X) \setminus \Delta, \quad (1.38)$$

which implies that  $(X \times X) \setminus \Delta$  is open. Hence,  $\Delta$  is closed. Conversely, suppose  $\Delta$  is closed. Let  $x, y \in X$  with  $x \neq y$ . Then  $(x, y) \in (X \times X) \setminus \Delta$ , which is open. Thus, there exists a basis element  $U \times V$  such that

$$(x, y) \in U \times V \subseteq (X \times X) \setminus \Delta. \quad (1.39)$$

This implies that  $x \in U$ ,  $y \in V$ , and  $U \cap V = \emptyset$ . Hence,  $X$  is Hausdorff.

# Chapter 2

# MAPPINGS

## 2.1 Continuous Functions

Recall the definition of a continuous function on  $\mathbb{R}$ ; a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  was said to be continuous at a point  $x \in \mathbb{R}$  if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta. \quad (2.1)$$

We can abstract this definition to metric spaces as follows.

**Definition 2.1.** A function  $f : (X, d_X) \rightarrow (Y, d_Y)$  between two metric spaces is said to be *continuous* at a point  $x \in X$  if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever } d_X(x, y) < \delta. \quad (2.2)$$

It is continuous at every point of  $X$  if it is continuous at each  $x \in X$ .

Also recall that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  was continuous at  $x \in \mathbb{R}$  if and only if for every open set  $V \subseteq \mathbb{R}$  containing  $f(x)$ , the preimage  $f^{-1}(V)$  is an open set in  $\mathbb{R}$  containing  $x$ . We can use this characterization to define continuity in topological spaces.

**Definition 2.2.** A function  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  between two topological spaces is said to be *continuous* at a point  $x \in X$  if for every open set  $V \in \tau_Y$  containing  $f(x)$ , the preimage  $f^{-1}(V)$  is an open set in  $\tau_X$  containing  $x$ . That is,

$$\forall V \in \tau_Y, f(x) \in V \implies x \in f^{-1}(V) \in \tau_X. \quad (2.3)$$

It is continuous at every point of  $X$  if it is continuous at each  $x \in X$ . That is,

$$\forall V \in \tau_Y \implies f^{-1}(V) \in \tau_X. \quad (2.4)$$

The above statement can be made more tight via bases:  $f$  is continuous if and only if the preimage of every basis element of  $Y$  is an open set in  $X$ .

**Lemma 2.3.** A function  $f : (X, d_X) \rightarrow (Y, d_Y)$  between two metric spaces is continuous if and only if  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is continuous, where  $\tau_X$  and  $\tau_Y$  are the topologies induced by the metrics  $d_X$  and  $d_Y$  respectively.

*Proof.* Suppose  $f : (X, d_X) \rightarrow (Y, d_Y)$  is continuous. Let  $V \in \tau_Y$  be an open set in  $Y$ . Then, for every  $y \in V$ , there exists an  $\varepsilon_y > 0$  such that  $B_Y(y, \varepsilon_y) \subseteq V$ . Since  $f$  is continuous, for every  $x \in f^{-1}(V)$ , there exists a  $\delta_x > 0$  such that

$$f(B_X(x, \delta_x)) \subseteq B_Y(f(x), \varepsilon_{f(x)}) \subseteq V. \quad (2.5)$$

Thus,  $B_X(x, \delta_x) \subseteq f^{-1}(V)$ . Hence,  $f^{-1}(V)$  is open in  $X$  and so  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is continuous.

For the converse, suppose  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is continuous. That is, for every open set  $V \in \tau_Y$ , the preimage  $f^{-1}(V)$  is an open set in  $\tau_X$ . Let  $\varepsilon > 0$  be given. Consider the open ball  $B_Y(f(x), \varepsilon) \in \tau_Y$ . Since  $f$  is continuous,  $f^{-1}(B_Y(f(x), \varepsilon))$  is open in  $X$  and contains  $x$ . Thus, there exists a  $\delta > 0$  such that  $B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon))$ . Hence, for every  $y \in B_X(x, \delta)$ , we have

$$f(y) \in B_Y(f(x), \varepsilon), \quad (2.6)$$

proving that  $f : (X, d_X) \rightarrow (Y, d_Y)$  is continuous.  $\blacksquare$

**Lemma 2.4.** *Let  $(X, \tau)$  be a topological space and let  $A \subseteq X$ . Then the following hold.*

- (i) *For every sequence  $(a_n)_{n \in \mathbb{N}} \subseteq A$ , if  $a_n \rightarrow x$  in  $X$ , then  $x \in \overline{A}$ . That is, if  $x \in U$  for some open set  $U \in \tau$ , there exists a  $N(U) \equiv N \in \mathbb{N}$  such that  $a_n \in A \cap U$  for all  $n \geq N$ .*
- (ii) *The converse holds if  $X$  is a metric space.*
- (iii) *In general, the converse need not hold.*

*Proof.* We prove the second statement only. Suppose  $x \in \overline{A}$ . Then  $B(x, \frac{1}{n}) \cap A \neq \emptyset$  for every  $n \in \mathbb{N}$ . Thus, we can choose  $a_n \in B(x, \frac{1}{n}) \cap A$  for each  $n \in \mathbb{N}$ . Then, for every  $\varepsilon > 0$ , there exists a  $N \equiv N(\varepsilon) \in \mathbb{N}$  such that  $\frac{1}{N} < \varepsilon$ . Thus, for every  $n \geq N$ , we have

$$d(a_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon, \quad (2.7)$$

proving that  $a_n \rightarrow x$  in  $X$ .  $\blacksquare$

There is a third definition for metric spaces. A function  $f : (X, d_X) \rightarrow (Y, d_Y)$  between two metric spaces is said to be *continuous* at a point  $x \in X$  if for every sequence  $(x_n)_{n \in \mathbb{N}} \subseteq X$  such that  $x_n \rightarrow x$  in  $X$ , we have  $f(x_n) \rightarrow f(x)$  in  $Y$ . In fact, the open-set definition implies the sequential definition.

*Proof.* Assume the open-set definition of continuity. Let  $(x_n)_{n \in \mathbb{N}} \subseteq X$  be a sequence such that  $x_n \rightarrow x$  in  $X$ . Let  $\varepsilon > 0$  be given. Consider the open ball  $B_Y(f(x), \varepsilon) \in \tau_Y$ . Since  $f$  is continuous,  $f^{-1}(B_Y(f(x), \varepsilon))$  is open in  $X$  and contains  $x$ . Thus, there exists a  $\delta > 0$  such that  $B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon))$ . Since  $x_n \rightarrow x$  in  $X$ , there exists a  $N \equiv N(\delta) \in \mathbb{N}$  such that for every  $n \geq N$ , we have

$$x_n \in B_X(x, \delta) \implies f(x_n) \in B_Y(f(x), \varepsilon), \quad (2.8)$$

proving that  $f(x_n) \rightarrow f(x)$  in  $Y$ .  $\blacksquare$

For metric spaces, the sequential definition also implies the open-set definition.

**Example 2.5.** Recall  $\mathbb{R}_l$ , which was generated from the basis  $\mathcal{B} = \{[a, b) : a < b, a, b \in \mathbb{R}\}$ . Consider the identity function  $f : \mathbb{R} \rightarrow \mathbb{R}_l$ , with  $x \mapsto x$ . Then  $[0, 1)$  is open in  $\mathbb{R}_l$ , but its preimage under  $f$  is  $[0, 1)$ , which is not open in  $\mathbb{R}$ .

**Theorem 2.6.** *Let  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  be a function between two topological spaces. Then the following are equivalent.*

- (i)  *$f$  is (open-set) continuous.*
- (ii) *For every closed  $C \subseteq Y$ , the preimage  $f^{-1}(C)$  is closed in  $X$ .*
- (iii) *For every  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$ .*
- (iv) *For all  $f(x) \in V$  with  $V \in \tau_Y$ , there exists  $U \in \tau_X$  such that  $x \in U$  and  $f(U) \subseteq V$ .*

January 20th.

*Proof.* We first show that (i) and (ii) are equivalent. Suppose  $C$  is closed in  $Y$ . Then,  $Y \setminus C$  is open in  $Y$ . Since  $f$  is continuous, the preimage  $f^{-1}(Y \setminus C)$  is open in  $X$ . But,

$$f^{-1}(Y \setminus C) = X \setminus f^{-1}(C). \quad (2.9)$$

Thus,  $f^{-1}(C)$  is closed in  $X$ . One can follow the converse argument to show that (ii) implies (i).

For (i) implies (iii), let  $A \subseteq X$  and let  $x \in \overline{A}$ . Let  $V$  be an open neighbourhood of  $f(x)$  in  $Y$ . Since  $f$  is continuous,  $f^{-1}(V)$  is an open neighbourhood of  $x$  in  $X$ . Thus,  $f^{-1}(V) \cap A \neq \emptyset$ . Let  $a \in f^{-1}(V) \cap A$ . Then,  $f(a) \in V$  and  $f(a) \in f(A)$ . Since  $V$  was arbitrary, we must have  $V \cap f(A) \neq \emptyset$ . Thus,  $f(x) \in \overline{f(A)}$ , proving that  $f(\overline{A}) \subseteq \overline{f(A)}$ . For (iii) implies (ii), let  $V$  be a closed set in  $Y$ . Then

$$f^{-1}(V) \subseteq X \implies f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} = V \quad (2.10)$$

which tells us  $\overline{f^{-1}(V)} \subseteq f^{-1}(V)$ , proving that  $f^{-1}(V)$  is closed in  $X$ .

For (i) implies (iv), let  $f(x) \in V$  with  $V \in \tau_Y$ . Since  $f$  is continuous,  $f^{-1}(V)$  is open in  $X$  and contains  $x$ . Thus, we can take  $U = f^{-1}(V)$ , proving (iv). For (iv) implies (i), let  $V \in \tau_Y$ . For every  $x \in f^{-1}(V)$ , we have  $f(x) \in V$ . By (iv), there exists an open set  $U_x \in \tau_X$  such that  $x \in U_x$  and  $f(U_x) \subseteq V$ . Thus, taking the union over all  $x \in f^{-1}(V)$  gives  $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$ , proving that  $f^{-1}(V)$  is open in  $X$ . ■

### 2.1.1 Rules of Continuous Functions

Note that the constant function  $f : X \rightarrow Y$  defined by  $f(x) = y_0$  for some fixed  $y_0 \in Y$  is continuous. Also, the identity function  $\text{id}_X : X \rightarrow X$  defined by  $\text{id}_X(x) = x$  is continuous.

Let  $X$  be a topological space, and  $A$  be a subset of  $X$  with the subspace topology. Then the inclusion map  $i : A \hookrightarrow X$  defined by  $i(a) = a$  is continuous. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous functions between topological spaces, then the composition  $g \circ f : X \rightarrow Z$  defined by  $(g \circ f)(x) = g(f(x))$  is also continuous.

**Theorem 2.7.** *Let  $f_1 : Z \rightarrow X$  and  $f_2 : Z \rightarrow Y$  be functions between topological spaces. Then the function  $f : Z \rightarrow X \times Y$  defined by  $f(z) = (f_1(z), f_2(z))$  is continuous if and only if both  $f_1$  and  $f_2$  are continuous.*

It is important to note that a function  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(B)$  is open in  $X$  for every basis element  $B$  of  $Y$ .

*Proof.* For the forward implication, it is enough to show that  $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$  is open in  $Z$ , where  $U$  and  $V$  are open sets in  $X$  and  $Y$  respectively.  $f$  is continuous implies that  $f^{-1}(U \times V)$  is open in  $Z$ . In particular, both  $f_1^{-1}(U)$  and  $f_2^{-1}(V)$  are open in  $Z$ , proving that both  $f_1$  and  $f_2$  are continuous.

Conversely, suppose both  $f_1$  and  $f_2$  are continuous. Let  $U \times V$  be a basis element of  $X \times Y$ , where  $U$  and  $V$  are open sets in  $X$  and  $Y$  respectively. By continuity of  $f_1$  and  $f_2$ , both  $f_1^{-1}(U)$  and  $f_2^{-1}(V)$  are open in  $Z$ . Thus  $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$  is open in  $Z$ , proving that  $f$  is continuous. ■

**Lemma 2.8.** *Let  $(X, d)$  be a metric space, and let  $f : X \rightarrow \mathbb{R}$  be a continuous function. Suppose  $f(x) \neq 0$  for some  $x \in X$ . Then there exists  $r > 0$  such that  $f(y) \neq 0$  for all  $y \in B(x, r)$ . Moreover, the map  $g : B(x, r) \rightarrow \mathbb{R}$  defined by  $g(y) = \frac{1}{f(y)}$  is continuous.*

*Proof.* Suppose such an  $r$  does not exist. Then for every natural  $n \in \mathbb{N}$ , there exists a point  $x_n \in B(x, \frac{1}{n})$  such that  $f(x_n) = 0$ . Thus, the sequence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$  in  $X$ . Since  $f$  is continuous, the sequence  $(f(x_n))_{n \in \mathbb{N}}$  converges to  $f(x)$  in  $\mathbb{R}$ . But  $f(x_n) = 0$  for all  $n \in \mathbb{N}$ , so  $(f(x_n))_{n \in \mathbb{N}}$  is the constant sequence 0, which converges to 0. Thus, we must have  $f(x) = 0$ , a contradiction. Hence, there exists  $r > 0$  such that  $f(y) \neq 0$  for all  $y \in B(x, r)$ .

To show continuity of  $g$ , let  $(y_n)_{n \geq 1} \subseteq B(x, r)$  be a sequence such that  $y_n \rightarrow y$  in  $B(x, r)$ . Since  $f$  is continuous, we have  $f(y_n) \rightarrow f(y)$  in  $\mathbb{R}$ . Since  $f(y) \neq 0$ , there exists a natural  $N \in \mathbb{N}$  such that for every  $n \geq N$ , we have  $f(y_n) \neq 0$ . Thus, for every  $n \geq N$ , we have  $g(y_n) = \frac{1}{f(y_n)}$ . Since the function  $h : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  defined by  $h(t) = \frac{1}{t}$  is continuous, we have  $g(y_n) = h(f(y_n)) \rightarrow h(f(y)) = g(y)$  in  $\mathbb{R}$ , proving that  $g$  is continuous. ■

For a metric space  $(X, d)$ , we define  $C(X, \mathbb{R})$  to be the set of all continuous functions from  $X$  to  $\mathbb{R}$ . It can be easily seen that  $C(X, \mathbb{R})$  is a vector space over  $\mathbb{R}$  under pointwise addition and scalar multiplication. Moreover, if we define multiplication of functions pointwise, then  $C(X, \mathbb{R})$  is an algebra over  $\mathbb{R}$ . An example is any polynomial map  $p : \mathbb{R}^n \rightarrow \mathbb{R}$ ; in such a case,  $p \in C(\mathbb{R}^n, \mathbb{R})$ .

If we denote  $M_2(\mathbb{R})$  to be the set of all  $2 \times 2$  real matrices, then matrices can be viewed as points in  $\mathbb{R}^4$ , and the same Euclidean topology can be induced on  $M_2(\mathbb{R})$ . Thus, we can consider continuous functions from  $M_2(\mathbb{R})$  to  $\mathbb{R}$ . An example is the determinant function  $\det : M_2(\mathbb{R}) \rightarrow \mathbb{R}$  defined by  $(a, b, c, d) \mapsto ad - bc$ , which is a polynomial map and hence continuous. Since this is a continuous map, and  $\mathbb{R} \setminus \{0\}$  is open in  $\mathbb{R}$ , the preimage  $\det^{-1}(\mathbb{R} \setminus \{0\})$  is open in  $M_2(\mathbb{R})$ . But  $\det^{-1}(\mathbb{R} \setminus \{0\})$  is precisely the set of all invertible  $2 \times 2$  real matrices  $GL_2(\mathbb{R})$ . Thus, we have shown that  $GL_2(\mathbb{R})$  is an open subset of  $M_2(\mathbb{R})$ . The set  $SL_2(\mathbb{R})$  of all  $2 \times 2$  real matrices with determinant 1, however, is closed in  $M_2(\mathbb{R})$ , since it is the preimage of the closed set  $\{1\}$  under the continuous determinant function.

**Example 2.9.** Let us classify all  $f : \mathbb{R} \rightarrow \mathbb{R}$  that are continuous and are also group homomorphisms. Since  $f$  is continuous, we only need to determine  $f : \mathbb{R} \rightarrow \mathbb{R}$  are of the form  $f(x) = \lambda x$  for some fixed  $\lambda \in \mathbb{R}$ . Let  $\lambda := f(1)$ . Then, for every  $m \in \mathbb{Z}$ ,  $f(m) = mf(1) = m\lambda$ . For every  $n \in \mathbb{N}$ , we have  $nf\left(\frac{m}{n}\right) = f(n \cdot \frac{m}{n}) = f(m) = m\lambda$ , so  $f\left(\frac{m}{n}\right) = \frac{m}{n}\lambda$ . Thus, for every rational number  $q$ , we have  $f(q) = q\lambda$ . Since the rationals are dense in  $\mathbb{R}$  and  $f$  is continuous, we must have  $f(x) = x\lambda$  for all real numbers  $x$ . Thus, all continuous group homomorphisms from  $\mathbb{R}$  to  $\mathbb{R}$  are of the form  $f(x) = x\lambda$ .

## 2.2 Product Topology

January 22nd.

The concept of the product topology can be extended to arbitrary products. Let  $\{(X_n, \tau_n) \mid n \geq 1\}$  be a collection of topological spaces. Consider the Cartesian product

$$\prod_{n=1}^{\infty} X_n = \{(x_1, x_2, \dots) \mid x_n \in X_n \text{ for all } n \geq 1\}. \quad (2.11)$$

The product topology on  $\prod_{n=1}^{\infty} X_n$  is the topology generated by the basis

$$\mathcal{B} = \{U_1 \times U_2 \times \dots \mid U_n \in \tau_n \text{ for all } n \geq 1, U_i \neq \emptyset \text{ for finitely many } i\}. \quad (2.12)$$

We require that  $U_i \neq \emptyset$  for only finitely many  $i$  to ensure that this is still a basis. This is known as the box topology. If this restriction were not in place, then the fact that countable intersections of open sets may not be open would imply that this is not a topology. A second option is to use the basis

$$\mathcal{B}' = \{U_1 \times U_2 \times \dots \mid U_n \in \tau_n \text{ for all } n \geq 1, U_i = X_i \text{ for all but finitely many } i\}. \quad (2.13)$$

This is known as the product topology. Note that for finite products, both definitions coincide. In the infinite case, the product topology is more interesting than the box topology. The product topology also extends to a collection of topological spaces indexed by an arbitrary set, such as  $\{(X_\alpha, \tau_\alpha) \mid \alpha \in \Lambda\}$  for some set  $\Lambda$ .

**Lemma 2.10.** If  $\{(X_n, \tau_n) \mid n \geq 1\}$  is a collection of Hausdorff topological spaces, then the product space  $\prod_{n=1}^{\infty} X_n$  with the product topology is also Hausdorff.

*Proof.* Let  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$  be two distinct points in  $\prod_{n=1}^{\infty} X_n$ . Then, there exists some  $k \geq 1$  such that  $x_k \neq y_k$ . Since  $X_k$  is Hausdorff, there exist disjoint open sets  $U_k, V_k \in \tau_k$  such that  $x_k \in U_k$  and  $y_k \in V_k$ . Now, consider the open sets

$$U = X_1 \times X_2 \times \dots \times U_k \times X_{k+1} \times \dots, \quad (2.14)$$

and

$$V = X_1 \times X_2 \times \dots \times V_k \times X_{k+1} \times \dots. \quad (2.15)$$

Then,  $U$  and  $V$  are open in the product topology, and they are disjoint since  $U_k$  and  $V_k$  are disjoint. Moreover,  $x \in U$  and  $y \in V$ , proving that  $\prod_{n=1}^{\infty} X_n$  is Hausdorff. ■

**Lemma 2.11.** *The function  $f : Z \rightarrow \prod_{n=1}^{\infty} X_n$  defined by  $f(z) = (f_1(z), f_2(z), \dots)$  is continuous if and only if each component function  $f_n : Z \rightarrow X_n$  is continuous for all  $n \geq 1$ . Here,  $\prod_{n=1}^{\infty} X_n$  is equipped with the product topology.*

*Proof.* For the forward implication, let  $n \geq 1$  be given. Consider an open set  $U_n \in \tau_n$ . Then, the set

$$U = X_1 \times X_2 \times \dots \times U_n \times X_{n+1} \times \dots \quad (2.16)$$

is open in the product topology. Since  $f$  is continuous, the preimage  $f^{-1}(U)$  is open in  $Z$ . But,

$$f^{-1}(U) = f_n^{-1}(U_n), \quad (2.17)$$

proving that  $f_n$  is continuous.

Conversely, suppose each component function  $f_n$  is continuous for all  $n \geq 1$ . Let  $U = U_1 \times U_2 \times \dots$  be a basis element of the product topology, where  $U_n \in \tau_n$  for all  $n \geq 1$  and  $U_i = X_i$  for all but finitely many  $i$ . Then,

$$f^{-1}(U) = \bigcap_{n=1}^{\infty} f_n^{-1}(U_n) = \bigcap_{i=1}^k f_i^{-1}(U_i), \quad (2.18)$$

where  $k$  is such that  $U_i = X_i$  for all  $i > k$ . Since each  $f_i$  is continuous, each  $f_i^{-1}(U_i)$  is open in  $Z$ . Thus,  $f^{-1}(U)$  is open in  $Z$ , proving that  $f$  is continuous. ■

Note that the above lemma does not hold if the product topology is replaced by the box topology. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}^{\mathbb{N}}$  defined by  $f(x) = (x, x, x, \dots)$ . Each component function  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f_n(x) = x$  is continuous. However,  $f$  is not continuous when  $\mathbb{R}^{\mathbb{N}}$  is equipped with the box topology. To see this, consider the open set  $(-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$  in the box topology. The preimage under  $f$  is  $\{0\}$ , which is not open in  $\mathbb{R}$ . Thus,  $f$  is not continuous. However, the continuity of  $f$  implies the continuity of each component function  $f_n$ , so the converse implication still holds for the box topology.

*January 23rd.*

Consider  $(\mathbb{R}, \bar{d})$  where  $\bar{d}(x, y) = \min\{|x - y|, 1\}$ . This metric generates the usual topology on  $\mathbb{R}$ . Then, on  $\mathbb{R}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{R}$ , we can consider the metric

$$\rho(x, y) = \sup_{n \geq 1} \bar{d}(x_n, y_n), \quad (2.19)$$

where  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$ . One can verify this is a metric. The topology generated by the metric space  $(\mathbb{R}^{\mathbb{N}}, \rho)$  is termed the *uniform topology* on  $\mathbb{R}^{\mathbb{N}}$ . We now wish to determine the relationship between the uniform topology, product topology, and box topology on  $\mathbb{R}^{\mathbb{N}}$ . To determine this, we consider a few results.

**Lemma 2.12.**  $B_{\rho}(x, \varepsilon) \subsetneq \prod_{n=1}^{\infty} (x_i - \varepsilon, x_i + \varepsilon)$  for every  $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$  and  $(1 >) \varepsilon > 0$ .

*Proof.* Let  $y = (y_1, y_2, \dots) \in B_{\rho}(x, \varepsilon)$ . Then, by definition of  $\rho$ , we have  $\bar{d}(x_n, y_n) < \varepsilon$  for all  $n \geq 1$ . Since  $\varepsilon < 1$ , this implies that  $|x_n - y_n| < \varepsilon$  for all  $n \geq 1$ . Thus,  $y \in \prod_{n=1}^{\infty} (x_i - \varepsilon, x_i + \varepsilon)$ , proving that  $B_{\rho}(x, \varepsilon) \subseteq \prod_{n=1}^{\infty} (x_i - \varepsilon, x_i + \varepsilon)$ .

To see that the inclusion is strict, consider the point  $z = (z_1, z_2, \dots)$  defined by  $z_i = x_i - \varepsilon + \frac{1}{n+i}$  for all  $i \geq 1$ , for  $n >> 1$  large enough. Then,  $z \in \prod_{n=1}^{\infty} (x_i - \varepsilon, x_i + \varepsilon)$ , but  $\sup_{i \geq 1} \bar{d}(x_i, z_i) = \varepsilon$ , so  $z \notin B_{\rho}(x, \varepsilon)$ . Thus, the inclusion is strict. ■

**Lemma 2.13.** *On  $\mathbb{R}^{\mathbb{N}}$ , the product topology is strictly coarser than the uniform topology, which is strictly coarser than the box topology.*

*Proof.* We first show that  $(x - \varepsilon, x + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$  is open in the uniform topology. Let  $y = (y_1, y_2, \dots) \in (x - \varepsilon, x + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$ . Then, there exists a  $\delta > 0$  such that  $(y_1 - \delta, y_1 + \delta) \subseteq (x - \varepsilon, x + \varepsilon)$ . Thus,  $B(y, \delta) \subseteq (x - \varepsilon, x + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$ , proving that this set is open in the uniform topology. Since basis elements of the product topology are finite intersections of such sets, they are also open in the uniform topology. Thus, the product topology is coarser than the uniform topology. ■

January 27th.

**Theorem 2.14.** On  $\mathbb{R}^{\mathbb{N}}$ , with  $D(x, y) := \sup_{n \geq 1} \frac{\bar{d}(x_n, y_n)}{n}$  where  $\bar{d}(x, y) = \min\{|x - y|, 1\}$ , the topology induced by  $D$  is the product topology.

*Proof.* For the forward implication, we wish to show that given an open ball  $B_D(x, \varepsilon)$ , and  $y \in B_D(x, \varepsilon)$ , that there exists a set  $U$  open in the product topology such that  $y \in U \subseteq B_D(x, \varepsilon)$ . Choose  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \frac{\varepsilon}{4}$ . Now we claim that

$$(x_1 - \frac{\varepsilon}{3}, x_1 + \frac{\varepsilon}{3}) \times (x_2 - \frac{\varepsilon}{3}, x_2 + \frac{\varepsilon}{3}) \times \cdots \times (x_{N+1} - \frac{\varepsilon}{3}, x_{N+1} + \frac{\varepsilon}{3}) \times \mathbb{R} \times \mathbb{R} \times \cdots \subseteq B_D(x, \varepsilon). \quad (2.20)$$

To this end, let  $y = (y_1, y_2, \dots)$  be in the left-hand side. Then for every  $1 \leq i \leq N-1$ , we have  $|y_i - x_i| < \frac{2\varepsilon}{3}$ , so  $\bar{d}(x_i, y_i)/i \leq \bar{d}(x_i, y_i) < \frac{2\varepsilon}{3} < \varepsilon$ . Moreover, for all  $i \geq N$ ,  $\bar{d}(x_i, y_i)/i \leq 1/i \leq 1/N < \frac{\varepsilon}{4} < \varepsilon$ . Thus,  $\sup_{i \geq 1} \bar{d}(x_i, y_i)/i = D(x, y) \leq \frac{2\varepsilon}{3} < \varepsilon$ , proving the claim. Hence, we have found an open set in the product topology containing  $x$  and contained in  $U$ .

For the converse, let  $U = \prod_{i \in \mathbb{N}} U_i$  be open in the product topology, where  $U_i$  is open in  $\mathbb{R}$  for all  $i \geq 1$  and  $U_i = \mathbb{R}$  for all but finitely many  $i$ . It is enough to show containment for these sets, since they form a basis for the product topology. Thus, we wish to show that given any  $x \in U$ , there exists an  $\varepsilon > 0$  such that  $B_D(x, \varepsilon) \subseteq U$ . Let  $\alpha$  denote an arbitrary index such that  $U_\alpha \neq \mathbb{R}$  (there are only finitely many such indices). Let  $x \in U$ . Then  $x_\alpha \in U_\alpha$ , so there exists  $\varepsilon_\alpha > 0$  such that  $(x_\alpha - \varepsilon_\alpha, x_\alpha + \varepsilon_\alpha) \subseteq U_\alpha$ . Now, let  $\varepsilon = \min_\alpha \frac{\varepsilon_\alpha}{\alpha}$ , where the minimum is taken over all indices  $\alpha$  such that  $U_\alpha \neq \mathbb{R}$ . We claim that this  $\varepsilon$  works. To see this, let  $y = (y_1, y_2, \dots) \in B_D(x, \varepsilon)$ . Then, for every index  $\alpha$  such that  $U_\alpha \neq \mathbb{R}$ , we have

$$|y_\alpha - x_\alpha| \leq \bar{d}(x_\alpha, y_\alpha) < \alpha\varepsilon \leq \varepsilon_\alpha, \quad (2.21)$$

so  $y_\alpha \in (x_\alpha - \varepsilon_\alpha, x_\alpha + \varepsilon_\alpha) \subseteq U_\alpha$ . For all other indices  $\beta$  such that  $U_\beta = \mathbb{R}$ , we have  $y_\beta \in U_\beta$  trivially. Thus,  $y \in U$ , proving that  $B_D(x, \varepsilon) \subseteq U$ . ■

The box topology, however, is not metrizable.

**Lemma 2.15.** The box topology on  $\mathbb{R}^{\mathbb{N}}$  is not metrizable.

To show this, recall that in a metric space if  $A$  is a subset of a metric space  $(X, d)$ , and  $(a_n)_{n \geq 1}$  is a sequence in  $A$  that converges to some  $x \in X$ , then  $x \in \overline{A}$ . The converse also holds in metric spaces.

*Proof.* Let  $A = \{(x_1, x_2, \dots) \mid x_i > 0 \text{ for all } i \geq 1\} \subseteq \mathbb{R}^{\mathbb{N}}$  with the box topology. Note that  $A$  is open in the box topology. Now consider the point  $x = (0, 0, 0, \dots) \in \mathbb{R}^{\mathbb{N}}$ . We claim that  $x \in \overline{A}$ . To see this, let  $U = U_1 \times U_2 \times \dots$  be an open neighbourhood of  $x$  in the box topology. Then, for each  $i \geq 1$ , there exists  $\varepsilon_i > 0$  such that  $(-\varepsilon_i, \varepsilon_i) \subseteq U_i$ . Now, consider the point  $y = (y_1, y_2, \dots)$  defined by  $y_i = \frac{\varepsilon_i}{2} > 0$  for all  $i \geq 1$ . Then,  $y \in A \cap U$ , proving that every open neighbourhood of  $x$  intersects  $A$ . Thus,  $x \in \overline{A}$ . However, there does not exist a sequence  $(a_n)_{n \geq 1} \subseteq A$  such that  $a_n \rightarrow x$  in the box topology. To see this, suppose such a sequence exists. Then, for each  $i \geq 1$ , consider the open neighbourhood  $U^{(i)} = U_1^{(i)} \times U_2^{(i)} \times \dots$  of  $x$  defined by

$$U_j^{(i)} = \begin{cases} (-\frac{1}{i}, \frac{1}{i}) & j = i, \\ \mathbb{R} & j \neq i. \end{cases} \quad (2.22)$$

Since  $a_n \rightarrow x$  in the box topology, there exists a natural  $N_i \in \mathbb{N}$  such that for every  $n \geq N_i$ , we have  $a_n \in U^{(i)}$ . In particular, this implies that for every  $n \geq N_i$ , the  $i$ -th coordinate of  $a_n$  satisfies  $|(a_n)_i| < \frac{1}{i}$ . Now, consider the sequence of natural numbers  $(N_i)_{i \geq 1}$ . Since this is an increasing sequence, we have  $N_i \leq N_j$  for all  $j \geq i$ . Thus, for every  $j \geq i$ , we have  $|(a_{N_j})_i| < \frac{1}{i}$ . In particular, this implies that  $(a_{N_j})_i \geq 0$  for all  $j \geq i$ , since  $a_{N_j} \in A$ . Thus, for each fixed  $i \geq 1$ , the sequence  $((a_{N_j})_i)_{j \geq i}$  is a sequence of non-negative real numbers that converges to 0. However, this does not guarantee that the entire sequence  $(a_n)_{n \geq 1}$  converges to  $x$  in the box topology, since the convergence must hold uniformly across all coordinates. This contradicts our assumption that such a sequence exists. Hence, there does not exist a sequence  $(a_n)_{n \geq 1} \subseteq A$  such that  $a_n \rightarrow x$  in the box topology, proving that the box topology is not metrizable. ■

## 2.3 Connectedness

The *distance from a set A* of a point  $x$  is defined as

$$d_A(x) := \inf\{d(x, a) \mid a \in A\}. \quad (2.23)$$

Note that  $d_A(x) = 0$  if and only if  $x \in \overline{A}$ . Moreover, the map  $\varphi : X \rightarrow \mathbb{R}$  defined by  $\varphi(x) = d_A(x)$  is continuous.

**Lemma 2.16.** Let  $(X, d)$  be a metric space, and  $A, B \subseteq X$  be disjoint closed sets. Then there exists a continuous map  $f : X \rightarrow [0, 1]$  such that  $f(a) = 0$  for all  $a \in A$  and  $f(b) = 1$  for all  $b \in B$ .

**Definition 2.17.** For a topological space  $(X, \tau)$ , a *separation* of  $X$  is a pair of disjoint non-empty open sets  $U, V \in \tau$  such that  $X = U \cup V$ . If no such separation exists, then  $X$  is said to be *connected*.

Note that  $X$  is connected if and only if the only non-empty clopen subset of  $X$  is  $X$  itself. For example,  $\mathbb{Q}$  is not connected in  $\mathbb{R}$  since  $(-\infty, \sqrt{2}) \cap \mathbb{Q}, (\sqrt{2}, \infty) \cap \mathbb{Q}$  is a separation of  $\mathbb{Q}$ .

**Lemma 2.18.** Suppose  $C$  and  $D$  form a separation of a topological space  $X$ . If  $Y$  is a connected subspace of  $X$ , then either  $Y \subseteq C$  or  $Y \subseteq D$ .

*Proof.* Suppose not. Then, there exist points  $y_1, y_2 \in Y$  such that  $y_1 \in C$  and  $y_2 \in D$ . Since  $C$  and  $D$  are open in  $X$ , the sets  $C \cap Y$  and  $D \cap Y$  are open in the subspace topology on  $Y$ . Moreover, they are disjoint and non-empty, and  $Y = (C \cap Y) \cup (D \cap Y)$ . Thus,  $C \cap Y$  and  $D \cap Y$  form a separation of  $Y$ , contradicting the connectedness of  $Y$ . Hence, either  $Y \subseteq C$  or  $Y \subseteq D$ . ■

**Theorem 2.19.** Suppose  $X_\alpha$  are connected subspaces of a topological space  $X$  for all  $\alpha$  in some index set  $\Lambda$ . If  $\bigcap_{\alpha \in \Lambda} X_\alpha \neq \emptyset$ , then  $\bigcup_{\alpha \in \Lambda} X_\alpha$  is connected.

*Proof.* Let  $x \in \bigcap_{\alpha \in \Lambda} X_\alpha$ . Suppose, for contradiction, that there exists a non-empty set  $U \subseteq \bigcup_{\alpha \in \Lambda} X_\alpha$  that is clopen in the subspace topology. That is,  $U \cap X_\alpha$  is open and closed in  $X_\alpha$  for all  $\alpha \in \Lambda$ . Since  $X_\alpha$  is connected, we must have either  $U \cap X_\alpha = \emptyset$  or  $U \cap X_\alpha = X_\alpha$  for all  $\alpha \in \Lambda$ . But since  $x \in \bigcap_{\alpha \in \Lambda} X_\alpha$ , we must have  $x \in U$  or  $x \notin U$ . Thus, either  $U = \emptyset$  or  $U = \bigcup_{\alpha \in \Lambda} X_\alpha$ , proving that  $\bigcup_{\alpha \in \Lambda} X_\alpha$  is connected. ■

**Lemma 2.20.** Let  $f : X \rightarrow Y$  be a continuous map. If  $X$  is connected, then  $f(X)$  is also connected.

*Proof.* Suppose not. Then, there exist disjoint non-empty open sets  $U, V \subseteq Y$  such that  $f(X) = U \cup V$ . Since  $f$  is continuous, the preimages  $f^{-1}(U)$  and  $f^{-1}(V)$  are open in  $X$ . Moreover, they are disjoint and non-empty, and  $X = f^{-1}(U) \cup f^{-1}(V)$ . Thus,  $f^{-1}(U)$  and  $f^{-1}(V)$  form a separation of  $X$ , contradicting the connectedness of  $X$ . Hence,  $f(X)$  is connected. ■

**Definition 2.21.** Two topological spaces  $X$  and  $Y$  are *homeomorphic* if there exists a bijective continuous map  $f : X \rightarrow Y$  such that the inverse map  $f^{-1} : Y \rightarrow X$  is also continuous. Such a map  $f$  is called a *homeomorphism*.

January 30th.

**Theorem 2.22.** If  $X_i$  are connected topological spaces for all  $1 \leq i \leq n$ , then the product space  $\prod_{i=1}^n X_i$  is also connected.

*Proof.* Note that  $X_1 \times \{b\}$  is connected for  $b \in X_2$ , since it is homeomorphic to  $X_1$ . Now, let  $a \in X_1$  be arbitrary. Then, the set  $\{a\} \times X_2$  is connected. Thus, the union  $X_1 \times X_2$  is connected, since  $(X_1 \times \{b\}) \cap (\{a\} \times X_2) = \{(a, b)\} \neq \emptyset$ . Repeating this argument inductively, we have that  $\prod_{i=1}^n X_i$  is connected. ■

**Theorem 2.23.** If  $A \subseteq X$  is connected and  $A \subseteq B \subseteq \overline{A}$ , then  $B$  is also connected.

*Proof.* Assume that  $B$  is disconnected. That is, there exists a clopen set  $U$  in  $B$  such that  $B = U \cup U^c$ . Without the loss of generality, suppose  $\emptyset \neq U \cap A$  is clopen in  $A$ . Then  $U \cap A = A$  or  $A \subseteq U$ . But then  $\overline{A} \subseteq \overline{U} = U$ , so  $B \subseteq U$ , a contradiction. Hence,  $B$  is connected. ■

We will now show that  $\mathbb{R}^{\mathbb{N}}$  with the product topology is connected, but not with the box topology. With the product topology equipped, let  $A_n = \{(x_1, \dots, x_n, 0, \dots) \mid x_i \in \mathbb{R}\}$ . Then  $\mathbb{R}^{\mathbb{N}} = \bigcup_{n=1}^{\infty} A_n$ , and each  $A_n$  is homeomorphic to  $\mathbb{R}^n$ , which is connected. Moreover,  $A_n \subseteq A_{n+1}$  for all  $n \geq 1$ , so by the previous theorem,  $\mathbb{R}^{\mathbb{N}}$  with the product topology is connected. For the box topology, let  $\mathbb{R}^{\mathbb{N}} = B \cup U$ , where  $B$  is the set of bounded sequences, and  $U$  is the set of unbounded sequences. To show that  $B$  is open, let  $x = (x_1, x_2, \dots) \in B$ . Then there exists  $K > 0$  such that  $|x_i| < K$  for all  $i \geq 1$ . Finally,  $x_i \in (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \dots$  is an open neighbourhood of  $x$  contained in  $B$ , proving that  $B$  is open. A similar argument shows that  $U$  is open. Since both  $B$  and  $U$  are non-empty, they form a separation of  $\mathbb{R}^{\mathbb{N}}$  with the box topology, proving that it is not connected.

**Definition 2.24.** Let  $a \neq b \in X$ . We say that  $\gamma$  is a *path* between  $a$  and  $b$  if  $\gamma : [0, 1] \rightarrow X$  is continuous, with  $\gamma(0) = a$  and  $\gamma(1) = b$ . If such a path exists, then  $a$  and  $b$  are said to be path connected. A topological space  $X$  is *path connected* if every pair of points in  $X$  are path connected.

**Lemma 2.25.** Suppose there exists a  $a_0 \in X$  such that  $a_0$  is path connected to every point in  $X$ . Then,  $X$  is path connected.

*Proof.* Let  $a, b \in X$  be arbitrary. By assumption, there exist paths  $\gamma_1 : [0, 1] \rightarrow X$  and  $\gamma_2 : [0, 1] \rightarrow X$  such that  $\gamma_1(0) = a_0$ ,  $\gamma_1(1) = a$ ,  $\gamma_2(0) = a_0$ , and  $\gamma_2(1) = b$ . Now, define the map  $\gamma : [0, 1] \rightarrow X$  by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t-1) & \frac{1}{2} < t \leq 1. \end{cases} \quad (2.24)$$

Then,  $\gamma$  is continuous, and  $\gamma(0) = a$ ,  $\gamma(1) = b$ . Thus,  $a$  and  $b$  are path connected. Since  $a, b$  were arbitrary,  $X$  is path connected. ■

**Theorem 2.26.** If  $X$  is path connected, then  $X$  is connected.

*Proof.* Suppose  $X$  is not connected. Then, there exists a separation  $U, V$  of  $X$  such that  $U \cap V = \emptyset$ . Let  $a \in U$  and  $b \in V$ . Since  $X$  is path connected, there exists a path  $\gamma : [0, 1] \rightarrow X$  such that  $\gamma(0) = a$  and  $\gamma(1) = b$ . Consider the preimages  $\gamma^{-1}(U)$  and  $\gamma^{-1}(V)$ . These sets are open in  $[0, 1]$ , disjoint, non-empty, and their union is  $[0, 1]$ . Thus, they form a separation of  $[0, 1]$ , contradicting the connectedness of  $[0, 1]$ . Hence,  $X$  is connected. ■

February 3rd.

Note that in the above proof we implicitly assumed the fact that  $[0, 1]$  is connected. To show that it is connected, we require the following lemma which is left as an exercise.

**Lemma 2.27.** A topological space  $X$  is connected if and only if every continuous map  $f : X \rightarrow \{\pm 1\}$  is constant.

**Theorem 2.28.** The interval  $[0, 1]$  is connected.

*Proof.* Suppose  $f : [0, 1] \rightarrow \{\pm 1\}$  is continuous, and not constant. Then there exist  $a, b \in [0, 1]$  such that  $f(a) = 1$  and  $f(b) = -1$ . Without loss of generality, assume  $a < b$ . By the intermediate value theorem, there exists some  $c \in (a, b)$  such that  $f(c) = 0$ , contradicting the fact that the codomain of  $f$  is  $\{\pm 1\}$ . Thus, every continuous map  $f : [0, 1] \rightarrow \{\pm 1\}$  is constant, proving that  $[0, 1]$  is connected. ■

*Alternate proof.* Suppose  $U \subseteq [0, 1]$  is non-empty, open, and closed in the subspace topology. Without loss of generality, let  $0 \in U$  (otherwise, take  $V = U^c$ ). Now define  $E = \{x \in [0, 1] \mid [0, x] \subseteq U\}$ . Since  $0 \in U$ , we have  $0 \in E$ , so  $E$  is non-empty. There now exists a natural  $N \in \mathbb{N}$  such that  $[0, \frac{1}{n}] \subseteq U$  for all  $n \geq N$ , so  $\frac{1}{N} \in E$ . Thus,  $E$  is bounded above by 1, so let  $c = \sup E$ . We claim that  $c = 1$ . Suppose not. Since  $U$  is closed in  $[0, 1]$ , we have  $c \in U$ . Thus, there exists some  $\varepsilon > 0$  such that  $(c - \varepsilon, c + \varepsilon) \cap [0, 1] \subseteq U$ . But since  $c = \sup E$ , there exists some  $x \in E$  such that  $c - \varepsilon < x \leq c$ . Then,  $[0, x] \subseteq U$ , and  $(c - \varepsilon, c + \varepsilon) \cap [0, 1] \subseteq U$  implies that  $[0, c + \frac{\varepsilon}{2}] \subseteq U$ , contradicting the fact that  $c$  is an upper bound for  $E$ . Thus,  $c = 1$ , so  $[0, 1] \subseteq U$ , proving that  $U = [0, 1]$ . Hence,  $[0, 1]$  is connected. ■

**Example 2.29.** We can use to show that  $GL_n(\mathbb{R})$  is not connected. Note that the map  $\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R} \setminus \{0\}$  is continuous. Since  $\mathbb{R} \setminus \{0\}$  is not connected,  $GL_n(\mathbb{R})$  is also not connected. Similarly, the hypersphere  $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$  is connected for all  $n \geq 1$ , since the map  $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$  defined by  $f(x) = \frac{x}{\|x\|}$  is continuous, and  $\mathbb{R}^{n+1} \setminus \{0\}$  is connected for all  $n \geq 1$ .

**Example 2.30.** We show that  $\mathbb{R}$  and  $\mathbb{R}^2$  cannot be homeomorphic. Suppose there exists a homeomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}^2$ . Let  $a \in \mathbb{R}$  be arbitrary, and consider the set  $\mathbb{R}^2 \setminus \{f(a)\}$ . Note that this set is connected, since it is homeomorphic to  $\mathbb{R}^2 \setminus \{0\}$ , which is connected. However, the set  $\mathbb{R} \setminus \{a\}$  is not connected, since it can be separated into  $(-\infty, a)$  and  $(a, \infty)$ . But since  $f$  is a homeomorphism, the image of  $\mathbb{R} \setminus \{a\}$  under  $f$  must also be disconnected, a contradiction. Thus, no such homeomorphism exists, proving that  $\mathbb{R}$  and  $\mathbb{R}^2$  are not homeomorphic.

### 2.3.1 Components

On a topological space  $X$ , define an equivalence relation  $\sim$  on  $X$  by  $x \sim y$  if and only if there exists a connected subspace of  $X$  containing both  $x$  and  $y$ . The equivalence classes under this relation are called the *components* of  $X$ .

**Theorem 2.31.** *The components of  $X$  are connected, disjoint subspaces such that their union is  $X$ . Any connected subspace of  $X$  intersects only one component.*

*Proof.* Let  $C$  be a component of  $X$ . Let  $x_0 \in C$  be arbitrary. Then, by definition of the equivalence relation, for every  $x \in C$ , there exists a connected subspace  $V_x$  of  $X$  such that  $x_0, x \in V_x$ . Thus, we have

$$C = \bigcup_{x \in C} V_x, \quad (2.25)$$

and since each  $V_x$  is connected and they all share the point  $x_0$ , by a previous theorem,  $C$  is connected. The components are disjoint simply by the definition of an equivalence relation, and their union is  $X$  since every point in  $X$  is contained in some component. Finally, suppose  $A \subseteq X$  is connected, and intersects two components  $C_1$  and  $C_2$ . Then, there exist points  $a_1 \in A \cap C_1$  and  $a_2 \in A \cap C_2$ . Since  $A$  is connected, there exists a connected subspace of  $X$  containing both  $a_1$  and  $a_2$ . Thus, by definition of the equivalence relation, we must have  $C_1 = C_2$ , proving that any connected subspace of  $X$  intersects only one component. ■

Thus, the components may even be called the connected components of  $X$ .

**Theorem 2.32.** *Let  $U \subseteq \mathbb{R}^n$  be an open connected set. Then,  $U$  is path connected.*

*Proof.* Fix some  $x_0 \in U$ , and let  $A$  be the set of all points in  $U$  that are path connected to  $x_0$ . We wish to show that  $A = U$ . Note that  $A$  is non-empty since  $x_0 \in A$ . To show that  $A$  is open, let  $x \in A$  be arbitrary. Since  $U$  is open, there exists some  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq U$ . Now, for any  $y \in B(x, \varepsilon)$ , the line segment from  $x$  to  $y$  lies entirely within  $B(x, \varepsilon)$ , so there exists a path from  $x$  to  $y$ . Since there also exists a path from  $x_0$  to  $x$ , by concatenating these two paths, we have a path from  $x_0$  to  $y$ . Thus,  $y \in A$ , proving that  $B(x, \varepsilon) \subseteq A$ . Hence,  $A$  is open. To show that  $A$  is closed, we look at  $U \setminus A$ . Let  $x \in U \setminus A$  be arbitrary; there is no path from  $x_0$  to  $x$ . Since  $U$  is open, there exists some  $\delta > 0$  such that  $B(x, \delta) \subseteq U$ . Now, for any  $y \in B(x, \delta)$ , the line segment from  $x$  to  $y$  lies entirely within  $B(x, \delta)$ , so there exists a path

from  $x$  to  $y$ . If there existed a path from  $x_0$  to  $y$ , then by concatenating this path with the path from  $x$  to  $y$ , we would have a path from  $x_0$  to  $x$ , contradicting our assumption. Thus, there is no path from  $x_0$  to  $y$ , so  $y \in U \setminus A$ . Hence,  $B(x, \delta) \subseteq U \setminus A$ , proving that  $U \setminus A$  is open. Since  $U$  is connected, and  $A$  is non-empty, open, and closed in  $U$ , we must have  $A = U$ . Thus, every point in  $U$  is path connected to  $x_0$ , so  $U$  is path connected. ■

More generally, we have the following result.

**Theorem 2.33.** *Let  $X$  be a topological space that is connected. If each point  $x \in X$  has an open neighbourhood that is path connected, then  $X$  is path connected.*

From here, we say that  $X$  is locally connected at  $x_0$  if every open neighbourhood of  $x_0$  contains a connected open neighbourhood of  $x_0$ . If  $X$  is locally connected at every point  $x \in X$ , then  $X$  is said to be *locally connected*.

## 2.4 Compactness

Recall that  $\{A_\alpha\}_{\alpha \in \Lambda}$  is an open *covering* of  $X$  if  $\bigcup_{\alpha \in \Lambda} A_\alpha \supseteq X$ . A *subcover* is a subset of the covering that still covers  $X$ .

**Definition 2.34.** A topological space  $X$  is *compact* if every open covering of  $X$  has a finite subcovering.

**Theorem 2.35.** *Every compact subset of a Hausdorff space is closed.*

*Proof.* Let  $K \subseteq X$  be compact. We wish to show that  $K^c$  is open. Let  $x_0 \in X \setminus K$ . For each  $y \in K$ , since  $X$  is Hausdorff, there exist disjoint open neighbourhoods  $U_y$  of  $x_0$  and  $V_y$  of  $y$ . Then  $\{V_y\}_{y \in K}$  is an open covering of  $K$ , so there exists a finite subcovering  $\{V_{y_i}\}_{i=1}^n$ . Now, let  $U = \bigcap_{i=1}^n U_{y_i}$ . Then,  $U$  is an open neighbourhood of  $x_0$ , and for every  $i$ , we have  $U \cap V_{y_i} = \emptyset$ . Thus,  $U \cap K = \emptyset$ , proving that  $x_0 \in K^c$  has an open neighbourhood contained in  $K^c$ . Since  $x_0$  was arbitrary,  $K^c$  is open, so  $K$  is closed. ■

# Index

- $\ell^1$  norm, 1
- $p$ -adic absolute value, 2
- $p$ -adic metric space, 2
- absolute homogeneity, 1
- basis, 5
- boundary, 10
- closed ball, 2
- closed set, 6
- closure, 7
- coarser, 5
- cocountable topology, 4
- cofinite topology, 4
- compact, 20
- component, 19
- connected, 17
- continuous, 11, 12
- converge, 8
- covering, 20
- discrete metric, 1
- discrete topology, 4
- distance from a set, 17
- equivalent metrics, 3
- Euclidean norm, 1
- field of  $p$ -adic numbers, 2
- finer, 5
- Hausdorff property, 2
- homeomorphic, 17
- homeomorphism, 17
- indiscrete topology, 4
- interior, 4
- K-topology, 6
- limit point, 8
- locally connected, 20
- lower limit topology, 6
- metric, 1
- metric space, 1
- norm, 1
- open ball, 2
- open set, 2
- path, 18
- path connected, 18
- positive definiteness, 1
- product topology, 10
- separation, 17
- subspace topology, 4
- supremum norm, 2
- symmetry, 1
- T1, 9
- topological space, 3
- triangle inequality, 1
- ultrametric inequality, 2
- uniform topology, 15
- upper limit topology, 6