TOPOLOGY

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Contents

Chapter 1

METRIC AND TOPOLOGICAL SPACES

1.1 Metric Spaces and Examples

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A *metric space* is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function, called a *metric* on X, satisfying the following properties for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$ (*positive definiteness*).

(ii) $d(x, y) = d(y, x)$ (*symmetry*).

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Let us look at some examples of metric spaces.

Example 1.1. Any set X can be made into a metric space by defining the *discrete metric* d as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \tag{1.1}$$

It is easy to verify that d satisfies all the properties of a metric.

Example 1.2. Recall that a normed space $(V, \|\cdot\|)$ was a vector space V equipped with a *norm* $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and $\alpha \in \mathbb{F}$:

(i) $\|v\| = 0$ if and only if $v = 0$ (positive definiteness).

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ (*absolute homogeneity*).

(iii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).

Given a normed space $(V, \|\cdot\|)$, we can define a metric d on V as follows:

$$d(u, v) = \|u - v\| \quad \forall u, v \in V. \tag{1.2}$$

Yet again, it is straightforward to verify that d satisfies all the properties of a metric. Given a vector space V, we can have multiple norms on it, and hence multiple metrics. For example, consider the vector space $\mathbb{R}^n$. We have the following norms on $\mathbb{R}^n$:

- The ℓ^1 *norm*: $\|x\|_1 = \sum_{i=1}^n |x_i|$,
- the *Euclidean norm*: $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$,

- the *supremum norm*: $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$,

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Each of these norms induces a different metric on $\mathbb{R}^n$.

The notion of open and closed balls is also abstracted to metric spaces as follows.

Definition 1.3. Let (X, d) be a metric space. The *open ball* of radius $r > 0$ centered at a point $x \in X$ is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \tag{1.3}$$

and the *closed ball* of radius $r > 0$ centered at x is the set

$$B[x, r] = \{y \in X \mid d(x, y) \leq r\}. \tag{1.4}$$

Note that in the discrete metric space, $B(x, 1) = \{x\} = B(x, \frac{1}{2})$, and $B(x, 2) = X = B(y, 2)$ for any $x, y \in X$. Thus, $B(x, r) = B(y, \rho)$ does not imply that $x = y$ or $r = \rho$ in general.

Example 1.4. Let p be a prime, say $p = 3$. Define a function $|\cdot|_3 : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ as follows: for any non-zero integer m, write $m = 3^k m'$ where m' is not divisible by 3, and set $|m|_3 = 3^{-k}$. Also, set $|0|_3 = 0$. This function $|\cdot|_3$ is called the 3-adic absolute value on $\mathbb{Z}$. In general, for any prime p, the *p-adic absolute value* is defined similarly.
This 3-adic absolute value induces a norm d_3 on $\mathbb{Q}$ as follows:

$$|q|_3 = \begin{cases} 0 & \text{if } q = 0, \\ |m|_3 \,/\, |n|_3 & \text{if } q = m/n \text{ in lowest terms.} \end{cases} \tag{1.5}$$

This induces a metric on $\mathbb{Q}$ defined by $d_3(x, y) = |x - y|_3$ for all $x, y \in \mathbb{Q}$. This metric space $(\mathbb{Q}, d_3)$ is called the 3-adic metric space, and in general $(\mathbb{Q}, d_p)$ is called the *p-adic metric space*. The completion of $(\mathbb{Q}, d_p)$ gives us the *field of p-adic numbers*, denoted by $\mathbb{Q}_p$. This metric space has some interesting properties; for instance, the triangle inequality is strengthened to the *ultrametric inequality*:

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\} \quad \forall x, y, z \in \mathbb{Q}. \tag{1.6}$$

Lemma 1.5 (*Hausdorff property*)**.** *Let (X, d) be a metric space. For any distinct $x, y \in X$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$.*

Proof. Verify that choosing any $r \leq \frac{1}{2} d(x, y)$ works. ■

Let (X, d) be a metric space. Then a subset $A \subseteq X$ can also be made into a metric space by restricting the metric d to $A \times A$. In the metric space $(A, d|_{A\times A})$, the open balls are given by $B_A(x, r) = B_X(x, r) \cap A$ for all $x \in A$ and $r > 0$, where $B_X(x, r)$ is the open ball in (X, d).

Again, as before, the notion of open sets is abstracted to metric spaces as follows.

Definition 1.6. Let (X, d) be a metric space. A subset $U \subseteq X$ is said to be an *open set* if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

As a small lemma, one can show that every open ball in a metric space is an open set. As an exercise, show that the complement of the closed ball $B[x, r]^c = \{y \mid d(x, y) > r\}$ is also an open set.

Proposition 1.7. *Let (X, d) be a metric space. Let $\tau = \{U \subseteq X \mid U \text{ is open}\}$, that is, the collection of all open sets in X. Then the following hold true.*

(i) $\emptyset, X \in \tau$.

(ii) *For $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \tau$, we have $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$. That is, an arbitrary union of open sets is open.*

(iii) For $U_1, U_2, \ldots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of open sets is open.

Proof. The proof of the first property is trivial. For the second property, let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$. Thus, $\bigcup_{\alpha\in\Lambda} U_\alpha$ is open.

For the third property, let $x \in \bigcap_{i=1}^n U_i$. Then $x \in U_i$ for all $1 \le i \le n$. Since each U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for all $1 \le i \le n$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then we have

$$B(x, r) \subseteq B(x, r_i) \subseteq U_i \quad \forall 1 \le i \le n, \tag{1.7}$$

which implies that $B(x, r) \subseteq \bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ∎

1.2 Topological Spaces and Examples

A *topological space* is a pair (X, τ) where X is a set and τ is a collection of subsets of X satisfying the following properties:

(i) $\emptyset, X \in \tau$.

(ii) For $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \tau$, we have $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$. That is, an arbitrary union of sets in τ is in τ.

(iii) For $U_1, U_2, \ldots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of sets in τ is in τ.

These are the exact same properties that the collection of open sets in a metric space satisfy. Hence, every metric space (X, d) gives rise to a topological space (X, τ_d) where τ_d is the collection of all open sets in (X, d). Such a topology τ_d is called the topology induced by the metric d.

As a smaller example, let $X = \{0, 1, 2, 3, 4\}$ and consider the collection $\tau = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}\}$. Then the pair (X, τ) is *not* a topological space since $\{0, 1\} \cup \{2, 4\} = \{0, 1, 2, 4\} \notin \tau$. However, the pair (X, τ') where $\tau' = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}, \{0, 1, 2, 4\}\}$ is a topological space.

Description of open sets in $\mathbb{R}$

Theorem 1.8. *A non-empty open set in $\mathbb{R}$ is a countable union of pairwise disjoint open intervals.*

Proof. Let $U \subseteq \mathbb{R}$ be a non-empty open set. For each $x \in U$, define

$$I_x = \bigcup\{(a, b) \mid x \in (a, b) \subseteq U\}. \tag{1.8}$$

Note that $x \in I_x \subseteq U$. Let $a_x = \inf I_x$ and $b_x = \sup I_x$. We claim that $I_x = (a_x, b_x)$. For $a_x < z < b_x$, there exists $a, b \in I_x$ such that $a_x < a < z < b < b_x$. Since $z \in (a, b) \subseteq I_x$, we have $z \in I_x$. Thus, $(a_x, b_x) \subseteq I_x$. The other inclusion is trivial. Hence, $I_x = (a_x, b_x)$ is an open interval.

We now claim that if $x \neq y$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose that $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is an interval containing both x and y and contained in U. By the definition of I_x and I_y, we have $I_x \cup I_y \subseteq I_x$ and $I_x \cup I_y \subseteq I_y$. Thus, $I_x = I_y$.

Finally, let $U = \bigcup_{x\in U} I_x$. By the above claim, the collection $\{I_x \mid x \in U\}$ consists of pairwise disjoint open intervals. Since each I_x contains a rational number (by the density of $\mathbb{Q}$ in $\mathbb{R}$), for each I_x, we can choose a distinct rational number $q_x \in I_x$. This gives $I_x = I_{q_x}$. Thus, we have

$$U = \bigcup_{x\in U} I_x = \bigcup_{q\in\mathbb{Q}\cap U} I_q, \tag{1.9}$$

which is a countable union of pairwise disjoint open intervals. ∎

Definition 1.9. Let (X, d_1) and (X, d_2) be two metric spaces on the same set X. The metrics d_1 and d_2 are said to be *equivalent metrics*, $d_1 \sim d_2$, if open sets in (X, d_1) are exactly the open sets in (X, d_2).

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Definition 1.10. Let $A \subseteq X$ be a subset of a metric space (X, d). The *interior* of A, denoted by $\mathrm{Int}(A)$, is defined as

$$\mathrm{Int}(A) := \{x \in A \mid \exists r > 0 \text{ such that } B(x, r) \subseteq A\}. \tag{1.10}$$

For a general topological space, the interior of a set A is defined as the largest open set contained in A. For another definition, we have

$$\mathrm{Int}(A) := \{x \in A \mid \exists U \in \tau \text{ such that } x \in U \subseteq A\}. \tag{1.11}$$

Lemma 1.11. *The above definitions of the interior of a set in a topological space are equivalent.*

Proof. Suppose we affirm the second definition. Let $x \in \mathrm{Int}(A)$. Then there exists an open set $U_x \in \tau$ such that $x \in U_x \subseteq A$. Thus, $\bigcup_{x \in \mathrm{Int}(A)} U_x \subseteq A$ is contained in A and is open. Moreover, if $z \in \mathrm{Int}(A) \setminus \bigcup_{x \in \mathrm{Int}(A)} U_x$, then there exists an open set $V \in \tau$ such that $z \in V \subseteq A$. But then $z \in U_z \subseteq \bigcup_{x \in \mathrm{Int}(A)} U_x$, a contradiction. Thus, $\mathrm{Int}(A) = \bigcup_{x \in \mathrm{Int}(A)} U_x$ is the largest open set contained in A. If V is any open set contained in A, then for any $y \in V$, there exists an open set $V_y \in \tau$ such that $y \in V_y \subseteq A$. Thus, $y \in \mathrm{Int}(A)$, which implies that $V \subseteq \mathrm{Int}(A)$. Hence, the first definition holds. ■

As an example, with the standard topology on $\mathbb{R}$, we have $\mathrm{Int}([0, 1]) = (0, 1)$, $\mathrm{Int}((0, 1) \cup \{2\}) = (0, 1)$, and $\mathrm{Int}(\mathbb{Q}) = \emptyset$. Note that the interior of an open set is the set itself; $\mathrm{Int}(U) = U$ for any open set U.

In the spirit of an induced metric, the subspace topology is defined as follows.

Definition 1.12. Let (X, τ) be a topological space and let $Y \subseteq X$. The *subspace topology* on Y is defined as

$$\tau_Y = \{U \cap Y \mid U \in \tau\}. \tag{1.12}$$

One can verify that τ_Y is a topology on Y. Let us look at some examples of topological spaces.

Example 1.13. The *discrete topology* on a set X is the topology $\tau = \mathcal{P}(X)$, the power set of X. Every subset of X is open in this topology. In contrast, the *indiscrete topology* (trivial topology) on X is the topology $\tau = \{\emptyset, X\}$. Only the empty set and the whole set are open in this topology.

Example 1.14. Let X be any set. The *cofinite topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is finite}\}. \tag{1.13}$$

One can verify that τ is a topology on X; both $\emptyset$ and X are in τ. For an arbitrary union of sets in τ, if any one of them is X, then the union is X. Otherwise, the complement of the union is the intersection of finite sets, which is finite. For a finite intersection of sets in τ, the complement of the intersection is the finite union of finite sets, which is finite. Thus, τ is a topology on X.

Example 1.15. Let X be any set. The *cocountable topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is countable}\}. \tag{1.14}$$

One can verify that τ is a topology on X; both $\emptyset$ and X are in τ. For an arbitrary union of sets in τ, if any one of them is X, then the union is X. Otherwise, the complement of the union is the intersection of countable sets, which is countable. For a finite intersection of sets in τ, the complement of the intersection is the finite union of countable sets, which is countable. Thus, τ is a topology on X.

1.2.1 Basis

Definition 1.16. Let (X, τ) be a topological space. A collection $\mathcal{B} \subseteq \tau$ is said to be a *basis* for τ if

(i) For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.

(ii) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

For any set X, if we have a collection $\mathcal{B}$ of subsets of X satisfying the above two properties, then the collection

$$\tau = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \tag{1.15}$$

is a topology on X, and $\mathcal{B}$ is a basis for this topology.

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For example, choosing $\mathcal{B} = \mathcal{P}(X)$ gives us the discrete topology on X, and choosing $\mathcal{B} = \{X\}$ gives us the indiscrete topology on X.

Lemma 1.17. *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ. Then τ is the collection of all possible unions of elements in $\mathcal{B}$.*

Proof. Note that every union of elements in $\mathcal{B}$ is in τ by the definition of a topology; we need to show the other inclusion, that is, every set in τ can be expressed as a union of elements in $\mathcal{B}$. Let $U \in \tau$. For each $x \in U$, by the definition of a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Thus, we have

$$U = \bigcup_{x \in U} B_x, \tag{1.16}$$

which is a union of elements in $\mathcal{B}$. ■

Definition 1.18. Let (X, τ_1) and (X, τ_2) be two topological spaces on the same set X. We say $\tau_1 \supseteq \tau_2$, or that τ_1 is (strictly) *finer* than τ_2, if every open set in τ_2 is also an open set in τ_1. Conversely, we say τ_2 is (strictly) *coarser* than τ_1 if every open set in τ_1 is also an open set in τ_2, that is, $\tau_2 \subseteq \tau_1$.

Lemma 1.19. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies τ and τ' on the same set X. Then the following are equivalent:*

(i) $\tau' \supseteq \tau$.

(ii) *For all $x \in X$ and all $B \in \mathcal{B}$ such that $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. For the reverse implication, let $U \in \tau$. Then U can be written as $U = \bigcup_{\alpha \in \Lambda} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for all $\alpha \in \Lambda$. For each $\alpha \in \Lambda$ and each $x \in B_\alpha$, by the second property, there exists $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subseteq B_\alpha$. Thus, we have

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{x \in B_\alpha} B'_x, \tag{1.17}$$

which is a union of elements in $\mathcal{B}'$. Thus, $U \in \tau'$, and hence $\tau' \supseteq \tau$. For the forward implication, let $B \in \mathcal{B}$ and $x \in B$. Since $B \in \tau \subseteq \tau'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. ■

Let us look at some topologies generated by bases.

Example 1.20. Let $X = \mathbb{R}$ and $\mathcal{B} = \{[a, b) \mid a < b,\, a, b \in \mathbb{R}\}$. We claim that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Clearly, for every $x \in \mathbb{R}$, there exists $[x, x+1) \in \mathcal{B}$ such that $x \in [x, x+1)$. Now, let $B_1 = [a_1, b_1), B_2 = [a_2, b_2) \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Then we have three cases:

- If $a_1 \le a_2 \le x \le b_1 \le b_2$, then choose $B_3 = [a_2, b_1)$.

- If $a_2 \leq a_1 \leq x \leq b_2 \leq b_1$, then choose $B_3 = [a_1, b_2)$.
- If $a_1 \leq x \leq b_2 \leq b_1$ (the case $a_2 \leq x \leq b_1 \leq b_2$ is similar), then choose $B_3 = [x, b_2)$.

In all cases, we have $x \in B_3 \subseteq B_1 \cap B_2$. Thus, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$, called the *lower limit topology* denoted by $\mathbb{R}_l$. Similarly, the collection $\mathcal{B}' = \{(a, b] \mid a < b,\ a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$, called the *upper limit topology* denoted by $\mathbb{R}_u$.

Example 1.21. Ket $K = \{1/n \mid n \in \mathbb{N}\}$. Consider the collection

$$\mathcal{B} = \{(a, b), (a, b) \setminus K \mid a < b,\ a, b \in \mathbb{R}\}. \tag{1.18}$$

Verify that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$, called the *K-topology* on $\mathbb{R}$, denoted by $\mathbb{R}_K$.

Lemma 1.22. *The standard topology on $\mathbb{R}$ is strictly finer than $\mathbb{R}_l$, and also strictly finer than $\mathbb{R}_K$. However, $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.*

Proof. For $\mathbb{R}_l$: Let $x \in \mathbb{R}$ and let $B = (a, b) \in \mathcal{B}$ such that $x \in B$. Then choose $B' = [x, b) \in \mathcal{B}$ such that $x \in B' \subseteq B$. Thus, by the previous lemma, the standard topology on $\mathbb{R}$ is finer than $\mathbb{R}_l$. To see that the inclusion is strict, note that $[a, b)$ is open in $\mathbb{R}_l$ but not in the standard topology on $\mathbb{R}$.

For $\mathbb{R}_K$: Note that inclusion is trivial since every basis element of $\mathbb{R}_K$ is also a basis element of the standard topology on $\mathbb{R}$. To see that the inclusion is strict, note that $(-1, 1) \setminus K$ is open in $\mathbb{R}_K$ but not in the standard topology on $\mathbb{R}$.

For non-comparability of $\mathbb{R}_l$ and $\mathbb{R}_K$: Note that $[5, 6)$ is open in $\mathbb{R}_l$ but not in $\mathbb{R}_K$ since it cannot be expressed as a union of basis elements of $\mathbb{R}_K$. Also, note that $(-1, 1) \setminus K$ is open in $\mathbb{R}_K$ but not in $\mathbb{R}_l$ since it cannot be expressed as a union of basis elements of $\mathbb{R}_l$. ∎

1.3 Closed Sets

Definition 1.23. Let (X, τ) be a topological space. A subset $A \subseteq X$ is said to be a *closed set* if its complement $A^c = X \setminus A$ is an open set.

Using De Morgan's laws, one can easily verify the following properties of closed sets in a topological space.

Proposition 1.24. *Let (X, τ) be a topological space. Then the following hold true.*

(i) $\emptyset$ and X are closed sets.

(ii) For any collection $\{A_\alpha\}_{\alpha \in \Lambda}$ of closed sets, we have $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a closed set. That is, an arbitrary intersection of closed sets is closed.

(iii) For closed sets $A_1, A_2, \ldots, A_n$, we have $\bigcup_{i=1}^n A_i$ is a closed set. That is, a finite union of closed sets is closed.

Lemma 1.25. *Any finite set of a metric space (X, d) is closed.*

Proof. Note that it is enough to show that a singleton set $\{x\}$ is closed for any $x \in X$. Let $y \in \{x\}^c$. Then $y \neq x$, and by the Hausdorff property, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$. Thus, $B(y, r) \subseteq \{x\}^c$, which implies that $\{x\}^c$ is open. Hence, $\{x\}$ is closed. ∎

Similar to the open sets, we can define closed sets in a subspace as follows.

Theorem 1.26. *Let (X, τ) be a topological space and let $Y \subseteq X$ be a subspace with the subspace topology τ_Y. Then a set $C \subseteq Y$ is closed in (Y, τ_Y) if and only if there exists a closed set D in (X, τ) such that $C = D \cap Y$.*

Proof. Let C be closed in (Y, τ_Y). Then $C^c = Y \setminus C$ is open in (Y, τ_Y). Thus, there exists an open set $U \in \tau$ such that $C^c = U \cap Y$. Let $D = U^c$, which is closed in (X, τ). Then we have

$$C = Y \setminus C^c = Y \setminus (U \cap Y) = Y \cap U^c = Y \cap D. \tag{1.19}$$

Conversely, let D be closed in (X, τ) and let $C = D \cap Y$. Then D^c is open in (X, τ). Thus, we have

$$C^c = Y \setminus C = Y \setminus (D \cap Y) = Y \cap D^c, \tag{1.20}$$

which is open in (Y, τ_Y). Hence, C is closed in (Y, τ_Y). ■

Analogous to the interior of a set, we have the notion of the closure of a set.

Definition 1.27. Let (X, τ) be a topological space and let $A \subseteq X$. The *closure* of A, denoted by $\overline{A}$, is defined as the smallest closed set containing A. That is,

$$\overline{A} := \bigcap \{C \subseteq X \mid C \text{ is closed and } A \subseteq C\}. \tag{1.21}$$

Theorem 1.28. *The closure of a set A in a topological space (X, τ) is equivalent to saying*

$$\overline{A} = \{x \in X \mid \forall U \in \tau \text{ with } x \in U,\, U \cap A \neq \emptyset\}. \tag{1.22}$$

Proof. Let $x \in \overline{A}$ with $x \in U \in \tau$. Suppose for the sake of contradiction that $U \cap A = \emptyset$. Then we have $A \subseteq U^c$, where U^c is closed. This must then imply that $\overline{A} \subseteq U^c$, which is a contradiction since $x \in U$. Thus, we have $U \cap A \neq \emptyset$.

For the converse inclusion, let $x \notin \overline{A}$. Then there exists a closed set C such that $A \subseteq C$ but $x \notin C$. Thus, $x \in C^c$, where C^c is open. Since $A \subseteq C$, we have $C^c \cap A = \emptyset$. Hence, there exists an open set C^c containing x such that $C^c \cap A = \emptyset$. ■

Instead of using "$\forall U \in \tau$" in the above theorem, one can equivalently use "$\forall B \in \mathcal{B}$" where $\mathcal{B}$ is a basis for the topology τ.

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Lemma 1.29. *A is closed if and only if $A = \overline{A}$. Moreover, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.*

Proof. The first assertion is left as an exercise. For the second assertion, let $x \in \overline{A}$. Then for every open set U containing x, we have $U \cap A \neq \emptyset$. Since $A \subseteq B$, we have $U \cap B \neq \emptyset$. Thus, $x \in \overline{B}$. ■

Theorem 1.30. *Let $Y \subseteq X$ be a subspace of a topological space (X, τ). For any $A \subseteq Y$, we have*

$$\overline{A}^{Y} = \overline{A} \cap Y, \tag{1.23}$$

where $\overline{A}^{Y}$ is the closure of A in the subspace (Y, τ_Y) and $\overline{A}$ is the closure of A in (X, τ).

Proof. We have

$$\overline{A}^{Y} = \{y \in Y \mid y \in U \cap Y, U \in \tau, U \cap Y \cap A \neq \emptyset\} \subseteq \overline{A} \cap Y. \tag{1.24}$$

Now let $y_0 \in \overline{A} \cap Y$. Then for every $U \in \tau$ such that $y_0 \in U$, we have $U \cap A \neq \emptyset$. Thus, taking intersection over all closed sets containing A in (X, τ) and then intersecting with Y, we have

$$\bigcap_{\substack{C \subseteq X \\ C \text{ closed} \\ A \subseteq C}} (C \cap Y) = \overline{A} \cap Y. \tag{1.25}$$

Thus, $y_0 \in \overline{A}^Y$. Hence, we have $\overline{A}^Y = \overline{A} \cap Y$. ∎

1.4 Convergence and Hausdorff Spaces

Definition 1.31. Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a *limit point* of A if for every open set $U \in \tau$ containing x, we have $(U \setminus \{x\}) \cap A \neq \emptyset$.

Let us denote the set of all limit points of A by A'. Then $A' \subseteq \overline{A}$ since for every open set U containing a limit point x, we have $U \cap A \neq \emptyset$. Note that nothing can be said about the relation between A and A'.

For example, if $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ with the standard topology, then $A' = \{0\}$ and $\overline{A} = A \cup \{0\}$. If $A = (0, 1) \cup \{2\}$, then $A' = [0, 1]$ and $\overline{A} = [0, 1] \cup \{2\}$. If $A = \mathbb{Q} \subseteq \mathbb{R}$ with the standard topology, then $A' = \mathbb{R}$ and $\overline{A} = \mathbb{R}$. There is a nice characterization of the closure of a set using limit points.

Theorem 1.32. *Let (X, τ) be a topological space and let $A \subseteq X$. Then*

$$\overline{A} = A \cup A'. \tag{1.26}$$

Proof. One direction is easy since we have already seen that $A, A' \subseteq \overline{A}$. For the other direction, let $x \in \overline{A}$. If $x \in A$, then we are done. So suppose that $x \notin A$. Then for every open set U containing x, we have $U \cap A \neq \emptyset$. Since $x \notin A$, we have $(U \setminus \{x\}) \cap A \neq \emptyset$. Thus, x is a limit point of A, and hence $x \in A'$. Therefore, we have $\overline{A} \subseteq A \cup A'$, and hence the result follows. ∎

Corollary 1.33. *A set A is closed if and only if $A' \subseteq A$.*

We are now ready to define convergence in a (metric) topological space.

Definition 1.34. Let (X, d) be a metric space. A sequence $(x_n)_{n \geq 1} \subseteq X$ is said to *converge* to a point $x \in X$ if for every $\varepsilon > 0$, there exists a natural $N(\varepsilon) \equiv N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \forall n \geq N. \tag{1.27}$$

Equivalently, the sequence $(d(x_n, x))_{n \geq 1} \subseteq \mathbb{R}$ converges to 0 in the usual sense.

In the topological sense, we say $(x_n)_{n \geq 1}$ converges to x if for every open set U containing x, there exists a natural $N(U) \equiv N \in \mathbb{N}$ such that

$$x_n \in U \quad \forall n \geq N. \tag{1.28}$$

Note that the limit of a sequence in a metric space is unique. However, in a general topological space, the limit of a sequence need not be unique. For example, consider the set X with the indiscrete topology. Then every sequence in X converges to every point in X.

Theorem 1.35. *Let (X, d) be a metric space and let $(x_n)_{n \geq 1} \subseteq X$ be a sequence converging to both x and y in X. Then $x = y$.*

Proof. For every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon/2 \quad \forall n \geq N_1. \tag{1.29}$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, y) < \varepsilon/2 \quad \forall n \geq N_2. \tag{1.30}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{1.31}$$

Since $\varepsilon > 0$ is arbitrary, we have $d(x, y) = 0$, which implies that $x = y$. ■

The concept of Hausdorff property can be generalized to topological spaces as follows: for every $x, y \in X$ with $x \neq y$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Theorem 1.36. *Let (X, τ) be a Hausdorff topological space and let $(x_n)_{n\geq 1} \subseteq X$ be a sequence converging to both x and y in X. Then $x = y$.*

Proof. The proof is similar to that in the metric space case. For $x \neq y$, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since $(x_n)_{n\geq 1}$ converges to x, there exists $N_1 \in \mathbb{N}$ such that

$$x_n \in U \quad \forall n \geq N_1. \tag{1.32}$$

Similarly, since $(x_n)_{n\geq 1}$ converges to y, there exists $N_2 \in \mathbb{N}$ such that

$$x_n \in V \quad \forall n \geq N_2. \tag{1.33}$$

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have $x_n \in U$ and $x_n \in V$, which implies that $x_n \in U \cap V$. This is a contradiction since $U \cap V = \emptyset$. Hence, we must have $x = y$. ■

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Lemma 1.37. *Let (X, τ) be a Hausdorff topological space. Then every finite subset of X is closed.*

Proof. It is enough to show that a singleton set $\{x\}$ is closed for any $x \in X$. Let $y \in \{x\}^c$. Then $y \neq x$, and by the Hausdorff property, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, $V \subseteq \{x\}^c$, which implies that $\{x\}^c$ is open. Hence, $\{x\}$ is closed. ■

We ask this: is it true that if every finite subset of a topological space (X, τ) is closed, then (X, τ) is Hausdorff? The answer is no, since a counterexample is provided by the cofinite topology on an infinite set X. In this topology, every finite subset is closed, but the space is not Hausdorff.

Definition 1.38. We say that a topological space (X, τ) is *T1* if finite subsets of X are closed.

Theorem 1.39. *Suppose (X, τ) is a T1 topological space. Let $A \subseteq X$. Then $x \in A'$ if and only if for every open set $U \in \tau$ containing x, we have $U \cap A$ is an infinite set.*

Proof. The reverse implication is obvious by the very definition of A'. For the forward implication, let $x \in A'$ and let $U \in \tau$ be any open set containing x. Suppose for the sake of contradiction that $U \cap A$ is a finite set. Since (X, τ) is T1, finite subsets of X are closed. Thus, $(U \cap A)^c$ is open. Moreover, we have

$$x \in U \cap (U \cap A)^c, \quad (U \cap (U \cap A)^c) \cap A = \emptyset, \tag{1.34}$$

which is a contradiction since $x \in A'$. Hence, $U \cap A$ is an infinite set. ■

Let us look at some examples of Hausdorff spaces.

Example 1.40. If X is Hausdorff, then the subspace $Y \subseteq X$ with the subspace topology is also Hausdorff. This is because for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there exist open sets $U, V \in \tau$ such that $y_1 \in U$, $y_2 \in V$, and $U \cap V = \emptyset$. Thus, $y_1 \in U \cap Y$, $y_2 \in V \cap Y$, and $(U \cap Y) \cap (V \cap Y) = \emptyset$.

Example 1.41. For a space (X, τ), and a subset $A \subseteq X$, the *boundary* of A is defined as $\mathrm{Bd}(A) = \overline{A} \cap \overline{X \setminus A}$. Then $\mathrm{Bd}(A)$ and $\mathrm{Int}(A)$ are disoint; indeed, let $x \in \mathrm{Bd}(A) \cap \mathrm{Int}(A)$. Then there exists an open set $U \in \tau$ such that $x \in U \subseteq A$. Since $x \in \mathrm{Bd}(A)$, we have $x \in \overline{X \setminus A}$. Thus, for every open set $V \in \tau$ containing x, we have $V \cap (X \setminus A) \neq \emptyset$. In particular, taking $V = U$, we have $U \cap (X \setminus A) \neq \emptyset$, which is a contradiction since $U \subseteq A$. Hence, we must have $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \emptyset$.
Moreover, we have $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$. The reverse inclusion is trivial since $\mathrm{Bd}(A) \subseteq \overline{A}$ and $\mathrm{Int}(A) \subseteq A \subseteq \overline{A}$. For the forward inclusion, let $x \in \overline{A}$. If $x \in \mathrm{Bd}(A)$, then we are done. So suppose that $x \notin \mathrm{Bd}(A)$. Then $x \notin \overline{X \setminus A}$, which implies that there exists an open set $U \in \tau$ such that $x \in U$ and $U \cap (X \setminus A) = \emptyset$. Thus, we have $U \subseteq A$, which implies that $x \in \mathrm{Int}(A)$. Therefore, we have $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.
Finally, if $\mathrm{Bd}(A) = \emptyset$, then A is both open and closed. Indeed, since $\overline{A} = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$, we have $\overline{A} = \mathrm{Int}(A)$. Thus, A is closed. Also, since $\mathrm{Bd}(A) = \emptyset$, we have $\overline{X \setminus A} = \mathrm{Int}(X \setminus A)$. Thus, $X \setminus A$ is closed, which implies that A is open.

1.5 Product Spaces

For two metric spaces (X, d_X) and (Y, d_Y), we can define a metric on the product set $X \times Y$ as follows:

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2).\} \tag{1.35}$$

A basis of $X \times Y$ with the product metric is given by

$$\mathcal{B} = \{B_X(x, r) \times B_Y(y, r) \mid x \in X, y \in Y, r > 0\}. \tag{1.36}$$

In particular, if (X, τ_X) and (Y, τ_Y) are two topological spaces, then we can define a topology on $X \times Y$, called the *product topology*, by taking the basis

$$\mathcal{B} = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}. \tag{1.37}$$

Then, the topology $\tau_{X\times Y}$ generated by the basis $\mathcal{B}$ is called the product topology on $X \times Y$. Verify that $\tau_{X\times Y}$ is indeed a topology on $X \times Y$. Note that if $\tau_{X\times Y}$ was taken to be $\{U \times V \mid U \in \tau_X, V \in \tau_Y\}$, then it would not be a topology since it is not closed under arbitrary unions. Also worth noting is that if X and Y are both Hausdorff, then so is $X \times Y$ with the product topology.

Example 1.42. X is Hausdorff if and only if the diagonal set $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed in $X \times X$ with the product topology; suppose X is Hausdorff. Let $(x, y) \in (X \times X) \setminus \Delta$. Then $x \neq y$, and by the Hausdorff property, there exist open sets $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, we have

$$(U \times V) \cap \Delta = \emptyset, \quad (x, y) \in U \times V \subseteq (X \times X) \setminus \Delta, \tag{1.38}$$

which implies that $(X \times X) \setminus \Delta$ is open. Hence, Δ is closed. Conversely, suppose Δ is closed. Let $x, y \in X$ with $x \neq y$. Then $(x, y) \in (X \times X) \setminus \Delta$, which is open. Thus, there exists a basis element $U \times V$ such that

$$(x, y) \in U \times V \subseteq (X \times X) \setminus \Delta. \tag{1.39}$$

This implies that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Hence, X is Hausdorff.

Chapter 2

MAPPINGS

2.1 Continuous Functions

Recall the definition of a continuous function on $\mathbb{R}$; a function $f : \mathbb{R} \to \mathbb{R}$ was said to be continuous at a point $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta. \tag{2.1}$$

We can abstract this definition to metric spaces as follows.

Definition 2.1. A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be *continuous* at a point $x \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever } d_X(x, y) < \delta. \tag{2.2}$$

It is continuous at every point of X if it is continuous at each $x \in X$.

Also recall that a function $f : \mathbb{R} \to \mathbb{R}$ was continuous at $x \in \mathbb{R}$ if and only if for every open set $V \subseteq \mathbb{R}$ containing $f(x)$, the preimage $f^{-1}(V)$ is an open set in $\mathbb{R}$ containing x. We can use this characterization to define continuity in topological spaces.

Definition 2.2. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ between two topological spaces is said to be *continuous* at a point $x \in X$ if for every open set $V \in \tau_Y$ containing $f(x)$, the preimage $f^{-1}(V)$ is an open set in τ_X containing x. That is,

$$\forall V \in \tau_Y, f(x) \in V \implies x \in f^{-1}(V) \in \tau_X. \tag{2.3}$$

It is continuous at every point of X if it is continuous at each $x \in X$. That is,

$$\forall V \in \tau_Y \implies f^{-1}(V) \in \tau_X. \tag{2.4}$$

The above statement can be made more tight via bases: f is continuous if and only if the preimage of every basis element of Y is an open set in X.

Lemma 2.3. *A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is continuous if and only if $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, where τ_X and τ_Y are the topologies induced by the metrics d_X and d_Y respectively.*

Proof. Suppose $f : (X, d_X) \to (Y, d_Y)$ is continuous. Let $V \in \tau_Y$ be an open set in Y. Then, for every $y \in V$, there exists an $\varepsilon_y > 0$ such that $B_Y(y, \varepsilon_y) \subseteq V$. Since f is continuous, for every $x \in f^{-1}(V)$, there exists a $\delta_x > 0$ such that

$$f(B_X(x, \delta_x)) \subseteq B_Y(f(x), \varepsilon_{f(x)}) \subseteq V. \tag{2.5}$$

Thus, $B_X(x, \delta_x) \subseteq f^{-1}(V)$. Hence, $f^{-1}(V)$ is open in X and so $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous.

For the converse, suppose $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous. That is, for every open set $V \in \tau_Y$, the preimage $f^{-1}(V)$ is an open set in τ_X. Let $\varepsilon > 0$ be given. Consider the open ball $B_Y(f(x), \varepsilon) \in \tau_Y$. Since f is continuous, $f^{-1}(B_Y(f(x), \varepsilon))$ is open in X and contains x. Thus, there exists a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon))$. Hence, for every $y \in B_X(x, \delta)$, we have

$$f(y) \in B_Y(f(x), \varepsilon), \tag{2.6}$$

proving that $f : (X, d_X) \to (Y, d_Y)$ is continuous. ■

Lemma 2.4. *Let (X, τ) be a topological space and let $A \subseteq X$. Then the following hold.*

(i) For every sequence $(a_n)_{n \in \mathbb{N}} \subseteq A$, if $a_n \to x$ in X, then $x \in \overline{A}$. That is, if $x \in U$ for some open set $U \in \tau$, there exists a $N(U) \equiv N \in \mathbb{N}$ such that $a_n \in A \cap U$ for all $n \geq N$.

(ii) The converse holds if X is a metric space.

(iii) In general, the converse need not hold.

Proof. We prove the second statement only. Suppose $x \in \overline{A}$. Then $B(x, \frac{1}{n}) \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Thus, we can choose $a_n \in B(x, \frac{1}{n}) \cap A$ for each $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there exists a $N \equiv N(\varepsilon) \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Thus, for every $n \geq N$, we have

$$d(a_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon, \tag{2.7}$$

proving that $a_n \to x$ in X. ■

There is a third definition for metric spaces. A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be *continuous* at a point $x \in X$ if for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y. In fact, the open-set definition implies the sequential definition.

Proof. Assume the open-set definition of continuity. Let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence such that $x_n \to x$ in X. Let $\varepsilon > 0$ be given. Consider the open ball $B_Y(f(x), \varepsilon) \in \tau_Y$. Since f is continuous, $f^{-1}(B_Y(f(x), \varepsilon))$ is open in X and contains x. Thus, there exists a $\delta > 0$ such that $B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \varepsilon))$. Since $x_n \to x$ in X, there exists a $N \equiv N(\delta) \in \mathbb{N}$ such that for every $n \geq N$, we have

$$x_n \in B_X(x, \delta) \implies f(x_n) \in B_Y(f(x), \varepsilon), \tag{2.8}$$

proving that $f(x_n) \to f(x)$ in Y. ■

For metric spaces, the sequential definition also implies the open-set definition.

Example 2.5. Recall $\mathbb{R}_l$, which was generated from the basis $\mathcal{B} = \{[a, b) : a < b, a, b \in \mathbb{R}\}$. Consider the identity function $f : \mathbb{R} \to \mathbb{R}_l$, with $x \mapsto x$. Then $[0, 1)$ is open in $\mathbb{R}_l$, but its preimage under f is $[0, 1)$, which is not open in $\mathbb{R}$.

Theorem 2.6. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function between two topological spaces. Then the following are equivalent.*

(i) f is (open-set) continuous.

(ii) For every closed $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

(iii) For every $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

(iv) For all $f(x) \in V$ with $V \in \tau_Y$, there exists $U \in \tau_X$ such that $x \in U$ and $f(U) \subseteq V$.

January 20th.

Proof. We first show that (i) and (ii) are equivalent. Suppose C is closed in Y. Then, $Y \setminus C$ is open in Y. Since f is continuous, the preimage $f^{-1}(Y \setminus C)$ is open in X. But,

$$f^{-1}(Y \setminus C) = X \setminus f^{-1}(C). \tag{2.9}$$

Thus, $f^{-1}(C)$ is closed in X. One can follow the converse argument to show that (ii) implies (i).

For (i) implies (iii), let $A \subseteq X$ and let $x \in \overline{A}$. Let V be an open neighbourhood of $f(x)$ in Y. Since f is continuous, $f^{-1}(V)$ is an open neighbourhood of x in X. Thus, $f^{-1}(V) \cap A \neq \emptyset$. Let $a \in f^{-1}(V) \cap A$. Then, $f(a) \in V$ and $f(a) \in f(A)$. Since V was arbitrary, we must have $V \cap f(A) \neq \emptyset$. Thus, $f(x) \in \overline{f(A)}$, proving that $f(\overline{A}) \subseteq \overline{f(A)}$. For (iii) implies (ii), let V be a closed set in Y. Then

$$f^{-1}(V) \subseteq X \implies f(\overline{f^{-1}(V)}) \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} = V \tag{2.10}$$

which tells us $\overline{f^{-1}(V)} \subseteq f^{-1}(V)$, proving that $f^{-1}(V)$ is closed in X.

For (i) implies (iv), let $f(x) \in V$ with $V \in \tau_Y$. Since f is continuous, $f^{-1}(V)$ is open in X and contains x. Thus, we can take $U = f^{-1}(V)$, proving (iv). For (iv) implies (i), let $V \in \tau_Y$. For every $x \in f^{-1}(V)$, we have $f(x) \in V$. By (iv), there exists an open set $U_x \in \tau_X$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Thus, taking the union over all $x \in f^{-1}(V)$ gives $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$, proving that $f^{-1}(V)$ is open in X. ■

2.1.1 Rules of Continuous Functions

Note that the constant function $f : X \to Y$ defined by $f(x) = y_0$ for some fixed $y_0 \in Y$ is continuous. Also, the identity function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ is continuous.

Let X be a topological space, and A be a subset of X with the subspace topology. Then the inclusion map $i : A \hookrightarrow X$ defined by $i(a) = a$ is continuous. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions between topological spaces, then the composition $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ is also continuous.

Theorem 2.7. *Let $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions between topological spaces. Then the function $f : Z \to X \times Y$ defined by $f(z) = (f_1(z), f_2(z))$ is continuous if and only if both f_1 and f_2 are continuous.*

It is important to note that a function $f : X \to Y$ is continuousif and only if $f^{-1}(B)$ is open in X for every basis element B of Y.

Proof. For the forward implication, it is enough to show that $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in Z, where U and V are open sets in X and Y respectively. f is continuous implies that $f^{-1}(U \times V)$ is open in Z. In particular, both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in Z, proving that both f_1 and f_2 are continuous.

Conversely, suppose both f_1 and f_2 are continuous. Let $U \times V$ be a basis element of $X \times Y$, where U and V are open sets in X and Y respectively. By continuity of f_1 and f_2, both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in Z. Thus $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ is open in Z, proving that f is continuous. ■

Lemma 2.8. *Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Suppose $f(x) \neq 0$ for some $x \in X$. Then there exists $r > 0$ such that $f(y) \neq 0$ for all $y \in B(x, r)$. Moreover, the map $g : B(x, r) \to \mathbb{R}$ defined by $g(y) = \frac{1}{f(y)}$ is continuous.*

Proof. Suppose such an r does not exist. Then for every natural $n \in \mathbb{N}$, there exists a point $x_n \in B(x, \frac{1}{n})$ such that $f(x_n) = 0$. Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in X. Since f is continuous, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in $\mathbb{R}$. But $f(x_n) = 0$ for all $n \in \mathbb{N}$, so $(f(x_n))_{n\in\mathbb{N}}$ is the constant sequence 0, which converges to 0. Thus, we must have $f(x) = 0$, a contradiction. Hence, there exists $r > 0$ such that $f(y) \neq 0$ for all $y \in B(x, r)$.

To show continuity of g, let $(y_n)_{n\geq 1} \subseteq B(x, r)$ be a sequence such that $y_n \to y$ in $B(x, r)$. Since f is continuous, we have $f(y_n) \to f(y)$ in $\mathbb{R}$. Since $f(y) \neq 0$, there exists a natural $N \in \mathbb{N}$ such that for every $n \geq N$, we have $f(y_n) \neq 0$. Thus, for every $n \geq N$, we have $g(y_n) = \frac{1}{f(y_n)}$. Since the function $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $h(t) = \frac{1}{t}$ is continuous, we have $g(y_n) = h(f(y_n)) \to h(f(y)) = g(y)$ in $\mathbb{R}$, proving that g is continuous. ■

For a metric space (X, d), we define $C(X, \mathbb{R})$ to be the set of all continuous functions from X to $\mathbb{R}$. It can be easily seen that $C(X, \mathbb{R})$ is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication. Moreover, if we define multiplication of functions pointwise, then $C(X, \mathbb{R})$ is an algebra over $\mathbb{R}$. An example is any polynomial map $p : \mathbb{R}^n \to \mathbb{R}$; in such a case, $p \in C(\mathbb{R}^n, \mathbb{R})$.

If we denote $M_2(\mathbb{R})$ to be the set of all 2×2 real matrices, then matrices can be viewed as points in $\mathbb{R}^4$, and the same Euclidean topology can be induced on $M_2(\mathbb{R})$. Thus, we can consider continuous functions from $M_2(\mathbb{R})$ to $\mathbb{R}$. An example is the determinant function $\det : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $(a, b, c, d) \mapsto ad - bc$, which is a polynomial map and hence continuous. Since this is a continuous map, and $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$, the preimage $\det^{-1}(\mathbb{R} \setminus \{0\})$ is open in $M_2(\mathbb{R})$. But $\det^{-1}(\mathbb{R} \setminus \{0\})$ is precisely the set of all invertible 2×2 real matrices $GL_2(\mathbb{R})$. Thus, we have shown that $GL_2(\mathbb{R})$ is an open subset of $M_2(\mathbb{R})$. The set $SL_2(\mathbb{R})$ of all 2×2 real matrices with determinant 1, however, is closed in $M_2(\mathbb{R})$, since it is the preimage of the closed set $\{1\}$ under the continuous determinant function.

Example 2.9. Let us classify all $f : \mathbb{R} \to \mathbb{R}$ that are continuous and are also group homomorphisms. Since f is continuous, we only need to determine f $\mathbb{R}$ to $\mathbb{R}$ are of the form $f(x) = \lambda x$ for some fixedon the rationals. Let $\lambda := f(1)$. Then, for every $m \in \mathbb{Z}$, $f(m) = mf(1) = m\lambda$. For every $n \in \mathbb{N}$, we have $nf(\frac{m}{n}) = f(n \cdot \frac{m}{n}) = f(m) = m\lambda$, so $f(\frac{m}{n}) = \frac{m}{n}\lambda$. Thus, for every rational number q, we have $f(q) = q\lambda$. Since the rationals are dense in $\mathbb{R}$ and f is continuous, we must have $f(x) = x\lambda$ for all real numbers x. Thus, all continuous group homomorphisms from $\lambda \in \mathbb{R}$.

2.2 Product Topology

January 22nd.

The concept of the product topology can be extended to arbitrary products. Let $\{(X_n, \tau_n) \mid n \geq 1\}$ be a collection of topological spaces. Consider the Cartesian product

$$\prod_{n=1}^{\infty} X_n = \{(x_1, x_2, \ldots) \mid x_n \in X_n \text{ for all } n \geq 1\}. \tag{2.11}$$

The product topology on $\prod_{n=1}^{\infty} X_n$ is the topology generated by the basis

$$\mathcal{B} = \{U_1 \times U_2 \times \ldots \mid U_n \in \tau_n \text{ for all } n \geq 1,\ U_i \neq \emptyset \text{ for finitely many } i\}. \tag{2.12}$$

We require that $U_i \neq \emptyset$ for only finitely many i to ensure that this is still a basis. This is known as the box topology. If this restriction were not in place, then the fact that countable intersections of open sets may not be open would imply that this is not a topology. A second option is to use the basis

$$\mathcal{B} = \{U_1 \times U_2 \times \ldots \mid U_n \in \tau_n \text{ for all } n \geq 1,\ U_i = X_i \text{ for all but finitely many } i\}. \tag{2.13}$$

This is known as the product topology. Note that for finite products, both definitions coincide. In the infinite case, the product topology is more interesting than the box topology. The product topology also extends to a collection of topological spaces indexed by an arbitrary set, such as $\{(X_\alpha, \tau_\alpha) \mid \alpha \in \Lambda\}$ for some set Λ.

Lemma 2.10. *If $\{(X_n, \tau_n) \mid n \geq 1\}$ is a collection of Hausdorff topological spaces, then the product space $\prod_{n=1}^{\infty} X_n$ with the product topology is also Hausdorff.*

Proof. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two distinct points in $\prod_{n=1}^{\infty} X_n$. Then, there exists some $k \geq 1$ such that $x_k \neq y_k$. Since X_k is Hausdorff, there exist disjoint open sets $U_k, V_k \in \tau_k$ such that $x_k \in U_k$ and $y_k \in V_k$. Now, consider the open sets

$$U = X_1 \times X_2 \times \ldots \times U_k \times X_{k+1} \times \ldots, \tag{2.14}$$

and

$$V = X_1 \times X_2 \times \ldots \times V_k \times X_{k+1} \times \ldots. \tag{2.15}$$

Then, U and V are open in the product topology, and they are disjoint since U_k and V_k are disjoint. Moreover, $x \in U$ and $y \in V$, proving that $\prod_{n=1}^{\infty} X_n$ is Hausdorff. ∎

Lemma 2.11. *The function $f : Z \to \prod_{n=1}^{\infty} X_n$ defined by $f(z) = (f_1(z), f_2(z), \ldots)$ is continuous if and only if each component function $f_n : Z \to X_n$ is continuous for all $n \geq 1$. Here, $\prod_{n=1}^{\infty} X_n$ is equipped with the product topology.*

Proof. For the forward implication, let $n \geq 1$ be given. Consider an open set $U_n \in \tau_n$. Then, the set

$$U = X_1 \times X_2 \times \ldots \times U_n \times X_{n+1} \times \ldots \tag{2.16}$$

is open in the product topology. Since f is continuous, the preimage $f^{-1}(U)$ is open in Z. But,

$$f^{-1}(U) = f_n^{-1}(U_n), \tag{2.17}$$

proving that f_n is continuous.

Conversely, suppose each component function f_n is continuous for all $n \geq 1$. Let $U = U_1 \times U_2 \times \ldots$ be a basis element of the product topology, where $U_n \in \tau_n$ for all $n \geq 1$ and $U_i = X_i$ for all but finitely many i. Then,

$$f^{-1}(U) = \bigcap_{n=1}^{\infty} f_n^{-1}(U_n) = \bigcap_{i=1}^{k} f_i^{-1}(U_i), \tag{2.18}$$

where k is such that $U_i = X_i$ for all $i > k$. Since each f_i is continuous, each $f_i^{-1}(U_i)$ is open in Z. Thus, $f^{-1}(U)$ is open in Z, proving that f is continuous. ■

Note that the above lemma does not hold if the product topology is replaced by the box topology. Consider the function $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ defined by $f(x) = (x, x, x, \ldots)$. Each component function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x$ is continuous. However, f is not continuous when $\mathbb{R}^{\mathbb{N}}$ is equipped with the box topology. To see this, consider the open set $(-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \ldots$ in the box topology. The preimage under f is $\{0\}$, which is not open in $\mathbb{R}$. Thus, f is not continuous. However, the continuity of f implies the continuity of each component function f_n, so the converse implication still holds for the box topology.

January 23rd.

Consider $(\mathbb{R}, \bar{d})$ where $\bar{d}(x, y) = \min\{|x - y|, 1\}$. This metric generates the usual topology on $\mathbb{R}$. Then, on $\mathbb{R}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{R}$, we can consider the metric

$$\rho(x, y) = \sup_{n \geq 1} \bar{d}(x_n, y_n), \tag{2.19}$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. One can verify this is a metric. The topology generated by the metric space $(\mathbb{R}^{\mathbb{N}}, \rho)$ is termed the *uniform topology* on $\mathbb{R}^{\mathbb{N}}$. We now wish to determine the relationship between the uniform topology, product topology, and box topology on $\mathbb{R}^{\mathbb{N}}$. To determine this, we consider a few results.

Lemma 2.12. *$B_\rho(x, \varepsilon) \subsetneq \prod_{n=1}^{\infty}(x_i - \varepsilon, x_i + \varepsilon)$ for every $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ and $(1 >)\varepsilon > 0$.*

Proof. Let $y = (y_1, y_2, \ldots) \in B_\rho(x, \varepsilon)$. Then, by definition of ρ, we have $\bar{d}(x_n, y_n) < \varepsilon$ for all $n \geq 1$. Since $\varepsilon < 1$, this implies that $|x_n - y_n| < \varepsilon$ for all $n \geq 1$. Thus, $y \in \prod_{n=1}^{\infty}(x_i - \varepsilon, x_i + \varepsilon)$, proving that $B_\rho(x, \varepsilon) \subseteq \prod_{n=1}^{\infty}(x_i - \varepsilon, x_i + \varepsilon)$.

To see that the inclusion is strict, consider the point $z = (z_1, z_2, \ldots)$ defined by $z_i = x_i - \varepsilon + \frac{1}{n+i}$ for all $i \geq 1$, for $n >> 1$ large enough. Then, $z \in \prod_{n=1}^{\infty}(x_i - \varepsilon, x_i + \varepsilon)$, but $\sup_{i \geq 1} \bar{d}(x_i, z_i) = \varepsilon$, so $z \notin B_\rho(x, \varepsilon)$. Thus, the inclusion is strict. ■

Lemma 2.13. *On $\mathbb{R}^{\mathbb{N}}$, the product topology is strictly coarser than the uniform topology, which is strictly coarser than the box topology.*

Proof. We first show that $(x - \varepsilon, x + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$ is open in the uniform topology. Let $y = (y_1, y_2, \ldots) \in (x - \varepsilon, x + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$. Then, there exists a $\delta > 0$ such that $(y_1 - \delta, y_1 + \delta) \subseteq (x - \varepsilon, x + \varepsilon)$. Thus, $B(y, \delta) \subseteq (x - \varepsilon, x + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$, proving that this set is open in the uniform topology. Since basis elements of the product topology are finite intersections of such sets, they are also open in the uniform topology. Thus, the product topology is coarser than the uniform topology. ■

January 27th.

Theorem 2.14. *On $\mathbb{R}^{\mathbb{N}}$, with $D(x,y) := \sup_{n\geq 1} \frac{\bar{d}(x_n,y_n)}{n}$ where $\bar{d}(x,y) = \min\{|x-y|, 1\}$, the topology induced by D is the product topology.*

Proof. For the forward implication, we wish to show that given an open ball $B_D(x,\varepsilon)$, and $y \in B_D(x,\varepsilon)$, that there exists a set U open in the product topology such that $y \in U \subseteq B_D(x,\varepsilon)$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{4}$. Now we claim that

$$(x_1 - \frac{\varepsilon}{3}, x_1 + \frac{\varepsilon}{3}) \times (x_2 - \frac{\varepsilon}{3}, x_2 + \frac{\varepsilon}{3}) \times \cdots \times (x_{N+1} - \frac{\varepsilon}{3}, x_{N+1} + \frac{\varepsilon}{3}) \times \mathbb{R} \times \mathbb{R} \times \cdots \subseteq B_D(x,\varepsilon). \tag{2.20}$$

To this end, let $y = (y_1, y_2, \ldots)$ be in the left-hand side. Then for every $1 \leq i \leq N-1$, we have $|y_i - x_i| < \frac{2\varepsilon}{3}$, so $\bar{d}(x_i,y_i)/i \leq \bar{d}(x_i,y_i) < \frac{2\varepsilon}{3} < \varepsilon$. Moreover, for all $i \geq N$, $\bar{d}(x_i,y_i)/i \leq 1/i \leq 1/N < \frac{\varepsilon}{4} < \varepsilon$. Thus, $\sup_{i\geq 1} \bar{d}(x_i,y_i)/i = D(x,y) \leq \frac{2\varepsilon}{3} < \varepsilon$, proving the claim. Hence, we have found an open set in the product topology containing x and contained in U.

For the converse, let $U = \prod_{i\in\mathbb{N}} U_i$ be open in the product topology, where U_i is open in $\mathbb{R}$ for all $i \geq 1$ and $U_i = \mathbb{R}$ for all but finitely many i. It is enough to containment for these sets, since they form a basis for the product topology. Thus, we wish to show that given any $x \in U$, there exists an $\varepsilon > 0$ such that $B_D(x,\varepsilon) \subseteq U$. Let α denote an arbitrary index such that $U_\alpha \neq \mathbb{R}$ (there are only finitely many such indices). Let $x \in U$. Then $x_\alpha \in U_\alpha$, so there exists $\varepsilon_\alpha > 0$ such that $(x_\alpha - \varepsilon_\alpha, x_\alpha + \varepsilon_\alpha) \subseteq U_\alpha$. Now, let $\varepsilon = \min_\alpha \frac{\varepsilon_\alpha}{\alpha}$, where the minimum is taken over all indices α such that $U_\alpha \neq \mathbb{R}$. We claim that this ε works. To see this, let $y = (y_1, y_2, \ldots) \in B_D(x,\varepsilon)$. Then, for every index α such that $U_\alpha \neq \mathbb{R}$, we have

$$|y_\alpha - x_\alpha| \leq \bar{d}(x_\alpha, y_\alpha) < \alpha\varepsilon \leq \varepsilon_\alpha, \tag{2.21}$$

so $y_\alpha \in (x_\alpha - \varepsilon_\alpha, x_\alpha + \varepsilon_\alpha) \subseteq U_\alpha$. For all other indices β such that $U_\beta = \mathbb{R}$, we have $y_\beta \in U_\beta$ trivially. Thus, $y \in U$, proving that $B_D(x,\varepsilon) \subseteq U$. ■

The box topology, however, is not metrizable.

Lemma 2.15. *The box topology on $\mathbb{R}^{\mathbb{N}}$ is not metrizable.*

To show this, recall that in a metric space if A is a subset of a metric space (X,d), and $(a_n)_{n\geq 1}$ is a sequence in A that converges to some $x \in X$, then $x \in \overline{A}$. The converse also holds in metric spaces.

Proof. Let $A = \{(x_1, x_2, \ldots) \mid x_i > 0 \text{ for all } i \geq 1\} \subseteq \mathbb{R}^{\mathbb{N}}$ with the box topology. Note that A is open in the box topology. Now consider the point $x = (0,0,0,\ldots) \in \mathbb{R}^{\mathbb{N}}$. We claim that $x \in \overline{A}$. To see this, let $U = U_1 \times U_2 \times \ldots$ be an open neighbourhood of x in the box topology. Then, for each $i \geq 1$, there exists $\varepsilon_i > 0$ such that $(-\varepsilon_i, \varepsilon_i) \subseteq U_i$. Now, consider the point $y = (y_1, y_2, \ldots)$ defined by $y_i = \frac{\varepsilon_i}{2} > 0$ for all $i \geq 1$. Then, $y \in A \cap U$, proving that every open neighbourhood of x intersects A. Thus, $x \in \overline{A}$. However, there does not exist a sequence $(a_n)_{n\geq 1} \subseteq A$ such that $a_n \to x$ in the box topology. To see this, suppose such a sequence exists. Then, for each $i \geq 1$, consider the open neighbourhood $U^{(i)} = U_1^{(i)} \times U_2^{(i)} \times \ldots$ of x defined by

$$U_j^{(i)} = \begin{cases} (-\frac{1}{i}, \frac{1}{i}) & j = i, \\ \mathbb{R} & j \neq i. \end{cases} \tag{2.22}$$

Since $a_n \to x$ in the box topology, there exists a natural $N_i \in \mathbb{N}$ such that for every $n \geq N_i$, we have $a_n \in U^{(i)}$. In particular, this implies that for every $n \geq N_i$, the i-th coordinate of a_n satisfies $|(a_n)_i| < \frac{1}{i}$. Now, consider the sequence of natural numbers $(N_i)_{i\geq 1}$. Since this is an increasing sequence, we have $N_i \leq N_j$ for all $j \geq i$. Thus, for every $j \geq i$, we have $\left|(a_{N_j})_i\right| < \frac{1}{i}$. In particular, this implies that $(a_{N_j})_i \geq 0$ for all $j \geq i$, since $a_{N_j} \in A$. Thus, for each fixed $i \geq 1$, the sequence $((a_{N_j})_i)_{j\geq i}$ is a sequence of non-negative real numbers that converges to 0. However, this does not guarantee that the entire sequence $(a_n)_{n\geq 1}$ converges to x in the box topology, since the convergence must hold uniformly across all coordinates. This contradicts our assumption that such a sequence exists. Hence, there does not exist a sequence $(a_n)_{n\geq 1} \subseteq A$ such that $a_n \to x$ in the box topology, proving that the box topology is not metrizable. ■

2.3 Connectedness

The *distance from a set* A of a point x is defined as

$$d_A(x) := \inf\{d(x,a) \mid a \in A\}. \tag{2.23}$$

Note that $d_A(x) = 0$ if and only if $x \in \overline{A}$. Moreover, the map $\varphi : X \to \mathbb{R}$ defined by $\varphi(x) = d_A(x)$ is continuous.

Lemma 2.16. *Let (X,d) be a metric space, and $A, B \subseteq X$ be disjoint closed sets. Then there exists a continuous map $f : X \to [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Definition 2.17. For a topological space (X,τ), a *separation* of X is a pair of disjoint non-empty open sets $U, V \in \tau$ such that $X = U \cup V$. If no such separation exists, then X is said to be *connected*.

Note that X is connected if and only if the only non-empty clopen subset of X is X itself. For example, $\mathbb{Q}$ is not connected in $\mathbb{R}$ since $(-\infty, \sqrt{2}) \cap \mathbb{Q}$, $(\sqrt{2}, \infty) \cap \mathbb{Q}$ is a separation of $\mathbb{Q}$.

Lemma 2.18. *Suppose C and D form a separation of a topological space X. If Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.*

Proof. Suppose not. Then, there exist points $y_1, y_2 \in Y$ such that $y_1 \in C$ and $y_2 \in D$. Since C and D are open in X, the sets $C \cap Y$ and $D \cap Y$ are open in the subspace topology on Y. Moreover, they are disjoint and non-empty, and $Y = (C \cap Y) \cup (D \cap Y)$. Thus, $C \cap Y$ and $D \cap Y$ form a separation of Y, contradicting the connectedness of Y. Hence, either $Y \subseteq C$ or $Y \subseteq D$. ∎

Theorem 2.19. *Suppose X_α are connected subspaces of a topological space X for all α in some index set Λ. If $\bigcap_{\alpha\in\Lambda} X_\alpha \neq \emptyset$, then $\bigcup_{\alpha\in\Lambda} X_\alpha$ is connected.*

Proof. Let $x \in \bigcap_\alpha X_\alpha$. Suppose, for contradicion, that there exists a non-empty set $U \subseteq \bigcup_\alpha X_\alpha$ that is clopen in the subspace topology. That is, $U \cap X_\alpha$ is open and closed in X_α for all $\alpha \in \Lambda$. Since X_α is connected, we must have either $U \cap X_\alpha = \emptyset$ or $U \cap X_\alpha = X_\alpha$ for all $\alpha \in \Lambda$. But since $x \in \bigcap_\alpha X_\alpha$, we must have $x \in U$ or $x \notin U$. Thus, either $U = \emptyset$ or $U = \bigcup_\alpha X_\alpha$, proving that $\bigcup_\alpha X_\alpha$ is connected. ∎

Lemma 2.20. *Let $f : X \to Y$ be a continuous map. If X is connected, then $f(X)$ is also connected.*

Proof. Suppose not. Then, there exist disjoint non-empty open sets $U, V \subseteq Y$ such that $f(X) = U \cup V$. Since f is continuous, the preimages $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Moreover, they are disjoint and non-empty, and $X = f^{-1}(U) \cup f^{-1}(V)$. Thus, $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of X, contradicting the connectedness of X. Hence, $f(X)$ is connected. ∎

Definition 2.21. Two topological spaces X and Y are *homeomorphic* if there exists a bijective continuous map $f : X \to Y$ such that the inverse map $f^{-1} : Y \to X$ is also continuous. Such a map f is called a *homeomorphism*.

January 30th.

Theorem 2.22. *If X_i are connected topological spaces for all $1 \leq i \leq n$, then the product space $\prod_{i=1}^n X_i$ is also connected.*

Proof. Note that $X_1 \times \{b\}$ is connected for $b \in X_2$, since it is homeomorphic to X_1. Now, let $a \in X_1$ be arbitrary. Then, the set $\{a\} \times X_2$ is connected. Thus, the union $X_1 \times X_2$ is connected, since $(X_1 \times \{b\}) \cap (\{a\} \times X_2) = \{(a,b)\} \neq \emptyset$. Repeating this argument inductively, we have that $\prod_{i=1}^n X_i$ is connected. ∎

Theorem 2.23. *If $A \subseteq X$ is connected and $A \subseteq B \subseteq \overline{A}$, then B is also connected.*

Proof. Assume that B is disconnected. That is, there exists a clopen set U in B such that $B = U \cup U^c$. Without the loss of generality, suppose $\emptyset \neq U \cap A$ is clopen in A. Then $U \cap A = A$ or $A \subseteq U$. But then $\overline{A} \subseteq \overline{U} = U$, so $B \subseteq U$, a contradiction. Hence, B is connected. ■

We will now show that $\mathbb{R}^{\mathbb{N}}$ with the product topology is connected, but not with the box topology. With the product topology equipped, let $A_n = \{(x_1, \ldots, x_n, 0, \ldots) \mid x_i \in \mathbb{R}\}$. Then $\mathbb{R}^{\mathbb{N}} = \bigcup_{n=1}^{\infty} A_n$, and each A_n is homeomorphic to $\mathbb{R}^n$, which is connected. Moreover, $A_n \subseteq A_{n+1}$ for all $n \geq 1$, so by the previous theorem, $\mathbb{R}^{\mathbb{N}}$ with the product topology is connected. For the box topology, let $\mathbb{R}^{\mathbb{N}} = B \cup U$, where B is the set of bounded sequences, and U is the set of unbounded sequences. To show that B is open, let $x = (x_1, x_2, \ldots) \in B$. Then there exists $K > 0$ such that $|x_i| < K$ for all $i \geq 1$. Finally, $x_i \in (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \ldots$ is an open neighbourhood of x contained in B, proving that B is open. A similar argument shows that U is open. Since both B and U are non-empty, they form a separation of $\mathbb{R}^{\mathbb{N}}$ with the box topology, proving that it is not connected.

Definition 2.24. Let $a \neq b \in X$. We say that γ is a *path* between a and b if $\gamma : [0,1] \to X$ is continuous, with $\gamma(0) = a$ and $\gamma(1) = b$. If such a path exists, then a and b are said to be path connected. A topological space X is *path connected* if every pair of points in X are path connected.

Lemma 2.25. *Supose there exists a $a_0 \in X$ such that a_0 is path connected to every point in X. Then, X is path connected.*

Proof. Let $a, b \in X$ be arbitrary. By assumption, there exist paths $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$ such that $\gamma_1(0) = a_0$, $\gamma_1(1) = a$, $\gamma_2(0) = a_0$, and $\gamma_2(1) = b$. Now, define the map $\gamma : [0,1] \to X$ by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t-1) & \frac{1}{2} < t \leq 1. \end{cases} \tag{2.24}$$

Then, γ is continuous, and $\gamma(0) = a$, $\gamma(1) = b$. Thus, a and b are path connected. Since a, b were arbitrary, X is path connected. ■

Theorem 2.26. *If X is path connected, then X is connected.*

Proof. Suppose X is not connected. Then, there exists a separation U, V of X such that $U \cap V = \emptyset$. Let $a \in U$ and $b \in V$. Since X is path connected, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Consider the preimages $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. These sets are open in $[0,1]$, disjoint, non-empty, and their union is $[0,1]$. Thus, they form a separation of $[0,1]$, contradicting the connectedness of $[0,1]$. Hence, X is connected. ■

February 3rd.

Note that in the above proof we implicitly assumed the fact that $[0,1]$ is connected. To show that it is connected, we require the following lemma which is left as an exercise.

Lemma 2.27. *A topological space X is connected if and only if every continuous map $f : X \to \{\pm 1\}$ is constant.*

Theorem 2.28. *The interval $[0,1]$ is connected.*

Proof. Suppose $f : [0,1] \to \{\pm 1\}$ is continuous, and not constant. Then there exist $a, b \in [0,1]$ such that $f(a) = 1$ and $f(b) = -1$. Without loss of generality, assume $a < b$. By the intermediate value theorem, there exists some $c \in (a, b)$ such that $f(c) = 0$, contradicting the fact that the codomain of f is $\{\pm 1\}$. Thus, every continuous map $f : [0,1] \to \{\pm 1\}$ is constant, proving that $[0,1]$ is connected. ■

Alternate proof. Suppose $U \subseteq [0,1]$ is non-empty, open, and closed in the subspace topology. Without loss of generality, let $0 \in U$ (otherwise, take $V = U^c$). Now define $E = \{x \in [0,1] \mid [0,x] \subseteq U\}$. Since $0 \in U$, we have $0 \in E$, so E is non-empty. There now exists a natural $N \in \mathbb{N}$ such that $[0, \frac{1}{n}] \subseteq U$ for all $n \geq N$, so $\frac{1}{N} \in E$. Thus, E is bounded above by 1, so let $c = \sup E$. We claim that $c = 1$. Suppose not. Since U is closed in $[0,1]$, we have $c \in U$. Thus, there exists some $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \cap [0,1] \subseteq U$. But since $c = \sup E$, there exists some $x \in E$ such that $c - \varepsilon < x \leq c$. Then, $[0,x] \subseteq U$, and $(c-\varepsilon, c+\varepsilon) \cap [0,1] \subseteq U$ implies that $[0, c + \frac{\varepsilon}{2}] \subseteq U$, contradicting the fact that c is an upper bound for E. Thus, $c = 1$, so $[0,1] \subseteq U$, proving that $U = [0,1]$. Hence, $[0,1]$ is connected. ■

Example 2.29. We can use to show that $GL_n(\mathbb{R})$ is not connected. Note that the map $\det : GL_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is continuous. Since $\mathbb{R} \setminus \{0\}$ is not connected, $GL_n(\mathbb{R})$ is also not connected. Similarly, the hypersphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is connected for all $n \geq 1$, since the map $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by $f(x) = \frac{x}{\|x\|}$ is continuous, and $\mathbb{R}^{n+1} \setminus \{0\}$ is connected for all $n \geq 1$.

Example 2.30. We show that $\mathbb{R}$ and $\mathbb{R}^2$ cannot be homeomorphic. Suppose there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$. Let $a \in \mathbb{R}$ be arbitrary, and consider the set $\mathbb{R}^2 \setminus \{f(a)\}$. Note that this set is connected, since it is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$, which is connected. However, the set $\mathbb{R} \setminus \{a\}$ is not connected, since it can be separated into $(-\infty, a)$ and (a, ∞). But since f is a homeomorphism, the image of $\mathbb{R} \setminus \{a\}$ under f must also be disconnected, a contradiction. Thus, no such homeomorphism exists, proving that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

2.3.1 Components

On a topological space X, define an equivalence relation $\sim$ on X by $x \sim y$ if and only if there exists a connected subspace of X containing both x and y. The equivalence classes under this relation are called the *components* of X.

Theorem 2.31. *The components of X are connected, disjoint subspaces such that their union is X. Any connected subspace of X intersects only one component.*

Proof. Let C be a component of X. Let $x_0 \in C$ be arbitrary. Then, by definition of the equivalence relation, for every $x \in C$, there exists a connected subspace V_x of X such that $x_0, x \in V_x$. Thus, we have

$$C = \bigcup_{x \in C} V_x, \tag{2.25}$$

and since each V_x is connected and they all share the point x_0, by a previous theorem, C is connected. The components are disjoint simply by the definition of an equivalence relation, and their union is X since every point in X is contained in some component. Finally, suppose $A \subseteq X$ is connected, and intersects two components C_1 and C_2. Then, there exist points $a_1 \in A \cap C_1$ and $a_2 \in A \cap C_2$. Since A is connected, there exists a connected subspace of X containing both a_1 and a_2. Thus, by definition of the equivalence relation, we must have $C_1 = C_2$, proving that any connected subspace of X intersects only one component. ■

Thus, the components may even be called the connected components of X.

Theorem 2.32. *Let $U \subseteq \mathbb{R}^n$ be an open connected set. Then, U is path connected.*

Proof. Fix some $x_0 \in U$, and let A be the set of all points in U that are path connected to x_0. We wish to show that $A = U$. Note that A is non-empty since $x_0 \in A$. To show that A is open, let $x \in A$ be arbitrary. Since U is open, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Now, for any $y \in B(x, \varepsilon)$, the line segment from x to y lies entirely within $B(x, \varepsilon)$, so there exists a path from x to y. Since there also exists a path from x_0 to x, by concatenating these two paths, we have a path from x_0 to y. Thus, $y \in A$, proving that $B(x, \varepsilon) \subseteq A$. Hence, A is open. To show that A is closed, we look at $U \setminus A$. Let $x \in U \setminus A$ be arbitrary; there is no path from x_0 to x. Since U is open, there exists some $\delta > 0$ such that $B(x, \delta) \subseteq U$. Now, for any $y \in B(x, \delta)$, the line segment from x to y lies entirely within $B(x, \delta)$, so there exists a path from x to y. If there existed a path from x_0 to y, then by concatenating this path with the path from x to y, we would have a path from x_0 to x, contradicting our assumption. Thus, there is no path from x_0 to y, so $y \in U \setminus A$. Hence, $B(x, \delta) \subseteq U \setminus A$, proving that $U \setminus A$ is open. Since U is connected, and A is non-empty, open, and closed in U, we must have $A = U$. Thus, every point in U is path connected to x_0, so U is path connected. ■

More generally, we have the following result.

Theorem 2.33. *Let X be a topological space that is connected. If each point $x \in X$ has an open neighbourhood that is path connected, then X is path connected.*

From here, we say that X is locally connected at x_0 if every open neighbourhood of x_0 contains a connected open neighbourhood of x_0. If X is locally connected at every point $x \in X$, then X is said to be *locally connected*.

2.4 Compactness

Recall that $\{A_\alpha\}_{\alpha\in\Lambda}$ is an open *covering* of X if $\bigcup_{\alpha\in\Lambda} A_\alpha \supseteq X$. A subcover is a subset of the covering that still covers X.

Definition 2.34. A topological space X is *compact* if every open covering of X has a finite subcovering.

Theorem 2.35. *Every compact subset of a Hausdorff space is closed.*

Proof. Let $K \subseteq X$ be compact. We wish to show that K^c is open. Let $x_0 \in X \setminus K$. For each $y \in K$, since X is Hausdorff, there exist disjoint open neighbourhoods U_y of x_0 and V_y of y. Then $\{V_y\}_{y\in K}$ is an open covering of K, so there exists a finite subcovering $\{V_{y_i}\}_{i=1}^n$. Now, let $U = \bigcap_{i=1}^n U_{y_i}$. Then, U is an open neighbourhood of x_0, and for every i, we have $U \cap V_{y_i} = \emptyset$. Thus, $U \cap K = \emptyset$, proving that $x_0 \in K^c$ has an open neighbourhood contained in K^c. Since x_0 was arbitrary, K^c is open, so K is closed. ■

Lemma 2.36. *Let $F \subseteq K$ where F is closed and K is compact. Then, F is also compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of F. Then, $\{U_\alpha\}_{\alpha\in\Lambda} \cup \{K \setminus F\}$ is an open covering of K, so there exists a finite subcovering $\{U_{\alpha_i}\}_{i=1}^n \cup \{K \setminus F\}$. Since F is not covered by $K \setminus F$, we have that $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcovering of F. Thus, F is compact. ■

Theorem 2.37. *$[a, b]$ is compact for all $a < b \in \mathbb{R}$.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of $[a, b]$. Suppose it is not compact. Choose $a \le a_1 < b_1 \le b$ such that $I_1 = [a_1, b_1]$ has no finite subcovering, and $|b_1 - a_1| \le \frac{1}{2} |b - a|$. Now, choose $a_1 \le a_2 < b_2 \le b_1$ such that $I_2 = [a_2, b_2]$ has no finite subcovering, and $|b_2 - a_2| \le \frac{1}{2} |b_1 - a_1|$. Continuing this process, we have a sequence of nested closed intervals $I_n = [a_n, b_n]$ such that I_n has no finite subcovering for all $n \ge 1$, and $|b_n - a_n| \le \frac{1}{2^{n-1}} |b - a|$ for all $n \ge 1$. By the nested interval property, there exists some $c \in [a, b]$ such that $\sup a_k = c = \inf b_k$. Then $c \in U_{\alpha_0}$ for some $\alpha_0 \in \Lambda$, since $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open covering of $[a, b]$. Thus, there exists some $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U_{\alpha_0}$. However, there exists some $N \in \mathbb{N}$ such that for all $n \ge N$, we have $|b_n - a_n| < \varepsilon$, so $I_N \subseteq (c - \varepsilon, c + \varepsilon) \subseteq U_{\alpha_0}$. This contradicts the fact that I_N has no finite subcovering. Hence, $[a, b]$ is compact. ■

Lemma 2.38. *Let $f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.*

Proof. Let $f(K) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$ be an open covering of $f(K)$. Then, $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is an open covering of K, so there exists a finite subcovering $\{f^{-1}(U_{\alpha_i})\}_{i=1}^n$. Thus, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcovering of $f(K)$, proving that $f(K)$ is compact. ■

Theorem 2.39. *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We wish to show that the inverse map $f^{-1} : Y \to X$ is continuous. Let $V \subseteq X$ be closed. Then, since X is compact, V is also compact. Thus, by the previous lemma, $f(V)$ is compact. Since Y is Hausdorff, $f(V)$ is closed. Hence, the preimage of every closed set under f^{-1} is closed, proving that f^{-1} is continuous. Thus, f is a homeomorphism. ■

Theorem 2.40. *If X and Y are compact topological spaces, then the product space $X \times Y$ is also compact.*

Proof. Note that for $x_0 \in X$, $x_0 \times Y$ is homeomorphic to Y, so it is compact. We need to show that if N is an open set such that $x_0 \times Y \subset N \subset X \times Y$, then there exists a neighbourhood W of x_0 in X such that $W \times Y \subseteq N$. Let $X \times Y = \bigcup_\alpha A_\alpha$ be an open covering of $X \times Y$. By compactness of $x_0 \times Y$, there exists a finite subcovering $\bigcup_{i \in \Lambda_{x_0}} A_{\alpha_i}$ where Λ_{x_0} is a finite index set. Then $x_0 \times Y \subseteq W_{x_0} \times Y \subseteq \bigcup_{i \in \Lambda_{x_0}} A_{\alpha_i}$, where $W_{x_0} = \bigcap_{i \in \Lambda_{x_0}} \pi_X(A_{\alpha_i})$ is an open neighbourhood of x_0 in X. Now, for each $x \in X$, we can repeat this process to obtain an open neighbourhood W_x of x in X such that $W_x \times Y$ is covered by a finite number of sets from the open covering. Then, $\{W_x\}_{x \in X}$ is an open covering of X, so by compactness of X, there exists a finite subcovering $\{W_{x_j}\}_{j=1}^m$. Thus, we have $X \times Y = \bigcup_{j=1}^m (W_{x_j} \times Y)$, and each $W_{x_j} \times Y$ is covered by a finite number of sets from the original open covering. Hence, $X \times Y$ has a finite subcovering, proving that it is compact. ■

February 6th.

Definition 2.41. A collection $\mathcal{C}$ of subsets of a topological space X has the *finite intersection property* if for every finite subcollection $\{C_i\}_{i=1}^n \subseteq \mathcal{C}$, we have $\bigcap_{i=1}^n C_i \neq \emptyset$.

Theorem 2.42. *X is compact if and only if every collection $\mathcal{C}$ of closed subsets of X with the finite intersection property satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.*

Proof. Suppose X is compact. Let $\mathcal{C}$ be a collection of closed subsets of X with the finite intersection property but $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Then, $X = \bigcup_{C \in \mathcal{C}} C^c$, so there exists a finite subcovering $\{C_i^c\}_{i=1}^n$ of X. This implies that $\bigcap_{i=1}^n C_i = \emptyset$, contradicting the finite intersection property. Hence, $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. Conversely, assume that every collection $\mathcal{C}$ of closed subsets of X with the finite intersection property satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$, and let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of X with no finite subcovering. Then, $\{U_\alpha^c\}_{\alpha \in \Lambda}$ is a collection of closed subsets of X with the finite intersection property, since for every finite subcollection $\{U_{\alpha_i}^c\}_{i=1}^n$, we have $\bigcap_{i=1}^n U_{\alpha_i}^c = \left(\bigcup_{i=1}^n U_{\alpha_i}\right)^c \neq \emptyset$ (if we did have $\bigcap_{i=1}^n U_{\alpha_i}^c = \emptyset$, then $\{U_{\alpha_i}\}_{i=1}^n$ would be a finite subcovering of X, contradicting our assumption). Thus, there exists some $x \in X$ such that $x \in \bigcap_{\alpha \in \Lambda} U_\alpha^c$, contradicting the fact that $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open covering of X. Hence, every open covering of X has a finite subcovering, proving that X is compact. ■

Recall that for a subset A of a metric space (X, d), the distance of $x \in X$ from A is defined as $d_A(x) = \inf\{d(x, a) \mid a \in A\}$. The mapping $x \mapsto d_A(x)$ is continuous. We can also define the *diameter* of A, when it is bounded, as $\operatorname{diam}(A) = \sup\{d(a, b) \mid a, b \in A\}$.

Lemma 2.43 (*Lebesgue's number lemma*)**.** *Let $\mathcal{A}$ be an open convering of a compact metric space (X, d). Then there exists some $\delta > 0$ such that every subset of X with diameter less than δ is contained in some set $A \in \mathcal{A}$.*

Such a δ is called a *Lebesgue number* of the covering $\mathcal{A}$.

Proof. Let $X = \bigcup_{i=1}^k A_i$, with $A_i \in \mathcal{A}$ (assume that $A_i \neq X$, since it is trivial otherwise); this is possible since $\mathcal{A}$ is an open covering and X is compact. Define, for each x, $f(x) = \frac{1}{n} \sum_{i=1}^n d_{A_i^c}(x)$. Then, f is continuous, and $f(x) > 0$ for all $x \in X$. Since $f : X \to (0, \infty)$ and X is compact, $f(X)$ must be compact in $(0, \infty)$, so there exists some $\delta > 0$ such that $f(X) \subseteq [\delta, K]$. We claim that this δ is a Lebesgue number of $\mathcal{A}$. Let M be a subset of X with $\operatorname{diam}(M) = \delta$. Take any $m \in M$. Then $M \subseteq B(m, \delta)$. Hence, we must show that $B(m,\delta) \subseteq A_i$ for some i. That is, we wish to show that $d_{A_i^c}(m) \geq \delta$ for some i. Suppose not. Then

$$f(m) = \frac{1}{n}\sum_{i=1}^{n} d_{A_i^c}(m) < \frac{1}{n}\sum_{i=1}^{n} \delta = \delta, \tag{2.26}$$

which contradicts the fact that $f(X) \subseteq [\delta, K]$. Thus, there exists some i such that $d_{A_i^c}(m) \geq \delta$, so $B(m,\delta) \subseteq A_i$, proving that δ is a Lebesgue number of $\mathcal{A}$. ■

February 10th.

Theorem 2.44. *Suppose X is a non-empty and compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. Assume X is as such and countable. Then, we can write $X = \{x_1, x_2, \ldots\}$. Note that X cannot be finite. Start with x_1. Since X has no isolated points, there must exist some $y \neq x_1$. Via the Hausdorff property, there exist disjoint open neighbourhoods U_1 of x_1 and V_1 of y. Moreover, $x_i \notin \bar{U}_1$, since if it were so, it would contradict the fact that U_1 and V_1 are disjoint. We now claim that there exists an open $U_2 \subseteq U_1$ such that $x_2 \notin \bar{U}_2$. Let $x_2 \neq y \in U_1$. Then, there exist disjoint open neighbourhoods W_2 of y and W_1 of x_2. Now $U_2 = U_1 \cap W_2$ is open. Moreover, $x_2 \notin \bar{U}_2$, since if it were so, it would contradict the fact that W_1 and W_2 are disjoint. We can repeat this process to obtain a sequence of nested open sets $U_1 \supseteq U_2 \supseteq \ldots$ such that for each $n \geq 1$, we have $x_n \notin \bar{U}_n$. Rewriting, we have a sequence of nested closed sets $\bar{U}_1 \supseteq \bar{U}_2 \supseteq \ldots$. Also note that any finite intersection of these sets is non-empty, since $\bar{U}_n \supseteq \bar{U}_m$ for all $n < m$, and the space X is compact. Thus, by the finite intersection property, we have $\bigcap_{n=1}^{\infty} \bar{U}_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} \bar{U}_n$. Then, $x \neq x_n$ for all n, since if $x = x_n$ for some n, then $x \in \bar{U}_n$, contradicting the fact that $x_n \notin \bar{U}_n$. This contradicts the fact that $X = \{x_1, x_2, \ldots\}$, so X must be uncountable. ■

A similar problem is to show that if X is a non-empty compact Hausdorff space, and $\{A_n\}$ is a collection of closed sets in X such that $\operatorname{Int} A_n = \emptyset$ for all n, then $\bigcup_{n=1}^{\infty} A_n \neq X$. This can be shown by a similar argument to the above proof, where we construct a sequence of nested open sets $U_1 \supseteq U_2 \supseteq \ldots$ such that for each $n \geq 1$, we have $A_n \cap \bar{U}_n = \emptyset$. We can then use the finite intersection property to obtain some $x \in \bigcap_{n=1}^{\infty} \bar{U}_n$, and show that x cannot be contained in any of the sets A_n, contradicting the fact that $\bigcup_{n=1}^{\infty} A_n = X$. This is also known as the *Baire category theorem*.

2.4.1 Sequential Compactness

Definition 2.45. A topological space X is *sequentially compact* if every sequence in X has a convergent subsequence.

For example, $[a,b] \subseteq \mathbb{R}$ is sequentially compact for all $a < b \in \mathbb{R}$, since every sequence in $[a,b]$ has a subsequence that converges to some point in $[a,b]$ via the Bolzano-Weierstrass theorem.

Theorem 2.46. *Let (X,d) be a metric space. Then, X is compact if and only if X is sequentially compact.*

Proof of ($\Rightarrow$). Suppose X is compact. Let $\{x_n\}$ be a sequence in X. We wish to show that $\{x_n\}$ has a convergent subsequence. If $\{x_n\}$ has only finitely many distinct terms, then there exists some $x \in X$ such that $x_n = x$ for infinitely many n, so the constant subsequence $x, x, x, \ldots$ converges to x. Otherwise, $\{x_n\}$ has infinitely many distinct terms; assume that $\{x_n\}$ as a set has no limit point, since if it did, we would be done. Thus, $A = \{x_n\}_{n\geq 1}$ is closed in X. Since X is compact, A is also compact. Thus, there exist open sets $\{U_x\}_{x\in A}$ such that $U_x \cap A = \{x\}$ for all $x \in A$. Then, $\{U_x\}_{x\in A}$ is an open covering of A. In particular, $X \subseteq \bigcup_{x\in A} U_x \cup (X \setminus A)$, so there exists a finite subcovering $\{U_{x_i}\}_{i=1}^{k} \cup (X \setminus A)$. Since A is not covered by $X \setminus A$, we have that $\{U_{x_i}\}_{i=1}^{k}$ is a finite subcovering of A. But since A is infinite, there exists some x_n such that $x_n \in U_{x_i}$ for some i, contradicting the fact that $U_{x_i} \cap A = \{x_i\}$. Hence, $\{x_n\}$ must have a limit point, so there exists a subsequence of $\{x_n\}$ that converges to this limit point. Thus, X is sequentially compact. ■

For the converse, we will show it via a few lemmas.

Lemma 2.47. *Let X be a sequentially compact topological space. Then Lebegue's number lemma holds for X.*

Proof. Let $\mathcal{A}$ be an open covering of X. Suppose there is no Lebesgue number of $\mathcal{A}$. Then, for every $n \in \mathbb{N}$, there exists some subset D_n of X such that $\operatorname{diam}(M_n) < \frac{1}{n}$, but D_n is not contained in any set in $\mathcal{A}$. We may assume $D_n = B(x_n, 1/2n)$ for some $x_n \in X$. Since X is sequentially compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some point $x \in X$. Then, $x \in A$ for some $A \in \mathcal{A}$, since $\mathcal{A}$ is an open covering of X. Thus, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq A$. However, there exists some $N \in \mathbb{N}$ such that for all $k \geq N$, we have $\frac{1}{2n_k} < \varepsilon$, so $D_{n_k} = B(x_{n_k}, 1/2n_k) \subseteq B(x, \varepsilon) \subseteq A$, contradicting the fact that D_n is not contained in any set in $\mathcal{A}$ for all n. Hence, there exists a Lebesgue number of $\mathcal{A}$. ■

Lemma 2.48. *Let (X, d) be a sequentially compact metric space. Then, for all $\varepsilon > 0$, the open cover $\{B(x, \varepsilon) \mid x \in X\}$ has a finite subcover.*

Proof. Let $\varepsilon > 0$ be arbitrary. Suppose there is no finite subcover of $\{B(x, \varepsilon) \mid x \in X\}$. Then, for every $n \in \mathbb{N}$, there exists some $x_n \in X$ such that $x_n \notin \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$. Since X is sequentially compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some point $x \in X$. Then, there exists some $N \in \mathbb{N}$ such that for all $k \geq N$, we have $d(x, x_{n_k}) < \varepsilon$, so $x_{n_k} \in B(x, \varepsilon)$ for all $k \geq N$. However, since $\{x_n\}$ is a sequence of distinct points and $x_n \notin \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$ for all n, we have that $x_{n_k} \notin \bigcup_{i=1}^{N-1} B(x_i, \varepsilon)$ for all $k \geq N$, contradicting the fact that $x_{n_k} \in B(x, \varepsilon)$ for all $k \geq N$. Hence, there exists a finite subcover of $\{B(x, \varepsilon) \mid x \in X\}$. ■

We may prove the converse now.

Proof of ($\Leftarrow$). Suppose X is sequentially compact. Let $\mathcal{A}$ be an open covering of X. By Lebesgue's number lemma, there exists some $\delta > 0$ such that every subset of X with diameter less than δ is contained in some set in $\mathcal{A}$. By the previous lemma, there exists a finite subcover $\{B(x_i, \delta/2)\}_{i=1}^{n}$ of $\{B(x, \delta/2) \mid x \in X\}$. Since each $B(x_i, \delta/2)$ is contained in some set in $\mathcal{A}$, we have that $\mathcal{A}$ has a finite subcovering. Thus, X is compact. ■

2.4.2 Local Compactness and Compactification

Definition 2.49. A space X is called locally compact at $x \in X$, if there exists an open U and compact K such that $x \in U \subseteq K$. The space X is *locally compact* if it is locally compact at every point $x \in X$.

Note that if X is compact, then X is locally compact, since we can take $U = K = X$.

Theorem 2.50. *X is locally compact and Hausdorff if and only if there exists a topological space Y with $X \subseteq Y$ such that $Y \setminus X$ consists of a single point, and Y is compact and Hausdorff.*

Proof. Define $Y = X \cup \{\infty\}$, where ∞ is a symbol not in X. Let the topology on Y be defined as follows: the open sets of Y are exactly the open sets of X, together with the sets of the form $\{\infty\} \cup (X \setminus K)$ where K is a compact subset of X. Let the open sets of the X be of 'type 1', and the sets of the form $\{\infty\} \cup (X \setminus K)$ be of 'type 2'. The reader may verify that this is a valid topology on Y. We further claim that Y is compact and Hausdorff. We first show Hausdorff; if $x, y \in X \subseteq Y$, then Hausdorffness of X works. So, let $x \in X$ and $y = \infty$. Since X is locally compact at x, there exists an open U and compact K such that $x \in U \subseteq K$. Then, U and $\{\infty\} \cup (X \setminus K)$ are disjoint open neighbourhoods of x and ∞, respectively. Thus, Y is Hausdorff.

Now, we show compactness. Let $Y \subseteq \bigcup_{\alpha \in \Lambda_1} A_\alpha^{(1)} \cup \bigcup_{\alpha \in \Lambda_2} A_\alpha^{(2)}$ be an open covering of Y, where $A_\alpha^{(1)}$ are the type 1 open sets and $A_\alpha^{(2)}$ are the type 2 open sets. Since $\infty \in Y$, there exists some $\alpha_0 \in \Lambda_2$ such that $\infty \in A_{\alpha_0}^{(2)}$. Then, $A_{\alpha_0}^{(2)} = \{\infty\} \cup (X \setminus K)$ for some compact K. Thus, K is covered by $\{A_\alpha^{(1)}\}_{\alpha \in \Lambda_1}$, so there exists a finite subcovering $\{A_{\alpha_i}^{(1)}\}_{i=1}^{n}$ of K. Now, we have $Y = A_{\alpha_0}^{(2)} \cup \bigcup_{i=1}^{n} A_{\alpha_i}^{(1)}$, so there exists a finite subcovering of Y. Hence, Y is compact. ■

February 17th.

This movtivates the following definition.

Definition 2.51. Let X be a topological space, and Y a compact space. Y is termed a *compactification* of X if X is a dense subspace of Y.

In general, there are many ways to construct compactifications of a given space X. The simplest one is the *one-point compactification*, which we observed in the above proof; if X is locally compact and Hausdorff, then we can construct a compactification Y of X by adding a single point ∞ to X, and defining the topology on Y as exactly the open sets of X together with the sets of the form $\{\infty\} \cup (X \setminus K)$ where K is a compact subset of X.

Suppose $f : X_1 \to X_2$ is a continuous map between two locally compact Hausdorff spaces. Let X_1^+ and X_2^+ be the one-point compactifications of X_1 and X_2, respectively. Then, we can extend f to a continuous map $f^+ : X_1^+ \to X_2^+$ by defining $f^+(x) = f(x)$ for all $x \in X_1$, and $f^+(\infty) = \infty$. The reader may verify that this is a well-defined continuous map. One can also show that f^+ is continuous if and only if $f^{-1}(K)$ is compact for every compact $K \subseteq X_2$. Moreover, if f is a homeomorphism, then so is f^+.

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