ELECTRODYNAMICS

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Fourth Semester

Contents

Chapter 1

ELECTROSTATICS

1.1 Introduction

January 5th.

The bulk of classical mechanics was developed by Newton in the 17th century. It was mostly concerned with the motion and trajectories of macroscopic objects under the influence of forces. This theory, however, breaks down at very high speeds when v/c approaches unity. This necessitated the special theory of relativity by Einstein in 1905. Electrodynamics is a part of classical mechanics, but sits at the crossroads of both special relativity and quantum mechanics; it does sit in classical mechanics yet violates many Newtonian principles. The idea of Einstein's special relativity paper was born with these conflicts of electrodynamics with Newtonian mechanics.

Another area where classical mechanics fails is at very small length scales. The laws of classical mechanics are unable to explain phenomena at atomic and subatomic scales, leading to the development of quantum mechanics in the early 20th century. Quantum mechanics and special relaitivity were later unified into quantum field theory. Under this, electrodynamics was reformulated as quantum electrodynamics.

1.1.1 Forces

Every force in nature can be classified into four fundamental interactions:

- Gravitational force
- Electromagnetic force
- Strong force
- Weak force

The *strong force* is reponsible for holding the nucleus of an atom together, while the *weak force* is responsible for nuclear decay. They are both short-range forces, acting only at subatomic distances. The *gravitational force* is the weakest of the four fundamental forces, but it has an infinite range and acts attractively on all masses. The *electromagnetic force* is much stronger than gravity and also has an infinite range, but it can be both attractive and repulsive, acting on charged particles.

If we set two electrons one metre apart, the gravitational force between them is approximately 10^{-42} times weaker than the electromagnetic force. This stark contrast highlights the relative weakness of gravity compared to electromagnetism at the scale of elementary particles. The electromagnetic force has an important aspect in that it unifies electricity and magnetism, which were once thought to be separate phenomena.

1.1.2 Electrostatics

The field of *electrostatics* deals with the study of electric charges at rest. The goal is, given a set of source charges with their positions and magnitudes, to determine the force on a test charge with a given position and magnitude. Moreover, we want to determine the trajectory of the test charge. The principle

of superposition plays a crucial role in electrostatics, allowing us to calculate the net electric field or force by summing the contributions from individual charges. In general, both source and test charges can be in motion; the net force is not so simple to calculate in that case, since the individual forces also depends on the velocities of the charges and even their accelerations.

It is not sufficient to just know positions and velocities at present time; we also need to know them at an earlier time due to the fact that electromagnetic interactions "news" travel at a finite velocity. We simplify this situation greatly. First, we assume that all source charges are at rest at fixed positions; they are stationary. Second, we assume that any electromagnetic effects propagate instantaneously.

With these assumptions, we can focus on the electrostatic forces between charges. The fundamental law governing these interactions is *Coulomb's law*. Let there be a point charge q located at position $\mathbf{r}'$ and a test charge Q located at position $\mathbf{r}$. If $\boldsymbol{\imath} = \mathbf{r} - \mathbf{r}'$ is the displacement vector from the source charge to the test charge, then the force $\mathbf{F}$ on the test charge due to the source charge is given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\imath^2}\hat{\boldsymbol{\imath}}, \tag{1.1}$$

where ϵ_0 is the *permittivity of free space*, a fundamental constant with a value of approximately 8.854×10^{-12} C^2/Nm^2 in SI units. The unit vector $\hat{\boldsymbol{\imath}}$ points from the source charge to the test charge, indicating the direction of the force. The force is attractive if the charges have opposite signs and repulsive if they have the same sign. We may also rewrite $\mathbf{F}$ in terms of the *electric field* $\mathbf{E}$ defined by $\mathbf{F} = Q\mathbf{E}$. Thus, over a set of source charges q_i, the electric field at position $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\imath_i^2}\hat{\boldsymbol{\imath}}_i, \tag{1.2}$$

where $\boldsymbol{\imath}_i$ is defined similarly as above for each source charge. For a continuous distribution of source charges, typically represented by a linear density λ, surface density σ or volume density ρ, an integral appropriately replaces the summation. In this case, we consider $\mathrm{d}q = \lambda\,\mathrm{d}l$, $\mathrm{d}q = \sigma\,\mathrm{d}A$ or $\mathrm{d}q = \rho\,\mathrm{d}V$ as the infinitesimal charge elements, and integrate over the entire distribution to find the electric field at the point of interest. For example, for a volume charge distribution, the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathrm{d}q}{\imath^2}\hat{\boldsymbol{\imath}} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\imath^2}\hat{\boldsymbol{\imath}}\,\mathrm{d}V'. \tag{1.3}$$

1.2 Divergence of E: Gauss's Law and Electric Flux

The electric field $\mathbf{E}(\mathbf{r})$ is a *vector field*, meaning that at every point in space, it assigns a vector quantity. We can represent this field by assigning an 'arrow' at each point in space, where the direction of the arrow indicates the direction of the electric field vector, and the length of the arrow represents the magnitude of the field at that point. This gets messy. To visualise the electric field, we often use *field lines*, which are imaginary lines that represent the direction of the electric field. The density of these lines indicates the strength of the field; closer lines correspond to a stronger field. Field lines originate from positive charges and terminate on negative charges, providing a visual representation of how the electric field behaves in space.

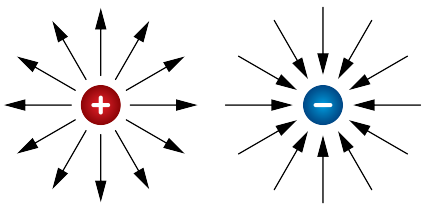

Figure 1.1: The electric field lines around a positive and a negative point charge.

Associated with the electric field is the concept of *electric flux*. Electric flux quantifies the amount of electric field passing through a given surface S. It is defined as

$$\Phi_E = \int_S \mathbf{E}\cdot\mathrm{d}\mathbf{a}, \tag{1.4}$$

where d**a** is an infinitesimal area element on the surface S, and the dot product $\mathbf{E} \cdot \mathrm{d}\mathbf{a}$ represents the component of the electric field passing through that area element. For example, let us consider a point charge Q at the origin and a spherical surface of radius R centred at the origin. We wish to find the electric flux through this spherical surface S. Converting to spherical coordinates, we have

$$\Phi_E = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{4\pi\epsilon_0} \int \frac{Q}{R^2} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{Q}{4\pi\epsilon_0} \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = \frac{Q}{\epsilon_0}. \tag{1.5}$$

Here, $\oint$ represents a surface integral over a closed surface. *Gauss's law* generalizes this result as

$$\Phi_E = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}, \tag{1.6}$$

where Q_{enc} is the total charge enclosed within the surface S. Using the divergence theorem, we have

$$\int_V \nabla \cdot \mathbf{E} \, \mathrm{d}V' = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \int_V \frac{\rho(\mathbf{r}')}{\epsilon_0} \, \mathrm{d}V'. \tag{1.7}$$

Since this holds for any arbitrary volume V, we obtain the differential form of Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{1.8}$$

January 7th.

In some sense, Coulomb's law is more fundamental than Gauss's law, despite both being equivalent. Coulomb's law follows from the inverse-square nature of the electric field, which in turn arises from the three-dimensional nature of space. If the electric field were to fall off with distance r in a different manner, say as $1/r^3$, then Gauss's law would not hold in its current form.

Let us now explicitly derive the *differential form of Gauss's law*. Note that we had

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\cancel{r^2}} \hat{\cancel{\mathbf{r}}} \, \mathrm{d}V'. \tag{1.9}$$

Taking the divergence of both sides, we have

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \nabla \cdot \left(\frac{\hat{\cancel{\mathbf{r}}}}{\cancel{r^2}} \right) \, \mathrm{d}V' \tag{1.10}$$

since the divergence operator acts on $\mathbf{r}$, not $\mathbf{r}'$ (a fixed quantity). Using the cartesian coordinates now gets messy quickly, so spherical coordinates are preferred; in these coordinates, the divergence of any vector field $\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$ is given by

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} A_\phi. \tag{1.11}$$

Applying this to $\mathbf{A} = \hat{\mathbf{r}} / |r|^2$, we find that $\nabla \cdot \left(\hat{\mathbf{r}} / |r|^2 \right) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \cdot \frac{1}{r^2}) = 0$ for $\mathbf{r} \neq 0$. Let us integrate this over the volume of a sphere of radius R centred at the origin:

$$\int_V \nabla \cdot \mathbf{A} \, \mathrm{d}V' = \oint_S \mathbf{A} \cdot \mathrm{d}\mathbf{a} = \oint_S \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a} = \int_0^{2\pi} \int_0^{\pi} \frac{1}{R^2} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = 4\pi. \tag{1.12}$$

This is a *contradictory* result, since we found that the divergence is zero everywhere except at the origin, so the integration should be zero. The resolution to this was provided by Dirac, who introduced the concept of the *Dirac delta function* $\delta(x)$. This is not a function in the traditional sense, but rather something known as a distribution. It is defined such that it is zero everywhere except at $x = 0$, where it is *infinitely* large, and its integral over the entire real line is equal to one:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) \, \mathrm{d}x = 1. \tag{1.13}$$

One may also note that $\int_{-\infty}^{\infty} \delta(x-a)\,\mathrm{d}x = 1$ for any real number a. If $f(x)$ is a 'well-behaved' function, then

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a). \tag{1.14}$$

The Dirac delta function can be generalised to higher dimensions; in three dimensions, it is defined as $\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. Using this, we can write

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}). \tag{1.15}$$

This generalizes to

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = 4\pi\delta^3(\boldsymbol{\imath}) \tag{1.16}$$

where the divergence is taken with respect to $\mathbf{r}$. Thus, the divergence of the electric field becomes

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)\,\mathrm{d}V' = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')4\pi\delta^3(\boldsymbol{\imath})\,\mathrm{d}V' = \frac{1}{\epsilon_0}\int \rho(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V' = \frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{1.17}$$

This recovers Gauss's law in differential form.

Common charge densities

The charge distributions we encounter in electrostatics are often highly symmetric, allowing us to exploit this symmetry to simplify calculations.

1. *Point charge*: A point charge is an idealized model of a charged particle with negligible size. The charge density for a point charge q located at the origin is simply given by

$$\rho(\mathbf{r}) = q\,\delta^3(\mathbf{r}). \tag{1.18}$$

2. *Dipole*: A dipole consists of two equal and opposite point charges separated by a small distance. For a point charge $+q$ at the origin and a point charge $-q$ at position $\mathbf{a}$, the charge density is given by

$$\rho(\mathbf{r}) = q\,\delta^3(\mathbf{r}) - q\,\delta^3(\mathbf{r}-\mathbf{a}). \tag{1.19}$$

3. *Hollow sphere*: Consider a uniformly charged thin spherical shell of radius R and total charge Q. The surface charge density σ is given by $\sigma = \frac{Q}{4\pi R^2}$. The volume charge density is then

$$\rho(\mathbf{r}) = \sigma\,\delta(r-R) = \frac{Q}{4\pi R^2}\,\delta(r-R). \tag{1.20}$$

   To derive this, we note that $\rho(\mathbf{r})$ must be singular at the surface of the sphere, so it must be proportional to $\delta(r-R)$. To find the proportionality constant, we integrate over all space to ensure the total charge is Q:

$$Q = \int \rho(\mathbf{r})\,\mathrm{d}V' = 4\pi\int_0^{\infty} r^2\,\rho(r)\,\mathrm{d}r = 4\pi k\int_0^{\infty} r^2\,\delta(r-R)\,\mathrm{d}r = 4\pi k R^2 \implies k = \frac{Q}{4\pi R^2}. \tag{1.21}$$

1.3 Curl of E: Electric Potential

Having discussed the divergence of the electric field, we now turn our attention to its curl; that is, we wish to compute $\nabla \times \mathbf{E}$. Consider the case of a point charge q at the origin. Stokes' theorem gives us

$$\int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{a} = \oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{1.22}$$

Plugging in the expression for $\mathbf{E}$, and using the fact that $\mathrm{d}\mathbf{l} = \mathrm{d}r\hat{\mathbf{r}} + r\,\mathrm{d}\theta\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\hat{\phi}$, let us integrate the line integral from $\mathbf{a}$ to $\mathbf{b}$ along some arbitrary path C:

$$\int_C \mathbf{E}\cdot\mathrm{d}\mathbf{l} = \frac{q}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{r^2}\hat{\mathbf{r}}\cdot\mathrm{d}\mathbf{l} = \frac{q}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}\,\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right). \tag{1.23}$$

This result is independent of the path taken between points $\mathbf{a}$ and $\mathbf{b}$. Therefore, for any closed loop C, the line integral evaluates to zero, that is, $\oint \mathbf{E}\cdot\mathrm{d}\mathbf{l} = 0$. Since this holds for any arbitrary surface S bounded by the loop C, we conclude that

$$\nabla\times\mathbf{E} = \mathbf{0}. \tag{1.24}$$

In fact, this result holds for any charge distribution and not just point charges. Note that this is a static situation; if charges were moving, then a time-varying magnetic field would be induced, leading to a non-zero curl of the electric field which we shall study later. We can also derive this result via computing the curl directly. Consider again a point charge q at the origin. In cartesian coordinates, the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^3} = \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}\right). \tag{1.25}$$

Calculating the curl in cartesian coordinates is relatively straightforward, albeit a bit tedious via the determinant method:

$$\nabla\times\mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}. \tag{1.26}$$

With this, one easily finds that $\nabla\times\mathbf{E} = \mathbf{0}$ for $\mathbf{r}\neq 0$. The singularity at the origin can be handled similarly as before using the Dirac delta function. Again, the result can be generalised to any (static) charge distribution by integrating over the entire distribution.

Since $\int \mathbf{E}\cdot\mathrm{d}\mathbf{l}$ was found to be independent of the path taken, we are fit to define a scalar function $V(\mathbf{r})$ known as the *electric potential* such that

$$V(\mathbf{r}) = -\int_O^{\mathbf{r}} \mathbf{E}\cdot\mathrm{d}\mathbf{l}, \tag{1.27}$$

where O is some standard point, often taken to be at infinity. This choice of reference point is arbitrary, as only the *potential difference* between two points is physically meaningful:

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_O^{\mathbf{b}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} + \int_O^{\mathbf{a}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot\mathrm{d}\mathbf{l}. \tag{1.28}$$

What is also true is that

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} \implies \mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}\right). \tag{1.29}$$

The advantage of working with the electric potential V instead of the electric field $\mathbf{E}$ is that V is a scalar function, making calculations often simpler. Note that this also implies V is arbitrary up to an additive constant, since adding a constant to V does not change $\mathbf{E}$. The principle of superposition also holds for electric potentials; the total potential due to a set of source charges is simply the algebraic sum of the potentials due to each individual charge. Thus, we have derived two fundamental properties of the electrostatic field:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \text{and} \quad \nabla\times\mathbf{E} = \mathbf{0}. \tag{1.30}$$

These are differential equations that govern the behaviour of the electrostatic field $\mathbf{E}$ in the presence of a charge distribution ρ. Using the relation $\mathbf{E} = -\nabla V$, we can rewrite the differential equations in terms of the electric potential V:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \quad \text{and} \quad \nabla\times\nabla V = \mathbf{0}. \tag{1.31}$$

Here, $\nabla^2 = \nabla \cdot \nabla$ is the Laplacian operator. The first equation is known as *Poisson's equation*, while the second is an identity that holds for any scalar function. If there are no charges present in a region, that is, $\rho = 0$, then Poisson's equation reduces to *Laplace's equation*, $\nabla^2 V = 0$.

January 12th.

Suppose $\mathbf{E} \sim \frac{1}{r^3}\hat{\mathbf{r}}$, that is, the electric field falls off as the cube of the distance from a point charge. Does it still hold true that $\nabla \times \mathbf{E} = \mathbf{0}$? Yes, since if a force only depends on the vector joining two points and not on the path taken, then the force must be conservative, and its curl must be zero, giving $q\nabla \times \mathbf{E} = \mathbf{0}$. However, Gauss's law would not hold in this case.

Let us take a look at how the potential V takes form. Start from a point charge q at the origin. Then

$$V = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int_{\infty}^{\mathbf{r}} \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\,\mathrm{d}r = \frac{q}{4\pi\epsilon_0 r}. \tag{1.32}$$

Thus, the potentia due to a point charge at $\mathbf{r}'$ is given by

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 \mathscr{r}}. \tag{1.33}$$

For a set of point charges q_i at positions $\mathbf{r}'_i$, the potential is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i}. \tag{1.34}$$

For a continuous distribution of charge with volume density $\rho(\mathbf{r}')$, the potential is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathrm{d}q}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}V'. \tag{1.35}$$

This is the formal solution to Poisson's equation.

One could compute the potential due to a uniformly charged spherical shell, but it is easier to use Gauss's law to find the electric field first, and then integrate to find the potential. Let us look at the case of a plate; consider a uniformly charged plate if radius R and surface charge density σ. The potential at a point along the axis of the plate at a distance z from its centre is given by

$$V(z) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}A' = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{r'\,\mathrm{d}r'\,\mathrm{d}\phi'}{\sqrt{r'^2+z^2}} = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - z\right). \tag{1.36}$$

Thus, the electric field is given by

$$E_z = -\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right). \tag{1.37}$$

1.3.1 Electrostatic Boundary Conditions

Let us look at the above example when the plate is infinite; consider an infinite plane with uniform surface charge density σ. By symmetry, the electric field must point directly away from the plane (if $\sigma > 0$) or towards the plane (if $\sigma < 0$). Using Gauss's law, we can find the magnitude of the electric field. Consider a cylindrical Gaussian surface that straddles the plane, with its flat faces parallel to the plane. The flux through the curved surface is zero since the electric field is perpendicular to it. The flux through the two flat faces (cyclindrical surface) is given by

$$\Phi_E = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \implies 2EA = \frac{\sigma A}{\epsilon_0} \implies E = \frac{\sigma}{2\epsilon_0}. \tag{1.38}$$

Note that there is a discontinuity in the electric field as we cross the plane. Just above the plane, the electric field is $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$, while just below the plane, it is $\mathbf{E} = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit normal vector pointing away from the plane.

Now consider an arbitrary surface with a (not necessarily uniform) surface charge density σ. Consider a similar cylindrical Gaussian surface that straddles the surface; this cylinder is infinitesimal, with height 2ϵ and cross-sectional area A. Here, we have

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{\sigma A}{\epsilon_0} \implies (E_\perp^{\text{above}} - E_\perp^{\text{below}})A = \frac{\sigma A}{\epsilon_0} \implies E_\perp^{\text{above}} - E_\perp^{\text{below}} = \frac{\sigma}{\epsilon_0}, \tag{1.39}$$

implying that the normal component of the electric field has a discontinuity across a surface charge. Here, $E_{\perp}^{\text{above}}$ and $E_{\perp}^{\text{below}}$ are the normal components of the electric field just above and just below the surface, respectively.

Through this surface, condier a small rectangular loop that pierces the surface; the loop has two sides parallel to the surface of length l and two sides perpendicular to the surface of height ϵ. We have

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0 \implies E_{\parallel}^{\text{above}} l - E_{\parallel}^{\text{below}} l = 0 \implies E_{\parallel}^{\text{above}} = E_{\parallel}^{\text{below}}, \tag{1.40}$$

implying that the tangential component of the electric field is continuous across the surface. Thus, we can conclude that

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}. \tag{1.41}$$

We discuss what happens to the potential V; since $\mathbf{E} = -\nabla V$, we have

$$-\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}_{\text{above}} \cdot \mathrm{d}\mathbf{l} + \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}_{\text{below}} \cdot \mathrm{d}\mathbf{l} = -\frac{\sigma}{\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \hat{\mathbf{n}} \cdot \mathrm{d}\mathbf{l} = -\frac{\sigma}{\epsilon_0}\epsilon + \frac{\sigma}{\epsilon_0}\epsilon = 0, \tag{1.42}$$

implying that

$$V_{\text{above}} - V_{\text{below}} = 0. \tag{1.43}$$

Thus, the electric potential is continuous across a surface charge, even though the electric field may be discontinuous. If we replace $\mathbf{E}$ with $-\nabla V$ in the earlier boundary condition for $\mathbf{E}$, we find that

$$\hat{\mathbf{n}} \cdot (-\nabla V_{\text{above}} + \nabla V_{\text{below}}) = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} \implies \left(-\frac{\partial V}{\partial n}\right)_{\text{above}} + \left(\frac{\partial V}{\partial n}\right)_{\text{below}} = \frac{\sigma}{\epsilon_0}. \tag{1.44}$$

1.4 Work and Energy

Recall that for a conservative force field, the work done in moving a particle from point $\mathbf{a}$ to point $\mathbf{b}$ is given by

$$W_{\mathbf{a}\to\mathbf{b}} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot \mathrm{d}\mathbf{l}. \tag{1.45}$$

Since the electrostatic force is conservative, for a test charge Q in an electric field $\mathbf{E}$, the work done by an external agent in moving the charge from point $\mathbf{a}$ to point $\mathbf{b}$ is given by

$$W_{\mathbf{a}\to\mathbf{b}} = -\int_{\mathbf{a}}^{\mathbf{b}} Q\mathbf{E} \cdot \mathrm{d}\mathbf{l} = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = Q(V(\mathbf{b}) - V(\mathbf{a})). \tag{1.46}$$

Thus, we define

$$W = QV. \tag{1.47}$$

Let us take up an example to illustrate this work done; let there be n point charges $q_1, q_2, \ldots, q_n$ in some configuration. We wish to find the work done in bringing these charges in from infinity to their respective positions. The work done in bringing in the first charge q_1 is zero, since there are no other charges present. The work done in bringing in the second charge q_2 is given by

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}. \tag{1.48}$$

Here, r_{12} is the distance between charges q_1 and q_2. The work done in bringing in the third charge q_3 is given by

$$W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right). \tag{1.49}$$

Continuing in this manner, the total work done in assembling the configuration of n charges is given by

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{1 \le j < i} \frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}}. \tag{1.50}$$

To write in terms of the electric potential, we have

$$W = \frac{1}{2} \sum_i q_i V(\mathbf{r}_i), \quad \text{where} \quad V(\mathbf{r}_i) = \sum_{j \neq i} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}}. \tag{1.51}$$

For a continuous charge distribution,

$$W = \frac{1}{2} \int \rho(\mathbf{r}) V(\mathbf{r}) \, \mathrm{d}V'. \tag{1.52}$$

From Gauss's law,

$$W = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V \, \mathrm{d}V'. \tag{1.53}$$

Here, we make use of the vector identity

$$\nabla \cdot (f\mathbf{A}) = (\nabla \cdot \mathbf{A}) f + \mathbf{A} \cdot (\nabla f). \tag{1.54}$$

This identity will be used throughout the course. Using this, we have

$$W = \frac{\epsilon_0}{2} \int \nabla \cdot (V\mathbf{E}) \, \mathrm{d}V' - \frac{\epsilon_0}{2} \int \mathbf{E} \cdot (\nabla V) \, \mathrm{d}V' = \frac{\epsilon_0}{2} \oint V\mathbf{E} \cdot \mathrm{d}\mathbf{a} + \frac{\epsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}V'. \tag{1.55}$$

The surface integral vanishes if we take the surface to be at infinity, since both $V\mathbf{E}$ will fall off sufficiently fast. Thus, we have

$$W = \frac{\epsilon_0}{2} \int E^2 \, \mathrm{d}V'. \tag{1.56}$$

January 14th.

Here, W may also be interpreted as the energy of the charge distribution. Note that the energy density (energy per unit volume) of the electrostatic field is given by

$$\frac{\epsilon_0}{2} E^2. \tag{1.57}$$

We have two expressions for the work done as $\frac{1}{2}\epsilon_0 \int E^2 \, \mathrm{d}V'$ and $\frac{1}{2} \sum_i q_i V(\mathbf{r}_i)$. The first of these expressions is always positive, while the second expression can be negative if there are opposite charges present. This is due to the fact that in the second expression, charges are taken as given; they do not take into account the energy required to assemble the charges themselves. On the contrary, in the first expression, $\rho(\mathbf{r})$ is treated as a continuous distribution, and the self-energy of assembling the charge distribution is included.

Another important feature is that since $W \propto E^2$, the energy does not obey the principle of superposition. That is, if we have two electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ due to two different charge distributions, then the total energy is not given by the sum of the individual energies but rather as

$$W = \frac{\epsilon_0}{2} \int (\mathbf{E}_1 + \mathbf{E}_2)^2 \, \mathrm{d}V' = \frac{\epsilon_0}{2} \int (E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2) \, \mathrm{d}V'. \tag{1.58}$$

1.4.1 Conductors and Induced Charges

The property of a *conductor* is that the electrons (and the positive holes) are free to move within the material. When a conductor is placed in an external electric field, the free charges rearrange themselves in such a way that the electric field inside the conductor is zero. This is because if there were a non-zero electric field inside the conductor, it would exert a force on the free charges, causing them to move until they reach an equilibrium state where the internal electric field is nullified. Thus, in electrostatic equilibrium, the electric field inside a conductor is zero.

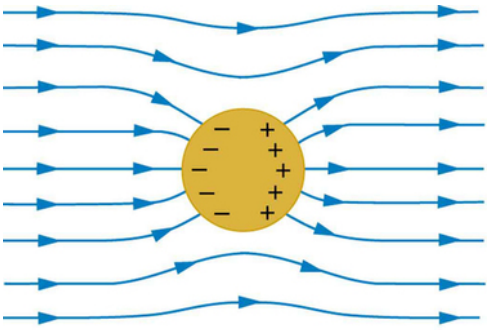

Figure 1.2: A conductor placed in an external electric field. Free charges within the conductor rearrange themselves to cancel the field inside the conductor.

That is, if the field applied was $\mathbf{E}_0$, then the induced field $\mathbf{E}_{\text{ind}}$ within the conductor is such that $\mathbf{E}_0 + \mathbf{E}_{\text{ind}} = \mathbf{0}$. Thus, $\mathbf{E} = \mathbf{0}$ within a conductor. This also implies that the charge density ρ is also zero within the conductor, since from Gauss's law, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$. Moreover, a conductor is an equipotential region; that is, the electric potential V is constant throughout the conductor. This is because if there were a potential difference between two points within the conductor, it would create an electric field that would cause charges to move, contradicting the assumption of electrostatic equilibrium. In mathematical terms,

$$V = -\int \mathbf{E} \cdot \mathrm{d}\mathbf{l} = V(\mathbf{b}) - V(\mathbf{a}) = 0 \implies V(\mathbf{a}) = V(\mathbf{b}). \tag{1.59}$$

Another property worth noting is that the electric field $\mathbf{E}$ just outside the surface of a conductor must be perpendicular to the surface: we know that V is constant on the surface. If we take $\mathrm{d}\mathbf{r}$ to be a small displacement along the surface, then

$$\nabla V \cdot \mathrm{d}\mathbf{r} = 0 \implies \nabla V \perp \mathrm{d}\mathbf{r}. \tag{1.60}$$

Now suppose we are dealing with a cavity within a conductor, as shown in Figure 1.3. If there are no charges within the cavity, then the electric field within the cavity is zero. This is because if there were a non-zero electric field within the cavity, it would create a potential difference between points on the surface of the cavity. However, since the surface of the cavity is part of the conductor, and the conductor is an equipotential region, this would lead to a contradiction. Thus, the electric field within the cavity must also be zero if there are no charges present within it.

However, let us suppose a point charge q is placed within the cavity.

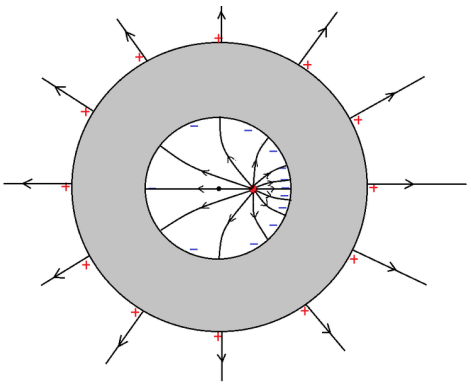

Figure 1.3: A conductor placed in an external electric field. Free charges within the conductor rearrange themselves to cancel the field inside the conductor.

The presence of this charge induces a redistribution of charges on the inner surface of the cavity, resulting in an induced charge distribution that ensures the electric field within the conductor remains zero. If we denote q_{ind} as the total induced charge on the inner surface of the cavity, we can determine its value using Gauss's law. Consider a Gaussian surface that lies just inside the conductor, enclosing the cavity and the point charge q. Since the electric field within the conductor is zero, the flux through this

Gaussian surface is also zero. According to Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{q + q_{\text{ind}}}{\epsilon_0} = 0 \implies q_{\text{ind}} = -q. \tag{1.61}$$

Thus, the total induced charge on the inner surface of the cavity is equal in magnitude but opposite in sign to the charge placed within the cavity. This induced charge distribution creates an electric field that cancels out the field produced by the charge q within the conductor, maintaining the condition of electrostatic equilibrium. If the Gaussian surface is taken just outside the conductor, then the total enclosed charge is q, since the conductor as a whole is electrically neutral. Thus, the electric field outside the conductor is as if the conductor were not present at all, and is completely independent of the location or shape of the cavity within the conductor.

Surface Charge and Force on a Conductor

January 19th.

Consider a conductor with a surface charge density σ. The electric field just outside the surface of the conductor is given by Gauss's law as

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}. \tag{1.62}$$

From here, one can see that

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}. \tag{1.63}$$

In the presence of an external electric field, the conductor experiences a force. The force per unit area (pressure) on the surface of the conductor due to the electric field is given by

$$\mathbf{f} = \sigma \mathbf{E}. \tag{1.64}$$

The questions is since the electric field is discontinuous across the surface of the conductor, which value of $\mathbf{E}$ should we use to compute the force? For a surface distribution, note that $\mathbf{E}_{\text{total}} = \mathbf{E}_{\text{patch}} + \mathbf{E}_{\text{other}}$, where $\mathbf{E}_{\text{patch}}$ is the field due to the surface charge patch itself, and $\mathbf{E}_{\text{other}}$ is the field due to all other charges. Thus, just above and below the surface of the conductor, we have

$$\mathbf{E}_{\text{total}}^{\text{above}} = \mathbf{E}_{\text{patch}}^{\text{above}} + \mathbf{E}_{\text{other}}, \quad \text{and} \quad \mathbf{E}_{\text{total}}^{\text{below}} = \mathbf{E}_{\text{patch}}^{\text{below}} + \mathbf{E}_{\text{other}}. \tag{1.65}$$

Noting that $\mathbf{E}_{\text{patch}}^{\text{above}} = \frac{1}{2\epsilon_0}\sigma\hat{\mathbf{n}}$ and $\mathbf{E}_{\text{patch}}^{\text{below}} = -\frac{1}{2\epsilon_0}\sigma\hat{\mathbf{n}}$, we have

$$\mathbf{E}_{\text{other}} = \frac{1}{2}\left(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}\right). \tag{1.66}$$

This argument shows that the effective electric field that exerts a force on the surface charge is the average of the fields just above and just below the surface; this argument applies to any surface charge distribution. So, since $\mathbf{E}_{\text{below}} = \mathbf{0}$ for a conductor, we have

$$\mathbf{f} = \sigma\left(\frac{1}{2}\mathbf{E}_{\text{above}}\right) = \frac{1}{2}\frac{\sigma^2}{\epsilon_0}\hat{\mathbf{n}}. \tag{1.67}$$

The force per unit area on the surface of a conductor due to the electric field is given by

$$P = \frac{1}{2}\frac{\sigma^2}{\epsilon_0} = \frac{1}{2}\epsilon_0 E^2. \tag{1.68}$$

1.5 Properties of Laplace's Equation

Let us now focus on Laplace's equation, $\nabla^2 V = 0$. This is a second-order partial differential equation. If this equation were one-dimensional, then

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = 0 \implies V(x) = mx + c, \tag{1.69}$$

where m and c are constants. To determine these constants, we need two boundary conditions, for example, the values of V at two different points. In such a case, $V(x)$ is the average of $V(x+a)$ and $V(x-a)$:

$$V(x) = \frac{1}{2}\left(V(x+a) + V(x-a)\right). \tag{1.70}$$

This solution has no local maxima or minima; the maximum and minimum values of V occur at the boundaries.

January 21st.

In three dimensions, we claim that these properties still hold. Let us suppose that there is a sphere of radius R centred at the origin, and a point charge q is placed at a distance of z along the z-axis from the origin. We wish to find the potential on the surface of the sphere due to this point charge. For an arbitrary point P on the surface of the sphere, we have

$$V_P = \frac{1}{4\pi\epsilon_0}\frac{q}{\ell} = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{R^2+z^2-2Rz\cos\theta}} \tag{1.71}$$

where θ is the angle between the z-axis and the line joining the origin to point P. The average potential on the surface of the sphere is given by

$$\begin{aligned} V_{\text{avg}} = \frac{1}{4\pi R^2}\oint V_P \,\mathrm{d}a &= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{R^2\sin\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}}\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{q}{8\pi\epsilon_0}\int_0^{\pi}\frac{\sin\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}}\,\mathrm{d}\theta = \frac{q}{4\pi\epsilon_0 R} \quad (\text{for } z<R). \end{aligned} \tag{1.72}$$

That is, V_{avg} is exactly V at the centre of the sphere. This can be generalized through superposition by adding more point charges. Again, there is no local maxima or minima of V; the maximum and minimum values of V occur at the boundaries. Since V can have no local minimum, an equilibrium configuration of charges cannot be stable; any small perturbation will cause the charges to move away from the equilibrium position. This is known as *Earnshaw's theorem*.

1.5.1 Boundary Conditions and Uniqueness

To solve Laplace's equation, we need to specify boundary conditions. The first uniqueness theorem is needed here.

Theorem 1.1 (The *first uniqueness theorem*)**.** *The solution to Laplace's equation in a region of volume $\mathscr{v}$ is uniquely determined by specifying V on the boundary surface S of the volume.*

Assume there are two possible distinct solutions V_1 and V_2 obeying the same boundary conditions. Then $\nabla^2 V_1 = 0 = \nabla^2 V_2$, and thus $\nabla^2(V_1 - V_2) = 0$. Let $U = V_1 - V_2$; then $\nabla^2 U = 0$ with $U = 0$ on the boundary surface S. Since extreme values can only occur on the boundary, and $U = 0$ on the boundary, we have $U = 0$ everywhere within the volume $\mathscr{v}$. Thus, $V_1 = V_2$ everywhere within the volume $\mathscr{v}$, proving uniqueness.

Suppose we have a hollow conductor with no charges inside it. The potential on the surface of the conductor is constant, say V_0. By the first uniqueness theorem, the potential within the hollow region is uniquely determined by the boundary condition $V = V_0$ on the surface of the conductor. A constant function $V = V_0$ within the hollow region satisfies Laplace's equation and the boundary condition. Thus, by uniqueness, $V = V_0$ everywhere within the hollow region, and thus $\mathbf{E} = -\nabla V = \mathbf{0}$ within the hollow region.

We now ask whether we can extend this to Poisson's equation $\nabla^2 V = -\frac{\rho}{\epsilon_0}$. Assume two solutions V_1 and V_2 satisfying the same boundary conditions. Then $\nabla^2(V_1 - V_2) = 0$. Letting $U = V_1 - V_2$, we have $\nabla^2 U = 0$ with $U = 0$ on the boundary surface S. By the same argument as before, we have $U = 0$ everywhere within the volume $\mathscr{v}$, proving uniqueness for Poisson's equation as well. The potential in a region $\mathscr{v}$ is uniquely determined if either ρ is specified within $\mathscr{v}$ and V is specified on the boundary surface S. This forms the basis for the second uniqueness theorem.

Theorem 1.2 (The *second uniqueness theorem*)**.** *In a volume $\mathscr{V}$ filled with charge density ρ, and containing conductors, the electric field $\mathbf{E}$ is uniquely determined by the total charge on the conductors.*

Let us assume two distinct solutions $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfying the same boundary conditions. Then $\nabla \cdot \mathbf{E}_1 = \frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E}_2$. With a Gaussian surface around the i-th conductor with charge Q_i, we have

$$\oint \mathbf{E}_1 \cdot \mathrm{d}\mathbf{a} = \frac{Q_i}{\epsilon_0} = \oint \mathbf{E}_2 \cdot \mathrm{d}\mathbf{a}. \tag{1.73}$$

Defining $\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$, we have $\nabla \cdot \mathbf{E} = 0$, and $\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = 0$ for any Gaussian surface around each conductor. Since conducting surfaces must be equipotential surfaces, we can conclude that the potential V, associated with $\mathbf{E}$, is (not necessarily the same) constant on each conductor. Taking the divergence of $V\mathbf{E}$, we have

$$\nabla \cdot (V\mathbf{E}) = V(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot (\nabla V) = V(0) + \mathbf{E} \cdot (-\mathbf{E}) = -E^2. \tag{1.74}$$

Integrating over the volume $\mathscr{V}$, we have

$$\int_{\mathscr{V}} \nabla \cdot (V\mathbf{E}) \, \mathrm{d}V' = \oint_S V\mathbf{E} \cdot \mathrm{d}\mathbf{a} = -\int_{\mathscr{V}} E^2 \, \mathrm{d}V'. \tag{1.75}$$

Since V is constant on S, and $\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = 0$, the left-hand side is zero. But the right-hand side is negative since $E^2 \geq 0$. Thus, we must have $\mathbf{E} = \mathbf{0}$ everywhere within the volume $\mathscr{V}$, proving uniqueness(?: proof incomplete).

1.5.2 The Method of Images

The *method of images* is a mathematical technique used to solve electrostatic problems involving conductors by replacing the conductors with imaginary charges (image charges) that replicate the boundary conditions imposed by the conductors. This method simplifies the problem by allowing us to work with point charges in free space rather than dealing with complex conductor geometries directly. Consider a point charge q located at a distance d above an infinite grounded conducting plane ($z = 0$). We wish to find the potential V at a point $P(x, y)$ in a region above the plane ($z \geq 0$). To solve this problem using the method of images, we introduce an imaginary charge $-q$ at a distance d below the plane and remove the conducting plane from the problem. The potential at point P due to both the real charge q and the image charge $-q$ is given by

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z + d)^2}}. \tag{1.76}$$

This potential satisfies Laplace's equation in the region above the plane and meets the boundary condition that $V = 0$ on the conducting plane ($z = 0$). Moreover, $V(x, y, z)$ approaches zero as $z \to \infty$, which is consistent with the physical expectation that the potential should vanish far away from the charges. Uniquness theorems guarantee that this is the correct solution to the original problem with the conducting plane. To find the charge density induced on the conduting plane, we have

$$\begin{aligned} \sigma(x, y) = \epsilon_0 E = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} &= \frac{1}{4\pi\epsilon_0}\left(\frac{-q(z - d)}{(x^2 + y^2 + (z - d)^2)^{3/2}} + \frac{q(z + d)}{(x^2 + y^2 + (z + d)^2)^{3/2}}\right) \\ &= -\frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}. \end{aligned} \tag{1.77}$$

January 28th.

Since we now have the charge density on the conducting plane, we can compute the total induced charge on the plane as

$$Q = \int \sigma(x, y) \, \mathrm{d}a = \int_0^{2\pi} \int_0^{\infty} -\frac{qd}{2\pi(r^2 + d^2)^{3/2}} r \, \mathrm{d}r \, \mathrm{d}\phi = -q. \tag{1.78}$$

This shows that the total induced charge on the conducting plane is equal in magnitude but opposite in sign to the original point charge q, which is consistent with the principle of conservation of charge.

We can also compute the electric field at any point above the conducting plane by taking the negative gradient of the potential:

$$E = -\frac{\partial V}{\partial z} = -\frac{1}{4\pi\epsilon_0}\left(-\frac{q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right). \tag{1.79}$$

Note that there is a singularity in the electric field at the location of the point charge q, as expected. Thus, to calculate the force on the point charge q due to the induced charges on the conducting plane, we drop the first term (self-force) and only consider the contribution from the image charge $-q$. The force on the point charge q (at $(0,0,d)$) is given by

$$\mathbf{F} = q\mathbf{E} = q\cdot\left(-\frac{1}{4\pi\epsilon_0}\frac{2qd}{((2d)^2)^{3/2}}\hat{\mathbf{z}}\right) = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{\mathbf{z}}. \tag{1.80}$$

This is the same force the point charge q would experience due to the image charge $-q$ located at $(0,0,-d)$. We have thus essentially ignored the sheet of induced charge and replaced it with the image charge $-q$ located at $(0,0,-d)$. The question now is whether one can repeat this argument for energies. The work done, in this argument, is given by

$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i) = \frac{1}{2}\left(-\frac{q^2}{(4\pi\epsilon_0)2d} - \frac{q^2}{(4\pi\epsilon_0)2d}\right) = -\frac{q^2}{8\pi\epsilon_0 d}. \tag{1.81}$$

This is *not* the correct answer, and is off by a factor of 2. We look at the alternate expression for work done:

$$W = \frac{\epsilon_0}{2}\int E^2\,\mathrm{d}V' \tag{1.82}$$

If we compute the work done in brining q from infinity to its position at $(0,0,d)$,

$$W = \int \mathbf{F}\cdot\mathrm{d}\mathbf{l} = \int_{-\infty}^{d}\frac{q^2}{4\pi\epsilon_0(2z)^2}\hat{\mathbf{z}}\cdot\mathrm{d}z\hat{\mathbf{z}} = -\frac{q^2}{16\pi\epsilon_0 d}. \tag{1.83}$$

Let us look at another image problem. Suppose we have a point charge q located at a distance a from the centre of a grounded conducting sphere of radius R (with $a > R$). We wish to find $V(\mathbf{r})$ for $r > R$. We place an image charge q' at a distance b from the centre of the sphere along the line joining the centre of the sphere to the point charge q, where

$$q' = -\frac{R}{a}q, \quad b = \frac{R^2}{a}. \tag{1.84}$$

For a point P outside the sphere, let $\not{z}'$ denote the distance from q' and $\not{z}$ denote the distance from q. The potential at point P due to both the real charge q and the image charge q' is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\not{z}} + \frac{q'}{\not{z}'}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right) \tag{1.85}$$

where θ is the angle between the line joining the centre of the sphere to point P and the line joining the centre of the sphere to the point charge q. Plugging in the values of q' and b, we get

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right). \tag{1.86}$$

One can verify that $V(R,\theta) = 0$, satisfying the boundary condition on the surface of the sphere. Uniqueness theorems guarantee that this is the correct solution to the original problem with the conducting sphere. Using $\sigma = -\epsilon_0\partial V/\partial n$, we have

$$\sigma(\theta) = -\epsilon_0\left.\frac{\partial V}{\partial r}\right|_{r=R}. \tag{1.87}$$

One can verify that

$$q_{\text{ind}} = \int\sigma(\theta)\,\mathrm{d}a = q' = -\frac{R}{a}q. \tag{1.88}$$

Let us now modify the problem slightly, by placing the conducting sphere at a potential V_0 instead of grounding it. One can solve this problem by simply adding another point charge q'' at the centre of the sphere, with $q'' = 4\pi\epsilon_0 R V_0$. Another variation of the problem is to maintain neutrality of the conducting sphere. This can be achieved by placing a charge $q'' = -q'$ at the centre of the sphere. The force between the point charge q and the neutral sphere is then

$$F = \frac{q}{4\pi\epsilon_0}\left(\frac{q'}{(a-b)^2} + \frac{q''}{a^2}\right). \tag{1.89}$$

1.5.3 Solving the Laplace Equation by Separation of Variables

February 2nd.

Consider Laplace's equation in Cartesian coordinates;

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1.90}$$

along with boundary conditions. Here, we seek solutions of the form $V(x,y,z) = X(x)Y(y)Z(z)$. First, we look at a specific example. Consider two infinite metal grounded plates at $y = 0$ and $y = a$, parallel to the xz-plane. Also $x = 0$ is closed off by an infinite strip of width a, maintained at a specific potential $V_0(y)$. We wish to obtain $V(x,y)$ in the space between the plates. This is essentially a two-dimensional problem due to symmetry in the z-direction. Thus, Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1.91}$$

subject to the boundary conditions $V(x,0) = 0 = V(x,a)$ for all $x \geq 0$, and $V(0,y) = V_0(y)$ for $0 \leq y \leq a$. Also, $V(x,y) \to 0$ as $x \to \infty$. We look for solutions of the form $V(x,y) = X(x)Y(y)$. Plugging this into Laplace's equation, we have

$$Y(y)\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + X(x)\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = 0 \implies \frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = k^2. \tag{1.92}$$

Thus, we have two ordinary differential equations

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - k^2 X = 0, \quad \text{and} \quad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + k^2 Y = 0. \tag{1.93}$$

A solution for $X(x)$ looks like $X = Ae^{kx} + Be^{-kx}$, and a solution for $Y(y)$ looks like $Y = C\sin ky + D\cos ky$. Since $V(x,y) \to 0$ as $x \to \infty$, we must have $A = 0$. Also, since $V(x,0) = 0$, we must have $D = 0$. The constraint $V(x,a) = 0$ implies that $\sin ka = 0$, giving $ka = n\pi$ for $n = 1, 2, 3, \ldots$. Finally, absorbing constants, we have

$$V(x,y) = Ce^{-kx}\sin\frac{n\pi}{a}y. \tag{1.94}$$

However, $V_0(y) = C\sin\frac{n\pi}{a}y$ is not general enough to represent an arbitrary function. Since each value of n gives a solution, we can superpose these solutions via linearity to obtain the general solution

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x}\sin\frac{n\pi}{a}y \tag{1.95}$$

with

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}y. \tag{1.96}$$

This is a Fourier sine series, where $\int_0^a \sin\frac{n\pi}{a}y\sin\frac{m\pi}{a}y\,\mathrm{d}y = \frac{a}{2}\delta_{nm}$. If we multiply both sides by $\sin\frac{m\pi}{a}y$ and integrate from 0 to a, we have

$$\int_0^a V_0(y)\sin\frac{m\pi}{a}y\,\mathrm{d}y = \sum_{n=1}^{\infty} C_n\int_0^a \sin\frac{n\pi}{a}y\sin\frac{m\pi}{a}y\,\mathrm{d}y = \frac{a}{2}C_m \implies C_m = \frac{2}{a}\int_0^a V_0(y)\sin\frac{m\pi}{a}y\,\mathrm{d}y. \tag{1.97}$$

For example, if $V_0(y) = V_0$ (a constant), then

$$\frac{2V_0}{n\pi}(1-\cos n\pi) = c_n = \begin{cases} \frac{4V_0}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \tag{1.98}$$

Thus, the potential between the plates is given by

$$V(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4V_0}{n\pi} e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y. \tag{1.99}$$

Now consider the case of two infinitely long grounded metal plates at $y = 0$ and $y = a$, connected at $x = \pm b$ by strips held at potential V_0. We wish to find $V(x,y)$ in the region between the plates. Here, the boundary conditions are $V(x,0) = 0 = V(x,a)$ and $V(-b,y) = V_0 = V(b,y)$. Via separation of variables $V(x,y) = X(x)Y(y)$, we get the solution $V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$. Applying the boundary conditions $V(x,0) = 0$ and $V(x,a) = 0$, we have $D = 0$ and $k = \frac{n\pi}{a}$ for $n = 1, 2, 3, \ldots$. We cannot use $A = 0$, but reflection about the origin gives $V(-x,y) = V(x,y)$, so $A = B$. Thus, letting $2A$ be replaced by A, we have

$$V(x,y) = A\cosh\frac{n\pi}{a}x\sin\frac{n\pi}{a}y. \tag{1.100}$$

or the general solution as

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cosh\frac{n\pi}{a}x\sin\frac{n\pi}{a}y. \tag{1.101}$$

Plugging $x = b$ or $x = -b$ gives

$$V_0 = \sum_{n=1}^{\infty} C_n \cosh\frac{n\pi}{a}b\sin\frac{n\pi}{a}y. \tag{1.102}$$

Multiplying both sides by $\sin\frac{m\pi}{a}y$ and integrating from 0 to a, and finally rearranging gives

$$C_m = \frac{2V_0}{a\cosh\frac{m\pi}{a}b}\int_0^a \sin\frac{m\pi}{a}y\,\mathrm{d}y = \frac{2V_0}{m\pi\cosh\frac{m\pi}{a}b}(1-\cos m\pi) = \begin{cases} \frac{4V_0}{m\pi\cosh\frac{m\pi}{a}b}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases} \tag{1.103}$$

February 9th.

Consider another example of an infinitely long rectangular pipe with sides a, b grounded but one end maintained at a potential $V_0(y,z)$. Here the boundary conditions are

$$V(x,0,z) = V(x,a,z) = V(x,y,0) = V(x,y,b) = 0, \quad V(0,y,z) = V_0(y,z), \quad V(x,y,z) \to 0 \text{ as } x \to \infty. \tag{1.104}$$

Via separation of variables $V(x,y,z) = X(x)Y(y)Z(z)$, the general solution turns out to be

$$V(x,y,z) = \sum_{n,m} C_{n,m} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}x}\sin\frac{n\pi}{a}y\sin\frac{m\pi}{b}z \tag{1.105}$$

We are left with the task of determining the coefficients $C_{n,m}$. Plugging in $x = 0$ gives

$$V_0(y,z) = \sum_{n,m} C_{n,m}\sin\frac{n\pi}{a}y\sin\frac{m\pi}{b}z. \tag{1.106}$$

Multiplying both sides by $\sin\frac{n'\pi}{a}y\sin\frac{m'\pi}{b}z$ and integrating from 0 to a and from 0 to b, we have

$$C_{n',m'} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin\frac{n'\pi}{a}y\sin\frac{m'\pi}{b}z\,\mathrm{d}y\,\mathrm{d}z. \tag{1.107}$$

1.5.4 Solving the Laplace Equation in Spherical Coordinates

We now look at problems dealing with spherical symmetry, rather than cartesian coordinates. The equation $\nabla^2 V = 0$ in spherical coordinates is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0. \tag{1.108}$$

Assuming azimuthal symmetry and separating variables $V(r,\theta) = R(r)\Theta(\theta)$, we have

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}V}{\mathrm{d}r}\right) + \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}V}{\mathrm{d}\theta}\right) = 0. \tag{1.109}$$

The first term depends only on r, and the second term depends only on θ. Thus, both terms must be equal to a constant, say $l(l+1)$. Thus, we have two ordinary differential equations

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}V}{\mathrm{d}r}\right) = l(l+1), \quad \text{and} \quad \frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}V}{\mathrm{d}\theta}\right) = -l(l+1). \tag{1.110}$$

The first of these has two linearly independent solutions r^l and $1/r^{l+1}$; the general solution for R is thus

$$R(r) = Ar^l + \frac{B}{r^{l+1}}. \tag{1.111}$$

For the second equation, we can make the substitution $x = \cos\theta$, giving

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}\Theta}{\mathrm{d}x} + l(l+1)\Theta = 0 \tag{1.112}$$

which is known as the Legendre equation, with solution given by $P_l(x)$ and $Q_l(x)$, the Legendre functions of the first and second kind respectively. The second solution blows up at $x = \pm 1$ at $\theta = 0$ and $\theta = \pi$, so we discard it. If we require l to be an integer, then $P_l(x)$ is a polynomial of degree l, known as the Legendre polynomial. For example, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, and so on. Rodrigues' formula gives a general expression for $P_l(x)$ as

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l. \tag{1.113}$$

If we impose the boundary condition that $V \to 0$ as $r \to \infty$, then we must have $A = 0$. Thus, the general solution to Laplace's equation in spherical coordinates with azimuthal symmetry is given by

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta). \tag{1.114}$$

For example, consider when $V(\theta)$ is specified on the surface of a hollow sphere of radius R. We fish to find V inside. Since V must be finite at the origin, we must have $B_l = 0$ for all l. Thus, the solution is given by

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{1.115}$$

Note that the Legendre polynomials are orthogonal with respect to the inner product $\int_{-1}^{1} P_l(x)P_{l'}(x)\,\mathrm{d}x = \frac{2}{2l+1}\delta_{l,l'}$. Thus we can determine the coefficients A_l by multiplying both sides by $P_{l'}(\cos\theta)$ and integrating from $\theta = 0$ to $\theta = \pi$, giving

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta. \tag{1.116}$$

If we further specify $V_0(\theta) = k\sin^2\frac{\theta}{2} = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}(P_0(\cos\theta) - P_1(\cos\theta))$, then we have

$$A_l = \frac{k(2l+1)}{4R^l}\int_0^{\pi}(P_0(\cos\theta) - P_1(\cos\theta))P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta = \begin{cases}\frac{k}{2}, & l=0,\\ -\frac{k}{2R}, & l=1,\\ 0, & l\geq 2.\end{cases} \tag{1.117}$$

Finally, the solution comes out to be

$$V(r,\theta) = \frac{k}{2} - \frac{kr}{2R}\cos\theta. \tag{1.118}$$

If we change the problem to finding $V(r,\theta)$ for $r \geq R$, then finiteness at infinity implies that $A_l = 0$ for all l, giving

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{1.119}$$

Everything else proceeds similarly.

Consider an uncharged metal sphere placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The sphere has to be an equipotential surface, so we set $V = 0$ on the surface of the sphere. Boundary conditions are $V = 0$ at $r = R$, $V = -E_0 z$ for $r \gg R$. We had obtained the general solution to Laplace's equation in spherical coordinates with azimuthal symmetry as $V(r,\theta) = \sum_l (A_l r^l + B_l/r^{l+1}) P_l(\cos\theta)$. The first boundary condition implies that $A_l R^l + B_l/R^{l+1} = 0$ for all l, giving $B_l = -A_l R^{2l+1}$. Thus,

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta), \quad -E_0 r\cos\theta = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \text{ for } r \gg R. \tag{1.120}$$

Since $P_1(\cos\theta) = \cos\theta$, we have $A_1 = -E_0$, and $A_l = 0$ for $l \neq 1$. Thus, the potential is given by

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{1.121}$$

February 11th.

We now wish to find the surface charge density $\sigma(R,\theta)$ on the sphere. This can be obtained via

$$\sigma(R,\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} = 3\epsilon_0 E_0 \cos\theta. \tag{1.122}$$

Note that this is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$, which is consistent with the fact that the electric field points in the positive z-direction.

1.6 Multipole Expansion, and the Dipole

Given a potential energy U, we can expand it via the Taylor expansion as

$$U(x) = U(x_0) + \left.\frac{\mathrm{d}U}{\mathrm{d}x}\right|_{x=x_0}(x-x_0) + \frac{1}{2}\left.\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\right|_{x=x_0}(x-x_0)^2 + \cdots \tag{1.123}$$

Note that if $\mathrm{d}U/\mathrm{d}x\big|_{x=x_0} = 0$, then the first non-zero term in the expansion is the second derivative term, and near this equilibrium point, one can approximate $U(x) \approx \frac{1}{2}kx^2$; any such potential would look like a harmonic oscillator near the equilibrium point. The idea is to use this expansion to approximate the potential at large distances from the source charge distribution.

Consider the case of a *dipole*, which consists of two point charges $+q$ and $-q$ separated by a distance d, say placed at $y = \pm d/2$ in the xy-plane. We wish to find the potential at a point P located at a distance r from the origin, making an angle θ with the positive z-axis. If $\mathscr{r}_+$ and $\mathscr{r}_-$ denote the distances from the positive and negative charges respectively, then the potential at P is given by

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right). \tag{1.124}$$

If $r \gg d$, we can rewrite $\mathscr{r}_\pm$ as

$$\mathscr{r}_\pm^2 = r^2 + \frac{d^2}{4} \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right). \tag{1.125}$$

Reciprocals gives

$$\frac{1}{\mathscr{r}_\pm} \approx \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \implies \frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \approx \frac{d\cos\theta}{r^2}. \tag{1.126}$$

Thus, the potential at P is approximately given by

$$V(\mathbf{r}) = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \tag{1.127}$$

where $\mathbf{p} = q\mathbf{d}$ is defined as the *dipole moment* of the dipole.

Now consider the case of a charged body. The potential, of course, is given by

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0\,\mathscr{r}}\,\mathrm{d}V' \tag{1.128}$$

where $\mathscr{r} = \mathbf{r} - \mathbf{r}'$. If θ' denotes the angle between $\mathbf{r}$ and $\mathbf{r}'$, then

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta' = r^2\left(1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta'\right). \tag{1.129}$$

Now define $\epsilon = (\frac{r'}{r})^2 - 2(\frac{r'}{r})\cos\theta'$, so that $\mathscr{r} = \sqrt{1+\epsilon}$. Thus, $\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} =$

$$\frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right) = \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' + \frac{r'^2}{r^2}\frac{3\cos^2\theta' - 1}{2} + \frac{r'^3}{r^3}\frac{5\cos^3\theta' - 3\cos\theta'}{2} + \cdots\right) \tag{1.130}$$

where the last expression comes from rearranging the terms in the Taylor expansion. One may notice that the Legendre polynomials $P_l(\cos\theta')$, with $P_0 = 1$, $P_1(\cos\theta') = \cos\theta'$, $P_2(\cos\theta') = \frac{1}{2}(3\cos^2\theta' - 1)$, and so on, appear in the expansion. In fact, one has

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta'). \tag{1.131}$$

With this, the potential at $\mathbf{r}$ can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\rho(\mathbf{r}')\,\mathrm{d}V'. \tag{1.132}$$

This is known as the *multipole expansion* of the potential. The first term in the expansion is called the monopole term, and is given by $\frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ where $Q = \int \rho(\mathbf{r}')\,\mathrm{d}V'$ is the total charge. Thus, even for a very complex body, the potential at large distances looks like that of a point charge Q. The second term is called the dipole term, where the potential drops off as $V \sim 1/r^2$. Similarly, the third term is called the quadrupole term, and the fourth term is called the octupole term, and so on. If we examine the dipole term, we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int r'\cos\theta'\rho(\mathbf{r}')\,\mathrm{d}V' = \frac{1}{4\pi\epsilon_0}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}V' = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} \tag{1.133}$$

where $\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}V'$ is defined as the dipole moment of the charge distribution. This definition is a little abstract, since there is no actual dipole in the charge distribution. In particular, the dipole moment $\mathbf{p}$ depends heavily on the geometry of the charge distribution for a collection of point charges. In such a case, it becomes

$$\mathbf{p} = \sum_i q_i\mathbf{r}_i' \tag{1.134}$$

since $\rho(\mathbf{r}') = \sum_i q_i\delta(\mathbf{r}' - \mathbf{r}_i')$. For example, for a dipole consisting of two point charges $+q$ and $-q$ separated by a distance d, we have $\mathbf{p} = q\mathbf{d}$, which is consistent with our previous definition of the dipole moment. Morevoer, the monopole moment does not depend on the choice of origin, but the dipole moment does.

A point charge at the origin represents a pure monopole. For a point charge shifted from the origin, say a charge q shifted a distance d along the positive y-axis, we have $Q = q$, $\mathbf{p} = qd\hat{\mathbf{y}}$.

We have obtained the expression

$$V(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}. \tag{1.135}$$

Thus, the electric field is given by $\mathbf{E} = -\nabla V$, which in spherical coordinates with azimuthal symmetry is given by

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}. \tag{1.136}$$

1.7 Electric Fields in Matter

February 16th.

Recall that a conductor, by definition, is a material that contains free charges that can move under the influence of an electric field. On the other hand, we have *dielectric*s or *insulator*s, which have bound charges. Upon applying an electric field $\mathbf{E}$ to a dielectric, the positive and negative charges in the dielectric are displaced in opposite directions, giving rise to an induced dipole moment. This phenomenon is known as polarization.

To solidify our theory, we work with a very crude model of an atom, consisting of a positive charge $+q$ as the nucleus, and a negative electron cloud with charge $-q$ surrounding the nucleus. The electron cloud is not a point charge, but rather a uniform distribution of charge on a sphere of radius a centred at the nucleus. Now suppose we apply an electric field $\mathbf{E}$ to this atom. Equilibrium conditions give $E_{\text{ext}} = E_e$, where E_{ext} is the external electric field, and E_e is the electric field due to the induced dipole moment. If the nucleus is displaced by a distance d from the centre of the electron cloud, then

$$E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} = E. \tag{1.137}$$

Note that the induced dipole moment is given by $\mathbf{p} = \alpha\mathbf{E}$, where α is called the *atomic polarizability*. In our crude model, we get $\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 \mathscr{v}$ where $\mathscr{v}$ is the volume of the electron cloud. Again, this is a very crude model, and assumptions such as the uniform distribution of charge on the electron cloud are not very realistic. In reality, the induced dipole moment may sometimes not even be aligned with $\mathbf{E}$, and polarizability depends on the direction of the applied electric field. Thus, α is actually a tensor, and we have $\mathbf{p} = \alpha\mathbf{E}$ where α is a 3×3 matrix. For now, we only work with scalar polarizability.

Now consider a dielectric material consisting of many atoms. We may model it as a collection of many tiny dipoles. We introduce the vector $\mathbf{P}$, called the *polarization*, which is defined as the dipole moment per unit volume. Recall that for a single dipole, the potential at a point $\mathbf{r}$ is given by $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$. For our case, we can generlize this to

$$V(\mathbf{r}) = \int_{\mathscr{v}} \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{P}}{4\pi\epsilon_0 \mathscr{r}^2}\,\mathrm{d}V'. \tag{1.138}$$

Of course, we can stop here, but we can also rewrite this in a more suggestive form. We know that

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{1.139}$$

Using $\mathscr{r} = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, we can verify this by direct differentiation. Note that $\nabla'(\frac{1}{\mathscr{r}}) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$, where ∇' denotes differentiation with respect to the primed coordinates. Using this, we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathscr{v}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)\,\mathrm{d}V' = \frac{1}{4\pi\epsilon_0}\int_{\mathscr{v}} \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)\,\mathrm{d}V' - \frac{1}{4\pi\epsilon_0}\int_{\mathscr{v}}\frac{\nabla'\cdot\mathbf{P}}{\mathscr{r}}\,\mathrm{d}V'. \tag{1.140}$$

Via the divergence theorem, we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\mathbf{P}}{\mathscr{r}}\cdot\mathrm{d}\mathbf{a} - \frac{1}{4\pi\epsilon_0}\int_{\mathscr{v}}\frac{\nabla'\cdot\mathbf{P}}{\mathscr{r}}\,\mathrm{d}V'. \tag{1.141}$$

Thus, the potential can be thought of as the potential due to a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ and a volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$. Finally, we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b}{\mathscr{r}} \, \mathrm{d}a + \frac{1}{4\pi\epsilon_0} \int_{\mathscr{v}} \frac{\rho_b}{\mathscr{r}} \, \mathrm{d}V'. \tag{1.142}$$

For an example, consider a uniformly polarized sphere of radius R with polarization $\mathbf{P} = P_0\hat{\mathbf{z}}$. The surface charge density is given by $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P_0 \cos\theta$, and the volume charge density is given by $\rho_b = -\nabla \cdot \mathbf{P} = 0$. Thus, the potential at a point for which $r \leq R$ is given by

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{1.143}$$

and for which $r \geq R$ is given by

$$V(r, \theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta). \tag{1.144}$$

Demanding continuity of V at $r = R$ gives $B_l = A_l R^{2l+1}$. Moreover, the discontinuity of the electric field at $r = R$ gives

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} \right)_{r=R} = -\frac{\sigma_b}{\epsilon_0} = -\frac{P_0 \cos\theta}{\epsilon_0}. \tag{1.145}$$

Solving via utilising the orthogonality of the Legendre polynomials gives $A_1 = \frac{P_0}{3\epsilon_0}$, and $A_l = 0$ for $l \neq 1$. Thus, the potential is given by

$$V(r, \theta) = \begin{cases} \dfrac{P_0 r \cos\theta}{3\epsilon_0}, & r \leq R, \\ \dfrac{P_0 R^3 \cos\theta}{3\epsilon_0 r^2}, & r \geq R. \end{cases} \tag{1.146}$$

Index