

TOPOLOGY

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Chapter 1

METRIC AND TOPOLOGICAL SPACES

1.1 Metric Spaces and Examples

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A *metric space* is a pair (X, d) where X is a set and $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is a function, called a *metric* on X , satisfying the following properties for all $x, y, z \in X$:

- (i) $d(x, y) = 0$ if and only if $x = y$ (*positive definiteness*).
- (ii) $d(x, y) = d(y, x)$ (*symmetry*).
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Let us look at some examples of metric spaces.

Example 1.1. Any set X can be made into a metric space by defining the *discrete metric* d as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \quad (1.1)$$

It is easy to verify that d satisfies all the properties of a metric.

Example 1.2. Recall that a normed space $(V, \|\cdot\|)$ was a vector space V equipped with a *norm* $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following properties for all $u, v \in V$ and $\alpha \in \mathbb{F}$:

- (i) $\|v\| = 0$ if and only if $v = 0$ (*positive definiteness*).
- (ii) $\|\alpha v\| = |\alpha| \|v\|$ (*absolute homogeneity*).
- (iii) $\|u + v\| \leq \|u\| + \|v\|$ (*triangle inequality*).

Given a normed space $(V, \|\cdot\|)$, we can define a metric d on V as follows:

$$d(u, v) = \|u - v\| \quad \forall u, v \in V. \quad (1.2)$$

Yet again, it is straightforward to verify that d satisfies all the properties of a metric. Given a vector space V , we can have multiple norms on it, and hence multiple metrics. For example, consider the vector space \mathbb{R}^n . We have the following norms on \mathbb{R}^n :

- The ℓ^1 norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$,
- the *Euclidean norm*: $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$,

- the *supremum norm*: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$,

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Each of these norms induces a different metric on \mathbb{R}^n .

The notion of open and closed balls is also abstracted to metric spaces as follows.

Definition 1.3. Let (X, d) be a metric space. The *open ball* of radius $r > 0$ centered at a point $x \in X$ is the set

$$B(x, r) = \{y \in X \mid d(x, y) < r\}, \quad (1.3)$$

and the *closed ball* of radius $r > 0$ centered at x is the set

$$B[x, r] = \{y \in X \mid d(x, y) \leq r\}. \quad (1.4)$$

Note that in the discrete metric space, $B(x, 1) = \{x\} = B(x, \frac{1}{2})$, and $B(x, 2) = X = B(y, 2)$ for any $x, y \in X$. Thus, $B(x, r) = B(y, \rho)$ does not imply that $x = y$ or $r = \rho$ in general.

Example 1.4. Let p be a prime, say $p = 3$. Define a function $|\cdot|_3 : \mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ as follows: for any non-zero integer m , write $m = 3^k m'$ where m' is not divisible by 3, and set $|m|_3 = 3^{-k}$. Also, set $|0|_3 = 0$. This function $|\cdot|_3$ is called the 3-adic absolute value on \mathbb{Z} . In general, for any prime p , the *p-adic absolute value* is defined similarly.

This 3-adic absolute value induces a norm d_3 on \mathbb{Q} as follows:

$$|q|_3 = \begin{cases} 0 & \text{if } q = 0, \\ |m|_3 / |n|_3 & \text{if } q = m/n \text{ in lowest terms.} \end{cases} \quad (1.5)$$

This induces a metric on \mathbb{Q} defined by $d_3(x, y) = |x - y|_3$ for all $x, y \in \mathbb{Q}$. This metric space (\mathbb{Q}, d_3) is called the 3-adic metric space, and in general (\mathbb{Q}, d_p) is called the *p-adic metric space*. The completion of (\mathbb{Q}, d_p) gives us the *field of p-adic numbers*, denoted by \mathbb{Q}_p . This metric space has some interesting properties; for instance, the triangle inequality is strengthened to the *ultrametric inequality*:

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\} \quad \forall x, y, z \in \mathbb{Q}. \quad (1.6)$$

Lemma 1.5 (Hausdorff property). Let (X, d) be a metric space. For any distinct $x, y \in X$, there exists $r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$.

Proof. Verify that choosing any $r \leq \frac{1}{2}d(x, y)$ works. ■

Let (X, d) be a metric space. Then a subset $A \subseteq X$ can also be made into a metric space by restricting the metric d to $A \times A$. In the metric space $(A, d|_{A \times A})$, the open balls are given by $B_A(x, r) = B_X(x, r) \cap A$ for all $x \in A$ and $r > 0$, where $B_X(x, r)$ is the open ball in (X, d) .

Again, as before, the notion of open sets is abstracted to metric spaces as follows.

Definition 1.6. Let (X, d) be a metric space. A subset $U \subseteq X$ is said to be an *open set* if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

As a small lemma, one can show that every open ball in a metric space is an open set. As an exercise, show that the complement of the closed ball $B[x, r]^c = \{y \mid d(x, y) > r\}$ is also an open set.

Proposition 1.7. Let (X, d) be a metric space. Let $\tau = \{U \subseteq X \mid U \text{ is open}\}$, that is, the collection of all open sets in X . Then the following hold true.

- (i) $\emptyset, X \in \tau$.
- (ii) For $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$, we have $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$. That is, an arbitrary union of open sets is open.

(iii) For $U_1, U_2, \dots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of open sets is open.

Proof. The proof of the first property is trivial. For the second property, let $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$. Then there exists some $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$. Thus, $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open.

For the third property, let $x \in \bigcap_{i=1}^n U_i$. Then $x \in U_i$ for all $1 \leq i \leq n$. Since each U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for all $1 \leq i \leq n$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then we have

$$B(x, r) \subseteq B(x, r_i) \subseteq U_i \quad \forall 1 \leq i \leq n, \quad (1.7)$$

which implies that $B(x, r) \subseteq \bigcap_{i=1}^n U_i$. Thus, $\bigcap_{i=1}^n U_i$ is open. ■

1.2 Topological Spaces and Examples

A *topological space* is a pair (X, τ) where X is a set and τ is a collection of subsets of X satisfying the following properties:

- (i) $\emptyset, X \in \tau$.
- (ii) For $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau$, we have $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$. That is, an arbitrary union of sets in τ is in τ .
- (iii) For $U_1, U_2, \dots, U_n \in \tau$, we have $\bigcap_{i=1}^n U_i \in \tau$. That is, a finite intersection of sets in τ is in τ .

These are the exact same properties that the collection of open sets in a metric space satisfy. Hence, every metric space (X, d) gives rise to a topological space (X, τ_d) where τ_d is the collection of all open sets in (X, d) . Such a topology τ_d is called the topology induced by the metric d .

As a smaller example, let $X = \{0, 1, 2, 3, 4\}$ and consider the collection $\tau = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}\}$. Then the pair (X, τ) is *not* a topological space since $\{0, 1\} \cup \{2, 4\} = \{0, 1, 2, 4\} \notin \tau$. However, the pair (X, τ') where $\tau' = \{\emptyset, X, \{0\}, \{0, 1\}, \{2, 4\}, \{0, 1, 2, 4\}\}$ is a topological space.

Description of open sets in \mathbb{R}

Theorem 1.8. A non-empty open set in \mathbb{R} is a countable union of pairwise disjoint open intervals.

Proof. Let $U \subseteq \mathbb{R}$ be a non-empty open set. For each $x \in U$, define

$$I_x = \bigcup \{(a, b) \mid x \in (a, b) \subseteq U\}. \quad (1.8)$$

Note that $x \in I_x \subseteq U$. Let $a_x = \inf I_x$ and $b_x = \sup I_x$. We claim that $I_x = (a_x, b_x)$. For $a_x < z < b_x$, there exists $a, b \in I_x$ such that $a_x < a < z < b < b_x$. Since $z \in (a, b) \subseteq I_x$, we have $z \in I_x$. Thus, $(a_x, b_x) \subseteq I_x$. The other inclusion is trivial. Hence, $I_x = (a_x, b_x)$ is an open interval.

We now claim that if $x \neq y$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Suppose that $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is an interval containing both x and y and contained in U . By the definition of I_x and I_y , we have $I_x \cup I_y \subseteq I_x$ and $I_x \cup I_y \subseteq I_y$. Thus, $I_x = I_y$.

Finally, let $U = \bigcup_{x \in U} I_x$. By the above claim, the collection $\{I_x \mid x \in U\}$ consists of pairwise disjoint open intervals. Since each I_x contains a rational number (by the density of \mathbb{Q} in \mathbb{R}), for each I_x , we can choose a distinct rational number $q_x \in I_x$. This gives $I_x = I_{q_x}$. Thus, we have

$$U = \bigcup_{x \in U} I_x = \bigcup_{q \in \mathbb{Q} \cap U} I_q, \quad (1.9)$$

which is a countable union of pairwise disjoint open intervals. ■

Definition 1.9. Let (X, d_1) and (X, d_2) be two metric spaces on the same set X . The metrics d_1 and d_2 are said to be *equivalent metrics*, $d_1 \sim d_2$, if open sets in (X, d_1) are exactly the open sets in (X, d_2) .

January 8th.

Definition 1.10. Let $A \subseteq X$ be a subset of a metric space (X, d) . The *interior* of A , denoted by $\text{Int}(A)$, is defined as

$$\text{Int}(A) := \{x \in A \mid \exists r > 0 \text{ such that } B(x, r) \subseteq A\}. \quad (1.10)$$

For a general topological space, the interior of a set A is defined as the largest open set contained in A . For another definition, we have

$$\text{Int}(A) := \{x \in A \mid \exists U \in \tau \text{ such that } x \in U \subseteq A\}. \quad (1.11)$$

Lemma 1.11. *The above definitions of the interior of a set in a topological space are equivalent.*

Proof. Suppose we affirm the second definition. Let $x \in \text{Int}(A)$. Then there exists an open set $U_x \in \tau$ such that $x \in U_x \subseteq A$. Thus, $\bigcup_{x \in \text{Int}(A)} U_x \subseteq A$ is contained in A and is open. Moreover, if $z \in \text{Int}(A) \setminus \bigcup_{x \in \text{Int}(A)} U_x$, then there exists an open set $V \in \tau$ such that $z \in V \subseteq A$. But then $z \in U_z \subseteq \bigcup_{x \in \text{Int}(A)} U_x$, a contradiction. Thus, $\text{Int}(A) = \bigcup_{x \in \text{Int}(A)} U_x$ is the largest open set contained in A . If V is any open set contained in A , then for any $y \in V$, there exists an open set $V_y \in \tau$ such that $y \in V_y \subseteq A$. Thus, $y \in \text{Int}(A)$, which implies that $V \subseteq \text{Int}(A)$. Hence, the first definition holds. ■

As an example, with the standard topology on \mathbb{R} , we have $\text{Int}([0, 1]) = (0, 1)$, $\text{Int}((0, 1) \cup \{2\}) = (0, 1)$, and $\text{Int}(\mathbb{Q}) = \emptyset$. Note that the interior of an open set is the set itself; $\text{Int}(U) = U$ for any open set U .

In the spirit of an induced metric, the subspace topology is defined as follows.

Definition 1.12. Let (X, τ) be a topological space and let $Y \subseteq X$. The *subspace topology* on Y is defined as

$$\tau_Y = \{U \cap Y \mid U \in \tau\}. \quad (1.12)$$

One can verify that τ_Y is a topology on Y . Let us look at some examples of topological spaces.

Example 1.13. The *discrete topology* on a set X is the topology $\tau = \mathcal{P}(X)$, the power set of X . Every subset of X is open in this topology. In contrast, the *indiscrete topology* (trivial topology) on X is the topology $\tau = \{\emptyset, X\}$. Only the empty set and the whole set are open in this topology.

Example 1.14. Let X be any set. The *cofinite topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is finite}\}. \quad (1.13)$$

One can verify that τ is a topology on X ; both \emptyset and X are in τ . For an arbitrary union of sets in τ , if any one of them is X , then the union is X . Otherwise, the complement of the union is the intersection of finite sets, which is finite. For a finite intersection of sets in τ , the complement of the intersection is the finite union of finite sets, which is finite. Thus, τ is a topology on X .

Example 1.15. Let X be any set. The *cocountable topology* on X is defined as

$$\tau = \{U \subseteq X \mid U = \emptyset \text{ or } U^c \text{ is countable}\}. \quad (1.14)$$

One can verify that τ is a topology on X ; both \emptyset and X are in τ . For an arbitrary union of sets in τ , if any one of them is X , then the union is X . Otherwise, the complement of the union is the intersection of countable sets, which is countable. For a finite intersection of sets in τ , the complement of the intersection is the finite union of countable sets, which is countable. Thus, τ is a topology on X .

1.2.1 Basis

Definition 1.16. Let (X, τ) be a topological space. A collection $\mathcal{B} \subseteq \tau$ is said to be a *basis* for τ if

- (i) For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
- (ii) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

For any set X , if we have a collection \mathcal{B} of subsets of X satisfying the above two properties, then the collection

$$\tau = \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\} \quad (1.15)$$

is a topology on X , and \mathcal{B} is a basis for this topology.

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