REAL ANALYSIS II

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Second Semester

List of Symbols

$[a, b]$, the set of all real numbers x such that $a \leq x \leq b$.

$\mathbb{N} = \{1, 2, \ldots\}$, the set of all natural numbers.

$\mathbb{Z}_+$, defined as $\mathbb{N} \cup \{0\}$.

$\mathcal{B}[a, b]$, the set of all boundary functions defined as $\{f : [a, b] \to \mathbb{R}\}$. It is a vector space (also an algebra) over $\mathbb{R}$.

$\mathcal{P}[a, b]$, the set of all partitions of the set $[a, b]$.

I_j, the j^{th} subinterval of $[a, b]$, controlled by a partition set.

$L(f, P)$, the lower Riemann sum for a function f and partition P.

$U(f, P)$, the upper Riemann sum for a function f and partition P.

$\underline{\int_a^b} f$, the lower Riemann integration for a function f.

$\overline{\int_a^b} f$, the upper Riemann integration for a function f.

$\mathcal{R}[a, b]$, the set of all Riemann integrable functions over the set $[a, b]$.

Contents

Chapter 1

THE RIEMANN INTEGRAL

1.1 On The Path of Definitions

January 6th.

Definition 1.1. A *partition* of $[a, b]$ are all the points $a = x_0 < x_1 < \ldots < x_n = b$. These points within are termed *nodes*, and there are $n-1$ of them. The set I_j, defined by $[x_{j-1}, x_j]$ denotes the j^{th} subinterval.

Definition 1.2. If $I = (a, b), [a, b], (a, b], [b, a)$, then the *length of the interval* I is denoted by $b - a$.

Denote by $\mathcal{P}[a, b]$, the set of all partition sets of $[a, b]$. For $P \in \mathcal{P}[a, b]$, with $n-1$ nodes, the length of $[a, b]$ will be $|[a, b]| = \sum_{j=1}^{n} I_j$. We also note that for all $P, \tilde{P} \in \mathcal{P}[a, b]$, $P \cup \tilde{P} \in \mathcal{P}[a, b]$. Note that here we consider n to be finite.

Example 1.3. The set $\{\frac{1}{n}\}_{n\geq 1} \cup \{0\}$ does not belong to the set of all partitions of the unit interval, $\mathcal{P}[0, 1]$.

Let $f \in \mathcal{B}[a, b]$, and $P \in \mathcal{P}[a, b]$. Suppose P has the nodes $a = x_0 < x_1 < \ldots < x_n = b$. For all $j = 1, \ldots n$, define $m_j = \inf_{x\in I_j} f(x)$ and $M_j = \sup_{x\in I_j} f(x)$. Finally, denote by m the value of $\inf_{x\in[a,b]} f(x)$ and M to be $\sup_{x\in[a,b]} f(x)$. These are all real values.

Note that for all valid j, $m \leq m_j \leq M_j \leq M$ always holds. This must mean that

$$m\,|I_j| \leq m_j\,|I_j| \leq M_j\,|I_j| \leq M\,|I_j|$$
$$m(b-a) \leq \sum_{j=1}^{n} m_j\,|I_j| \leq \sum_{j=1}^{n} M_j\,|I_j| \leq M(b-a). \tag{1.1}$$

Definition 1.4. Let $f \in \mathcal{B}[a, b]$. For P $(a = x_0, x_1, \ldots, x_n = b) \in \mathcal{P}[a, b]$, the *lower Riemann sum* and the *upper Riemann sum* are defined as

$$L(f, P) = \sum_{j=1}^{n} m_j\,|I_j| \text{ and } U(f, P) = \sum_{j=1}^{n} M_j\,|I_j|\,, \tag{1.2}$$

respectively. Thus, $m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)\ \forall\ P \in \mathcal{P}[a, b]$.

Remark 1.5. Clearly, $L(f, P), U(f, P) \in \mathbb{R}$ for all paritions $P \in \mathcal{P}[a, b]$ and all boundary functions $f \in \mathcal{B}[a, b]$. In fact, $L(f, P), U(f, P) \in [m(b-a), M(b-a)]$.

Definition 1.6. For $f \in \mathcal{B}[a, b]$, the *lower Riemann integration* is defined as

$$\underline{\int_a^b} f = \sup\{L(f, P)|P \in \mathcal{P}[a, b]\}. \tag{1.3}$$

Subsequently, the *upper Riemann integration* is defined as

$$\overline{\int_a^b} f = \inf\{U(f, P)|P \in \mathcal{P}[a, b]\}. \tag{1.4}$$

Remark 1.7. Note that both $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ belong to the set $[m(b-a), M(b-a)]$.

Definition 1.8. A function $f \in \mathcal{B}[a,b]$ is *Riemann integrable* if the lower and the upper Riemann integration are equal, that is, $\underline{\int_a^b} f = \overline{\int_a^b} f$. We denote this value by $\int_a^b f$, and call it the integration of f over $[a,b]$. We then say that $f \in \mathcal{R}[a,b]$.

Appendices

Chapter A

Appendix

Extra content goes here.

Appendix

Index