

LINEAR ALGEBRA II

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List of Symbols

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Chapter 1

PERMUTATION GROUPS

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Let S_n denote the set of all bijections (permutations) on the set $\{1, 2, \dots, n\}$. If $\sigma, \tau \in S_n$, let us define $\sigma\tau$ to be the bijection defined as

$$(\sigma\tau)(i) = \sigma(\tau(i)) \forall 1 \leq i \leq n. \quad (1.1)$$

This gives us a binary operation on S_n which is associative, and S_n will then contain the identity permutation 1 such that $\sigma 1 = 1\sigma = \sigma$ for all $\sigma \in S_n$. For every such σ , we can also find a $\sigma^{-1} \in S_n$ such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1$. The set S_n equipped with this binary operation, thus, forms a group. In this case, we call S_n as the *symmetric group* of degree n . We now define a cycle in regards to permutations.

Definition 1.1. A *cycle* is a string of positive integers, say (i_1, i_2, \dots, i_k) , which represents the permutation $\sigma \in S_n$ (with $k \leq n$) such that $\sigma(i_j) = i_{j+1}$ for all $1 \leq j \leq k-1$, and $\sigma(i_k) = i_1$, and fixes all other integers.

We also note that S_3 is the smallest Abelian group possible, upto isomorphism. S_3 is one of the only two groups of order 6, and can be written as

$$S_3 = \{1, \sigma = (1, 2, 3), \sigma^2 = (1, 3, 2), \tau = (1, 2), \sigma\tau = (1, 3), \tau\sigma = (2, 3)\}. \quad (1.2)$$

Some other observations arise. We find that $\sigma^3 = \tau^2 = 1$, and that $\tau\sigma = \sigma^2\tau$. We notice another fact via this σ ;

Remark 1.2. A k -cycle $\sigma = (i_1, i_2, \dots, i_k)$ is of order k , that is, $\sigma^k = 1$.

Definition 1.3. Two cycles in S_n are called disjoint if they have no integer in common.

We note that if σ and τ are two disjoint cycles in S_n then σ and τ commute, that is, $\sigma\tau = \tau\sigma$.

Proposition 1.4. Every σ in S_n can be written uniquely as a product of disjoint cycles.

Every cycle can be written as a product of 2-cycles. 2-cycles are called *transpositions*. This can easily be seen as

$$(a_1, a_2, \dots, a_n) = (a_1, a_n)(a_1, a_{n-1}) \cdots (a_1, a_3)(a_1, a_2). \quad (1.3)$$

1.1 Even and Odd Permutations

Let x_1, x_2, \dots, x_n be indeterminates, and let

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j). \quad (1.4)$$

Let $\sigma \in S_n$, and define

$$\sigma(\Delta) = \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)}). \quad (1.5)$$

We find that $\sigma(\Delta) = \pm\Delta$. Based on this, we classify permutations as odd or even.

Definition 1.5. A permutation σ is said to be an *even permutation* if $\sigma(\Delta) = \Delta$, and is said to be an *odd permutation* if $\sigma(\Delta) = -\Delta$. The sign of a permutation σ , denoted by $\epsilon(\sigma)$, is $+1$ if σ is even, and is -1 if σ is odd. So, $\sigma(\Delta) = \epsilon(\sigma)\Delta$.

Proposition 1.6. The map $\epsilon : S_n \rightarrow \{-1, +1\}$, where $\epsilon(\sigma)$ is the sign of σ , is a homomorphism, that is, $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$ for all $\sigma, \tau \in S_n$.

Proof. Start with $\tau(\Delta)$;

$$\tau(\Delta) = \prod_{1 \leq i < j \leq n} (x_{\tau(i)} - x_{\tau(j)}). \quad (1.6)$$

Let there be k factors of this polynomial where $\tau(i) > \tau(j)$ with $i < j$. We find that $\tau(\Delta) = (-1)^k \Delta$, and so, $\epsilon(\tau) = (-1)^k$. Now, $\sigma\tau(\Delta)$ has exactly k factors of the form $x_{\sigma(j)} - x_{\sigma(i)}$, with $j > i$. Bringing out a factor $(-1)^k$, we find that $\sigma\tau(\Delta)$ has all factors of the form $x_{\sigma(i)} - x_{\sigma(j)}$, with $j > i$. Thus,

$$\epsilon(\sigma\tau)\Delta = \sigma\tau(\Delta) = (-1)^k \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)}) = (-1)^k \sigma(\Delta) = (-1)^k \epsilon(\sigma)\Delta = \epsilon(\tau)\epsilon(\sigma)\Delta. \quad (1.7)$$

Cancelling out the Δ , we find $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$. ■

ϵ is a homomorphism to an Abelian group, so $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau) = \epsilon(\tau)\epsilon(\sigma)$.

Proposition 1.7. If $\lambda = (i, j)$ is a transposition, then $\epsilon(\lambda) = -1$.

Proof. If $\lambda = (1, 2) \in S_n$, it is easy to show that

$$\lambda(\Delta) = (x_1 - x_2) \cdots (x_1 - x_n)(x_2 - x_3) \cdots (x_2 - x_n) \cdots = (-1)(\Delta). \quad (1.8)$$

Now, if $\sigma = (i, j)$, with $(i, j) \neq (1, 2)$, then $(i, j) = \lambda(1, 2)\lambda$ where λ interchanges 1 and i , and interchanges 2 and j . Using that fact that ϵ is a homomorphism, $\epsilon(\sigma) = -1$. ■

A cycle σ of length k is an even permutation if and only if k is odd. This is because it can be decomposed into $k - 1$ transpositions, and we would then have $\epsilon(\sigma) = (-1)^{k-1} = 1$ (using the fact that ϵ is a homomorphism). Some more corollaries of the previous proposition include the fact that ϵ is a surjective map, and that $\epsilon(\sigma^{-1}) = \epsilon(\sigma)$.

If, for $\sigma \in S_n$, σ can be decomposed as $\sigma_1\sigma_2 \cdots \sigma_k$, where σ_i is a m_i -cycle, then $\epsilon(\sigma_i) = (-1)^{m_i-1}$, and $\epsilon(\sigma) = (-1)^{(\sum m_i) - k}$.

Proposition 1.8. σ is an odd permutation if and only if the number of cycles of even length in its cycle decomposition is odd.

Definition 1.9. If $A = (a_{ij})$ is a square matrix of order n , then the *determinant* of A is defined as

$$\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \quad (1.9)$$

Using this definition of the determinant of a square matrix, one may derive the usual determinant properties with ease.

Appendices

Chapter A

Appendix

Extra content goes here.

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