

# **PROBABILITY THEORY II**

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Second Semester

# List of Symbols

$\Omega$ , a sample space.

$\omega$ , an element of a sample space.

$EX$ , the expectation of the random variable  $X$ .

$\text{Var}X$ , the variance of the random variable  $X$ .

$N(\mu, \sigma^2)$ , a normal distribution with expectation  $\mu$  and variance  $\sigma^2$ .

$N_n(k)$ , the number of paths from  $(0, 0)$  to  $(n, k)$  in a simple random walk.

$N_n^+(k)$ , the number of paths from  $(0, 0)$  to  $(n, k)$  through strictly positive values in a random walk.

$p_k^X$ , the probability mass function for a random variable  $X$ .

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## Chapter 1

# RANDOM WALKS AND MISC. RESULTS

January 3rd.

We first start with some initial statements. Let  $\Omega$  be a countable state space, and let each  $\omega \in \Omega$  have a probability  $P(\omega)$  associated with it.

**Lemma 1.1.** *For random variables  $X, Y$  such that  $X(\omega) \leq Y(\omega)$  for all  $\omega \in \Omega$ . Then,  $EX \leq EY$ .*

*Proof.* This can easily be seen by summing over all terms via the alternate definition of the expectation,

$$EX = \sum_{\omega \in \Omega} X(\omega)P(\omega) \leq \sum_{\omega \in \Omega} Y(\omega)P(\omega) = EY. \quad (1.1)$$

■

We now state Markov's inequality.

**Theorem 1.2** (Markov's inequality). *If  $X$  is a non-negative random variable, then for  $a > 0$ , we have*

$$P(X > a) \leq \frac{EX}{a}. \quad (1.2)$$

*Proof.* Define an indicator function  $I_a(\omega)$  as 1 if  $X(\omega) \geq a$ , and 0 if otherwise. We then have

$$I_a(\omega) \leq \frac{X(\omega)}{a} \implies P(X \geq a) = EI_a \leq \frac{1}{a}EX. \quad (1.3)$$

■

**Remark 1.3.** A better upper bound here may be found by starting with  $I_a(\omega)X(\omega)$  instead of just  $X(\omega)$ .

If we have  $X \sim N(0, 1)$ , then we can find an upper bound for its probability density function.

$$P(X > a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \leq \int_a^\infty \frac{1}{\sqrt{2\pi}} \frac{x}{a} e^{-\frac{x^2}{2}} dx = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}a}. \quad (1.4)$$

Note that  $X$  here is a random variable over a continuous state space; the previous lemma and Markov's inequality also work here. We are to show them for the continuous case instead of the discrete one.

*Proof.* Here, we have  $0 \leq X(\omega) \leq Y(\omega)$  for all  $\omega$  in our continuous state space  $\Omega$ . We see that  $\{X > x\} \subseteq \{Y > x\} \implies P(X > x) \leq P(Y > x)$ . Integrating both sides gives us  $EX \leq EY$ . ■

**Theorem 1.4** (Chebyshev's inequality). *Let  $X$  be a random variable with finite mean  $\mu = EX$  and finite variance  $\sigma^2 = \text{Var}(X)$ . Then for  $a > 0$ ,*

$$P(|X - \mu| > a) \leq \frac{\text{Var}(X)}{a^2}. \quad (1.5)$$

*Proof.* Start with the proof of Markov's inequality, replacing the indicator function with one that's unity when  $|X - \mu| \geq a$ . ■

**Example 1.5.** Suppose  $X_1, X_2, \dots, X_n$  are  $n$  independent and identically distributed random variables, with  $EX_i = \mu$  and  $\text{Var}X_i = \sigma^2$ . If  $S_n = \sum X_i$ , we then have

$$P(|S_n - n\mu| > a) \leq \frac{\text{Var}S_n}{a^2} = \frac{n\sigma^2}{a^2}. \quad (1.6)$$

If we replace  $a$  with  $n^{\frac{1}{2}+\varepsilon}$ , we then have

$$P(|S_n - n\mu| > n^{\frac{1}{2}+\varepsilon}) \leq \frac{\sigma^2}{n^{2\varepsilon}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.7)$$

**Proposition 1.6.** If  $\text{Var}(X) = 0$ , then  $P(X = EX) = 1$ .

*Proof.* For all  $\varepsilon > 0$ , we have

$$P(|X - EX| > \varepsilon) \leq \frac{\text{Var}X}{\varepsilon^2} = 0. \quad (1.8)$$

Define  $A_n$  as  $\{|X - EX| > \frac{1}{n}\}$ . Taking  $P(\bigcup A_n) = \lim_{n \rightarrow \infty} P(A_n)$ , the proof follows. ■

## 1.1 The Law of Large Numbers

We start by stating the weak law of large numbers.

**Theorem 1.7** (*Weak law of large numbers*). Let  $\{X_k\}_{k \geq 1}$  be a sequence of independent and identically distributed random variables with  $E|X_i| < \infty$ . Let  $\mu = EX_i$ . Then for any  $a > 0$ ,

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{X_1 + X_2 + \dots + X_n}{n} - \mu\right| > a\right) = 0. \quad (1.9)$$

*Proof.* For now, let us assume that  $\Omega$  is countable. We begin with the case where the variance of  $X_i$ ,  $\sigma^2$ , is finite. Fix  $a > 0$ , and let  $S_n = X_1 + X_2 + \dots + X_n$ . Then,

$$P\left(\left|\frac{S_n}{n} - \mu\right| > a\right) = P(|S_n - n\mu| > na) \leq \frac{\text{Var}S_n}{n^2a^2} = \frac{n\sigma^2}{n^2a^2} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.10)$$

We now focus the case when the variance,  $\sigma^2$ , is infinite. Assume that the expected value,  $\mu$ , is 0; if it were non-zero, we would then instead work with  $X_i - \mu$ . Let  $\delta > 0$ ; we shall choose a particular  $\delta$  later. For each  $n$ , define  $n$  pairs of random variables,  $U_1, V_1, \dots, U_n, V_n$ , as  $U_k = X_k, V_k = 0$  if  $|X_k| \leq \delta n$ , and  $U_k = 0, V_k = X_k$  if  $|X_k| > \delta n$ .  $X_k$  can be rewritten as  $U_k + V_k$ . We then have

$$\{|X_1 + \dots + X_n| \geq na\} \subseteq \{|U_1 + \dots + U_n| \geq \frac{na}{2}\} \cup \{|V_1 + \dots + V_n| \geq \frac{na}{2}\} \quad (1.11)$$

$$\implies P(|X_1 + \dots + X_n| \geq na) \leq P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) + P\left(|V_1 + \dots + V_n| \geq \frac{na}{2}\right). \quad (1.12)$$

We focus on the first term on the right hand side. The  $U_i$ 's are independently and identically distributed, so

$$P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) \leq \frac{4E[|U_1 + \dots + U_n|^2]}{a^2n^2} = \frac{4}{a^2n^2} (\text{Var}(U_1 + \dots + U_n) + (nEU_i)^2). \quad (1.13)$$

For the variance, we have

$$\text{Var}(U_1 + \dots + U_n) = n\text{Var}U_i \leq nEU_i^2 \leq nE[|U_i||U_i|] \leq \delta n^2 E[|U_i|] \quad (1.14)$$

which transforms the previous equation as

$$P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) \leq \frac{4}{a^2n^2} (\delta n^2 E[|U_i|] + (nEU_i)^2). \quad (1.15)$$

A lemma (to be proven later) states that  $E[|U_i|] = E[|X_i|]$  as  $n \rightarrow \infty$ , and  $EU_i = EX_i = 0$  too. So,

$$P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) \leq \frac{4}{a^2n^2} (\delta n^2 E[|U_i|] + (nEU_i)^2) \leq \frac{4\delta E[|U_i|]}{a^2} + \frac{4}{a^2} (EU_i)^2. \quad (1.16)$$

For the second term on the right hand side, begin with

$$\begin{aligned} P(V_1 + \dots + V_n \neq 0) &\leq P(\{V_1 \neq 0\} \cup \dots \cup \{V_n \neq 0\}) \leq nP(V_i \neq 0) = n \sum_{|x| > \delta n} P(X_i = x) \\ &\leq n \sum_{|x| > \delta n} \frac{|x|}{\delta n} P(X_i = x) = \frac{1}{\delta} E[|V_i|]. \end{aligned} \quad (1.17)$$

The rightmost term here tends to 0 as  $n \rightarrow \infty$ . Now choose  $\delta$  to be  $\frac{\varepsilon a^2}{6E[|X_i|]}$ , and then choose  $N$  to be large enough such that for all  $n > N$ , both the terms are smaller than  $\frac{\varepsilon}{2}$ . ■

*January 7th.*

We now prove the lemma called upon earlier.

**Lemma 1.8.** *If  $X$  is a discrete random variable and takes values  $y_1, y_2, \dots, y_k$ , and  $E[|X|] < \infty$ , then  $\lim_{n \rightarrow \infty} E[|X| \cdot 1_{|X| \leq n}] = E[|X|]$ .*

*Proof.* Notice that the terms on the left hand side and right hand side are  $\sum_{y_k: |y_k| \leq n}$  and  $\sum_{y_k} |y_k| P(Y = y_k)$ . The condition for convergence may now be applied. ■

The above equation, begin inside absolute braces, must imply that the term  $E[X \cdot 1_{|X| \leq n}]$  must also absolutely converge to  $EX$ .

## 1.2 Simple Random Walk

Let  $X_1, X_2, \dots$  be independent and identically distributed random variables, with  $X_i = 1$  with probability  $\frac{1}{2}$  and  $X_i = -1$  with probability  $\frac{1}{2}$ . Now define  $S_0 = 0$  and  $S_n = \sum_{i=1}^n X_i$ . The sequence  $(S_n)_{n \geq 0}$  is a *simple random walk*.

Note that  $S_0 = k_0 = 0, S_1 = k_1, \dots, S_n = k_n$  can occur if and only if  $|k_i - k_{i+1}| = 1$  for all  $0 \leq i \leq n-1$ . The sequence  $(k_n)_{n \geq 0}$  is a *simple path* of the simple random walk. By the event  $\{S_n = k\}$ , we are concerned with the event that the random walk visits  $k$  at step  $n$ . If  $(k_n)_{n \geq 0}$  is given we have  $X_i = k_i - k_{i-1}$ . Because the  $X_i$ 's are independent and identically distributed, each event  $\{X_1 = l_1, X_2 = l_2, \dots, X_n = l_n\}$ , where  $l_i = \pm 1$ , is equally likely with probability  $\frac{1}{2^n}$ . Thus,

$$P(S_n = k) = \frac{N_n(k)}{2^n} \quad (1.18)$$

where  $N_n(k)$  is defined as the number of distinct of path that start at 0 and end at  $k$  at step  $n$ . We also define  $N_n^+(k)$  to be the number of distinct paths that end at  $k$  at step  $n$  and stay above the  $x$ -axis up to time  $n-1$ . The probability of the corresponding event is

$$P(\{S_1 > 0, S_2 > 0, \dots, S_{n-1} > 0, S_n = k\}) = \frac{N_n^+(k)}{2^n}. \quad (1.19)$$

**Lemma 1.9.** *Suppose  $a, a', b, b'$  are integers, with  $0 \leq a < a'$ . Then the number of distinct path from  $(a, b)$  to  $(a', b')$  depends only on  $a' - a = n$  and  $b' - b = k$ , and is given by  $\binom{n+k}{2}$ .*

*Proof.* Notice that we need  $x+1$ 's and  $y-1$ 's to appear, satisfying  $x+y = a' - a$  and  $x-y = b' - b$ . Solving, we get  $x = \frac{n+k}{2}$  and  $y = \frac{n-k}{2}$ . Thus, the number of paths is given by  $\binom{n+k}{2}$ . ■

Using this lemma, we find that  $N_n(k) = \binom{n+k}{2}$ . The following convention is now followed; if  $t$  is not an integer, then  $\binom{n}{t} = 0$ .

**Lemma 1.10** (The *method of images*). *Suppose  $a, a', b, b'$  are integers, with  $0 \leq a < a'$  and  $b, b' > 0$ . Then the number of distinct paths from  $(a, b)$  to  $(a', b')$  that intersect the  $x$ -axis is equal to the number of paths from  $(a, -b)$  to  $(a', b')$ .*

*Proof.* Consider any path  $(b = k_0, k_1, \dots, k_{n-1}, k_n = b')$ , from  $(a, b)$  to  $(a', b')$ , that intersects the  $x$ -axis. Let  $j$  be the smallest index for which  $k_j = 0$ . For ease, denote  $(a, b)$  by  $A$ ,  $(a', b')$  by  $A'$ ,  $(a+j, 0)$  by  $B$ , and  $(a, -b)$  by  $A''$ . Reflect the segment from  $A$  to  $B$  about the  $x$ -axis to obtain a 'mirrored-path' from  $A''$  to  $B$ ;  $(-b = -k_0, -k_1, \dots, -k_{j-1}, k_j = 0, k_{j+1}, \dots, k_n = b')$ . There is now a one-to-one correspondence between the paths from  $A$  to  $A'$  that intersect the  $x$ -axis, and the paths from  $A''$  to  $A'$ . ■

We can now easily compute  $N_n^+(k)$ ; it simply the number of paths from  $(1, 1)$  to  $(n, k)$  that do not intersect the  $x$ -axis.

**Theorem 1.11** (*Ballot theorem*). *The number of paths that progress from  $(0, 0)$  to  $(n, k)$  through strictly positive values is given by  $N_n^+(k) = \frac{k}{n} N_n(k)$ .*

*Proof.* We have

$$\begin{aligned} N_n^+(k) &= \text{number of paths from } (1, 1) \text{ to } (n, k) - \text{number of such paths that intersect the } x\text{-axis} \\ &= N_{n-1}(k-1) - N_{n-1}(k+1) \\ &= \binom{n-1}{\frac{n+k}{2}-1} - \binom{n-1}{\frac{n+k}{2}} = \frac{k}{n} \binom{n}{\frac{n+k}{2}} = \frac{k}{n} N_n(k). \end{aligned} \quad (1.20)$$

■

Suppose  $n = 2\nu$ . Define  $u_{2\nu}$  to be  $P(S_{2\nu} = 0) = \frac{\binom{2\nu}{\nu}}{2^{2\nu}}$ . The question we ask is to compute the probability that the first return to 0, if at all, occurs after step  $n$ . It can be found out as

$$P(\text{first return to } 0 \dots) = P(S_1 \neq 0, S_2 \neq 0, \dots, S_{2\nu} \neq 0) \quad (1.21)$$

$$= P(S_1 > 0, \dots, S_{2\nu} > 0) + P(S_1 < 0, \dots, S_{2\nu} < 0)$$

$$= 2P(S_1 > 0, \dots, S_{2\nu} > 0)$$

$$= 2 \sum_{k \text{ even}, k > 0} P(S_1 > 0, \dots, S_{2\nu-1} > 0, S_{2\nu} = k)$$

$$= \frac{2}{2^{2\nu}} \sum_{k \text{ even}, k > 0} N_{2\nu}^+(k)$$

$$= \frac{2}{2^{2\nu}} \sum_{k \text{ even}, k > 0} N_{2\nu-1}(k-1) - N_{2\nu-1}(k+1)$$

$$= \frac{2}{2^{2\nu}} N_{2\nu-1}(1) = u_{2\nu}. \quad (1.22)$$

We state this down as a lemma.

**Lemma 1.12** (*Basic lemma*). *For  $n$  even, the probability that the first return to 0, if at all, occurs after step  $n$  is the same as the probability that the location at step  $n$  is 0. For  $n$  odd, it is the probability that the location at step  $n-1$  is 0.*

We ask another question; for a fixed  $n$ , where does the random walk achieve its first maximum upto time  $n$ ? For this, denote by  $M_n$  the index  $m$  at which the walk  $S_0, S_1, \dots, S_n$ , over  $n$  steps, achieves its maximum for the first time.

For  $0 < m < n$ ,  $M_n = m$  if and only if  $S_m > S_0, S_m > S_1, \dots, S_m > S_{m-1}$  and  $S_m \geq S_{m+1}, S_m \geq S_{m+2}, \dots, S_m \geq S_n$ . Notice that the first of these two conditions depends only on  $X_1, X_2, \dots, X_m$ , and the second condition depends only on  $X_{m+1}, X_{m+2}, \dots, X_n$ . So,  $P(M_n = m) = P(S_m > S_0, S_m > S_1, \dots, S_m > S_{m-1}) \cdot P(S_m \geq S_{m+1}, S_m \geq S_{m+2}, \dots, S_m \geq S_n)$ .

The key idea here is to consider the *reversed walk*; define a new walk with  $X'_1 = X_m, X'_2 = X_{m-1}, \dots, X'_m = X_1$ . Also define  $S'_k = X'_1 + \dots + X'_k$ . From here, we can deduce that  $S_m > S_{m-i}$  is true if and only if  $X_m + \dots + X_{m-i} > 0$  is true, which is true if and only if  $S'_i > 0$  is true. So,  $P(S_m > S_0, S_m > S_1, \dots, S_m > S_{m-1}) = P(S'_1 > 0, S'_2 > 0, \dots, S'_m > 0)$ . If we now define  $S''_k = X_{m+1} + \dots + X_{m+k}$ , we have

$$\begin{aligned} P(S_m \geq S_{m+1}, S_m \geq S_{m+2}, \dots, S_m \geq S_n) &= P(X_{m+1} \leq 0, X_{m+1} + X_{m+2} \leq 0, \dots, X_{m+1} + \dots + X_n \leq 0) \\ &= P(S''_1 \leq 0, S''_2 \leq 0, \dots, S''_{n-m} \leq 0) \\ &= P(S''_1 \geq 0, S''_2 \geq 0, \dots, S''_{n-m} \geq 0) \end{aligned}$$

The first of the terms discussed,  $P(S'_1 > 0, S'_2 > 0, \dots, S'_m > 0)$ , can be computed for  $m = 2\nu, 2\nu + 1$ ; it is simply  $\frac{1}{2} u_{2\nu}$ . For the latter of these terms, we introduce a new random variable  $\tilde{X}$  which has the same distribution as the  $X_i$ 's and is independent. Also define  $\tilde{S}_i$  to be  $\tilde{X} + X_1 + \dots + X_{i-1}$  and  $\tilde{S}_0$  to be 0.

We then have

$$\begin{aligned}
\frac{1}{2}P(S_0 \geq 0, \dots, S_{n-m} \geq 0) &= P(\tilde{X} = 1) \cdot P(S_0 \geq 0, \dots, S_{n-m} \geq 0) \\
&= P(\tilde{X} = 1, S_0 \geq 0, S_0 \geq 0, \dots, S_{n-m} \geq 0) \\
&= P(\tilde{S}_1 = 1, \tilde{S}_2 > 0, \dots, S_{n-m+1} > 0) \\
&= P(S_1 > 0, S_2 > 0, \dots, S_{n-m+1} > 0).
\end{aligned} \tag{1.23}$$

Thus, we get

$$P(M_n = m) = \frac{1}{2}u_{2k}u_{2\nu-2k} \tag{1.24}$$

where  $m$  is of the form  $2k$  or  $2k + 1$ , and  $n$  is of the form  $2\nu$ , with  $1 < k < \nu$ .

*January 10th.*

Plugging in  $m = 0$ , we get  $P(M_n = 0) = P(S_1 \leq 0, \dots, S_{2\nu} \leq 0) = \frac{1}{2}u_{2\nu}$ . For  $m = n$ , we have  $P(M_n = n) = P(S_1 \leq 0, \dots, S_{2\nu} \leq 0) = \frac{1}{2}u_{2\nu}$ . Let us first compute  $u_{2k}$ .

$$\begin{aligned}
u_{2k} &= P(2k = 0) = \frac{\binom{2k}{k}}{2^{2k}} = \frac{(2k)!}{(k!)^2 2^{2k}} \\
&\sim \frac{(2k)^{2k+\frac{1}{2}} e^{-2k} \sqrt{2\pi}}{(\sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k})^2 2^{2k}} = \frac{1}{\sqrt{\pi k}}.
\end{aligned} \tag{1.25}$$

For  $0 < a < b < 1$ , we have

$$\begin{aligned}
P(an \leq M_n \leq bn) &= \sum_{m=an}^{bn} P(M_n = m) = \sum_{k=a\nu}^{b\nu} u_{2k}u_{2\nu-2k} \\
&\sim \sum_{k=a\nu}^{b\nu} \frac{1}{\sqrt{\pi k}} \frac{1}{\sqrt{\pi(\nu-k)}} = \sum_{k=a\nu}^{b\nu} \frac{1}{\nu \sqrt{\pi \frac{k}{\nu}} \sqrt{\pi(1-\frac{k}{\nu})}} \\
&\rightarrow \frac{1}{\pi} \int_a^b \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi} (\arcsin \sqrt{b} - \arcsin \sqrt{a}).
\end{aligned} \tag{1.26}$$

In fact, this is the *arcsin law for maxima*; for  $0 \leq t \leq 1$ , we have

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n}{n} \leq t\right) = \frac{2}{\pi} \arcsin \sqrt{t}. \tag{1.27}$$

If we look at this as a cumulative density function, the probability density function becomes  $\frac{d}{dt} \frac{2}{\pi} \arcsin \sqrt{t} = \frac{1}{\pi \sqrt{t(1-t)}}$ .

We are now interested in  $\tilde{M}_n$ , the last time when maximum up to time  $n$  is attained. We can just look at the walk backwards again; in this case, we get

$$P\left(\frac{\tilde{M}_n}{n} \leq t\right) = P\left(\frac{n - \tilde{M}_n}{n} \leq t\right) \rightarrow \frac{2}{\pi} \arcsin \sqrt{t}. \tag{1.28}$$

We now ask the probability that the random walk of  $n = 2\nu$  steps last visit 0 at time  $2k$ . We denote by  $K_n$  the location of the last return to 0 in a walk of  $n$  steps. Now look at

$$\begin{aligned}
\alpha_{2k, 2\nu} &= P(K_n = 2k) = P(S_{2k} = 0, S_{2k+1} \neq 0, \dots, S_{2\nu} \neq 0) \\
&= P(S_{2k} = 0) \cdot P(X_{2k+1} \neq 0, \dots, X_{2k+1} + \dots + X_{2\nu} \neq 0) \\
&= P(S_{2k} = 0) \cdot P(S_1 \neq 0, \dots, S_{2\nu-2k} \neq 0) = u_{2k}u_{2\nu-2k}.
\end{aligned} \tag{1.29}$$

We can also state an *arcsin law for last visit* here; for  $0 < t < 1$

$$\lim_{n \rightarrow \infty} P(K_n \leq tn) = \frac{2}{\pi} \arcsin \sqrt{t}. \tag{1.30}$$



If we set the an additional limit that says  $t$  tends to 0, replacing  $t$  by an arbitrary  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} P(K_n = 0) = 0. \quad (1.31)$$

Given enough time, a simple random walk must return to 0.

Denote by  $f_{2n}$  the probability that the first return to 0 occurs at time  $2n$ .

$$\begin{aligned} f_{2n} &= P(S_1 \neq 0, \dots, S_{2n-1} \neq 0, S_{2n} = 0) \\ &= P(S_1 \neq 0, \dots, S_{2n-1} \neq 0) - P(S_1 \neq 0, \dots, S_{2n} \neq 0) \\ &= P(S_1 \neq 0, \dots, S_{2n-2} \neq 0) - P(S_1 \neq 0, \dots, S_{2n} \neq 0) \\ &= u_{2n-2} - u_{2n} = \frac{1}{2n-1} u_{2n}. \end{aligned} \quad (1.32)$$

**Lemma 1.13.** *With the usual notation,*

$$u_{2n} = f_2 u_{2n-2} + f_4 u_{2n-4} + \dots + f_{2n} u_0. \quad (1.33)$$

*Proof.* We have

$$\begin{aligned} P(S_{2n} = 0) &= \sum_{k=1}^n P(S_{2n} = 0, \text{ first return at } 2k) \\ &= \sum_{k=1}^n P(\text{first return at } 2k) \cdot P(S_{2n} = 0 \mid \text{first return at } 2k) \\ \implies P(S_n = 0) &= \sum_{k=1}^n f_{2k} u_{2n-2k}. \end{aligned} \quad (1.34)$$

■

**Theorem 1.14.** *The probability that in the time interval 0 to  $n = 2\nu$ , the random walk spends  $2k$  amount of time on the positive side and  $2\nu - 2k$  amount of time on the negative side is  $\alpha_{2k, 2\nu}$ .*

**Corollary 1.15.** *For  $0 < t < 1$ ,*

$$P(\text{random walk spends less than } tn \text{ time on positive side}) \rightarrow \frac{2}{\pi} \arcsin \sqrt{t}. \quad (1.35)$$

*Proof.* This is the proof of the theorem. We introduce  $b_{2k, 2\nu}$ ; it is defined as the probability that the random walk of length  $2\nu$  and  $2k$  sides above the  $x$ -axis. We need to show that  $b_{2k, 2\nu} = \alpha_{2k, 2\nu}$ . We have

$$b_{2\nu, 2\nu} = P(S_1 \geq 0, S_2 \geq 0, \dots, S_{2\nu} \geq 0) = u_{2\nu}, \quad (1.36)$$

$$b_{0, 2\nu} = P(S_1 \leq 0, \dots, S_{2\nu} \leq 0) = u_{2\nu}. \quad (1.37)$$

We are left to prove it for  $1 \leq k \leq \nu - 1$ . Assume that exactly  $2k$  out of  $2\nu$  time are spent above the  $x$ -axis, with  $1 \leq k \leq \nu - 1$ . Suppose first return to 0 occurs at time  $2r < 2\nu$ . We deal in cases.

- Case I:  $2r$  time units upto first return are on the positive side. Then,  $r \leq k \leq \nu - 1$ . The time from  $2r$  to  $2\nu$  has to be above the  $x$ -axis,  $2k - 2\nu$  time. The number of such paths is  $(\frac{1}{2} 2^{2r} f_{2r})(2^{2\nu-2r} b_{2k-2r, 2\nu-2r})$ .
- The  $2r$  time units upto the first return are on the negative side. The nubmer of such paths is  $(\frac{1}{2} 2^{2r} f_{2r})(2^{2\nu-2r} b_{2k, 2\nu-2r})$ . Also,  $\nu - r \geq k$ .

Thus, we have

$$b_{2k, 2\nu} = \frac{1}{2} \sum_{r=1}^k f_{2r} b_{2k-2r, 2\nu-2r} + \frac{1}{2} \sum_{r=1}^{\nu-k} f_{2r} b_{2k, 2\nu-2r}. \quad (1.38)$$

We now proceed with induction on  $\nu$ . We have already shown this for  $\nu = 1$ ; assume that this is true for  $\nu \leq V - 1$ . By induction,

$$\begin{aligned} b_{2k, 2V} &= \frac{1}{2} \sum_{r=1}^k f_{2r} \alpha_{2k-2r, 2V-2r} + \frac{1}{2} \sum_{r=1}^{V-k} f_{2r} \alpha_{2k, 2V-2r} \\ &= \frac{1}{2} u_{2V-2k} \sum_{r=1}^k f_{2r} u_{2k-2r} + \frac{1}{2} u_{2k} \sum_{r=1}^{V-k} f_{2r} u_{2V-2k-2r} \\ &= u_{2k} u_{2V-2k} = \alpha_{2k, 2\nu}. \end{aligned} \quad (1.39)$$

■

January 17th.

**Theorem 1.16** (*Weirstrass's polynomial approximation.*). Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Then for every  $\varepsilon > 0$ , there is a polynomial  $P$ , dependent on  $f$  and  $\varepsilon$ , such that

$$|f(x) - P(x)| < \varepsilon \text{ for all } x \in [0, 1]. \quad (1.40)$$

**Remark 1.17.** Any continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  is bounded and uniformly continuous. This fact will be useful in proving the previous theorem.

*Proof.* Start with  $X_1, X_2, \dots$  which are independent and identically distributed Bernoulli random variables,  $\text{Ber}(x)$ . Let  $S_n = X_1 + X_2 + \dots + X_n$ . From the weak law of large numbers, we know that  $\frac{S_n}{n}$  is approximately  $x$ . We can expect that  $f(x)$  will also be approximately  $f(\frac{S_n}{n})$ . We now have

$$\begin{aligned} f_n(x) &= Ef\left(\frac{S_n}{n}\right) = \sum_{j=0}^n f\left(\frac{j}{n}\right) P(S_n = j) \\ &= \sum_{j=0}^n f\left(\frac{j}{n}\right) \binom{n}{j} x^j (1-x)^{n-j}. \end{aligned} \quad (1.41)$$

This is now a polynomial; we wish to see how close this is to  $f$ . Define  $A_\delta$  to be  $\{j : |\frac{j}{n} - x| \leq \delta\}$

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \sum_{j=0}^n \left( f\left(\frac{j}{n}\right) - f(x) \right) P(S_n = j) \right| \\ &= \left| \sum_{j \in A_\delta} \left( f\left(\frac{j}{n}\right) - f(x) \right) + \sum_{j \notin A_\delta} \left( f\left(\frac{j}{n}\right) - f(x) \right) \right| P(S_n = j) \\ &\leq \sum_{j \in A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j) + \sum_{j \notin A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j). \end{aligned} \quad (1.42)$$

We have two terms to deal with now. For the first term, choose  $\delta > 0$  such that  $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ ; this  $\delta$  can be chosen since  $f$  is uniformly continuous. Similarly, also choose  $M = \sup_{x \in [0, 1]} |f(x)|$ .  $M$  is finite since  $f$  is bounded. Thus, we have

$$\sum_{j \in A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j) \leq \sum_{j \in A_\delta} \varepsilon P(S_n = j) \leq \varepsilon \quad (1.43)$$

and

$$\sum_{j \notin A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j) \leq 2MP \left( \left| \frac{S_n}{n} - x \right| > \delta \right) \leq 2M \frac{\text{Var}(S_n)}{n^2 \delta^2} = \frac{2Mnx(1-x)}{n^2 \delta^2}. \quad (1.44)$$

Combining the two, and choosing  $n$  large enough, we have

$$|f_n(x) - f(x)| \leq \varepsilon + \frac{2Mx(1-x)}{n\delta^2} \leq \varepsilon + \frac{M}{2n\delta^2} \leq 2\varepsilon. \quad (1.45)$$

■

### 1.3 Erdős-Renyi Random Graph

We first discuss the setup; start with  $n$  vertices of an empty graph. For any pair of points  $(i, j)$ , with  $i \neq j$ , join these vertices with an edge with probability  $p$  independently for all such pairs. Such a graph is denoted by  $G_{n,p}$ .

A collection of three points  $S = \{i, j, k\}$  form a triangle if  $G_{n,p}$  has the edges  $\{i, j\}$ ,  $\{j, k\}$ , and  $\{i, k\}$ . We question the probability that such a graph has no formed triangles. Can we find  $p = p_n$  such that

triangles begin to appear at  $p_n$ ? Let  $S$  be any set of three vertices. Define  $X_S$  to be the indicator function; 1 if  $S$  forms a triangle, and 0 otherwise. We note that  $X_S \sim \text{Ber}(p^3)$ . We note that

$$EX_S = p^3, \text{Var}X_S = p^3(1 - p^3) \leq p^3.$$

Denote by  $N$  the number of triangles in the graph  $G_{n,p}$ . Clearly,

$$N = \sum_{S:|S|=3} X_S, \quad EN = \binom{n}{3} p^3 < n^3 p^3, \quad \text{Var}N = \sum_S \text{Var}X_S + \sum_S \sum_{T \neq S} \text{Cov}(X_S X_T) \leq n^3 p^3 + n^4 p^5$$

Also,  $P(N \geq 1) \leq EN < n^3 p^3$ . If  $p = p_n < \frac{1}{n}$ , then  $P(N \geq 1) \rightarrow 0$  as  $n \rightarrow \infty$ . We discuss this for  $p > \frac{1}{n}$ . We have

$$P(N = 0) \leq P(|N - EN| \geq EN) \leq \frac{\text{Var}N}{(EN)^2} \leq \frac{(n^3 p^3 + n^4 p^5)}{\frac{n^6 p^6}{100}} \leq \frac{100}{n^3 p^3} + \frac{100}{n(np)} \rightarrow 0. \quad (1.46)$$

We can state this as a theorem.

**Theorem 1.18.** *Consider  $G_{n,p_n}$ . Let  $E$  be the event that the graph is triangle free. We then have*

$$P(E) \rightarrow \begin{cases} 0 & \text{if } \frac{p_n}{\frac{1}{n}} \rightarrow \infty, \\ 1 & \text{if } \frac{p_n}{\frac{1}{n}} \rightarrow 0. \end{cases} \quad (1.47)$$

Now suppose that  $\frac{np_n}{\frac{1}{n}} C > 0$  as  $n \rightarrow \infty$ . Then we have

$$N \approx \text{Poisson}\left(\frac{C^3}{6}\right). \quad (1.48)$$

*January 21st.*

**Remark 1.19.** For this next ‘game’, we will think of  $X_i$ ’s as the winnings in game  $i$  and  $\mu$  to be the entrance fees for a game.

**Definition 1.20.** Suppose that  $X_1, X_2, \dots$  are independent, but not necessarily identically distributed. Let  $S_n = X_1 + \dots + X_n$ . We say a game with accumulated entrance fees  $\{\alpha_n, n \geq 1\}$  is fair if

$$P\left(\left|\frac{S_n}{\alpha_n} - 1\right| > \varepsilon\right) \rightarrow 0 \quad (1.49)$$

for all  $\varepsilon > 0$ .

Using this definition of ‘fair’, we look at an example.

**Example 1.21.** This is the St. Petersburg’s paradox. This is the game; toss a coin repeatedly until the first head is observed. If this head occurs at the  $k^{\text{th}}$  toss, the amount paid out is  $X = 2^k$ . Let us find a fair accumulated entrance fees. In this case,

$$EX = \sum_{k=1}^{\infty} \frac{1}{2^k} 2^k = \infty. \quad (1.50)$$

Suppose we play this game  $n$  times. We are to find a fair accumulated sum  $\{\alpha_n\}$  such that

$$P(|S_n - \alpha_n| > \varepsilon \alpha_n) \rightarrow 0. \quad (1.51)$$

To find this, we will define

$$\begin{aligned} U_j &= X_j 1_{\{X_j \leq a_n\}}, \\ V_j &= X_j 1_{\{X_j > a_n\}}. \end{aligned}$$

$a_n$  shall be determined later. Note that  $S_n = X_1 + \dots + X_n = U_1 + \dots + U_n + V_1 + \dots + V_n$ . Then,

$$P(|S_n - \alpha_n| > \varepsilon \alpha_n) \leq P(|U_1 + \dots + U_n - \alpha_n| > \frac{1}{2} \varepsilon \alpha_n) + P(|V_1 + \dots + V_n| > \frac{1}{2} \varepsilon \alpha_n). \quad (1.52)$$

We first bound the second term on the right hand side. We have

$$P(|V_1 + \dots + V_n| > \frac{1}{2}\varepsilon\alpha_n) \leq P(\bigcup_{i=1}^n \{V_i \neq 0\}) \leq nP(V_1 \neq 0) = nP(X_1 > a_n) \quad (1.53)$$

$$= \sum_{2^k > a_n} P(X = 2^k) \leq \frac{2n}{a_n}. \quad (1.54)$$

Thus, we will require that  $a_n \gg n$ . Also,

$$EU_1 = \sum_{k \leq \log_2 a_n} 2^k \cdot 2^{-k} = \lfloor \log_2 a_n \rfloor, \quad \text{Var} U_1 \leq E[U_1^2] = \sum_{k \leq \log_2 a_n} (2^k)^2 \cdot 2^{-k} = 2^{\lfloor \log_2 a_n \rfloor + 1} - 1 < 2a_n. \quad (1.55)$$

$\frac{1}{n}(U_1 + \dots + U_n) \approx EU_j = \lfloor \log_2 a_n \rfloor$ , so we should choose

$$\alpha_n = nEU_j = n \lfloor \log_2 a_n \rfloor. \quad (1.56)$$

This gives us

$$P(|U_1 + \dots + U_n - \alpha_n| > \frac{1}{2}\varepsilon\alpha_n) \leq \frac{n(2a_n)}{\frac{1}{4}\varepsilon^2\alpha_n^2}. \quad (1.57)$$

Thus, we have another condition where we require that  $\frac{na_n}{\alpha_n^2} \rightarrow 0$ . The conditions we require are

$$\frac{n}{a_n} \rightarrow 0 \text{ and } \frac{na_n}{n^2(\log_2 a_n)^2} \rightarrow 0.$$

The sequence  $\{a_n\}$  defined as  $a_n = n \log_2 n$  satisfies these properties. The sequence  $\alpha_n$  is thus

$$\alpha_n = n \log_2 a_n = n \log_2 n + n \log_2 \log_2 n. \quad (1.58)$$



## Chapter 2

# GENERATING FUNCTIONS

January 24th.

**Definition 2.1.** For a sequence  $\{a_n\}_{n \geq 0}$ , the *generating function* of  $\{a_n\}$  is given as

$$A(s) = \sum_{n=0}^{\infty} a_n s^n \quad (2.1)$$

for some  $-s_0 < s < s_0$ .

For this probability course, we will be interested in a particular form; for a random variable  $X$  that takes values  $k = 0, 1, \dots$ , the function we look at is

$$\sum_{k=0}^{\infty} P(X = k) s^k \text{ for } -1 \leq s \leq 1. \quad (2.2)$$

Suppose we have two sequences  $\{a_n\}$  and  $\{b_n\}$  with generating functions  $A(s)$  and  $B(s)$ , respectively. If we define a new sequence  $\{c_n\}$  as

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0 \text{ for all } n \geq 0, \quad (2.3)$$

then the sequence  $\{c_n\}$  is termed the *convolution* of the sequences  $\{a_n\}$  and  $\{b_n\}$ , and we shall denote it as

$$\{c_n\} = \{a_n\} * \{b_n\}.$$

Note that this convolution operation is both associative and commutative. We are now interested in finding the generating function of  $\{c_n\}$ . We have

$$\begin{aligned} C(s) &= \sum_{n=0}^{\infty} c_n s^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) s^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n a_k s^k b_{n-k} s^{n-k} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k s^k b_m s^m \\ \implies C(s) &= \left( \sum_{k=0}^{\infty} a_k s^k \right) \cdot \left( \sum_{m=0}^{\infty} b_m s^m \right) = A(s) \cdot B(s). \end{aligned} \quad (2.4)$$

We state this down as a theorem.

**Theorem 2.2.**  $C(s) = A(s) \cdot B(s)$  when  $\{c_n\} = \{a_n\} * \{b_n\}$ .

Suppose  $X$  takes values in  $\mathbb{Z}_+ = \{0, 1, \dots\}$ . Denote  $P(X = k)$  as  $p_k$ . The generating function is, thus,

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} p_k s^k = E[s^X].$$

Also,

$$\mathcal{P}(1) = 1, \quad (2.5)$$

$$\mathcal{P}'(1) = \sum_{k=1}^{\infty} k p_k s^{k-1} \big|_{s=1} = EX. \quad (2.6)$$

Also note that

$$E[X^2] = \sum_{k=0}^{\infty} k^2 p_k = \sum k(k-1)p_k + \sum k p_k = \mathcal{P}''(1) + \mathcal{P}'(1) \quad (2.7)$$

which gives us the variance of  $X$  as

$$\text{Var}X = E[X^2] - (EX)^2 = \mathcal{P}''(1) + \mathcal{P}'(1) - (\mathcal{P}'(1))^2. \quad (2.8)$$

The individual probabilities of  $X = k$  may also be found as

$$p_k = P(X = k) = \frac{1}{k!} \cdot \frac{d^k}{ds^k} \mathcal{P}(s) \big|_{s=0}. \quad (2.9)$$

Now suppose that  $X$  and  $Y$  are two independent variables, taking values in  $\mathbb{Z}_+$ . Let  $Z = X + Y$ . We ask the probability that  $Z$  equals  $k$ . We can find this as

$$P(Z = k) = \sum_{m=0}^k P(X = m, Y = k - m) = \sum_{m=0}^k P(X = m) \cdot P(Y = k - m). \quad (2.10)$$

Therefore, denoting  $p_k^{(X)}$  to be the probability mass function of  $X$ , we have

$$\{p_k^{(Z)}\} = \{p_k^{(X)}\} * \{p_k^{(Y)}\} \implies \mathcal{P}^{(Z)}(s) = \mathcal{P}^{(X)}(s) \cdot \mathcal{P}^{(Y)}(s). \quad (2.11)$$

There is an easier way to see the last equation; we could have started with  $Es^Z = E[s^X \cdot s^Y] = E[s^X]E[s^Y]$ .

If we have  $S_n = X_1 + X_2 + \dots + X_n$ , where the  $X_i$ 's are independently distributed taking values in  $\mathbb{Z}_+$ , it can be shown that

$$\{p_k^{(S_n)}\} = \{p_k^{(X)}\}^{n*} \quad (2.12)$$

**Example 2.3.** Let us compute the generating function of  $X \sim \text{Bin}(n, p)$ . We have

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} P(X = k) s^k = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} s^k = ((1-p) + ps)^n. \quad (2.13)$$

This is the generating function of the binomial distribution. Clearly,

$$\begin{aligned} EX &= \mathcal{P}'(1) = np, \\ \text{Var}X &= \mathcal{P}''(1) + \mathcal{P}'(1) - (\mathcal{P}'(1))^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p). \end{aligned}$$

Note that using this generating function, we can also show that  $\text{Bin}(n, p) + \text{Bin}(m, p) = \text{Bin}(m+n, p)$  when the former terms are independent.

**Example 2.4.** We look at  $X \sim \text{Poisson}(\lambda)$ . We have

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda + \lambda s}. \quad (2.14)$$

For this, we can also verify  $EX = \text{Var}X = \lambda$ . We can also show that  $\text{Poisson}(\lambda) + \text{Poisson}(\mu) = \text{Poisson}(\lambda + \mu)$  when the former terms are independent.

**Example 2.5.** We look at  $X \sim \text{Geo}(p)$ . Denote  $1-p$  as  $q$ . The generating function is given as

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} p q^k s^k = \frac{p}{1-qs}. \quad (2.15)$$

As an extension, let  $X_k$  denote the number of failures between the  $(k-1)^{\text{th}}$  and  $k^{\text{th}}$  successes. If we denote  $S_r = X_1 + X_2 + \dots + X_r$ , we find that  $S_r \sim \text{NB}(p, r)$ . From direct computation, we know that

$$P(S_r = k) = \binom{r+k-1}{k} q^k p^r \text{ for } k = 0, 1, \dots$$

Let us compute this in another way;  $S_r$  is the sum of independent geomtric random variables with parameter  $p$ . We have

$$\mathcal{P}^{(S_r)}(s) = \left( \frac{p}{1-qs} \right)^r = p^r (1-qs)^{-r} = p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qs)^k \quad (2.16)$$

which tells us that

$$P(S_r = k) = p^r \binom{-r}{k} (-q)^k. \quad (2.17)$$

## 2.1 Random Walks, with Generating Functions

Here, we consider the paths that have a right step with probability  $p$  and a left step with probability  $q = 1-p$ . We first look at the waiting time for the first gain, that is, the event  $\{S_1 \leq 0, S_2 \leq 0, \dots, S_{n-1} \leq 0, S_n = 1\}$  (Event (\*)). Denote the probability of this event by  $\phi_n$ , and its generating function by  $\Phi(s)$ . Note that  $\phi_0 = 0$  and  $\phi_1 = p$  lead to trivial cases. We focus on  $n > 1$ .

We must have  $S_1 = -1$  (Event (1)). Denote, by  $\nu < n$ , the first return to 0 (Event (2)).  $\nu$  only depends on  $X_0, X_1, \dots, X_\nu$ . We need another  $n-\nu$  steps to reach 1; this depends on  $X_{\nu+1}, X_{\nu+2}, \dots, X_n$  (Event (3)). For some  $n > 1$ , Event (\*) occurs if and only Event (1)  $\cap$  Event (2)  $\cap$  Event (3) occurs for some  $\nu < n$ . The point here is that the three events are independent. For some fixed  $\nu < n$ ,

$$P(\text{Event (1)}) = q, P(\text{Event (2)}) = \phi_{\nu-1}, P(\text{Event (3)}) = \phi_{n-\nu}. \quad (2.18)$$

Thus,

$$\phi_n = \sum_{\nu=2}^{n-1} q \phi_{\nu-1} \phi_{n-\nu}. \quad (2.19)$$

We have

$$\Phi(s) - ps = \sum_{n=2}^{\infty} \phi_n s^n = q \sum_{n=2}^{\infty} (\phi_1 \phi_{n-2} + \dots + \phi_{n-2} \phi_1) s^n = qs \sum_{n=1}^{\infty} \phi_n^2 s^n = qs(\Phi(s))^2 \quad (2.20)$$

$$\implies \Phi(s) - ps = qs(\Phi(s))^2. \quad (2.21)$$

This is a standard quadratic; solving gives us

$$\Phi(s) = \frac{1 \pm \sqrt{1-4pqs^2}}{2qs}. \quad (2.22)$$

The solution with the '+' is rejected; if it was valid, then plugging in  $s < 1$  would give us  $\Phi(s) > 1$ , which is impossible. We expand this using the binomial theorem,

$$\Phi(s) = \frac{1}{2qs} \left( 1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4pqs^2)^k \right) = \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \frac{(-1)^{k-1} (4pq)^k}{2q} s^{2k-1} \quad (2.23)$$

which tells us that

$$\phi_{2k-1} = \frac{(-1)^{k-1}}{2q} \binom{\frac{1}{2}}{k} (4pq)^k, \phi_{2k} = 0. \quad (2.24)$$

Thus,

$$\Phi(1) = \sum \phi_n = \frac{1 - \sqrt{1-4pq}}{2q} = \frac{1 - |p-q|}{2q} = \begin{cases} \frac{p}{q} & \text{if } p < q, \\ 1 & \text{if } p \geq q. \end{cases}$$



This gives the probability that, at some point of the random walk, the displacement 1 is reached.

Similarly, for displacement  $S_n$ , we have

$$P(S_n \leq 0 \ \forall n) = \begin{cases} \frac{q-p}{p} & \text{if } p < q, \\ 0 & \text{if } p \geq q. \end{cases}$$

*January 28th.*

Recall that we used  $u_k$  denote the probability that the random walk returns to zero at step  $k$ . For unequal left-right step probabilities,

$$u_k = P(S_k = 0) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \binom{2k}{k} p^n q^n & \text{if } k = 2n. \end{cases}$$

Thus, the generating function for this is

$$U(s) = \sum_{n=0}^{\infty} u_{2n} s^{2n} = \sum_{n=0}^{\infty} \binom{2n}{n} (pq s^2)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4pq s^2)^n = \frac{1}{\sqrt{1-4pq s^2}}. \quad (2.25)$$

Denote, by  $f_{2n}$ , the probability that the first return to zero occurs at step  $2n$ , for some  $n \geq 1$ . In fact, it consists of subevents; if  $X_1 = 1$ , denote it by  $f_{2n}^+$  and if  $X_1 = -1$ , denote it by  $f_{2n}^-$ . If we also recall the definition of our  $\phi_n$ ,

$$f_{2n}^- = P(X_1 = -1, S_2 < 0, S_3 < 0, \dots, S_{2n-1} < 0, S_{2n} = 0) = q\phi_{2n-1}. \quad (2.26)$$

The generating function of  $\{f_{2n}^-\}$  will be given as

$$F^-(s) = \sum_{n=1}^{\infty} f_{2n}^- s^{2n} = q \sum_{n=1}^{\infty} \phi_{2n-1} s^{2n} = qs \sum_{n=1}^{\infty} \phi_{2n-1} s^{2n-1} = qs\Phi(s) = \frac{1}{2}(1 - \sqrt{1-4pq s^2}). \quad (2.27)$$

It can be shown that  $f_{2n}^+$  is just  $f_{2n}^-$  with the probabilities reversed (check!). The generating function of  $\{f_{2n}^+\}$  is given as

$$F^+(s) = \sum_{n=0}^{\infty} f_{2n}^+ s^{2n} = \frac{1}{2}(1 - \sqrt{1-4pq s^2}). \quad (2.28)$$

Adding both of these, we get

$$F(s) = F^+(s) + F^-(s) = 1 - \sqrt{1-4pq s^2} = 1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4pq s^2)^n \quad (2.29)$$

$$\implies f_{2n} = (-1)^{n+1} \binom{\frac{1}{2}}{n} (4pq)^n. \quad (2.30)$$

$F(1)$  gives us the probability that walk eventually returns to zero,

$$F(1) = \sum_{n=0}^{\infty} f_{2n} = 1 - \sqrt{1-4pq} = 1 - |p - q|. \quad (2.31)$$

$F'(1)$  gives us the expected time of return to zero,

$$F'(s) = -\frac{1}{2}(1-4pq s^2)^{-\frac{1}{2}}(-8pq s). \quad (2.32)$$

If  $p = q = \frac{1}{2}$ , then

$$F'(1) = \lim_{s \rightarrow 1^-} F'(s) = \infty.$$

The basic lemma can be proved using the generating functions.

## 2.2 Simple Random Walks in Higher Dimensions

Consider the walk in the dimension  $d$ . A walker starts at the origin in the lattice  $\mathbb{Z}^d$ . The random variables  $X_1, X_2, \dots$  are independent and identically distributed with probabilities

$$P(X_i = -e_d) + \dots + P(X_i = -e_2) + P(X_i = -e_1) + P(X_i = e_1) = P(X_i = e_2) + \dots + P(X_i = e_d) = \frac{1}{2d}.$$

for all valid  $i$ . The random walk here is defined as  $S_n = X_1 + \dots + X_n$ . We ask the probability that  $S_n$  returns to the origin. Denote by  $u_{2n}$  the probability that  $S_{2n} = 0$ , and denote by  $f_{2n}$  the probability that the first return to the origin occurs at time  $2n$ . By conditioning,

$$u_{2n} = \sum_{k=0}^n f_{2k} u_{2n-2k}. \quad (2.33)$$

If  $U(s)$  and  $F(s)$  are the appropriate generating functions, then we can show that

$$U(s) - 1 = F(s)U(s) \implies U(s) = \frac{1}{1 - F(s)}. \quad (2.34)$$

Both  $U(s)$  and  $F(s)$  are convergent for  $|s| < 1$ . For each  $N$ ,

$$\sum_{n=0}^N u_{2n} \leq \lim_{s \rightarrow 1^-} U(s) \leq \sum_{n=0}^{\infty} u_{2n}. \quad (2.35)$$

**Lemma 2.6.** *A random walk on  $\mathbb{Z}^d$  return to the origin with probability 1 if and only if  $\sum u_{2n} = \infty$ .*

*Proof.* Suppose  $F(1) < 1$ . Then,  $\lim_{s \rightarrow 1^-} U(s) < \infty$  and, consequently,  $\sum_{n=0}^{\infty} u_{2n} < \infty$ . The converse can be proved by reversing the steps.  $\blacksquare$

The lemma tells us that to see the probability that the random walk returns to the origin, we only need to compute  $\sum_{n=0}^{\infty} u_{2n}$ .

For  $d = 2$ , we need the number of  $e_i$  jumps to be equal to the number of  $-e_i$  jumps for  $i = 1, 2$ . We have

$$\begin{aligned} u_{2n} &= \frac{1}{4^{2n}} \sum_{j=0}^n \binom{2n}{j} \binom{2n-j}{j} \binom{2n-2j}{n-j} \binom{n-j}{n-j} = \frac{1}{4^{2n}} \binom{2n}{n} \sum_{j=0}^n \binom{n}{j}^2 = \frac{1}{4^{2n}} \binom{2n}{n}^2 \\ &\sim \frac{2}{2\pi} \frac{n^{4n+1}}{n^{4n+2}} = \frac{1}{\pi n}. \end{aligned} \quad (2.36)$$

Since this is any asymptotic relationship,  $u_{2n} \geq \frac{(1-\varepsilon)}{\pi n}$  for large  $n$ . Thus, we can show  $\sum u_{2n} = \infty$ .

For  $d = 3$ ,

$$\begin{aligned} u_{2n} &= \frac{1}{6^{2n}} \sum_{j,k=0; j+k \leq n}^n \frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!} = \frac{1}{6^{2n}} \sum_{j,k=0; j+k \leq n}^{\infty} \frac{(2n)!}{(j!)^2(k!)^2((n-j-k)!)^2} \\ &= \frac{1}{2^{2n}} \binom{2n}{n} \sum_{j,k; j+k \leq n} \left( \frac{n!}{j!k!(n-j-k)!} \frac{1}{3^n} \right)^2. \end{aligned} \quad (2.37)$$

$\frac{1}{2^{2n}} \binom{2n}{n}$  behaves asymptotically as  $\frac{1}{\sqrt{\pi n}}$ . For the rest of the term,

$$\sum_{j,k; j+k \leq n} \left( \frac{n!}{j!k!(n-j-k)!} \frac{1}{3^n} \right)^2 \leq t_n \sum_{j,k; j+k \leq n} \frac{n!}{j!k!(n-j-k)!} \frac{1}{3^n} \quad (2.38)$$

where  $t_n = \max_{j,k; j+k \leq n} \frac{n!}{j!k!(n-j-k)!}$ . The maximum is attained roughly when  $j, k \approx \frac{n}{3}$ . Also, the summation behaving as the upper bound is just unity. Thus,

$$\sum_{j,k; j+k \leq n} \left( \frac{n!}{j!k!(n-j-k)!} \frac{1}{3^n} \right)^2 \leq t_n \approx \frac{n!}{((\frac{n}{3})!)^3 3^n} \sim \frac{C}{n} \quad (2.39)$$

for some constant  $C$ . Therefore,

$$u_{2n} \leq \frac{C^*}{n^{\frac{3}{2}}} \implies \sum u_{2n} < \infty \implies F(1) < 1. \quad (2.40)$$

**Theorem 2.7** (*Polya*). *A random walk in 1 or 2 dimensions will always return to the origin with probability 1. A random walk in more than 2 dimensions has a positive probability of never returning to the origin.*

## 2.3

January 31st.

Recall that in the first course, we studied that if  $X_n \sim \text{Bin}(n, p_n)$  with  $np_n \rightarrow \lambda$  as  $n \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} P(X_n = k) = P(\text{Poisson}(\lambda) = k) \text{ for } k \geq 0. \quad (2.41)$$

We now extend upon this idea.

**Theorem 2.8** (*Continuity theorem*). *Suppor for each  $n$  the sequence  $a_{0,n}, a_{1,n}, \dots$  is a probability distribution, that is,*

$$a_{k,n} \geq 0 \text{ for all } k \text{ and } \sum_{k=0}^{\infty} a_{k,n} = 1. \quad (2.42)$$

Let  $A^{(n)}(s)$  denote the generating function for  $\{a_{k,n}\}_{k \geq 0}$ , that is,

$$A^{(n)}(s) = \sum_{k=0}^{\infty} a_{k,n} s^k \text{ for all } n. \quad (2.43)$$

Then  $a_k = \lim_{n \rightarrow \infty} a_{k,n}$  exists for all  $k$  (statement  $\star$ ) if and only if  $A(s) = \lim_{n \rightarrow \infty} A^{(n)}(s)$  exists for all  $0 < s < 1$  (statement  $\star\star$ ). In this case,  $A(s) = \sum_{k=0}^{\infty} a_k s^k$ .

*Proof.* Assume statement  $\star$ . Thus,  $|a_{k,n} - a_k| \leq 1$  for all  $n$  large enough. If we now fix  $0 < s < 1$ , then for some  $K$  and a fixed  $\varepsilon > 0$ , we have

$$\left| A^{(n)}(s) - A(s) \right| = \left| \sum_{k=0}^{\infty} a_{k,n} s^k - \sum_{k=0}^{\infty} a_k s^k \right| \quad (2.44)$$

$$\begin{aligned} &= \left| \sum_{k=0}^K a_{k,n} s^k + \sum_{k=K+1}^{\infty} a_{k,n} s^k - \sum_{k=0}^K a_k s^k - \sum_{k=K+1}^{\infty} a_k s^k \right| \\ &\leq \left| \sum_{k=0}^K a_{k,n} s^k - \sum_{k=0}^K a_k s^k \right| + \left| \sum_{k=K+1}^{\infty} (a_{k,n} - a_k) s^k \right| \\ &\leq \left| \sum_{k=0}^K a_{k,n} s^k - \sum_{k=0}^K a_k s^k \right| + \frac{s^{K+1}}{1-s}. \end{aligned} \quad (2.45)$$

We can choose  $K$  such that the second term becomes less than  $\varepsilon$ , and we can choose  $N$  such that for all  $n \geq N$ , the first term becomes smaller than  $\varepsilon$ . Therefore, the entire term becomes less than  $2\varepsilon$ .

For the converse, assume statement  $\star\star$ .  $A(s)$  is monotonic in  $s$ ;  $A(0) = \lim_{s \rightarrow 0^-} A(s)$ . We sandwich as follows—

$$\begin{aligned} a_{0,n} &\leq A^{(n)}(s) \leq a_{0,n} + \frac{s}{1-s} \\ \implies A^{(n)}(s) - \frac{s}{1-s} &\leq a_{0,n} \leq A^{(n)}(s). \end{aligned}$$

Letting  $n$  grow to infinity,

$$A(s) - \frac{s}{1-s} \leq \liminf_{n \rightarrow \infty} a_{0,n} \leq \limsup_{n \rightarrow \infty} a_{0,n} \leq A(s). \quad (2.46)$$

If  $s \rightarrow 0$ , note that  $\lim_{n \rightarrow \infty} a_{0,n} = A(0)$ . Now define

$$B^{(n)}(s) = \frac{A^{(n)}(s) - a_{0,n}}{s} \rightarrow \frac{A(s) - A(0)}{s} \rightarrow A'(0). \quad (2.47)$$

Working similarly,

$$a_{1,n} \leq B^{(n)}(s) \leq a_{1,n} + \frac{s}{1-s} \quad (2.48)$$

$$\implies B^{(n)}(s) - \frac{s}{1-s} \leq a_{1,n} \leq B^{(n)}(s). \quad (2.49)$$

If we again proceed as shown, we will get  $B(0) = \lim_{n \rightarrow \infty} a_{1,n}$  and  $a_{1,n} \rightarrow A'(0) = a_1$ . Thus, induction is in play here.  $\blacksquare$

**Example 2.9.** Let us work with the binomial distribution example given before. We have  $X_n \sim \text{Bin}(n, p_n)$  with  $np_n \rightarrow \lambda$ . We have

$$\begin{aligned} A^{(n)}(s) &= \sum_{k=0}^{\infty} P(X_n = k) s^k = ((1 - p_n) + p_n s)^n = (1 + p_n(s - 1))^n \\ \implies \lim_{n \rightarrow \infty} A^{(n)}(s) &= \lim_{n \rightarrow \infty} \left(1 + \frac{np_n}{n}(s - 1)\right)^n = e^{\lambda(s-1)} = E[s^{\text{Poisson}(\lambda)}]. \end{aligned} \quad (2.50)$$

Thus, we have shown the prior statement.

**Example 2.10.** We have  $X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)}$  independent, with  $X_i^{(n)} \sim \text{Ber}(p_i^{(n)})$  for  $1 \leq i \leq n$ . Let  $S_n = X_1^{(n)} + X_2^{(n)} + \dots + X_n^{(n)}$ . We have

$$E[s^{S_n}] = \prod_{i=1}^n E[s^{X_i^{(n)}}] = \prod_{i=1}^n \left((1 - p_i^{(n)}) + p_i^{(n)} s\right) = \exp\left(\ln \prod_{i=1}^n (\dots)\right). \quad (2.51)$$

Assume that  $\lim_{n \rightarrow \infty} \sum_{i=1}^n p_i^{(n)} = \lambda$  and  $\lim_{n \rightarrow \infty} \max_i p_i^{(n)} = 0$ . Thus,

$$\begin{aligned} \exp\left(\sum_{i=1}^n \ln(1 + p_i^{(n)}(s - 1))\right) &= \exp\left(\sum_{i=1}^n p_i^{(n)}(s - 1) - \frac{(p_i^{(n)}(s - 1))^2}{2} + \dots\right) \\ &= \exp\left((s - 1) \sum_{i=1}^n p_i^{(n)} - \sum_{i=1}^n o(p_i^{(n)}(s - 1))\right) \end{aligned} \quad (2.52)$$

$$\rightarrow e^{\lambda(s-1)}. \quad (2.53)$$

**Example 2.11.** Let  $X^{(n)} \sim \text{NB}(r_n, p)_n$ , the number of successes before the  $r_n^{\text{th}}$  success in trials with success probability  $p_n$ . Let  $p_n \rightarrow 1$  and  $r_n \rightarrow \infty$  such that  $r_n(1 - p_n) \rightarrow \lambda$ , where  $\lambda$  is fixed. We would then have  $P(X^{(n)} = k) \rightarrow P(\text{Poisson}(\lambda) = k)$ .

## 2.4 Gambler's Ruin

We take a look at a gambler, who has starting capital  $z$ . His probability of a success (+1) is  $p$ , and of a failure (−1) is  $q$ . We ask the probability  $q_z$  that the gambler reaches 0 before  $a$  when he starts at capital  $z$ . Note that  $q_z$  satisfies

$$q_z = pq_{z+1} + qq_{z-1} \text{ for } 1 < z < a - 1 \text{ (statement } \star), \text{ with } q_0 = 1, q_a = 0 \text{ (statement } \star\star). \quad (2.54)$$

We look at two cases, beginning with the case when  $p \neq q$ . Note that  $q_z = 1$  for  $1 \leq z \leq a - 1$  solves for statement  $\star$ , ignoring statement  $\star\star$  and ignoring probability for now.  $q_z = (\frac{q}{p})^z$  for  $1 \leq z \leq a - 1$  also solves for statement  $\star$ . Therefore,  $A + B(\frac{q}{p})^z$  solves statement  $\star$ . Now, we plug in the boundary conditions given by statement  $\star\star$ . Solving the equations  $A + B = 1$  and  $A + B(\frac{q}{p})^a = 0$  gives us

$$B = \frac{1}{1 - (\frac{q}{p})^a}, \quad A = 1 - \frac{1}{1 - (\frac{q}{p})^a}. \quad (2.55)$$

Plugging this in, gives us

$$q_z = \frac{(\frac{q}{p})^a - (\frac{q}{p})^z}{(\frac{q}{p})^a - 1}. \quad (2.56)$$

Note that we were working the case when  $p \neq q$ . For  $p = q$ , this solution does not work.

We work the case for when  $p = q = \frac{1}{2}$ . Again,  $q_z = 1$  for  $1 \leq z \leq a - 1$  satisfies statement  $\star$ . We also find that  $q_z = z$  for  $1 \leq z \leq a - 1$  also satisfies this statement. Hence, we look for  $A + Bz$  which satisfies boundary condition given by statement  $\star\star$ . Solving, this gives us

$$q_z = 1 - \frac{z}{a}. \quad (2.57)$$

Note that we are yet to show  $p_z + q_z = 1$ . If we instead focus on a *second* gambler playing against our gambler, we would have a gambler with capital  $a - z$ , and probability of success  $q$  and probability of failure  $p$ . Replacing  $z$  by  $a - z$  and  $q$  by  $p$  and  $p$  by  $q$  in our formed equations would give us  $p_z + q_z = 1$ .

Let us intuitively look at our equations with a table of examples.

$p$	$q$	$z$	$a$	$q_z$
0.45	0.55	9	10	0.21
0.45	0.55	90	100	0.866
0.45	0.55	99	100	0.182
0.5	0.5	9	10	0.1
0.5	0.5	90	100	0.1
0.5	0.5	99	100	0.01

Table 2.1: Probability of ruin ( $q_z$ ) given initial parameters.

Note that the expected net gain is given by

$$(a - z)(1 - q_z) - zq_z = a(1 - q_z) - z. \quad (2.58)$$

If we plug in this into our first three rows of our table, we would have  $-1.1$ ,  $-77$ ,  $-18$ . If one is gambling under such condition, we must start with big capital  $z$  and low target  $a - z$ .

### 2.4.1 Duration of the Game

We look at  $D_z$ , the expected duration of a game starting at  $z$ ; the expected time before the gambler hits  $a$  or 0. The linear recurrence satisfied here is

$$D_z = pD_{z+1} + qD_{z-1} + 1 \text{ with boundary conditions } D_0 = 0, D_a = 0. \quad (2.59)$$

For  $p \neq q$ ,

$$D_z = \frac{z}{q - p} - \frac{a}{q - p} \left( \frac{1 - (\frac{q}{p})^z}{1 - (\frac{q}{p})^a} \right). \quad (2.60)$$

For  $p = q = \frac{1}{2}$ ,

$$D_z = z(a - z). \quad (2.61)$$

## Chapter 3

# JOINT CONTINUOUS DISTRIBUTIONS

### 3.1 Introduction

February 4th.

Recall that  $X : \Omega \rightarrow \mathbb{R}$  is *continuous random variable* if it has a probability density function  $f_X : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ . In this case, if  $A \subseteq \mathbb{R}$ , then

$$P(X \in A) = \int_A f_X(x)dx. \quad (3.1)$$

For a minute  $dx$ ,

$$P(X \in [x, x + dx]) \approx f_X(x)dx. \quad (3.2)$$

Two random variables  $X$  and  $Y$  are termed *jointly continuous* if there exists a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for  $A \subseteq \mathbb{R}^2$ ,

$$P((X, Y) \in A) = \iint_A f(x, y)dx dy. \quad (3.3)$$

In this case,  $f$  is termed the *joint probability density function* of  $X$  and  $Y$ . In particular, if  $B, C \subseteq \mathbb{R}$ , then

$$P(X \in C, Y \in B) = P((X, Y) \in C \times B) = \int_B \int_C f(x, y)dx dy. \quad (3.4)$$

The *joint cumulative density function* is given as

$$F(a, b) = P(X \leq a, Y \leq b) = P((X, Y) \in (-\infty, a] \times (-\infty, b]) = \int_{-\infty}^b \int_{-\infty}^a f(x, y)dx dy. \quad (3.5)$$

There is, again, joint analagous versions of the single random variables cases;

$$\frac{\partial^2}{\partial a \partial b} F(a, b) = f(a, b) \quad (3.6)$$

and

$$P(X \in [a, a + da], Y \in [b, b + db]) \approx f(a, b)dadb. \quad (3.7)$$

Note that  $(X, Y)$  being jointly continuous implies that both  $X$  and  $Y$  are continuous. Indeed, if  $A \subseteq \mathbb{R}$ , then

$$P(X \in A) = P(X \in A, Y \in \mathbb{R}) = \int_A \int_{\mathbb{R}} f(x, y)dy dx = \int_A f_X(x)dx. \quad (3.8)$$

In this case, the inner intergral becomes the probability density funciton of  $X$ .

**Example 3.1.** We are given the joint probability density function of  $X$  and  $Y$  as

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & \text{if } 0 < x < \infty, 0 < y < \infty, \\ 0 & \text{if otherwise.} \end{cases}$$

We are to compute  $P(X > 1, Y < 1)$ ,  $P(X < Y)$  and  $P(X < a)$ . This is left as an exercise.

**Example 3.2.** Suppose  $(X, Y)$  represents a random points inside a circle of radius  $R$ . The probability density function is given by

$$f(x, y) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x^2 + y^2 \leq R^2, \\ 0 & \text{if otherwise.} \end{cases}$$

Compute  $f_X, f_Y$  and  $f_D$  where  $D = \sqrt{X^2 + Y^2}$ .

We call  $X$  and  $Y$  independent if, for  $A, B \subseteq \mathbb{R}$ ,

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B). \quad (3.9)$$

If we take  $A = (-\infty, a], B = (-\infty, b]$ , then

$$F(a, b) = F_X(a)F_Y(b) \implies f(a, b) = f_X(a)f_Y(b).$$

In fact, all three conditions are equivalent. Note that everything discussed so far may be extended to more than two random variables. If  $A \subseteq \mathbb{R}^n$  and  $(X_1, X_2, \dots, X_n)$  are jointly continuous, then

$$P((X_1, X_2, \dots, X_n) \in A) = \int \cdots \int_A f(x_1, x_2, \dots, x_n) dx_1 \cdots dx_n \quad (3.10)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ .

**Example 3.3** (*Buffin's needle problem*). Suppose we have a table of width  $D$  and length sufficient. We throw a needle of length  $L \leq D$  randomly (and necessarily) on this table. We are to find the probability that the needle will land over either side of the table separated by width  $D$ . Denote by  $X$  the distance of the midpoint of the needle from the nearest edge, and denote by  $\theta$  the angle it subtends with respect to the vertical.

Note that  $X \in [0, \frac{D}{2}]$ , and  $\theta \in [0, \frac{\pi}{2}]$ . By the procedure, one can assume these are independent. We look at the probability that the length of half the needle is greater than the length of the hypotenuse determined by  $X$  and  $\theta$ —

$$P\left(\frac{L}{2} > \frac{X}{\cos \theta}\right) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2} \cos \theta} \frac{2}{D} \cdot \frac{2}{\pi} dx d\theta = \frac{2L}{D\pi} \int_0^{\frac{\pi}{2}} \cos \theta d\theta = \frac{2L}{D\pi}. \quad (3.11)$$

**Proposition 3.4.** *The jointly continuous random variables  $X$  and  $Y$  are independent if and only if the probability density function can be factored into functions of  $x$  and  $y$  respectively.*

Suppose that  $X$  and  $Y$  are independent and jointly continuous random variables. Then  $f(x, y) = f_X(x)f_Y(y)$ . Let us compute the cumulative density function of  $X + Y$ .

$$P(X+Y \leq a) = \int_{-\infty}^{\infty} \int_{-\infty}^{a-x} f_X(x)f_Y(y) dy dx = \int_{-\infty}^{\infty} f_X(x)F_Y(a-x) dx = \int_{-\infty}^{\infty} f_Y(y)F_X(a-y) dy. \quad (3.12)$$

Differentiating to give the probability density function of  $X + Y$ ,

$$\frac{d}{da} F_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(x)f_Y(a-x) dx = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y) dy. \quad (3.13)$$

This is the convolution of  $f_X$  and  $f_Y$ .

*February 11th.*

Suppose we have  $U_1, U_2, \dots, U_n$  which are independent and have the distribution  $\text{Uniform}(0, 1)$ . From  $\{1, 2, \dots, n\}$ , if we choose a subset of size  $k$ , then each subset of size  $k$  has probability  $\frac{1}{\binom{n}{k}}$ . We will generate  $I_1, I_2, \dots, I_n$ , where exactly  $k$  of them are 1.  $I_1$  is defined as

$$I_1 = \begin{cases} 1 & \text{if } U_1 < \frac{k}{n}, \\ 0 & \text{if otherwise.} \end{cases} \quad (3.14)$$

Once  $I_1, I_2 < \dots, I_i$  are determined, we can define

$$I_{i+1} = \begin{cases} 1 & \text{if } U_{i+1} < \frac{k - (I_1 + I_2 + \dots + I_i)}{n - i}, \\ 0 & \text{if otherwise.} \end{cases} \quad (3.15)$$

Notice that if  $I_1 + I_2 + \dots + I_i = k$ , then  $I_{i+1}, I_{i+2}, \dots = 0$ . Note that  $P(I_1 = 1) = \frac{k}{n}$ , and

$$P(I_{i+1} = 1 | I_1, I_2, \dots, I_i) = \frac{k - \sum_{r=1}^i I_r}{n - i}, \text{ for } 1 < i \leq n. \quad (3.16)$$

If we do induction on  $k + n$ , we know the case  $k + n = 2$  to be true ( $k = 1, n = 1$ ). We assume for all the cases  $k + n \leq l$ , and suppose that  $k + n = l + 1$ . Consider any subset of size  $k$ , say,  $i_1 \leq i_2 \leq \dots \leq i_k$ .

- Case I, where  $i_1 = 1$ . Here,

$$\begin{aligned} & P(I_{i_1} = I_{i_2} = \dots = I_{i_k} = 1, I_j = 0 \text{ otherwise}) \\ &= P(I_1 = 1) \cdot P(I_{i_2} = \dots = I_{i_k} = 1, I_j = 0 \text{ otherwise} | I_1 = 1) \\ &= \frac{k}{n} \frac{1}{\binom{n-1}{k-1}} = \frac{1}{\binom{n}{k}}. \end{aligned}$$

- Case II, where  $i_1 > 1$ . In this case,

$$\begin{aligned} & P(I_{i_1} = 0, I_{i_2} = \dots = I_{i_k} = 1, I_j = 0 \text{ otherwise}) \\ &= P(I_1 = 0) \cdot P(I_{i_2} = \dots = I_{i_k} = 1, I_j = 0 \text{ otherwise} | I_1 = 0) \\ &= \left(1 - \frac{k}{n}\right) \frac{1}{\binom{n-1}{k}} = \frac{1}{\binom{n}{k}}. \end{aligned}$$

## 3.2 Some Distributions

### 3.2.1 Gamma Random Variable

We have  $X \sim \text{Gamma}(t, \lambda)$  where  $\lambda > 0$  and  $t > 0$ . The probability density function is defined as

$$f_X(x) = \frac{\lambda e^{-\lambda y} (\lambda y)^{t-1}}{\Gamma(t)} \text{ for } 0 < y < \infty. \quad (3.17)$$

Here,  $\Gamma$  is the gamma function.

**Proposition 3.5.** *If  $X \sim \text{Gamma}(s, \lambda)$  and  $Y \sim \text{Gamma}(t, \lambda)$ , then  $X + Y \sim \text{Gamma}(s + t, \lambda)$ .*

*Proof.* Via convolution, we know that

$$\begin{aligned} f_{X+Y}(a) &= \int_0^a f_X(a-y) f_Y(y) dy \\ &= \frac{1}{\Gamma(t)\Gamma(s)} \int_0^a \lambda e^{-\lambda(a-y)} (\lambda(a-y))^{t-1} \lambda e^{-\lambda y} (\lambda y)^{s-1} dy \\ &= \frac{\lambda e^{-\lambda a} (\lambda a)^{s+t-1}}{\Gamma(t+s)} \frac{\Gamma(t+s)}{\Gamma(t)\Gamma(s)} \int_0^a \frac{(a-y)^{t-1} y^{s-1}}{a^{s+t-1}} dy \\ &= \frac{\lambda e^{-\lambda a} (\lambda a)^{s+t-1}}{\Gamma(t+s)} \frac{\Gamma(t+s)}{\Gamma(t)\Gamma(s)} \int_0^1 (1-u)^{t-1} u^{s-1} du. \end{aligned}$$

If we integrate this probability distribution function, we must have

$$\int_0^\infty f_{X+Y}(a) da = 1 \quad (3.18)$$

$$\implies \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} \int_0^1 (1-u)^{t-1} u^{s-1} du = 1. \quad (3.19)$$

This gives rise to the Beta function, defined as  $\int_0^1 (1-u)^{t-1} u^{s-1} du = B(t, s) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$ . ■



One corollary that can be inferred from here is that if  $X_1, X_2, \dots, X_n$  are independent  $\text{Gamma}(t_i, \lambda)$  distributions, then  $\sum_{i=1}^n X_i \sim \text{Gamma}(\sum_{i=1}^n t_i, \lambda)$ . We also notice that  $X \sim \text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$ . If we take the  $X_i$ 's to be all the exponential distribution, then  $S_n = X_1 + X_2 + \dots + X_n \sim \text{Gamma}(n, \lambda)$ . The density of  $S_n$  is given by

$$g_n(y) = \frac{\lambda(\lambda y)^{n-1} e^{-\lambda y}}{\Gamma(n)} = \frac{\lambda(\lambda y)^{n-1} e^{-\lambda y}}{(n-1)!} \text{ for } 0 < y < \infty \quad (3.20)$$

with the cumulative distribution function

$$G_n(y) = \frac{1}{(n-1)!} \int_0^y \lambda(\lambda a)^{n-1} e^{-\lambda a} da. \quad (3.21)$$

**Example 3.6.** Suppose we have buses that each take time  $X_i \sim \text{Exp}(\lambda)$ . Suppose we fix a time  $t$ , and define  $N(t)$  to be the number of buses seen up to time  $t$ . We ask the probability  $P(N(t) = n)$ .

$$P(N(t) = n) = P(X_1 + X_2 + \dots + X_n \leq t, X_1 + X_2 + \dots + X_{n+1} > t) \quad (3.22)$$

$$= P(X_1 + \dots + X_n \leq t) - P(X_1 + \dots + X_n \leq t, X_1 + \dots + X_{n+1} \leq t)$$

$$= P(X_1 + \dots + X_n \leq t) - P(X_1 + \dots + X_{n+1} \leq t)$$

$$= G_n(t) - G_{n+1}(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \quad (3.23)$$

Notice that there is a unique  $k$  for which  $S_{k-1} < t \leq S_k$ . Define  $X_k$  to be  $S_k - S_{k-1}$ .

**Proposition 3.7.** The  $X_k$  satisfying  $S_{k-1} < t \leq S_k$  has density

$$v_t(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } 0 < x \leq t, \\ \lambda(1 + \lambda t) e^{-\lambda x} & \text{if } x > t. \end{cases} \quad (3.24)$$

*Proof.* Let  $k$  denote the index for which  $S_{k-1} < t \leq S_k$ . Define  $L_t = S_k - S_{k-1}$ . Let us first compute the cumulative distribution function.

- Case I, when  $x < t$ . Note that  $L_t \leq x \iff$  there exist  $n, y$  such that  $t - x \leq y < t$ ,  $S_n = y$ ,  $t - y \leq X_{n+1} \leq x$ .

$$\begin{aligned} P(L_t \leq x) &= \sum_{n=1}^{\infty} \int_{t-x}^t \int_{t-y}^x f_{S_n, X_{n+1}}(y, z) dz dy \\ &= \sum_{n=1}^{\infty} \int_{t-x}^t \int_{t-y}^x g_n(y) f_{X_{n+1}}(z) dz dy \\ &= \sum_{n=1}^{\infty} \int_{t-x}^t g_n(y) [e^{-\lambda(t-y)} - e^{-\lambda x}] dy \\ &= \lambda \int_{t-x}^t [e^{-\lambda(t-y)} - e^{-\lambda x}] dy. \end{aligned} \quad (3.25)$$

- Case II, when  $x > t$ . Note that

$$\{L_t \leq x\} = \{t < S_1 \leq x\} \cup \bigcup_{n=1}^{\infty} \{\text{bus } n \text{ arrives at } y < t \text{ and } t - y < X_{n+1} \leq x\}. \quad (3.26)$$

These are all disjoint events.  $P(t < S_1 \leq x) = e^{-\lambda t} - e^{-\lambda x}$ , and

$$P(\text{bus } n \text{ arrives at } y < t \text{ and } t - y < X_{n+1} \leq x) = \int_0^t \int_{t-y}^x g_n(y) f_{X_{n+1}}(z) dz dy. \quad (3.27)$$

Adding up the probabilities of the disjoint events, we have

$$\begin{aligned} P(L_t \leq x) &= e^{-\lambda t} - e^{-\lambda x} + \sum_{n=1}^{\infty} \int_0^t g_n(y) [e^{-\lambda(t-y)} - e^{-\lambda x}] dy \\ &= e^{-\lambda t} - e^{-\lambda x} + \lambda \int_0^t [e^{-\lambda(t-y)} - e^{-\lambda x}] dy \\ &= e^{-\lambda t} - e^{-\lambda x} + 1 - e^{-\lambda t} - \lambda t e^{-\lambda x}. \end{aligned} \quad (3.28)$$

■

### 3.3 Conditional Distribution

February 25th.

Let  $X$  and  $Y$  have a joint probability density function given by  $f(x, y)$ . The *conditional probability density function* of  $X$  given  $Y = y$  is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad (3.29)$$

and it is defined for  $f_Y(y) > 0$ . The motivation behind defining this is as follows—

$$f_{X|Y}(x|y)dx = \frac{f(x, y)dxdy}{f_Y(y)dy} = \frac{P(x \leq X \leq x+dx, y \leq Y \leq y+dy)}{P(y \leq Y \leq y+dy)} = P(x \leq X \leq x+dx | y \leq Y \leq y+dy). \quad (3.30)$$

If one is given the conditional probability density function, we can do the following computation as well,

$$P(X \in A | Y = y) = \int_A f_{X|Y}(x|y)dx. \quad (3.31)$$

We can also make sense of a *conditional cumulative distribution function* of  $X$  given  $Y = Y$ .

$$F_{X|Y}(a|y) = P(X \leq a | Y = y) = \int_{-\infty}^a f_{X|Y}(x|y)dx \quad (3.32)$$

**Remark 3.8.** If  $X$  and  $Y$  are independent, then the joint density factorizes into the product of the marginals which results in

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x). \quad (3.33)$$

**Example 3.9.** Suppose the joint density of  $X$  and  $Y$  is given as

$$f(x, y) = \begin{cases} e^{-\frac{x}{y}} e^{-y} y^{-1} & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{if otherwise.} \end{cases} \quad (3.34)$$

Let us compute  $P(X > 1 | Y = y)$  for  $0 < y < \infty$ .

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}} e^{-y}}{y \int_0^\infty \frac{e^{-\frac{x}{y}} e^{-y}}{y} dx} = \frac{1}{y} e^{-\frac{x}{y}} \text{ for } 0 < x < \infty. \quad (3.35)$$

Thus,

$$P(X > 1 | Y = y) = \int_1^\infty \frac{1}{y} e^{-\frac{x}{y}} dx = e^{-\frac{1}{y}}. \quad (3.36)$$

#### 3.3.1 The $t$ -distribution

Suppose we have  $Y \sim \chi_n^2 \equiv \text{Gamma}(\frac{n}{2}, \frac{1}{2})$  and  $Z \sim N(0, 1)$  with both independent. Then the  $t$ -distribution with  $n$  degrees of freedom is defined as

$$T = \frac{Z}{\sqrt{Y/n}} = \sqrt{n} \frac{Z}{\sqrt{Y}}. \quad (3.37)$$

If a  $Y = y$  is fixed, then  $T = \sqrt{\frac{n}{y}} Z \sim N(0, \frac{n}{y})$ . Thus,

$$f_{T|Y}(t|y) = \frac{1}{\sqrt{2\pi \frac{n}{y}}} e^{-\frac{1}{2} \frac{t^2 y}{n}} \quad (3.38)$$

$$\implies f_{T,Y}(t, y) = f_{T|Y}(t|y) f_Y(y) = \frac{1}{\sqrt{2\pi \frac{n}{y}}} e^{-\frac{1}{2} \frac{t^2 y}{n}} \left( \frac{e^{-\frac{y}{2}} y^{\frac{n}{2}-1}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \right) \text{ for } t \in \mathbb{R}, y > 0. \quad (3.39)$$

Thus, the probability density function for  $T$  can be found out as

$$f_T(t) = \int_0^\infty f_{T|Y}(t|y)dy = \int_0^\infty \frac{1}{\sqrt{2\pi n}} \cdot \frac{e^{-\frac{1}{2}(1+\frac{t^2}{n})y} y^{\frac{n-1}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} dy. \quad (3.40)$$

If we let  $c = \frac{1}{2}(1 + \frac{t^2}{n})$  and make the change of variable from  $x$  to  $cy$ , the intergral transforms as

$$\frac{c^{-\frac{n+1}{2}}}{\sqrt{\pi n} 2^{\frac{n+1}{2}} \Gamma(\frac{n}{2})} \int_0^\infty e^{-x} x^{\frac{(n-1)}{2}} dx = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n} \Gamma(\frac{n}{2})} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}. \quad (3.41)$$

Note that as  $n \rightarrow \infty$ ,

$$f_T(t) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}. \quad (3.42)$$

### 3.3.2 The Bivariate Normal Distribution

Five parameters are used here—  $\mu_x, \mu_y \in \mathbb{R}$ ,  $\sigma_x, \sigma_y > 0$ , and  $-1 < \rho < 1$ . The joint probability density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right)\right). \quad (3.43)$$

We find the conditional density of  $X$  given  $Y = y$ .

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x, y)}{f_Y(y)} = c_1(y)f(x, y) \\ &= c_2(y) \exp\left(-\frac{1}{2(1-\rho^2)} \left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\frac{x(y-\mu_y)}{\sigma_x\sigma_y}\right)\right) \\ &= c_2(y) \exp\left(-\frac{1}{2(1-\rho^2)\sigma_x^2} \left(x^2 - 2x\mu_x + \mu_x^2 - 2\rho\frac{\sigma_x}{\sigma_y}x(y-\mu_y)\right)\right) \\ &= c_3(y) \exp\left(-\frac{1}{2(1-\rho^2)\sigma_x^2} \left(x^2 - 2x(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y))\right)\right) \\ &= c_4(y) \exp\left(-\frac{1}{2(1-\rho^2)\sigma_x^2} \left(x - \left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)\right)^2\right). \end{aligned} \quad (3.44)$$

The integral of the last term must be 1, so we can conclude that it is the probability density function must be that of the normal distribution.

$$X|Y = y \sim N\left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y), \sigma_x^2(1 - \rho^2)\right), \quad (3.46)$$

$$Y|X = x \sim N\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)\right). \quad (3.47)$$

Note that  $X$  and  $Y$  are independent if and only if  $\rho = 0$ . Also,

$$f_X(x) = \frac{f(x, y)}{f_{Y|X}(y|x)} = \frac{1}{\sqrt{2\pi}\sigma_x^2} \exp\left(-\frac{1}{2\sigma_x^2}(x - \mu_x)^2\right). \quad (3.48)$$

## 3.4 Order Statistics

*February 28th.*

Suppose we have  $X_1, X_2, \dots, X_n$  independent and identically distributed continuous random variables with common probability density function  $f$  and cumulative distribution function  $F$ . We order the  $X_i$ 's in such a way that  $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ . These are termed the *order statistics* corresponding to  $X_1, X_2, \dots, X_n$ , and  $X_{(k)}$  is termed the  $k^{\text{th}}$  order statistic. Note that  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  takes the

values  $x_1 \leq x_2 \leq \dots \leq x_n$  if and only if for some permutation  $(i_1, i_2, \dots, i_n)$  of  $(1, 2, \dots, n)$  we have  $X_1 = x_{i_1}, X_2 = x_{i_2}, \dots, X_n = x_{i_n}$ . Also, for sufficiently small  $\varepsilon > 0$ ,

$$\begin{aligned} P\left(x_{i_1} - \frac{\varepsilon}{2} < X_1 < x_{i_1} + \frac{\varepsilon}{2}, x_{i_2} - \frac{\varepsilon}{2} < X_2 < x_{i_2} + \frac{\varepsilon}{2}, \dots, x_{i_n} - \frac{\varepsilon}{2} < X_n < x_{i_n} + \frac{\varepsilon}{2}\right) \\ \approx \varepsilon^n f(x_{i_1})f(x_{i_2}) \cdots f(x_{i_n}) = \varepsilon^n f(x_1)f(x_2) \cdots f(x_n). \end{aligned} \quad (3.49)$$

Therefore, for  $x_1 < x_2 < \dots < x_n$ , we have

$$\begin{aligned} P\left(x_1 - \frac{\varepsilon}{2} < X_{(1)} < x_1 + \frac{\varepsilon}{2}, x_2 - \frac{\varepsilon}{2} < X_{(2)} < x_2 + \frac{\varepsilon}{2}, \dots, x_n - \frac{\varepsilon}{2} < X_{(n)} < x_n + \frac{\varepsilon}{2}\right) \\ \approx n! \varepsilon^n f(x_1) \cdots f(x_n) \end{aligned} \quad (3.50)$$

$$\implies f_{(X_{(1)}, X_{(2)}, \dots, X_{(n)})}(x_1, x_2, \dots, x_n) = n! f(x_1)f(x_2) \cdots f(x_n). \quad (3.51)$$

**Example 3.10.** Three people are distributed on a 1 mile long rong, uniformly. Fix  $d \leq \frac{1}{2}$ . We are to find the probability that no two people are less distance  $d$  apart. Note that the probability density function is  $f_{(X_{(1)}, X_{(2)}, X_{(3)})}(x_1, x_2, x_3) = 6$  for  $0 \leq x_1 < x_2 < x_3 \leq 1$ . The probability is then given us

$$P(X_{(2)} - X_{(1)} \geq d, X_{(3)} - X_{(2)} \geq d) = \int_0^1 \int_{x_1+d}^1 \int_{x_2+d}^1 6 dx_3 dx_2 dx_1 = (1 - 2d)^3. \quad (3.52)$$

The marginal density of  $X_{(k)}$  is given as  $f_{X_{(k)}}(y_k) =$

$$\int \cdots \int_{x_1 < \dots < x_{k-1} < y_k < x_{k+1} < \dots < x_n} f_{(X_{(1)}, \dots, X_{(n)})}(x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n) dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n. \quad (3.53)$$

If we note that

$$F_{(X_{(n)})}(y) = P(X_{(n)} \leq y) = P(X_1 \leq y, \dots, X_n \leq y) = [F(y)]^n \quad (3.54)$$

then

$$f_{X_{(n)}}(y) = n[F(y)]^{n-1}f(y). \quad (3.55)$$

Similarly,

$$1 - F_{X_{(1)}}(y) = P(X_{(1)} > y) = P(X_1 > y, X_2 > y, \dots, X_n > y) = [1 - F(y)]^n \quad (3.56)$$

gives us

$$f_{X_{(1)}}(y) = n[1 - F(y)]^{n-1}f(y). \quad (3.57)$$

For the other marginal densities, we work as follows—

$$\begin{aligned} F_{X_{(k)}}(y) &= P(X_{(k)} \leq y) = P(\text{at least } k \text{ of } X_1, \dots, X_n \leq y) \\ &= \sum_{j=k}^n P(\text{exactly } j \text{ of } X_1, \dots, X_n \leq y) \\ &= \sum_{j=k}^n \binom{n}{j} F(y)^j (1 - F(y))^{n-j}. \end{aligned} \quad (3.58)$$

Differentiating this to find the probability density function is a tedious task. We work around this. Note that for  $X_{(k)}$  to attain the value  $y_k$ , one of the  $X_1, \dots, X_k$  should equal  $y_k$ ,  $k-1$  should be less than  $y_k$ , and  $n-k$  should be greater than  $y_k$ . For a fixed partition to satisfy these conditions, we have

$$P(X_{i_1}, X_{i_2}, \dots, X_{i_{k-1}} < y_k, X_{i_k} = y_k, X_{j_1}, X_{j_2}, \dots, X_{j_{n-k}} > y) = F(y)^{k-1} f(y) (1 - F(y))^{n-k}. \quad (3.59)$$

Thus, we can choose any partition in the above mentioned way to get

$$f_{X_{(k)}}(y) = \binom{n}{k-1, n-k, 1} F(y)^{k-1} f(y) (1 - F(y))^{n-k}. \quad (3.60)$$

For  $i < j$  the joint density function of  $(X_{(i)}, X_{(j)})$ ,  $f_{X_{(i)}X_{(j)}}(y_i, y_j)$  exists for  $y_i < y_j$ . From the same intuitive reasoning as before, we have

$$f_{X_{(i)}X_{(j)}}(y_i, y_j) = \binom{n}{i-1, 1, j-i-1, 1, n-j} F(y_i)^{i-1} f(y_i) (F(y_j) - F(y_i))^{j-i-1} f(y_j) (1 - F(y_j))^{n-j}. \quad (3.61)$$

**Example 3.11.** We are to find the cumulative distribution function of  $R = X_{(n)} - X_{(1)}$ , the range of the  $X_i$ 's. We simply have

$$\begin{aligned} P(R \leq a) &= P(X_{(n)} - X_{(1)} \leq a) = \iint_{x_n - x_1 \leq a} f_{X_{(1)}X_{(n)}}(x_1, x_n) dx_1 dx_n \\ &= \int_{-\infty}^{\infty} \int_{x_1}^{x_1+a} \frac{n!}{(n-2)!} [F(x_n) - F(x_1)]^{n-2} f(x_1) f(x_n) dx_n dx_1 \end{aligned} \quad (3.62)$$

Making the substitution  $y = F(x_n) - F(x_1)$  with  $dy = f(x_n) dx_n$  will help compute the function. The computation is left as an exercise to the reader.

*March 4th.*

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed as  $\text{Exp}(\alpha)$ . These will denote the service times of  $n$  counters in a post office, commencing at time 0. Thus,  $X_{(i)}$  will denote the time of the  $i^{\text{th}}$  discharge. We have

$$P(X_{(1)} > t) = P(X_1 > t, \dots, X_n > t) = \prod_{i=1}^n P(X_i > t) = e^{-n\alpha t} \implies X_{(1)} \sim \text{Exp}(n\alpha). \quad (3.63)$$

Also,

$$P(X_{(n)} \leq t) = P(X_1 \leq t, \dots, X_n \leq t) = \prod_{i=1}^n P(X_i \leq t) = (1 - e^{-\alpha t})^n.$$

Intuitively, from the memoryless property of the exponential distribution, we can infer a proposition.

**Proposition 3.12.** *The  $n$  variables  $X_{(1)}, X_{(2)} - X_{(1)}, \dots, X_{(n)} - X_{(n-1)}$  are independent, and the density of  $X_{(k+1)} - X_{(k)}$  is  $(n-k)\alpha e^{-(n-k)\alpha t}$ .*

*Proof.* We show the case for  $n = 3$ . The proof can then be generalized. We have

$$f_{X_{(1)}X_{(2)}X_{(3)}}(z_1, z_2, z_3) = 6f_{X_1, X_2, X_3}(z_1, z_2, z_3) = 6\alpha^3 e^{-\alpha} e^{-\alpha(z_1+z_2+z_3)}. \quad (3.64)$$

Also,

$$\begin{aligned} P(X_{(1)} > t_1, X_{(2)} - X_{(1)} > t_2, X_{(3)} - X_{(2)} > t_3) &= 6 \int_{t_1}^{\infty} \alpha e^{-\alpha z_1} \int_{z_1+t_2}^{\infty} \alpha e^{-\alpha z_2} \int_{z_2+t_3}^{\infty} \alpha e^{-\alpha z_3} dz_3 dz_2 dz_1 \\ &= e^{-\alpha t_3} e^{-2\alpha t_2} e^{-3\alpha t_1}. \end{aligned} \quad (3.65)$$

Therefore,  $X_{(1)} \sim \text{Exp}(3\alpha)$ ,  $X_{(2)} - X_{(1)} \sim \text{Exp}(2\alpha)$ , and  $X_{(3)} - X_{(2)} \sim \text{Exp}(\alpha)$ . ■

We ask a few questions; suppose  $A$  and  $B$  are currently being served at the office, and a third clerk  $C$  enters. The probability that  $C$  leaves last is one-half due to the memoryless property of the exponential distribution. To find the total time spent by  $C$ , we have  $T = X_{(1)} + Z$ , where  $Z$  is the distribution of  $C$ 's time spent being served. The distribution of  $T$  is given as

$$f(t) = \int_{\mathbb{R}} f_{X_{(1)}}(x) f_Z(t-x) dx = \int_0^t 2\alpha e^{-2\alpha x} \alpha e^{-\alpha(t-x)} dx = 2\alpha e^{-\alpha t} (1 - e^{-\alpha t}).$$

Let us also compute the distribution of the time of last departure,  $\tilde{T}$ . Note that  $C$  enters only when one of  $A$  or  $B$  is served. The first discharge has distribution  $Z \sim \text{Exp}(2\alpha)$ . When  $C$  enters, we have 2 independent  $\text{Exp}(\alpha)$  random variables, where the last discharge is given as  $X_{(2)}$ . Note that  $X_1, X_2$  are independent of  $Z$ . From the previous question, we know that  $X_{(2)} = X_{(2)} - X_{(1)} + X_{(1)}$  has distribution

$\text{Exp}(2\alpha) + \text{Exp}(\alpha)$ , and density  $2\alpha e^{-\alpha t}(1 - e^{-\alpha t})$ . Therefore, the distribution of  $\tilde{T}$  matches that of  $Z + X_{(2)}$ . Integrating gives us

$$f_{\tilde{T}}(\tilde{t}) = 4\alpha(e^{-\alpha\tilde{t}} - e^{-2\alpha\tilde{t}}) = \alpha\tilde{t}e^{-2\alpha\tilde{t}}. \quad (3.66)$$

Moving away from the exponential distribution, let us consider  $X_1, X_2, \dots, X_n$  to be independent and identically distributed to  $\text{Unif}[0, 1]$ . Note that  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  partition the interval  $[0, 1]$  into subintervals of length  $l_1 = X_{(1)}, l_2 = X_{(2)} - X_{(1)}, \dots, l_n = X_{(n)} - X_{(n-1)}$ . The lengths  $l_1, l_2, \dots, l_n$  are not independent since they have to satisfy  $\sum l_i = 1$ . The distributions of  $l_i$ 's turn out to be identical.

As another exercise, let  $X_1, X_2$  be independent and randomly chosen uniformly on the unit circle. Let  $r(x, y)$  denote the clockwise circular distance from points  $x$  to  $y$  on the unit circle. Note that  $r(X_1, X_2)$  has the same distribution as  $r(X_2, X_1)$ , computed as  $\text{Unif}[0, 1]$ .

### 3.5 Joint Distribution of Functions of Random Variables

*March 7th.*

Suppose  $(X_1, X_2)$  has joint probability density function  $f_{X_1 X_2}$ , and suppose  $Y_1 = g_1(X_1, X_2)$  and  $Y_2 = g_2(X_1, X_2)$  for some functions  $g_1, g_2$ . We assume that  $g_1$  and  $g_2$  satisfy the following conditions:

- The equations  $y_1 = g_1(x_1, x_2)$  and  $y_2 = g_2(x_1, x_2)$  can be uniquely solved for  $x_1$  and  $x_2$  in terms of  $y_1$  and  $y_2$ , say,  $x_1 = h_1(y_1, y_2)$  and  $x_2 = h_2(y_1, y_2)$ .
- $g_1$  and  $g_2$  have continuous partial derivatives at all  $(x_1, x_2)$  and the determinant of the *Jacobian matrix* is

$$J(x_1, x_2) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} = \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \neq 0. \quad (3.67)$$

Then  $(Y_1, Y_2)$  has the joint density

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1, x_2) |J(x_1, x_2)|^{-1} \quad (3.68)$$

where  $x_1 = h_1(y_1, y_2)$  and  $x_2 = h_2(y_1, y_2)$ . The proof of this comes from multivariate analysis.

**Remark 3.13.** The above can be extended to more than 2 random variables. Suppose that we have the probability density function  $f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n)$  and we have  $Y_1 = g_1(X_1, X_2, \dots, X_n), Y_2 = g_2(X_1, X_2, \dots, X_n), \dots, Y_n = g_n(X_1, X_2, \dots, X_n)$ . We have to assume that all the  $g_i$ 's have continuous partial derivatives, and we have to assume that

$$J(x_1, x_2, \dots, x_n) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \dots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} \neq 0 \text{ for all } (x_1, x_2, \dots, x_n). \quad (3.69)$$

We also suppose that that  $y_1 = g_1(x_1, \dots, x_n), \dots, y_n = g_n(x_1, \dots, x_n)$  has a unique solution given by  $x_1 = h_1(y_1, \dots, y_n), \dots, x_n = h_n(y_1, \dots, y_n)$ . The joint density function of  $(Y_1, \dots, Y_n)$  is then given by

$$f_{Y_1 Y_2 \dots Y_n}(y_1, y_2, \dots, y_n) = f_{X_1 X_2 \dots X_n}(x_1, x_2, \dots, x_n) |J(x_1, \dots, x_n)|^{-1} \quad (3.70)$$

where  $x_i = h_i(y_1, \dots, y_n)$ .

#### 3.5.1 Conditional Expectation and Variance

Let  $(X, Y)$  have a joint probability density function  $f(x, y)$ , and suppose  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function.

**Proposition 3.14.** *The expectation of  $g(X, Y)$  is given as*

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dy dx.$$

**Remark 3.15.** By this proposition, if we pick  $g(x, y) = x$  for all  $(x, y) \in \mathbb{R}^2$ , we get

$$EX = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dy dx = \int_{-\infty}^{\infty} x f_X(x) dx.$$

*Proof.* Assume that  $g(x, y) \geq 0$  for all  $(x, y) \in \mathbb{R}^2$ . We have

$$\begin{aligned} E[g(X, Y)] &= \int_0^{\infty} P(g(X, Y) > t) dt \\ &= \int_0^{\infty} \left( \iint_{\{(x, y): g(x, y) > t\}} f(x, y) dy dx \right) dt \\ &= \iint \int_0^{g(x, y)} f(x, y) dt dy dx \\ &= \iint g(x, y) f(x, y) dy dx. \end{aligned} \quad (3.71)$$

For a general  $g$ , we simply work with  $g = g^+ - g^-$  where  $g^+(x, y) = \max\{g(x, y), 0\}$  and  $g^-(x, y) = \max\{-g(x, y), 0\}$ . ■

If  $(X, Y)$  have a joint probability density function  $f(x, y)$ , then it can be shown that  $E[X + Y] = EX + EY$ .

**Example 3.16.** Suppose  $X, Y \sim \text{Unif}[0, L]$ . We compute  $E|X - Y|$ . Noting that the joint probability density function is  $f(x, y) = \frac{1}{L^2}$  for all  $(x, y) \in [0, L] \times [0, L]$ , we have

$$\begin{aligned} E|X - Y| &= \int_0^L \int_0^L \frac{|x - y|}{L^2} dy dx = \frac{1}{L^2} \int_0^L \left( \int_0^x (x - y) dy + \int_x^L (y - x) dy \right) dx \\ &= \frac{1}{L^2} \int_0^L \left( \frac{x^2}{2} + \frac{(L - x)^2}{2} \right) dx = \frac{L}{3}. \end{aligned} \quad (3.72)$$

Recall that the conditional density of  $X$  given  $Y = y$  is given as

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

provided that  $f_Y(y) > 0$ .

**Definition 3.17.** The conditional expectation is given as

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

provided that  $f_Y(y) > 0$ .

**Example 3.18.** We compute  $E[X|Y = y]$  for  $f(x, y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y}$  for  $x, y > 0$ . It can be shown that the marginal density  $f_Y(y)$  is  $f_Y(y) = e^{-y}$ . Thus, the conditional density is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}} e^{-y}}{y e^{-y}} = \frac{1}{y} e^{-\frac{x}{y}}. \quad (3.73)$$

The conditional expectation is then given as

$$E[X|Y = y] = \int_0^{\infty} \frac{x e^{-\frac{x}{y}}}{y} dx = y.$$

It can be shown that

$$E[g(X)|Y = y] = \int g(x) f_{X|Y}(x|y) dx. \quad (3.74)$$

**Proposition 3.19.** *We have*

$$EX = E[E[X|Y]].$$

*Proof.* Since  $E[X|Y = y] = \int x f_{X|Y}(x|y) dx$  is a function of  $y$ , we have

$$\begin{aligned} E[E[X|Y]] &= \int_{-\infty}^{\infty} f_Y(y) \left( \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \right) dy = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{\infty} x \frac{f(x, y)}{f_Y(y)} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dy dx = \int_{-\infty}^{\infty} x f_X(x) dx = EX. \end{aligned}$$

■

The conditional variance is also given as

$$\text{Var}(X|Y = y) = E[X^2|Y = y] - (E[X|Y = y])^2 = \int x^2 f_{X|Y}(x|y) dx - \left( \int x f_{X|Y}(x|y) dx \right)^2. \quad (3.75)$$

Also recall the conditional variance formula—

$$\begin{aligned} \text{Var}(X) &= E[(X - EX)^2] = E[(X - E[X|Y] + E[X|Y] - EX)^2] \\ &= E[(X - E[X|Y])^2 + (E[X|Y] - EX)^2 + 2(X - E[X|Y])(E[X|Y] - EX)] \\ &= E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) + 0. \end{aligned} \quad (3.76)$$

**Example 3.20.** We look at the bivariate normal distribution. The probability density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left( -\frac{1}{2(1-\rho^2)} \left( \left( \frac{x-\mu_x}{\sigma_x} \right)^2 + \left( \frac{y-\mu_y}{\sigma_y} \right)^2 - 2\rho \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right) \right). \quad (3.77)$$

It can be shown that the correlation between  $X$  and  $Y$  is indeed  $\rho$ , that is,  $\frac{E(XY) - \mu_x\mu_y}{\sigma_x\sigma_y} = \rho$ .  $E(XY)$  is computed by considering  $E(XY) = E[E(XY|Y)]$ .

**Example 3.21.** Suppose  $(S, R)$  has the following distributions;  $R|S = s \sim N(s, 1)$  and  $S \sim N(\mu, \sigma^2)$ . We wish to compute  $E[S|R = r]$  and  $\text{Var}(S|R = r)$ . The former can be found out by first computing

$$f_{S|R}(s|r) = \frac{f_{S,R}(s, r)}{f_R(r)} = \frac{f_{R|S}(r|s)f_S(s)}{f_R(r)}. \quad (3.78)$$

*March 18th.*

For random variables  $X$  and  $Y$  which are jointly distributed, the following hold true:

- $E[(X - a)^2] \geq E[(X - EX)^2]$ ,
- $E[(Y - g(X))^2] \geq E[(Y - E[Y|X])^2]$ .

The best predictor of  $Y$  using  $X$  is  $E[Y|X]$ .

**Example 3.22.** Suppose  $X$  is a continuous random variable that we wish to discretize. We first fix an increasing set of numbers  $\dots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \dots$  such that  $\lim_{i \rightarrow \infty} a_i = \infty$  and  $\lim_{i \rightarrow -\infty} a_i = -\infty$ . Let  $Y$  be a random variable taking the value  $y_i$  when  $a_i < X \leq a_{i+1}$ . We wish to choose  $y_i$  that minimizes  $E[(X - Y)^2]$ . For these optimal  $y_i$ 's one can also show that  $EY = EX$  and  $\text{Var}Y = \text{Var}X - E[(X - Y)^2]$ .

Start with

$$\begin{aligned} E[(X - Y)^2] &= \sum_i E[(X - Y)^2 | a_i < X \leq a_{i+1}] P(a_i < X \leq a_{i+1}) \\ &= \sum_i E[(X - y_i)^2 | a_i < X \leq a_{i+1}] P(a_i < X \leq a_{i+1}). \end{aligned} \quad (3.79)$$



It is enough to minimize  $E[(X - y_i)^2 | a_i < X \leq a_{i+1}]$ .  $y_i$  is then computed to be

$$y_i = E[X | a_i < X \leq a_{i+1}] = \frac{E[X \cdot 1_{\{a_i < X \leq a_{i+1}\}}]}{P(a_i < X \leq a_{i+1})} = \frac{\int_{a_i}^{a_{i+1}} x f_X(x) dx}{\int_{a_i}^{a_{i+1}} f_X(x) dx}. \quad (3.80)$$

To show that the expectations of  $Y$  and  $X$  are equal, we have

$$EY = \sum_i y_i P(Y = y_i) = \sum_i E[X | a_i < X \leq a_{i+1}] P(a_i < X \leq a_{i+1}) = EX. \quad (3.81)$$

Define a random variable  $I$  to be  $I = i$  if  $a_i < X \leq a_{i+1}$ . Then

$$\text{Var} X = \text{Var}(E[X|I]) + E[\text{Var}(X|I)] \quad (3.82)$$

can be used. We can compute  $E[X|I]$  as  $y_I = Y$  and  $\text{Var}(X|I)$  as  $E[(X - Y)^2 | I]$ . Thus, the variance equality follows.

## Chapter 4

# CONVERGENCE OF RANDOM VARIABLES

### 4.1 Types of Convergence

We begin with a probability space  $(\Omega, P)$ , and define a sequence of random variables  $X_n : \Omega \rightarrow \mathbb{R}$  for  $n \geq 1$  and also have another  $X : \Omega \rightarrow \mathbb{R}$ .

#### 4.1.1 Almost Sure Convergence

**Definition 4.1.** We first discuss the notion of *almost sure convergence*. A sequence of random variables  $X_n$  converges almost surely to  $X$ , that is,  $X_n \xrightarrow{\text{a.s.}} X$  if

$$P(\{\omega : X_n(\omega) \rightarrow X(\omega)\}) = 1.$$

**Example 4.2.** Suppose that  $(\Omega, P) = ([0, 1], \text{Unif})$ . Suppose that  $q_n$  is an enumeration of the rationals in  $[0, 1]$ . We define  $X(\omega) = 1$  and  $X_n(\omega) = 1$  if  $\omega \in [0, 1] \setminus \{q_n\}$  and 0 otherwise. We find that

$$A = \{\omega : X_n(\omega) \rightarrow X(\omega)\} = [0, 1] \quad (4.1)$$

since for any rational  $\omega$ , further enough terms lead to  $X_n(\omega) = 0$  and for any irrational  $\omega$ , we always have  $X_n(\omega) = 1$ . Thus,  $P(A) = 1$ .

**Example 4.3.** As a continuation to the previous example, if we define the sequence of random variables as

$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in \{q_1, \dots, q_n\}, \\ 1 & \text{if otherwise,} \end{cases} \quad (4.2)$$

we find that  $\{\omega : X_n(\omega) \rightarrow X(\omega)\} = [0, 1] \setminus \mathbb{Q}$ . The sequence converges to 0 for rational  $\omega$ . But, in our probability space,  $P([0, 1] \setminus \mathbb{Q}) = 1$ .

**Example 4.4.** If we define the sequence of random variables as  $X_n = n1_{[0, \frac{1}{n}]}$ , we find that  $X_n \xrightarrow{\text{a.s.}} 0$ . However,  $EX_n = 1$  and  $E[0] = 0$ . The expectation is not preserved in almost sure convergence

#### 4.1.2 Convergence in Probability

**Definition 4.5.** We discuss the notion of *convergence in probability*. We say that a sequence of random variables  $X_n$  converges in probability to  $X$ , that is,  $X_n \xrightarrow{P} X$  if for every  $\varepsilon > 0$ ,

$$P(|X_n - X| > \varepsilon) \rightarrow 0.$$

To show that a sequence of random variables  $X_n$  converges in probability to  $X$ , it is enough to show that  $P(|X_n - X| > \frac{1}{k}) \rightarrow 0$  for all  $k \geq 1$ .

**Theorem 4.6.**  $X_n \xrightarrow{a.s.} X$  implies  $X_n \xrightarrow{P} X$ .

*Proof.* We have

$$\begin{aligned} \{\omega : X_n(\omega) \rightarrow X(\omega)\} &= \{\omega : \text{for all } k \geq 1, \exists N \text{ such that for all } n \geq N, |X_n(\omega) - X(\omega)| \leq \frac{1}{k}\} \\ &= \bigcap_{k \geq 1} \bigcup_N \bigcap_{n \geq N} \{|X_n - X| \leq \frac{1}{k}\}. \end{aligned} \quad (4.3)$$

The probability of this set is 1 due to almost sure convergence. Since the intersection of these events has a probability of 1, each inside event must also have a probability of 1; thus,

$$P\left(\bigcup_N \bigcap_{n \geq N} \{|X_n - X| \leq \frac{1}{k}\}\right) = 1. \quad (4.4)$$

We note that  $\bigcap_{n \geq N} \{|X_n - X| \leq \frac{1}{k}\}$  are events increasing in  $N$ . Therefore,

$$P\left(\bigcap_{n \geq N} \{|X_n - X| \leq \frac{1}{k}\}\right) \rightarrow 1 \implies P(|X_n - X| \leq \frac{1}{k}) \rightarrow 1. \quad (4.5)$$

■

However, the converse is not true.

*March 21st.*

**Theorem 4.7.** Let  $X_1, X_2, \dots$  be a sequence of random variables. If  $EX_n \rightarrow c$  and  $\text{Var}(X_n) \rightarrow 0$ , then  $X_n \xrightarrow{P} c$ .

*Proof.* We have

$$P(|X_n - c| > \varepsilon) = P(|X_n - EX_n + EX_n - c| > \varepsilon) \leq P(|X_n - EX_n| + |EX_n - c| > \varepsilon). \quad (4.6)$$

Choose a natural  $N$  such that  $|EX_n - c| < \frac{\varepsilon}{2}$ , which gives us

$$P(|X_n - c| > \varepsilon) \leq P(|X_n - EX_n| > \frac{\varepsilon}{2}) \leq \frac{E[|X_n - EX_n|^2]}{\varepsilon^2/4} = \text{Var}X_n \frac{4}{\varepsilon^2} \rightarrow 0. \quad (4.7)$$

■

A consequence of this is the weak law of large numbers.

**Corollary 4.8.** Let  $X_1, X_2, \dots$  be independent and identically distributed random variables, with  $EX_1 = \mu$  and  $\text{Var}X_1 = \sigma^2 < \infty$ . Let  $\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}$ . Then  $\bar{X}_n \xrightarrow{P} \mu$ .

*Proof.* We note that  $E\bar{X}_n = \mu$  and  $\text{Var}\bar{X}_n = \frac{1}{n}\sigma^2$ . The above theorem proves the law. ■

**Example 4.9.** We put  $n$  balls randomly into  $n$  boxes. Here, let  $X_n$  denote the number of empty boxes. We are interested in  $Y_n = \frac{X_n}{n}$ , the proportion of empty boxes. We claim that  $Y_n$  converges to  $e^{-1}$ . To compute the mean, we write  $EX_n = E(\sum_{i=1}^n 1_{A_i})$ , where  $A_i$  is the event that the  $i^{\text{th}}$  box is empty. Thus, this is  $EX_n = \sum_{i=1}^n \frac{(n-1)^n}{n^n} = n(1 - \frac{1}{n})^n \implies EY_n = (1 - \frac{1}{n})^n \rightarrow \frac{1}{e}$ .

For the variance,  $\text{Var}X_n = \sum_{i,j=1}^n \text{Cov}(1_{A_i}, 1_{A_j})$ . For  $i = j$ , this is

$$\text{Cov}(1_{A_i}, 1_{A_j}) = E[1_{A_i}] - (E[1_{A_i}])^2 = P(A_i)(1 - P(A_i)) = \frac{(n-1)^n}{n^n} \left(1 - \frac{(n-1)^n}{n^n}\right). \quad (4.8)$$

For  $i \neq j$ , we have

$$\text{Cov}(1_{A_i}, 1_{A_j}) = E[1_{A_i} 1_{A_j}] - P(A_i)P(A_j) = \left(\frac{n-2}{n}\right)^n - \left(\frac{n-1}{n}\right)^{2n}. \quad (4.9)$$

Thus, the final variance is calculated as

$$\text{Var}X_n = n \left(\frac{n-1}{n}\right)^n \left(1 - \left(\frac{n-1}{n}\right)^n\right) + n(n-1) \left(\left(\frac{n-2}{n}\right)^n - \left(\frac{n-1}{n}\right)^{2n}\right) \quad (4.10)$$

$$\implies \text{Var}Y_n = \frac{1}{n^2} \text{Var}X_n \rightarrow 0. \quad (4.11)$$

Thus,  $Y_n \xrightarrow{P} e^{-1}$ .

**Example 4.10.** We toss a  $p$ -coin, and look at  $X_n$ , the number of head runs in  $n$  tosses. We claim that  $Y_n = \frac{1}{n}X_n \xrightarrow{p} p(1-p)$ . We rewrite  $X_n$  as  $X_n = \sum_{i=1}^n 1_{A_i}$  where  $A_i$  is the event that a head run starts at position  $i$ . We have

$$EX_n = \sum_{i=1}^n P(A_i) = P(A_1) + \sum_{i=2}^n P(A_i) = p + (n-1)(1-p)p \implies EY_n \rightarrow p(1-p). \quad (4.12)$$

For the variance,

$$\text{Var}X_n = \sum_{i,j=1}^n \text{Cov}(1_{A_i}, 1_{A_j}) = \sum_{i=1}^n \text{Var}1_{A_i} + 2 \sum_{i<j} \text{Cov}(1_{A_i}, 1_{A_j}). \quad (4.13)$$

The variance of a single indicator,  $\text{Var}1_{A_i}$  is  $p - p^2$  if  $i = 1$ , and  $(1-p)p - ((1-p)p)^2$  if  $i > 1$ . For the covariance, if  $|i-j| \geq 2$ , the events  $A_i$  and  $A_j$  are clearly independent, so  $\text{Cov}(1_{A_i}, 1_{A_j}) = 0$ .

$$\text{Var}Y_n = \frac{1}{n^2} \text{Var}X_n = \frac{1}{n^2} (p - p^2 + (n-1)((1-p)p - (1-p)^2 p^2)) + \frac{1}{n^2} (\dots) \rightarrow 0. \quad (4.14)$$

**Example 4.11.** We revisit the coupon collector problem; there are  $n$  types of coupons, and each purchase results in one of the  $n$  types with equal probability.  $T_n$  denotes the number of purchases to get  $n$  types. The claim is that  $\frac{1}{n \log n} T_n \xrightarrow{p} 1$ .

Clearly,  $T_1 = 1$ ,  $T_2 - T_1 \sim \text{Geom}(\frac{n-1}{n})$ ,  $T_3 - T_2 \sim \text{Geom}(\frac{n-2}{n})$ , etc. These are all independent quantities, and the expectation of a geometric distribution is  $\frac{1}{p}$ . Thus,

$$\begin{aligned} T_n &= T_1 + (T_2 - T_1) + \dots + (T_n - T_{n-1}) \\ \implies ET_n &= n \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \implies \frac{1}{n \log n} ET_n = \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\log n} \rightarrow 1. \end{aligned} \quad (4.15)$$

and the variance is

$$\frac{1}{n^2 (\log n)^2} \text{Var}T_n = \frac{1}{(n \log n)^2} n \sum_{i=1}^n \frac{(i-1)}{(n-(i-1))^2} \leq \frac{1}{n^2 (\log n)^2} n^2 \sum_{i=1}^n \frac{1}{i^2} \rightarrow 0. \quad (4.16)$$

**Theorem 4.12** (The *strong law of large numbers*). Let  $X_1, X_2, \dots$  be independent and identically distributed random variables with  $EX_1 = \mu \in (-\infty, \infty)$ . Let  $\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$ . Then  $\bar{X}_n \xrightarrow{\text{a.s.}} \mu$ .

*Proof.* We prove this assuming that  $K = EX_1^4 < \infty$ . Without the loss of generality, let  $\mu = 0$ . Denote  $S_n = \sum_{i=1}^n X_i$ . The terms inside  $ES_n^4$  are any one of the form  $X_i^4, X_i^3 X_j, X_i^2 X_j^2, X_i^2 X_j X_k, X_i X_j X_k X_l$ , where  $i, j, k, l$  are distinct. We have assumed that  $\mu = 0$ , so

$$E[X_i^3 X_j] = E[X_i^3] EX_j = 0, \quad E[X_i^2 X_j X_k] = E[X_i^2] EX_j EX_k = 0, \quad E[X_i X_j X_k X_l] = 0. \quad (4.17)$$

Thus, the only remaining terms are

$$\begin{aligned} ES_n^4 &= n EX_1^4 + \binom{n}{2} \binom{4}{2} (EX_1^2)^2 \\ \implies E\bar{X}_n^4 &\leq \frac{1}{n^3} EX_1^4 + \frac{3}{n^2} (EX_1^2)^2 \implies \sum_{n=1}^{\infty} E\bar{X}_n^4 = \sum_{n=1}^{\infty} \frac{1}{n^3} EX_1^4 + \frac{3}{n^2} (EX_1^2)^2 < \infty. \end{aligned} \quad (4.18)$$

The expectation of the infinite sum is finite, which means

$$E \left( \sum_{n=1}^{\infty} \bar{X}_n^4 \right) < \infty \implies P \left( \sum_{n=1}^{\infty} \bar{X}_n^4 < \infty \right) = 1 \implies P(\bar{X}_n \rightarrow 0) = 1. \quad (4.19)$$

■

### 4.1.3 Convergence in Distribution

March 28th.

**Definition 4.13.** We discuss the notion of *convergence in distribution*. Suppose  $X_1, X_2, \dots$  are a sequence of random variables, and let  $X$  be another random variables. Denote the cumulative distribution function of  $X_n$  by  $F_n$  and that of  $X$  by  $F$ . We say that the sequence of random variables  $X_n$  converges to  $X$  if  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$  for all  $x$  at which  $F$  is continuous.

We write  $X_n \Rightarrow X$  or  $F_n \Rightarrow F$ , or even  $X_n \xrightarrow{d} X$  if  $X_n$  converges to  $X$  in distribution.

**Remark 4.14.** When  $X$  has a density function  $f$ , then  $F$  is continuous everywhere and we need the sequence to converge everywhere.

When  $F$  is continuous, we have  $P(a < X_n \leq b) = P(X_n \leq b) - P(X_n \leq a) = F_n(b) - F_n(a) \rightarrow F(b) - F(a)$ . Similarly, we can show that  $P(a \leq X_n \leq b) \rightarrow P(a < X \leq b)$  since  $P(X = a) = 0$ ; we wish to show that  $P(X_n = a) = 0$  is  $X_n$  converges to a  $X$  with continuous  $F$ . This can be shown as

$$P(X_n = a) \leq P(a - \varepsilon < X_n \leq a) = F_n(a) - F_n(a - \varepsilon) \implies \limsup_{n \rightarrow \infty} P(X_n = a) \leq F(a) - F(a - \varepsilon).$$

Letting  $\varepsilon \downarrow 0$ , we find  $\limsup_{n \rightarrow \infty} P(X_n = a) = 0 \implies \lim_{n \rightarrow \infty} P(X_n = a) = 0$ .

**Example 4.15.** Suppose  $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$  and  $X \sim \text{Poisson}(\lambda)$ . We have already seen that  $P(X_n = k) \rightarrow P(X = k)$ . For  $x \geq 0$ , we have

$$\lim_{n \rightarrow \infty} P(X_n \leq x) = \lim_{n \rightarrow \infty} \sum_{0 \leq k \leq x} P(X_n = k) = \sum_{0 \leq k \leq x} \lim_{n \rightarrow \infty} P(X_n = k) = \sum_{0 \leq k \leq x} P(X = k) = P(X \leq x). \quad (4.20)$$

Suppose  $X_1, X_2, \dots$  are independent and identically distributed random variables with finite mean  $\mu$  and finite variance  $\sigma^2$ . From the law of large numbers, we know that  $\frac{S_n}{n} \rightarrow \mu$  where  $S_n$  are the partial sums. The central limit theorem roughly states that  $\frac{S_n}{n} = \mu + O(\frac{1}{\sqrt{n}})$ . We state this more formally.

**Theorem 4.16** (The *central limit theorem*). *Let  $X_1, X_2, \dots$  be independent and identically distributed random variables with with quantities defined as above. The central limit theorem states that*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow N(0, 1).$$

For  $-\infty < a \leq b \leq \infty$ , we have

$$P\left(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b\right) \xrightarrow{n \rightarrow \infty} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

The left hand side may be rewritten as  $P(n\mu + a\sigma\sqrt{n} \leq S_n \leq n\mu + b\sigma\sqrt{n})$  and the right hand side may be rewritten as  $P(m\mu + a\sigma\sqrt{m} \leq Z_1 + \dots + Z_m \leq m\mu + b\sigma\sqrt{m})$  where  $Z_1, \dots, Z_m \sim N(\mu, \sigma^2)$  are independent.

**Example 4.17.** Suppose a fair coin is tossed a million times, and denote  $S$  to be the number of heads. This is the sum of a million independent and identically distributed  $\text{Ber}(\frac{1}{2})$  variables. Here,  $\mu = \frac{1}{2}$  and  $\sigma^2 = \frac{1}{4}$ . In this case,  $n\mu = 500,000$  and  $\sigma\sqrt{n} = 500$ . The central limit theorem states that  $\frac{S - 500,000}{500} \approx N(0, 1)$ . Using  $a = -2.576$  and  $b = 2.576$ , we have

$$P(498712 \leq S \leq 501288) \approx \int_{-2.576}^{2.576} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.99. \quad (4.21)$$

Suppose  $g : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $g(x) = e^{-\frac{1}{x}}$  for  $x > 0$ , and  $g(x) = 0$  for  $x \leq 0$ . For  $x \neq 0$ , clearly  $g(x)$  is infinitely differentiable. We shall show that  $g(x)$  is also infinitely differentiable at  $x = 0$  with  $g^{(k)}(0) = 0$  always. We have

$$\lim_{x \uparrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \uparrow 0} \frac{0}{x} = 0 \quad (4.22)$$

and

$$\lim_{x \downarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \downarrow 0} \frac{e^{-\frac{1}{x}}}{x} = 0. \quad (4.23)$$

Thus,  $g(x)$  is differentiable at 0, and there exists a  $C$  such that  $|g'(x)| \leq C$  for all  $x \in \mathbb{R}$ . For any  $k^{\text{th}}$  derivative of  $g(x)$  for  $x > 0$ , we obtain  $g^{(k)}(x) = e^{-\frac{1}{x}} p_{2k}(\frac{1}{x})$  where  $p_{2k}$  is a polynomial of degree  $2k$ . Again, the only problem is at 0. By the induction hypothesis,  $g^{(k-1)}(0) = 0$ . We have

$$\lim_{x \downarrow 0} \frac{g^{(k-1)}(x) - g^{(k-1)}(0)}{x - 0} = \lim_{x \downarrow 0} \frac{g^{(k-1)}(x)}{x} = \lim_{x \downarrow 0} \frac{e^{-\frac{1}{x}} p_{2(k-1)}(\frac{1}{x})}{x} = 0 \quad (4.24)$$

and left hand limit is easily seen to be zero. Hence,  $g^{(k)}(0) = 0$  and there always exists a  $C_k$  such that  $|g^{(k)}(x)| \leq C_k$  for all  $x \in \mathbb{R}$ . Thus,  $g \in C_b^\infty$ , the set of functions on  $\mathbb{R}$  that are infinitely differentiable with bounded derivatives, including the  $0^{\text{th}}$  derivative.

For any  $-\infty < a < b < \infty$ , let  $g_{a,b}(x) = g(x-a)g(b-x)$ . We note that  $g_{a,b} \equiv 0$  on  $\mathbb{R} \setminus [a, b]$  and  $g_{a,b} \in C_b^\infty$ . Also,  $g > 0$  on  $(a, b)$ . Define  $C_{a,b}$  to be the integral  $\int_{\mathbb{R}} g_{a,b}(x) dx > 0$ . Lastly, defining  $h_{a,b}(x) = \frac{1}{C_{a,b}} g_{a,b}(x)$  gives a probability density function. Denote the cumulative distribution function by  $H_{a,b}$ .

$$H_{a,b}(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a], \\ \text{strictly increasing} & \text{if } x \in (a, b), \\ 1 & \text{if } x \in [b, \infty). \end{cases} \quad (4.25)$$

We may also note that  $H_{a,b} \in C_b^\infty$ . We have defined this distribution for the following lemma.

**Lemma 4.18.** *Let  $X$  be a random variable and suppose  $X_1, X_2, \dots$  be a sequence of random variables such that for each  $f \in C_b^\infty$  we have  $Ef(X_n) \rightarrow Ef(X)$ . Then  $X_n \rightrightarrows X$ .*

*Proof.* Let  $F_n$  be the cumulative distribution function of  $X_n$  and  $F$  be that of  $X$ . Fix  $t \in \mathbb{R}$  at which  $F$  is continuous, and also fix  $s < t$ . We need to show  $F_n(t) \xrightarrow{n \rightarrow \infty} F(t)$ . Consider  $H_{s,t}$  defined as above;  $H_{s,t} \in C_b^\infty$ . Look at  $1 - H_{s,t} \leq 1_{(-\infty, t]}$ . We then have

$$\begin{aligned} E[1 - H_{s,t}(X_n)] &\leq E[1_{(-\infty, t]}(X_n)] = F_n(t) \\ \implies E[1 - H_{s,t}(X)] &= \lim_{n \rightarrow \infty} E[1 - H_{s,t}(X_n)] \leq \liminf_{n \rightarrow \infty} F_n(t) \end{aligned} \quad (4.26)$$

This gives us  $F(s) \leq \liminf_{n \rightarrow \infty} F_n(t)$ . Letting  $s \uparrow t$ , we get  $F(t) \leq \liminf_{n \rightarrow \infty} F_n(t)$ .

Similarly, consider  $H_{t,s}$  for  $s > t$ . We get

$$\limsup_{n \rightarrow \infty} F_n(t) = \limsup_{n \rightarrow \infty} E[1_{(\infty, t]}(X_n)] \leq \limsup_{n \rightarrow \infty} E[1 - H_{t,s}(X_n)] = E[1 - H_{t,s}(X)] \leq E[1_{(-\infty, s]}(X)] = F(s). \quad (4.27)$$

Letting  $s \downarrow t$ , we have  $\limsup_{n \rightarrow \infty} F_n(t) \leq F(t)$ . Also considering  $F(t) \leq \liminf_{n \rightarrow \infty} F_n(t)$ , we obtain  $F_n(t) \rightarrow F(t)$ . ■



# Appendices





## Chapter A

# Appendix

Extra content goes here.



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