

PROBABILITY THEORY II

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Second Semester

List of Symbols

Ω , a sample space.

ω , an element of a sample space.

EX , the expectation of the random variable X .

$\text{Var}X$, the variance of the random variable X .

$N(\mu, \sigma^2)$, a normal distribution with expectation μ and variance σ^2 .

$N_n(k)$, the number of paths from $(0, 0)$ to (n, k) in a simple random walk.

$N_n^+(k)$, the number of paths from $(0, 0)$ to (n, k) through strictly positive values in a random walk.

p_k^X , the probability mass function for a random variable X .

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Chapter 1

RANDOM WALKS AND MISC. RESULTS

January 3rd.

We first start with some initial statements. Let Ω be a countable state space, and let each $\omega \in \Omega$ have a probability $P(\omega)$ associated with it.

Lemma 1.1. *For random variables X, Y such that $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. Then, $EX \leq EY$.*

Proof. This can easily be seen by summing over all terms via the alternate definition of the expectation,

$$EX = \sum_{\omega \in \Omega} X(\omega)P(\omega) \leq \sum_{\omega \in \Omega} Y(\omega)P(\omega) = EY. \quad (1.1)$$

■

We now state Markov's inequality.

Theorem 1.2 (Markov's inequality). *If X is a non-negative random variable, then for $a > 0$, we have*

$$P(X > a) \leq \frac{EX}{a}. \quad (1.2)$$

Proof. Define an indicator function $I_a(\omega)$ as 1 if $X(\omega) \geq a$, and 0 if otherwise. We then have

$$I_a(\omega) \leq \frac{X(\omega)}{a} \implies P(X \geq a) = EI_a \leq \frac{1}{a}EX. \quad (1.3)$$

■

Remark 1.3. A better upper bound here may be found by starting with $I_a(\omega)X(\omega)$ instead of just $X(\omega)$.

If we have $X \sim N(0, 1)$, then we can find an upper bound for its probability density function.

$$P(X > a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \leq \int_a^\infty \frac{1}{\sqrt{2\pi}} \frac{x}{a} e^{-\frac{x^2}{2}} dx = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}a}. \quad (1.4)$$

Note that X here is a random variable over a continuous state space; the previous lemma and Markov's inequality also work here. We are to show them for the continuous case instead of the discrete one.

Proof. Here, we have $0 \leq X(\omega) \leq Y(\omega)$ for all ω in our continuous state space Ω . We see that $\{X > x\} \subseteq \{Y > x\} \implies P(X > x) \leq P(Y > x)$. Integrating both sides gives us $EX \leq EY$. ■

Theorem 1.4 (Chebyshev's inequality). *Let X be a random variable with finite mean $\mu = EX$ and finite variance $\sigma^2 = \text{Var}(X)$. Then for $a > 0$,*

$$P(|X - \mu| > a) \leq \frac{\text{Var}(X)}{a^2}. \quad (1.5)$$

Proof. Start with the proof of Markov's inequality, replacing the indicator function with one that's unity when $|X - \mu| \geq a$. ■

Example 1.5. Suppose X_1, X_2, \dots, X_n are n independent and identically distributed random variables, with $EX_i = \mu$ and $\text{Var}X_i = \sigma^2$. If $S_n = \sum X_i$, we then have

$$P(|S_n - n\mu| > a) \leq \frac{\text{Var}S_n}{a^2} = \frac{n\sigma^2}{a^2}. \quad (1.6)$$

If we replace a with $n^{\frac{1}{2}+\varepsilon}$, we then have

$$P(|S_n - n\mu| > n^{\frac{1}{2}+\varepsilon}) \leq \frac{\sigma^2}{n^{2\varepsilon}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.7)$$

Proposition 1.6. If $\text{Var}(X) = 0$, then $P(X = EX) = 1$.

Proof. For all $\varepsilon > 0$, we have

$$P(|X - EX| > \varepsilon) \leq \frac{\text{Var}X}{\varepsilon^2} = 0. \quad (1.8)$$

Define A_n as $\{|X - EX| > \frac{1}{n}\}$. Taking $P(\bigcup A_n) = \lim_{n \rightarrow \infty} P(A_n)$, the proof follows. ■

1.1 The Law of Large Numbers

We start by stating the weak law of large numbers.

Theorem 1.7 (*Weak law of large numbers*). Let $\{X_k\}_{k \geq 1}$ be a sequence of independent and identically distributed random variables with $E|X_i| < \infty$. Let $\mu = EX_i$. Then for any $a > 0$,

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{X_1 + X_2 + \dots + X_n}{n} - \mu\right| > a\right) = 0. \quad (1.9)$$

Proof. For now, let us assume that Ω is countable. We begin with the case where the variance of X_i , σ^2 , is finite. Fix $a > 0$, and let $S_n = X_1 + X_2 + \dots + X_n$. Then,

$$P\left(\left|\frac{S_n}{n} - \mu\right| > a\right) = P(|S_n - n\mu| > na) \leq \frac{\text{Var}S_n}{n^2a^2} = \frac{n\sigma^2}{n^2a^2} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.10)$$

We now focus the case when the variance, σ^2 , is infinite. Assume that the expected value, μ , is 0; if it were non-zero, we would then instead work with $X_i - \mu$. Let $\delta > 0$; we shall choose a particular δ later. For each n , define n pairs of random variables, $U_1, V_1, \dots, U_n, V_n$, as $U_k = X_k, V_k = 0$ if $|X_k| \leq \delta n$, and $U_k = 0, V_k = X_k$ if $|X_k| > \delta n$. X_k can be rewritten as $U_k + V_k$. We then have

$$\{|X_1 + \dots + X_n| \geq na\} \subseteq \{|U_1 + \dots + U_n| \geq \frac{na}{2}\} \cup \{|V_1 + \dots + V_n| \geq \frac{na}{2}\} \quad (1.11)$$

$$\implies P(|X_1 + \dots + X_n| \geq na) \leq P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) + P\left(|V_1 + \dots + V_n| \geq \frac{na}{2}\right). \quad (1.12)$$

We focus on the first term on the right hand side. The U_i 's are independently and identically distributed, so

$$P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) \leq \frac{4E[|U_1 + \dots + U_n|^2]}{a^2n^2} = \frac{4}{a^2n^2} (\text{Var}(U_1 + \dots + U_n) + (nEU_i)^2). \quad (1.13)$$

For the variance, we have

$$\text{Var}(U_1 + \dots + U_n) = n\text{Var}U_i \leq nEU_i^2 \leq nE[|U_i||U_i|] \leq \delta n^2 E[|U_i|] \quad (1.14)$$

which transforms the previous equation as

$$P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) \leq \frac{4}{a^2n^2} (\delta n^2 E[|U_i|] + (nEU_i)^2). \quad (1.15)$$

A lemma (to be proven later) states that $E[|U_i|] = E[|X_i|]$ as $n \rightarrow \infty$, and $EU_i = EX_i = 0$ too. So,

$$P\left(|U_1 + \dots + U_n| \geq \frac{na}{2}\right) \leq \frac{4}{a^2n^2} (\delta n^2 E[|U_i|] + (nEU_i)^2) \leq \frac{4\delta E[|U_i|]}{a^2} + \frac{4}{a^2} (EU_i)^2. \quad (1.16)$$

For the second term on the right hand side, begin with

$$\begin{aligned} P(V_1 + \dots + V_n \neq 0) &\leq P(\{V_1 \neq 0\} \cup \dots \cup \{V_n \neq 0\}) \leq nP(V_i \neq 0) = n \sum_{|x| > \delta n} P(X_i = x) \\ &\leq n \sum_{|x| > \delta n} \frac{|x|}{\delta n} P(X_i = x) = \frac{1}{\delta} E[|V_i|]. \end{aligned} \quad (1.17)$$

The rightmost term here tends to 0 as $n \rightarrow \infty$. Now choose δ to be $\frac{\varepsilon a^2}{6E[|X_i|]}$, and then choose N to be large enough such that for all $n > N$, both the terms are smaller than $\frac{\varepsilon}{2}$. ■

January 7th.

We now prove the lemma called upon earlier.

Lemma 1.8. *If X is a discrete random variable and takes values y_1, y_2, \dots, y_k , and $E[|X|] < \infty$, then $\lim_{n \rightarrow \infty} E[|X| \cdot 1_{|X| \leq n}] = E[|X|]$.*

Proof. Notice that the terms on the left hand side and right hand side are $\sum_{y_k: |y_k| \leq n}$ and $\sum_{y_k} |y_k| P(Y = y_k)$. The condition for convergence may now be applied. ■

The above equation, begin inside absolute braces, must imply that the term $E[X \cdot 1_{|X| \leq n}]$ must also absolutely converge to EX .

1.2 Simple Random Walk

Let X_1, X_2, \dots be independent and identically distributed random variables, with $X_i = 1$ with probability $\frac{1}{2}$ and $X_i = -1$ with probability $\frac{1}{2}$. Now define $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. The sequence $(S_n)_{n \geq 0}$ is a *simple random walk*.

Note that $S_0 = k_0 = 0, S_1 = k_1, \dots, S_n = k_n$ can occur if and only if $|k_i - k_{i+1}| = 1$ for all $0 \leq i \leq n-1$. The sequence $(k_n)_{n \geq 0}$ is a *simple path* of the simple random walk. By the event $\{S_n = k\}$, we are concerned with the event that the random walk visits k at step n . If $(k_n)_{n \geq 0}$ is given we have $X_i = k_i - k_{i-1}$. Because the X_i 's are independent and identically distributed, each event $\{X_1 = l_1, X_2 = l_2, \dots, X_n = l_n\}$, where $l_i = \pm 1$, is equally likely with probability $\frac{1}{2^n}$. Thus,

$$P(S_n = k) = \frac{N_n(k)}{2^n} \quad (1.18)$$

where $N_n(k)$ is defined as the number of distinct of path that start at 0 and end at k at step n . We also define $N_n^+(k)$ to be the number of distinct paths that end at k at step n and stay above the x -axis up to time $n-1$. The probability of the corresponding event is

$$P(\{S_1 > 0, S_2 > 0, \dots, S_{n-1} > 0, S_n = k\}) = \frac{N_n^+(k)}{2^n}. \quad (1.19)$$

Lemma 1.9. *Suppose a, a', b, b' are integers, with $0 \leq a < a'$. Then the number of distinct path from (a, b) to (a', b') depends only on $a' - a = n$ and $b' - b = k$, and is given by $\binom{n+k}{2}$.*

Proof. Notice that we need $x+1$'s and $y-1$'s to appear, satisfying $x+y = a' - a$ and $x-y = b' - b$. Solving, we get $x = \frac{n+k}{2}$ and $y = \frac{n-k}{2}$. Thus, the number of paths is given by $\binom{n+k}{2}$. ■

Using this lemma, we find that $N_n(k) = \binom{n+k}{2}$. The following convention is now followed; if t is not an integer, then $\binom{n}{t} = 0$.

Lemma 1.10 (The *method of images*). *Suppose a, a', b, b' are integers, with $0 \leq a < a'$ and $b, b' > 0$. Then the number of distinct paths from (a, b) to (a', b') that intersect the x -axis is equal to the number of paths from $(a, -b)$ to (a', b') .*

Proof. Consider any path $(b = k_0, k_1, \dots, k_{n-1}, k_n = b')$, from (a, b) to (a', b') , that intersects the x -axis. Let j be the smallest index for which $k_j = 0$. For ease, denote (a, b) by A , (a', b') by A' , $(a+j, 0)$ by B , and $(a, -b)$ by A'' . Reflect the segment from A to B about the x -axis to obtain a 'mirrored-path' from A'' to B ; $(-b = -k_0, -k_1, \dots, -k_{j-1}, k_j = 0, k_{j+1}, \dots, k_n = b')$. There is now a one-to-one correspondence between the paths from A to A' that intersect the x -axis, and the paths from A'' to A' . ■

We can now easily compute $N_n^+(k)$; it simply the number of paths from $(1, 1)$ to (n, k) that do not intersect the x -axis.

Theorem 1.11 (*Ballot theorem*). *The number of paths that progress from $(0, 0)$ to (n, k) through strictly positive values is given by $N_n^+(k) = \frac{k}{n} N_n(k)$.*

Proof. We have

$$\begin{aligned} N_n^+(k) &= \text{number of paths from } (1, 1) \text{ to } (n, k) - \text{number of such paths that intersect the } x\text{-axis} \\ &= N_{n-1}(k-1) - N_{n-1}(k+1) \\ &= \binom{n-1}{\frac{n+k}{2}-1} - \binom{n-1}{\frac{n+k}{2}} = \frac{k}{n} \binom{n}{\frac{n+k}{2}} = \frac{k}{n} N_n(k). \end{aligned} \quad (1.20)$$

■

Suppose $n = 2\nu$. Define $u_{2\nu}$ to be $P(S_{2\nu} = 0) = \frac{\binom{2\nu}{\nu}}{2^{2\nu}}$. The question we ask is to compute the probability that the first return to 0, if at all, occurs after step n . It can be found out as

$$P(\text{first return to } 0 \dots) = P(S_1 \neq 0, S_2 \neq 0, \dots, S_{2\nu} \neq 0) \quad (1.21)$$

$$= P(S_1 > 0, \dots, S_{2\nu} > 0) + P(S_1 < 0, \dots, S_{2\nu} < 0)$$

$$= 2P(S_1 > 0, \dots, S_{2\nu} > 0)$$

$$= 2 \sum_{k \text{ even}, k > 0} P(S_1 > 0, \dots, S_{2\nu-1} > 0, S_{2\nu} = k)$$

$$= \frac{2}{2^{2\nu}} \sum_{k \text{ even}, k > 0} N_{2\nu}^+(k)$$

$$= \frac{2}{2^{2\nu}} \sum_{k \text{ even}, k > 0} N_{2\nu-1}(k-1) - N_{2\nu-1}(k+1)$$

$$= \frac{2}{2^{2\nu}} N_{2\nu-1}(1) = u_{2\nu}. \quad (1.22)$$

We state this down as a lemma.

Lemma 1.12 (*Basic lemma*). *For n even, the probability that the first return to 0, if at all, occurs after step n is the same as the probability that the location at step n is 0. For n odd, it is the probability that the location at step $n-1$ is 0.*

We ask another question; for a fixed n , where does the random walk achieve its first maximum upto time n ? For this, denote by M_n the index m at which the walk S_0, S_1, \dots, S_n , over n steps, achieves its maximum for the first time.

For $0 < m < n$, $M_n = m$ if and only if $S_m > S_0, S_m > S_1, \dots, S_m > S_{m-1}$ and $S_m \geq S_{m+1}, S_m \geq S_{m+2}, \dots, S_m \geq S_n$. Notice that the first of these two conditions depends only on X_1, X_2, \dots, X_m , and the second condition depends only on $X_{m+1}, X_{m+2}, \dots, X_n$. So, $P(M_n = m) = P(S_m > S_0, S_m > S_1, \dots, S_m > S_{m-1}) \cdot P(S_m \geq S_{m+1}, S_m \geq S_{m+2}, \dots, S_m \geq S_n)$.

The key idea here is to consider the *reversed walk*; define a new walk with $X'_1 = X_m, X'_2 = X_{m-1}, \dots, X'_m = X_1$. Also define $S'_k = X'_1 + \dots + X'_k$. From here, we can deduce that $S_m > S_{m-i}$ is true if and only if $X_m + \dots + X_{m-i} > 0$ is true, which is true if and only if $S'_i > 0$ is true. So, $P(S_m > S_0, S_m > S_1, \dots, S_m > S_{m-1}) = P(S'_1 > 0, S'_2 > 0, \dots, S'_m > 0)$. If we now define $S''_k = X_{m+1} + \dots + X_{m+k}$, we have

$$\begin{aligned} P(S_m \geq S_{m+1}, S_m \geq S_{m+2}, \dots, S_m \geq S_n) &= P(X_{m+1} \leq 0, X_{m+1} + X_{m+2} \leq 0, \dots, X_{m+1} + \dots + X_n \leq 0) \\ &= P(S''_1 \leq 0, S''_2 \leq 0, \dots, S''_{n-m} \leq 0) \\ &= P(S''_1 \geq 0, S''_2 \geq 0, \dots, S''_{n-m} \geq 0) \end{aligned}$$

The first of the terms discussed, $P(S'_1 > 0, S'_2 > 0, \dots, S'_m > 0)$, can be computed for $m = 2\nu, 2\nu + 1$; it is simply $\frac{1}{2} u_{2\nu}$. For the latter of these terms, we introduce a new random variable \tilde{X} which has the same distribution as the X_i 's and is independent. Also define \tilde{S}_i to be $\tilde{X} + X_1 + \dots + X_{i-1}$ and \tilde{S}_0 to be 0.

We then have

$$\begin{aligned}
\frac{1}{2}P(S_0 \geq 0, \dots, S_{n-m} \geq 0) &= P(\tilde{X} = 1) \cdot P(S_0 \geq 0, \dots, S_{n-m} \geq 0) \\
&= P(\tilde{X} = 1, S_0 \geq 0, S_0 \geq 0, \dots, S_{n-m} \geq 0) \\
&= P(\tilde{S}_1 = 1, \tilde{S}_2 > 0, \dots, S_{n-m+1} > 0) \\
&= P(S_1 > 0, S_2 > 0, \dots, S_{n-m+1} > 0).
\end{aligned} \tag{1.23}$$

Thus, we get

$$P(M_n = m) = \frac{1}{2}u_{2k}u_{2\nu-2k} \tag{1.24}$$

where m is of the form $2k$ or $2k + 1$, and n is of the form 2ν , with $1 < k < \nu$.

January 10th.

Plugging in $m = 0$, we get $P(M_n = 0) = P(S_1 \leq 0, \dots, S_{2\nu} \leq 0) = \frac{1}{2}u_{2\nu}$. For $m = n$, we have $P(M_n = n) = P(S_1 \leq 0, \dots, S_{2\nu} \leq 0) = \frac{1}{2}u_{2\nu}$. Let us first compute u_{2k} .

$$\begin{aligned}
u_{2k} &= P(2k = 0) = \frac{\binom{2k}{k}}{2^{2k}} = \frac{(2k)!}{(k!)^2 2^{2k}} \\
&\sim \frac{(2k)^{2k+\frac{1}{2}} e^{-2k} \sqrt{2\pi}}{(\sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k})^2 2^{2k}} = \frac{1}{\sqrt{\pi k}}.
\end{aligned} \tag{1.25}$$

For $0 < a < b < 1$, we have

$$\begin{aligned}
P(an \leq M_n \leq bn) &= \sum_{m=an}^{bn} P(M_n = m) = \sum_{k=a\nu}^{b\nu} u_{2k}u_{2\nu-2k} \\
&\sim \sum_{k=a\nu}^{b\nu} \frac{1}{\sqrt{\pi k}} \frac{1}{\sqrt{\pi(\nu-k)}} = \sum_{k=a\nu}^{b\nu} \frac{1}{\nu \sqrt{\pi \frac{k}{\nu}} \sqrt{\pi(1-\frac{k}{\nu})}} \\
&\rightarrow \frac{1}{\pi} \int_a^b \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi} (\arcsin \sqrt{b} - \arcsin \sqrt{a}).
\end{aligned} \tag{1.26}$$

In fact, this is the *arcsin law for maxima*; for $0 \leq t \leq 1$, we have

$$\lim_{n \rightarrow \infty} P\left(\frac{M_n}{n} \leq t\right) = \frac{2}{\pi} \arcsin \sqrt{t}. \tag{1.27}$$

If we look at this as a cumulative density function, the probability density function becomes $\frac{d}{dt} \frac{2}{\pi} \arcsin \sqrt{t} = \frac{1}{\pi \sqrt{t(1-t)}}$.

We are now interested in \tilde{M}_n , the last time when maximum up to time n is attained. We can just look at the walk backwards again; in this case, we get

$$P\left(\frac{\tilde{M}_n}{n} \leq t\right) = P\left(\frac{n - \tilde{M}_n}{n} \leq t\right) \rightarrow \frac{2}{\pi} \arcsin \sqrt{t}. \tag{1.28}$$

We now ask the probability that the random walk of $n = 2\nu$ steps last visit 0 at time $2k$. We denote by K_n the location of the last return to 0 in a walk of n steps. Now look at

$$\begin{aligned}
\alpha_{2k, 2\nu} &= P(K_n = 2k) = P(S_{2k} = 0, S_{2k+1} \neq 0, \dots, S_{2\nu} \neq 0) \\
&= P(S_{2k} = 0) \cdot P(X_{2k+1} \neq 0, \dots, X_{2k+1} + \dots + X_{2\nu} \neq 0) \\
&= P(S_{2k} = 0) \cdot P(S_1 \neq 0, \dots, S_{2\nu-2k} \neq 0) = u_{2k}u_{2\nu-2k}.
\end{aligned} \tag{1.29}$$

We can also state an *arcsin law for last visit* here; for $0 < t < 1$

$$\lim_{n \rightarrow \infty} P(K_n \leq tn) = \frac{2}{\pi} \arcsin \sqrt{t}. \tag{1.30}$$

If we set the an additional limit that says t tends to 0, replacing t by an arbitrary $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} P(K_n = 0) = 0. \quad (1.31)$$

Given enough time, a simple random walk must return to 0.

Denote by f_{2n} the probability that the first return to 0 occurs at time $2n$.

$$\begin{aligned} f_{2n} &= P(S_1 \neq 0, \dots, S_{2n-1} \neq 0, S_{2n} = 0) \\ &= P(S_1 \neq 0, \dots, S_{2n-1} \neq 0) - P(S_1 \neq 0, \dots, S_{2n} \neq 0) \\ &= P(S_1 \neq 0, \dots, S_{2n-2} \neq 0) - P(S_1 \neq 0, \dots, S_{2n} \neq 0) \\ &= u_{2n-2} - u_{2n} = \frac{1}{2n-1} u_{2n}. \end{aligned} \quad (1.32)$$

Lemma 1.13. *With the usual notation,*

$$u_{2n} = f_2 u_{2n-2} + f_4 u_{2n-4} + \dots + f_{2n} u_0. \quad (1.33)$$

Proof. We have

$$\begin{aligned} P(S_{2n} = 0) &= \sum_{k=1}^n P(S_{2n} = 0, \text{ first return at } 2k) \\ &= \sum_{k=1}^n P(\text{first return at } 2k) \cdot P(S_{2n} = 0 \mid \text{first return at } 2k) \\ \implies P(S_n = 0) &= \sum_{k=1}^n f_{2k} u_{2n-2k}. \end{aligned} \quad (1.34)$$

■

Theorem 1.14. *The probability that in the time interval 0 to $n = 2\nu$, the random walk spends $2k$ amount of time on the positive side and $2\nu - 2k$ amount of time on the negative side is $\alpha_{2k, 2\nu}$.*

Corollary 1.15. *For $0 < t < 1$,*

$$P(\text{random walk spends less than } tn \text{ time on positive side}) \rightarrow \frac{2}{\pi} \arcsin \sqrt{t}. \quad (1.35)$$

Proof. This is the proof of the theorem. We introduce $b_{2k, 2\nu}$; it is defined as the probability that the random walk of length 2ν and $2k$ sides above the x -axis. We need to show that $b_{2k, 2\nu} = \alpha_{2k, 2\nu}$. We have

$$b_{2\nu, 2\nu} = P(S_1 \geq 0, S_2 \geq 0, \dots, S_{2\nu} \geq 0) = u_{2\nu}, \quad (1.36)$$

$$b_{0, 2\nu} = P(S_1 \leq 0, \dots, S_{2\nu} \leq 0) = u_{2\nu}. \quad (1.37)$$

We are left to prove it for $1 \leq k \leq \nu - 1$. Assume that exactly $2k$ out of 2ν time are spent above the x -axis, with $1 \leq k \leq \nu - 1$. Suppose first return to 0 occurs at time $2r < 2\nu$. We deal in cases.

- Case I: $2r$ time units upto first return are on the positive side. Then, $r \leq k \leq \nu - 1$. The time from $2r$ to 2ν has to be above the x -axis, $2k - 2\nu$ time. The number of such paths is $(\frac{1}{2} 2^{2r} f_{2r})(2^{2\nu-2r} b_{2k-2r, 2\nu-2r})$.
- The $2r$ time units upto the first return are on the negative side. The nubmer of such paths is $(\frac{1}{2} 2^{2r} f_{2r})(2^{2\nu-2r} b_{2k, 2\nu-2r})$. Also, $\nu - r \geq k$.

Thus, we have

$$b_{2k, 2\nu} = \frac{1}{2} \sum_{r=1}^k f_{2r} b_{2k-2r, 2\nu-2r} + \frac{1}{2} \sum_{r=1}^{\nu-k} f_{2r} b_{2k, 2\nu-2r}. \quad (1.38)$$

We now proceed with induction on ν . We have already shown this for $\nu = 1$; assume that this is true for $\nu \leq V - 1$. By induction,

$$\begin{aligned} b_{2k, 2V} &= \frac{1}{2} \sum_{r=1}^k f_{2r} \alpha_{2k-2r, 2V-2r} + \frac{1}{2} \sum_{r=1}^{V-k} f_{2r} \alpha_{2k, 2V-2r} \\ &= \frac{1}{2} u_{2V-2k} \sum_{r=1}^k f_{2r} u_{2k-2r} + \frac{1}{2} u_{2k} \sum_{r=1}^{V-k} f_{2r} u_{2V-2k-2r} \\ &= u_{2k} u_{2V-2k} = \alpha_{2k, 2\nu}. \end{aligned} \quad (1.39)$$

■

January 17th.

Theorem 1.16 (*Weirstrass's polynomial approximation.*). Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Then for every $\varepsilon > 0$, there is a polynomial P , dependent on f and ε , such that

$$|f(x) - P(x)| < \varepsilon \text{ for all } x \in [0, 1]. \quad (1.40)$$

Remark 1.17. Any continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is bounded and uniformly continuous. This fact will be useful in proving the previous theorem.

Proof. Start with X_1, X_2, \dots which are independent and identically distributed Bernoulli random variables, $\text{Ber}(x)$. Let $S_n = X_1 + X_2 + \dots + X_n$. From the weak law of large numbers, we know that $\frac{S_n}{n}$ is approximately x . We can expect that $f(x)$ will also be approximately $f(\frac{S_n}{n})$. We now have

$$\begin{aligned} f_n(x) &= Ef\left(\frac{S_n}{n}\right) = \sum_{j=0}^n f\left(\frac{j}{n}\right)P(S_n = j) \\ &= \sum_{j=0}^n f\left(\frac{j}{n}\right)\binom{n}{j}x^j(1-x)^{n-j}. \end{aligned} \quad (1.41)$$

This is now a polynomial; we wish to see how close this is to f . Define A_δ to be $\{j : |\frac{j}{n} - x| \leq \delta\}$

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \sum_{j=0}^n \left(f\left(\frac{j}{n}\right) - f(x) \right) P(S_n = j) \right| \\ &= \left| \sum_{j \in A_\delta} \left(f\left(\frac{j}{n}\right) - f(x) \right) + \sum_{j \notin A_\delta} \left(f\left(\frac{j}{n}\right) - f(x) \right) \right| P(S_n = j) \\ &\leq \sum_{j \in A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j) + \sum_{j \notin A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j). \end{aligned} \quad (1.42)$$

We have two terms to deal with now. For the first term, choose $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$; this δ can be chosen since f is uniformly continuous. Similarly, also choose $M = \sup_{x \in [0, 1]} |f(x)|$. M is finite since f is bounded. Thus, we have

$$\sum_{j \in A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j) \leq \sum_{j \in A_\delta} \varepsilon P(S_n = j) \leq \varepsilon \quad (1.43)$$

and

$$\sum_{j \notin A_\delta} \left| f\left(\frac{j}{n}\right) - f(x) \right| P(S_n = j) \leq 2MP\left(\left|\frac{S_n}{n} - x\right| > \delta\right) \leq 2M \frac{\text{Var}(S_n)}{n^2\delta^2} = \frac{2Mnx(1-x)}{n^2\delta^2}. \quad (1.44)$$

Combining the two, and choosing n large enough, we have

$$|f_n(x) - f(x)| \leq \varepsilon + \frac{2Mx(1-x)}{n\delta^2} \leq \varepsilon + \frac{M}{2n\delta^2} \leq 2\varepsilon. \quad (1.45)$$

■

1.3 Erdős-Renyi Random Graph

We first discuss the setup; start with n vertices of an empty graph. For any pair of points (i, j) , with $i \neq j$, join these vertices with an edge with probability p independently for all such pairs. Such a graph is denoted by $G_{n,p}$.

A collection of three points $S = \{i, j, k\}$ form a triangle if $G_{n,p}$ has the edges $\{i, j\}$, $\{j, k\}$, and $\{i, k\}$. We question the probability that such a graph has no formed triangles. Can we find $p = p_n$ such that

triangles begin to appear at p_n ? Let S be any set of three vertices. Define X_S to be the indicator function; 1 if S forms a triangle, and 0 otherwise. We note that $X_S \sim \text{Ber}(p^3)$. We note that

$$EX_S = p^3, \text{Var}X_S = p^3(1 - p^3) \leq p^3.$$

Denote by N the number of triangles in the graph $G_{n,p}$. Clearly,

$$N = \sum_{S:|S|=3} X_S, \quad EN = \binom{n}{3} p^3 < n^3 p^3, \quad \text{Var}N = \sum_S \text{Var}X_S + \sum_S \sum_{T \neq S} \text{Cov}(X_S X_T) \leq n^3 p^3 + n^4 p^5$$

Also, $P(N \geq 1) \leq EN < n^3 p^3$. If $p = p_n < \frac{1}{n}$, then $P(N \geq 1) \rightarrow 0$ as $n \rightarrow \infty$. We discuss this for $p > \frac{1}{n}$. We have

$$P(N = 0) \leq P(|N - EN| \geq EN) \leq \frac{\text{Var}N}{(EN)^2} \leq \frac{(n^3 p^3 + n^4 p^5)}{\frac{n^6 p^6}{100}} \leq \frac{100}{n^3 p^3} + \frac{100}{n(np)} \rightarrow 0. \quad (1.46)$$

We can state this as a theorem.

Theorem 1.18. *Consider G_{n,p_n} . Let E be the event that the graph is triangle free. We then have*

$$P(E) \rightarrow \begin{cases} 0 & \text{if } \frac{p_n}{\frac{1}{n}} \rightarrow \infty, \\ 1 & \text{if } \frac{p_n}{\frac{1}{n}} \rightarrow 0. \end{cases} \quad (1.47)$$

Now suppose that $\frac{np_n}{\frac{1}{n}} C > 0$ as $n \rightarrow \infty$. Then we have

$$N \approx \text{Poisson}\left(\frac{C^3}{6}\right). \quad (1.48)$$

January 21st.

Remark 1.19. For this next ‘game’, we will think of X_i ’s as the winnings in game i and μ to be the entrance fees for a game.

Definition 1.20. Suppose that X_1, X_2, \dots are independent, but not necessarily identically distributed. Let $S_n = X_1 + \dots + X_n$. We say a game with accumulated entrance fees $\{\alpha_n, n \geq 1\}$ is fair if

$$P\left(\left|\frac{S_n}{\alpha_n} - 1\right| > \varepsilon\right) \rightarrow 0 \quad (1.49)$$

for all $\varepsilon > 0$.

Using this definition of ‘fair’, we look at an example.

Example 1.21. This is the St. Petersburg’s paradox. This is the game; toss a coin repeatedly until the first head is observed. If this head occurs at the k^{th} toss, the amount paid out is $X = 2^k$. Let us find a fair accumulated entrance fees. In this case,

$$EX = \sum_{k=1}^{\infty} \frac{1}{2^k} 2^k = \infty. \quad (1.50)$$

Suppose we play this game n times. We are to find a fair accumulated sum $\{\alpha_n\}$ such that

$$P(|S_n - \alpha_n| > \varepsilon \alpha_n) \rightarrow 0. \quad (1.51)$$

To find this, we will define

$$\begin{aligned} U_j &= X_j 1_{\{X_j \leq a_n\}}, \\ V_j &= X_j 1_{\{X_j > a_n\}}. \end{aligned}$$

a_n shall be determined later. Note that $S_n = X_1 + \dots + X_n = U_1 + \dots + U_n + V_1 + \dots + V_n$. Then,

$$P(|S_n - \alpha_n| > \varepsilon \alpha_n) \leq P(|U_1 + \dots + U_n - \alpha_n| > \frac{1}{2} \varepsilon \alpha_n) + P(|V_1 + \dots + V_n| > \frac{1}{2} \varepsilon \alpha_n). \quad (1.52)$$

We first bound the second term on the right hand side. We have

$$P(|V_1 + \dots + V_n| > \frac{1}{2}\varepsilon\alpha_n) \leq P(\bigcup_{i=1}^n \{V_i \neq 0\}) \leq nP(V_1 \neq 0) = nP(X_1 > a_n) \quad (1.53)$$

$$= \sum_{2^k > a_n} P(X = 2^k) \leq \frac{2n}{a_n}. \quad (1.54)$$

Thus, we will require that $a_n \gg n$. Also,

$$EU_1 = \sum_{k \leq \log_2 a_n} 2^k \cdot 2^{-k} = \lfloor \log_2 a_n \rfloor, \quad \text{Var} U_1 \leq E[U_1^2] = \sum_{k \leq \log_2 a_n} (2^k)^2 \cdot 2^{-k} = 2^{\lfloor \log_2 a_n \rfloor + 1} - 1 < 2a_n. \quad (1.55)$$

$\frac{1}{n}(U_1 + \dots + U_n) \approx EU_j = \lfloor \log_2 a_n \rfloor$, so we should choose

$$\alpha_n = nEU_j = n \lfloor \log_2 a_n \rfloor. \quad (1.56)$$

This gives us

$$P(|U_1 + \dots + U_n - \alpha_n| > \frac{1}{2}\varepsilon\alpha_n) \leq \frac{n(2a_n)}{\frac{1}{4}\varepsilon^2\alpha_n^2}. \quad (1.57)$$

Thus, we have another condition where we require that $\frac{na_n}{\alpha_n^2} \rightarrow 0$. The conditions we require are

$$\frac{n}{a_n} \rightarrow 0 \text{ and } \frac{na_n}{n^2(\log_2 a_n)^2} \rightarrow 0.$$

The sequence $\{a_n\}$ defined as $a_n = n \log_2 n$ satisfies these properties. The sequence α_n is thus

$$\alpha_n = n \log_2 a_n = n \log_2 n + n \log_2 \log_2 n. \quad (1.58)$$

Chapter 2

GENERATING FUNCTIONS

January 24th.

Definition 2.1. For a sequence $\{a_n\}_{n \geq 0}$, the *generating function* of $\{a_n\}$ is given as

$$A(s) = \sum_{n=0}^{\infty} a_n s^n \quad (2.1)$$

for some $-s_0 < s < s_0$.

For this probability course, we will be interested in a particular form; for a random variable X that takes values $k = 0, 1, \dots$, the function we look at is

$$\sum_{k=0}^{\infty} P(X = k) s^k \text{ for } -1 \leq s \leq 1. \quad (2.2)$$

Suppose we have two sequences $\{a_n\}$ and $\{b_n\}$ with generating functions $A(s)$ and $B(s)$, respectively. If we define a new sequence $\{c_n\}$ as

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0 \text{ for all } n \geq 0, \quad (2.3)$$

then the sequence $\{c_n\}$ is termed the *convolution* of the sequences $\{a_n\}$ and $\{b_n\}$, and we shall denote it as

$$\{c_n\} = \{a_n\} * \{b_n\}.$$

Note that this convolution operation is both associative and commutative. We are now interested in finding the generating function of $\{c_n\}$. We have

$$\begin{aligned} C(s) &= \sum_{n=0}^{\infty} c_n s^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right) s^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n a_k s^k b_{n-k} s^{n-k} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k s^k b_m s^m \\ \implies C(s) &= \left(\sum_{k=0}^{\infty} a_k s^k \right) \cdot \left(\sum_{m=0}^{\infty} b_m s^m \right) = A(s) \cdot B(s). \end{aligned} \quad (2.4)$$

We state this down as a theorem.

Theorem 2.2. $C(s) = A(s) \cdot B(s)$ when $\{c_n\} = \{a_n\} * \{b_n\}$.

Suppose X takes values in $\mathbb{Z}_+ = \{0, 1, \dots\}$. Denote $P(X = k)$ as p_k . The generating function is, thus,

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} p_k s^k = E[s^X].$$

Also,

$$\mathcal{P}(1) = 1, \quad (2.5)$$

$$\mathcal{P}'(1) = \sum_{k=1}^{\infty} k p_k s^{k-1} \big|_{s=1} = EX. \quad (2.6)$$

Also note that

$$E[X^2] = \sum_{k=0}^{\infty} k^2 p_k = \sum k(k-1)p_k + \sum k p_k = \mathcal{P}''(1) + \mathcal{P}'(1) \quad (2.7)$$

which gives us the variance of X as

$$\text{Var}X = E[X^2] - (EX)^2 = \mathcal{P}''(1) + \mathcal{P}'(1) - (\mathcal{P}'(1))^2. \quad (2.8)$$

The individual probabilities of $X = k$ may also be found as

$$p_k = P(X = k) = \frac{1}{k!} \cdot \frac{d^k}{ds^k} \mathcal{P}(s) \big|_{s=0}. \quad (2.9)$$

Now suppose that X and Y are two independent variables, taking values in \mathbb{Z}_+ . Let $Z = X + Y$. We ask the probability that Z equals k . We can find this as

$$P(Z = k) = \sum_{m=0}^k P(X = m, Y = k - m) = \sum_{m=0}^k P(X = m) \cdot P(Y = k - m). \quad (2.10)$$

Therefore, denoting $p_k^{(X)}$ to be the probability mass function of X , we have

$$\{p_k^{(Z)}\} = \{p_k^{(X)}\} * \{p_k^{(Y)}\} \implies \mathcal{P}^{(Z)}(s) = \mathcal{P}^{(X)}(s) \cdot \mathcal{P}^{(Y)}(s). \quad (2.11)$$

There is an easier way to see the last equation; we could have started with $Es^Z = E[s^X \cdot s^Y] = E[s^X]E[s^Y]$.

If we have $S_n = X_1 + X_2 + \dots + X_n$, where the X_i 's are independently distributed taking values in \mathbb{Z}_+ , it can be shown that

$$\{p_k^{(S_n)}\} = \{p_k^{(X)}\}^{n*} \quad (2.12)$$

Example 2.3. Let us compute the generating function of $X \sim \text{Bin}(n, p)$. We have

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} P(X = k) s^k = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} s^k = ((1-p) + ps)^n. \quad (2.13)$$

This is the generating function of the binomial distribution. Clearly,

$$\begin{aligned} EX &= \mathcal{P}'(1) = np, \\ \text{Var}X &= \mathcal{P}''(1) + \mathcal{P}'(1) - (\mathcal{P}'(1))^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p). \end{aligned}$$

Note that using this generating function, we can also show that $\text{Bin}(n, p) + \text{Bin}(m, p) = \text{Bin}(m+n, p)$ when the former terms are independent.

Example 2.4. We look at $X \sim \text{Poisson}(\lambda)$. We have

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda + \lambda s}. \quad (2.14)$$

For this, we can also verify $EX = \text{Var}X = \lambda$. We can also show that $\text{Poisson}(\lambda) + \text{Poisson}(\mu) = \text{Poisson}(\lambda + \mu)$ when the former terms are independent.

Example 2.5. We look at $X \sim \text{Geo}(p)$. Denote $1-p$ as q . The generating function is given as

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} p q^k s^k = \frac{p}{1-qs}. \quad (2.15)$$

As an extension, let X_k denote the number of failures between the $(k-1)^{\text{th}}$ and k^{th} successes. If we denote $S_r = X_1 + X_2 + \dots + X_r$, we find that $S_r \sim \text{NB}(p, r)$. From direct computation, we know that

$$P(S_r = k) = \binom{r+k-1}{k} q^k p^r \text{ for } k = 0, 1, \dots$$

Let us compute this in another way; S_r is the sum of independent geomtric random variables with parameter p . We have

$$\mathcal{P}^{(S_r)}(s) = \left(\frac{p}{1-qs} \right)^r = p^r (1-qs)^{-r} = p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qs)^k \quad (2.16)$$

which tells us that

$$P(S_r = k) = p^r \binom{-r}{k} (-q)^k. \quad (2.17)$$

2.1 Random Walks, with Generating Functions

Here, we consider the paths that have a right step with probability p and a left step with probability $q = 1-p$. We first look at the waiting time for the first gain, that is, the event $\{S_1 \leq 0, S_2 \leq 0, \dots, S_{n-1} \leq 0, S_n = 1\}$ (Event (*)). Denote the probability of this event by ϕ_n , and its generating function by $\Phi(s)$. Note that $\phi_0 = 0$ and $\phi_1 = p$ lead to trivial cases. We focus on $n > 1$.

We must have $S_1 = -1$ (Event (1)). Denote, by $\nu < n$, the first return to 0 (Event (2)). ν only depends on X_0, X_1, \dots, X_ν . We need another $n-\nu$ steps to reach 1; this depends on $X_{\nu+1}, X_{\nu+2}, \dots, X_n$ (Event (3)). For some $n > 1$, Event (*) occurs if and only if Event (1) \cap Event (2) \cap Event (3) occurs for some $\nu < n$. The point here is that the three events are independent. For some fixed $\nu < n$,

$$P(\text{Event (1)}) = q, P(\text{Event (2)}) = \phi_{\nu-1}, P(\text{Event (3)}) = \phi_{n-\nu}. \quad (2.18)$$

Thus,

$$\phi_n = \sum_{\nu=2}^{n-1} q \phi_{\nu-1} \phi_{n-\nu}. \quad (2.19)$$

We have

$$\Phi(s) - ps = \sum_{n=2}^{\infty} \phi_n s^n = q \sum_{n=2}^{\infty} (\phi_1 \phi_{n-2} + \dots + \phi_{n-2} \phi_1) s^n = qs \sum_{n=1}^{\infty} \phi_n^2 s^n = qs(\Phi(s))^2 \quad (2.20)$$

$$\implies \Phi(s) - ps = qs(\Phi(s))^2. \quad (2.21)$$

This is a standard quadratic; solving gives us

$$\Phi(s) = \frac{1 \pm \sqrt{1-4pqs^2}}{2qs}. \quad (2.22)$$

The solution with the '+' is rejected; if it was valid, then plugging in $s < 1$ would give us $\Phi(s) > 1$, which is impossible. We expand this using the binomial theorem,

$$\Phi(s) = \frac{1}{2qs} \left(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4pqs^2)^k \right) = \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \frac{(-1)^{k-1} (4pq)^k}{2q} s^{2k-1} \quad (2.23)$$

which tells us that

$$\phi_{2k-1} = \frac{(-1)^{k-1}}{2q} \binom{\frac{1}{2}}{k} (4pq)^k, \phi_{2k} = 0. \quad (2.24)$$

Thus,

$$\Phi(1) = \sum \phi_n = \frac{1 - \sqrt{1-4pq}}{2q} = \frac{1 - |p-q|}{2q} = \begin{cases} \frac{p}{q} & \text{if } p < q, \\ 1 & \text{if } p \geq q. \end{cases}$$

This gives the probability that, at some point of the random walk, the displacement 1 is reached.

Similarly, for displacement S_n , we have

$$P(S_n \leq 0 \forall n) = \begin{cases} \frac{q-p}{p} & \text{if } p < q, \\ 0 & \text{if } p \geq q. \end{cases}$$

Appendices

Chapter A

Appendix

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