REAL ANALYSIS II

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Second Semester

List of Symbols

$[a, b]$, the set of all real numbers x such that $a \leq x \leq b$.

$\mathbb{N} = \{1, 2, \ldots\}$, the set of all natural numbers.

$\mathbb{Z}_+$, defined as $\mathbb{N} \cup \{0\}$.

$\mathcal{B}[a, b]$, the set of all boundary functions defined as $\{f : [a, b] \to \mathbb{R}\}$. It is a vector space (also an algebra) over $\mathbb{R}$.

$\mathcal{P}[a, b]$, the set of all partitions of the set $[a, b]$.

I_j, the j^{th} subinterval of $[a, b]$, controlled by a partition set.

$L(f, P)$, the lower Riemann sum for a function f and partition P.

$U(f, P)$, the upper Riemann sum for a function f and partition P.

$\underline{\int_a^b} f$, the lower Riemann integration for a function f.

$\overline{\int_a^b} f$, the upper Riemann integration for a function f.

$\mathcal{R}[a, b]$, the set of all Riemann integrable functions over the set $[a, b]$.

$\mathcal{C}[a, b]$, the set of all continuous functions over the set $[a, b]$.

$\{a_n\}_{n \geq 1}$, a sequence of real numbers.

T_P, a tag set for the partition P.

$\mathcal{D}$, the set of all differentiable functions.

$\mathcal{F}(\mathbb{R})$, the set of all real functions.

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Chapter 1

THE RIEMANN INTEGRAL

1.1 On The Path of Definitions

January 6th.

Definition 1.1. A *partition* of $[a, b]$ are all the points $a = x_0 < x_1 < \ldots < x_n = b$. These points within are termed *nodes*, and there are $n-1$ of them. The set I_j, defined by $[x_{j-1}, x_j]$ denotes the j^{th} subinterval.

Definition 1.2. If $I = (a, b), [a, b], (a, b], [b, a)$, then the *length of the interval* I is denoted by $b - a$.

Denote by $\mathcal{P}[a, b]$, the set of all partition sets of $[a, b]$. For $P \in \mathcal{P}[a, b]$, with $n-1$ nodes, the length of $[a, b]$ will be $|[a, b]| = \sum_{j=1}^{n} I_j$. We also note that for all $P, \tilde{P} \in \mathcal{P}[a, b]$, $P \cup \tilde{P} \in \mathcal{P}[a, b]$. Note that here we consider n to be finite.

Example 1.3. The set $\{\frac{1}{n}\}_{n\geq 1} \cup \{0\}$ does not belong to the set of all partitions of the unit interval, $\mathcal{P}[0, 1]$.

Let $f \in \mathcal{B}[a, b]$, and $P \in \mathcal{P}[a, b]$. Suppose P has the nodes $a = x_0 < x_1 < \ldots < x_n = b$. For all $j = 1, \ldots n$, define $m_j = \inf_{x\in I_j} f(x)$ and $M_j = \sup_{x\in I_j} f(x)$. Finally, denote by m the value of $\inf_{x\in[a,b]} f(x)$ and M to be $\sup_{x\in[a,b]} f(x)$. These are all real values.

Note that for all valid j, $m \leq m_j \leq M_j \leq M$ always holds. This must mean that

$$m\,|I_j| \leq m_j\,|I_j| \leq M_j\,|I_j| \leq M\,|I_j|$$
$$m(b-a) \leq \sum_{j=1}^{n} m_j\,|I_j| \leq \sum_{j=1}^{n} M_j\,|I_j| \leq M(b-a). \tag{1.1}$$

Definition 1.4. Let $f \in \mathcal{B}[a, b]$. For P $(a = x_0, x_1, \ldots, x_n = b) \in \mathcal{P}[a, b]$, the *lower Riemann sum* and the *upper Riemann sum* are defined as

$$L(f, P) = \sum_{j=1}^{n} m_j\,|I_j| \text{ and } U(f, P) = \sum_{j=1}^{n} M_j\,|I_j|, \tag{1.2}$$

respectively. Thus, $m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)\ \forall\ P \in \mathcal{P}[a, b]$.

Remark 1.5. Clearly, $L(f, P), U(f, P) \in \mathbb{R}$ for all paritions $P \in \mathcal{P}[a, b]$ and all boundary functions $f \in \mathcal{B}[a, b]$. In fact, $L(f, P), U(f, P) \in [m(b-a), M(b-a)]$.

Definition 1.6. For $f \in \mathcal{B}[a, b]$, the *lower Riemann integration* is defined as

$$\int_{\underline{a}}^{b} f = \sup\{L(f, P) | P \in \mathcal{P}[a, b]\}. \tag{1.3}$$

Subsequently, the *upper Riemann integration* is defined as

$$\int_{a}^{\bar{b}} f = \inf\{U(f, P) | P \in \mathcal{P}[a, b]\}. \tag{1.4}$$

Remark 1.7. Note that both $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ belong to the set $[m(b-a), M(b-a)]$.

Definition 1.8. A function $f \in \mathcal{B}[a,b]$ is *Riemann integrable* if the lower and the upper Riemann integration are equal, that is, $\underline{\int_a^b} f = \overline{\int_a^b} f$. We denote this value by $\int_a^b f$, and call it the integration of f over $[a,b]$. We then say that $f \in \mathcal{R}[a,b]$.

January 8th.

Example 1.9. Note that $\mathcal{R}[a,b] \subseteq \mathcal{B}[a,b]$. In fact, it is a proper subset; for there exists the *Dirichlet function* $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{if otherwise.} \end{cases} \tag{1.5}$$

Clearly, f is a boundary function but not a continuous one. Now pick a partition P with $x_0 = 0 < x_1 < \ldots < x_n = 1$. Now, $m_j = 0\ \forall \implies L(f,P) = 0\ \forall\ P \implies \underline{\int_0^1} f = 0$. If we consider that M_j is always 1, we get $\overline{\int_0^1} f = 1$. f does not belong to the set of Riemann integrable functions.

Example 1.10. We show that $\mathcal{R}[a,b]$ is not empty. Simply pick $f : [a,b] \to \mathbb{R}$ defined by $f(x) = c$ for all valid x. For every partition of this interval, we $m_j = M_j = c$ for all j. Finally, after computing the lower Riemann and upper Riemann sums, we get $\underline{\int_a^b} f = \overline{\int_a^b} f = c(b-a)$.

Example 1.11. There exists a function $f \in \mathcal{B}[a,b]$ such that $|f| \in \mathcal{R}[a,b]$ but $f \notin \mathcal{R}[a,b]$. Indeed, simply pick a modification of the Dirichlet function defined as

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Q} \cap [a,b], \\ 1 & \text{if otherwise.} \end{cases} \tag{1.6}$$

Definition 1.12. Let $P, \tilde{P} \in \mathcal{P}[a,b]$. We say $\tilde{P} \supset P$, or $\tilde{P}$ is a *refinement* of P if the nodes of P are a subset of the nodes of $\tilde{P}$.

Example 1.13. For all $P, \tilde{P} \in \mathcal{P}[a,b]$, we have $P \cup \tilde{P} \supset P, \tilde{P}$.

Proposition 1.14. *Let $f \in \mathcal{B}[a,b]$, and $P, \tilde{P} \in cP[a,b]$. Suppose $\tilde{P} \supset P$. Then*

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P). \tag{1.7}$$

Proof. Note that it is sufficient to prove the first inequality; the second one is already true and the third one is analagous to the first one. Set $\tilde{P} = P \cup \{\tilde{x}\}$ with $\tilde{x} \notin P$, and set P as $a = x_0 < x_1 < \ldots < x_n = b$. As $\tilde{x}$ is not part of P, there must exist some $j \in \{1, \ldots, n\}$ such that $\tilde{x} \in (x_{j-1}, x_j)$.

For this j, let $\tilde{m}_{j-1} = \inf_{[x_{j-1},\tilde{x}]} f$ and let $\tilde{m}_j = \inf_{[\tilde{x},x_j]} f$. Therefore, we shall have

$$\begin{aligned} L(f,\tilde{P}) - L(f,P) &= \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - x_{j-1}) \\ &= \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - \tilde{x}) - m_j(\tilde{x} - x_{j-1}) \\ &= (\tilde{m}_j - m_j)(x_j - \tilde{x}) + (\tilde{m}_{j-1} - m_j)(\tilde{x} - x_{j-1}) \geq 0 \\ \implies L(f,\tilde{P}) - L(f,P) &\geq 0. \end{aligned} \tag{1.8}$$

Induction may now be applied to make any refinement $\tilde{P}$ of P. A similar logic applies to the upper Riemann sums. ■

Corollary 1.15. *Let $f \in \mathcal{B}[a,b]$. Then, for all $P, Q \in \mathcal{P}[a,b]$, $L(f,P) \leq U(f,Q)$.*

Proof. Choose $R = P \cup Q$ to be a refinement of both P and Q. Applying the previous proposition, we simply get $L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,Q)$. ■

Corollary 1.16. *For all $f \in \mathcal{B}[a,b]$, $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ is always true.*

Proof. The lower Riemann intergral is the supremum of all the lower Riemann sums, so it must be the lowest upper bound, and, thus, has to be lesser than the upper Riemann sums. Similarly, the upper Riemann integral is greater than the lower Riemann sums. Consequently, we get the desired inequality. ■

1.2 Classification and Computation

We now discuss the classification of Riemann integrable functions, and the computation of the Riemann integral.

Theorem 1.17. *Let $f \in \mathcal{B}[a,b]$. Then, for f to be Riemann intergrable, the only neccessary and sufficient condition is $\underline{\int}_a^b f \geq \overline{\int}_a^b f$.*

Proof. If the condition is satisfied, then we must conclude that the Riemann integrals have to be equal. The converse follows the opposite argument. ■

Theorem 1.18. *Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$, there exists a $P \in \mathcal{P}[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.*

Proof. We first prove the converse; assume that for every $\varepsilon > 0$, there exists a P satisfying $U(f,P) - L(f,P) < \varepsilon$. Now,

$$L(f,P) \leq \underline{\int} f \leq \overline{\int} f \leq U(f,P) < \varepsilon + L(f,P) \leq \varepsilon + \underline{\int} f$$

$$\implies \overline{\int} f - \underline{\int} f < \varepsilon \; \forall \, \varepsilon < 0 \tag{1.9}$$

$$\implies \overline{\int} f = \underline{\int} f. \tag{1.10}$$

To show that every Riemann integrable function satisfies this property, let $f \in \mathcal{R}[a,b]$ and $\varepsilon > 0$. The Riemann integrals are a supremum and an infimum, so there must exist a $P_1 \in \mathcal{P}[a,b]$ such that $L(f,P_1) > \int f - \frac{\varepsilon}{2}$ and a $P_2 \in \mathcal{P}[a,b]$ such that $U(f,P_1) < \int f + \frac{\varepsilon}{2}$. Now choose P to be $P_1 \cup P_2$, a refinement of both P_1 and P_2. Therefore,

$$U(f,P) \leq U(f,P_2) < \int f + \frac{\varepsilon}{2} < (L(f,P_1) + \frac{\varepsilon}{2}) + \frac{\varepsilon}{2} \leq L(f,P) + \varepsilon$$

$$\implies U(f,P) - L(f,P) < \varepsilon. \tag{1.11}$$

■

Definition 1.19. Let P be a partition with $a = x_0 < x_1 < \ldots < x_n = b$. We define the *norm* of P, or the *mesh* of P, as $\|P\| = \max_j \{x_j - x_{j-1}\}$.

Theorem 1.20 (*Darboux's theorem*)**.** *Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $U(f,P) - L(f,P) < \varepsilon$ for all $P \in \mathcal{P}[a,b]$ with $\|P\| < \delta$.*

Remark 1.21. To prove this, we define $\eta : \mathcal{P}[a,b] \to \mathbb{R}_{\geq 0}$ by $\eta(P) = U(f,P) - L(f,P)$ for all $P \in \mathcal{P}[a,b]$.

Proof. The proof of the converse is trivial and follows the same reasoning as before. To show that every Riemann integrable function satisfies this property, let $f \in \mathcal{R}[a,b]$ and $\varepsilon > 0$. There exists a $\tilde{P} \in \mathcal{P}[a,b]$ such that $U(f,\tilde{P}) - L(f,\tilde{P}) < \frac{\varepsilon}{2}$. Denote the number of nodes in $\tilde{P}$ by p, and set $\delta = \frac{\varepsilon}{8pM}$, where M is the supremum of f over $[a,b]$. Pick $P \in \mathcal{P}[a,b]$ and assume that $\|P\| < \delta$. Now set $\hat{P} = P \cup \tilde{P}$; $\hat{P}$ has at most p points that are not in P.

For now, assume that $p = 1$. Then, $\tilde{P} = P \cup \{\tilde{x}\}$ with $\tilde{x} \notin P$. Thus, with variables defined as before,

$$L(f,\hat{P}) - L(f,P) = (\tilde{m}_j - m_j)(x_j - \tilde{x}) + (\tilde{m}_{j-1} - m_j)(\tilde{x} - x_{j-1}). \tag{1.12}$$

Notice that $(\tilde{m}_j - m_j), (\tilde{m}_{j-1} - m_j) < 2M$ and $(x_j - \tilde{x}), (\tilde{x} - x_{j-1}) < \delta$. In fact, in general, for an arbitrary p, we have

$$L(f,\hat{P}) - L(f,P) < 4pM\delta = \frac{\varepsilon}{2}. \tag{1.13}$$

A similar story unfolds for the upper sums,

$$U(f,P) - U(f,\hat{P}) < \frac{\varepsilon}{2}. \tag{1.14}$$

Together, the equations combine to form

$$U(f,P) - L(f,P) < \varepsilon + U(f,\hat{P}) - L(f,\hat{P}) < \varepsilon + U(f,\tilde{P}) - L(f,\tilde{P}) < 2\varepsilon. \tag{1.15}$$

■

January 13th.
We now wish to answer two questions; which elements reside in the set $\mathcal{R}[a,b]$, and the value of the Riemann integral $\int_a^b f$ for some $f \in \mathcal{R}[a,b]$.

Let us first wonder whether $\mathcal{C}[a,b]$, the set of continuous functions, is a subset of $\mathcal{R}[a,b]$. In fact, this is true.

Theorem 1.22. $\mathcal{C}[a,b] \subseteq \mathcal{R}[a,b]$.

Proof. Fix $f \in \mathcal{C}[a,b]$; thus, $f : [a,b] \to \mathbb{R}$ is also uniformly continuous. For all $\varepsilon > 0$, there eixsts $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \text{ for all } |x-y| < \delta. \tag{1.16}$$

Pick $P \in \mathcal{P}[a,b]$ such that $\|P\| < \delta$, and fix such a $P : a = x_0 < x_1 < \ldots < x_n = b$. Thus,

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (M_j - m_j)\,|I_j|\,. \tag{1.17}$$

Now, $f|_{I_j} : I_j \to \mathbb{R}$ is a continuous function for all valid j. Therefore, there exist $\eta_j, \zeta_j \in I_j$ such that $f(\eta_j) = M_j$ and $f(\zeta_j) = m_j$. $M_j - m_j$ can be rewritten as $f(\eta_j) - f(\zeta_j)$. As $|\eta_j - \zeta_j| < \delta$, it follows that

$$M_j - m_j < \frac{\varepsilon}{b-a} \text{ for all } j \tag{1.18}$$

$$\implies (M_j - m_j)\,|I_j| < \frac{\varepsilon}{b-a}\,|I_j|$$

$$\implies \sum_{j=1}^{n} (M_j - m_j)\,|I_j| < \varepsilon. \tag{1.19}$$

By Darboux's theorem, f is Riemann integrable. ■

We now wish to compute $\int_a^b f$. Our first attempt at this is the following theorem.

Theorem 1.23. *Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if there exists a sequence $\{P_n\}_{n\geq 1} \subseteq \mathcal{P}[a,b]$ such that*

$$\lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0. \tag{1.20}$$

Moreover, in this case,

$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n). \tag{1.21}$$

Proof. Let us first assume that f is Riemann integrable. Thus, for $\varepsilon = \frac{1}{n}$, there exists $P_n \in \mathcal{P}[a,b]$ such that

$$0 \leq U(f,P) - L(f,P) < \frac{1}{n} \implies U(f,P) - L(f,P) \to 0. \tag{1.22}$$

For the converse, let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$0 \leq U(f,P_n) - L(f,P_n) < \varepsilon \text{ for all } n \geq N. \tag{1.23}$$

Pick n to be N. Thus, $U(f,P_N) - L(f,P_N) < \varepsilon$ must imply that $f \in \mathcal{R}[a,b]$.

Let us now show the computation of the integral. We have

$$0 \leq U(f,P_n) - \overline{\int_a^b} = U(f,P_n) - \underline{\int_a^b} f \leq U(f,P_n) - L(f,P_n) \to 0 \implies \tag{1.24}$$

$$\implies U(f,P_n) \to \int_a^b f. \tag{1.25}$$

Similarly,

$$0 \leq \underline{\int_a^b} f - L(f,P_n) = \int_a^b f - U(f,P_n) + U(f,P_n) - L(f,P_n) \to 0 \tag{1.26}$$

$$\implies L(f,P_n) \to \int_a^b f. \tag{1.27}$$

■

Remark 1.24. Let $f \in \mathcal{B}[a,b]$, and let $\{P_n\}_{n\geq 1} \subseteq \mathcal{P}[a,b]$. If $L(f,P_n) \to \lambda$ and if $U(f,P_n) \to \lambda$. We then must have $f \in \mathcal{R}[a,b]$ and $\int_a^b = \lambda$. This is reminiscent of Newton's method of integration.

Example 1.25. Let us compute $\int_0^1 f$ where $f(x) = x^2$ on $[0,1]$. For all $n \in \mathbb{N}$, consider the partitions $P_n : 0 = x_0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \ldots < x_n = \frac{n}{n} = 1$. Thus, $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$ for all $j = 1, \ldots, n$. We then have $M_j = (\frac{j}{n})^2$ and $m_j = (\frac{j-1}{n})^2$. The sums can be computed as

$$U(f,P_n) = \sum_{j=1}^{n} \frac{1}{n} \cdot \frac{j^2}{n^2} = \frac{1}{n^3}\sum_{j=1}^{n} j^2 = \frac{(n+1)(2n+1)}{6n^2} \to \frac{1}{3}, \tag{1.28}$$

$$L(f,P_n) = \sum_{j=1}^{n} \frac{1}{n} \cdot \frac{(j-1)^2}{n^2} = \frac{1}{n^3}\sum_{j=1}^{n} (j-1)^2 = \frac{(n-1)(2n-1)}{6n^2} \to \frac{1}{3}. \tag{1.29}$$

Both sums converge to $\frac{1}{3}$; f is Riemann integrable and $\int_0^1 f = \frac{1}{3}$.

Example 1.26. We show that $\mathcal{C}[a,b]$ is a proper subset of $\mathcal{R}[a,b]$. Let us consider the function $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x = \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \tag{1.30}$$

Fix $\varepsilon > 0$. Choose the partition $P_\varepsilon : 0 < \frac{1}{2} - \varepsilon < \frac{1}{2} + \varepsilon < 1$. We then have $m_1 = 1 = M_1$, $m_2 = 0 = m_3$, and $M_2 = 1$, $M_3 = 0$. Therefore, we have

$$L(f,P) = \frac{1}{2} - \varepsilon,\ U(f,P) = \frac{1}{2} + \varepsilon. \tag{1.31}$$

Finally,

$$U(f,P) - L(f,P) = 2\varepsilon < 3\varepsilon. \tag{1.32}$$

f is Riemann integrable, but is not a continuous function.

January 15th.

We now discuss a more refined way of computing the Riemann integral.

Definition 1.27. Let $P : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. A *tag* of P is a function $T_P : \{I_j\}_{j=1}^n \to [a,b]$ such that $T_p(I_j) \in I_j$ for all $j = 1, 2, \ldots, n$. In other words, $T_p = \{\zeta_j\}_{j=1}^n$ such that $\zeta_j \in I_j$ for all valid j.

Definition 1.28. Let $f \in \mathcal{B}[a,b]$, $P \in \mathcal{P}[a,b]$, and T_P be a tag set. The *Riemann sum* of f with respect (P, T_P) is

$$S(f,P) = \sum_{j=1}^{n} f(\zeta_j)\,|I_j|\,. \tag{1.33}$$

What good is Riemann sum and why the need for defining it? Let us fix $f \in \mathcal{B}[a,b]$, $P \in \mathcal{P}[a,b]$, and $T_P = \{\zeta_j\}_{j=1}^n$. We now have $m_j \leq f(\zeta_j) \leq M_j$ for all valid j. Multiplying by the subintervals $|I_j|$ and summing over all j's gives us

$$L(f,P) \leq S(f,P) \leq U(f,P) \tag{1.34}$$

for any tag set T_P. This gives us a better condition as if both the lower and upper sum collapse, then the Riemann sum will give us a value, say λ, for any tag set T_P.

What do we hope for? We wish to show that $L(f,P) \to \lambda$ as $\|P\| \to 0$. Let us write this more formally.

Definition 1.29. Given $f \in \mathcal{B}[a,b]$ and $\lambda \to \mathbb{R}$, we say

$$\lim_{\|P\|\to 0} S(f,P) = \lambda \tag{1.35}$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(f,P) - \lambda| < \varepsilon \text{ for all } P \in \mathcal{P}[a,b] \text{ satisfying } \|P\| < \delta \text{ for any } T_P. \tag{1.36}$$

Note that if such λ exists, it is unique.

Theorem 1.30. *Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\lim_{\|P\|\to 0} S(f,P) = \lambda$. Also, in this case, $\int_a^b f = \lambda$.*

Proof. Assume f is Riemann integrable. Let $\lambda = \int_a^b f$, For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$U(f,P) - L(f,P) < \varepsilon \text{ for all } \|P\| < \delta.$$

We know that $L(f,P) \leq S(f,P) \leq U(f,P)$ for all tag sets T_P. Now,

$$L(f,P) \geq U(f,P) - \varepsilon \geq \overline{\int} f - \varepsilon = \lambda - \varepsilon. \tag{1.37}$$

Similarly,

$$U(f,P) < \varepsilon + L(f,P) \leq \varepsilon + \underline{\int} f = \varepsilon + \lambda. \tag{1.38}$$

Thus, we have, for all $P \in \mathcal{P}[a,b]$ satisfying $\|P\| < \delta$ and for all T_P, we have

$$\lambda - \varepsilon < S(f,P) \leq \lambda + \varepsilon \implies |S(f,P) - \lambda| < \varepsilon \implies \lim_{\|P\|\to 0} S(f,P) = \lambda.$$

For the converse, let $\lambda = \lim_{\|P\|\to 0} S(f,P)$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(f,P) - \lambda| < \frac{\varepsilon}{3}$$

for all $|P| < \delta$ and for all T_P. Note that $S(f,P)$ is just $\sum_{j=1}^n f(\zeta_j)\,|I_j|$. By taking the infimum and supremum of tag sets for a fixed P over their valid intervals, we have

$$\lambda - \frac{\varepsilon}{3} < L(f,P) < \lambda + \frac{\varepsilon}{3},\ \lambda - \frac{\varepsilon}{3} < U(f,P) < \lambda + \frac{\varepsilon}{3}. \tag{1.39}$$

Finally, we can now minimize $U(f,P) - L(f,P)$ for Darboux's criteria—

$$U(f,P) - L(f,P) < \lambda + \frac{\varepsilon}{3} - \lambda + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon \implies f \in \mathcal{R}[a,b]. \tag{1.40}$$

Finally, for all $\|P\| < \delta$, we have

$$\lambda - \frac{\varepsilon}{3} < L(f,P) \leq \underline{\int} f = \int f = \overline{\int} f < U(f,P) < \lambda + \frac{\varepsilon}{3} \implies \lambda = \int f. \tag{1.41}$$

■

We are now done with the classification and computation of the Riemann integral.

Theorem 1.31. *Let $f \in \mathcal{R}[a,b]$ and let $\{P_n\} \subseteq \mathcal{P}[a,b]$ such that $\|P_n\| \to 0$. Then*

$$\lim_{n\to\infty} S(f,P_n) = \int_a^b f \tag{1.42}$$

for all tag sets T_{P_n}.

Proof. f is Riemann integrable. For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\|P\| < \delta$, $U(f,P) - L(f,P) < \varepsilon$. There also exists a natural N such that $\|P_n\| < \delta$ for all $n \geq N$. This tells us that $U(f,P_n) - L(f,P_n) < \varepsilon$ for all $n \geq N$. We rewrite this as

$$U(f,P_n) - \int f + \int f - L(f,P_n) < \varepsilon \text{ for all } n \geq N. \tag{1.43}$$

Pairing up the terms on the left, we find that they are non-negative, so each pair individually must be less than ε—

$$0 \leq U(f,P_n) - \int f < \varepsilon \text{ and } 0 \leq \int f - L(f,P_n) < \varepsilon \tag{1.44}$$

Using this equation, we can finally write

$$\int f - \varepsilon < L(f,P_n) \leq S(f,P_n) \leq U(f,P_n) < \int f + \varepsilon \implies \lim_{n\to\infty} S(f,P_n) = \int f. \tag{1.45}$$

Again, this is regardless of the choice of tag sets. ■

1.3 School Integration Rocks

Let us now connect to Newton's definition of integration. Pick $f \in \mathcal{C}[a,b]$. For any $n \in \mathbb{N}$, consider the partition $P_n : a = x_0 < x_1 < \ldots < x_n = b$ such that $x_j - x_{j-1} = \frac{b-a}{n}$. This is the standard school partition. Note that $\|P_n\| = \frac{b-a}{n} \to 0$. For all tag sets $\{\zeta_j^{(n)}\}$ of P_n, we find that

$$\int_a^b f = \lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j^{(n)}) \frac{b-a}{n}. \tag{1.46}$$

In school, the tag set was generally chosen as the left endpoints or right endpoints of the subintervals. The left endpoints tag set is

$$\zeta_j = a + \frac{b-a}{n}(j-1). \tag{1.47}$$

Chapter 2

ANTIDERIVATIVES

January 20th.

Let us first summarise; let f be a bounded function on the interval $[a, b]$. Then, the following are equivalent—

- $f \in \mathcal{R}[a, b]$,
- For $\varepsilon > 0$, there exists $P \in \mathcal{P}[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$,
- For $\varepsilon > 0$, there exists $\delta > 0$ such that for all $P \in \mathcal{P}[a, b]$ satisfying $\|P\| < \delta$, $U(f, P) - L(f, P) < \varepsilon$,
- There exists $\{P_n\} \subset \mathcal{P}[a, b]$ such that $U(f, P_n) - L(f, P_n) \to 0$,
- There exists $\{P_n\} \subset \mathcal{P}[a, b]$ such that $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$,
- $\lim_{\|P\|\to 0} S(f, P) = \lambda$.

2.1 Spaces and Algebras

Note that $\mathcal{B}[a, b]$ is a vector space; a linear combination of two elements in it is also part of this set. If we now define a multiplication of two vectors in it as $(f \cdot g)(x) = f(x) \cdot g(x)$ for $f, g \in \mathcal{B}[a, b]$, we see that $f \cdot g$ is also a vector in this space. Thus, we have made $\mathcal{B}[a, b]$ into an algebra.

We now question if $\mathcal{R}[a, b]$ is an algebra, or even a vector space. We begin by defining $\mathcal{I} : \mathcal{R}[a, b] \to \mathbb{R}$ by $\mathcal{I}(f) = \int_a^b f$. We can ask the following questions:

- whether $\mathcal{I}$ is linear; $\mathcal{I}(rf + g) = r\mathcal{I}(f) + \mathcal{I}(g)$, for $r \in \mathbb{R}$,
- whether $\mathcal{I}$ is multiplicative; $\mathcal{I}(f \cdot g) = \mathcal{I}(f) \cdot \mathcal{I}(g)$.

Other questions can also be asked; is $\mathcal{I}$ a monotonic function, or even a homomorphism if $\mathcal{R}[a, b]$ proves to be a vector space.

Given $f \in \mathcal{B}[a, b]$ and $P \in \mathcal{P}[a, b]$, we have

$$M_j - m_j = \sup\{|f(x) - f(y)| : x, y \in I_j\}. \tag{2.1}$$

We denote this value by $\operatorname{osc}_{I_j} f$, the oscillation of f over I_j. If we adopt this notation, we would then have

$$U(f, P) - L(f, P) = \sum_{j=1}^{n} \operatorname{osc}_{I_j} f \cdot |I_j| \,. \tag{2.2}$$

2.1.1 Results on $\mathcal{I}$ and $\mathcal{R}[a,b]$

Assume that, here, $f, g \in \mathcal{R}[a,b]$ and $r \in \mathbb{R}$.

1. Coming back, let us prove that $\mathcal{I}$ is, in fact, linear.

   Proof. First, we show that $rf + g$ is Riemann integrable for $f, g \in \mathcal{R}[a,b]$. For any partition $P \in \mathcal{P}[a,b]$, we have

   $$\begin{aligned} S(rf+g,P) &= \sum_{j=1}^{n} (rf+g)(\zeta_j)\,|I_j| \\ &= r\sum_{j=1}^{n} f(\zeta_j)\,|I_j| + \sum_{j=1}^{n} g(\zeta_j)\,|I_h| = rS(f,P) + S(g,P). \end{aligned} \tag{2.3}$$

   This result is regardless of choice of tag set T_P. Thus, $rf + g \in \mathcal{R}[a,b]$. Now, we show the linearity of $\mathcal{I}$.

   $$\begin{aligned} \left|S(rf+g,P) - r\int f - \int g\right| &= \left|r(S(f,P) - \int f) + (S(g,P) - \int g)\right| \\ &\leq |r|\left|S(f,P) - \int f\right| + \left|S(g,P) - \int g\right| \end{aligned} \tag{2.4}$$

   $$\implies S(rf+g,P) \to r\int f + \int g \text{ as } \|P\| \to 0. \tag{2.5}$$

   ■

2. We now show that $f \cdot g \in \mathcal{R}[a,b]$ for f and g Riemann integrable. We first show this for f^2.

   Proof. We show that $f^2 \in \mathcal{R}[a,b]$. We have

   $$\begin{aligned} \left|f^2(x) - f^2(y)\right| &= |f(x) + f(y)|\,|f(x) - f(y)| \leq 2M\,|f(x) - f(y)| \\ \implies \sum_j \operatorname{osc}_{I_j} |f|^2 \cdot |I_j| &\leq \sum_j 2M \cdot \operatorname{osc}_{I_j} f \cdot |I_j| \\ U(f^2,P) - L(f^2,P) &\leq 2M(U(f,P) - L(f,P)). \end{aligned} \tag{2.6}$$

   Since f is Riemann integrable, $U(f,P) - L(f,P)$ can be lowered to less than $\varepsilon > 0$, we can make the left hand term less than ε. Thus, $f^2 \in \mathcal{R}[a,b]$. ■

   Despite this, $\mathcal{I}(f^2) \neq (\mathcal{I}(f))^2$; $\mathcal{I}$ is not multiplicative.

3. Let us show $f \cdot g \in \mathcal{R}[a,b]$.

   Proof. Break down $f \cdot g$ as

   $$f \cdot g = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right). \tag{2.7}$$

   From the previous results, the right hand side is Riemann integrable. Thus, $f \cdot g \in \mathcal{R}[a,b]$. ■

4. If $f(x) \geq 0$ for all $x \in [a,b]$, then $\mathcal{I}(f) \geq 0$. This result is left as an exercise to the reader.

5. If $f \geq g$, then $\mathcal{I}(f) \geq \mathcal{I}(g)$. This result is also left as an exercise to the reader.

6. If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$. Moreover, $|\mathcal{I}(f)| \leq \mathcal{I}(|f|)$.

   Proof. Start with

   $$|f(x)| - |f(y)| \leq |f(x) - f(y)|. \tag{2.8}$$

   Therefore, for all $P \in \mathcal{P}[a,b]$, $\operatorname{osc}_{I_j} |f| \leq \operatorname{osc}_{I_j} f$ for all valid j. Thus,

   $$U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P) \tag{2.9}$$

   tells us that $|f|$ is also Riemann integrable. Using the fact that $\mathcal{I}$ is monotonous,

   $$-|f| \leq f \leq |f| \implies -\int |f| \leq \int f \leq \int |f| \implies |\mathcal{I}(f)| \leq \mathcal{I}(|f|). \tag{2.10}$$

   ■

7. We have $\max\{f,g\}, \min\{f,g\} \in \mathcal{R}[a,b]$. The proof of this result is left as an exercise to the reader.
8. If $\frac{1}{g} \in \mathcal{B}[a,b]$, then $\frac{1}{g} \in \mathcal{R}[a,b]$. This would also imply that $\frac{f}{g} \in \mathcal{R}[a,b]$.

   Proof. Denote $1/\tilde{M} = \sup_{[a,b]} \frac{1}{g}$. Then,

   $$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| \leq \frac{1}{\tilde{M}^2}\left|g(x) - g(y)\right|. \tag{2.11}$$

   We can then proceed by using the oscillations. ■

9. Let $a < c < b$. Then $f|_{[a,c]} \in \mathcal{R}[a,c]$ and $f|_{[c,b]} \in \mathcal{R}[c,b]$. Moreover, $\int_a^c f + \int_c^b f = \int_a^b f$.

   Proof. For $\varepsilon > 0$, there exists $P \in \mathcal{P}[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Let, without loss of generality, $c \in P$. If not, we could refine P as $P \cup \{c\}$. We then have $P : x_0 = a < x_1 < \ldots < x_m = c < \ldots < x_n = b$. The nodes from x_0 to x_m form a partition $P_1 \in \mathcal{P}[a,b]$, and the nodes x_m to x_n form a partition $P_2 \in \mathcal{P}[c,b]$. Thus,

   $$(U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) < \varepsilon. \tag{2.12}$$

   Both of these pairs of terms are non-negative, so each pair individually is less than ε. Thus, $f|_{[a,c]} \in \mathcal{R}[a,c]$ and $f|_{[c,b]} \in \mathcal{R}[c,b]$. Now let $\lambda_1 = \int_a^c f$ and $\lambda_2 = \int_c^b f$. Then,

   $$\int_a^b f \geq L(f,P) = L(f,P_1) + L(f,P_2) > U(f,P_1) - \varepsilon + U(f,P_2) - \varepsilon \geq \lambda_1 + \lambda_2 - 2\varepsilon \tag{2.13}$$

   and

   $$\int_a^b f \leq U(f,P) = U(f,P_1) + U(f,P_2) < L(f,P_1) + \varepsilon + L(f,P_2) + \varepsilon \leq \lambda_1 + \lambda_2 + 2\varepsilon. \tag{2.14}$$

   Both inequalities imply that $\left|\int_a^b f - (\lambda_1 + \lambda_2)\right| \leq 2\varepsilon$. ■

2.1.2 Results on $f \in \mathcal{R}[a,b]$

Assume that, here, $f \in \mathcal{R}[a,b]$.

(a) $m(b-a) \leq \int_a^b f \leq M(b-a)$. This result is left as an exercise to the reader.

(b) If $f \in \mathcal{C}[a,b]$, then there exists $c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f$. This is known as the *mean value theorem.*

   Proof. Simply start with $m(b-a) \leq \int_a^b f \leq M(b-a)$, and divide everything by $b - a \neq 0$. There exists η and ζ such that $f(\eta) = m$ and $f(\zeta) = M$, so f takes every value in between. ■

Theorem 2.1. *Let $f : [a,b] \to [c,d]$ and $g : [c,d] \to \mathbb{R}$, and let $f \in \mathcal{R}[a,b]$ and $g \in \mathcal{C}[c,d]$. Then $g \circ f \in \mathcal{R}[a,b]$. Note that for this to be satisfied, g must be continuous on top of Riemann integrable.*

Proof. Clearly, $g \circ f \in \mathcal{B}[a,b]$. By uniform continuity, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|g(x) - g(y)| < \frac{\varepsilon}{2(b-a)} \text{ for all } |x-y| < \delta. \tag{2.15}$$

Also, there exists $P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \frac{\varepsilon\delta}{4M} \tag{2.16}$$

where $M = \sup_{[c,d]} g(y)$. We claim that $U(g \circ f, P) - L(g \circ f, P) < \varepsilon$. Let n be the number of intervals of P. Write $J = \{1, 2, \ldots, n\} = J_1 \sqcup J_2$, where we define

$$J_1 = \{j \in J : \operatorname{osc}_{I_j} f < \delta\},\ J_2 = \{j \in J : \operatorname{osc}_{I_j} f \geq \delta\}.$$

Now, if $j \in J_1$, then

$$\begin{align}
&|f(x) - f(y)| < \delta \text{ for all } x, y \in I_j \tag{2.17}\\
\implies &|g(f(x)) = g(f(y))| < \frac{\varepsilon}{2(b-a)} \text{ for all } x, y \in I_j \notag\\
\implies \sup_{x,y \in I_j} &|g(f(x)) - g(f(y))| = \operatorname{osc}_{I_j} g \circ f \leq \frac{\varepsilon}{2(b-a)} \notag\\
\implies &\sum_{j \in J_1} \operatorname{osc}_{I_j} g \circ f \cdot |I_j| \leq \frac{\varepsilon}{2(b-a)} \cdot \sum_{j \in J_1} |I_j| \leq \frac{\varepsilon}{2}. \tag{2.18}
\end{align}$$

Now, if $j \in J_2$, then

$$\begin{align}
&\operatorname{osc}_{I_j} g \circ f \leq 2M \tag{2.19}\\
\implies &\sum_{j \in J_2} \operatorname{osc}_{I_j} g \circ f \cdot |I_j| \leq 2M \cdot \sum_{j \in J_2} |I_j| \leq 2M \cdot \sum_{j \in J_2} |I_j| \operatorname{osc}_{I_j} \cdot \frac{1}{\delta} \notag\\
&\leq \frac{2M}{\delta} \cdot \sum_{j \in J} |I_J| \operatorname{osc}_{I_j} f = \frac{2M}{\delta}(U(f,P) - L(f,P)) < \frac{2M}{\delta} \cdot \frac{\varepsilon\delta}{4M} \tag{2.20}\\
\implies &\sum_{j \in J_2} \operatorname{osc}_{I_j} g \circ f \cdot |I_j| < \frac{\varepsilon}{2} \tag{2.21}
\end{align}$$

Combining both inequalities, we have

$$U(g \circ f, P) - L(g \circ f, P) = \sum_{j \in J_1} \operatorname{osc}_{I_j} g \circ f \cdot |I_j| + \sum_{j \in J_2} \operatorname{osc}_{I_j} g \circ f \cdot |I_j| < \varepsilon. \tag{2.22}$$

■

Example 2.2. Let $f \in \mathcal{R}[a,b]$. From the previous theorem, $e^f, \sin f, \cos f \in \mathcal{R}[a,b]$ and for $f \geq 0$, $f^{\frac{1}{n}} \in \mathcal{R}[a,b]$.

Theorem 2.3. *Let $f, g \in \mathcal{B}[a,b]$. If $f(x) = g(x)$ for all x but finitely many, then $f \in \mathcal{R}[a,b]$ if and only if $g \in \mathcal{R}[a,b]$. Moreover, in this case, $\int_a^b f = \int_a^b g$.*

So, if $f \equiv 0$ except for some finitely many points in $[a,b]$, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = 0$.

Proof. Without loss of generality, let $c \in [a,b]$ and $f(x) = g(x)$ for all $x \in [a,b] \backslash \{c\}$, and $f(c) \neq g(c)$. Note that it is enough to prove $\overline{\int} f = \overline{\int} g$ and $\underline{\int} f = \underline{\int} g$. Let $\tilde{M} \geq \sup f, \sup g$. For $\varepsilon > 0$, there exists $P \in \mathcal{P}[a,b]$ such that

$$U(f,P) < \frac{\varepsilon}{2} + \overline{\int} f. \tag{2.23}$$

Now set $\delta = \frac{\varepsilon}{8\tilde{M}}$. Let $\tilde{P} \supset P$ such that $\left\|\tilde{P}\right\| < \delta$. Now $f \equiv g$ except for $x = c$. Let $\{\tilde{I_j}\}_{j=1}^n$ be the subintervals of $\tilde{P}$. Let p (and possibly $p+1$) be such that $f(x) \neq g(x)$ on I_p (and possibly on I_{p+1}). Note that

$$\begin{aligned}
\left|\sup_{I_j} f - \sup_{I_j}\right| g &= 0 \text{ for all } j \neq p, p+1 \text{ or}\\
&\leq 2\tilde{M} \text{ for } j = p, p+1.
\end{aligned}$$

Thus,

$$\begin{align}
&\left|U(f,\tilde{P}) - U(g,\tilde{P})\right| \leq \sum_{j=1}^n |I_j| \cdot \left|\sup_{I_j} f - \sup_{I_j} g\right| = \sum_{j=p,p+1} |I_j| \cdot \left|\sup_{I_j} f - \sup_{I_j} g\right| \leq 4\delta\tilde{M} \tag{2.24}\\
\implies &\left|U(f,\tilde{P}) - U(g,\tilde{P})\right| < \frac{\varepsilon}{2}. \tag{2.25}
\end{align}$$

Using this, we have

$$\begin{aligned}
&\overline{\int} g \leq U(g,\tilde{P}) < U(f,\tilde{P}) + \frac{\varepsilon}{2} \leq U(f,P) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \overline{\int} f + \frac{\varepsilon}{2} \\
\implies &\overline{\int} g < \overline{\int} f + \varepsilon \text{ for all } \varepsilon > 0 \\
\implies &\overline{\int} g \leq \overline{\int} f. \qquad (2.26)
\end{aligned}$$

Note that if we switch f and g around, we would get $\overline{\int} f \leq \overline{\int} g$. Thus, we must have

$$\overline{\int} f = \overline{\int} g.$$

■

2.2 The Fundamental Theorem of Calculus

To start off, we will define the set(s) $\mathcal{D}$ as the set of all differentiable functions, and $\mathcal{F}(\mathbb{R})$ is the set of all real valued functions. Recall that we define the integral function as

$$\mathcal{I} : \mathcal{R}[a,b] \to \mathcal{F}(\mathbb{R}) \text{ defined as } \mathcal{I}(f)(x) = \int_a^x f. \tag{2.27}$$

We will also define differentiation as a function,

$$\frac{d}{dx} : \mathcal{D} \to \mathcal{F}(\mathbb{R}) \text{ defined as } \frac{d}{dx}(f) = \frac{df}{dx}. \tag{2.28}$$

The fundamental theorem of calculus, roughly, states that both $\frac{d}{dx} \circ \mathcal{I}$ and $\mathcal{I} \circ \frac{d}{dx}$ are the identity function. There is a little trouble with this rough statement as the composition here makes not much sense.

Definition 2.4. Let $S \subseteq \mathbb{R}$, and let $f : S \to \mathbb{R}$ be a function. A differentiable function F is called the *antiderivative* of f if $f(x) = F'(x)$ for all $x \in S$.

Example 2.5. We state some antiderivatives here.

(a) The antiderivative of x is $\frac{1}{2}x^2$.
(b) Polynomials have an antiderivative.
(c) Continuous functions have an antiderivative.
(d) The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 0$ if $x \geq 0$ and $f(x) = 1$ if $x < 0$ does not have an antiderivative.

Theorem 2.6 (Darboux.)**.** *Let $f : (a,b) \to \mathbb{R}$ be differentiable, and let $a < a_0 < b_0 < b$. If $f'(a_0) < r < f'(b_0)$, then there exists $c_0 \in (a_0, b_0)$ such that $f'(c_0) = r$.*

Proof. Construct a new function as $g(x) = -f(x) + rx$ for all $x \in (a,b)$. Note that $g : (a,b) \to \mathbb{R}$ is differentiable, and $g'(x) = -f'(x) + r$ for all $x \in (a,b)$. Also, $g'(a_0) > 0$ and $g'(b_0) < 0$. Now, $g|_{[a_0,b_0]}$ is uniformly continuous. Using the sign of the derivatives, we have

$$g(a_0 + h) - g(a_0) > 0 \text{ for some small } h > 0, \tag{2.29}$$

$$g(b_0 + h) - g(b_0) > 0 \text{ for some small } h < 0. \tag{2.30}$$

From these two equations, we can see that g attains a maximum at some $c_0 \in (a_0, b_0)$. This implies that $g'(c_0) = 0$ and $f'(c_0) = r$. ■

January 27th.

Theorem 2.7 (The *first fundamental theorem of calculus*)**.** *Let $f \in \mathcal{R}[a,b]$, $F \in \mathcal{C}[a,b]$, and let F be an antiderivative of f on (a,b), that is, $F'(x) = f(x)$ for all $x \in (a,b)$. Then*

$$\int_a^b f = F(b) - F(a). \tag{2.31}$$

Proof. Let $P \in \mathcal{P}[a,b]$ defined as $P : a = x_0 < x_1 < \ldots < x_n = b$. Now,

$$F(b) - F(a) = \sum_{j=1}^{n} F(x_j) - F(x_{j-1}).$$

Since $F \in \mathcal{C}[a,b]$, we have $F|_{[x_{j-1},x_j]} \in \mathcal{C}[x_{j-1},x_j] \cap \mathcal{D}(x_{j-1},x_j)$. Thus, from the intermediate value theorem, there exists some $\zeta_j \in (x_{j-1},x_j)$ such that

$$F(x_j) - F(x_{j-1}) = F'(\zeta_j) \cdot (x_j - x_{j-1}) = f(\zeta_j) \cdot (x_j - x_{j-1}). \tag{2.32}$$

We choose these ζ_j's to be our tag set. Thus, we shall have

$$\begin{aligned} F(b) - F(a) &= \sum_{j=1}^{n} F(x_j) - F(x_{j-1}) = \sum_{j=1}^{n} f(\zeta_j) \cdot |I_j| \\ \implies L(f,P) &\leq F(b) - F(a) \leq U(f,P) \text{ for all } P \in \mathcal{P}[a,b] \\ \implies F(b) - F(a) &= \int_a^b f. \end{aligned} \tag{2.33}$$

■

Remark 2.8. Note that the first fundamental theorem of calculus implies that $\int_a^b f$ can be computed simply by finding an antiderivative of f. We now ask where the antiderivative is, and even if it exists. The second fundamental theorem of calculus answers this.

Theorem 2.9 (The *second fundamental theorem of calculus*)**.** *Let $f \in \mathcal{R}[a,b]$. Define $F : [a,b] \to \mathbb{R}$ as $F(x) = \int_a^x f(t)dt$ for all $x \in [a,b]$. Then,*

(a) $F \in \mathcal{C}[a,b]$,

(b) if f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$,

(c) if f is continuous from the right at a, then $F'_+(a) = f(a)$.

Corollary 2.10. *Let $f \in \mathcal{C}[a,b]$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x). \tag{2.34}$$

This is an interesting corollary that comes in handy. We now prove the theorem.

Proof. Set $M = \sup_{x \in [a,b]} |f(x)|$. Now,

$$|F(x) - F(y)| = \left|\int_x^y f(t)dt\right|. \tag{2.35}$$

Now start with

$$\begin{aligned} -M &\leq f(t) \leq M \text{ for all } t \in [a,b] \\ \implies -M(y-x) &\leq \int_x^y f(t)dt \leq M(y-x) \\ \implies |F(x) - F(y)| = \left|\int_x^y f(t)dt\right| &\leq M\,|x-y| \text{ for all } x,y \in [a,b]. \end{aligned} \tag{2.36}$$

Any function that satisfies thie property is a *lipschitz function*. It can be shown that any lipschitz function is continuous. This part of the proof is left as an exercise. The first part is now proved.

Let f be continuous on some $x_0 \in (a,b)$. For some $x \neq x_0$, we can start with

$$\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x} f(t)dt - \frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)dt = \frac{1}{x-x_0}\int_{x_0}^{x}(f(t)-f(x_0))dt. \tag{2.37}$$

f is continuous at x_0, so for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t)-f(x_0)| < \varepsilon \text{ for all } |t-x_0| < \delta.$$

Therefore,

$$\begin{aligned}\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| &= \left|\frac{1}{x-x_0}\int_{x_0}^{x}(f(t)-f(x_0))dt\right| \leq \frac{1}{|x-x_0|}\left|\int_{x_0}^{x}|f(t)-f(x_0)|\,dt\right| \\ &< \frac{1}{|x-x_0|}\left|\int_{x_0}^{x}\varepsilon dt\right| = \varepsilon \text{ for all } x \in (x_0-\delta, x_0+\delta)\backslash\{x_0\}. \end{aligned}\tag{2.38}$$

This must imply that

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = f(x_0). \tag{2.39}$$

■

Here is an interesting example.

Example 2.11. Define $f : [0,2] \to \mathbb{R}$ by $f = \chi_{[0,1]}$. Clearly, $f \in \mathcal{R}[0,2]$. Let $F : [0,2] \to \mathbb{R}$ be defined by $F = \int_0^x f$. We have

$$\begin{aligned}\text{for all } x \in [0,1],\, F(x) &= \int_0^x \chi_{[0,1]} = \int_0^x 1 = x, \\ \text{for all } x \in (1,2],\, F(x) &= \int_0^x \chi_{[0,1]} = \int_0^1 1 + \int_1^x 0 = 1.\end{aligned}$$

The corollary guarantees that the antiderivative of a continuous function is differentiable; if our function is Riemann integrable but not continuous, the antiderivative may not be differentiable.

2.2.1 Some Interesting Methods

Theorem 2.12 (The method of *integration by parts*)**.** *Let $f, g \in \mathcal{D}[a,b]$ and $f', g' \in \mathcal{R}[a,b]$. Then,*

$$\int_a^b f'g + \int_a^b fg' = f(b)g(b) - f(a)g(a). \tag{2.40}$$

Proof. Simply start with $\int_a^b (fg)'$. ■

Theorem 2.13 (The method of *change of variable*)**.** *Let $u \in \mathcal{D}[a,b]$, $u' \in \mathcal{R}[a,b]$, and let $f \in \mathcal{C}(u([a,b]))$. Then,*

$$\int_a^b f(u(t))u'(t)dt = \int_{u(a)}^{u(b)} f(x)dx. \tag{2.41}$$

Proof. The proof for a constant function u is trivial. So, assume that u is a non-constant function. Note that $(f \circ u)u'$ in Riemann integrable on $[a,b]$. Define $F : u([a,b]) \to \mathbb{R}$ as

$$F(x) = \int_{u(a)}^{x} f(t)dt \text{ for all } t \in u([a,b]). \tag{2.42}$$

The second fundamental theorem of calculus tells us that $F' = f$ on $u([a,b])$. Now for $t \in [a,b]$, $(F \circ u)'(t) = F'(u(t))u'(t)$. By the first fundmental theorem of calculus,

$$\int_a^b F'(u(t))u'(t)dt = \int_a^b (F\circ u)'(t)dt \tag{2.43}$$

$$\implies \int_a^b f(u(t))u'(t)dt = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f(t)dt. \tag{2.44}$$

■

Chapter 3

IMPROPER INTEGRALS

February 3rd.

We first state some ‘assumptions’ on Riemann intgerals, which we now consider to be *proper integrals*.

1. The function to be integrated was bounded, that is, $f \in \mathcal{B}[a, b]$.
2. The interval on which the function was to be integrated was bounded.

We want to talk about integrals of functions that *don’t* follow these assumptions.

1. Evaluating $\int_a^b f$ even if f is not bounded on $[a, b]$.
2. Evaluating $\int_a^\infty f$, $\int_{-\infty}^b f$, or $\int_{-\infty}^\infty f$.

3.1 Improper Integrals of Type I

Definition 3.1 (An *improper integral of type I.*)**.** Let $f \notin \mathcal{R}[a, b]$, and $f \in \mathcal{R}[c, b]$ for all $a < c < b$. If

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f \; \left(= \int_a^b f \right) \tag{3.1}$$

exists, then we say that the integral $\int_a^b f$ converges to this value, and diverges otherwise. The convergence is equivalent to saying that $\lim_{c \to a^+} \int_c^b f$ exists.

Similarly, if $f \notin \mathcal{R}[a, b]$ and $f \in \mathcal{R}[a, c]$ for all $a < c < b$, and if

$$\lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f \; \left(= \int_a^b f \right) \tag{3.2}$$

exists, then we say that the integral $\int_a^b f$ converges to this value, and diverges otherwise. The convergence is equivalent to saying that $\lim_{c \to b^-} \int_a^c f$ exists.

If f properly diverges at a point $a < c < b$ in $[a, b]$, then we define

$$\int_a^b f = \int_a^c f + \int_c^b f \tag{3.3}$$

provided that *either* of the right hand side integrals exist.

Example 3.2. We wish to compute $\int_0^1 \frac{1}{x^2} dx$. This is an improper integral of type I; the function is unbounded at $x = 0$. Therefore, for small $\varepsilon > 0$,

$$\int_\varepsilon^1 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_\varepsilon^1 = \frac{1}{\varepsilon} - 1. \tag{3.4}$$

Taking the limit as ε tends to zero from the positive side, we see that the limit diverges to positive infinity. The improper integral diverges.

Example 3.3. We wish to compute $\int_0^1 \frac{1}{\sqrt{x}}dx$. For $\varepsilon > 0$, small,

$$\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_\varepsilon^1 = 2 - 2\sqrt{\varepsilon} \to 2 \text{ as } \varepsilon \to 0^+. \tag{3.5}$$

Thus, $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$.

Example 3.4. We wish to compute $\int_0^2 \frac{1}{2x-x^2}dx$. Let us split this integral as $\int_0^2 f = \int_0^1 + \int_1^2$. For $\varepsilon > 0$, small,

$$\int_1^{2-\varepsilon} \frac{1}{x(2-x)}dx = \left[\frac{1}{2}\ln\left(\frac{x}{2-x}\right)\right]_1^{2-\varepsilon} = \frac{1}{2}\ln\left(\frac{2-\varepsilon}{\varepsilon}\right) - \frac{1}{2}\ln 1. \tag{3.6}$$

This integral diverges, so the entire integral must diverge.

In general,

Example 3.5. Let $p > 0$. We wish to evaluate $\int_0^1 \frac{1}{x^p}dx$. For $\varepsilon > 0$, small,

$$\int_\varepsilon^1 \frac{1}{x^p}dx = \begin{cases} \left[\frac{x^{1-p}}{1-p}\right]_\varepsilon^1 & \text{if } p \neq 1, \\ \left[\ln x\right]_\varepsilon^1 & \text{if } p = 1 \end{cases} = \begin{cases} \frac{1}{1-p}(1-\varepsilon^{1-p}) & \text{if } p \neq 1, \\ -\ln\varepsilon & \text{if } p = 1. \end{cases} \tag{3.7}$$

If we apply $\lim_{\varepsilon\to 0^+}$, we see that $\int_0^1 \frac{1}{x^p}$ converges to $\frac{1}{1-p}$ if $0 < p < 1$, and diverges otherwise.

3.1.1 Tests of Convergence

We now discuss when an improper integral of type I is convergent or not.

Theorem 3.6 (The *comparison test I*)**.** *Let $0 \leq f(x) \leq g(x)$ for all $x \in [a,b)$. Assume that $\int_a^b f$ and $\int_a^b g$ are improper integrals solely due to the point $x = b$.*

1. *If $\int_a^b g$ converges, then $\int_a^b f$ converges.*
2. *If $\int_a^b f$ diverges, then $\int_a^b g$ diverges.*

Proof. We prove part the first part only. Set $G(x) = \int_a^x g$ for all $x \in [a,b)$. As $\int_a^b g$ converges, and G is non-decreasing function, we have

$$\int_a^b g = \sup\{\int_a^x g \,|x \in [a,b)\}.$$

Now, from the inequality, since the integral function is monotonous,

$$0 \leq \int_a^x f \leq \int_a^x g \text{ for all } x \in [a,b)$$

$$\implies 0 \leq \sup_{x\in[a,b)} \int_a^x f \leq \int_a^b g < \infty \tag{3.8}$$

$$\implies \lim_{c\to b^-} \int_a^c f < \infty. \tag{3.9}$$

■

Example 3.7. We question whether $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p}dx$ converges for $p > 0$. Firstly, note that

$$0 \leq \sin x \leq 1 \implies 0 \leq \frac{\sin x}{x^p} \leq \frac{1}{x^{p-1}} \text{ for all } x \in (0, \frac{\pi}{2}].$$

Note that the integral of the rightmost term converges for $1 < p < 2$. Also, $\int_0^1 \frac{1}{x^p}dx$ also converges for $0 < p < 1$. The case for $p = 1$ is also convergent. Hence, $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p}$ converges if $0 < p < 2$.

Theorem 3.8 (The *limit comparison test I*)**.** *Let $f(x), g(x) \geq 0$ for all $x \in [a,b)$. Suppose*

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = l \neq 0, \infty.$$

Then the improper integrals $\int_a^b f$ and $\int_a^b g$ converge, or diverge, together.

Proof. We know $0 < l < \infty$. Pick $\varepsilon > 0$ such that $l - \varepsilon > 0$. There exists $c \in [a,b)$ such that

$$\begin{aligned} &\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon \text{ for all } x \in (c,b) \\ \implies & l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \text{ for all } x \in (c,b). \end{aligned} \tag{3.10}$$

Therefore, $0 < (l - \varepsilon)g(x) < f(x)$ for all $x \in (c,b)$. By the comparison test - I, if $\int_a^b f$, that is, $\int_c^b f$ converges, then $\int_c^b (l-\varepsilon)g$, that is, $\int_a^b (l-\varepsilon)g$ converges. This tells us that $\int_a^b g$ converges. To show that $\int_a^b g$ converges implies that converge of $\int_a^b f$ is left as an exercise to the reader. ■

February 6th.

Definition 3.9. An improper integral $\int_a^b f$ is absolutely convergent if the improper integral $\int_a^b |f|$ converges.

Theorem 3.10. *The absolute convergence of an improper integral implies the convergence of the improper integral.*

Proof. Let $\int_a^b f$ be an improper integral at a. Now, $-|f(x)| \leq f(x) \leq |f(x)|$ implies that $0 \leq |f(x)| + f(x) \leq 2|f(x)|$ for all $x \in (a,b]$. Therefore, for all $a < c < b$,

$$0 \leq \int_c^b (|f(x)| + f(x))dx \leq 2 \int_c^b |f(x)|\, dx. \tag{3.11}$$

If the rightmost integral converges, then by the comparison test, $\int_a^b |f| + f$ converges. Finally,

$$\begin{aligned} \int_c^b f(x)dx &= \int_c^b (|f(x)| + f(x))dx - \int_c^b |f(x)|\, dx \\ \implies \lim_{c \to a^+} \int_c^b f(x)dx &= \int_a^b (|f(x)| + f(x))dx - \int_a^b |f(x)|\, dx. \end{aligned} \tag{3.12}$$

■

3.2 Improper Integrals of Type II

Definition 3.11 (An *improper integral of type II.*)**.** Fix $a \in \mathbb{R}$ and let $f \in \mathcal{R}[a,r]$ for all $r > a$. If $\lim_{r \to \infty} \int_a^r f$ exists, then we say that $\int_a^\infty f$ converges and we write

$$\int_a^\infty f = \lim_{r \to \infty} \int_a^r f. \tag{3.13}$$

If the limit diverges, then we say that the integral diverges. Similarly, we define

$$\int_{-\infty}^b = \lim_{r \to \infty} \int_{-r}^a f \tag{3.14}$$

provided that the limit exists.

Definition 3.12. Let $f \in \mathcal{R}[a,b]$ for all $a < b \in \mathbb{R}$. If there exists a $c \in \mathbb{R}$ such that $\int_{-\infty}^c f$ and $\int_c^\infty f$ exist, then we say that $\int_{-\infty}^\infty f$ exists and define

$$\int_{-\infty}^\infty f = \int_{-\infty}^c f + \int_c^\infty f. \tag{3.15}$$

Note that the value of the integral, if it exists, is independent of the choice of c; this result is left as an exercise to the reader.

Example 3.13. We wish to compute $\int_0^\infty \sin x dx$. Note that, by the fundamental theorem of calculus,

$$\int_0^r \sin x dx = [-\cos x]_0^r = 1 - \cos r. \tag{3.16}$$

The right hand side does not converge as $r \to \infty$, so we conclude that the integral does not converge.

Example 3.14. We wish to compute $\int_{-\infty}^0 e^{-x} dx$. Note that $e^{-x} \in \mathcal{R}[-r, 0]$ for all $r > 0$. Also,

$$\int_{-r}^0 e^{-x} dx = [-e^{-x}]_{-r}^0 = e^r - 1 \to \infty \text{ as } r \to \infty. \tag{3.17}$$

Therefore, the integral diverges.

Example 3.15. We wish to compute $\int_{-\infty}^\infty \frac{dx}{1+x^2}$. Let us take $c = 0$ to be out 'split'; evaluating the integrals, we have

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{r\to\infty} \int_0^r \frac{dx}{1+x^2} = [\arctan x]_0^r = \arctan r \to \frac{\pi}{2} \text{ as } r \to \infty, \tag{3.18}$$

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{r\to\infty} \int_{-r}^0 \frac{dx}{1+x^2} = [\arctan x]_{-r}^0 = \arctan r \to \frac{\pi}{2} \text{ as } r \to \infty. \tag{3.19}$$

Thus,

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \tag{3.20}$$

We say that $f \in \mathcal{R}[a, \infty)$ if $f \in \mathcal{R}[a, r]$ for all $r > a$; this is just another notation.

3.2.1 Tests of Convergence

Theorem 3.16 (The *comparison test II*)**.** *Let $a \in \mathbb{R}$ and let $f, g \in \mathcal{R}[a, \infty)$. Suppose that $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$.*

1. *If $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges.*
2. *If $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.*

Proof. For all $t > a$, it follows that

$$0 \le \int_a^t f(x)dx \le \int_a^t g(x)dx. \tag{3.21}$$

Denote the two depicted integrals by $F(t)$ and $G(t)$ respectively, for all $t > a$. The result follows. ■

Example 3.17. We question whether $\int_a^\infty \frac{dx}{e^x+1}$ converges. Note that $\frac{1}{e^x+1} \le e^{-x}$ for all $x \in [a, \infty)$. Evaluting the integral of the right hand side function,

$$\int_a^\infty e^{-x} dx = \lim_{r\to\infty} \int_a^r e^{-x} dx = \lim_{r\to\infty} (e^{-a} - e^{-r}) = e^{-a}. \tag{3.22}$$

It converges, so our original integral also converges.

February 10th.

Theorem 3.18 (The *limit comparison test II*)**.** *Let $f, g \in \mathcal{R}[a, \infty)$ and suppose $f(x), g(x) \ge 0$ for all $x \in [0, \infty)$. Also suppose*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0.$$

Then the imporper integrals $\int_a^\infty f$ and $\int_a^\infty g$ converge, or diverge, together.

Proof. The proof is left as an exercise to the reader. ■

Example 3.19. We wish to test the convergence of $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$. Let $f(x) = \frac{1}{x\sqrt{x^2+1}}$ and let $g(x) = \frac{1}{x^2}$. We see that

$$\frac{f(x)}{g(x)} = \frac{1}{\sqrt{1+\frac{1}{x^2}}} \to 1 > 0 \text{ as } x \to \infty. \tag{3.23}$$

As $\int_1^\infty \frac{1}{x^2}dx$ is convegent, then by the limit comparison test, $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ must also converge.

Theorem 3.20. *Let $f \in \mathcal{R}[a,\infty)$. If $\int_a^\infty |f|$ converges, then $\int_a^\infty f$ also converges. The converse is not true.*

Chapter 4

REFINING OF CONVERGENCE

Recall the following:

1. The *Cauchy limit criterion*; $\lim_{x\to a} f(x)$ exists if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that
$$|f(x_1) - f(x_2)| < \varepsilon \text{ for all } x_1, x_2 \in (a-\delta, a+\delta)\backslash\{a\}. \tag{4.1}$$
2. We say $\lim_{x\to\infty} f(x) = l$ if for every $\varepsilon > 0$, there exiss $M > 0$ such that
$$|f(x) - l| < \varepsilon \text{ for all } x > M. \tag{4.2}$$
3. Also, $\lim_{x\to\infty} f(x) = l$ if and only if for every $\varepsilon > 0$, there exists $M > 0$ such that
$$|f(x_1) - f(x_2)| < \varepsilon \text{ for all } x_1, x_2 > M. \tag{4.3}$$

Theorem 4.1 (*Cauchy test for convergence of integrals*)**.** *Let $f \in \mathcal{R}[a,\infty)$. Then $\int_a^\infty f$ converges if and only if for every $\varepsilon > 0$, there exists $M > 0$ such that*
$$\left|\int_{R_1}^{R_2} f\right| < \varepsilon \textit{ for all } R_1, R_2 > M.$$

Proof. We can easily see that $\int_a^\infty f$ converges $\iff$ $\lim_{R\to\infty}\int_a^R f$ converges $\iff$ for every $\varepsilon > 0$, there exists $M > 0$ such that
$$\left|\int_a^{R_1} f - \int_a^{R_2} f\right| < \varepsilon \text{ for all } R_1, R_2 > M. \tag{4.4}$$
But the left hand side is just $\int_{R_1}^{R_2} f$. ■

Theorem 4.2. *Let $\int_a^b f$ be an improper integral at b. Then $\int_a^b f$ exists if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $a < b - \delta$ and*
$$\left|\int_{x_1}^{x_2} f\right| < \varepsilon \textit{ for all } x_1, x_2 \in (b-\delta, b).$$

Proof. The proof is left as an exercise to the reader. ■

Theorem 4.3 (*Absolute convergence test*)**.** *Let $\varphi \in \mathcal{B}[a,\infty) \cap \mathcal{R}[a,\infty)$. If $\int_a^\infty f$ is absolutely convergent, then $\int_a^\infty \varphi f$ is also absolutely convergent.*

Proof. Set M to be $\sup_{x\in[a,\infty)} |\varphi(x)|$. Then, we must have $|(\varphi f)(x)| \leq M\,|f(x)|$ for all $x \in [a,\infty)$. By the limit comparison test, we are done. ■

4.1 The Mean Value Theorems

Theorem 4.4 (The *first mean value theorem*)**.** *Let $f, g \in \mathcal{R}[a,b]$ and suppose that f keeps the same sign over $[a,b]$. Then there exists $\zeta \in [\inf g, \sup g]$ such that*

$$\int_a^b fg = \zeta \int_a^b f.$$

Remark 4.5. Before proving the first mean value theorem, we have a remark.

1. If $g \in \mathcal{C}[a,b]$, then $\int_a^b fg = g(c)\int_a^b f$ for some $c \in [a,b]$.
2. If $f \equiv 1$, and $g \in \mathcal{C}[a,b]$, then there exists $c \in [a,b]$ such that

$$g(c) = \frac{1}{b-a}\int_a^b g.$$

Proof. Let $m = \inf_{[a,b]} g$ and let $M = \sup_{[a,b]} g$. Thus, $m \leq g(x) \leq M$ for all $x \in [a,b]$. Without loss of generality, assume that $f > 0$. The inequality implies that

$$mf(x) \leq f(x)g(x) \leq Mf(x) \implies m\int_a^b f \leq \int_a^b fg \leq M\int_a^b f. \tag{4.5}$$

Thus, there exists a $\zeta \in [m, M]$ such that $\int_a^b fg = \zeta \int_a^b f$. ■

Example 4.6. Let $r \in (0,1)$. Then,

$$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-rx^2)}} \leq \frac{\pi}{6}\frac{1}{\sqrt{1-\frac{r}{4}}}. \tag{4.6}$$

To show this, let $f(x) = \frac{1}{\sqrt{1-x^2}}$ and let $g(x) = \frac{1}{\sqrt{1-rx^2}}$ for all $x \in [0, \frac{1}{2}]$. Clearly, $f, g \in \mathcal{C}[0, \frac{1}{2}]$ and $f > 0$. Let us denote $\int_0^{\frac{1}{2}} fg$ by I. So, by the first mean value theorem, $I = g(c)\int_0^{\frac{1}{2}} f = g(c)\frac{\pi}{6}$ for some $c \in [0, \frac{1}{2}]$. For $c \leq \frac{1}{2}$, notice that $c^2 \leq \frac{1}{4} \implies rc^2 \leq \frac{r}{4}$ which will give us $g(c) \leq \frac{1}{\sqrt{1-\frac{r}{4}}}$. For the left inequality, we observe $0 \leq f \leq fg \implies \frac{\pi}{6} \leq \int_0^{\frac{1}{2}} fg = I$.

February 12th.

Lemma 4.7 (*Abel's lemma*)**.** *Fix $\alpha, \beta \in \mathbb{R}$ and let $\{\omega_j\}_{j=1}^n \subseteq \mathbb{R}$. If*

$$\alpha \leq \sum_{j=1}^m \omega_j \leq \beta \text{ for all } m = 1, 2, \ldots, n,$$

then for all decreasing $a_1 \leq a_2 \leq \ldots \leq a_n \leq 0$, we have

$$a_1\alpha \leq \sum_{j=1}^n a_j\omega_j \leq a_1\beta.$$

Proof. Set $S_m = \sum_{j=1}^m \omega_j$. We are given that $\alpha \leq S_m \leq \beta$ for all $m = 1, 2, \ldots, n$. Thus,

$$\begin{aligned}
\sum_{j=1}^n a_j\omega_j &= a_1S_1 + a_2(S_2 - S_1) + \ldots + a_n(S_n - S_{n-1}) \\
&= (a_1 - a_2)S_1 + (a_2 - a_3)S_2 + \ldots + (a_{n-1} - a_n)S_{n-1} + a_nS_n \\
&\leq \beta(a_1 - a_2 + a_2 - a_3 + \ldots - a_n + a_n) = \beta a_1 \text{ and} &&(4.7)\\
&\geq \alpha(a_1 - a_2 + a_2 - a_3 + \ldots - a_n + a_n) = \alpha a_1. &&(4.8)
\end{aligned}$$

Thus, we have shown both the inequalities. ■

Theorem 4.8 (The *Bonnet form of the second mean value theorem*). *Let $f, \varphi \in \mathcal{R}[a,b]$. Assume that $\varphi \geq 0$ and monotonically decreasing. Then there exists $\zeta \in [a,b]$ such that*

$$\int_a^b \varphi f = \varphi(a) \int_a^\zeta f.$$

Proof. Fix a partition $P \in \mathcal{P}[a,b]$ defined as $P : a = x_0 < x_1 < \ldots < x_n = b$. Consider a tag set $\{\zeta_j\}_{j \in \{1,\ldots,n\}}$ of P. Assume that $a = \zeta_1$. Now,

$$m_j(x_j - x_{j-1}) \leq \int_{x_{j-1}}^{x_j} f \leq M(x_j - x_{j-1}) \text{ for all } j \text{ and} \tag{4.9}$$

$$m_j(x_j - x_{j-1}) \leq f(\zeta_j)\,|I_j| \leq M(x_j - x_{j-1}) \text{ for all } j. \tag{4.10}$$

These equations imply

$$\sum_{j=1}^t m_j\,|I_j| \leq \int_a^{x_t} f \leq \sum_{j=1}^t M_j\,|I_j| \text{ for all } t \text{ and} \tag{4.11}$$

$$\sum_{j=1}^t m_j\,|I_j| \leq \sum_{j=1}^t f(\zeta_j)\,|I_j| \leq \sum_{j=1}^t M_j\,|I_j| \text{ for all } t \tag{4.12}$$

$$\implies \left| \int_a^{x_t} f - \sum_{j=1}^t f(\zeta_j)\,|I_j| \right| \leq \sum_{j=1}^t (M_j - m_j)\,|I_j| \text{ for all } t = 1,\ldots,n$$

$$\implies \left| \int_a^{x_t} f - \sum_{j=1}^t f(\zeta_j)\,|I_j| \right| \leq \operatorname{osc}_P f \tag{4.13}$$

$$\implies \int_a^{x_t} f - \operatorname{osc}_P f \leq \sum_{j=1}^t f(\zeta_j)\,|I_j| \leq \int_a^{x_t} f + \operatorname{osc}_P f. \tag{4.14}$$

Recall that the map $F : [a,b] \to \mathbb{R}$ defined by $x \mapsto \int_a^x f$ is continuous on $[a,b]$. Set $\delta_1 = \inf_{[a,b]} F$ and set $\delta_2 = \sup_{[a,b]} F$. Thus,

$$\delta_1 - \operatorname{osc}_P f \leq \sum_{j=1}^t f(\zeta_j)\,|I_j| \leq \delta_2 + \operatorname{osc}_P f. \tag{4.15}$$

Set $\omega_j = f(\zeta_j)\,|I_j|$ and $a_j = \varphi(\zeta_j)$. for all $j = 1,\ldots,n$. By assumption, $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. By Abel's lemma,

$$\varphi(\alpha)(\delta_1 - \operatorname{osc}_P f) \leq \sum_{j=1}^n \omega_j a_j \leq \varphi(a)(\delta_2 + \operatorname{osc}_P f). \tag{4.16}$$

As $\|P\| \to 0$, we shall have

$$\varphi(a)\delta_1 \leq \int_a^b f\varphi \leq \varphi(a)\delta_2 \tag{4.17}$$

$$\implies \delta_1 \leq \frac{1}{\varphi(a)} \int_a^b f\varphi \leq \delta_2. \tag{4.18}$$

Recall that the δ_1 and δ_2 are simply the infimum and supremum of the continuous function F over $[a,b]$, so there exists a $\zeta \in [a,b]$ such that $F(\zeta)$ equals the middle value, or,

$$\varphi(a) \int_a^\zeta f = \int_a^b \varphi f. \tag{4.19}$$

■

Theorem 4.9 (The *Weierstrass form of the second mean value thoerem*)**.** *Let $f, \varphi \in \mathcal{R}[a,b]$ and suppose that φ is montonic. Then there exists a $\zeta \in [a,b]$ such that*

$$\int_a^b \varphi f = \varphi(a) \int_a^\zeta f + \varphi(b) \int_\zeta^b f.$$

Proof. Let, without loss of generality, φ be increasing. Set $\tilde{\varphi}(x) = \varphi(b) - \varphi(x)$ for all $x \in [a,b]$. By Bonnet form of the second mean value theorem,

$$\int_a^b \tilde{\varphi} f = \tilde{\varphi}(a) \int_a^\zeta f \text{ for some } \varphi \in [a,b] \tag{4.20}$$

$$\implies \varphi(b) \int_a^b f - \int_a^b \varphi f = (\varphi(b) - \varphi(a)) \int_a^\zeta f$$

$$\implies \int_a \varphi f = \varphi(a) \int_a^\zeta f + \varphi(b) \int_\zeta^b f. \tag{4.21}$$

■

4.2 More Tests for Improper Integrals of Type II

Theorem 4.10 (*Abel's test*)**.** *Let $\varphi \in \mathcal{B}[a,\infty) \cap \mathcal{R}[a,\infty)$ be a monotonic function. If $\int_a^\infty f$ converges, then $\int_a^\infty \varphi f$ also converges.*

Proof. Let $R_2 > R_1 > a$. By the second mean value theorem,

$$\int_{R_1}^{R_2} \varphi f = \varphi(R_1) \int_{R_1}^\zeta f + \varphi(R_2) \int_\zeta^{R_2} f \tag{4.22}$$

for some $\zeta(R_1, R_2) \in [R_1, R_2]$. Let $M = \sup_{[a,\infty)} |\varphi|$. Fix $\varepsilon > 0$. As $\int_a^\infty f$ converges, there exists a real $R_0 > 0$ such that

$$\left| \int_{B_1}^{B_2} f \right| < \frac{\varepsilon}{2M} \text{ for all } B_1, B_2 > R_0. \tag{4.23}$$

Thus, for all $R_1, R_2 > R_0$, we have

$$\left| \int_{R_1}^{R_2} \varphi f \right| = \left| \varphi(R_1) \int_{R_1}^\zeta + \varphi(R_2) \int_\zeta^{R_2} f \right| \leq M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} = \varepsilon. \tag{4.24}$$

Hence, by the Cauchy criterion, $\int_a^\infty \varphi f$ converges. ■

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Theorem 4.11 (*Dirichlet test*)**.** *Let $\varphi \in \mathcal{B}[a,\infty)$ be a monotonic function and let $\lim_{x\to\infty} \varphi(x) = 0$. Suppose $f \in \mathcal{R}[a,\infty)$ and $x \mapsto \int_a^x f$ is bounded on $[a,\infty)$. Then, $\int_a^\infty \varphi f$ converges.*

Proof. Define M to be $\sup_{x\in[a,\infty)} \left| \int_a^x f \right|$. For $\varepsilon \to 0$, as $\{\varphi(x) \to 0 \text{ as } x \to \infty\}$, there exists $m_0 \in \mathbb{R}$ such that

$$|\varphi(x)| < \frac{\varepsilon}{4M} \text{ for all } x \geq m_0. \tag{4.25}$$

Therefore, for all $R_1, R_2 > m_0$, there exists a ζ in between R_1 and R_2 such that

$$\int_{R_1}^{R_2} \varphi f = \varphi(R_1) \int_{R_1}^\zeta f + \varphi(R_2) \int_\zeta^{R_2} f \tag{4.26}$$

$$\begin{aligned} \implies \left| \int_{R_1}^{R_2} \varphi f \right| &\leq |\varphi(R_1)| \left| \int_{R_1}^\zeta f \right| + |\varphi(R_2)| \left| \int_\zeta^{R_2} f \right| < \frac{\varepsilon}{4M} \left(\int_{R_1}^\zeta f + \int_\zeta^{R_2} f \right) \\ &< \frac{\varepsilon}{4M} \left(\left| \int_a^\zeta f \right| + \left| \int_a^{R_1} f \right| + \left| \int_a^{R_2} f \right| + \left| \int_a^\zeta f \right| \right) < \varepsilon \text{ for all } R_1, R_2 > m_0. \end{aligned} \tag{4.27}$$

By Cauchy criterion, $\int_a^\infty \varphi f$ converges. ■

Example 4.12. We wish to determine whether $\int_1^\infty \frac{1}{x}\sin x \log x dx$ converges. Here, we shall let $f(x) = \sin x$ and $\varphi(x) = \frac{\log x}{x}$. We know that φ is decreasing monotonically to 0 as $x \to \infty$. Also, $\left|\int_1^x \sin t dt\right| = |\cos x - \cos 1| \leq 2$. Thus, $x \mapsto \int_1^x \sin t$ is uniformly bounded tells us that, by the Dirichlet test, the integral converges.

Example 4.13. We determine the p for which $\int_1^\infty \frac{\sin x}{x^p} dx$ converges. We let $f(x) = \sin x$ and $\varphi(x) = \frac{1}{x^p}$ and focus only on the case when $p > 0$. Notice that φ monotonically decreases to 0 as $x \to \infty$. Therefore, $\int_1^\infty \frac{\sin x}{x^p}$ converges for all $p > 0$.

Note that since $\int_0^1 \frac{\sin x}{x} dx$ converges, $\int_0^\infty \frac{\sin x}{x} dx$ also converges. By the comparison test, $\int_1^\infty \frac{\sin x}{x^p} dx$ is absolutely convergent for all $p > 1$. We question whether the integral absolutely converges for $0 < p \leq 1$.

Let us fix $0 < p \leq 1$. Now, $|\sin x| \geq \sin^2 x$ for all $x \in \mathbb{R}$. Therefore,

$$\left|\frac{\sin x}{x^p}\right| \geq \frac{\sin^2 x}{x^p} = \frac{1 - \cos 2x}{2x^p} = \frac{1}{2x^p} - \frac{\cos 2x}{2x^p}. \tag{4.28}$$

But the left term on the right hand side converges if and only $p > 1$. Thus, $\int_1^\infty \left|\frac{\sin x}{x^p}\right| dx$ diverges for $0 < p \leq 1$. We provide a more slick proof of this.

Set $f(x) = \frac{\sin x}{x}$. We claim that $\int_0^\infty |f|$ is not convergent. Fix a natural n. Therefore,

$$\int_0^{n\pi} |f| = \int_0^{\pi} f + \int_{\pi}^{2\pi} f + \ldots + \int_{(n-1)\pi}^{n\pi} f. \tag{4.29}$$

For all $1 \leq m \leq n$,

$$\int_{(m-1)\pi}^{m\pi} |f| = \int_{(m-1)\pi}^{m\pi} \frac{|\sin x|}{x} dx = \int_0^{\pi} \frac{\sin x}{(m-1)\pi + x} dx \tag{4.30}$$

Now for all $x \in [0, \pi]$, $(m-1)\pi + x \leq m\pi \implies \frac{1}{(m-1)\pi + x} \geq \frac{1}{m\pi}$

$$\implies \int_{(m-1)\pi}^{m\pi} |f| \geq \int_0^{\pi} \frac{\sin x}{m\pi} dx = \frac{2}{m\pi} \text{ for all } m = 1, 2, \ldots, n. \tag{4.31}$$

Therefore,

$$\int_0^{\pi} |f| \geq \frac{2}{\pi} \sum_{m=1}^{n} \frac{1}{m}. \tag{4.32}$$

This tells us that $\lim_{n\to\infty} \int_0^{n\pi} |f| = \infty$.

Example 4.14. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ (-1)^{n+1}(n+1) & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}] \text{ for some } n \in \mathbb{N}. \end{cases} \tag{4.33}$$

Clearly, $\int_0^1 |f|$ and $\int_0^1 f$ are improper integrals of type I at 0. Pick $0 < \varepsilon < 1$. Let n be a natural number such that $\frac{1}{n+1} < \varepsilon \leq \frac{1}{n}$. Thus,

$$\int_\varepsilon^1 |f| = \int_\varepsilon^{\frac{1}{n}} |f| + \int_{\frac{1}{n}}^{\frac{1}{n-1}} |f| + \ldots + \int_{\frac{1}{3}}^{\frac{1}{2}} |f| + \int_{\frac{1}{2}}^{1} |f| \tag{4.34}$$

$$= (n+1)\left(\frac{1}{n} - \varepsilon\right) + n\left(\frac{1}{n-1} - \frac{1}{n}\right) + \ldots + 3\left(\frac{1}{2} - \frac{1}{3}\right) + 2\left(1 - \frac{1}{2}\right)$$

$$= (n+1)\left(\frac{1}{n} - \varepsilon\right) + \sum_{m=1}^{n-1} \frac{1}{m} > \sum_{m=1}^{n-1} \frac{1}{m}. \tag{4.35}$$

As $\varepsilon \to 0^+$, $n \to \infty$ implies that $\int_0^1 |f|$ diverges. Again,

$$\int_\varepsilon^1 f = \int_\varepsilon^{\frac{1}{n}} (-1)^{n+1}(n+1)dx + \int_{\frac{1}{n}}^{\frac{1}{n-1}} (-1)^n n dx + \ldots + \int_{\frac{1}{2}}^1 2dx \tag{4.36}$$

$$= (-1)^{n+1}(n+1)\left(\frac{1}{n} - \varepsilon\right) + \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{m}$$

$$\implies \left| \int_\varepsilon^1 f - \sum_{m=1}^{n-1} \frac{(-1)^{m-1}}{m} \right| = (n+1)\left(\frac{1}{n} - \varepsilon\right) < (n+1)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n} \tag{4.37}$$

$$\implies \int_0^1 f = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}. \tag{4.38}$$

Theorem 4.15 (*Cauchy-Maclaurin test*)**.** *Let $f : [1, \infty) \to \mathbb{R}$ be a decreasing function with $f > 0$. Then the improper integral $\int_1^\infty f$ converges if and only if the the infinite series $\sum_{n=1}^\infty f(n)$ converges.*

4.3 The Gamma Function

Recall that for all natural $n \geq 1$, $n!$ is defined as $n \cdot (n-1) \cdots 2 \cdot 1$. We wish to extend this notation of the factorial for all $x \in \mathbb{R}$. We know that

$$\int_0^\infty t^n e^{-t} dt = n! \text{ for all natural } n \geq 1. \tag{4.39}$$

Thus, we propose this as the extended definition.

Definition 4.16. For all $x > 0$, the *Gamma function* Γ is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{4.40}$$

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Theorem 4.17. *$\Gamma(x)$ converges for all $x > 0$.*

Proof. Rewrite the function as $\Gamma(x) = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt$. The two integrals are termed as $\gamma_1(x)$ and $\gamma_2(x)$ respectively. We first prove the convergence of $\gamma_1(x)$.

For $0 \leq t \leq 1$, $e^{-t} \leq 1$ holds. Note that $t^{x-1}e^{-t}$ is dominated by t^{x-1}, and the integral $\int_0^1 t^{x-1}dt$ converges to $\frac{1}{x}$ for $x > 0$. Thus, $\gamma_1(x)$ converges.

To show the convergence of $\gamma_2(x)$, we first note that $\lim_{t\to\infty} t^{x+1}e^{-t} = 0$. Thus, there must exist some $M > 0$ such that $t^{x+1}e^{-t} < 1$ holds for all $t \geq M$, which implies that $t^{x-1}e^{-t} < \frac{1}{t^2}$. As $\int_1^\infty \frac{1}{t^2}$ converges, $\gamma_2(x)$ must also converge by the comparison test. ■

Theorem 4.18. *$\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.*

Proof. For all $r_1, r_2 > 0$, $\int_{r_1}^{r_2} t^{x-1}e^{-t}dt$ represents $\Gamma(x)$ as r_2 tends to infinity and r_1 tends to 0. Using the method of intergration by parts, we have

$$\int_{r_1}^{r_2} t^x e^{-t} dt = \left[-t^x e^{-t}\right]_{t=r_1}^{r_2} + x\int_{r_1}^{r_2} t^{x-1}e^{-t}dt = (r_1^x e^{-r_1} - r_2^x e^{-r_2}) + x\int_{r_1}^{r_2} t^{x-1}e^{-t}dt. \tag{4.41}$$

The left term on the right hand side tends to 0 as r_1 tends to 0 and r_2 tends to infinity, and the integral tends to $x\Gamma(x)$. The left hand side tends to $\Gamma(x+1)$. Thus, $\Gamma(x+1) = x\Gamma(x)$. ■

4.4 Cauchy's Principle Value

Definition 4.19. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then *Cauchy's principle value* for $\int_{-\infty}^\infty f$ is defined as $\lim_{R\to\infty} \int_{-R}^R f$, if the limit exists.

If the indefinite integral exists, then Cauchy's principle value exists and equals the indefinite integral. However, the converse is not true.

Example 4.20. Let us look at the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{x}{1+x^2}$ for all real x. Cauchy's principle value is given as

$$\begin{aligned} \int_{-R}^{R} f = \frac{1}{2}\int_{-R}^{R} \frac{d(1+x^2)}{1+x^2} = \frac{1}{2}(\log(1+R^2) - \log(1+R^2)) = 0 \text{ for all } R > 0 \\ \implies \lim_{R\to\infty} \int_{-R}^{R} f = 0. \end{aligned} \tag{4.42}$$

However, the indefinite integral does not converge, and does not exist.

Chapter 5

SEQUENCE OF FUNCTIONS

Recall that a sequence of reals $\{x_n\} \subseteq \mathbb{R}$ converges to a value $x \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \text{ for all } n \geq N. \tag{5.1}$$

The Cauchy criterion is also equivalent. We now define some notation. S will represent a subset of $\mathbb{R}$, $S \subseteq \mathbb{R}$, and $\mathcal{F}(S)$ will represent the set of all functions acting on S, $\mathcal{F}(s) = \{f : S \to \mathbb{R}\}$. Our goal here is to replace the real points x_n by functions $f_n \in \mathcal{F}(S)$. To begin, we must define a 'distance' between functions in $\mathcal{F}(S)$. This can be done in two ways.

Pick a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{F}(S)$. Then for all $x \in S$, note that $\{f_n(x)\} \subseteq \mathbb{R}$. Thus, we bring back the absolute function $|.|$ and introduce it to the sequence $\{f_n(x)\}$ as our 'distance'. This is the first way. The second way, which is harder, is done by introducing an original definition for $|.|$ acting on functions and use the same definition.

5.1 Convergence in Sequence of Functions

Definition 5.1 (The *pointwise convergence of a sequence of functions*)**.** Let $\{f_n\} \subseteq \mathcal{F}(S)$ and $f \in \mathcal{F}(S)$. We say that f_n converges to f pointwise if for each $x \in S$, $f_n(x)$ converges to $f(x)$ $(\star)$. Thus, we will use $|f_n - f| < \varepsilon$ to denote $(\star)$.

Note that $(\star)$ is equivalent to saying that for every $\varepsilon > 0$, there exists a natural N such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. But here, N is both a function of ε and x which poses a problem.

Example 5.2. Let $S = [0, 1]$ and define $f_n : S \to \mathbb{R}$ to be $f_n(x) = x^n$ for all natural n and all $x \in S$. Then for all $x \in [0, 1)$, $f_n(x) = x^n$ converges to 0 as n tends to infinity, and for $x = 1$, $f_n(x) = 1$ converges to 1 as n tends to infinity. If we set $f : S \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases} \tag{5.2}$$

Then f_n will converge to f pointwise. Note that f_n is both a continuous and infinite differentiable function for all natural n, but f is not even a continuous function. We see that there is a need for a better definition of convergence of sequence of functions.

Example 5.3. For $S = [0, \infty)$, define $f_n : S \to \mathbb{R}$ as $f_n(x) = \frac{1}{x+n}$. As $\frac{1}{x+n} \leq \frac{1}{n}$, we find that $f_n(x)$ converges to 0 as n tends to infinity for all $x \in [0, \infty)$. Thus, f_n converges to the zero function f, pointwise. However, for all $x \in [0, \infty)$,

$$\begin{aligned} |f_n(x) - 0| &= \frac{1}{x+n} < \frac{1}{n} \\ \implies |f_n(x) - f(x)| &< \varepsilon \text{ for all } n > \frac{1}{\varepsilon} \end{aligned} \tag{5.3}$$

implies that the choice of N is independent of x. This gives us an idea for the new definition of convergence.

Definition 5.4 (The *uniform convergence of a sequence of functions*)**.** Let $\{f_n\} \subseteq \mathcal{F}(S)$ and $f \in \mathcal{F}(S)$. We say $\{f_n\}$ converges to f uniformly if for every $\varepsilon > 0$, there exists a natural N such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N \text{ and } x \in S. \tag{5.4}$$

Remark 5.5. Note that the condition here is equivalent to saying that $f(x) - \varepsilon \leq f_n(x) \leq f(x) + \varepsilon$ for all $n \geq N$ and $x \in S$.

Example 5.6. It can be shown that $\{x^n\} \subseteq \mathcal{F}([0,1])$ is not uniformly convergent, but the sequence $\{\frac{1}{x+n}\} \subseteq \mathcal{F}([0,\infty))$ uniformly converges.

Remark 5.7. The uniform convergence of f_n to f implies the pointwise convergence of f_n to f.

Despite these definitions, it is unclear what our notion of distance even is.

Definition 5.8. $\mathcal{B}(S)$ denotes the set of all bounded functions from S to $\mathbb{R}$, that is, $\mathcal{B}(S) = \{f : S \to \mathbb{R}; f \text{ is bounded}\}$.

We may note that $\mathcal{B}(S)$ is a vector space, and even an algebra. We will make use of this fact to define the distance.

Definition 5.9. For all $f \in \mathcal{B}(S)$, we define the *norm of a function* f by $\|f\| = \sup_{x\in S} |f(x)|$.

Remark 5.10. We note that $\|.\|$ is a function $\|.\| : \mathcal{B}(S) \to \mathbb{R}_{\geq 0}$. It also satisfies the properties of the distance;

- $\|f\| = 0$ if and only if $f \equiv 0$,
- $\|f + g\| \leq \|f\| + \|g\|$,
- $\|\alpha f\| = |\alpha| \, \|f\|$,
- $\|fg\| \leq \|f\| \, \|g\|$.

The non-negativity and the first three properties are reminiscent of the absolute value function defined on the reals. The last property, however, involves an inequality and the norm function is termed *submultiplicative* due to this.

Definition 5.11. For all $f, g \in \mathcal{B}(S)$, we define the *distance between two functions* f and g as $d(f, g) = \|f - g\|$.

Remark 5.12. Again, we note that d is a function $d : \mathcal{B}(S) \times \mathcal{B}(S) \to R_{\geq 0}$. Other properties include

- $d(f, g) = \sup_{x\in S} |f(x) - g(x)|$,
- $d(f, g) \leq d(f, h) + d(h, g)$ for all $h \in \mathcal{B}(S)$,
- $d(f, g) = 0$ if and only if $f = g$.

We modify our definition of uniform convergence as an equivalent statement.

Remark 5.13. Let $\{f_n\} \subseteq \mathcal{F}(S)$ and $f \in \mathcal{F}(S)$. f_n converges to f uniformly on S if for every $\varepsilon > 0$, there exists a natural N such that

$$\|f_n - f\| < \varepsilon \text{ for all } n \geq N. \tag{5.5}$$

This statement is equivalent to the previous definition.

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The following example depicts a useful trick, where we truncate that 'bad part' of the function to give us a sequence of functions that we intuitively know converges to the desired function.

Example 5.14. Let $S = [-1, 1]$. Define $\{f_n\} \subseteq \mathcal{F}(S)$ by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq \frac{1}{n}, \\ |x| & \text{if } \frac{1}{n} < |x| \leq 1. \end{cases} \tag{5.6}$$

Note that f_n converges to f pointwise, where $f(x) = |x|$. This is not too hard to show. We question the uniform convergence. Computing the norm gives us $\|f_n - f\| = \sup_{x\in[-1,1]} |f_n(x) - f(x)| = \frac{1}{n}$, where the supremum is achieved at $x = 0$. Clearly, this can be minimized. Thus, f_n also uniformly converges to f.

Theorem 5.15 (The *Cauchy criterion for a sequence of functions*)**.** *Let $\{f_n\} \subseteq \mathcal{F}(S)$. Then $\{f_n\}$ converges uniformly if and only if for every $\varepsilon > 0$, there exists a natural $N \in \mathbb{N}$ such that*

$$\|f_n - f_m\| < \varepsilon \text{ for all } n, m \geq N.$$

Proof. Let f_n uniformly converge to f. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f\| < \frac{\varepsilon}{2}$ for all $n \geq N$. Thus, for all $n, m \geq N$,

$$\|f_n - f_m\| = \|(f_n - f) - (f_m - f)\| \leq \|f_n - f\| + \|f_m - f\| < \varepsilon. \tag{5.7}$$

For the converse, we have, for every $\varepsilon > 0$ and some $N \in \mathbb{N}$,

$$\begin{aligned} \|f_n - f_m\| &< \frac{\varepsilon}{2} \text{ for all } n, m \in \mathbb{N} \\ \implies \sup_{x \in S} |f_n(x) - f_m(x)| &< \frac{\varepsilon}{2} \end{aligned} \tag{5.8}$$

The last inequality is true for any evaluation at $x \in S$, so the sequence $\{f_n(x)\}$ is Cauchy for all $x \in S$. This implies that there exists a $f \in \mathcal{F}(S)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$. This implies the pointwise convergence of f_n to f. To show the uniform convergence, we 'blow up' the m. We have

$$\begin{aligned} f_n(x) - \frac{\varepsilon}{2} \leq f_m(x) &\leq f_n(x) + \frac{\varepsilon}{2} \text{ for all } x \in S,\ n, m \geq N \\ \implies f_n(x) - \frac{\varepsilon}{2} \leq f(x) &\leq f_n(x) + \frac{\varepsilon}{2} \\ \implies |f_n(x) - f(x)| &< \frac{\varepsilon}{2} \text{ for all } x \in S,\ n \geq N \\ \implies \|f_n - f\| &< \frac{\varepsilon}{2} \text{ for all } n \geq N. \end{aligned} \tag{5.9}$$

■

Example 5.16. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. For all $n \geq 1$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \{r_1, r_2, \ldots, r_n\}, \\ 1 & \text{if } x \notin \{r_1, r_2, \ldots, r_n\}. \end{cases} \tag{5.10}$$

It can be shown that f_n converges to the Dirichlet function, pointwise. We question the uniform convergence. Note that for a fixed $n \in \mathbb{N}$, note that $f_n(r_{n+1}) = 1$ and $f_{n+1}(r_{n+1}) = 0$. Thus, $\|f_n - f_{n+1}\| = 1$, which cannot be minimized. This violates the Cauchy criterion of uniform convergence.

Theorem 5.17 (The *M-test*)**.** *Let f_n be a sequence of functions converging to f, pointwise. For all $n \in \mathbb{N}$, set $M_n = \sup_{x\in S} |f_n(x) - f(x)|$. Then f_n converges to f uniformly if and only if M_n converges to 0.*

Proof. This statement is trivial if you consider the fact that $M_n = \|f_n - f\|$. ■

Example 5.18. For all $n \in \mathbb{N}$, define $f_n(x) = \frac{nx}{1+n^2x^2}$ on the interval $S = [0, 1]$. Again, it can be shown with ease that f_n converges to $f \equiv 0$, pointwise. We use the M-test to check the uniform convergence. Thus,

$$M_n = \sup_{x\in[0,1]} \left| \frac{nx}{1 + n^2x^2} - 0 \right| = \sup_{x\in[0,1]} |f_n(x)| = \frac{1}{2} \text{ for all } n \in \mathbb{N}. \tag{5.11}$$

Certainly, $\frac{1}{2}$ does not converge to 0. The function does not uniformly converge.

5.2 Properties of Limit Convergence

We note that boundedness is not preserved under the notion of pointwise convergence; one may show by picking the example

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ \frac{1}{x} & \text{if } \frac{1}{n} < x \leq 1. \end{cases} \tag{5.12}$$

Riemann integrability is also not preserved under pointwise convergence due to Example 5.16. The sequence $\{x^n\} \subseteq \mathcal{F}([0,1])$ shows that continuity and differentiability are also not preserved in pointiwse convergence. In fact, due to the above sequence, not even limits are preserved! We infer that pointwise convergence is not a 'good' form of convergence to study. We shift our focus to uniform convergence.

Theorem 5.19. *Let $x_0 \in S$, and let $\{f_n\} \cup \{f\} \subseteq \mathcal{F}(S \backslash \{x_0\})$. Suppose that f_n converges uniformly to f. If $\lim_{x\to x_0} f_n(x)$ exists for all $n \in \mathbb{N}$, then $\lim_{x\to x_0} f(x)$ exists. Moreover,*

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x).$$

Proof. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \frac{\varepsilon}{2}$ for all $n, m \geq N$ on $S\backslash\{x_0\}$. For all $n \in \mathbb{N}$, let $a_n = \lim_{x\to x_0} f_n(x)$. Therefore, $|a_n - a_m| = \lim_{x\to x_0} |f_n(x) - f_m(x)|$; there exists $a \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = a \left(= \lim_{n\to\infty} \lim_{x\to x_0} f_n(x) \right) < \frac{\varepsilon}{2} \text{ for all } n, m \geq N.$$

Now, there exists $n_0 \in \mathbb{N}$ such that $\|f_n - f\| < \frac{\varepsilon}{3}$ for all $n \geq n_0$ on $S\backslash\{x_0\}$. Also, there exists $\tilde{n}_0 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{3}$ for all $n \geq \tilde{n}_0$. Let $\hat{n} = \max\{n_0, \tilde{n}_0\}$. We know that $\lim_{x\to x_0} f_{\hat{n}}(x) = a_{\hat{n}}$ exists. Therefore, there exists $\delta > 0$ such that $|f_{\hat{n}}(x) - a_{\hat{n}}| < \frac{\varepsilon}{3}$ for all $0 < |x - x_0| < \delta$. Thus, combining these three, we have

$$|f(x) - a| \leq |f(x) - f_{\hat{n}}(x)| + |f_{\hat{n}}(x) - a_{\hat{n}}| + |a_{\hat{n}} - a| < \varepsilon \text{ for all } 0 < |x - x_0| < \delta.$$

This implies that $\lim_{x\to x_0} f(x) = a$. ■

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Corollary 5.20. *Let f_n converge uniformly to f on S. Let $x_0 \in S$ and suppose that f_n is continuous at x_0 for all n. Then f is continuous at x_0.*

Proof. Recall that $f_n(x_0) = \lim_{x\to x_0} f_n(x)$ for all n. The above theorem may be applied. ■

Remark 5.21. If $\{f_n\} \subseteq \mathcal{C}[0,1]$ and f_n converges to f, uniformly, then $f \in \mathcal{C}[0,1]$.

Theorem 5.22. *Let $\{f_n\} \subseteq \mathcal{B}(S)$ and suppose that f_n converges uniformly to f. Then $f \in \mathcal{B}(S)$.*

Proof. For every $\varepsilon > 0$, there exists a natural N such that $\|f_n - f\| < \varepsilon$ for all $n \geq N$. Therefore, for all $x \in S$,

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < \varepsilon + \|f_N\| \tag{5.13}$$

$$\implies \|f\| \leq \varepsilon + \|f_N\|. \tag{5.14}$$

f is bounded. ■

Corollary 5.23. *If $\{f_n\} \subseteq \mathcal{B}(S)$ and suppose that f_n uniformly converges to f. Then f_n is also uniformly bounded.*

Proof. Note that $\|f_n\| \leq \|f_n - f\| + \|f\|$. We are done by the above theorem. ■

Theorem 5.24. *Let $\{f_n\} \subseteq \mathcal{R}[a,b]$, and suppose f_n uniformly converges to f. Then $f \in \mathcal{R}[a,b]$. Moreover,*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n.$$

Proof. For every $\varepsilon > 0$, there exists a natural N such that $\|f_N - f\| < \frac{\varepsilon}{b-a}$. This gives us $f(x) - \frac{\varepsilon}{b-a} < f_N(x) < f(x) + \frac{\varepsilon}{b-a}$ for all $x \in [a,b]$. Now, $f_N \in \mathcal{R}[a,b]$ implies that there exists a partition $P \in \mathcal{P}[a,b]$ such that $U(f_N, P) - L(f_N, P) < \varepsilon$. But for all $x \in [a,b]$, we have

$$f(x) < f_N(x) + \frac{\varepsilon}{b-a} \implies U(f, P) < U(f_N, P) + \varepsilon, \tag{5.15}$$

$$f(x) > f_N(x) - \frac{\varepsilon}{b-a} \implies L(f, P) > L(f_N, P) - \varepsilon. \tag{5.16}$$

Subtracting $L(f,P)$ from $U(f,P)$ and making use of the inequalities, we get

$$U(f,P) - L(f,P) < U(f_N,P) - L(f_N,P) + 2\varepsilon < 3\varepsilon. \tag{5.17}$$

Thus, f is Riemann integrable on $[a,b]$. We now show the commutativity of the limit and the integral. Note that since f_n uniformly converges to f, for every $\varepsilon > 0$, there exists a natural n_0 such that $\|f_n - f\| < \frac{\varepsilon}{b-a}$ for all $n \geq n_0$. For all $x \in [a,b]$, we have

$$\left|\int_a^x f - \int_a^x f_n\right| = \left|\int_a^x (f - f_n)\right| \leq \int_a^x |f - f_n| < \frac{\varepsilon}{b-a}|x-a| < \varepsilon. \tag{5.18}$$

■

However, despite so many properties being conserved, the following example talks about the derivative.

Example 5.25. Define $\{f_n\} \subseteq \mathcal{F}(\mathbb{R})$ by $f_n(x) = \frac{x}{1+nx^2}$ for all $x \in \mathbb{R}$. For $x \neq 0$, we have

$$|f_n(x)| = \left|\frac{x}{1+nx^2}\right| \leq \frac{1}{2\sqrt{n}}. \tag{5.19}$$

Thus, f_n converges uniformly to the null function. We note that both f_n and the zero funciton are differentiable. But there is a flaw here. Differentiating $f_n(x)$ gives us $f_n'(x) = \frac{-nx^2+1}{(1+nx^2)^2}$. For $x \neq 0$, this derivative tends to 0 as n tends to infinity. However, when $x = 0$, the derivative is (tends to) 1! We see that f_n' converges to F pointwise where $F(x) = 0$ for $x \neq 0$ and $F(0) = 1$. The limit and derivative do *not* commute in general.

This example suggests that unlike the previous theorems, the theorem regarding differentiability will turn out to be slightly uglier.

Theorem 5.26. *Let $\{f_n\}$ be a sequence of differentiable functions on $[a,b]$. Suppose*

- *there exists $x_0 \in [a,b]$ such that $\{f_n(x_0)\}$ converges, and*
- *$\{f_n'\}$ converges uniformly on $[a,b]$.*

Then $\{f_n\}$ converges uniformly on $[a,b]$ to a differentiable funciton f. Moreover,

$$f'(x) = \lim_{n\to\infty} f_n'(x) \text{ for all } x \in [a,b].$$

Proof. For every $\varepsilon > 0$, there exists a natural N such that both $\|f_n' - f_m'\| < \frac{\varepsilon}{2(b-a)}$ and $|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$ hold true for all $n, m \geq N$. We now apply the mean value theorem to $f_n - f_m$; fix $x \in [a,b]\backslash\{x_0\}$ to get

$$(f_n - f_m)(x) = (f_n - f_m)(x_0) + (x - x_0)(f_n - f_m)'(\zeta) \tag{5.20}$$

from some ζ between x and x_0. This tells us that

$$|(f_n - f_m)(x)| \leq |(f_n - f_m)(x_0)| + |x - x_0|\,|f_n'(\zeta) - f_m'(\zeta)| < \frac{\varepsilon}{2} + |x - x_0|\frac{\varepsilon}{2(b-a)} < \varepsilon. \tag{5.21}$$

This tells us that $\|f_n - f_m\| < \varepsilon$ for all $n, m \geq N$. Thus, f_n converges uniformly to some function f on $[a,b]$. We now claim that f is differentiable and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$. Fix $\tilde{x} \in [a,b]$, and set $F_n(x) = \frac{f_n(x) - f_n(\tilde{x})}{x - \tilde{x}}$ and $F(x) = \frac{f(x) - f(\tilde{x})}{x - \tilde{x}}$ for all $x \neq \tilde{x}$. Note that F_n converges to F, pointwise, and $\lim_{x\to\tilde{x}} F_n(x) = f_n'(\tilde{x})$ for all n.

Again, by the mean value theorem, there exists $\zeta \in (x, \tilde{x})$ such that

$$\begin{aligned}
(f_n - f_m)(x) - (f_n - f_m)(\tilde{x}) &= (f_n'(\zeta) - f_m'(\zeta))(\tilde{x} - x) &(5.22)\\
\implies F_n(x) - F_m(x) &= f_n'(\zeta) - f_m'(\zeta)\\
\implies |F_n(x) - F_m(x)| &< \frac{\varepsilon}{2(b-a)} \text{ for all } n, m \geq N \text{ and } x \neq \tilde{x}\\
\implies \|F_n - F_m\| &< \frac{\varepsilon}{2(b-a)}. &(5.23)
\end{aligned}$$

Thus, $\{F_n\}$ is uniformly convergent on $[a,b]\backslash\{\tilde{x}\}$. This tells us that

$$\lim_{x\to\tilde{x}}\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\lim_{x\to\tilde{x}} F_n(x) \implies \lim_{x\to\tilde{x}} F(x) = f'(\tilde{x}) = \lim_{n\to\infty} f_n'(\tilde{x}). \tag{5.24}$$

■

Example 5.27. Let $f_n(x) = \frac{\sin nx}{n}$ for $x \in [0,1]$. Note that f_n converges to the zero function (f), uniformly. Thus, f' is also the zero function. But $f_n'(x) = \cos nx$ and $\{f_n'(x)\}$ does not converge for all $x \neq 0$.

5.3 Convergence in Series of Functions

Definition 5.28. Let $\{f_n\} \subseteq \mathcal{F}(S)$. Then the *formal sum*

$$f_1 + f_2 + \ldots = \sum_{n=1}^{\infty} f_n$$

is called a series of functions on S. Also, for all $x \in S$, we define

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \ldots$$

Remark 5.29. Given a series $\sum_{n=1}^{\infty} f_n$ on S, we define the n^{th} partial sum as $S_n = \sum_{k=1}^{n} f_k$ for all natural n. Note that $\{S_n\} \subseteq \mathcal{F}(S)$.

Definition 5.30. Let $\{f_n\} \subseteq \mathcal{F}(S)$ with $f \in cF(S)$. We say that $\sum_{n=1}^{\infty} f_n = f$ pointwise if $\sum_{n=1}^{\infty} f_n(x) = f(x)$ for all $x \in S$; if $S_n \to f$ uniformly then we say $\sum_{n=1}^{\infty} f_n = f$ uniformly.

Remark 5.31. The uniform convergence of a series of functions implies its pointwise convergence.

Example 5.32. Let $S = [0,1)$, and look at the series $\sum_{n=0}^{\infty} x^n$. We note that $S_n(x) = \sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} \to \frac{1}{1-x}$ for all $x \in S$. This implies that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ pointwise on $[0,1)$. Since $S_n \in \mathcal{B}(S)$, if $S_n \to \frac{1}{1-x}$ uniformly, this will imply that $\frac{1}{1-x} \in \mathcal{B}(S)$, which is a contradiction. Thus, uniform convergenc cannot occur.

Theorem 5.33 (Cauchy criterion)**.** *Let $\{f_n\} \subseteq \mathcal{F}(S)$. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on S if and only if for every $\varepsilon > 0$, there exists a natural N such that*

$$\left\| \sum_{k=n+1}^{m} f_k \right\| < \varepsilon \textit{ for all } m > n \geq N.$$

Proof. The proof is left as an exercise to the reader. ■

Corollary 5.34. *If $\sum f_n$ converges uniformly, then $\|f_n\| \to 0$.*

Example 5.35. Let $0 < \varepsilon < 1$. Consider the series $\sum_{n=0}^{\infty} x^n$ on $S = [-\varepsilon, \varepsilon]$. We begin by appealing to the Cauchy criterion; for $n > m$ and for any $x \in S$,

$$|S_n(x) - S_m(x)| = \frac{|x^m - x^n|}{|1-x|} \leq \frac{2\,|x^m|}{|1-x|} \leq \frac{2\,|x^m|}{1-|x|} \leq \frac{2\varepsilon^m}{1-\varepsilon} \implies \|S_n - S_m\| < \frac{2\varepsilon^m}{1-\varepsilon}. \tag{5.25}$$

For a given $\varepsilon > 0$, we can find an m to minimize the norm. Thus, $\{S_n\}$ satisfies the Cauchy criterion; $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $x \in [-\varepsilon, \varepsilon]$. We notice that this series is pointwise convergent on $(-1,1)$ and uniformly on any compact subset of $(-1,1)$.

Theorem 5.36 (M-test)**.** *Let $\{f_n\} \subseteq \mathcal{F}(S)$ and suppose $\|f_n\| \leq M_n$ for all natural n. If $\sum M_n < \infty$, then $\sum_{f_n}$ is uniformly convergent.*

Proof. $\left\|\sum_{k=n+1}^{m} f_k\right\| \leq \sum_{k=n+1}^{m} \|f_k\| \leq \sum_{k=n+1}^{m} M_k$. ■

Theorem 5.37. *Let $\sum_{n=1}^{\infty} f_n = f$ uniformly on $S \backslash \{x_0\}$. If $\lim_{x \to x_0} f_n(x_0)$ exists for all n, then $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x)$; the limit and the sum commute.*

Proof. Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ for all $x \in S \backslash \{x_0\}$. We assume that $\lim_{x \to x_0} S_n(x)$. Since $S_n \to \sum_{n=1}^{\infty} f_n = f$, it follows. The commutativity also is seen easily. ■

Corollary 5.38. *Let $\{f_n\}, f$ be from $\mathcal{F}(S)$. Let $\sum f_n = f$ uniformly on S.*

- *If $\{f_n\} \subseteq \mathcal{B}(S)$, then $f \in \mathcal{B}(S)$.*
- *If $\{f_n\} \subseteq \mathcal{C}(S)$, then $f \in \mathcal{C}(S)$.*
- *If $\{f_n\} \subseteq \mathcal{R}[a,b]$, then $f \in \mathcal{R}[a,b]$ and the integral and sum commute.*

Theorem 5.39 (*Dini's result*)**.** *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $\{f_n\} \subseteq \mathcal{C}(S)$. Also let $f_n \to f$ pointwise and $f \in \mathcal{C}(S)$. If $\{f_n\}$ is monotonic in n, then $f_n \to f$ uniformly.*

The proof of this theorem is out of the scope of this course.
March 10th.

Theorem 5.40 (Abel's test)**.** *Let $\sum f_n$ be uniformly convergent on S and suppose $\{g_n\}$ is a uniformly bounded montone sequence of functions on S. Then $\sum f_n g_n$ is uniformly convergent on S.*

Proof. Let $\sup \|g_n\| = M < \infty$. Set S_n to be the partial sum $\sum_{j=1}^n f_n \in \mathcal{F}(S)$ for all natural n. We appeal to the Cauchy criterion;

$$\sum_{j=n+1}^{m} f_j(x)g_j(x) = (S_m(x) - S_n(x))g_{n+1}(x) + \sum_{j=n+1}^{m} (S_m(x) - S_j(x))(g_{j+1}(x) - g_j(x)) \text{ for all } x \in S. \tag{5.26}$$

For $\varepsilon > 0$, there exists a natural N such that $\|S_m - S_n\| < \varepsilon$ for all $m > n \geq N$. Therefore, for all $m > n \geq N$,

$$\left| \sum_{j=n+1}^{m} f_j(x)g_j(x) \right| \leq \varepsilon \cdot M + \varepsilon \sum_{j=n+1}^{m} |g_{j+1}(x) - g_j(x)| \text{ for all } x \in S. \tag{5.27}$$

The sum on the right equals $|g_{n+1}(x) - g_m(x)|$ as g_n either increases or decreases in n. This term is bounded by $2M$ giving us

$$\left| \sum_{j=n+1}^{m} f_j(x)g_j(x) \right| \leq \varepsilon M + 2\varepsilon M = 3\varepsilon M. \tag{5.28}$$

Thus, the sum if uniformly convergent. ■

Example 5.41. We consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$ on $\mathbb{R}_{\geq 0}$. The series of constant functions $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is uniformly convergent. Also, $n > m \implies e^{-nx} \leq e^{-mx}$ for all $x \in [0, \infty)$. Thus, by Abel's test, $\sum \frac{(-1)^n}{n} e^{-nx}$ is uniformly convergent on this interval.

Example 5.42. Look at the series $\sum \frac{(-1)^n}{n} |x|^n$ is uniformly convergent on the interval $[-1, 1]$. We can utilise the previous example to show that this series is uniformly convergent.

Theorem 5.43 (Dirichlet test)**.** *Let $\{f_n\}, \{g_n\} \subseteq \mathcal{F}(S)$ Suppose the following—*

- *The sequence of partial sums $\{S_n\}$ $(\sum_{j=1}^n f_j)$ is uniformly bounded, and*
- *$g_n \geq 0$ for all n and g_n uniformly converges to 0.*

Then $\sum_{n=1}^{\infty} f_n g_n$ is uniformly convergent on S.

Proof. There exists $M > 0$ such that $\|S_n\| \leq M$ for all n. Thus, $\left\| \sum_{j=n+1}^m f_j \right\| = \|S_m - S_n\| \leq 2M$. This implies that

$$-2M \leq \sum_{j=n+1}^{m} f_j(x) \leq 2M \text{ for all } x \in S. \tag{5.29}$$

Now, for all $x \in S$, $g_{n+1}(x) \geq g_{n+2}(x) \geq \ldots \geq g_m(x) \geq 0$. By Abel's lemma,

$$-2Mg_{n+1}(x) \leq \sum_{j=n+1}^{m} f_j(x)g_j(x) \leq 2Mg_{n+1}(x) \text{ for all } x \in S$$

$$\implies \left\| \sum_{j=n+1}^{m} f_j g_j \right\| \leq 2M \|g_{n+1}\|. \tag{5.30}$$

As $\|g_n\| \to 0$, it follows that the sum is uniformly bounded on S. ■

Example 5.44. Consider the series of functions $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$ defined on all $x \neq 2m\pi$. Label this set as $S = \mathbb{R} \backslash \{2m\pi : m \in \mathbb{Z}\}$. Set $f_n(x) = \cos nx$ for all n. Set $S_n(x) = \sum_{j=1}^{n} f_j(x)$ for all $x \in S$ and $n \in N$. For a fixed $x \in S$,

$$|S_n(x)| = \left| \frac{\sin(\frac{n+1}{2}x) - \sin\frac{x}{2}}{2\sin\frac{x}{2}} \right| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}. \tag{5.31}$$

Also, $\frac{1}{n}$ is decreasing and converges to 0. By the Dirichlet test for real numbers, $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$ converges pointwise on S. For $0 < \varepsilon < \pi$, we have $\|S_n\| \leq \frac{1}{\sin\frac{\varepsilon}{2}}$ on $[\varepsilon, 2\pi - \varepsilon]$, and extend it to S periodically. Hence, $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$ converges uniformly on $[\varepsilon, 2\pi - \varepsilon]$, extended periodically.

Example 5.45. Let $0 < \varepsilon < 1$. We know that $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$ converges uniformly on $[-\varepsilon, \varepsilon]$. This tells us that for all $x \in [-\varepsilon, \varepsilon]$,

$$\int_0^x \frac{1}{1-t} dt = \int_0^x \sum_{n=0}^{\infty} t^n dt = \sum_{n=0}^{\infty} \int_0^x t^n dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\implies \log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for all } x \in (-1, 1). \tag{5.32}$$

The sum is uniformly convergent on compact subsets of $(-1, 1)$.

Example 5.46. Again, we know that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for all $x \in (-1, 1)$. We apply the same trick as in the previous example;

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for all } x \in (-1, 1). \tag{5.33}$$

The sum is uniformly convergent on compact subsets of $(-1, 1)$. If we set $x = \frac{1}{2}, \frac{1}{3}$, we get

$$\sum \frac{(-1)^n}{2n+1} \left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right) = \arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\frac{\frac{1}{2}+\frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \frac{\pi}{4}. \tag{5.34}$$

Example 5.47. Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos\frac{x}{n}$. Pick $x_0 = 0$; this gives $\sum \frac{(-1)^n}{n} < \infty$. Also, $f_n(x) = \frac{(-1)^n}{n} \cos\frac{x}{n} \implies f_n'(x) = \frac{(-1)^{n+1}}{n^2} \sin\frac{x}{n}$. Then $\sum f_n' = \sum \frac{(-1)^{n+1}}{n^2} \sin\frac{x}{n}$ is uniformly convergent on $\mathbb{R}$. Therefore, the original sum is uniformly convergent on $\mathbb{R}$ *and* is a differentiable function on $\mathbb{R}$. Also, the sum and differentiating operator commute.

5.4 Power Series

Let $c \in \mathbb{R}$ and let $\{a_n\}_{n\geq 0} \subseteq \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} a_n (x-c)^n \tag{5.35}$$

is called a *power series* around the centre c with coefficients $\{a_n\}_{n\geq 0}$.

Remark 5.48. The following are worth noting;

- The sum is in fact a series of functions, with $f_n(x) = a_n(x-c)^n$.
- The sum refers to a formal sum.
- Polynomials from $\mathbb{R}[x]$ admit a power series.
- Let $S = \{x \in \mathbb{R} : \sum a_n(x-c)^n \text{ converges}\}$. Since $c \in S$, S is non-empty. Therefore, there exists $f \in \mathcal{F}(S)$ such that $f(x) = \sum a_n(x-c)^n$ on S.

Theorem 5.49. *If a power series converges at $x_0 \in \mathbb{R} \setminus \{c\}$, then the power series converges absolutely for all $x \in \mathbb{R}$ such that $|x - c| < |x_0 - c|$. If a power series diverges at $x_1 \in \mathbb{R}$, then the power series diverges for all $x \in \mathbb{R}$ such that $|x - c| > |x_1 - c|$.*

Proof. Let $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converge. Then, there exists a natural N such that $a_n(x_0-c)^n < \frac{1}{2}$ for all $n \geq N$. Let us assume that $|x-c| < |x_0-c|$. Therefore, we have

$$|a_n(x-c)^n| = |a_n(x_0-c)^n| \left|\frac{x-c}{x_0-c}\right|^n = |a_n(x_0-c)^n|\, r^n \leq \frac{1}{2} r^n \text{ for all } n \geq N \tag{5.36}$$

where r is defiend as $r = \left|\frac{x-c}{x_0-c}\right|$. From the assumption, $0 < r < 1$. Thus, by the M-test, the power series $\sum_{n=0}^{\infty} |a_n(x-c)^n| < \infty$. The divergence is shown in a similar manner. ■

Theorem 5.50 (*Cauchy-Hadamard theorem*)**.** *Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ and set $\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$. Then the power series is absolutely convergent for all x such that $|x-c| < R$ and is divergent for all $|x-c| > R$. Also, the power series is uniformly convergent on any closed subset of $(c-R, c+R)$.*

As a small remark, when the limit superior evaluates to ∞, we take R to be 0. If the limit superior evaluates to 0, we take R to be ∞.

Proof. Set $\alpha_n = |a_n(x-c)^n|$. This shows that $|\alpha_n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}} |x-c|$. We are done by the root test. Now pick $\varepsilon > 0$ such that $c < c+R-\varepsilon < c+R$, and let $|x-c| \leq R-\varepsilon$. Then we know $\sum_{n=0}^{\infty} |a_n(R-\varepsilon)|^n < \infty$. Thus there exists a natural N such that $|a_n(x-c)^n| < \frac{1}{2}$ for all $n \geq N$. But for all $|x-c| \leq R-\varepsilon$, $|a_n(x-c)^n| \leq |a_n(R-\varepsilon)^n|$. By the M-test, the power series converges uniformly on $|x-c| \leq R-\varepsilon$. ■

Definition 5.51. In the above result, R is defined as the *radius of convergence*. The set $(c-R, c+R)$ is termed the *region of convergence*.

Let $\alpha_n \geq 0$. If $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n}$ exists, then $\limsup \alpha_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n}$. This proves to be a helpful result.

Corollary 5.52. *If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists, then $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.*

Example 5.53. Let $p \in \mathbb{R}[x]$, and fix $c \in \mathbb{R}$. Then the radius of convergence is infinity.

Example 5.54. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. If we set $a_n = \frac{1}{n!} \implies \frac{a_{n+1}}{a_n} = \frac{1}{n+1}$. This tends to 0, so the radius of convergence R is infinity. This power series absolutely converges everywhere and uniformly converges on compact subsets of $\mathbb{R}$. We define this as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{5.37}$$

Example 5.55. Consider the power series $\frac{1}{3} - x + \frac{x^2}{3^2} + \frac{x^4}{3^4} - x^5 + \ldots$ In this case, the coefficients are

$$a_n = \begin{cases} \frac{1}{3} & \text{if } n = 0, \\ -1 & \text{if } n \text{ is odd}, \\ \frac{1}{3^n} & \text{if } n \text{ is even and } n > 0. \end{cases} \tag{5.38}$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist here, so we turn to the limit superior. In this case, $\limsup |a_n|^{\frac{1}{n}} = 1$. Thus, the radius of convergence of the power series is 1, with $(-1, 1)$ as the region of convergence.

Example 5.56. We look at $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} x^n$. The limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ evaluates to 1, the $(-1, 1)$ is the interval of convergence. We note that the power series converges for $x = 1$ but diverges for $x = -1$.

Henceforth, without loss of generality, we shall take the centre c to be 0. The interval of convergence then becomes $(-R, R)$.

Theorem 5.57. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. Then, for all $x \in (-R, R)$,*

$$\int_0^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Proof. The proof is left as an exercise to the reader. ■

Definition 5.58. Given a power series $\sum_{n=0}^{\infty} a_n x^n$, call $\sum_{n=1}^{\infty} n a_n x^{n-1}$ its *derived power series.*

It can be shown that the radius of convergence of the power series, R, exactly equals the radius of convergence of the dervied power series, R_d.

Theorem 5.59. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. Then f is differentiable on $(-R, R)$, and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for all $x \in (-R, R)$.*

Proof. The proof is left as an exercise to the reader. ■

Corollary 5.60. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. Then f is infinitely differentiable on $(-R, R)$. Moreover, $a_n = \frac{f^{(n)}(0)}{n!}$ for all $n \geq 0$.*

Corollary 5.61. *$\sum_{n=0}^{\infty} a_n x^n = 0$ on $(-R, R)$ if and only if $a_n = 0$ for all n.*

Definition 5.62. A function $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is said to be a *real-analytic function* if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in (-\varepsilon, \varepsilon)$.

By this definition, a power series is real-analytic. However, the converse may not be true. Simply consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{5.39}$$

Here, $f^{(n)}(0) = 0$ for all n implies that f is not analytic.

5.4.1 Analytic Functions

March 21st.

A better definition is this; a function $f : (a, b) \to \mathbb{R}$ (or Θ, an open set as the domain) is analytic at point $c \in (a, b)$ if $f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$ for all $x \in (c - \delta, c + \delta)$ for some $\delta > 0$. A function $f : \Theta \to \mathbb{R}$ is said to be analytic if it is analytic at c for all $c \in \Theta$. f being analytic implies that f is smooth.

Example 5.63. The function $f(x) = \frac{1}{1-x}$ is analytic at $\mathbb{R} \setminus \{1\}$. Let $c \neq 1$. Then

$$f(x) = \frac{1}{1-x} = \frac{1}{1 - \frac{x-c}{1-c}} \cdot \frac{1}{1-c} = \sum_{n=0}^{\infty} \frac{1}{(1-c)^{n+1}} (x-c)^n \text{ if } |x - c| < |1 - c|, \tag{5.40}$$

that is, f admits a power series on $(c - |1 - c|, c + |1 - c|)$.

Definition 5.64. Given $f : \Theta \to \mathbb{R}$, we define $\mathcal{Z}(f) = \{x \in \Theta : f(x) = 0\}$.

If $p \in \mathbb{R}[x]$, we know that $\#\mathcal{Z}(p) < \infty$. We question whether $\mathcal{Z}$(power series) is finite, countably infinite, or uncountably.

Remark 5.65. If $f : (a, b) \to \mathbb{R}$ is analytic, then it is continuous on (a, b). This continuity forces $f^{-1}(\{0\}) = \mathcal{Z}(f)$ to be closed.

Theorem 5.66. *Let $f : (a, b) \to \mathbb{R}$ be analytic and non-zero. Then $\mathcal{Z}(f)$ is a set of isolated points of (a, b), which is equivalent to saying $\mathcal{Z}(f)$ cannot have limit points in (a, b).*

Proof. Let $\mathcal{O} = \{x \in (a, b) : f^{(n)}(x) = 0 \text{ for all } n = 0, 1, \ldots\}$. We claim that $\mathcal{O}$ is both open and closed in (a, b). Pick $c \in \mathcal{O}$; then $f^{(n)}(c) = 0$ for all n and $f \equiv 0$ in a neighbourhood of c. This implies that c is an interior point of $\mathcal{O}$ and that $\mathcal{O}$ is open. We now show that the complement, $(a, b) \setminus \mathcal{O}$ is also open. Let $c \in (a, b) \setminus \mathcal{O}$. There then exists a natural n such that $f^{(n)}(c) \neq 0$. But $f^{(n)} : (a, b) \to \mathbb{R}$ is an analytic function; it is a continuous function so $f^n(c)$ is non-zero in a neighbourhood of c. This shows that $(a, b) \setminus \mathcal{O}$ is open. But f is non-zero, so we must have $\mathcal{O} = \emptyset$.

If $c \in \mathcal{Z}(f)$, then $f(x) = (x - c)^n \times (a_n + a_{n+1}(x - c) + \ldots)$ in a neigbourhood of c, with $a_n \neq 0$. This shows that $f(x) = (x - c)^n g(x)$ where g is analytic and $g(c) \neq 0$. g does not have a zero in a neighbourhood of c. This implies that f does not vanish in this neighbourhood except at $x = c$. This shows that c is an isolated point of $\mathcal{Z}(f)$. ■

Theorem 5.67 (*Abel's theorem*)**.** *Let the series $\sum_{n=0}^{\infty} a_n$ is convergent. Then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, 1]$.*

Proof. Let $f_n(x) = a_n$ for all $n \in N$ and $x \in [0,1]$. Certainly, $\sum f_n(x)$ is uniformly convergent on $[0,1]$. Now let $g_n(x) = x^n$ for all $x \in [0,1]$. This sequence of functions $\{g_n\}$ is decreasing. Therefore, by Abel's test, $\sum f_n(x)g_n(x)$ is uniformly convergent on $[0,1]$. ■

Corollary 5.68. *If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\lim_{x\to 1^-}(\sum a_n x^n) = \sum_{n=0}^{\infty} a_n$.

Proof. Note that $\sum_{n=0}^{\infty} a_n x^n = \lim_{n\to\infty} S_n(x)$. This is uniform on $[0,1]$. Then,

$$\lim_{x\to 1^-}(\sum a_n x^n) = \lim_{x\to 1^-}\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\lim_{x\to 1^-} S_n(x) = \sum a_n. \tag{5.41}$$

■

Example 5.69. Recall that $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$. From here, $\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ for all $x \in (-1,1)$. By Abel's theorem, $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Chapter 6

MISCELLANEOUS RESULTS

6.1 DOUBLE SEQUENCES & SERIES

A double sequence is simply $f : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ with $f(m,n) = a_{m,n}$. The sequence $\{a_{m,n}\}$ to termed a double sequence. We ask if the m- and n- limits commute; whether $\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}$.

Example 6.1. Choose the double sequence defined as $a_{m,n} = \frac{n}{m+n}$. Here, $\lim_{n\to\infty} a_{m,n} = 1$ and $\lim_{m\to\infty} a_{m,n} = 0$. The commuting of the limits is not true even at the basic level.

6.1.1 Double Sequences

Definition 6.2. We say that a double sequence $\{a_{m,n}\}$ converges to a real $a \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a natural N such that

$$|a_{m,n} - a| < \varepsilon \text{ for all } m, n \geq N.$$

Example 6.3. We have $a_{m,n} = (-1)^{m+n}\left(\frac{1}{m} + \frac{1}{n}\right)$. Then $a_{m,n} \to 0$ as $m, n \to \infty$. But for all m, $\{a_{m,n}\}$ does not converge and for all n, $\{a_{m,n}\}$ doest not converge.

Theorem 6.4. *Let the double sequence $a_{m,n} \to a$ as $m, n \to \infty$. If $\lim_{n\to\infty} a_{m,n}$ exists for all m, then $\lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} = 0$.*

Proof. Define $\alpha_m = \lim_{n\to\infty} a_{m,n}$. We want to show that $\alpha_m \to a$. For every $\varepsilon > 0$, there exists a natural N such that $|a_{m,n} - a| < \frac{\varepsilon}{2}$ for all $m, n \geq N$. Also, for all m, there exists a natural $N(m)$ such that $|a_{m,n} - \alpha_m| < \frac{\varepsilon}{2}$ for all $n \geq N(m)$. For all $m \geq N$, choose $n \geq N(m)$ so that

$$|\alpha_m - a| \leq |\alpha_m - a_{m,n}| + |a_{m,n} - a| < \varepsilon. \tag{6.1}$$

■

Theorem 6.5 (*Cauchy Stolz criterion*)**.** *The double sequence $\{a_{m,n}\}$ converges if and only if for every $\varepsilon > 0$, there exists a natural N such that*

$$|a_{m,n} - a_{p,q}| < \varepsilon \textit{ for all } m, n \geq p, q \geq N.$$

Proof. We only prove the converse. Let $\alpha_n = a_{n,n}$. for all n. The condition implies that $\{\alpha_n\}$ is Cauchy, that is, $|\alpha_n - \alpha_p| < \frac{\varepsilon}{2}$ for all $n, p \geq N$. Thus, $\alpha_m \to a$ and there exists a natural N_0 such that $|\alpha_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N_0$. Therefore, for all $m, n \geq \max\{N_0, N\}$, we have

$$|a_{m,n} - a| \leq |a_{m,n} - a_{n,n}| + |a_{n,n} - a| < \varepsilon.$$

■

6.1.2 Double Series

March 24th.

Partial sums for double series are defined as $S_{m,n} = \sum_{1\leq i\leq m, 1\leq j\leq n} a_{i,j}$, the $(m,n)^{\text{th}}$ partial sum.

Definition 6.6. If $\lim_{m,n\to\infty} S_{m,n} = a$ exists, we say that the *double series* converges to $\sum_{m,n=1}^{\infty} a_{m,n} = a$.

It can be shown that if $\sum a_{m,n}$ converges, then $\lim_{m,n\to\infty} a_{m,n} = 0$. If the terms are always non-negative, that is, $a_{m,n} \geq 0$, then $\sum a_{m,n}$ converges if and only if the double sequence $\{S_{m,n}\}_{m,n=0}^{\infty}$ is bounded.

Example 6.7. Let $\alpha, \beta > 1$. Then the infinite double series $\sum_{m,n=1}^{\infty} \frac{1}{\alpha^m \beta^n}$ is convergent. The partial sums here are

$$S_{m,n} = \left(\sum_{i=1}^{m} \frac{1}{\alpha^i}\right)\left(\sum_{j=1}^{n} \frac{1}{\beta^j}\right). \tag{6.2}$$

As $\alpha, \beta > 1$, both the series $\sum_{n=1}^{\infty} \frac{1}{\alpha_n}$ and $\sum_{n=1}^{\infty} \frac{1}{\beta^n}$ converges. The partial sums are bounded, and the double series is convergent.

Theorem 6.8 (*Fubini-Torelli theorem*)**.** *A double series $\sum_{m,n=1}^{\infty} a_{m,n}$ is absolutely convergent if and only if one, and hence all, of the following holds—*

- $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{m,n}| < \infty$,
- $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{m,n}| < \infty$.

Proof. We look at the partial sums $S_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$, and also define $r_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{i,j}|$. Let $\sum_{m,n=1}^{\infty} |a_{m,n}| < \infty$. For every $\varepsilon > 0$, there exists a natural N such that $|r_{m,n} - r_{p,q}| < \varepsilon$ for all $m \geq p \geq N$ and $n \geq q \geq N$. This implies that

$$|S_{m,n} - S_{p,q}| = \left|\sum_{\substack{p+1\leq i\leq m,\\ q+1\leq j\leq n}} a_{i,j}\right| \leq \sum_{\substack{p+1\leq i\leq m,\\ q+1\leq j\leq n}} |a_{ij}| = |r_{m,n} - r_{p,q}| < \varepsilon. \tag{6.3}$$

The sequence $\{S_{m,n}\}$ satisfies the Cauchy criterion. Thus, $a = \sum_{m,n=1}^{\infty} a_{m,n}$ exists. Set $r = \sup_{m,n} r_{m,n} < \infty$. Now, for all $i \in \mathbb{N}$ and $n \in \mathbb{N}$, $\sum_{j=1}^{n} |a_{i,j}| \leq r_{i,n} \leq r$. This implies that

$$\sum_{j=1}^{\infty} |a_{i,j}| < \infty \text{ for all } i \in \mathbb{N} \implies \sum_{j=1}^{\infty} a_{i,j} \qquad\qquad < \infty \text{ for all } i \in \mathbb{N}. \tag{6.4}$$

For all $m \in \mathbb{N}$, set $\alpha_m = \sum_{i=1}^{m}\sum_{j=1}^{\infty} a_{ij}$. As $\sum_{m,n=1}^{\infty} a_{m,n} = a$, for every $\varepsilon > 0$, there exists a natural N such that $|S_{m,n} - a| < \varepsilon$ for all $m, n \geq N$, that is,

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} - a\right| < \varepsilon \text{ for all } m, n \geq N \implies |\alpha_m - a| \leq \varepsilon \text{ for all } m \geq N \tag{6.5}$$

which shows that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = a$. Similarly, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j} = a$. Therefore, $\sum_{m,n=1}^{\infty} a_{m,n} = \sum_m \sum_n a_{m,n} = \sum_n \sum_m a_{m,n}$.

Observe that $r = \sup_{m,n} r_{m,n} = \sum_{m,n} |a_{m,n}|$. For all m fixed, $r_{m,n}$ increases as n increases.

$$\implies \lim_{m\to\infty}\lim_{n\to\infty} r_{m,n} \leq r \implies \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{m,n}| < \infty. \tag{6.6}$$

Similarly, $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{m,n}|$ is also convergent.

The converse is left as an exercise to the reader. ■

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have a radius of convergence $R > 0$. Then for $a \in (-R, R)$, $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$ for all $|x - a| < R - |a|$. This result is also left as an exercise to the reader.

6.2 Weierstrass Approximation Theorem

We ask the following— given $f \in \mathcal{C}[0,1]$, does there exist a polynomial $p \in \mathbb{R}[x]$ such that f is *close* to p? Alternatively, given $f \in \mathcal{C}[0,1]$, does there exist $\{p_n\} \subseteq \mathbb{R}[x]$ such that $p_n \to f$; for every $\varepsilon > 0$, there exists $p \in \mathbb{R}[x]$ such that $\|f-p\| < \varepsilon$. In fact, this is always true for a continuous function.

The idea to show this is to introduce bump functions; for all $n \in \mathbb{N}$, we know that $\sum_{k=0}^{n} x^k(1-x)^{n-k} = 1$ for all $x \in [0,1]$

Definition 6.9. We adopt the notation $b_k^n(x) = \binom{n}{k} x^k(1-x)^{n-k}$ on $[0,1]$ for $0 \leq k \leq n$. These are termed the *Bernstein polynomial*s.

Remark 6.10. The following results can be shown:

- b_k^n has a unique maxima at $x = \frac{k}{n}$.
- $\sum_{k=0}^{n} b_k^n = 1$.
- $\deg b_k^n = n$.
- $b_k^n \geq 0$.

Definition 6.11. Let $f : [0,1] \to \mathbb{R}$ be a function. Define the *Bernstein polynomial of degree n* as

$$(B_n(f))(x) = \sum_{k=0}^{n} f(\frac{k}{n}) b_k^n(x) \text{ for all } x \in [0,1].$$

Remark 6.12. The following results may also be shown:

- B_n may be interpreted as $B_n : \mathcal{C}[0,1] \to \{\text{polynomials with degree} \leq n\}$.
- B_n is linear in continuous functions.
- If $f \geq g$, then $B_n(f) \geq B_n(g)$.
- $|f| \leq g \implies |B_n(f)| \leq B_n(g)$.
- $B_n(1) = 1$.
- $B_n(x) = x$.
- $B_n(x^2) = x^2 + \frac{x-x^2}{n}$.

Theorem 6.13 (*Weierstrass approximation theorem*)**.** *Given $f \in \mathcal{C}[0,1]$ and $\varepsilon > 0$, there exists $p \in \mathbb{R}[x]$ such that $\|f-p\| < \varepsilon$.*

Proof. Set M to be $\sup_{x\in[0,1]} |f(x)|$. Also, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ for all $|x-y| < \delta$. Pick and fix $a \in [0,1]$. For all $x \in [0,1]$, we claim that

$$|f(x) - f(a)| < \frac{\varepsilon}{2} + \frac{2M}{\delta^2}(x-a)^2. \tag{6.7}$$

Clearly, this is true for $|x-a| < \delta$; we let $|x-a| \geq \delta$. Then $|f(x)-f(a)| \leq 2M \leq 2M\frac{(x-a)^2}{\delta^2} + \frac{\varepsilon}{2}$. Thus the inequality is satisfied by $x \in [0,1]$. We apply the Bernstein polynomial B_n on both sides to get

$$\begin{aligned} |(B_n(f))(x) - f(a)| &\leq B_n\left(\frac{\varepsilon}{2} + \frac{2M}{\delta^2}(x-a)^2\right) = \frac{\varepsilon}{2} + \frac{2M}{\delta^2} B_n(x^2 - 2ax + a^2) \\ &= \frac{\varepsilon}{2} + \frac{2M}{\delta^2}(x^2 + \frac{x-x^2}{n} - 2ax + a^2). \end{aligned} \tag{6.8}$$

Setting $x = a$ gives us

$$\begin{aligned} |(B_n(f))(a) - f(a)| &\leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2}\left(\frac{a-a^2}{n}\right) \leq \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n} \\ \implies |(B_n(f))(x) - f(x)| &\leq \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}. \end{aligned} \tag{6.9}$$

Choose $N \geq \frac{M}{\delta^2 \varepsilon}$. So for all $n \geq N$, the term $|(B_n(f))(x) - f(x)|$ minmizes to less than ε for all $x \in [0,1]$. Thus,

$$\|B_n(f) - f\| < \varepsilon \text{ for all } n \geq N. \tag{6.10}$$

■

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