

LINEAR ALGEBRA II

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List of Symbols

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Chapter 1

PERMUTATION GROUPS

January 3rd.

Let S_n denote the set of all bijections (permutations) on the set $\{1, 2, \dots, n\}$. If $\sigma, \tau \in S_n$, let us define $\sigma\tau$ to be the bijection defined as

$$(\sigma\tau)(i) = \sigma(\tau(i)) \forall 1 \leq i \leq n. \quad (1.1)$$

This gives us a binary operation on S_n which is associative, and S_n will then contain the identity permutation 1 such that $\sigma 1 = 1\sigma = \sigma$ for all $\sigma \in S_n$. For every such σ , we can also find a $\sigma^{-1} \in S_n$ such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1$. The set S_n equipped with this binary operation, thus, forms a group. In this case, we call S_n as the *symmetric group* of degree n . We now define a cycle in regards to permutations.

Definition 1.1. A *cycle* is a string of positive integers, say (i_1, i_2, \dots, i_k) , which represents the permutation $\sigma \in S_n$ (with $k \leq n$) such that $\sigma(i_j) = i_{j+1}$ for all $1 \leq j \leq k-1$, and $\sigma(i_k) = i_1$, and fixes all other integers.

We also note that S_3 is the smallest Abelian group possible, upto isomorphism. S_3 is one of the only two groups of order 6, and can be written as

$$S_3 = \{1, \sigma = (1, 2, 3), \sigma^2 = (1, 3, 2), \tau = (1, 2), \sigma\tau = (1, 3), \tau\sigma = (2, 3)\}. \quad (1.2)$$

Some other observations arise. We find that $\sigma^3 = \tau^2 = 1$, and that $\tau\sigma = \sigma^2\tau$. We notice another fact via this σ ;

Remark 1.2. A k -cycle $\sigma = (i_1, i_2, \dots, i_k)$ is of order k , that is, $\sigma^k = 1$.

Definition 1.3. Two cycles in S_n are called disjoint if they have no integer in common.

We note that if σ and τ are two disjoint cycles in S_n then σ and τ commute, that is, $\sigma\tau = \tau\sigma$.

Proposition 1.4. Every σ in S_n can be written uniquely as a product of disjoint cycles.

Every cycle can be written as a product of 2-cycles. 2-cycles are called *transpositions*. This can easily be seen as

$$(a_1, a_2, \dots, a_n) = (a_1, a_n)(a_1, a_{n-1}) \cdots (a_1, a_3)(a_1, a_2). \quad (1.3)$$

1.1 Even and Odd Permutations

Let x_1, x_2, \dots, x_n be indeterminates, and let

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j). \quad (1.4)$$

Let $\sigma \in S_n$, and define

$$\sigma(\Delta) = \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)}). \quad (1.5)$$

We find that $\sigma(\Delta) = \pm\Delta$. Based on this, we classify permutations as odd or even.

Definition 1.5. A permutation σ is said to be an *even permutation* if $\sigma(\Delta) = \Delta$, and is said to be an *odd permutation* if $\sigma(\Delta) = -\Delta$. The sign of a permutation σ , denoted by $\epsilon(\sigma)$, is $+1$ if σ is even, and is -1 if σ is odd. So, $\sigma(\Delta) = \epsilon(\sigma)\Delta$.

Proposition 1.6. The map $\epsilon : S_n \rightarrow \{-1, +1\}$, where $\epsilon(\sigma)$ is the sign of σ , is a homomorphism, that is, $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$ for all $\sigma, \tau \in S_n$.

Proof. Start with $\tau(\Delta)$;

$$\tau(\Delta) = \prod_{1 \leq i < j \leq n} (x_{\tau(i)} - x_{\tau(j)}). \quad (1.6)$$

Let there be k factors of this polynomial where $\tau(i) > \tau(j)$ with $i < j$. We find that $\tau(\Delta) = (-1)^k \Delta$, and so, $\epsilon(\tau) = (-1)^k$. Now, $\sigma\tau(\Delta)$ has exactly k factors of the form $x_{\sigma(j)} - x_{\sigma(i)}$, with $j > i$. Bringing out a factor $(-1)^k$, we find that $\sigma\tau(\Delta)$ has all factors of the form $x_{\sigma(i)} - x_{\sigma(j)}$, with $j > i$. Thus,

$$\epsilon(\sigma\tau)\Delta = \sigma\tau(\Delta) = (-1)^k \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)}) = (-1)^k \sigma(\Delta) = (-1)^k \epsilon(\sigma)\Delta = \epsilon(\tau)\epsilon(\sigma)\Delta. \quad (1.7)$$

Cancelling out the Δ , we find $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$. ■

ϵ is a homomorphism to an Abelian group, so $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau) = \epsilon(\tau)\epsilon(\sigma)$.

Proposition 1.7. If $\lambda = (i, j)$ is a transposition, then $\epsilon(\lambda) = -1$.

Proof. If $\lambda = (1, 2) \in S_n$, it is easy to show that

$$\lambda(\Delta) = (x_1 - x_2) \cdots (x_1 - x_n)(x_2 - x_3) \cdots (x_2 - x_n) \cdots = (-1)(\Delta). \quad (1.8)$$

Now, if $\sigma = (i, j)$, with $(i, j) \neq (1, 2)$, then $(i, j) = \lambda(1, 2)\lambda$ where λ interchanges 1 and i , and interchanges 2 and j . Using that fact that ϵ is a homomorphism, $\epsilon(\sigma) = -1$. ■

A cycle σ of length k is an even permutation if and only if k is odd. This is because it can be decomposed into $k - 1$ transpositions, and we would then have $\epsilon(\sigma) = (-1)^{k-1} = 1$ (using the fact that ϵ is a homomorphism). Some more corollaries of the previous proposition include the fact that ϵ is a surjective map, and that $\epsilon(\sigma^{-1}) = \epsilon(\sigma)$.

If, for $\sigma \in S_n$, σ can be decomposed as $\sigma_1\sigma_2 \cdots \sigma_k$, where σ_i is a m_i -cycle, then $\epsilon(\sigma_i) = (-1)^{m_i-1}$, and $\epsilon(\sigma) = (-1)^{(\sum m_i) - k}$.

Proposition 1.8. σ is an odd permutation if and only if the number of cycles of even length in its cycle decomposition is odd.

1.2 The Determinant

Definition 1.9. If $A = (a_{ij})$ is a square matrix of order n , then the *determinant* of A is defined as

$$\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \quad (1.9)$$

Using this definition of the determinant of a square matrix, one may derive the usual determinant properties with ease.

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Remark 1.10. The following properties may be inferred:

- If A contains a row of zeroes, or a column of zeroes, then $\det A = 0$.
- $\det I_n = 1$.
- The determinant of a diagonal matrix is the product of the diagonal elements. This is because if $\sigma \in S_N$ is not the identity permutation, then there exists at least one element in the corresponding term where $i \neq \sigma(i)$, and $a_{i\sigma(i)}$ makes the term zero. For the identity transformation, it contains only those elements of the form a_{ii} .

Other non-trivial properties may also be shown with ease.

Corollary 1.11. *If A is an upper triangular matrix, then $\det A$ is the product of the diagonal entries.*

Proof. If $a_{1\sigma(1)} \cdots a_{n\sigma(n)} \neq 0$, then $a_{n\sigma(n)} \neq 0$, that is, $\sigma(n) = n$, as $a_{ni} = 0 \ \forall \ i < n$. Again, $a_{(n-1)\sigma(n-1)} \neq 0$ leads us to conclude that $\sigma(n-1) = n-1$ as σ is a bijection and has to lead to a non-zero element. By similar logic, $\sigma(i) = i$ for all valid i . So, σ is the identity permutation. ■

Corollary 1.12. *If A is a lower triangular matrix, then $\det A$ is the product of the diagonal entries.*

Proof. The proof of this is similar to the previous proof if we consider that the determinant of the transpose of a matrix is equal to the determinant of said matrix. ■

Theorem 1.13. *The determinant of a matrix is equal to the determinant of its transpose, that is, $\det A = \det A^t$ for a square matrix A .*

Proof. The proof is left as an exercise to the reader. ■

Proposition 1.14. *Let B be obtained from A by multiplying a row (or column) of A by a non-zero scalar, α . Then, $\det B = \alpha \det A$.*

Proof. The proof is left as an exercise to the reader. ■

Proposition 1.15. *If B is obtained from A by interchanging any two rows (or columns) of A , then $\det B = -\det A$.*

Proof. Let B be obtained from A by interchanging the rows k and l , with $k < l$. We then have

$$\begin{aligned} \det B &= \sum_{\sigma \in S_n} \epsilon(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{(k-1)\sigma(k-1)} a_{l\sigma(k)} a_{(k+1)\sigma(k+1)} \cdots a_{k\sigma(l)} \cdots a_{n\sigma(n)}. \end{aligned} \quad (1.10)$$

As σ runs through all elements in S_n , $\tau = \sigma(k, l)$ also runs through all S_n . Hence, via $\epsilon(\tau) = -\epsilon(\sigma)$, the equation now looks like

$$\det B = - \sum_{\tau \in S_n} \epsilon(\tau) a_{1\tau(1)} \cdots a_{l\tau(l)} \cdots a_{k\tau(k)} \cdots a_{n\tau(n)} = -\det A. \quad (1.11)$$

■

Proposition 1.16. *If two rows (or columns) of A are equal, then $\det A = 0$.*

Proof. Suppose that the rows k and l of A are equal. Interchanging will alter the determinant by -1 , so $\det A = -\det A \implies 2\det A = 0 \implies \det A = 0$ if $2 \neq 0$ in the field F from where the elements of A arrive.

If $2 = 0$ in F , that is, F is of characteristic 2, we pair the σ term in the expression of $\det A$ with the term τ where $\tau = \sigma(k, l)$. The terms corresponding to σ and τ in the expressions are the same, differing in only the sign. Hence, $\det A = 0$. ■

Theorem 1.17. *For a fixed k , let the row k of A be the sum of the two row vectors X^t and Y^t , that is, $a_{kj} = x_j + y_j$ for all $1 \leq j \leq n$. Then $\det A = \det B + \det C$ where B is obtained from A by replacing the row k of A by the row vector X^t , and C is obtained from A by replacing the row k of A by the row vector Y^t .*

Proof. We utilize the fact that $a_{kj} = x_j + y_j$. We have

$$\begin{aligned} \det A &= \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)} \cdots a_{n\sigma(n)} \\ &= \left(\sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots x_{\sigma(k)} \cdots a_{n\sigma(n)} \right) + \left(\sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots y_{\sigma(k)} \cdots a_{n\sigma(n)} \right) \\ &= \det B + \det C. \end{aligned}$$

■

Proposition 1.18. *If a scalar multiple of a row (or column) is added to a row (or column) of a matrix, the determinant remains unchanged.*

Proof. The proof follows immediately from the previously proved properties. ■

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Definition 1.19. For $a_{ij} \in A$, the *cofactor* of a_{ij} is $A_{ij} = (-1)^{i+j} \det M_{ij}$, where M_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column of A .

Lemma 1.20. *Fix k, j . If $a_{kl} = 0$ for all $l \neq j$, then $\det A = a_{kj} A_{kj}$.*

Proof. Take A to be a $n \times n$ matrix. We deal in cases.

- Case I: $k = j = n$. In the expansion of the determinant,

$$\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

only those σ 's survive where $\sigma(n) = n$. These σ 's can be thought of as permutations of S_{n-1} instead. The sign of $\sigma \in S_n$ and $\sigma \in S_{n-1}$ is the same as n is fixed. Thus, we get

$$a_{nn} \sum_{\sigma \in S_{n-1}} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{(n-1)\sigma(n-1)} = a_{nn} \det M_{nn} = (-1)^{n+n} a_{nn} A_{nn} = a_{nn} A_{nn}. \quad (1.12)$$

- Case II: $(k, j) \neq (n, n)$. We construct a matrix B by interchanging $n-k$ rows and $n-j$ columns to bring a_{ij} to the position (n, n) . Thus, we have $\det B = (-1)^{n-k+n-j} \det A = (-1)^{k+j} \det A$. But $B = a_{kj} \det M_{kj}$, so

$$\det A = (-1)^{k+j} a_{kj} \det M_{kj} = a_{kj} A_{kj}. \quad (1.13)$$

■

Theorem 1.21. *Let A be a $n \times n$ matrix, and let $1 \leq k \leq n$. Then, $\det A = \sum_{j=1}^n a_{kj} A_{kj}$, expansion by the k^{th} row.*

Proof. Write out the k^{th} row of A as $x_1^t + \cdots + x_n^t$, where $x_i = (0, \dots, 0, a_{ki}, 0, \dots, 0)^t$, and all the other rows remaining are the same. Writing the matrix A as the sum of n matrices where each matrix is the same as A but with a row that looks like x_i^t , we can easily show that $\det A = \sum_{j=1}^n a_{kj} A_{kj}$. ■

Example 1.22. Let $n \geq 1$, and let $A_n = \begin{pmatrix} a_1^{n-1} & a_1^{n-2} & \cdots & a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \cdots & a_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n^{n-1} & a_n^{n-2} & \cdots & a_n & 1 \end{pmatrix}$. Then, $\det A_n = \prod_{1 \leq i < j \leq n} (a_i - a_j)$.

Proof. If $a_i = a_j$ for some $i \neq j$, then $\det A_n = 0$ as two rows are then identical. Hence, assume that the a_i 's are distinct. Now construct

$$B_n = \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \cdots & a_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n^{n-1} & a_n^{n-2} & \cdots & a_n & 1 \end{pmatrix}. \quad (1.14)$$

Notice that $\det B_n \in F[x]$, where F is the field, and x is an indeterminate. $\det B$ is also of degree $(n-1)$; let us call this polynomial $f(x)$. Each of a_2, \dots, a_n are roots of $f(x)$, so $f(x)$ must be of the form $f(x) = C(x - a_2) \cdots (x - a_n)$. Equating coefficients of x^{n-1} , we get

$$C = \prod_{2 \leq i < j \leq n} (a_i - a_j) = \det \begin{pmatrix} a_2^{n-2} & \cdots & a_2 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ a_n^{n-2} & \cdots & a_n & a_1 \end{pmatrix}. \quad (1.15)$$

Thus, we must have

$$f(x) = \left(\prod_{2 \leq i < j \leq n} (a_i - a_j) \right) (x - a_2) \cdots (x - a_n) \quad (1.16)$$

$$\implies \det A_n = f(1) = \prod_{1 \leq i < j \leq n} (a_i - a_j). \quad (1.17)$$

■

Example 1.23. Show that there exists a unique polynomial of degree n that takes arbitrary prescribed values at the $(n + 1)$ points x_0, x_1, \dots, x_n .

Chapter 2

EIGENVECTORS AND EIGENVALUES

2.1 Linear Transformers and an Introduction

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of vector space V and $\mathcal{C} = (w_1, \dots, w_m)$ be a basis of a vector space W . As these are bases, given a $v \in V$, there exists a unique $X \in F^n$ such that $v = \mathcal{B}X$, called the *coordinate vector* of v with respect to the basis \mathcal{B} . We note that since the mapping from a $v \in V$ to a $X \in F^n$ is linear in nature and is bijection, the vector spaces V and F^n are isomorphic to each other. Similarly, a mapping that takes $w \in W$ to $Y \in F^m$ shows that W and F^m are isomorphic to each other.

Now suppose that there exists a linear map that takes $v \mapsto Tv$ with $v \in V$ and $Tv \in W$. This transformer T is with respect to the bases \mathcal{B} and \mathcal{C} of V and W , respectively. We construct the $m \times n$ matrix A so that the j^{th} column of A is the coordinate vector of Tv_j with respect to the basis \mathcal{C} . We will then have $T(\mathcal{B}) = \mathcal{C}A$. For any vector $v \in V$, we have

$$\begin{aligned} v &= \mathcal{B}X = v_1x_1 + \dots v_nx_n \\ \implies T(v) &= T(v_1)x_1 + \dots T(v_n)x_n = (T(v_1), \dots, T(v_n)) \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = T(\mathcal{B})X = (\mathcal{C}A)X \end{aligned} \quad (2.1)$$

$$= (w_1, \dots, w_m)AX; \quad (2.2)$$

the coordinate vector of Tv with respect to the basis AX . In fact, if we denote the isomorphism from V to F^n by $\phi_{\mathcal{B}}$ and the isomorphism from W to F^m by $\phi_{\mathcal{C}}$, we get $\phi_{\mathcal{C}} \circ T = (\text{mult. by } A) \circ \phi_{\mathcal{B}}$.

The next theorem will be divided into two parts.

Theorem 2.1. 1. *The vector space form. Let $T : V \rightarrow W$ be a linear mapping between finite dimensional vector spaces V and W , of dimensions n and m respectively. There are bases \mathcal{B} and \mathcal{C} of V and W respectively such that the matrix of T with respect to the bases \mathcal{B} and \mathcal{C} looks like*

$$\begin{pmatrix} I_r & O_{r \times (n-r)} \\ O_{(m-r) \times r} & O_{(m-r) \times (n-r)} \end{pmatrix}_{m \times n}.$$

2. *The matrix form. If A is a $m \times n$ matrix, then there exists an invertible matrix $Q_{m \times m}$ and an invertible matrix $P_{n \times n}$ such that $Q^{-1}AP$ is of the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where r is the rank of A .*

3. *In fact, both these forms of the theorem are equivalent.*

Proof. 1. Let (u_1, \dots, u_{n-r}) be a basis of $\ker T$. We can extend this to a basis \mathcal{B} by appending independent vectors that do not belong to the kernel of T , that is, $(v_1, \dots, v_r, u_1, \dots, u_{n-r})$. Let (Tv_1, \dots, Tv_r) be a basis of $\text{Im}T$. We can extend this to a basis of W , say $\mathcal{C} = (w_1, \dots, w_r, w_{r+1}, \dots, w_m)$, where $w_i = Tv_i$ for $1 \leq i \leq r$. These bases are the desired ones.

2. P is a sequence of column operations, multiplied to form a matrix, and Q^{-1} is a sequence of row operations, multiplied to form a matrix, that get the matrix A into the desired form. These are our desired P and Q .
3. Suppose the vector space form holds. Let A be a $m \times n$ matrix over F , with $A : F^n \rightarrow F^m$ defined as $X \mapsto AX$. There then exists a basis \mathcal{B} of F^n and a basis \mathcal{C} of F^m such that the linear map A with respect to the bases \mathcal{B} and \mathcal{C} has the desired matrix. We then have $\mathcal{B} = I_n P_{n \times n}$ and $\mathcal{C} = I_m Q_{m \times m}$, with both P and Q invertible. We claim that the matrix of the linear mapping A with respect to the bases \mathcal{B} and \mathcal{C} is $Q^{-1}AP$. ■

January 16th.

Proposition 2.2. 1. Let $T : V \rightarrow W$ be a linear map, and A the matrix of T with respect to the bases \mathcal{C} and \mathcal{C} of V and W respectively. Let \mathcal{B}' and \mathcal{C}' be new bases of V and W respectively, and let the change of basis matrices be given by $\mathcal{B}' = \mathcal{B}P$ and $\mathcal{C}' = \mathcal{C}Q$. Then the matrix of T with respect to \mathcal{B}' and \mathcal{C}' is $Q^{-1}AP$.

2. If $A' = Q_1^{-1}AP_1$, where P_1 and Q_1 are $n \times n$ and $m \times m$ invertible matrices, respectively, then A' is the matrix of T with respect to the bases $\mathcal{B}P_1$ and $\mathcal{C}Q_1$.

Proof. Let the coordinate vector of v with respect to the basis \mathcal{B}' be X' . We claim that the coordinate vector of Tv with respect to the basis \mathcal{C}' is Y' , where $Y' = (Q^{-1}AP)X'$. We assume that $\mathcal{B}' = \mathcal{B}P_{n \times n}$, $\mathcal{C}' = \mathcal{C}Q_{m \times m}$, and $T(\mathcal{B}) = \mathcal{C}A_{m \times n}$. If $v = \mathcal{B}X$, then $T(v) = \mathcal{C}(AX)$. If we let $v = \mathcal{B}'X' = v'_1x'_1 + \dots + v'_nx'_n$, then

$$T(v) = \mathcal{C}'Y' = (\mathcal{C}Q)' = \mathcal{C}(QY') = \mathcal{C}(APX') \implies QY' = APX' \implies Y' = (Q^{-1}AP)X' \quad (2.3)$$

To prove the second part, we will show that the first part implies it. Let $A_{m \times n}$ be a matrix. Let T_A be the linear map from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by multiplication by A , that is $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $X \mapsto AX$. By the first part, there exist bases $P_{n \times n}$ and $Q_{m \times m}$, both invertible, such that with respect P and Q , the matrix of T_A looks like $\begin{pmatrix} I & O \\ O & O \end{pmatrix}$, that is, $Q^{-1}AP = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$. ■

2.1.1 Linear Operators

Let $T : V_{\mathcal{B}} \rightarrow V_{\mathcal{B}}$. Let A be the matrix of T with respect to the basis \mathcal{B} . The other matrices of T with respect to new bases are $P^{-1}AP$, where $P_{n \times n}$ is invertible. Also, the fact that T is bijective, one-one, or onto are all equivalent for a finite dimensional vector space V .

2.1.2 Eigenvectors and Eigenvalues

Definition 2.3. A non-zero vector $v \in V$ is said to be an *eigenvector* of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$. If A is a $n \times n$ matrix, a non-zero column vector X is said to be an eigenvector of A if $AX = \lambda X$ for some $\lambda \in \mathbb{F}$. λ , in both these cases, is called the *eigenvalue* of v and X respectively.

Usually, we always disregard the zero vector being an eigenvector. If v is an eigenvector of $T : V \rightarrow V$, and $v = \mathcal{B}X$ with respect to some basis \mathcal{B} of V , then X is an eigenvector of the matrix of T with respect to the basis \mathcal{B} . In fact,

$$\mathcal{B}(AX) = (\mathcal{B}A)X = T(\mathcal{B})X = T(\mathcal{B}X) = Tv = \lambda v = \lambda \mathcal{B}X = \mathcal{B}(\lambda X) \implies AX = \lambda X. \quad (2.4)$$

The converse is also true; if X is an eigenvector of $A_{n \times n}$, then X is also an eigenvector of $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Proposition 2.4. 0 is an eigenvalue of $A_{n \times n}$ ($T : V \rightarrow V$) if and only if A (T) is non-invertible (not an isomorphism).

Suppose v is an eigenvector of $T : V \rightarrow V$ with eigenvalue λ . Let W be the subspace spanned by v . Then every vector $w \in W$ is an eigenvector of T with eigenvalue λ . The proof of this is left as an exercise.

Definition 2.5. Two matrices $A'_{n \times n}$ and $A_{n \times n}$ are called *similar matrices* if there exists an invertible matrix $P_{n \times n}$ such that $P^{-1}AP = A'$.

Again let $T : V \rightarrow V$ be a linear operator, and let $\mathcal{B} = (v_1, \dots, v_n)$. Suppose, with respect to the basis \mathcal{B} , the matrix of T is $\begin{pmatrix} \lambda_1 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{pmatrix}$. Then v_1 is an eigenvector with eigenvalue λ_1 .

2.2 Finding Eigenvalues and Eigenvectors

January 21st.

Let $T : V \rightarrow V$ and let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V . Then the matrix of T with respect to the basis \mathcal{B} is a diagonal matrix if and only if each of the basis elements is an eigenvector. An equivalent statement for matrices is that an $n \times n$ matrix A is similar to a diagonal matrix if and only if \mathbb{F}^n admits a basis consisting of eigenvectors of A . The proof of this is left as an exercise to the reader.

We can now discuss the computation. For a linear operator $T : V \rightarrow V$, λ is an eigenvalue of T if and only if there exists a non-zero vector v such that $Tv = \lambda v$. This can be rearranged to give

$$(\lambda I_v - T)v = 0. \quad (2.5)$$

We can now consider $\lambda I_v - T : V \rightarrow V$ to be a linear operator which maps $v \mapsto \lambda v - Tv$. If eigenvalues exist, this operator is a singular operator, that is, it contains a non-trivial kernel. The matrix of the operator $\lambda I_v - T$ comes out to be $\lambda I_n - A$, where A is the matrix of T with respect to the basis \mathcal{B} . This matrix is now singular, so we must have

$$\det(\lambda I_n - A) = 0. \quad (2.6)$$

The equation $\det(\lambda I_n - A)$ is called the *characteristic polynomial* of A , and also $T(?)$. The roots of this polynomial in λ which lie in \mathbb{F} are the eigenvalues of A , and T as well.

We would now like to show that similar matrices have the same eigenvalues, that is,

$$\det(\lambda I_n - P^{-1}AP) = \det(\lambda I_n - A). \quad (2.7)$$

This is simple to see as $\det(\lambda I_n - P^{-1}AP) = \det(P^{-1}(\lambda I_n - A)P) = \det P^{-1} \cdot \det(\lambda I_n - A) \cdot \det P = \det(\lambda I_n - A)$. The found out eigenvalues from this equation can then be put back and solved for v to get the corresponding eigenvectors.

Proposition 2.6. *Let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues of $T : V \rightarrow V$ and let v_1, \dots, v_r be the corresponding eigenvectors of T . Then (v_1, \dots, v_r) is a linearly independent set in V .*

Proof. We claim that this is true for $r = 1, 2$. Using a form of induction, we will assume the result for $r - 1$. Begin with

$$\begin{aligned} \alpha_1 v_1 + \dots + \alpha_r v_r &= 0 \\ \implies \alpha_1 T v_1 + \dots + \alpha_r T v_r &= 0 \\ \implies \alpha_1 \lambda_1 v_1 + \dots + \alpha_r \lambda_r v_r &= 0. \end{aligned} \quad (2.8)$$

Multiplying the first equation by λ_1 and subtracting it from the current equation, we have

$$\begin{aligned} (\alpha_2 \lambda_2 - \alpha_2 \lambda_1) v_2 + (\alpha_3 \lambda_3 - \alpha_3 \lambda_1) v_3 + \dots + (\alpha_r \lambda_r - \alpha_r \lambda_1) v_r &= 0 \\ \implies \alpha_2 (\lambda_2 - \lambda_1) + \alpha_3 (\lambda_3 - \lambda_1) v_3 + \dots + \alpha_r (\lambda_r - \lambda_1) v_r &= 0. \end{aligned} \quad (2.9)$$

By hypothesis, $\alpha_j (\lambda_j - \lambda_1) = 0$. As the eigenvalues are distinct, we must have $\alpha_j = 0$ for $j = 2, 3, \dots, r$. We are left with $\alpha_1 v_1 = 0$, which gives us $\alpha_1 = 0$. ■

When the n eigenvalues found of A are distinct, the corresponding eigenvectors v_1, \dots, v_n are linearly independent in \mathbb{F}^n , and hence $\mathcal{B} = (v_1, \dots, v_n)$ is a basis of \mathbb{F}^n . The matrix $P^{-1}AP$ is the matrix of the linear operator $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ with respect to the basis \mathcal{B} , with the column of P being the eigenvectors v_1, \dots, v_n . As \mathcal{B} consists of only eigenvectors, $P^{-1}AP$ is a diagonal matrix with the diagonal entries being the n eigenvalues.

We now define the determinant and trace for a linear operator. For such an operator T , $\text{tr} T = \text{tr} A$ where A is a matrix of T with respect to some arbitrary basis. Note that since $\text{tr}(P^{-1}AP) = \text{tr}(APP^{-1}) = \text{tr} A$, the choice of basis is not important. Similarly, we define $\det T = \det A$.

We can now have a closer look at the characteristic equation. To find the constant term of $\det(xI - A)$, we simply plug in $x = 0$ to give us $\det(-A) = (-1)^n \det A$. The coefficient of x^{n-1} in $\det(xI - A)$ is $-\text{tr} A$ as the coefficients of x^{n-1} come solely from the expansion of $(x - a_{11})(x - a_{22}) \cdots (x - a_{nn})$. Clearly, we can conclude that the sum of the eigenvalues is $\text{tr} A$ and the product of the eigenvalues is $\det A$.

2.2.1 Eigenspace

January 23rd.

For ease, let us denote $\chi_T(x)$ to mean $\det(xI - A)$. The *eigenspace* for a given eigenvalue λ is defined as

$$E_\lambda = \{v \in V : Tv = \lambda v\}. \quad (2.10)$$

This is a subspace of the vector space V . The *geometric multiplicity* of λ is defined as the dimension of E_λ . This geometric multiplicity of λ is always less than or equal to its algebraic multiplicity in $\chi_T(x)$. For recall, the *algebraic multiplicity* of λ is the highest power of $(x - \lambda)$ that divides $\chi_T(x)$.

Theorem 2.7. *Let λ be an eigenvalue of $T : V \rightarrow V$. Then the geometric multiplicity of λ is always less than or equal to its algebraic multiplicity.*

Proof. Let k be the geometric multiplicity of λ . Let (v_1, \dots, v_k) be an ordered basis of E_λ . Extend this to a basis $\mathcal{B} = (v_1, \dots, v_k, u_1, \dots, u_{n-k})$ of V . The matrix of T with respect to the basis \mathcal{B} is of the form $A = \begin{pmatrix} \lambda I_k & B \\ O & D \end{pmatrix}$. Thus, the characteristic polynomial looks like

$$\chi_T(x) = \det(xI_n - A) = \det \begin{pmatrix} (x - \lambda)I_k & -B \\ O & xI_{n-k} - D \end{pmatrix} = (x - \lambda)^k \cdot \det(xI_{n-k} - D). \quad (2.11)$$

This shows that $(x - \lambda)^k$ divides $\chi_T(x)$, so we must have an algebraic multiplicity greater than or equal to this k . ■

2.3 Diagonalizability

We first define what this means for a linear mapping from V to V .

Definition 2.8. A linear operator $T : V \rightarrow V$ is said to be a *diagonalizable linear operator* if there exists a basis of V consisting of eigenvectors of T . This means that the matrix of T with respect to this basis is a diagonal matrix and the matrix of T with respect to any other basis is similar to this diagonal matrix.

A similar definition works for matrices.

Definition 2.9. An $n \times n$ matrix A over \mathbb{F} is said to be a *diagonalizable matrix* if A is similar to a diagonal matrix. Equivalently, \mathbb{F}^n then admits a basis consisting of eigenvectors of A , thinking of $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ as a linear operator.

Now let us suppose that T is diagonalizable. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of T . There then exists an ordered basis consisting of eigenvectors of T and with respect to this basis, the matrix of T is a diagonal matrix with diagonal entries consisting solely of $\lambda_1, \lambda_2, \dots, \lambda_k$.

If λ_i is of algebraic multiplicity d_i , then the matrix of T looks like $\begin{pmatrix} \lambda_1 I_{d_1} & & & \\ & \lambda_2 I_{d_2} & & \\ & & \dots & \\ & & & \lambda_k I_{d_k} \end{pmatrix}$.

Thus, the characteristic polynomial then looks like $(x - \lambda_1)^{d_1} (x - \lambda_2)^{d_2} \dots (x - \lambda_k)^{d_k}$.

The geometric multiplicity of λ_i is the dimension of E_{λ_i} , that is, the nullity of the operator $(\lambda_i I_n - A)$. But here, $\ker(\lambda_i I_n - A) = d_i$, which is just the algebraic multiplicity of λ_i . Hence, if T is diagonalizable, then each eigenvalue of it has the same algebraic multiplicity and geometric multiplicity.

Proposition 2.10. *If $E_{\lambda_1}, \dots, E_{\lambda_k}$ are the eigenspaces corresponding to the distinct eigenvalues, say, $\lambda_1, \dots, \lambda_k$ of T , then $E = E_{\lambda_1} + \dots + E_{\lambda_k}$ is a direct sum.*

Proof. It is enough to show that $E_{\lambda_1}, \dots, E_{\lambda_k}$ are independent. Let $v_1 + v_2 + \dots + v_k = 0$, where $v_i \in E_{\lambda_i}$. As v_1, v_2, \dots, v_k come from distinct eigenspaces, they are linearly independent, and our equation must imply that $v_1 = \dots = v_k = 0$. ■

Proposition 2.11. *If T is a diagonalizable operator, and if $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of T , then*

$$V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}. \quad (2.12)$$

Proof. As T is diagonalizable, the algebraic and geometric multiplicities are equal for all the eigenvalues λ_i . Denote $\dim E_{\lambda_i} = d_i$. As $\chi_T(x)$ completely factors into linear factors, due to T being diagonalizable, we have $n = d_1 + \dots + d_k$. Also, $E_{\lambda_1} + \dots + E_{\lambda_k}$ is a direct sum, that is,

$$\dim(E_{\lambda_1} + \dots + E_{\lambda_k}) = \dim E_{\lambda_1} + \dots + \dim E_{\lambda_k} = n. \quad (2.13)$$

This direct sum is a subspace of V and has the dimension as V . This must mean that the direct sum is exactly V . ■

Theorem 2.12. *Let T be a linear operator on a finite dimensional vector space V , and let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of T . Also let E_{λ_i} be the eigenspace of λ_i . Then, the following are equivalent.*

- T is diagonalizable,
- $\chi_T(x) = (x - \lambda_1)^{d_1} \dots (x - \lambda_k)^{d_k}$ and $\dim E_{\lambda_i} = d_i$,
- $V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}$.

2.4 Polynomials

January 28th.

Let $\mathbb{F}[x]$ denote the set of all polynomials with coefficients coming from the field \mathbb{F} . With respect to the addition, it is an Abelian group. The multiplication here is associative, commutative, and distributive; there also exists a multiplicative identity. This makes $\mathbb{F}[x]$ into a commutative ring. Note that $\mathbb{F}[x]$ is also an infinite dimensional vector space over \mathbb{F} , since scalar multiplication is also defined. Together, these combine to form an algebra over the field.

Definition 2.13. Let $d \in \mathbb{F}[x]$ with $d \neq 0$. For $f \in \mathbb{F}[x]$, we say that d divides f if there exists a $q \in \mathbb{F}[x]$ such that $f = dq$ in $\mathbb{F}[x]$.

Corollary 2.14. *For $f \in \mathbb{F}[x]$, $f(c) = 0$ if and only if $x - c$ divides $f(x)$.*

Corollary 2.15. *A polynomial $f \in \mathbb{F}[x]$ of degree n has at most n roots in \mathbb{F} .*

Proof. The proof is by induction. Note that this is true for $n = 0, 1$. If α is a root, then $f(x) = (x - \alpha) \cdot q(x)$. As $q(x)$ is of degree $n - 1$, and all roots of $q(x)$ are root of $f(x)$, this follows by hypothesis. ■

Definition 2.16. An *ideal* of $\mathbb{F}[x]$ is a subspace of $\mathbb{F}[x]$, say I , such that if $f \in I$ and $g \in \mathbb{F}[x]$, then $fg \in I$.

Example 2.17. Let $f \in \mathbb{F}[x]$. Define $I_f = \langle f \rangle = \{fg : g \in \mathbb{F}[x]\}$. Note that I_f is called a *principal ideal*, that is, it is an ideal generated by a single element.

Theorem 2.18. *$\mathbb{F}[x]$ is a principal ideal domain, that is, every ideal in $\mathbb{F}[x]$ is a principal ideal.*

Proof. Let d be a polynomial of least degree in the ideal I , where I is a non-zero ideal. Let, without loss of generality, d be monic (if not, simply multiply it by a suitable scalar).

Let $f \in I$. Then there exists $q, r \in \mathbb{F}[x]$ such that $f = dq + r$ and either $r = 0$ or $\deg r < \deg d$. Note that since $f, d \in I$, $dq \in I$, so $f - dq \in I \implies r \in I$. As d was of minimal degree in I , we must have $r = 0$. Thus, $f = dq$ and, thus, $I = \langle d \rangle$. ■

If I is an ideal of $\mathbb{F}[x]$, then there exists a unique polynomial $d \in I$ such that $I = \langle d \rangle$.

2.4.1 Interaction with Linear Operators

Let $f \in \mathbb{F}[x]$, and let $T : V \rightarrow V$ be a linear mapping. If

$$f(x) = a_0 + a_1x + \dots + a_kx^k$$

with $a_k \neq 0$, we define

$$f(T) = a_0I_n + a_1T + \dots + a_kT^k.$$

Note that $f(T)$ is also a linear mapping from V to V . Let I be the set of all $f \in \mathbb{F}[x]$ such that $f(T)$ is the zero operator. All such polynomials are called *annihilators*. I satisfies the properties of a vector space; it is a subspace of the space of all polynomials. I is also an ideal of $\mathbb{F}[x]$.

Definition 2.19. The *minimal polynomial* of the linear operator $T : V \rightarrow V$ is the generator of the ideal of annihilators.

Denote the minimal polynomial by $m_T(x)$. So, $m_T(x)$ is

1. monic,
2. of least degree among all annihilators of T .

If A is a $n \times n$ matrix, the minimal polynomial of A is defined as the unique monic polynomial $m_A(x)$ of least degree such that $m_A(A) = O_{n \times n}$. It can be verified that if A is the matrix of a linear operator $T : V \rightarrow V$ and if $f \in \mathbb{F}[x]$, then the matrix of the operator $f(T) : V \rightarrow V$ is $f(A)$ with respect to the same basis. It follows that the minimal polynomial of T is same as the minimal polynomial of a matrix of T .

Note that T belongs to $\text{Hom}_{\mathbb{F}}(V, V)$, which is of dimension n^2 . Thus, $I, T, T^2, \dots, T^{n^2}$ is a linearly dependent set and there exist scalars a_0, a_2, \dots, a_{n^2} such that

$$a_0 I + a_1 T + a_2 T^2 + \dots + a_{n^2} T^{n^2} = O. \quad (2.14)$$

So, an annihilator of T is

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n^2} x^{n^2}$$

and we must have $\deg m_T(x) \leq n^2$.

Theorem 2.20. Let $T : V \rightarrow V$ with n the dimension of the space V . The characteristic polynomial of T and the minimal polynomial of T have the same roots, except (possibly) for the multiplicities.

Proof. We claim that $m_T(c) = 0$ if and only if c is an eigenvalue. Let $m_T(c) = 0$. Thus, $m_T(c) = (x - c) \cdot q(x)$, with $q \in \mathbb{F}[x]$ and $\deg q < \deg m$. Also, $q(T)$ is *not* the zero operator. So, there exists a $u \in V$ (non-zero vector) such that $q(T)(u) = v \neq 0$. Then,

$$0 = m(T)(u) = (T - cI) \cdot q(T)(u) = (T - cI)v \quad (2.15)$$

which shows that v is an eigenvector of T with eigenvalue c . So all roots of $m_T(x)$ are roots of the characteristic polynomial.

Conversely, let c be an eigenvalue of T . Say, $Tv = cv$ for some $v \neq 0$. Thus, $m_T(T)(v) = m(c)(v)$. But $m_T(T) = 0$ must mean that $0 = m(c)(v)$, and $m(c) = 0$. So every root of the characteristic polynomial is a root of the minimal polynomial. ■

January 30th.

Proposition 2.21. If λ is an eigenvalue of T , then $f(\lambda)$ is an eigenvalue of $f(T)$ for $f \in \mathbb{F}[x]$.

Proof. The proof is left as an exercise to the reader. ■

Proposition 2.22. Let $T : V \rightarrow V$ be a diagonalizable operator. The minimal polynomial is the product of distinct linear factors, that is, if

$$\chi_T(x) = \prod_{i=1}^k (x - \lambda_i)^{d_i}$$

where the λ_i 's are the distinct eigenvalues, then

$$m_T(x) = \prod_{i=1}^k (x - \lambda_i).$$

Proof. As T is a diagonalizable operator, there exists a basis of V consisting of eigenvectors of T , say $\mathcal{B} = (v_1, v_2, \dots, v_n)$. Note that $m_T(T)v_i = 0$ for all valid i . For each $v_i \in \mathcal{B}$, there exists a λ_i such that $(T - \lambda_i I)v_i = 0$, which tells us

$$m_T(T) = (T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_k I)v_i = 0. \quad (2.16)$$

Hence, $m_T(x)$ is an annihilator for T , and it is of minimal degree by the above theorem. ■

Theorem 2.23 (*Cayley-Hamilton theorem*). *Let $T : V \rightarrow V$ be a linear operator on a finite dimensional vector space V . If $\chi_T(x)$ is the characteristic polynomial of T , then $\chi_T(T) = 0$, that is, the characteristic polynomial annihilates T . Hence, the minimal polynomial of T divides the characteristic polynomial.*

Proof. Let $\mathcal{B} = (v_1, v_2, \dots, v_n)$ be a basis of V , and let $A = (a_{ij})$ be the matrix of T with respect to the basis \mathcal{B} . We have

$$\begin{aligned} a_{1j}v_1 + a_{2j}v_2 + \dots + a_{nj}v_n &= Tv_j \\ \implies -a_{1j}v_1 - a_{2j}v_2 - \dots + (T - a_{jj})v_j - a_{(j+1)(j)}v_{j+1} - \dots - a_{nj}v_n &= 0. \end{aligned} \quad (2.17)$$

This system of equations can be written as

$$B_{n \times n} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2.18)$$

where

$$B = \begin{pmatrix} T - a_{11}I & -a_{21} & \cdots & -a_{n1} \\ -a_{12} & T - a_{22}I & \cdots & -a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & T - a_{nn}I \end{pmatrix}. \quad (2.19)$$

Therefore $\det B = \chi_T(T)$. It is enough to show that $\det B = 0$ as an operator, that is, to show $\det B(b_i) = 0$ for all $v_i \in \mathcal{B}$. Let $(\text{adj} B)_{ij} = c_{ij}$, and $(B)_{ij} = b_{ij}$. Note that

$$\sum_{k=1}^n c_{ik} b_{kj} = \begin{cases} \det B & \text{if } i = j, \\ 0 & \text{if otherwise.} \end{cases}$$

Now,

$$\begin{aligned} \sum_{j=1}^n b_{kj} v_j &= 0 \text{ for all } 1 \leq k \leq n \\ \implies \sum_{j=1}^n b_{kj} v_j &= 0. \end{aligned}$$

Summing over all rows,

$$\begin{aligned} \sum_{k=1}^n \left(\sum_{j=1}^n c_{ik} b_{kj} v_j \right) &= 0 \\ \implies \sum_{j=1}^n \left(\sum_{k=1}^n c_{ik} b_{kj} \right) v_j &= 0. \end{aligned} \quad (2.20)$$

The left hand side is zero except for when $i = j$, in which case it is $\det B$ —

$$0 = \sum_{j=1}^n \left(\sum_{k=1}^n c_{ik} b_{kj} \right) v_j = (\det B) v_i \quad (2.21)$$

which implies that the operator $\det B$ is zero on all the basis vectors, and hence it is the zero vector. Thus, since $\chi_T(T) = \det B$, $\chi_T(T)$ is also the zero operator. ■

February 4th.

Proposition 2.24. *If the minimal polynomial $m_T(x) \in \mathbb{F}[x]$ of a linear operator $T : V \rightarrow V$ splits into distinct linear factors, then T is diagonalizable.*

Proof. Let $m_T(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k)$ where the λ_i 's are distinct. We are to show that $V = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}$. We wish to find polynomials $h_1(x), \dots, h_k(x)$ such that

1. $h_1(x) + \dots + h_k(x) = 1$,
2. $(x - \lambda_i) \cdot h_i(x)$ is divisible by $m_T(x)$ for all $1 \leq i \leq k$.

The second condition implies that $(T - \lambda_i I) \cdot h_i(T)$ is the zero operator. The first condition implies that $\sum_{i=1}^k h_i(T)(v) = v$. But the second condition again implies that $h_i(T)(v)$ is an eigenvector corresponding to λ_i , that is, $h_i(T)(v) \in E_{\lambda_i}$. If we can find these h_i 's satisfying the two conditions then we can say that V is the direct sum of the eigenspaces.

For $1 \leq i \leq k$, let $f_i(x) = \frac{m_T(x)}{(x - \lambda_i)} = \prod_{j \neq i} (x - \lambda_j)$. As the λ_i 's are distinct, the f_i 's are relatively prime, so there exist g_1, \dots, g_k such that

$$f_1(x)g_1(x) + \dots + f_k(x)g_k(x) = 1. \quad (2.22)$$

Let $h_i(x) = f_i(x)g_i(x)$ for all $1 \leq i \leq k$. Both the conditions hold, and the result follows. \blacksquare

Corollary 2.25. *Let $T : V \rightarrow V$ be a linear operator on a finite dimensional complex vector space such that $T^m = I$ for some positive integral m . Then T is diagonalizable.*

Proposition 2.26. *Let $T : V \rightarrow V$ be linear operator, and let U be an invariant subspace of T , that is, $T(U) \subseteq U$ (or equivalently, $T(u) \in U$ for all $u \in U$). The minimal polynomial $m_{T|_U}(x)$ of the operator $T|_U : U \rightarrow U$ divides the minimal polynomial $m_T(x)$ of the operator $T : V \rightarrow V$ in $\mathbb{F}[x]$.*

Proof. Note that $m_T(T)(u) = 0$ for all $u \in U$ as $U \subseteq V$. Thus, $m_T(T) = 0$ on the subspace U . So $m_T(x)$ annihilates $T|_U$. So, as $m_{T|_U}(x)$ is the minimal polynomial of $T|_U$, it should divide all annihilators of $T|_U$ and thus divides $m_T(x)$ in $\mathbb{F}[x]$. \blacksquare

2.5 Triangularizability

A similar definition works as in the case of diagonalizability.

Definition 2.27. A linear operator $T : V \rightarrow V$ is said to be a *triangularizable linear operator* if there exists a basis of V with respect to which the matrix of T is a triangular matrix, be it upper or lower.

If our basis is $\mathcal{B} = (v_1, v_2, \dots, v_n)$, then we can show that $Tv_k \in \text{span}(v_1, \dots, v_k)$.

Theorem 2.28. *A linear operator $T : V \rightarrow V$ is triangularizable if and only if the minimal polynomial splits into linear factors.*

Proof. Let $T : V \rightarrow V$ be triangularizable, that is, there exists a basis with respect to which the matrix of T is a triangular matrix, with diagonal entries $\lambda_1, \dots, \lambda_n$, say. Then the characteristic polynomial of T is $(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$ where the λ_i 's are not necessarily distinct. But $m_T(x)$ divides $\chi_T(x)$, hence is again a product of linear factors.

Conversely, let $m_T(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k)$ where the λ_i 's are not necessarily distinct. We prove by induction on the number of factors of $m_T(x)$. If $k = 1$, then $m_T(x) = x - \lambda_1$; as $m_T(T) = 0$, $T = \lambda_1 I$, matrix of T is the scalar matrix. Now let $k > 1$, and let the result hold for smaller positive integers. Let $U = \text{Im}(T - \lambda_k I)$. We find that U is a proper subspace of V . Note that U is an invariant subspace of T ; if we let $u = (T - \lambda_k I)(v)$ for some $v \in V$, then

$$T(u) = T(T - \lambda_k I)(v) = (T - \lambda_k I)T(v) \in U. \quad (2.23)$$

The minimal polynomial $m_{T|_U}(x)$ of $T|_U$ divides $m_T(x)$, and hence $m_{T|_U}(x) = (x - \alpha_1) \cdots (x - \alpha_l)$, where $l \leq k$, and $\alpha_1, \dots, \alpha_l \in \{\lambda_1, \dots, \lambda_k\}$. By hypothesis, $T|_U$ is triangularizable. So there exists a basis of U , say (u_1, \dots, u_m) with respect to which the matrix of $T|_U$ is a triangular matrix. So $T|_U(u_k) \in \text{span}(u_1, \dots, u_k)$. Extend this to a basis $\mathcal{B} = (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$. If we rewrite $Tv_j = (T - \lambda_k I)v_j + \lambda_k Iv_j$, we see that $(T - \lambda_k I)v_j \in U = \text{span}(u_1, \dots, u_m)$, and $Tv_j \in \text{span}(u_1, \dots, u_m, v_j)$; the matrix of T with respect to \mathcal{B} is a triangular matrix. \blacksquare

Corollary 2.29. *Every operator $T : V \rightarrow V$, where V is a complex finite dimensional vector space, is triangularizable.*

2.5.1 Determinant of Partitioned Matrices

Proposition 2.30. Let $\Gamma = \begin{pmatrix} A & O \\ O & I \end{pmatrix}$ or $\Gamma = \begin{pmatrix} I & O \\ O & A \end{pmatrix}$, where A is a square matrix. Then we necessarily have $\det \Gamma = \det A$.

Proof. Let Γ be of order $(n+1) \times (n+1)$ and A be of order $n \times n$. By definition,

$$\det \Gamma = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) g_{1\sigma(1)} \cdots g_{(n+1)\sigma(n+1)}. \quad (2.24)$$

Note that $g_{(n+1)\sigma(n+1)}$ is 1 if $\sigma(n+1) = n+1$, and 0 otherwise. Also, $\epsilon(\sigma)$ remains the same when σ is considered to be an element of S_n . Thus,

$$\det \Gamma = \sum_{\sigma \in S_n} \epsilon(\sigma) g_{1\sigma(1)} \cdots g_{n\sigma(n)} = \det A. \quad (2.25)$$

Iterating this, we get the desired result. A similar proof works for the other type of matrix stated. ■

Proposition 2.31. Let $\Gamma = \begin{pmatrix} A & B \\ O & D \end{pmatrix}$ where A and D are square matrices. Then we necessarily have $\det \Gamma = \det A \cdot \det D$.

Proof. Let the orders be $A_{k \times k}$, $D_{l \times l}$, $B_{k \times l}$ and $O_{l \times k}$. Note that Γ can be broken up as

$$\Gamma = \begin{pmatrix} I_k & O_{k \times l} \\ O_{l \times k} & D_{l \times l} \end{pmatrix} \begin{pmatrix} I_k & B_{k \times l} \\ O_{l \times k} & I_l \end{pmatrix} \begin{pmatrix} A_{k \times k} & O_{k \times l} \\ O_{l \times k} & I_l \end{pmatrix}. \quad (2.26)$$

The determinant is multiplicative, so $\det \Gamma = \det D \cdot \det A$ as the determinant of the middle matrix can be shown to be 1. ■

Proposition 2.32. Let $\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A and D are square matrices. If A is invertible, then we necessarily have $\det \Gamma = \det A \cdot \det(D - CA^{-1}B)$.

Proof. Again, we break down Γ .

$$\Gamma = \begin{pmatrix} I & O \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & O \\ O & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ O & I \end{pmatrix}. \quad (2.27)$$

From here, it is clear that $\det \Gamma = \det A \cdot \det(D - CA^{-1}B)$. ■

Proposition 2.33. Let $\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A and D are square matrices. If D is invertible, then we necessarily have $\det \Gamma = \det D \cdot \det(A - BD^{-1}C)$.

Proof. Yet again, we break down Γ .

$$\Gamma = \begin{pmatrix} I & BD^{-1} \\ O & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & O \\ O & D \end{pmatrix} \begin{pmatrix} I & O \\ D^{-1}C & I \end{pmatrix}. \quad (2.28)$$

From here, it is clear that $\det \Gamma = \det D \cdot \det(A - BD^{-1}C)$. ■

2.6 On the Characteristic and Minimal Polynomials

February 6th.

Theorem 2.34. Let A be a $m \times n$ matrix and B be a $n \times m$ matrix with $m \leq n$. Then,

$$\chi_{BA}(x) = x^{n-m} \chi_{AB}(x). \quad (2.29)$$

Proof. Note that there exist non-singular matrices P and Q such that $PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, where r is the rank of A . Partition $Q^{-1}BP^{-1}$ as $\begin{pmatrix} C & D \\ E & G \end{pmatrix}$ where C is of order $r \times r$, with the other submatrices being of appropriate order. Then,

$$PABP^{-1} = PAQQ^{-1}BP^{-1} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \begin{pmatrix} C_{r \times r} & D \\ E & G \end{pmatrix} = \begin{pmatrix} C & D \\ O & O \end{pmatrix}. \quad (2.30)$$

Similarly,

$$Q^{-1}BAQ = Q^{-1}BP^{-1}PAQ = \begin{pmatrix} C_{r \times r} & D \\ E & G \end{pmatrix} \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} C & O \\ E & O \end{pmatrix}. \quad (2.31)$$

Thus,

$$\chi_{AB}(x) = \chi_{PABP^{-1}}(x) = \det \begin{pmatrix} xI_r - C & -D \\ O & xI \end{pmatrix} = x^{m-r} \det(xI - C) \quad (2.32)$$

and

$$\chi_{BA}(x) = \chi_{Q^{-1}BAQ}(x) = \det \begin{pmatrix} xI - C & O \\ -E & xI \end{pmatrix} = x^{n-r} \det(xI - C) \quad (2.33)$$

which tells us that $\chi_{BA}(x) = x^{n-m} \chi_{AB}(x)$. ■

1. Suppose $T : V \rightarrow V$ and $U \subseteq V$.
 - (a) If $U \subseteq \ker T$, then U is T -invariant.
 - (b) If $T(V) \subseteq U$, then U is T -invariant.
2. If V_1, \dots, V_m are T -invariant subspaces, then $V_1 + \dots + V_m$ is T -invariant.
3. Let $P : V \rightarrow V$ be a linear operator such that $P^2 = P$; then the eigenvalues of P are 0 or 1.
4. Let $T : V \rightarrow V$ be an invertible operator. Then λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} .
5. Let $T : V \rightarrow V$, with $0 \neq v \in V$. Then $W = \text{span}(v, Tv, T^2v, \dots)$ is T -invariant, and is the smallest T -invariant subspace of V containing v .
6. Let $T : V \rightarrow V$ and $\text{rank} T = k \leq n$, where n is the dimension of V . Then T has at most $k + 1$ distinct eigenvalues.
- 7.

Appendices

Chapter A

Appendix

Extra content goes here.

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