GROUP THEORY

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Third Semester

List of Symbols

Placeholder

Contents

Chapter 1

INTRODUCTION TO GROUP THEORY

1.1 Set Theory

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We begin with some basic assumptions to introduce set theory. The symbol $\in$ is used to denote membership in a set. A statement using this in set theory may be stated as $x \in y$, which can be either true or false. Once we have developed this language to discuss sets, we can introduce some axioms.

Axiom 1.1. There exists a set with no elements, the *empty set* $\emptyset$.

Formally, the above axiom is $\exists x(\forall y(y \notin x))$.

Axiom 1.2. Two sets are equal if they have the same elements.

From the above two axioms, we can infer a unique empty set. A notion of subsets may also be declared.

Definition 1.3. We say the set A is a *subset* of the set B, denoted $A \subseteq B$, if every element of A is also an element of B.

We also have a bunch of similarity axioms stated below.

Axiom 1.4 (Similarity axioms)**.** We have the following:

1. If x, y are sets, then $\{x, y\} \Rightarrow \{x, \{x, y\}\}$ (not an ordered pair).
2. If A is a set, then $\bigcup A = \{x \mid \exists y \in A,\ x \in y\}$ is a set.
3. There exists a *power set* for every set; given a set A, there exists a set $P(A)$ such that for all $B \subseteq A$, $B \in P(A)$. Formally, $\forall A \exists P(A)(\forall B \subseteq A, B \in P(A))$.
4. The *infinite axiom*: Formally, $\exists I(\emptyset \in I \wedge \forall y \in I(P(y) \in I))$.
5. If A and B are sets, then $A \times B = \{(x, y) \mid x \in A, y \in B\}$ is a set.

Before discussing the last axiom, we define a relation on sets.

Definition 1.5. A *relation* R on a set A is a subset $R \subseteq A \times A$. If $(x, y) \in R$, we write xRy.

Axiom 1.6 (The *axiom of choice*)**.** Let A be a collection of non-empty and disjoint sets. Then there exists a set C consisting of exactly one element from each set in A.

Definition 1.7. A relation R on a set A is said to be:

- *reflexive* if $xRx \forall x \in A$,
- *symmetric* if $xRy \Rightarrow yRx$,
- *transitive* if $xRy \wedge yRz \Rightarrow xRz$,
- *antisymmetric* if $xRy \wedge yRx \Rightarrow x = y$.

Definition 1.8. A *partial order* on a set A is a reflexive, transitive, and antisymmetric relation on A.

Some examples of partially ordered sets include $(R, \leq)$, $(P(\mathbb{R}), \subseteq)$.

Definition 1.9. A *total order* R on a set A is a partial order such that for all $x, y \in A$, either xRy or yRx.

Again, $(R, \leq)$ is a totally ordered set, but not $(P(\mathbb{R}), \subseteq)$.

Definition 1.10. A total order $\leq$ on a set A is said to be a *well-order* if given any non-empty subset $B \subseteq A$, there exists $x \in B$ such that for all $y \in B$, $x \leq y$.

The below theorem may be derived from the above definitions and axioms.

Theorem 1.11 (The *well-ordering principle*)**.** *Every set can be well-ordered.*

We may note that the well-ordering principle and the axiom of choice are equivalent.

Definition 1.12. A *chain* in partially ordered set A, with relation $\prec$, is a subset of A which is totally ordered with respect to $\prec$.

Definition 1.13. Let $C \subseteq A$ be a subset in a partially ordered set $(A, \prec)$. An element $x \in A$ is an *upper bound* of C if for all $y \in C$, $y \prec x$.

Definition 1.14. An element $x \in A$ is a *maximal element* of a partially ordered set $(A, \prec)$ if for all $y \in A$, $x \prec y \Rightarrow x = y$.

Lemma 1.15 (*Zorn's lemma*)**.** *Let A be a set and let $\prec$ be a partial order on A such that every chain in A has an upper bound. Then A has a maximal element.*

Theorem 1.16. *The following are equialent:*

1. *The axiom of choice,*
2. *The well-ordering principle,*
3. *Zorn's lemma.*

Proof. We begin with 2. implies 3.; let A be a non-empty set. Consider

$$\mathcal{C} = \{(B, \leq) \mid B \subseteq A \text{ and } \leq \text{ is a well-order on } B\}. \tag{1.1}$$

We note that $\mathcal{C}$ is non-empty since if we pick $B = \{x\}$ for some $x \in A$, then $x \leq x$ and $(B, \leq) \in \mathcal{C}$. Let $(B, \leq), (C, \leq') \in \mathcal{C}$. We say $(B, \leq) \preceq (C, \leq')$ if there exists $y \in C$ such that

$$B = \{x \in C \mid x \lneq' y\} \ (= I(c, y)) \text{ and } \leq = \leq'|_B, \text{ or } (B, \leq) = (C, \leq') \tag{1.2}$$

Note that $\preceq$ is a partial order on $\mathcal{C}$ and is clearly reflexive.

For transitivity, if we take $B \preceq C$ and $C \preceq D$, then $B = C$ or $B = I(C, y)$ for some $y \in C$, and $C = D$ or $C = I(D, z)$ for some $z \in D$. If equality holds in either case, then clearly $B \preceq D$. If $B = I(C, y)$ and $C = I(D, z)$. Clearly, $B = I(D, y)$.

Now let $T = (\{(B_i, \leq_i) \mid i \in I\})$ be a chain in $\mathcal{C}$. Let $B = \bigcup_{i \in I} B_i$, and $\leq = \bigcup_{i \in I} \leq_i$. Note that this makes sense since if $x \in B_i$ and $y \in B_j$ with $B_i \preceq B_j$, then $x, y \in B_j$. So, we assign $x \leq y$ if $x \leq_j y$. Now let $C \subseteq B$ be non-empty. Also let $x \in C$; then $x \in B_i$ for some $i \in I$. Let $w = \min(B_i \cap C)$. We claim that $w = \min C$. For $y \in C$, if $y \in B_i$ then $w \leq y$. If $y \notin B_i$ then $y \in B_j \in T$. Since T is a chain, either $B_i \preceq B_j$ or $B_j \preceq B_i$; the latter is not possible since $y \notin B_i$. Thus, $B_i = I(B_j, z)$, for some $z \in B_j$, and for any $x \in B_i$, $w \leq x \leq y$.

So $(B, \leq) \in \mathcal{C}$ and it is an upper bound of T; to realize it is an upper bound, we show that $B_i \preceq B$ for all valid i. If $B_i = B$, we are done. Otherwise, let $x = \min(B \setminus B_i)$. Then $B_i = I(B, x)$, and $B_i \preceq B$. Thus, by Zorn's lemma, $\mathcal{C}$ has a maximal element— cal it $(M, \leq)$.

We now claim that $M = A$. If $M \subsetneq A$, then let $a \in A \setminus M$. If we let $\hat{M} = (M \cup \{a\}, \leq')$ where $x \leq' a$ for all $x \in M$, then $M = I(\hat{M}, a)$ but this is a contradiction to the fact that $(M, \leq)$ is a maximal element. Thus, $A = M$.

Next comes 1. implies 3. Let X be a partially ordered set such that every chain has an upper bound. Suppose X has no maximal element; we will utilise the axiom of choice to arise at a contradiction. For every chain T in X, there exists a strict upper bound c_T. Define a function f sending chains T in X to X as $f(T) = c_T \notin T$. Such a function f exists by the axiom of choice. A subset $A \subseteq X$ is called a *conforming subset* if A is well-ordered, with respect to order on X, and for all $x \in A$, $f(I(A, x)) = x$. We claim that if A and B are conforming subsets of X, then $A = B$ or one is the initial segment of the other. For now, let us take this claim to be true. We shall prove it later.

If $f(\emptyset) = x$ then $A = \{x\}$. Note that A is conforming. But $I(A, x) = \emptyset \implies f(I(A, x)) = x$. Let U be the union of all conforming subsets of X. Then U is conforming since if $x \in U$ then $x \in B$ for some B conforming and $x = f(I(B, x)) = f(I(U, x))$. Let $f(U) = w$. Define a new set $\tilde{U} = U \sqcup \{w\}$, which is well-ordered and conforming. Then $U = I(\tilde{U}, w)$, which is a contradiction.

Coming back to the claim, suppose $x \in A \setminus B$. We wish to show that $B = I(A, x)$ for some $x \in A$. Let $x = \min(A \setminus B)$. We claim that this x works. $I(A, x) \subseteq B$ holds since if $y \in A$ and $y < x$ then $y \in B$, or else $x \neq \min(A \setminus B)$. Suppose, now, that the equality does not hold. Take $y = \min(B \setminus I(A, x))$ and $z = \min(A \setminus I(B, y))$. We claim that $I(A, z) = I(B, y)$. Take $v \in I(A, z)$; then $v < z$ implies $v \in I(B, y)$ since $z = \min(A \setminus I(B, y))$. Taking $u \in I(B, y)$, we have $u \in I(A, x) \implies u < x$ since $y = \min(B \setminus I(A, x))$. If $z \leq u$, then $z \in I(A, x) \subseteq B \implies z \in I(B, y)$ contrading the fact that $z = \min(A \setminus I(B, y))$. Thus, $z > u$ and $y \in I(A, z)$. Finally, $z = f(I(A, z)) = f(I(B, y)) = y$ implies $z = x = y$. But this a contradiction since $x \in A \setminus B$ and $y \in B$. ■

Definition 1.17. A relation R on a set A is said to be an *equivalence relation* if it is reflexive, symmetric, and transitive. Let $x \in A$. Then $[x] = \{yRx \mid y \in A\} \subseteq A$ is called the *equivalence class* of x.

We note that $\bigcup_{x \in A}[x] = A$ and for $x, y \in A$, either $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Thus, we get a partition of A into equivalence classes.

Let I be an indexing set, and let A_i be sets for all $i \in I$. Then the existence of $\text{X}_{i \in I} A_i = \{f : I \to \dot{\bigsqcup} A_i \mid f(i) \in A_i \text{ for all } i \in I\}$ is another way of stating the axiom of choice.

Theorem 1.18 (The *principle of induction*)**.** *Let $S(n)$ be statements about the naturals $n \in \mathbb{N}$. Suppose $S(1)$ holds and for all $k \in \mathbb{N}$, $S(k) \Rightarrow S(k+1)$. Then $S(n)$ holds true for all $n \in \mathbb{N}$.*

Let I be a well-ordered set and let $S(i)$ be statements for all $i \in I$. Suppose that if $S(j)$ holds for all $j < i$, then $S(i)$ holds. Then $S(i)$ holds for all $i \in I$. This is the *principle of transfinite induction*, which is also equivalent to the axiom of choice. We now properly introduce the theory of groups.

1.2 Groups

We first define a group.

Definition 1.19. A *group* is a triple $(G, \cdot, e)$ where G is a set, $\cdot : G \times G \to G$ is a binary operation on G, and $e \in G$ is an element of G satisfying the following axioms:

- The property of *associativity*: For $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- The property of the *identity element*: For all $a \in G$, $a \cdot e = e \cdot a = a$. e is referred to as the identity element.
- The existence and property of the *inverse element*: For all $a \in G$, there exists $b \in G$ such that $a \cdot b = b \cdot a = e$.

In addition, $(G, \cdot, e)$ is also termed an *abelian group* if for all $a, b \in G$, $a \cdot b = b \cdot a$, that is, commutativity holds.

A group may also be rewritten as $(G, \cdot)$, or just G. Some examples include $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$. The set $(\mathbb{Q}, \cdot)$ is not a group since 0 does not have an inverse. However, $(\mathbb{Q}^*, \cdot)$ is a group, where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. All these groups are also abelian. An example of a non-abelian group is S_n, the set of all bijections from $\{1, 2, \ldots, n\}$ to itself, under the binary operation of composition of functions. Another non-abelian group is $(GL_n(\mathbb{R}), \cdot)$, for $n \geq 2$, the set of all invertible real $n \times n$ matrices.

July 24th.

From the axioms, arise basic properties related to groups.

Proposition 1.20. *Let* $(G, \cdot, e)$ *be a group.*

1. *Let* $a \in G$ *be such that* $a \cdot b = b$ *for all* $b \in G$. *Then* $a = e$; *the identity element is unique.*
2. *Each element* $a \in G$ *has a unique inverse. Thus, the inverse of* a *is then termed* a^{-1}.
3. $(a^{-1})^{-1} = a$ *holds for all* $a \in G$.
4. *For all* $a, b \in G$, $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.
5. *Let* $a \in G$ *be such that* $a \cdot b = b$ *for some* $b \in G$. *Then* $a = e$.

Proof.

1. Choose b to be e. Then $a \cdot e = e$ by hypothesis, and $a \cdot e = a$ by the property of the identity element. Thus, $a = e$.
2. Let $a \in G$ and $b \in G$ be such that $a \cdot b = b \cdot a = e$. Let $c \in G$ be also such that $c \cdot a = e$. Thus, $(c \cdot a) \cdot b = e \cdot b \Rightarrow c \cdot (a \cdot b) = e \cdot b \Rightarrow c \cdot e = b \Rightarrow c = b$.
3. Easy to see since $a^{-1} \cdot a = a \cdot a^{-1} = e$ which just means that the inverse of a^{-1} is a.
4. Also easy since $(a \cdot b) \cdot (b^{-1} \cdot a^{-1}) = (b^{-1} \cdot a^{-1}) \cdot (a \cdot b) = e$.
5. Finally, right multiplying b^{-1} leads to $a = a \cdot b \cdot b^{-1} = b \cdot b^{-1} = e$.

■

July 29th.

Definition 1.21. The *order of a group* G is the cardinality of the set G, and is denoted by $|G|$, $o(G)$, or $\mathrm{ord}(G)$. If $|G|$ is finite, we say G is a *finite group*.

We provide some examples.

Example 1.22.

- The *trivial group* is $G = \{e\}$, with $e \cdot e = e$. Here, $|G| = 1$, and it is the smallest possible finite group. Similarly, one can form a group with two elements as $G = \{e, a\}$, with $a \cdot a = e$ and $a \cdot e = e \cdot a = a$.

- Another important example is the set of all bijections of a set X, denoted by $S(X)$. It forms a group under composition. Here, if $f, g \in S(X)$, then $f \circ g \in S(X)$. Similarly, the bijection $\mathrm{id}_X(x) = x$ for all $x \in X$ is the identity element of $S(X)$. Associativity also holds, and the inverse of $f \in S(X)$ is simply the inverse mapping $f^{-1} \in S(X)$ to get $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}_X$. If $X = \{1, 2, \ldots, n\}$, then $S(X)$ is also denoted by S_n, with $|S_n| = n!$. If the set X is infinite, then so is $S(X)$.
- The set $\mathbb{Z}/n\mathbb{Z}$ is a group when equipped with the binary operation of addition $(+)$. Here, $|\mathbb{Z}/n\mathbb{Z}| = n$.
- The set $\mu_n = \{e^{2\pi i m/n} \mid 1 \leq m \leq n\}$ is a group with respect to multiplication. Again, $|\mu_n| = n$.

Order is also defined for elements.

Definition 1.23. Let $(G, \cdot, e)$ be a group. The *order of an element* $a \in G$, denoted $o(a)$, $\mathrm{ord}(a)$, or $|a|$, is the least $n \geq 1$ such that $a^n = e$. If no such n exists, then we term $|a| = \infty$.

Examples follow.

Example 1.24.
- In μ_n, $o(e^{2\pi i/n}) = n$.
- Similarly, in $\mathbb{Z}/n\mathbb{Z}$, $o([1]_n) = n$. For a general element $[a]_n \in \mathbb{Z}/n\mathbb{Z}$, the order is $o([a]_n) = \frac{n}{\gcd(a,n)}$.

Proposition 1.25. *Let G be a finite group. For all $a \in G$, $o(a)$ is finite.*

Proof. Let $a \in G$. We look at $a, a^2, a^3, \ldots \in G$. Since G is finite, not all are distinct; there exists $m > n$ such that $a^m = a^n$. Multiplying by a^{-n}, we have $a^{m-n} = a^{n-n} = e$, and the order of a is finite. ■

1.2.1 The S_n Group

To understand the order better, we look specifically at S_3.

Example 1.26. The elements in S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \tag{1.3}$$

Alternatively, the elements may be (correspondingly) written as

$$e, \begin{pmatrix} 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \text{ and } \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}. \tag{1.4}$$

It is easy to see that the orders of $e, \begin{pmatrix} 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ are $1, 2, 3$, respectively. The elements $\begin{pmatrix} 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \end{pmatrix}$, and $\begin{pmatrix} 1 & 3 \end{pmatrix}$ are termed transpositions. In general, an element $\sigma \in S_n$ is called a *transposition* if there exists $1 \leq a \neq b \leq n$ such that $\sigma(a) = b$ and $\sigma(b) = a$, but $\sigma(x) = x$ for all $x \notin \{a, b\}$.
An element $\sigma \in S_n$ is called a *cycle* if there exists distinct $1 \leq a_1, a_2, \ldots, a_m \leq n$ such that $\sigma(a_i) = a_{i+1}$ for $1 \leq i \leq m-1$, $\sigma(a_m) = a_1$, and $\sigma(x) = x$ for all $x \notin \{a_1, a2, \ldots, a_m\}$. Thus, a transposition is really just a cycle of length 2. If σ is a cycle of length m, then $o(\sigma) = m$.
In the above, $\sigma^i(a_1) = a_{i+1}$ if $i < m$. Thus, $\sigma^i \neq e$ for $i < m$. But for m-times composition, we have $\sigma^m(a_i) = a_i$ for all $1 \leq i \leq m$. Hence, the order of σ is really m.

Note that S_3 is non-abelian since $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$, but $\begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.

Definition 1.27. Let $\sigma, \tau \in S_n$ be cycles. They are called *disjoint cycles* if $\sigma = (a_1, \ldots, a_m)$ and $\tau = (b_1, \ldots, b_k)$, and $\{a_1, \ldots, a_m\} \cap \{b_1, \ldots, b_k\} = \emptyset$.

If σ and τ are disjoint cycles then they commute; that is, $\sigma \circ \tau = \tau \circ \sigma$.

Proposition 1.28. *Every element of S_n can be written as a product of disjoint cycles.*

Proof. Let $\sigma \in S_n$, and let k be the least positive integer such that $\sigma^k(1) = 1$. Then let $\tau_1 = \begin{pmatrix}1 & \sigma(1) & \sigma^2(1) & \cdots & \sigma^{k-1}(1)\end{pmatrix}$. Let S_1' be the *support* of τ_1, defined as $\operatorname{supp}(\tau_1) = \{1, \sigma(1), \dots, \sigma^{k-1}(1)\}$. If $S_1' = \{1, 2, \dots, n\}$, we are done. Otherwise, let $a_2 = \min(\{1, 2, \dots, n\} \setminus S_1')$. Let k_2 be the least positive integer such that $\sigma^{k_2}(a_2) = a_2$, and then let $\tau_2 = \begin{pmatrix}a_2 & \sigma(a_2) & \cdots & \sigma^{k_2-1}(a_2)\end{pmatrix}$. Then τ_2 is a cycle of length of k_2. Again, let $S_2' = \operatorname{supp}(\tau_2)$. We claim that $S_1' \cap S_2' = \emptyset$.

If $\sigma(a_2)$ were in S_1', then we would have $\sigma^i(i) = a_2 \in S_1'$, but a_2 was taken from $\{1, 2, \dots, n\} \setminus S_1'$. Similarly, if $\sigma^j(a_2) \in S_1'$, then a similar problem arises. Thus, the sets have to be disjoint.

Continue this way to get $\tau_1, \tau_2, \dots, \tau_l$ until $S_1' \cup S_2' \cup \cdots \cup S_k' = \{1, 2, \dots, n\}$. The process stops since $S_1', S_2', \dots, S_k'$ are non-empty. Thus, we conclude that $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_l$ is the disjoint cycle decomposition of σ. ∎

For ease of notation, we will write $\sigma \circ \tau$ as $\sigma\tau$.

Proposition 1.29. *Let $\sigma \in S_n$ and $\sigma = \tau_1\tau_2 \cdots \tau_k$ be a disjoint cycle decomposition of σ. Then, $|\sigma| = \operatorname{lcm}(|\tau_1|, |\tau_2|, \dots, |\tau_k|)$.*

Proof. The proof of this proposition is left as an exercise to the reader. ∎

1.3 Subgroups

We begin with the definition.

Definition 1.30. A non-empty subset H of a group $(G, \cdot)$ is called a *subgroup* if the following properties hold.

1. For all $a, b \in H$, $a \cdot b \in H$.
2. For all $a \in H$, $a^{-1} \in H$.

In such a scenario, we write $H \leqslant G$.

More properties of a subgroup can be inferred.

Proposition 1.31. *The following properties hold true for a subgroup $H \leqslant G$, where $(G, \cdot, e)$ is a group.*

1. *$e \in G$.*
2. *$(H, \cdot, e)$ is a group.*

Proof.
1. H is non-empty, so there exists $a \in G$ such that $a \in H$. From the definition, $a^{-1} \in H$ also. Since H is closed under the binary operation, we have $a \cdot a^{-1} = e \in H$.
2. We show that $(H, \cdot, e)$ satisfies the group axioms. From definition, $\cdot$ is an associative binary operaion on H. Also, e is the identity element in H. Again, from the definition, each $a \in H$ has an inverse $a^{-1} \in H$.

∎

Equivalently, H is a subgroup if the following holds.

Theorem 1.32. *Let G be a group and $H \subseteq G$ be non-empty. Then H is a subgroup of G if and only if $a \cdot b^{-1} \in H$ for all $a, b \in H$.*

Proof. The forward implication is left as an exercise to the reader. If $a \in H$ then $a \cdot a^{-1} \in H$ shows that $e \in H$. Since $e, a \in H$, $e \cdot a^{-1} = a^{-1} \in H$. If $a, b \in H$, then $a, b^{-1} \in H \implies a \cdot (b^{-1})^{-1} \in H \implies ab \in H$ ■

July 31st.

We look at some examples of subgroups.

Example 1.33.
- For any group G, $\{e\} \subseteq G$ is a subgroup. This is termed the *trivial group*.
- Any group G is a subgroup of itself.
- We have $(\mathbb{Z}, +) \leqslant (\mathbb{Q}, +) \leqslant (\mathbb{R}, +) \leqslant (\mathbb{C}, +)$. Similarly, $(\{\pm 1\}, \cdot) \leqslant (\mathbb{Q}^*, \cdot) \leqslant (\mathbb{R}^*, \cdot) \leqslant (\mathbb{C}^*, \cdot)$.
- If $H \leqslant G$ and $K \leqslant H$, then $K \leqslant G$.
- $\mu_n \leqslant (\mathbb{C}^*, \cdot)$ for all natural n.
- For $({}^{\mathbb{Z}}/_{6\mathbb{Z}}, +) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, the only possible subgroups are $\{\bar{0}\}$, $\{\bar{0}, \bar{3}\}$, $\{\bar{0}, \bar{2}, \bar{4}\}$, and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

1.3.1 Generation

Definition 1.34. Let G be a group and $S \subseteq G$ be a subset. We say S generates a subgroup H if H is the smallest subgroup of G containing S. We denote this as $\langle S \rangle = H$.

Remark 1.35. Let $H_1, H_2 \leqslant G$. Then $H_1 \cap H_2 \leqslant G$.

Proof. Note that $e \in H_1, H_2$, so $H_1 \cap H_2 \neq \emptyset$. Also, if $x, y \in H_1 \cap H_2$, then $xy^{-1} \in H_1 \cap H_2$. We are done. ■

Proposition 1.36. *For $S \subseteq G$, $\langle S \rangle$ always exists and is unique.*

Proof. Let $\Omega = \{H \leqslant G \mid S \subseteq H\}$. Since $G \in \Omega$, it is non-empty. Thus, we simply take $\langle S \rangle = \cap_{H \in \Omega} H$, which is the smallest subgroup containing S. ■

The above proof is merely of existence, and will be a hassle for constructing the generated group. The following proposition simplifies the construction process.

Proposition 1.37. *Let G be a group and $S \subseteq G$ be a subset. Then*

$$\langle S \rangle = H = \{a_1 \cdots a_n \mid a_i \in S \text{ or } a_i^{-1} \in S \text{ for } n \geq 1\} \cup \{e\}. \tag{1.5}$$

Proof. Note that $S \subseteq H$, so H is non-empty. Let $x, y \in H$. Then, $x = a_1 \cdots a_n$ with $a_i \in S$ or $a_i^{-1} \in S$. Similarly, $y = b_1 \cdots b_m$ with $b_j \in S$ or $b_j^{-1} \in S$. We then have

$$a_1 \cdots a_n b_m^{-1} \cdots b_1^{-1} \text{ with } a_i \in S \text{ or } a_i^{-1} \in S, \text{ and } (b_j^{-1})^{-1} \in S \text{ or } b_j^{-1} \in S. \tag{1.6}$$

Thus, $xy^{-1} \in H$ and $\langle S \rangle \subseteq H$. For the converse inclusion, it is enough to show that if H' is a subgroup such that $S \subseteq H'$, then $H \leqslant H'$. Suppose H' is such a subgroup. Then $a_1 \cdots a_n \in H^{-1}$ for $a_i \in S$ or $a_i^{-1} \in S$ since $a_i \in S \subseteq H' \implies a_i^{-1} \in H'$ and $x, y \in H' \implies xy \in H$. Hence, $H \leqslant H'$. ■

Definition 1.38. A group G is termed a *cyclic group* if there exists $a \in G$ such that $\langle \{a\} \rangle = G$. Usually, we prefer to write it as $\langle a \rangle = G$.

Example 1.39. ${}^{\mathbb{Z}}/_{n\mathbb{Z}} = \langle \bar{1} \rangle$ for all natural n.

Proposition 1.40. *The group S_n is generated by transpositions, for all $n \geq 1$.*

Proof. From **Proposition 1.28**, every $\sigma \in S_n$ can be written as $\sigma = \tau_1 \cdots \tau_k$ where $\tau_i \in S_n$ are cycles. So it is enough to show that every cycle is a product of transpositions. Suppose $\begin{pmatrix} i_1 & i_2 & \cdots & i_l \end{pmatrix}$ is such a cycle with $i_1, \ldots, i_l$ begin distinct elements of $\{1, 2, \ldots, n\}$. This can be rewritten simply as

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_l \end{pmatrix} = \begin{pmatrix} i_1 & i_l \end{pmatrix} \begin{pmatrix} i_1 & i_{l-1} \end{pmatrix} \cdots \begin{pmatrix} i_1 & i_3 \end{pmatrix} \begin{pmatrix} i_1 & i_2 \end{pmatrix}. \tag{1.7}$$

■

Example 1.41. Let us look at $S_3 = \{e, \begin{pmatrix} 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}\}$. Then the only possible subgroups are

- $\{e\}$,
- $\{e, \begin{pmatrix} 1 & 2 \end{pmatrix}\}$,
- $\{e, \begin{pmatrix} 2 & 3 \end{pmatrix}\}$,
- $\{e, \begin{pmatrix} 1 & 3 \end{pmatrix}\}$,
- $\{e, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}\}$, and
- S_3.

An important subgroup of S_n is A_n, defined as the set of all permutations in S_n with even parity; all permutations that can be written as the product of even number of transpositions. A_n is termed the *alternating group*. Similarly, D_n is also defined. The *dihedral group* D_n is the group of symmetries of a n-regular polygon, which includes rotations and reflections. Labelling the vertices as 1 through n, the rotations and reflections can really be seen as those permutations in S_n that leave the n-regular polygon as itself after permutation. For example, D_4 is the group $\{\sigma \in S_4 \mid \sigma(\square) \text{ is still a } \square\}$. If n is even, we can write

$$D_n = \langle \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix}, \begin{pmatrix} 1 & n \end{pmatrix} \begin{pmatrix} 2 & n-1 \end{pmatrix} \cdots \begin{pmatrix} \frac{n}{2} & \frac{n}{2}+1 \end{pmatrix} \rangle. \tag{1.8}$$

If n is odd, we have

$$D_n = \langle \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix}, \begin{pmatrix} 1 & n-1 \end{pmatrix} \begin{pmatrix} 2 & n-2 \end{pmatrix} \cdots \begin{pmatrix} \frac{n-1}{2} & \frac{n+1}{2} \end{pmatrix} \rangle. \tag{1.9}$$

Chapter 2

COSETS AND MORPHISMS

2.1 Cosets

We start with cosets.

Definition 2.1. Let $H \leqslant G$ and $x \in G$. A *left coset* of H generated by x is $xH = \{xh \mid h \in H\} \subseteq G$. The left coset need not be a subgroup of G. Similarly, a *right coset* of H generated by x is $Hx = \{hx \mid h \in H\} \subseteq G$. Again, the right coset need not be a subgroup

Let $H \leqslant G$. For $x, y \in G$, let us write $x \sim y$ if $x^{-1}y \in H$. Then $\sim$ is an equivalence relation. Moreover, $[x] = xH$ for all $x \in G$. Once we have proved, we will be able to partition our group.

Proof. Clearly, $\sim$ is reflexive since $x^{-1}x = e \in H$ for all $x \in G$. $\sim$ is symmetric since we have

$$x \sim y \implies x^{-1}y \in H \implies (x^{-1}y)^{-1}H \implies y^{-1}x \in H \implies y \sim x. \tag{2.1}$$

Finally, $\sim$ is also transitive since

$$x \sim y \text{ and } y \sim z \implies x^{-1}y, y^{-1}z \in H \implies x^{-1}y \cdot y^{-1}z = x^{-1}z \in H \implies x \sim z. \tag{2.2}$$

To show the latter result, we first have

$$y \in [x] \implies x \sim y \implies x^{-1}y \in H \implies xx^{-1}y = y \in xH \implies y \in xH. \tag{2.3}$$

So, $[x] \subseteq xH$. For the converse inclusion, we have

$$y \in xH \implies y = xh \text{ for some } h \in H \implies x^{-1}y = h \in H \implies y \in [x]. \tag{2.4}$$

Thus, $xH \subseteq [x]$ and $xH = [x]$. ■

The above results of cosets prove to be useful in the following theorem.

Theorem 2.2 (*Lagrange's theorem*)**.** *Let G be a finite group with $H \leqslant G$. Then $|H| \mid |G|$.*

Proof. For $x, y \in G$, if $xH \cap yH \neq \emptyset$, then we must have $xH = yH$. Also, $\bigcup_{x \in G} xH = G$. We now claim that $|xH| = |yH|$ for all $x, y \in G$. To show this, we let $f : xH \to yH$ be defined as $f(a) = yx^{-1}a$, and $g : yH \to xH$ be defined as $g(b) = xy^{-1}b$. Then f and g are inverses of each other since

$$(f \circ g)(b) = f(xy^{-1}b) = yx^{-1}xy^{-1}b = b \text{ and } (g \circ f)(a) = g(yx^{-1}a) = xy^{-1}yx^{-1}a = a. \tag{2.5}$$

Let $S = G/\sim$ (also denoted as G/H). Since $G = \bigcup_{A \in S} A$, we have $|A| = |H|$ for all $A \in S$, implying $|G| = |S|\,|H|$. ■

Corollary 2.3. *Let G be a finite group, with $a \in G$. Then $o(a) \mid |G|$.*

Proof. If $o(a) = n$, then $\langle a \rangle = \{a, a,^2, \dots, a^{n-1}, e\}$. Since this is a subgroup, we have $|\langle a \rangle| = n \mid |G|$ by Lagrange's theorem. ■

2.2 Mappings

August 5th.

We now study important mappings between groups and the types of mappings one can define.

Definition 2.4. A function $f : (G, *) \to (H, \circ)$, where $(G, *)$ and $(H, \circ)$ are groups, is said to be a (group) *homomorphism* if

$$f(x * y) = f(x) \circ f(y) \text{ for all } x, y \in G. \tag{2.6}$$

The following is a trivial example of a group homomorphism.

Example 2.5. For instance, the map $a \mapsto a$ in $(\mathbb{Z}, +) \to (\mathbb{Q}, +)$ is a group homomorphism, trivially. More generally, if $H \leqslant G$, then $a \mapsto a$, called the *inclusion map* is a group homomorphism.

Homomorphisms can be classified further if they inherit nicer properties.

Definition 2.6. The group homomorphism is also called an injective homomorphism, or a *monomorphism*, if the mapping is also injective. Similarly, it is also called a surjective homomorphism, or a *epimorphism*, if the mapping is also surjective. Finally, the group homomorpshim is termed an *isomorphism* if it is bijective.

Example 2.7.

1. The map $q : (\mathbb{Z}, +) \to (\mathbb{Z}/n\mathbb{Z}, +)$ defined as $q(a) = [a]_n$ for $n \geq 1$ is a group homomorphism. Specifically, it is an epimorphism.
2. $f : (G, *) \to (\{e\}, \cdot)$ with $f(g) = e$ for all $g \in G$ is another epimorphism. This is also a trivial homomorphism.
3. The scaling map $a \mapsto \lambda a$ in $\mathbb{Z} \to \mathbb{Z}$ is a monomorphism for $\lambda \in Z_{\geq 1}$. Similarly, $[a] \mapsto [\lambda a]$ in $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is group homomorphism. If $\gcd(n, \lambda) = 1$, then the map is also an isomorphism in this case.
4. The scaling map $f : \mathbb{Q} \to \mathbb{Q}$ with $f(a) = ca$ with $c \in \mathbb{Q}^*$ is an isomorphism. For $c = 0$, we get the trivial homomorphism.
5. From linear algebra, the map $T : (\mathbb{Q}^n, +) \to (\mathbb{Q}^n, +)$ with $T \in M_n(\mathbb{Q})$ defined as $v \mapsto Tv$ is also a group homomorphism. If $T \in GL_n(\mathbb{Q}) \subseteq M_n(\mathbb{Q})$, the map is also an isomorphism.
6. Towards more non-trivial examples, one can confirm that the map $\exp : (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$ defined as $x \mapsto e^x$ is a group homomorphism.

2.2.1 Properties

Arising from these structure-preserving mappings are some useful properties.

Proposition 2.8. *Let $f : (G, *) \to (H, \circ)$ be a group homomorphism. Then*

1. $f(e_G) = e_H$,
2. $f(a^n) = f(a)^n$, *and*
3. $f(a)^{-1} = f(a^{-1})$.

Proof. 1. Simply work as

$$f(e_G) = f(e_G * e_G) = f(e_G) \circ f(e_G) \implies e_H = f(e_G)^{-1} \circ f(e_G) = f_{e_G}. \tag{2.7}$$

2. We show the base case, then induction may be applied.

$$f(a^2) = f(a * a) = f(a) \circ f(a) = f(a)^2. \tag{2.8}$$

3. Again,

$$f(a^{-1}) \circ f(a) = f(a^{-1} * a) = f(e_G) = e_H = f(a)^{-1} \circ f(a) \implies f(a^{-1}) = f(a)^{-1}. \tag{2.9}$$

■

We show further some properties of bijective homomorphisms.

Proposition 2.9. *Let $f : (G, *) \to (H, \cdot)$ be a group isomorphism. Then*

1. *$f^{-1} : H \to G$ is a group isomorphism,*
2. *$o(x) = o(f(x))$ for all $x \in G$,*
3. *$|G| = |H|$, and*
4. *G is abelian if and only if H is abelian.*

Proof. 1. Fix $a, b \in H$, and let $x = f^{-1}(a)$ and $y \in f^{-1}(b)$. We want to show that $f^{-1}(a \cdot b) = f^{-1}(a) * f^{-1}(b)$. To this end, we have

$$f(f^{-1}(a) * f^{-1}(b)) = f(x * y) \Rightarrow f(f^{-1}(a)) \cdot f(f^{-1}(b)) = a \cdot b = f(x * y) \Rightarrow f^{-1}(a \cdot b) = x * y. \tag{2.10}$$

2. Let $o(x) = n$, where $x^m \neq e_G$ for $1 \leq m < n$ and $x^n = e_G$. This shows that $f(x)^n = f(x^n) = e_H$. Also, since $x^m \neq e_G$ for $1 \leq m < n$, we must have $f(x^m) \neq e_H$ for $1 \leq m < n$ as f is bijective. Thus, $o(f(x)) = o(x)$. If $o(x)$ were not finite, then $x^n \neq e_G$ for all $n \geq 1$ implies $f(x)^n \neq e_H$ for all $n \geq 1$.
3. This is trivial.
4. If G is abelian then $a * b = b * a$ for all $a, b \in G$. Applying f, we get $f(a * b) = f(b * a) \Rightarrow f(a) \cdot f(b) = f(b) \cdot f(a)$ for all $a, b \in G$. If we take $a = f^{-1}(x)$ and $b = f^{-1}(y)$, we get $x \cdot y = y \cdot x$ for all $x, y \in H$. For the converse implication, simply consider the isomorphism f^{-1}.

■

Essentially, in group theory, we consider two groups the same if they are isomorphic. Thus, we are equipped to classify groups up to isomorphism seeing as they share basically the same struture and properties. If two groups G and H are isomorphic, we denote it as $G \cong H$.

Proposition 2.10. *Let $(G, \cdot)$ be a group of order p, where p is prime. Then $G \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. Let $x \in G$ be a non-identity element. Then $o(x) = p$ since $o(x) \mid p$ and $o(x) \neq 1$. Define the map $f : \mathbb{Z}/n\mathbb{Z} \to G$ as $f(a) = x^a$. We show that this mapping is an isomorphism. For $\overline{a}, \overline{b} \in \mathbb{Z}/n\mathbb{Z}$, we have

$$f(\overline{a} + \overline{b}) = f(\overline{a + b}) = x^{a+b} = x^a \cdot x^b = f(\overline{a}) \cdot f(\overline{b}) \tag{2.11}$$

showing f is a group homomorphism. Moreover, $G = \langle x \rangle$ as $o(x) = p$, so G is also surjective. Hence, f is an isomorphism as G is finite. ■

Example 2.11. We find all the groups of order 4 upto isomorphism. The only two possibilities are $\mathbb{Z}/4\mathbb{Z}$, and $(\mathbb{Z}/2\mathbb{Z})^2$ with component-wise addition.

Example 2.12. We list down all the groups of order 6 upto isomorphism. Again, the only two possibilities are $\mathbb{Z}/6\mathbb{Z}$ and S_3.

August 7th.

Example 2.13. For groups of order 8, we have $\mathbb{Z}/8\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_4, and Q_8, the quaternions.

The *quaternions* Q_8 is the group $\{\pm 1, \pm i, \pm j, \pm k\}$ equipped with the multiplication operation such that

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1, \tag{2.12}$$
$$ij = k, \quad jk = i, \quad ki = j, \tag{2.13}$$
$$ji = -k, \quad kj = -i, \quad ik = -j. \tag{2.14}$$

2.2.2 Kernel and Image

Definition 2.14. For a group homomorphism $f : G \to H$, we define the *kernel* of f as $\ker f = \{g \in G \mid f(g) = e_H\}$. We also define the *image* of f as $f(G) = \operatorname{Im} f = \{f(g) \mid g \in G\}$.

The image and kernel are both subgroups; this is our proposition.

Proposition 2.15. *For a group homomorphism $f : G \to H$, $\ker f$ and $\operatorname{Im} f$ are subgroups of G and H respectively.*

Proof. Note that $\emptyset \neq \operatorname{Im} f \subseteq H$; let $a, b \in \operatorname{Im} f$. Then $f(x) = a$ and $f(y) = b$ for some $x, y \in G$. So, $ab = f(x)f(y) = f(xy) \in \operatorname{Im} f$, showing $ab \in \operatorname{Im} f$. Also, $a^{-1} = f(x)^{-1} = f(x^{-1}) \in \operatorname{Im} f$, showing $a^{-1} \in \operatorname{Im} f$. Thus, $\operatorname{Im} f \leqslant H$.

For the kernel, note that $e_g \in \ker f$ since $f(e_G) = e_H$. Let $x, y \in \ker f$. Then $f(x) = f(y) = e_H$ implying $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$. Thus, $xy^{-1} \in \ker f$, showing $\ker f \leqslant H$. ■

Remark 2.16. Let f be a group homomorphism.

1. If f is an isomorphism, then $\operatorname{Im} f = H$ and $\ker f = \{e_G\}$.
2. If f is a monomorphism, then $\ker f = \{e_G\}$.
3. If f is an epimorphism, then $\operatorname{Im} f = H$.

2.3 Normal Subgroups and Quotient Groups

Proposition 2.17. *Let G be a group and $H \leqslant G$ be a subgroup. Then the following are equivalent.*

1. *$gH \subseteq Hg$ for all $g \in G$,*
2. *$g^{-1}Hg \subseteq H$ for all $g \in G$,*
3. *$gH = Hg$ for all $g \in G$,*
4. *$g^{-1}Hg = H$ for all $g \in G$.*

Such a subgroup satisfying any (all) of the above conditions is termed a *normal subgroup* of G and is denoted by $H \trianglelefteq G$.

Proof. For 1. implies 2., we are given $g^{-1}H \subseteq Hg^{-1}$ for all $g \in G$. Let $x \in g^{-1}Hg$. Then $x = g^{-1}hg$ for some $h \in H$. Thus, $g^{-1}h \in g^{-1}H \subseteq Hg^{-1}$ which implies $g^{-1}h = h'g^{-1}$ for some $h' \in H$. But then $g^{-1}hg = h' \in H$, showing $x \in H$. Therefore, $g^{-1}Hg \subseteq H$.

For 2. implies 3., assume $g^{-1}Hg \subseteq H$ for all $g \in G$. Let $x \in gH$, that is, $x = gh$ for some $h \in H$. Write this as $x = ghg^{-1}g$. But $ghg^{-1} \in gHg^{-1} \subseteq H$, so $ghg^{-1} = h'$ for some $h' \in H$. Thus, $x = h'g \in Hg$. Similarly, if $x \in Hg$, then $x \in gH$. We conclude that $Hg = gH$ for all $g \in G$.

For 3. implies 4., we have $gH = Hg$ for all $g \in G$. Let $x \in g^{-1}Hg$, where $x = g^{-1}hg$ for some $h \in H$. Note that $hg = gh'$ for some $h' \in H$ since $gH = Hg$. Thus, $x = g^{-1}hg = g^{-1}(gh') = h' \in H$, giving us $g^{-1}Hg \subseteq H$.

Finally, for 4. implies 1., let $x \in gh$; there exists $h \in H$ such that $x = gh$. Thus, $x = ghg^{-1}g = h'g \in Hg$ since $gHg^{-1} = H$. Hence, $gH \subseteq Hg$. ■

Note that if G is abelian, then every subgroup is normal.

Proposition 2.18. *The following miscellaneous propositions hold true. Let G be a group.*

1. *If $g, h \in G$, then $\operatorname{ord}(ghg^{-1}) = \operatorname{ord}(h)$.*
2. *The mapping $\varphi_g : G \to G$ defined as $\varphi_h(h) = g^{-1}hg$ is an isomorphism for all $g \in G$. The inverse isomorphism is given by $\varphi_g^{-1} = \varphi_{g^{-1}}$.*
3. *Both G and $\{e\}$ are normal subgroups of G.*

Proof. The proofs of these are left as an exercise to the reader. ■

Proposition 2.19. *Let $f : G \to H$ be a group homomorphism. Then $\ker f \trianglelefteq G$.*

Proof. For $g \in G$, let $x \in g^{-1}\ker(f)g$; that is, $x = g^{-1}hg$ for some $h \in \ker f$. Then,

$$f(x) = f(g^{-1}hg) = f(g^{-1})f(h)f(g) = f(g)^{-1}e_H f(g) = e_H. \tag{2.15}$$

Thus, $x \in \ker f$, showing $g^{-1}\ker(f)g \subseteq \ker f$ for all $g \in G$; $\ker f \trianglelefteq G$. ■

Proposition 2.20. *Let G be a group with $H \leqslant G$ a subgroup. Then $H \trianglelefteq G$ if and only if for all $\varphi_g : G \to G$, we have $\varphi_g|_H : H \to H$, an isomorphism.*

Note that an isomorphism from a group to itself is called an *automorphism*. Thus, the above proposition equates to φ_g still remaning an automorphism when restricted to H.

Proof. The statement is simply equivalent to saying $g^{-1}Hg = H$ for all $g \in G$. ■

One also defines the notion of *product of groups*. Let G_1, G_2 be two groups. Then

$$G_1 \times G_2 := \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\} \tag{2.16}$$

is a group with the equipped operation defined as

$$(g_1, g_2) \cdot (g_1', g_2') = (g_1g_1', g_2g_2'). \tag{2.17}$$

Here, the identity element is (e_{G_1}, e_{G_2}) and the inverse of (g_1, g_2) is (g_1^{-1}, g_2^{-1}).

Let G be a group and H, K be subgroups of G. Let $HK = \{hk \mid h \in H, k \in K\}$. Then HK is a group if either H or K is a normal subgroup of G.

Proof. Let us assume $H \trianglelefteq G$. Take the elements $h_1k_1, h_2k_2 \in HK$. Since $H \trianglelefteq G$, $k_1H = Hk_1$. So, $k_1h_2 = h'k_1$ for some $h' \in H$ Thus,

$$h_1k_1h_2k_2 = h_1h'k_1k_2 \in HK. \tag{2.18}$$

Similarly, $(h_1k_1)^{-1} = k_1^{-1}h_1^{-1} = h'k_1^{-1} \in HK$ for some $h' \in H$, since $H \trianglelefteq G$ and $k_1^{-1}H = Hk_1^{-1}$. ■

August 12th.

We now get familiar with quotient groups.

Definition 2.21. Let G be a group and $H \trianglelefteq G$ a normal subgroup. The *quotient group* is defined as $G/H = \{gH \mid g \in G\}$ with the operation defined as $gH * g'H := (gg')H$ for all $g, g' \in G$.

Of course, it still remains to verify that the groups axioms are not violated and the operation is indeed well-defined.

Proof. Let $gH = kH$ and $g'H = k'H$ for $k, k' \in G$. We wish to show that $gg'H = kk'H$. Since $gH = kH$, we have $k^{-1}g \in H$. Similarly, $k'^{-1}g' \in H$, and $k'^{-1}g'(k^{-1}g) \in H$. Thus, $(kg')^{-1}gg' \in H$ and $gg'H = kg'H$. Hence, the operation is well-defined. We verify the group axioms now.

1. *Associativity:* We have

$$(gH * hH) * (kH) = (ghH) * kH = (gh)kH = g(hk)H = gH * (hkH) = (gH) * (gH * kH). \tag{2.19}$$

2. *Existence of Identity:* The identity here is $e_{G/H} = H$ since

$$gH * H = (ge)H = gH = (eg)H = H * gH. \tag{2.20}$$

3. *Existence of Inverse:* For $gH \in G/H$, we have $(gH)^{-1} = g^{-1}H$ since

$$(gH) * (g^{-1}H) = (gg^{-1})H = H = (g^{-1}g)H = (g^{-1}H) * (gH). \tag{2.21}$$

■

Note that the map $q : G \to G/H$ defined as $g \mapsto gH$ is a group epimorphism, with $\ker q = H$. The proof of showing surjectivity and preservation of group structure is left as an exercise the reader.

Example 2.22.

- As a familiar example, $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ is a normal subgroup, and the quotient group $\mathbb{Z}/n\mathbb{Z}$ is the group of integers modulo n.
- If one sets $H = \{e\} \trianglelefteq G$, then $G/H = \{\{g\} \mid g \in G\}$ is the group of singletons, and the quotient map $q : G \to G/H$ becomes an isomorphism.
- If $H = G \trianglelefteq G$, then $G/H = \{G\}$ is the trivial group.

2.3.1 Centre

Definition 2.23. The *centre of a group* G, denoted $Z(G)$, is the set of all elements in G that commute with every element of G:

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}. \tag{2.22}$$

One can show that $Z(G) \leqslant G$ always holds true. In fact, the centre is a subgroup of G.

Example 2.24.

- Since $Z(G)$ is a normal subgroup, the quotient group makes sense. However, in general, $Z(G/Z(G))$ is not trivial.
- G is abelian if and only if $Z(G) = G$.
- $Z(GL_2(\mathbb{R})) = \{\lambda I \mid \lambda \in \mathbb{R}^*\}$, where I is the identity matrix.

One can also define a centre for individual elements and subsets in a group.

Definition 2.25. The *centre of a subset* $H \subseteq G$ is defined as

$$C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}. \tag{2.23}$$

Note that $Z(G) \subseteq C_G(H)$ and $C_G(G) = Z(G)$. For $g \in G$, we define the *centralizer* of g as

$C_G(g) := C_G(\{g\})$. Additionally, if $H \leqslant G$, then

$$N_G(H) = \{g \in G \mid gH = Hg\} \tag{2.24}$$

is termed the *normalizer* of H in G.

Remark 2.26. The following may be shown, for a subset $H \subseteq G$.

- $C_G(H) = \bigcap_{g \in H} C_G(g)$.
- $C_G(H) \leqslant G$ holds.

The following may be shown, for a subgroup $H \leqslant G$.

- $C_G(H) \subseteq N_G(H)$ holds.
- $H \trianglelefteq N_G(H)$ holds.
- $N_G(H) \leqslant G$ holds.

The proofs of the above are left as an exercise to the reader.

2.4 The Isomorphism Theorems

These are important theorems that hold regarding isomorphisms. In particular, they describe the relationships between different quotient groups and subgroups. Before we encounter the actual theorems, we establish a minor result.

Proposition 2.27. *Let $f : G \to H$ be a group homomorphism and let $K = \ker f$. Then $K \trianglelefteq G$. Moreover, $K = \{e\}$ if and only if f is injective.*

Proof. The first part follows from the fact that $g^{-1}Kg \subseteq K$ for all $g \in G$. For the second part, if f is injective, then $\ker f = \{e\}$ since $f(g) = e_H$ implies $g = e_G$. Conversely, suppose $K = \{e\}$ and let $f(g) = f(g')$ for some $g, g' \in G$. Then

$$f(g^{-1}g') = f(g^{-1})f(g') = f(g)^{-1}f(g) = e_H \implies g^{-1}g' = e_G \implies g' = g. \tag{2.25}$$

■

2.4.1 First Isomorphism Theorem

Theorem 2.28 (The *first isomorphism theorem*)**.** *Let $f : G \to H$ be a group homomorphism. Then the map $\tilde{f} : G/K \to \operatorname{Im} f$ sending $gK \mapsto f(g)$ is a well-defined isomorphism where $K = \ker f$. Bluntly,*

$$G/\ker f \cong \operatorname{Im} f. \tag{2.26}$$

Proof. We first show that $\tilde{f}$ is well-defined. Suppose $gK = g'K$ for some $g, g' \in G$. Then

$$g^{-1}g' \in K \implies f(g^{-1}g') = e_H \implies f(g) = f(g'). \tag{2.27}$$

To show $\tilde{f}$ is a homomorphism, let $aK, bK \in G/K$. Then

$$\tilde{f}(aK \cdot bK) = \tilde{f}((ab)K) = f(ab) = f(a)f(b) = \tilde{f}(aK) \cdot \tilde{f}(bK). \tag{2.28}$$

Finally, we show $\tilde{f}$ is bijective. Let $h \in \operatorname{Im} f$. Then there exists $g \in G$ such that $f(g) = h$. We claim that $\tilde{f}(gK) = h$. Indeed,

$$\tilde{f}(gK) = f(g) = h. \tag{2.29}$$

Thus, $\tilde{f}$ is surjective. To show injectivity, suppose $\tilde{f}(gK) = \tilde{f}(g'K)$ for some $g, g' \in G$. Then

$$f(g) = f(g') \implies g^{-1}g' \in K \implies gK = g'K. \tag{2.30}$$

Therefore, $\tilde{f}$ is injective. We conclude that $\tilde{f}$ is a bijection. ■

August 14th.

Example 2.29. We discuss the *Heisenberg group*

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}. \tag{2.31}$$

The group operation here is matrix multiplication, and one can see that $H \leqslant SL_3(\mathbb{R})$. Here, the center of H is precisely

$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}. \tag{2.32}$$

If we look at the map $f : H \to \mathbb{R}^2$ that sends

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, c) \tag{2.33}$$

then f is a group homomorphism since

$$f(AA') = (a + a', c + c') = (a, c) + (a', c') = f(A) + f(A'). \tag{2.34}$$

Moreover, $\operatorname{Im} f = \mathbb{R}^2$ and $\ker f = Z(H)$. Hence, by the first isomorphism theorem, we have

$$H/Z(H) \cong \mathbb{R}^2 \tag{2.35}$$

with the map $\tilde{f} : H/Z(H) \to \mathbb{R}^2$ given by $\tilde{f}(AZ(H)) = f(A)$. Note that $\mathbb{R}^2$ is an abelian group, and so is $H/Z(H)$. Hence, $Z(H/Z(H)) = H/Z(H)$.

Commutator Subgroup

Definition 2.30. Let G be a group. Then

$$(G : G) := \langle \{ghg^{-1}h^{-1} \mid g, h \in G\} \rangle \tag{2.36}$$

is termed the *commutator subgroup* of G.

Of course, the name suggests that $(G : G)$ is a subgroup of G. In fact, it is actually a normal subgroup of G.

Proposition 2.31. *$(G : G)$ is a normal subgroup of G. Moreover $G/(G : G)$ is abelian. Let $f : G \to A$ be a group homomorphism with an abelian group A. Then $(G : G) \subseteq \ker f$ and there exists $\bar{f} : G/(G : G) \to A$ such that $f = \bar{f} \circ q$ where $q : G \to G/(G : G)$ is the quotient map.*

Proof. Let $x \in G$. Then we have

$$\begin{aligned} x^{-1}(ghg^{-1}h^{-1})x &= x^{-1}gxx^{-1}hxx^{-1}g^{-1}xx^{-1}h^{-1}x = (x^{-1}gx)(x^{-1}hx)(x^{-1}g^{-1}x)(x^{-1}h^{-1}x) \\ &= aba^{-1}b^{-1} \in (G : G) \text{ where } a = x^{-1}gx \text{ and } b = x^{-1}hx. \end{aligned} \tag{2.37}$$

Thus, if $S = \{ghg^{-1}h^{-1} \mid g,h \in G\}$, then $x^{-1}Sx \subseteq (G:G)$ for all $x \in G$. We now show that $(G:G)$ is a subgroup. Let $a \in (G:G)$. Then $a = b_1 \cdots b_n$, where $b_i \in S$, and

$$x^{-1}ax = x^{-1}b_1xx^{-1}b_2x\cdots x^{-1}b_nx \in (G:G) \tag{2.38}$$

since $x^{-1}b_ix \in (G:G)$ for all i. Thus, $x^{-1}(G:G)x \subseteq (G:G)$ for all $x \in G$, showing $(G:G) \trianglelefteq G$. Hereforth, in this example, let $C = (G:G)$. We now show G/C is abelian. This is simple enough since $ghg^{-1}h^{-1} = gh(hg)^{-1} \in C$ implies $gChC = ghC = hgC = hCgC$.

Now let $f : G \to A$ be a group homomorphism with A an abelian group. We then have

$$f(ghg^{-1}h^{-1}) = f(g)f(h)f(g^{-1})f(h^{-1}) = f(g)f(h)f(g)^{-1}f(h)^{-1} = e_A. \tag{2.39}$$

Thus, $S \subseteq \ker f$ implying $(G:G) \subseteq \ker f$. Finally, we show that $\bar{f} : G/C \to A$ defined as $gC \mapsto f(g)$ is an isomorphism. To show $\bar{f}$ is well-defined, we have $gC = hC \Leftrightarrow gh^{-1} \in C \subseteq \ker f$ showing $f(gh^{-1}) = e_A$ or $f(g) = f(h)$. To show a homomorphism, for $g,h \in G$, we have

$$\bar{f}(gChC) = f(gh) = f(g)f(h) = \bar{f}(gC)\bar{f}(hC). \tag{2.40}$$

Moreover, $\bar{f} \circ q(g) = \bar{f}(gC) = f(g)$ for all $g \in G$, so $\bar{f} \circ q = f$. ■

Remark 2.32. A few corollaries, we have

- If G is abelian then $(G:G) = \{e\}$.
- For H, the Heisenberg group, we have $(H:H) = Z(H)$.
- $(G/(G:G) : G/(G:G)) = \{e\}$.
- If $H \leqslant G$ then $|H| \mid |G|$. The *index* of H in G is defined as $[G:H] = |G/H|$. It is also equal to $\frac{|G|}{|H|}$ if $|G|$ is finite.

2.4.2 Second and Third Isomorphism Theorems

Theorem 2.33 (The *second ismorphism theorem*)**.** *Let G be a group and $A, B \leqslant G$ be subgroups such that $A \leqslant N_G(B)$. Then $AB \leqslant G$, $B \trianglelefteq AB$, and $A \cap B \trianglelefteq A$. Moreover,*

$$AB/B \cong A/(A \cap B). \tag{2.41}$$

Proof. Firstly, we have $A \leqslant N_G(B)$ and $B \trianglelefteq N_G(B) = \{g \in G \mid gB = Bg\}$. Thus, $AB \leqslant N_G(B)$. Of course, this also means $AB \leqslant G$. Also, $B \trianglelefteq AB$ since $B \trianglelefteq N_G(B)$ and $AB \subseteq N_G(B)$.

We now define a map $f : A \to AB/B$ that maps $a \mapsto aB$ since $a \in A \subseteq AB$. f is a group homomorphism. Thus, by the first isomorphism theorem,

$$\bar{f} : A/\ker f \to \operatorname{Im} f \tag{2.42}$$

is an isomorphism. It is enough to show that $\ker f = A \cap B$ and f is surjective. Again, simple to see since

$$\ker f = \{a \in A \mid aB = B\} = \{a \in A \mid a \in B\} = A \cap B; \tag{2.43}$$

for surjectivity, let $x \in AB/B$. Then $x = abB$ for some $a \in A$ and $b \in B$. But abB is simply aB so we simply have $abB = aB = f(a)$. ■

Corollary 2.34. *For G a group with $A, B \leqslant G$, we have $[AB : B] = [A : A \cap B]$.*

Finally, we move on to the third theorem.

Theorem 2.35 (The *third isomorphism theorem*)**.** *Let G be a group and let $H, K \trianglelefteq G$ such that*

$K \leqslant H$. Then $H/K \trianglelefteq G/K$ and

$$\frac{G/K}{H/K} \cong G/H. \tag{2.44}$$

Proof. We define a map $f : G/K \to G/H$ via $gK \mapsto gH$. This is well-defined since if $gK = g'K$, then $g^{-1}g' \in K \subseteq H$ and hence $gH = g'H$. We now show that f is a group homomorphism. For $gK, hK \in G/K$, we have

$$f(gKhK) = f(ghK) = ghH = gHhH = f(gK)f(hK). \tag{2.45}$$

f is also surjective as $gH = f(gK)$ for all $g \in G$. By the first isormorphism theorem, we have

$$\frac{G/K}{\ker f} \cong \operatorname{Im} f = G/H \tag{2.46}$$

where the isomorphism is given by the map $gK \ker f \mapsto gH$. Since $\ker f = \{gK \mid gH = H\} = \{gK \mid g \in H\} = H/K$, our proof is complete. ■

Chapter 3

GROUP ACTIONS

3.1 An Overview

Let G be a group and S be a set. A *(left) group action* or G-action on S is a function $\theta : G \times S \to S$ satisfying

1. $\theta(g_1 g_2, x) = \theta(g_1, \theta(g_2, x))$ for all $g_1, g_2 \in G$ and $x \in S$.
2. $\theta(e_G, x) = x$ for all $x \in S$.

In practice, we prefer to write θ simply as $(g, x) \mapsto gx$. Thus, the axioms are simply $g_1(g_2 x) = (g_1 g_2)x$ and $ex = x$ for all $g_1, g_2 \in G$ and $x \in S$. Unless there are multiple different group actions on S in context, we will stick with this notation instead. In this case, S is called a G-set.

Remark 3.1.

- Let $\varphi_g : S \to S$ sending $x \mapsto gx$ for all $g \in G$ and $x \in S$. Then φ_g is a bijection for each $g \in G$. This can be shown simply by considering the maps $\varphi_{g^{-1}} \circ \varphi_g$.
- Let G be a group and X be a G-set. Then the map $G \to \mathrm{Bij}(X)$ given by $g \mapsto \varphi_g$ is a group homomorphism. Here, $\mathrm{Bij}(X)$ denotes the set (group) of bijections from X to itself.

Index