# ANALYSIS OF SEVERAL VARIABLES

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# List of Symbols

Placeholder

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Chapter 1

# INTRODUCTION TO $\mathbb{R}^n$

## 1.1 Placeholder

We begin with a definition.

**Definition 1.1.** The space $\mathbb{R}^n$ is defined as $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ ($n$ times) $= \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$.

In the context of analysis, we will talk about open sets, closed sets, sequences, compact sets, and connected sets. In contrast, algebra considers $\mathbb{R}^n$ as a vector space with the operators $+$ and $\cdot$. Combining both these aspects results in the study of analysis of several variables. In this course, we will mainly focus on dealing with functions of the form $f : \mathbb{R}^n \to \mathbb{R}^m$, and talk about their properties such as continuity, differentiability, and integrability.

### 1.1.1 Algebraic and Analytic Structure

We note that $\mathbb{R}^n$ is also an inner product space with the following properties:

- $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ for all $x, y \in \mathbb{R}^n$.
- The set $\{e_i\}_{i=1}^{n}$ consisting of the unit vectors is an orthonormal basis for $\mathbb{R}^n$.
- The simplest maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are linear maps that send lines to lines.

**Example 1.2.** Suppose the function $f$ is a linear map from $\mathbb{R}$ to $\mathbb{R}$. This implies that $f(x) = xf(1)$ for all $x \in \mathbb{R}$. Thus, $f(x) = cx$ for all $x \in \mathbb{R}$, where $c$ is a constant. Conversely, if $c \in \mathbb{R}$, then $x \mapsto cx$ is a linear map. Therefore, we conclude that $\{f : \mathbb{R} \to \mathbb{R}, \text{linear}\} \leftrightarrow \mathbb{R}$, with a possible bijection given by $f \mapsto f(1)$.

In the above example, we note that 1 is not special; we could simply fix any $\alpha \in \mathbb{R} \setminus \{0\}$, and notice that $f(x) = \frac{x}{\alpha} f(\alpha)$ for all $x \in \mathbb{R}$. Here, replacing $f(1)$ by $f(\alpha)$ is a kind of 'change of variable'.

**Remark 1.3.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then, $Le_j = \sum_{i=1}^{m} a_{ij} e_i$ for all $j = 1, 2, \ldots, n$. We may write $L$ as $(a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{R})$, the set of all $m \times n$ matrices with real entries.

Coming to the analysis side, there is a need for defining a distance between points in $\mathbb{R}^n$. Previously, we have seen that the *norm* may be given as $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ for all $x \in \mathbb{R}^n$. We can use this norm to define our required distance function.

**Definition 1.4.** The *distance* function between two points $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as $d(x, y) = \|x - y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$ for all $x, y \in \mathbb{R}^n$.

For $n = 1$, we note that $d(x, y) = |x - y|$, from the previous analysis courses. $\mathbb{R}^n$ equipped with the function $d$ is called a *metric space*. Coming to the properties of the inner product, we have

- $\|x\| = \langle x, x\rangle^{1/2}$ for all $x \in \mathbb{R}^n$.
- $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathbb{R}^n$.
- The function $\langle , \rangle$ is linear with respect to the first and second arguments.

We also have the important Cauchy-Schwarz inequality.

**Theorem 1.5** (*Cauchy-Schwarz inequality*)**.** *For all $x, y \in \mathbb{R}^n$, we have $|\langle x, y\rangle| \leq \|x\| \, \|y\|$.*

*Proof.* Note that

$$0 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i y_j - x_j y_i)^2 = 2\left(\sum_{i,j} x_i^2 y_j^2 - \sum_{i,j} x_i x_j y_i y_j\right) = 2\left(\|x\|^2 \|y\|^2 - \langle x, y\rangle^2\right) \tag{1.1}$$

$$\implies |\langle x, y\rangle| \leq \|x\| \, \|y\| . \tag{1.2}$$

■

We note that equality occurs if and only if the first quantity in the above equation is zero, *i.e.*, if and only if $x_i y_j = x_j y_i$ for all $i, j$, or $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ for all $i, j$ showing that $x$ and $y$ are linearly dependent.

**Corollary 1.6** (*Triangle inequality*)**.** *For all $x, y \in \mathbb{R}^n$, we have $\|x + y\| \leq \|x\| + \|y\|$.*

*Proof.* We have

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \tag{1.3}$$

where the inequality follows from Cauchy-Schwarz. ■

The following will prove to be an important result.

**Theorem 1.7.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then, there exists a $M > 0$ such that $\|Lx\| \leq M \|x\|$ for all $x \in \mathbb{R}^n$.*

*Proof.* Rewriting $x$ as $x = \sum_{i=1}^{n} x_i e_i$, we have

$$Lx = \sum_{i=1}^{n} x_i L e_i$$

$$\implies \|Lx\| = \left\|\sum_{i=1}^{n} x_i L e_i\right\| \leq \sum_{i=1}^{n} |x_i| \, \|Le_i\| \leq \|x\| \sum_{i=1}^{n} \|Le_i\| = \|x\| \, M. \tag{1.4}$$

The first inequality follows from the triangle inequality, and the second from Cauchy-Schwarz. In the last step, $M$ is set to be $\sum_{i=1}^{n} \|Le_i\|$, which is a constant. ■

We also term $(\mathbb{R}^n, d)$ as a Euclidean metric space. There is now a need to define open sets in $\mathbb{R}^n$ to talk more about the analysis of several variables.

**Definition 1.8.** For $a \in \mathbb{R}^n$ and $r > 0$, the *open ball* centred at $a$ of radius $r$ is $B_r(a) := \{x \in \mathbb{R}^n : d(x, a) < r\}$, the set of all points in $\mathbb{R}^n$ that are at a distance less than $r$ from $a$.

From the notion of open balls, we can define open sets.

**Definition 1.9.** A set $S \subseteq \mathbb{R}^n$ is said to be an *open set* if for all $x \in S$, there exists an $r > 0$ such that $B_r(x) \subseteq S$.

We now bring the notion of convergence of sequences.

**Definition 1.10.** Let $\{x_m\} \subseteq \mathbb{R}^n$ be a sequence and $x \in \mathbb{R}^n$. We say that $\{x_m\}$ *converges* to $x$ if for every $\varepsilon > 0$, there exists a natural $N$ such that $\|x_m - x\| < \varepsilon$ for all $m \geq N$.

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