ANALYSIS OF SEVERAL VARIABLES

Jaydeb Sarkar, notes by Ramdas Singh

Third Semester

List of Symbols

Placeholder

Contents

Chapter 1

INTRODUCTION TO $\mathbb{R}^n$

1.1 Translation into Higher Dimensions

July 21st.

We begin with a definition.

Definition 1.1. The space $\mathbb{R}^n$ is defined as $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) $= \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$.

In the context of analysis, we will talk about open sets, closed sets, sequences, compact sets, and connected sets. In contrast, algebra considers $\mathbb{R}^n$ as a vector space with the operators $+$ and $\cdot$. Combining both these aspects results in the study of analysis of several variables. In this course, we will mainly focus on dealing with functions of the form $f : \mathbb{R}^n \to \mathbb{R}^m$, and talk about their properties such as continuity, differentiability, and integrability.

1.1.1 Algebraic and Analytic Structure

We note that $\mathbb{R}^n$ is also an inner product space with the following properties:

- $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ for all $x, y \in \mathbb{R}^n$.
- The set $\{e_i\}_{i=1}^n$ consisting of the unit vectors is an orthonormal basis for $\mathbb{R}^n$.
- The simplest maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are linear maps that send lines to lines.

Example 1.2. Suppose the function f is a linear map from $\mathbb{R}$ to $\mathbb{R}$. This implies that $f(x) = xf(1)$ for all $x \in \mathbb{R}$. Thus, $f(x) = cx$ for all $x \in \mathbb{R}$, where c is a constant. Conversely, if $c \in \mathbb{R}$, then $x \mapsto cx$ is a linear map. Therefore, we conclude that $\{f : \mathbb{R} \to \mathbb{R}, \text{linear}\} \leftrightarrow \mathbb{R}$, with a possible bijection given by $f \mapsto f(1)$.

In the above example, we note that 1 is not special; we could simply fix any $\alpha \in \mathbb{R} \setminus \{0\}$, and notice that $f(x) = \frac{x}{\alpha} f(\alpha)$ for all $x \in \mathbb{R}$. Here, replacing $f(1)$ by $f(\alpha)$ is a kind of 'change of variable'.

Remark 1.3. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then, $Le_j = \sum_{i=1}^m a_{ij} e_i$ for all $j = 1, 2, \ldots, n$. We may write L as $(a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{R})$, the set of all $m \times n$ matrices with real entries.

Coming to the analysis side, there is a need for defining a distance between points in $\mathbb{R}^n$. Previously, we have seen that the *norm* may be given as $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ for all $x \in \mathbb{R}^n$. We can use this norm to define our required distance function.

Definition 1.4. The *distance* function between two points $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as $d(x, y) = \|x - y\| = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$ for all $x, y \in \mathbb{R}^n$.

For $n = 1$, we note that $d(x, y) = |x - y|$, from the previous analysis courses. $\mathbb{R}^n$ equipped with the function d is called a *metric space*. Coming to the properties of the inner product, we have

- $\|x\| = \langle x, x\rangle^{1/2}$ for all $x \in \mathbb{R}^n$.
- $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathbb{R}^n$.
- The function $\langle , \rangle$ is linear with respect to the first and second arguments.

We also have the important Cauchy-Schwarz inequality.

Theorem 1.5 (*Cauchy-Schwarz inequality*)**.** *For all $x, y \in \mathbb{R}^n$, we have $|\langle x, y\rangle| \leq \|x\| \, \|y\|$.*

Proof. Note that

$$0 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}(x_iy_j - x_jy_i)^2 = 2\left(\sum_{i,j} x_i^2y_j^2 - \sum_{i,j} x_ix_jy_iy_j\right) = 2\left(\|x\|^2\|y\|^2 - \langle x, y\rangle^2\right) \tag{1.1}$$

$$\implies |\langle x, y\rangle| \leq \|x\| \, \|y\| . \tag{1.2}$$

■

We note that equality occurs if and only if the first quantity in the above equation is zero, *i.e.*, if and only if $x_iy_j = x_jy_i$ for all i, j, or $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ for all i, j showing that x and y are linearly dependent.

Corollary 1.6 (*Triangle inequality*)**.** *For all $x, y \in \mathbb{R}^n$, we have $\|x + y\| \leq \|x\| + \|y\|$.*

Proof. We have

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \tag{1.3}$$

where the inequality follows from Cauchy-Schwarz. ■

The following will prove to be an important result.

Theorem 1.7. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then, there exists a $M > 0$ such that $\|Lx\| \leq M\|x\|$ for all $x \in \mathbb{R}^n$.*

Proof. Rewriting x as $x = \sum_{i=1}^{n} x_ie_i$, we have

$$Lx = \sum_{i=1}^{n} x_iLe_i$$

$$\implies \|Lx\| = \left\|\sum_{i=1}^{n} x_iLe_i\right\| \leq \sum_{i=1}^{n} |x_i|\,\|Le_i\| \leq \|x\| \sum_{i=1}^{n} \|Le_i\| = \|x\| M. \tag{1.4}$$

The first inequality follows from the triangle inequality, and the second from Cauchy-Schwarz. In the last step, M is set to be $\sum_{i=1}^{n} \|Le_i\|$, which is a constant. ■

We also term $(\mathbb{R}^n, d)$ as a Euclidean metric space. There is now a need to define open sets in $\mathbb{R}^n$ to talk more about the analysis of several variables.

Definition 1.8. For $a \in \mathbb{R}^n$ and $r > 0$, the *open ball* centred at a of radius r is $B_r(a) := \{x \in \mathbb{R}^n : d(x, a) < r\}$, the set of all points in $\mathbb{R}^n$ that are at a distance less than r from a.

From the notion of open balls, we can define open sets.

Definition 1.9. A set $S \subseteq \mathbb{R}^n$ is said to be an *open set* if for all $x \in S$, there exists an $r > 0$ such that $B_r(x) \subseteq S$.

We now bring the notion of convergence of sequences.

Definition 1.10. Let $\{x_m\} \subseteq \mathbb{R}^n$ be a sequence and $x \in \mathbb{R}^n$. We say that $\{x_m\}$ *converges* to x if for every $\varepsilon > 0$, there exists a natural N such that $\|x_m - x\| < \varepsilon$ for all $m \geq N$.

July 23rd.

Definition 1.11. Let $S \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$. We say that a is a *limit point* of S if $S \cap (B_r(a) \setminus \{a\})$ is non-empty for all $r > 0$.

We introduce more notation; for all $i = 1, 2, \ldots, n$, the mapping $\Pi_i : \mathbb{R}^n \to \mathbb{R}$ is called the i^{th} *projection* where $\Pi_i(x) = x_i$. Note that $x = (x_1, x_2, \ldots, x_n) = (\Pi_1(x), \Pi_2(x), \ldots, \Pi_n(x))$. This notation allows us to formulate the following useful fact a little more neatly.

Theorem 1.12. *Let $\{x_m\} \subseteq \mathbb{R}^n$ be a sequence and $x \in \mathbb{R}^n$. Then $x_m \to x$ if and only if $\Pi_i(x_m) \to \Pi_i(x)$ for all $i = 1, 2, \ldots, n$.*

Proof. Suppose $x_m \to x$. Then for all $\varepsilon > 0$, there exists a natural N such that $\|x_m - x\| < \varepsilon$ for all $N \geq N$. Restating, we have

$$\sum_{i=1}^{n} (\Pi_i(x_m) - \Pi_i(x))^2 < \varepsilon^2 \text{ for all } n \geq N \tag{1.5}$$

$$\implies \text{For all } i,\ |\Pi_i(x_m) - \Pi_i(x)| < \varepsilon^2 \text{ for all } n \geq N. \tag{1.6}$$

For the converse, we simply work backwards with $\varepsilon/\sqrt{n}$ as our choice of epsilon. ■

For example, the sequence $\{(\frac{1}{n}, \frac{1}{2n+3})\}_{n=1}^{\infty}$ converges to $(0, 0)$. However, the sequence $\{(\frac{1}{n}, n^2)\}_{n=1}^{\infty}$ does not.

Definition 1.13. Let $S \subseteq \mathbb{R}^n$. $a \in S$ is termed an *interior point* of S if for some $r > 0$, $B_r(a) \subseteq S$ holds. Thus, a set S is open if a is an interior point for all $a \in S$. The *interior* of set S is defined as $\operatorname{int} S := \{a \in S \mid a \text{ is an interior point.}\}$.
If $a \in \operatorname{int}(S^c)$, then a is termed an *exterior point* of S. a is termed a *boundary point* if $B_r(a)$ meets both S and S^c for all $r > 0$. The set of *boundary points* of S is denoted as ∂S.

We also term a set $S \subseteq \mathbb{R}^n$ as a *closed set* if $\mathbb{R}^n \setminus S$ is open. The following facts will only be stated and will be left as an exercise to the reader:

- A set $C \subseteq \mathbb{R}^n$ is closed if and only if for all sequences $\{x_m\}_{m=1}^{\infty} \subseteq C$ that converge to x implies $x \in C$.
- The open ball $B_r(a)$ is an open set.
- The intersection of an arbitrary collection of closed sets is closed; likewise, the union of an arbitrary collection of open sets is open.
- The set $S \subseteq \mathbb{R}^n$ is open if and only if $S = \operatorname{int} S$.

Fix $O \subseteq \mathbb{R}^n$.

- O is open if and only if $O \cap \partial O = \emptyset$.
- O is closed if and only if $\partial O \subseteq O$.

For $S \subseteq \mathbb{R}^n$, we define the *closure* of set S as $\overline{S} = \operatorname{int} S \cup \partial S$.

- $S \subseteq \mathbb{R}^n$ is closed if and only if $\overline{S} = S$.
- Let $C_i \subseteq R$ be closed sets and $O_i \subseteq \mathbb{R}$ be open sets, for $i = 1, 2, \ldots, n$. Then $C_1 \times C_2 \times \cdots \times C_n \subseteq \mathbb{R}^n$ is a closed set, and $O_1 \times O_2 \times \cdots \times O_n \subseteq \mathbb{R}^n$ is an open set.
- The n *dimensional unit sphere* $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ is closed in $\mathbb{R}^n$.

Definition 1.14. For $S \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$, a is termed an *isolated point* if a is *not* a limit points; that there exists an $r > 0$ such that $S \cap (B_r(a) \setminus \{a\}) = \emptyset$.

With the pesky definitions and translation of one dimensional concept into being defined over several variables, we come to limits and continuity.

1.2 Limit and Continuity

Recall that given $f : (a, b) \setminus \{c\} \to \mathbb{R}$, we say that $\lim_{x \to c} f(x) = b$ if for every $\varepsilon 0$, there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all x satisfying $0 < |x - c| < \delta$. Note that in this definition of the limit, we have $f(x) \in B_\varepsilon(b)$ and $x \in B_\delta(c) \setminus \{c\}$; this can easily be rewritten as $f(B_\delta(c) \setminus \{c\}) \subseteq B_\varepsilon(b)$. However, for our definition we would not require f to be defined on an open set. We defind it over any arbitrary set.

Definition 1.15. Let $a \in S \subseteq \mathbb{R}^n$ be a limit point of S and let $f : S \setminus \{a\} \to \mathbb{R}^m$ be a function and $b \in \mathbb{R}^m$. We say $\lim_{x \to a} f(x) = b$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f((B_\delta(a) \setminus \{a\}) \cap S) \subseteq B_\varepsilon(b)$. Again, this is equivalent to saying that $\|f(x) - b\| < \varepsilon$ for all $x \in S \setminus \{a\}$ satisfying $\|x - a\| < \delta$.

It is important to get accustomed to the definition that works with open balls.

Remark 1.16. In the above definition, if we instead write $x - a = h$, then $\lim_{x \to a} f(x) = b$ is equivalent to saying that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|f(a + h) - b\| < \varepsilon$ for all $\|h\| < \delta$. We can further rewrite to get the usual notation of

$$\lim_{\|h\| \to 0} \|f(a + h) - b\| = 0. \tag{1.7}$$

Note that the above limit is in the real numbers, making it eaiser to deal with.

A notion of continuity also comes in handy.

Definition 1.17. For $S \subseteq \mathbb{R}^n$, let $f : S \to R^m$ with $a \in S$. We say f is *continuous* at a if $\lim_{x \to a} f(x) = f(a)$. In other words, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - f(a)\| < \varepsilon \text{ for all } x \in S \text{ satisfying } \|x - a\| < \delta \tag{1.8}$$

or

$$f(B_\delta(a) \cap S) \subseteq B_\varepsilon(f(a)). \tag{1.9}$$

Note that if a is an isolated point of S, then any $f : S \to \mathbb{R}^m$ is continuous at a since $f(\{a\}) \subseteq B_\varepsilon(f(a))$ holds true, trivially.

Remark 1.18. Similar to the previous remark, f is continuous at a if and only if

$$\lim_{\|h\| \to 0} \|f(a + h) - f(a)\| = 0. \tag{1.10}$$

Functions defined on $S \subseteq \mathbb{R}^n$ can be broken down into components; given $f : S \to R^m$, define $f_j := \Pi_j \circ f$ for all $j = 1, 2, \ldots, m$. Thus, f can be rewritten as $(f_1, f_2, \ldots, f_m)$. We can conclude that f is continuous at $a \in S$ if and only if $f_j : S \to \mathbb{R}$ is continuous at a for all $j = 1, 2, \ldots, m$. The proof of this observation is left as an exercise to the reader.

Theorem 1.19. *Let $a \in \mathbb{R}^n$ be a limit point of a set $S \subseteq \mathbb{R}^n$, with $b \in \mathbb{R}^m$ and $f : S \to R^m$ a function. Then, the following are equivalent—*

1. $\lim_{x\to a} f(x) = b$.
2. *If $\{x_p\} \subseteq S \setminus \{a\}$ and $x_p \to a$, then $f(x_p) \to b$.*
3. $\lim_{x\to a} \|f(x) - b\| = 0$.

The proof of this theorem is left as an exercise to the reader.

Definition 1.20. For a set $S \subseteq \mathbb{R}^n$, a function $f : S \to \mathbb{R}^m$ is termed a continuous function if f is continuous at a for all $a \in S$.

Theorem 1.21. *Let $f : S \to \mathbb{R}^m$ be a function, where $S \subseteq \mathbb{R}^n$. The following are, then, equivalent—*

1. *f is continuous.*
2. *For all $a \in S$ and $\{x_n\} \subseteq S$ with $x_n \to a$, we have $f(x_n) \to f(a)$.*
3. *For all open sets $O \subseteq \mathbb{R}^m$, the set $f^{-1}(O) \subseteq S$ is also open.*
4. *For all closed sets $C \subseteq \mathbb{R}^m$, the set $f^{-1}(C) \subseteq S$ is also closed.*

Proof. For 1. implies 3. , let $O \subseteq \mathbb{R}^m$ be open. Pick some $a \in f^{-1}(O)$. Then, since $f(a) \in O$, there exists $r > 0$ such that $B_r(f(a)) \subseteq O$. Also, f is continuous at a; for $\frac{r}{2} > 0$, there exists $\delta > 0$ such that

$$f(B_\delta(a)) \subseteq B_{\frac{r}{2}}(f(a)) \subseteq B_r(f(a)) \implies a \in B_\delta(a) \subseteq f^{-1}(B_r(f(a))) \subseteq f^{-1}(O). \tag{1.11}$$

Thus, $f^{-1}(O)$ is open. For 3. implies 1. , let $a \in S$. Fix $\varepsilon > 0$. Then the set $f^{-1}(B_\varepsilon(f(a)))$ is open; there exists a $\delta > 0$ such that $B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a)))$. ■

Index