DISCRETE MATHEMATICS I

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Third Semester

List of Symbols

Placeholder

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Chapter 1

DISCRETE STRUCTURES

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1.1 A Brief Introduction

July 22nd.

Discrete mathematics is primarily the study of tools for reasoning precisely the systematically about digital systems, logical problems, and combinatorial structures such as the integers, graphs, logical statements, and finite automata. Furthermore, combinatorics is the mathematics of counting and configuration; the counting, organizing, and analyzing discrete structures.

Bloch's principle, or Bloch's heuristics, states that every proposition on whose statement the actual infinity occurs can always be considered as a consequence of a proposition where it does not occur as a proposition on finite terms. The *Ramsey principle* states that complete disorder is impossible. In any sufficiently large structure, order or regularity must emerge. These two principles may be considered complimentary to each other.

1.2 Useful Methods

The method of *double counting* can be thought of a creative device or trick. Before strictly showing the statement, we utilise some examples.

Example 1.1. Suppose we wish to show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. We first ask how many ways can a subset be chosen from $\{1, 2, \ldots, n\}$. The first method is to build a subset by deciding whether we want i to be a part of our subset for $i \in \{1, 2, ..., n\}$. The second method is find the number of subsets of caridnality i for $i \in \{1, 2, \ldots, n\}$ and add up all the results. This leads us to conclude $2^n = \sum_{k=0}^{n} \binom{n}{k}$ after equating the answers from both methods.

Theorem 1.2 (The *q-binomial theorem*). *We use the following notation:*

$$\binom{n}{k}_q = \frac{(q^n - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1) \cdots (q - 1)}. \tag{1.1}$$

Simply stated, the q-binomial theorem is

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q z^k = \prod_{i=0}^{n-1} (1 + q^i z). \tag{1.2}$$

The proof of the above theorem is performed by double counting; counting the number of pairs (U, B) where U is a k-dimensional subspace of $\mathbb{F}_q^n$ and B is the flag of nested subspaces of U.

Reccurrence Relations and Generating Functions

Perhaps, the most important example of a recurrence relation is the Fibonacci sequence, where the terms in the sequence are defined as $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Let us find the *generating function* of this sequence; we start by creating

$$F(t) = F_0 + F_1 t + F_2 t^2 + F_3 t^3 + \cdots, \tag{1.3}$$

the generating function of $(F_n)_{n=0}^{\infty}$. We can then work as follows—

$$tF(t) = F_0 t + F_1 t^2 + \cdots,$$
$$t^2 F(t) = F_0 t^2 + \cdots,$$
$$\implies (1 - t - t^2)F(t) = t \implies F(t) = \frac{-t}{t^2 + t - 1}. \tag{1.4}$$

If we look at $F_{n+1} = 1 \cdot F_n + 1 \cdot F_{n-1}$ and $F_n = 1 \cdot F_n + 0 \cdot F_{n-1}$, we may notice a matrix as $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$. Back substituting multiple times leads us to conclude $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We can then diagonalize the centre matrix to decompose it as $P \begin{bmatrix} (1+\sqrt{5})^n/2^n & 0 \\ 0 & (1-\sqrt{5})^n/2^n \end{bmatrix} P^{-1}$; thus, the terms of the sequence are really linear combinations of the diagonal elements that appear.

Principle of Inclusion-Exclusion

July 24th.

Simply stated, for sets A and B, $\#(A \cup B) = \#A + \#B + \#(A \cap B)$. For three sets A, B and C, we have $\#(A \cup B \cup C) = \#A + \#B + \#C - \#(A \cap B) - \#(B \cap C) - \#(A \cap C) + \#(A \cap B \cap C)$. This can be extended to any finite number of finite sets.

Theorem 1.3 (The *principle of inclusion-exclusion*). *Let S be an N-set ($\#S = N$), and let $E_1, E_2, \ldots, E_r$ be, not necessarily distinct, subsets of S. For any subset M of the indexing set $\{1, 2, \ldots, r\}$, let $N(M)$ denote the number of elements of S in $\bigcap_{i \in M} E_i$, and for $0 \leq j \leq r$, define*

$$N_j = \sum_{\#M=j} N(M). \tag{1.5}$$

Then the number of elements of S not in any of the E_i's is

$$\#(S \setminus \bigcup_{i=1}^{r} E_i) = N - N_1 + N_2 - N_3 + \cdots + (-1)^r N_r. \tag{1.6}$$

Proof. For $x \in S$, define $M : S \to \{0, 1\}$ as $M(x) = 1$ if $x \in \bigcap_{i \in M} E_i$ and 0 otherwise. Thus,

$$\sum_{x \in S} M(x) = \#(\bigcap_{i \in M} E_i) = N(M) \implies N_j = \sum_{\#M=j} \sum_{x \in S} M(x) = \sum_{x \in S} \sum_{\#M=j} M(x). \tag{1.7}$$

The alternating sum then becomes

$$\sum_{x \in S} 1 - \sum_{x \in S} \sum_{\#M=1} M(x) + \cdots + (-1)^r \sum_{x \in S} \sum_{\#M=r} M(x) = \sum_{x \in S} \left(1 - \sum_{\#M=1} M(x) + \cdots + (-1)^r \sum_{\#M=r} M(x) \right). \tag{1.8}$$

Call the term within the parentheses as $F(x)$. We deal with cases; if $x \notin \bigcup_{i=1}^{r} E_i$, then $F(x) = 1$. If x is in exactly $k \geq 1$ of the sets $E_1, E_2, \ldots, E_r$, then

$$F(x) = 1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k} = (1-1)^k = 0. \tag{1.9}$$

This is independent of k; we conclude that the alternating sum reduces to the number of elements in S not in any of the E_i's. $\blacksquare$

Corollary 1.4. *Retaining notation from the previous theorem, if $S = \bigcup_{i=1}^r E_r$, then*

$$N = N_1 - N_2 + \cdots + (-1)^{r-1} N_r. \tag{1.10}$$

We look at some examples of the principle in use.

Example 1.5. Let d_n be the number of permutations π of the set $\{1, 2, \ldots, n\}$ such that $\pi(i) \neq i$ for all $1 \leq i \leq n$. Such a permutation is called a *derangement*, where no point is fixed. We wish to count all such permutations. Let the set of all permutations of the set be S. Let E_i denote the set of all permutations that fix i, for $1 \leq i \leq n$. Thus, S without $\bigcup_{i=1}^n E_i$ would then denote the set of all derangements. Making use of the principle, we have

$$\#(S \setminus \bigcup_{i=1}^n E_i) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$
$$\tag{1.11}$$

which is approximately $\frac{n!}{e}$ for larger n. Thus, the probability of choosing a derangemnet is e^{-1}.

Example 1.6. Suppose we have two sets X and Y with $\#X = n$ and $\#Y = k$. We ask how many surjective maps exists from X to Y. The set S, this time, is the set of all functions from X to Y, being Y^X. E_i denotes the set of functions from X to Y such that y_i is not in the image of X. The elments within S not in any of the E_i's are surjective maps. Clearly, $N_i = \binom{k}{i}(k-i)^n$, and the cardinality of $S \setminus \bigcup_{i=1}^k E_i$ is then

$$\sum_{i=0}^k (-1)^i \binom{k}{i}(k-i)^n. \tag{1.12}$$

Example 1.7. We wish to show that the expression $\sum_{i=0}^n (-1)^i \binom{n}{i}\binom{m+n-i}{k-i}$ evaluates to $\binom{m}{k}$ if $m \geq k$, and 0 otherwise. To this end, fix $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ and set $Z = X \cup Y$. We now ask how many k-subsets of Z consist of only points form Y. Let S be the set of all k-subsets of Z, and denote E_i to be the set of k-subsets of Z containing x_i for $1 \leq i \leq n$. The left hand side of our inclusion-exclusion principle evaluates to $\binom{m+n}{k}$. Each N_i evaluates to $\binom{n}{i}\binom{m+n-i}{k-i}$, proving our expression above.

The next example relates to the Euler totient function.

Example 1.8. Recall that, from the *fundamental theorem of arithmetic*, each natural number may be expressed uniquely (upto order) as the product of distinct primes raised to values, that is, $n = p_1^{a_1} \cdots p_r^{a_r}$. The *Euler totient function* $\phi : \mathbb{N} \to \mathbb{C}$ acts on the naturals and returns $\phi(n)$, the number of positive integers $k \leq n$ such that $\gcd(k, n) = 1$. Certainly, $\phi(p) = p - 1$ for a prime p. Our task is to find a closed form formula for $\phi(n)$.

Set $S = \{1, 2, \ldots, n\}$, and set E_i to be the set of integers in S divisible by p_i for $1 \leq i \leq r$. Clearly, the value $\#(S \setminus \bigcup_{i=1}^r E_i)$ returns the set of all numbers in $\{1, 2, \ldots, n\}$ coprime to n. Note that $N = n$. The value of N_1 is $\sum_{i=1}^r \frac{n}{p_i}$, the value of N_2 is is $\sum_{1 \leq i \leq j \leq r} \frac{n}{p_i p_j}$. The cloesd form formula then becomes

$$\phi(n) = \#(S \setminus \bigcup_{i=1}^r E_i) = n - n \sum_{i=1}^r \frac{1}{p_i} + n \sum_{1 \leq i \leq j \leq r} \frac{1}{p_i p_j} + \cdots = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$
$$\tag{1.13}$$

Number Theory

We continue with the Euler totient function.

Theorem 1.9. $\sum_{d|n} \phi(d) = n$.

Proof. For each integer $m \in \{1, 2, \ldots, n\}$, the value $\gcd(m, n)$ is a divides n. Fix a divisor d of n. The number of integers m such that $\gcd(m, n) = d$ is equal to the number of integers m such that $\gcd\left(\frac{m}{d}, \frac{n}{d}\right) = 1$, where $\frac{m}{d}$ runs over integers between 1 and $\frac{n}{d}$. Therefore, the number of such m is $\phi\left(\frac{n}{d}\right)$. Summing over all divisors d of n, we get:

$$n = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

which is the same as $\sum_{d|n} \phi(d)$. ∎

The *Möbius function* is defined as

$$\mu(d) := \begin{cases} 1, & \text{if } d \text{ is a product of even number of distinct primes,} \\ -1, & \text{if } d \text{ is a product of odd number of distinct primes,} \\ 0, & \text{if otherwise; the number } d \text{ is square-free.} \end{cases} \tag{1.14}$$

Theorem 1.10.

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if otherwise.} \end{cases} \tag{1.15}$$

Proof. For $n = 1$, it is clear. For $n > 1$, rewriting n as $p_1^{a_1} \cdots p_r^{a_r}$ helps us see that

$$\sum_{d|n} \mu(d) = 1 - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots = (1-1)^r = 0. \tag{1.16}$$

∎

This property of the Möbius function proves to be useful.

July 29th.

Theorem 1.11 (The *Möbius inversion formula*)**.** *Suppose we have two function* $f : \mathbb{N} \to \mathbb{R}$ *and* $g : \mathbb{N} \to \mathbb{R}$ *which relate as*

$$f(n) = \sum_{d|n} g(d). \tag{1.17}$$

Then the function g satisfies

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d). \tag{1.18}$$

Proof. We work as

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \left(\sum_{d'|d} g(d')\right) = \sum_{d|n,\, d'|d} \mu\left(\frac{n}{d}\right) g(d') = \sum_{d'|n,\, m|\frac{n}{d'}} g(d')\mu(m)$$

$$= \sum_{d'|n} \left(g(d') \left(\sum_{m|\frac{n}{d'}} \mu(m)\right)\right) = g(n). \tag{1.19}$$

∎

Example 1.12. Let N_n denote the number of distinct circular binary sequences of length n, up to rotation. That is, two sequences are considered the same if one is a rotation of the other. We aim to compute N_n explicitly.

Let $M(d)$ denote the number of aperiodic circular binary sequences of length d, meaning sequences that are not periodic with any smaller period. Note that each such aperiodic sequence of length d contributes to sequences of length n whenever $d \mid n$. Indeed, every binary circular sequence of length n can be viewed as made up of $\frac{n}{d}$ repetitions of a primitive block of length d.

Thus, we have:

$$N_n = \sum_{d \mid n} M(d).$$

Now consider the total number of binary strings of length n, which is 2^n. Each such string can be arranged in a circle in n different ways, one for each rotation. However, many of these circular sequences are identical under rotation, so we overcounted by a factor of the size of the symmetry group.

Let $f(n) = 2^n$ be the total number of binary strings of length n. Each such string is generated by repeating an aperiodic sequence of length d exactly $\frac{n}{d}$ times, for some $d \mid n$. Since each aperiodic circular sequence of length d has d rotations, we get:

$$f(n) = 2^n = \sum_{d \mid n} d \cdot M(d).$$

Applying Möbius inversion to this relation, we obtain:

$$n \cdot M(n) = \sum_{d \mid n} \mu(d) \cdot 2^{n/d},$$

and hence,

$$M(n) = \frac{1}{n} \sum_{d \mid n} \mu(d) \cdot 2^{n/d}.$$

Substituting back into the formula for N_n, we get:

$$N_n = \sum_{d \mid n} M(d) = \sum_{d \mid n} \frac{1}{d} \sum_{k \mid d} \mu(k) \cdot 2^{d/k}.$$

Interchanging the order of summation, we arrive at the classical formula:

$$N_n = \frac{1}{n} \sum_{d \mid n} \phi(d) \cdot 2^{n/d},$$

where ϕ is Euler's totient function.

Lemma 1.13 (*Burnside's lemma*). *Let G be a permutation group acting on some finite set X. Let $\psi(g)$ denote the number of points of X fixed by $g \in G$. Then the number of orbits of G is $\frac{1}{|G|} \sum_{g \in G} \psi(g)$.*

In the above, by the set of points fixed by $g \in G$, we mean the set $\{x \mid g \cdot x = x\}$. By the *orbit* of x, we mean the set $\{g \cdot x \mid g \in G\}$.

Such inversion formulae are common in discrete math. The following are some examples.

Example 1.14.
- For an integer n, $f(n) = \sum_{i=1}^{n} g(i)$ if and only if $g(n) = f(n) - f(n-1)$. This is known as a *telescoping sum*.

- For an integer n, $f(n) = \sum_{d \mid n} g(d)$ if and only if $g(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(d)$. This is the Möbius inversion formula.

- For a set S, $f(S) = \sum_{T \subseteq S} g(T)$ if and only if $g(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f(T)$.

Partially Ordered Sets

Definition 1.15. A *poset* S, or a *paritally ordered set*, is a (countable or finite) set of objects with a binary relation $\leq$ satisfying the following properties.

1. Reflexivity: $x \leq x$ for all $x \in S$.

2. Antisymmetry: if $x \leq y$ and $y \leq x$, for some $x, y \in S$, then $x = y$.

3. Transitivity: if $x \leq y$ and $y \leq z$, for some $x, y, z \in S$, then $x \leq z$.

Some examples are as follows.

Example 1.16.   1. $(\{1, 2, \ldots, n\}, \leq)$ is a poset, with $a \leq b$ if $b - a$ is a non-negative integer.

2. $(\{1, 2, \ldots, n\}, \leq_1)$ is also a poset, where $a \leq_1 b$ if $a \mid b$.

3. $(\mathcal{P}(\{1, 2, \ldots, n\}), \leq_2)$ is also a poset; here, $S \leq_2 T$ if $S \subseteq T$. The power set is also denoted as $2^{\{1,2,\ldots,n\}}$.

4. The set of partitions of $\{1, 2, \ldots, n\}$, equippied with the partial ordering of refinement. By a partition, we mean $\{S_1, S_2, \ldots, S_r\}$ such that $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{r} S_i = \{1, 2, \ldots, n\}$. Similarly, let $\{T_1, T_2, \ldots, T_d\}$ be another partition. Then $\{S_1, \ldots, S_r\} \leq \{T_1, \ldots, T_d\}$ if for $1 \leq i \leq r$, $S_i \subseteq T_k$ for some $1 \leq k \leq d$. Here, we term $\{S_1, \ldots, S_r\}$ a *refinement* of $\{T_1, \ldots, T_d\}$.

The idea of the Möbius function really comes from posets, where its definition is more generalized. Here, $\mu(d, n)$ can be thought of as a place-in for $\mu\left(\frac{n}{d}\right)$.

Definition 1.17. The *Möbius function of a poset* is defined as

$$\mu(x, y) = \begin{cases} 0 & \text{if } x \not\leq y, \\ 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x \lneq y. \end{cases} \tag{1.20}$$

Note that we need only compute $\mu(x, y)$ on all intervals $[x, y]$ $(x \leq y)$. We call an element y a *successor* of x if there exists no z satisfying $x \lneq z \lneq y$. Note that the successor may not be unique. Let us denote any successor by $\mathrm{succ}(x)$.

If g and f are two functions defined and related as

$$g(x) = \sum_{y \leq x} f(y). \tag{1.21}$$

We now introduce the *zeta function* ζ defined as $\zeta(x, y) = 1$ if $x \leq y$ and 0 otherwise. Thus, the above equation can be rewritten as

$$g(x) = \sum_{y \leq x} f(y) = \sum_{y \in S} \zeta(y, x) f(y). \tag{1.22}$$

August 5th.

Lemma 1.18. *A finite partial order can always be embedded in a total ordering; that is, there exists an indexing $S = \{x_1, \ldots, x_n\}$ such that $x_i \leq x_j$ in S implies $i \leq j$.*

As a proof outline, pick a maximal element x of S. Label it x_n. Repeat the process with $S \setminus \{x\}$, then proceed inductively. The embedding is then clear.

Thus, using the lemma, we can rewrite the relation between g and f in matrix form as

$$\begin{pmatrix} g(x_1) & \cdots & g(x_n) \end{pmatrix} = \begin{pmatrix} f(x_1) & \cdots & f(x_n) \end{pmatrix} \begin{pmatrix} \zeta(x_1, x_1) & \cdots & \zeta(x_1, x_n) \\ \vdots & \ddots & \vdots \\ \zeta(x_n, x_1) & \cdots & \zeta(x_n, x_n) \end{pmatrix}. \tag{1.23}$$

Since $\zeta(x_i, x_j) = 0$ when $i > j$, the matrix on the right is upper triangular. Also, all the diagonal entries are 1. Denote the above matrix on the right by Z. Note that $Z = I + N$ where I is the identity matrix and N is upper triangular with 0's on the diagonal. Z^{-1} can be computed by taking a power series, and noting that N^n is 0.

$$Z^{-1} = (I + N)^{-1} = I - N + N^2 - N^3 + \cdots + (-1)^{n-1} N^{n-1} \tag{1.24}$$

Let $M = [\mu(x_i, x_j)]$. We find the (x_i, x_j) entry of MZ as

$$\sum_{y \in P} M_{x_i, y} Z_{y, x_j} = \sum_{y \in P} \mu(x_i, y) \zeta(y, x_j) = \sum_{y \leq x_j} \mu(x_i, y) = \sum_{x_i \leq y \leq x_j} \mu(x_i, y). \tag{1.25}$$

Noting that $\mu(x_i, x_j) = -\sum_{x_i \leq z < x_j} \mu(x_i, z)$, we get the above expression to be 1 if $x_i = x_j$ and 0 otherwise. Thus, $MZ = I$. This is the used definition of the the Möbius functon. $ZM = I$ may be verified similarly.

Theorem 1.19. *Let $(P, \leq)$ be a finite poset, with $f, g : P \to \mathbb{Z}$ functions. Then,*

1. *$f(x) = \sum_{y \leq x} g(y)$ if and only if $g(x) = \sum_{y \leq x} \mu(y, x) f(y)$, and*

2. *$f(x) = \sum_{x \leq y} g(y)$ if and only if $g(x) = \sum_{x \leq y} \mu(x, y) f(y)$.*

This is the Möbius inversion formula *for a poset.*

Proof. We have

$$\sum_{y \leq x} \mu(y, x) f(y) = \sum_{y \leq x} \left(\sum_{z \leq y} \mu(y, x) g(z) \right) = \sum_{z \leq x} \sum_{z \leq y \leq x} \mu(y, x) g(z) = \sum_{z \leq x} g(z) \sum_{z \leq y \leq x} \mu(y, x) = g(x) \tag{1.26}$$

since $\mu(y, x)$ at the end will be zero if $z \neq x$. To show the converse, we have

$$\sum_{y \leq x} g(y) = \sum_{y \leq x} \sum_{z \leq y} \mu(z, y) f(z) = \sum_{z \leq x} f(z) \left(\sum_{z \leq y \leq x} \mu(z, y) \right) = f(x) \tag{1.27}$$

since $\mu(z, y)$ is zero if $z \neq x$. $\blacksquare$

Example 1.20. Verify that if the poset is the positive integers with the standard ordering $\leq$, then

$$\mu(i, j) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j - 1, \\ 0 & \text{if otherwise.} \end{cases} \tag{1.28}$$

Example 1.21. Let our poset be $(2^S, \subseteq)$ for a set S. For fixed subsets $U, T \in 2^S$, we have

$$\sum_{U \subseteq R \subseteq T} (-1)^{\#T - \#S} = \begin{cases} 1 & \text{if } U = T, \\ 0 & \text{if otherwise.} \end{cases} \tag{1.29}$$

To show this, without the loss of generality, we will assume that $U = \emptyset$. Denoting $\#T = n$, we have

$$\sum_{R \subseteq T} (-1)^{\#R} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0. \tag{1.30}$$

Here, $Z(R,T) = 1$ if $R \subseteq T$ and 0 otherwise. Also, $M(R,T) = (-1)^{\#T-\#R}$ if $R \subseteq T$ and 0 otherwise. Since $MZ = I$, $M(R,T)$ must be the Möbius function for $(2^S, \leq)$.

Chapter 2

RECURRENCE RELATIONS AND GENERATING FUNCTIONS

2.1 Generating Functions

August 7th.

We begin with ordinary ones.

> **Definition 2.1.** For a sequence $(a_n)_{n \geq 0} \subseteq \mathbb{R}$, the *ordinary generating function* associated with this sequence is defined as
>
> $$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots . \tag{2.1}$$

Note that we are not concerned with convergence right now. We deconstruct our abstraction of ideas into levels, starting with the first level as regarding ordinary generating functions as algebraic objects. One can multiply and add them to create new generating functions. The second level is regarding them as analytic objects, only if the radius of convergence is positive.

The above is known as *Z-transform*, where a sequence is mapped onto a function. When using the word transform, we generally mean a 'change of basis'; in this case, we are changing from a sequence space to a function space.

> **Definition 2.2.** For a sequence $(a_n)_{n \geq 0} \subseteq \mathbb{R}$, the *exponential generating function* associated with this sequence is defined as
>
> $$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \cdots . \tag{2.2}$$

Again, we have transformed from a sequence space to a function space. One can also transform from a random variable space to a function space.

> **Definition 2.3.** For a random variable X taking values in $\mathbb{R}$, the *moment generating function* associated with this random variable is defined as
>
> $$M_X(t) = E[e^{tX}] = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n = 1 + E[X]t + \frac{E[X^2]}{2!} t^2 + \cdots . \tag{2.3}$$

2.1.1   Algberaic Operaions

We give a kind of correspondence between algebraic operations and combinatorial interpretations.

1. Multiplying by x^k maps $a_0 + a_1 x + a_2 x^2 + \cdots$ to $a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \cdots$. This corresponds to shifting the sequence $(a_0, a_1, \ldots)$ right by k places. This is known as the *shift operator*.

2. Multiplication is also defined; for two functions $a_0 + a_1 x + a_2 x^2 + \cdots$ and $b_0 + b_1 x + b_2 x^2 + \cdots$, their product is given by $a_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots$. This corresponds to combining objects of size k and size $n - k$ chosen independently.

3. Differentiation maps $a_0 + a_1 x + a_2 x^2 + \cdots$ to $a_1 + 2a_2 x + 3a_3 x^2 + \cdots$. This corresponds to weighing the sequence values by their index, with a shift of one place to the right.

Example 2.4. Suppose we have k boxes labelled 1 through k, and box i contains r_i balls for $1 \leq i \leq k$. We wish to encode all possible configurations in a kind of book-keeping device. For a particular $(r_1, \ldots, r_k)$, we have

$$\sum_{r_i \geq 0} x_1^{r_1} \cdots x_k^{r_k} = (1 + x_1 + x_1^2 + \cdots)(1 + x^2 + x_2^2 + \cdots) \cdots (1 + x_k + x_k^2 + \cdots). \tag{2.4}$$

We find the number of partitions of n (balls) into k numbers (boxes), where each number if non-negative. Disregarding the order, we set all the x_i's equal to each other. Thus, we wish to find the coefficient of x^n where $r_1 + \cdots + r_k = n$. From the sum above, we have

$$(1 + x + x^2 + \cdots)^k = (1 - x)^{-k} = \sum_{j=0}^{\infty} \binom{k-1+j}{j} x^j x^j. \tag{2.5}$$

Therefore, the required coefficient is $\binom{k-1+n}{n}$.

We briefly introduce the idea of rings. A *ring* $(R, +, *)$ is a set R with two operations $+$ and $*$ such that $(R, +)$ is an abelian group, $(R, *)$ is a monoid, and the distributive law holds. Some examples of rings include $\mathbb{Z}$, $M_n(\mathbb{C})$, and $\mathbb{C}[x]$. Another example is the ring of formal power series $\mathbb{C}[[x]]$, which consists of all series of the form $a_0 + a_1 x + a_2 x^2 + \cdots$ where $a_i \in \mathbb{C}$. We ask which elements of this ring are invertible.

We claim that $a_0 + a_1 x + a_2 x^2 + \cdots$ is invertible if and only if $a_0 \neq 0$. We find $b_0 + b_1 x + b_2 x^2 + \cdots$ such that

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1. \tag{2.6}$$

This first gives us $a_0 b_0 = 1$, so $b_0 = \frac{1}{a_0}$. The next term gives us $a_0 b_1 + a_1 b_0 = 0$, so $b_1 = -\frac{a_1}{a_0^2}$. Continuing this process, we find that the coefficients of b can be expressed in terms of the coefficients of a as

$$b_n = -\frac{1}{a_0} \sum_{k=1}^{n} a_k b_{n-k}. \tag{2.7}$$

There is also the ring homomorphism $\text{ev}_z : \mathbb{C}[[x]] \to \mathbb{C}$ where $x \mapsto z$, with $z \in \mathbb{C}$.

Example 2.5. Let d_n denote the number of derangements of $\{1, 2, \ldots, n\}$. We consider a derangement Π of $\{1, 2, \ldots, n+1\}$ where

- Case I: $\Pi(n+1) = i$ and $\Pi(i) = n+1$ for some i. The number of such derangements is nd_{n-1}.

- Case II: $\Pi(n+1) = i$ and $\Pi(j) = n+1$ for some $i \neq j$. The number of such derangements is $d_{n+1} = nd_n$.

Thus, the total number of derangements is $d_{n+1} = n(d_n + d_{n-1})$. Here, $d_0 = 1$, $d_1 = 0$, and $d_2 = 1$. The exponential generating function, here, is

$$D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} \implies D'(x) = \sum_{n=1}^{\infty} nd_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} d_{n+1} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{d_n}{(n-1)!} x^n + \sum_{n=1}^{\infty} \frac{d_{n-1}}{(n-1)!} x^n. \tag{2.8}$$

This gives us

$$D'(x) = xD'(x) + xD(x) \implies \frac{D'(x)}{D(x)} = \frac{x}{1-x} \implies D(x) = C\left(\frac{e^{-x}}{1-x}\right). \tag{2.9}$$

Plugging in $x = 0$ givens $C = 1$. Thus, the exponential generating function is

$$D(x) = \frac{e^{-x}}{1-x} = (1 + x + x^2 + x^3 + \cdots)(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots). \tag{2.10}$$

The coefficient of x^n in the above series is $\frac{d_n}{n!}$, giving us

$$d_n = n! \sum_{k=0}^{n} a_k b_{n-k} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^n \frac{1}{n!}\right). \tag{2.11}$$

Example 2.6. Suppose we wish to find number of ways to make n change with the denominations 1, 2, and 5. We use generating functions. Thus,

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots) = \frac{1}{(1-x)(1-x^2)(1-x^5)}. \tag{2.12}$$

From above, taking the nth derivative of the fraction, dividng it by $n!$, and evaluating at $x = 0$ provides the number of ways to make change for n.

2.1.2 Extended Binomial Theorem

August 8th.

We begin by extending the definition of a binomial coefficient.

Definition 2.7. For any $u \in \mathbb{R}$ and positive integer k, we define the *extended binomial coefficient* as

$$\binom{u}{k} = \frac{u(u-1)(u-2)\cdots(u-k+1)}{k!}. \tag{2.13}$$

with $\binom{u}{0} = 1$.

Theorem 2.8. *For positive integers n and r, we have*

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}. \tag{2.14}$$

Proof. We simply have

$$\binom{-n}{r} = (-1)^r \frac{n(n+1)\cdots(n+r-1)}{r!} = (-1)^r \binom{n+r-1}{r}. \tag{2.15}$$

∎

Theorem 2.9 (The *extended binomial theorem*). *For any $u \in \mathbb{R}$ and positive integer k, we have*

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k. \tag{2.16}$$

.

The extended binomial theorem helps to compute coefficients of generating functions such like $\frac{1}{(1+x)^5}$ or even $\sqrt{1+x}$.

Example 2.10. We compute the coefficient of x^{2026} in the generating function $G(x) = \frac{1}{(1-x)^2(1+x)^2}$.
We break down as partial fractions to get

$$G(x) = \frac{1}{(1-x)^2(1+x)^2} = \frac{1}{4}\left(\frac{1}{1-x} + \frac{1}{(1-x)^2} + \frac{1}{1+x} + \frac{1}{(1+x)^2}\right) \tag{2.17}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\left((-1)^k\binom{-1}{k} + (-1)^k\binom{-2}{k} + \binom{-1}{k} + \binom{-2}{k}\right)x^k. \tag{2.18}$$

The odd terms vanish, so we set k to be even to get the coefficient of x^k as

$$\frac{1}{4}\left(2\binom{-1}{k} + 2\binom{-2}{k}\right) = 1 + \frac{k}{2}. \tag{2.19}$$

Setting $k = 2026$ gives the desired solution of 1014.

2.1.3 Bernoulli Numbers

Definition 2.11. In power series of $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k\frac{t^k}{k!}$, the coefficients B_k are known as the *Bernoulli numbers*.

Here, B_0 is defined to be 1. One can also recursively define them as $B_0 = 1$ and B_j for $j \geq 1$ such that $\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0$ for $n \geq 1$
One also has a useful formula for the Bernoulli numbers.

Theorem 2.12 (*Faulhaber's formula*). *We have*

$$\sum_{m=1}^{n} m^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}. \tag{2.20}$$

Setting $k = 1$, we get

$$1 + 2 + \cdots + n = \frac{1}{2}\left(\binom{2}{0}B_0(n+1)^2 + \binom{2}{1}B_1(n+1)\right) = \frac{1}{2}\left((n+1)^2 - (n+1)\right) = \frac{1}{2}(n+1)n. \tag{2.21}$$

Similarly, one may verify for $k = 2$ or $k = 3$.

Proof. To this end, we use the *Bernoulli polynomials* B_k which are interpreted as coefficients in

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \tag{2.22}$$

with $B_k(0) = B_k$. We claim that $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}$. We have

$$\frac{t}{e^t-1}e^{xt} = \left(\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}\right)e^{xt} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n}{k}B_k x^{n-k}\right)\frac{t^n}{n!}. \tag{2.23}$$

Summing gives us

$$\sum_{m=0}^{n-1}\frac{te^{mt}}{e^t-1} = \sum_{m=0}^{n-1}(1 + mt + m^2\frac{t^2}{2!} + \cdots) = \frac{t}{e^t-1}\left(n + \sum_{m=0}^{n-1}m\frac{t}{1!} + \cdots + \sum_{m=0}^{n-1}m^k\frac{t^k}{k!}\right) \tag{2.24}$$

$\blacksquare$

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