Examples of GLM
Logistic regression and Poisson regression

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Outline

1. **Logistic regression**
2. Log regression

Outline

1. **Logistic regression**
   - Modeling
   - Odds and odds ratio
   - Simple logistic regression
   - Multiple logistic regression

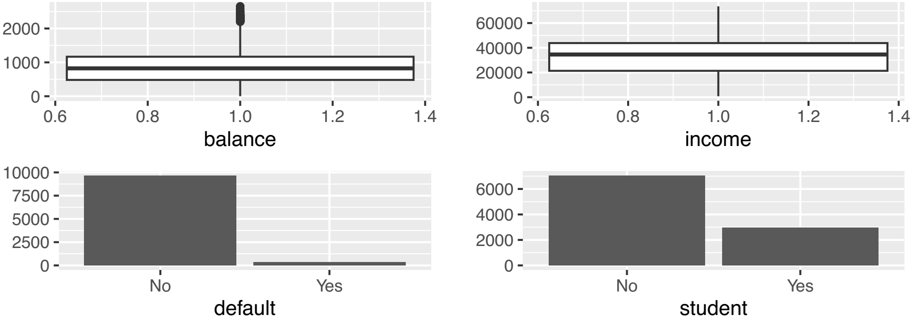
Context

- A **binary** response variable Y
- Explanatory variables: $x^{(1)}, \ldots, x^{(p)}$
- Example : Credit Card Default
  A simulated data set containing information on $n = 10000$ customers. The aim is to predict which customers will default on their credit card debt. We want to explain the binary variable *default* (1 if default, 0 otherwise) with the 3 following explanatory variables:
  - *student*: A factor with levels No and Yes indicating whether the customer is a student
  - *balance*: The average balance that the customer has remaining on their credit card after making their monthly payment
  - *income*: Income of customer

Example

```
data(Default)
attach(Default)
summary(Default)
```

```
 default     student       balance           income     
 No :9667   No :7056   Min.   :   0.0   Min.   :  772  
 Yes: 333   Yes:2944   1st Qu.: 481.7   1st Qu.:21340  
                       Median : 823.6   Median :34553  
                       Mean   : 835.4   Mean   :33517  
                       3rd Qu.:1166.3   3rd Qu.:43808  
                       Max.   :2654.3   Max.   :73554  
```



Modeling

loi de Bernoulli

- Random component: $Y_i|\mathbf{x}_i \sim \mathcal{B}(\pi(\mathbf{x}_i))$, $Y_1, \ldots, Y_n$ indep.

  π_i

- Link function g :

  $[0,1] \ni \pi \underset{F}{\overset{g}{\rightleftarrows}} u \in \mathbb{R}$

  - **logistic function**:

$$g^{-1}(u) = F(u) = \frac{e^u}{1+e^u} \iff g(\pi) = \ln\left(\frac{\pi}{1-\pi}\right) = \mathrm{logit}(\pi).$$

    In this case, the model is called **logistic model**.

  - **probit function**: F is the cdf of $\mathcal{N}(0,1)$ and $g = F^{-1}$ is the probit function.

    fonction de répartition.

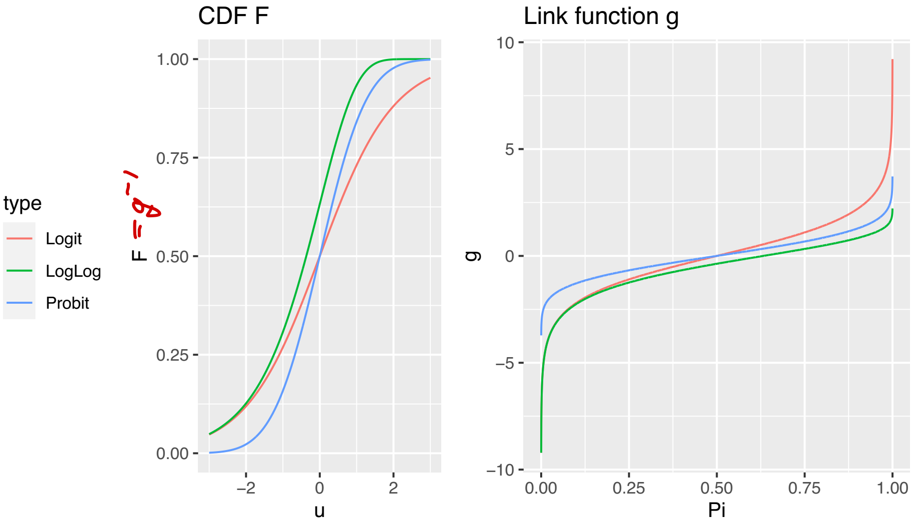
  - **Gompit or complementary log-log function**: F is the cdf of the Gompertz law

$$F(u) = 1 - \exp\left(-e^u\right) \iff g(\pi) = \ln[-\ln(1-\pi)],$$

    but this function is asymmetric.

- Partie linéaire $g(\pi_i) = \cancel{x_i}\,\theta$

Link functions



Outline

1. **Logistic regression**
   - Modeling
   - Odds and odds ratio
   - Simple logistic regression
   - Multiple logistic regression

Odds and odds ratio

- The **odds** for an individual $\mathbf{x}$ is

$$\text{odds}(\mathbf{x}) = \frac{\mathbb{P}(Y=1|\mathbf{x})}{\mathbb{P}(Y=0|\mathbf{x})} = \frac{\pi(\mathbf{x})}{1-\pi(\mathbf{x})} = \exp[\text{logit}(\pi(\mathbf{x}))]$$

- The **odds ratio** between two individuals $\mathbf{x}$ and $\tilde{\mathbf{x}}$ is defined as the ratio between their odds:

$$\text{OR}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{\text{odds}(\mathbf{x})}{\text{odds}(\tilde{\mathbf{x}})}.$$

- The **odds ratio** allow to measure the effect of an explanatory variable on the binary response variable.

Odds and odds ratio

The odds ratios can be used in several ways:

- Comparison of success probabilities between two individuals:

$$\left\{\begin{array}{lcl} \mathrm{OR}(\mathbf{x},\tilde{\mathbf{x}}) > 1 & \Leftrightarrow & \pi(\mathbf{x}) > \pi(\tilde{\mathbf{x}}) \\ \mathrm{OR}(\mathbf{x},\tilde{\mathbf{x}}) = 1 & \Leftrightarrow & \pi(\mathbf{x}) = \pi(\tilde{\mathbf{x}}) \\ \mathrm{OR}(\mathbf{x},\tilde{\mathbf{x}}) < 1 & \Leftrightarrow & \pi(\mathbf{x}) < \pi(\tilde{\mathbf{x}}) \end{array}\right.$$

- Effect of an explanatory variable:
  when $\mathrm{logit}[\pi(\mathbf{x})] = \theta_0 + \theta_1 x^{(1)} + \ldots + \theta_p x^{(p)}$, $= x\theta$

$$\mathrm{OR}(\mathbf{x},\tilde{\mathbf{x}}) = \prod_{j=1}^{p} \exp\left[\theta_j(x^{(j)} - \tilde{x}^{(j)})\right].$$

  If two individuals only differ on the j-th variable:

$$\mathrm{OR}(\mathbf{x},\tilde{\mathbf{x}}) = \exp\left[\theta_j(x^{(j)} - \tilde{x}^{(j)})\right].$$

1. $OR(x, \tilde{x}) > 1$

$$\frac{\pi(x)}{1-\pi(x)} > \frac{\pi(\tilde{x})}{1-\pi(\tilde{x})}$$

$$\pi(x) - \cancel{\pi(x)\pi(\tilde{x})} > \pi(\tilde{x}) - \cancel{\pi(x)\pi(\tilde{x})}$$

$$\pi(x) > \pi(\tilde{x})$$

2. $OR(x, \tilde{x}) = \dfrac{odds(x)}{odds(\tilde{x})}$

$$odds(x) = \exp\left[\text{logit}(\pi(x))\right]$$
$$= \exp\left[\theta_0 + \sum_{j=1}^{p} \theta_j \, x^{(j)}\right]$$
$$odds(\tilde{x}) = \exp\left[\theta_0 + \sum_{j=1}^{p} \theta_j \, \tilde{x}^{(j)}\right]$$

$$OR(x, \tilde{x}) = \exp\left[\sum_{j=1}^{p} \theta_j \left(x^{(j)} - \tilde{x}^{(j)}\right)\right]$$

Si on s'intéresse à θ_j :
$x^{(k)} = \tilde{x}^{(k)} \quad \forall k \neq j$

$$OR(x, \tilde{x}) = \exp\left[\theta_j \left(x^{(j)} - \tilde{x}^{(j)}\right)\right]$$

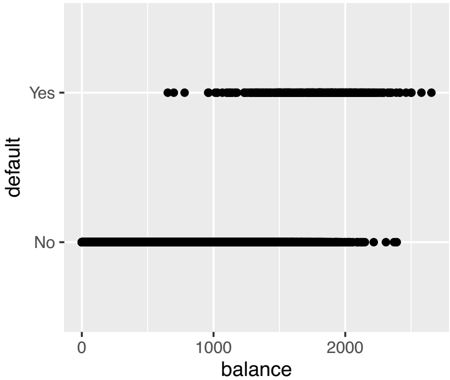
Outline

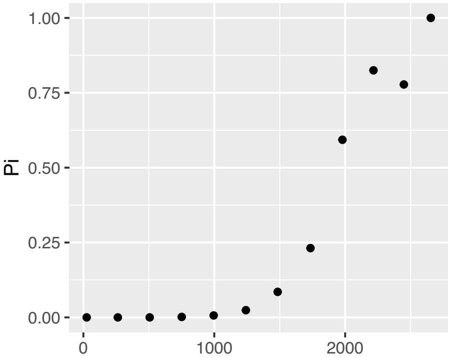
1. **Logistic regression**
   - Modeling
   - Odds and odds ratio
   - Simple logistic regression
   - Multiple logistic regression

- Goal: explain *default* with the variable *balance*.
- Model:

logit(π_i)

$$Y_i \sim \mathcal{B}(\underset{\pi_i}{\pi_\theta(\mathbf{x}_i)}) \text{ with } \ln\left(\frac{\pi_\theta(\mathbf{x_i})}{1-\pi_\theta(\mathbf{x_i})}\right) = \theta_0 + \underline{\theta_1 x_i}.$$





Parameter estimation

- The parameter vector $\theta = (\theta_0, \theta_1)'$ is estimated by maximum likelihood.
- The likelihood: $L(\underline{Y};\theta) = \prod_{i=1}^n \pi_\theta(\mathbf{x}_i)^{Y_i}[1-\pi_\theta(\mathbf{x}_i)]^{1-Y_i}$
- The log-likelihood:

$g(\pi_i) = \ln\left(\frac{\pi_i}{1-\pi_i}\right) = x_i\theta$

$$
\begin{aligned}
l(\underline{Y};\theta) &= \sum_{i=1}^n \left\{ Y_i \ln[\pi_\theta(\mathbf{x}_i)] + (1-Y_i)\ln[1-\pi_\theta(\mathbf{x}_i)] \right\} \\
&= \sum_{i=1}^n \left\{ Y_i \ln\left[F(\theta_0+\theta_1 x_i)\right] + (1-Y_i)\ln\left[1-F(\theta_0+\theta_1 x_i)\right] \right\}
\end{aligned}
$$

$\pi_i = F(x_i\theta)$

Rem : $\frac{F'(u)}{F(u)} = 1-F(u)$

- System to be solved:

cf poly détail calculs.

$$
\begin{cases}
\sum_{i=1}^n [Y_i - \pi_\theta(\mathbf{x}_i)] = 0 \iff \frac{\partial}{\partial\theta_0} l(\underline{Y};\theta) = 0 \\
\sum_{i=1}^n x_i [Y_i - \pi_\theta(\mathbf{x}_i)] = 0 \iff \frac{\partial}{\partial\theta_1} l(\underline{Y};\theta) = 0
\end{cases}
$$

Parameter estimation

- In order to use a Newton-Raphson or a Fisher-scoring algorithm, the Hassian matrix or the Fisher information matrix.

$$\mathbb{E}\left[-\frac{\partial^2 \ell(y,\theta)}{\partial\theta_j\partial\theta_k}\right]$$

$$\begin{cases} \dfrac{\partial^2 l(\underline{Y};\theta)}{\partial\theta_0^2} = -\displaystyle\sum_{i=1}^n F(\theta_0+\theta_1 x_i)[1-F(\theta_0+\theta_1 x_i)] = -\sum_{i=1}^n \pi_i(1-\pi_i) \\ \dfrac{\partial^2 l(\underline{Y};\theta)}{\partial\theta_1^2} = -\displaystyle\sum_{i=1}^n x_i^2 F(\theta_0+\theta_1 x_i)[1-F(\theta_0+\theta_1 x_i)] \\ \dfrac{\partial^2 l(\underline{Y};\theta)}{\partial\theta_0\partial\theta_1} = -\displaystyle\sum_{i=1}^n x_i F(\theta_0+\theta_1 x_i)[1-F(\theta_0+\theta_1 x_i)] \end{cases}$$

$$\mathcal{I}_n(\theta) = \begin{pmatrix} \displaystyle\sum_{i=1}^n \pi_\theta(\mathbf{x}_i)(1-\pi_\theta(\mathbf{x}_i)) & \displaystyle\sum_{i=1}^n x_i\pi_\theta(\mathbf{x}_i)(1-\pi_\theta(\mathbf{x}_i)) \\ \displaystyle\sum_{i=1}^n x_i\pi_\theta(\mathbf{x}_i)(1-\pi_\theta(\mathbf{x}_i)) & \displaystyle\sum_{i=1}^n x_i^2\pi_\theta(\mathbf{x}_i)(1-\pi_\theta(\mathbf{x}_i)) \end{pmatrix} = \left(X'WX\right),$$

with $W = \text{diag}\left[\pi_\theta(\mathbf{x}_1)(1-\pi_\theta(\mathbf{x}_1))\ ,\ \ldots\ ,\ \pi_\theta(\mathbf{x}_n)(1-\pi_\theta(\mathbf{x}_n))\right].$ $\Longrightarrow \hat{\theta}_{MV}$

$$I_n(\theta) = \begin{pmatrix} \sum_{i=1}^{n} \pi_i(1-\pi_i) & \sum_{i=1}^{n} x_i \pi_i(1-\pi_i) \\ \sum_{i=1}^{n} x_i \pi_i(1-\pi_i) & \sum_{i=1}^{n} x_i^2 \pi_i(1-\pi_i) \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & x_1 \\ | & | \\ 1 & x_n \end{pmatrix}$$

$$I_n(\theta) = X' W X$$

$$W = \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix}$$

$$X' W X = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} w_1 & & 0 \\ & \ddots & \\ 0 & & w_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ | & | \\ 1 & x_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{n} w_i & \sum_{i=1}^{n} x_i w_i \\ \sum_{i=1}^{n} x_i w_i & \sum_{i=1}^{n} x_i^2 w_i \end{pmatrix} \qquad w_i = \pi_i(1-\pi_i)$$

Example

```
glm.balance<-glm(default~balance,data=Default,family=binomial(link="logit"))
summary(glm.balance)
```

```
Call:
glm(formula = default ~ balance, family = binomial(link = "logit"),
    data = Default)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.2697  -0.1465  -0.0589  -0.0221   3.7589

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.065e+01  3.612e-01  -29.49   <2e-16 ***
balance      5.499e-03  2.204e-04   24.95   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 1596.5  on 9998  degrees of freedom
AIC: 1600.5

Number of Fisher Scoring iterations: 8
```

$\hat{\theta}_0^{obs}$

$\hat{\theta}_1^{obs}$

z-test

test de nullité de param.

$n-1$

$n-2$

$\mathcal{D}(M_0)$ $(M_0) : g(\pi_i) = \mu \ \forall i$

$\mathcal{D}(M)$

↳ pseudo R^2

$= 1 - \frac{\mathcal{D}(M)}{\mathcal{D}(M_0)}$

Example

```
import pandas as pd
import numpy as np
import statsmodels.api as sm
Defaultpy=r.Default
y=Defaultpy["default"].cat.codes
x=Defaultpy["balance"]
x_stat = sm.add_constant(x)
modelbalance = sm.Logit(y, x_stat).fit()
```

```
Optimization terminated successfully.
         Current function value: 0.079823
         Iterations 10
```

```
modelbalance.summary()
```

```
<class 'statsmodels.iolib.summary.Summary'>
"""
                           Logit Regression Results
==============================================================================
Dep. Variable:                      y   No. Observations:                10000
Model:                          Logit   Df Residuals:                     9998
Method:                           MLE   Df Model:                            1
Date:                Mar, 22 aoû 2023   Pseudo R-squ.:                  0.4534
Time:                        09:45:52   Log-Likelihood:                -798.23
converged:                       True   LL-Null:                       -1460.3
Covariance Type:            nonrobust   LLR p-value:                 6.233e-290
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const        -10.6513      0.361    -29.491      0.000     -11.359      -9.943
balance        0.0055      0.000     24.952      0.000       0.005       0.006
==============================================================================
```

Prediction

$$\pi_i \xrightarrow{g} \eta_i = x_i\theta$$

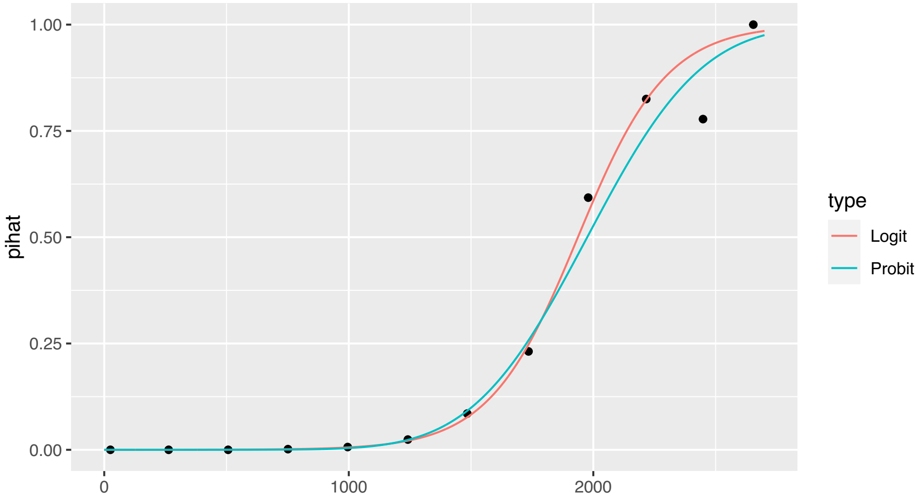
$$\hat{\pi}_i \xleftarrow{F = g^{-1}} \hat{\eta}_i = x_i\hat{\theta}$$

- Linear predictor: $\hat{\eta}_i = \hat{\theta}_0 + x_i\hat{\theta}_1$
- $\hat{\pi}(\mathbf{x}_i) = F(\hat{\eta}_i) = F(\hat{\theta}_0 + \hat{\theta}_1 x_i) = \pi_{\hat{\theta}}(\mathbf{x}_i)$.
- Adjusted values $\hat{Y}_i$ using the Bayes'rule :

$$\hat{Y}_i = \begin{cases} 1 & \text{if } \hat{\pi}(\mathbf{x}_i) > s \\ 0 & \text{otherwise.} \end{cases}$$

- For a new individual $\mathbf{x}_0 = (1, x_0)$, the fitted model allows to predict
  - a proportion (probability) $\hat{\pi}(\mathbf{x}_0) = F(\hat{\theta}_0 + \hat{\theta}_1 x_0)$
  - a predicted response $\hat{Y}_0 = \mathbb{1}_{\hat{\pi}(\mathbf{x}_0) > s}$.
- The threshold by default is $s = 0.5$

Prediction



Confidence interval

$\mathcal{I}_n(\hat\theta)^{1/2}(\hat\theta - \theta) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}_2(0_2, I_2)$

pour n assez grand, $\mathcal{I}_n(\hat\theta)^{1/2}(\hat\theta-\theta) \approx \mathcal{N}_2(0_2, I_2)$

$\hat\theta - \theta \approx \mathcal{N}_2(0_2, \mathcal{I}_n(\hat\theta)^{-1})$

- Goal: Construct a confidence interval for θ_j $\quad \hat\theta_j - \theta_j \approx \mathcal{N}(0, [\mathcal{I}_n(\hat\theta)^{-1}]_{jj})$
- Methods: Wald's method or method based on the likelihood ratio
- With Wald,

$$IC_{1-\alpha}(\theta_j) = \left[(\widehat{\theta}_{ML})_j \pm \underbrace{z_{1-\alpha/2}}_{} \sqrt{[\mathcal{I}_n(\widehat{\theta}_{ML})^{-1}]_{jj}}\right]$$

$1-\alpha/2$ quantile d'une $\mathcal{N}(0,1)$.

$\mathbb{P}\left(-z \le \frac{\hat\theta_j - \theta_j}{\sqrt{\cdots}} \le z\right) \xrightarrow[n\to+\infty]{} \mathbb{P}(-z \le Z \le z) = 1-\alpha$.

où $Z \sim \mathcal{N}(0,1)$.

Example

```
# likelihood ratio
confint(glm.balance)
```

```
                   2.5 %       97.5 %
(Intercept) -11.383288936 -9.966565064
balance       0.005078926  0.005943365
```

```
# Wald
confint.default(glm.balance)
```

```
                   2.5 %       97.5 %
(Intercept) -11.359186056 -9.943475172
balance       0.005066999  0.005930835
```

```
ci = modelbalance.conf_int(0.05)
print(ci)
```

```
                 0         1
const   -11.359208 -9.943453
balance   0.005067  0.005931
```

Test for the nullity of θ_j (Z-test)

- Hypotheses: $\mathcal{H}_0 : \theta_j = 0$ against $\mathcal{H}_1 : \theta_j \neq 0$
- Based on the result

$$\mathcal{I}_n(\hat{\theta}_{ML})^{1/2}(\hat{\theta}_{ML} - \theta) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0_2, I_2),$$

we prove that, under $\mathcal{H}_0$,

$$\frac{\hat{\theta}_j}{\hat{\sigma}_j} \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0,1) \text{ with } \hat{\sigma}_j = \sqrt{[\mathcal{I}(\hat{\theta}_{ML})^{-1}]_{jj}}$$

- Reject zone (asymptotic test of level α):

$$\mathbb{P}_{H_0}\left(\left| \frac{\hat{\theta}_j}{\hat{\sigma}_j} \right| > z \right)$$

$$\mathcal{R}_\alpha = \left\{ \left| \hat{\theta}_j / \hat{\sigma}_j \right| > z_{1-\alpha/2} \right\} \xrightarrow[n\to+\infty]{} \mathbb{P}(|Z| > z) = \alpha .$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of $\mathcal{N}(0,1)$. où $Z \sim \mathcal{N}(0,1)$.

Example

```
summary(glm.balance)
```

```
Call:
glm(formula = default ~ balance, family = binomial(link = "logit"),
    data = Default)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.2697  -0.1465  -0.0589  -0.0221   3.7589

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.065e+01  3.612e-01  -29.49   <2e-16 ***
balance      5.499e-03  2.204e-04   24.95   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 1596.5  on 9998  degrees of freedom
AIC: 1600.5

Number of Fisher Scoring iterations: 8
```

Since p-values $< 2e - 16$, we reject the nullity of both parameters.

Example

```
modelbalance.summary()
```

```
<class 'statsmodels.iolib.summary.Summary'>
"""
                           Logit Regression Results
==============================================================================
Dep. Variable:                      y   No. Observations:                10000
Model:                          Logit   Df Residuals:                     9998
Method:                           MLE   Df Model:                            1
Date:                Mar, 22 aoû 2023   Pseudo R-squ.:                  0.4534
Time:                        09:45:53   Log-Likelihood:                -798.23
converged:                       True   LL-Null:                       -1460.3
Covariance Type:            nonrobust   LLR p-value:                6.233e-290
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const        -10.6513      0.361    -29.491      0.000     -11.359      -9.943
balance        0.0055      0.000     24.952      0.000       0.005       0.006
==============================================================================

Possibly complete quasi-separation: A fraction 0.13 of observations can be
perfectly predicted. This might indicate that there is complete
quasi-separation. In this case some parameters will not be identified.
"""
```

Test for the nullity of θ_j

- It is also possible to consider a submodel testing
- Example for the nullity of θ_1

```
anova(glm(default~1,data=Default,family=binomial(link="logit")), glm.balance,test="Chisq")
```

```
Analysis of Deviance Table

Model 1: default ~ 1
Model 2: default ~ balance
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1      9999     2920.7
2      9998     1596.5  1   1324.2 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$T = \mathcal{D}(m_0) - \mathcal{D}(m) \xrightarrow[n \to +\infty]{\mathcal{L}} \chi^2(\ell-1)$

```
from scipy.stats import chi2
LR_stat = (-2)* (modelbalance.llnull - modelbalance.llf);
df = 1
pvalue = 1 - chi2(df).cdf(LR_stat);
print(LR_stat)
```

```
1324.1980279638472
```

```
print(pvalue)
```

```
0.0
```

With a qualitative explanatory variable

- Goal: Explain *default* with the variable *student* (2 levels).
- Possible models:

$$\begin{aligned}
\operatorname{logit}\left[\pi_\theta(\mathbf{x}_i)\right] &= \theta_0 + \theta_1 \mathbb{1}_{x_i=1} + \theta_2 \mathbb{1}_{x_i=0} \\
&= (\theta_0 + \theta_2) + (\theta_1 - \theta_2)\mathbb{1}_{x_i=1} + 0\ \mathbb{1}_{x_i=0}.
\end{aligned}$$

- $\Rightarrow$ non-identifiable model $\Rightarrow$ constraint on parameters.
- For example, if we assume that $\theta_2 = 0$, the model is

$$\operatorname{logit}[\pi_\theta(\mathbf{x}_i)] = \theta_0 + \theta_1 \mathbb{1}_{x_i=1}$$

- We can then use the same reasoning to estimate the parameters, construct confidence intervals, test the nullity of each parameter, ...

Example

- With R, the constraint by default is $\theta_2 = 0$:

```
glm.student = glm(default~student, data=Default, family=binomial)
summary(glm.student)
```

```
Call:
glm(formula = default ~ student, family = binomial, data = Default)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-0.2970  -0.2970  -0.2434  -0.2434   2.6585

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.50413    0.07071  -49.55  < 2e-16 ***
studentYes   0.40489    0.11502    3.52 0.000431 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 2908.7  on 9998  degrees of freedom
AIC: 2912.7

Number of Fisher Scoring iterations: 6
```

Example

```python
y=Defaultpy["default"].cat.codes
x=Defaultpy["student"].cat.codes
x_stat = sm.add_constant(x)
modelstudent = sm.Logit(y, x_stat).fit();
```

```
Optimization terminated successfully.
         Current function value: 0.145434
         Iterations 7
```

```python
modelstudent.summary()
```

```
<class 'statsmodels.iolib.summary.Summary'>
"""
                           Logit Regression Results
==============================================================================
Dep. Variable:                      y   No. Observations:                10000
Model:                          Logit   Df Residuals:                     9998
Method:                           MLE   Df Model:                            1
Date:                Mar, 22 aoû 2023   Pseudo R-squ.:                0.004097
Time:                        09:45:54   Log-Likelihood:                -1454.3
converged:                       True   LL-Null:                       -1460.3
Covariance Type:            nonrobust   LLR p-value:                 0.0005416
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -3.5041      0.071    -49.554      0.000      -3.643      -3.366
0              0.4049      0.115      3.520      0.000       0.179       0.630
==============================================================================
"""
```

Example

- Model: $Y_i \sim \mathcal{B}(\pi_\theta(\mathbf{x}_i))$ with

$$\text{logit}[\pi_\theta(\mathbf{x}_i)] = \theta_0 + \theta_1 \ \mathbb{1}_{student_i=1}$$

- Odds and odds ratio:

$$\begin{cases} \text{odds("student")} = e^{\theta_0+\theta_1} = 0.045 \\ \text{odds("non-student")} = e^{\theta_0} = 0.030 \\ \text{OR("student", "non-student")} = e^{\theta_1} = 1.5 \end{cases}$$

odds ("student") = 1.5 odds("non student")

Thus a student is 1.5 times more likely to be in default than a non-student.

```
exp(c(glm.student$coefficients,sum(glm.student$coefficients)))
```

```
(Intercept)  studentYes
 0.03007299  1.49913321  0.04508342
```

Outline

1. **Logistic regression**
   - Modeling
   - Odds and odds ratio
   - Simple logistic regression
   - Multiple logistic regression

Context

- A binary response variable Y
- p regressors $x^{(1)}, \dots, x^{(p)}$
- Example: $p = 3$ regressors
  - 1 qualitative variable (*student* $= x^{(1)}$)
  - 2 quantitative variables (*balance* $= x^{(2)}$, *income*$=x^{(3)}$)

- Model: $Y_i \sim \mathcal{B}(\pi_\theta(\mathbf{x}_i))$ indep. with

$$\text{logit}[\pi_\theta(\mathbf{x}_i)] = \theta_0 + \theta_1 \; \mathbb{1}_{x_i^{(1)}=1} + \theta_2 \; x_i^{(2)} + \theta_3 \; x_i^{(3)}$$

```
glm.additif<-glm(default~.,data=Default,family=binomial(link="logit"))
summary(glm.additif)
```

```
Call:
glm(formula = default ~ ., family = binomial(link = "logit"),
    data = Default)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.4691  -0.1418  -0.0557  -0.0203   3.7383

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.087e+01  4.923e-01 -22.080  < 2e-16 ***
studentYes  -6.468e-01  2.363e-01  -2.738  0.00619 **
balance      5.737e-03  2.319e-04  24.738  < 2e-16 ***
income       3.033e-06  8.203e-06   0.370  0.71152
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 1571.5  on 9996  degrees of freedom
AIC: 1579.5
```

Model without interaction

```
Defaultpy=r.DefaultBIS
y=Defaultpy["default"]
x=Defaultpy[Defaultpy.columns.drop("default")]
x_stat = sm.add_constant(x)
modeladditif = sm.Logit(y, x_stat).fit()
```

```
Optimization terminated successfully.
         Current function value: 0.078577
         Iterations 10
```

```
modeladditif.summary()
```

```
<class 'statsmodels.iolib.summary.Summary'>
"""
                           Logit Regression Results
==============================================================================
Dep. Variable:                default   No. Observations:                10000
Model:                          Logit   Df Residuals:                     9996
Method:                           MLE   Df Model:                            3
Date:                Mar, 22 aoû 2023   Pseudo R-squ.:                  0.4619
Time:                        09:45:54   Log-Likelihood:                -785.77
converged:                       True   LL-Null:                       -1460.3
Covariance Type:            nonrobust   LLR p-value:                3.257e-292
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const        -10.8690      0.492    -22.079      0.000     -11.834      -9.904
student       -0.6468      0.236     -2.738      0.006      -1.110      -0.184
balance        0.0057      0.000     24.737      0.000       0.005       0.006
income      3.033e-06    8.2e-06      0.370      0.712    -1.3e-05    1.91e-05
==============================================================================

Possibly complete quasi-separation: A fraction 0.15 of observations can be
```

Test of nullity of θ_j

- Test of nullity $\Longrightarrow$ Z-test
  In our example, p-value=0.71152 for the nullity of $\theta_3 \Longrightarrow$ we can remove the variable *income* from the model
- We can reach the same conclusion by a sub-model test

```
glm.sansincome<-glm(default~student+balance,data=Default,family=binomial(link="logit"))
anova(glm.sansincome,glm.additif,test="Chisq")
```

```
Analysis of Deviance Table

Model 1: default ~ student + balance
Model 2: default ~ student + balance + income
  Resid. Df Resid. Dev Df Deviance Pr(>Chi)
1      9997     1571.7
2      9996     1571.5  1  0.13677   0.7115
```

Test the nullity of θ_2 and θ_3 simul.

- We want to test the nullity of θ_2 and θ_3 simultaneously
- Method 1: sub-model testing
  - (M_0): $\text{logit}[\pi_\theta(\mathbf{x}_i)] = \theta_0 + \theta_1 \mathbb{1}_{x_i^{(1)}=1}$ $k_0 = 2$
  - (M_1): $\text{logit}[\pi_\theta(\mathbf{x}_i)] = \theta_0 + \theta_1 \ \mathbb{1}_{x_i^{(1)}=1} + \theta_2 \ x_i^{(2)} + \theta_3 \ x_i^{(3)}$ $k = 4$

$$T = \mathcal{D}(M_0) - \mathcal{D}(M_1) \underset{n\to+\infty}{\overset{\mathcal{L}}{\to}} \chi^2(4-2)$$

```
anova(glm.student,glm.additif,test="Chisq")
```

```
Analysis of Deviance Table

Model 1: default ~ student
Model 2: default ~ student + balance + income
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1      9998     2908.7
2      9996     1571.5  2   1337.1 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Test the nullity of θ_2 and θ_3 simul.

- We want to test the nullity of θ_2 and θ_3 simultaneously
- Method 2: Wald's test

$$\mathcal{H}_0 : C\theta = 0_2 \text{ against } \mathcal{H}_1 : C\theta \neq 0_2 \text{ with } C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Under $\mathcal{H}_0$,

$$T = (C\hat{\theta}_{ML})' \left[C\mathcal{I}_n(\hat{\theta}_{ML})^{-1} C' \right]^{-1} (C\hat{\theta}_{ML}) \xrightarrow[n\to+\infty]{\mathcal{L}} \chi^2(2).$$

```
hattheta <- glm.additif$coefficients
hatpi <- glm.additif$fitted.values
W <- diag(hatpi*(1-hatpi))
X <- cbind(rep(1,10000),student,balance,income)
In <- t(X) %*% W %*% X
C <- matrix(c(0,0,1,0,0,0,0,1),nrow=4)
t(t(C)%*%hattheta) %*% solve(t(C)%*%solve(In)%*%C) %*% (t(C)%*%hattheta) > qchisq(0.95,df=2)
```

```
     [,1]
[1,] TRUE
```

Variable selection

- More generally, a variable selection procedure can be implemented
- For instance, we can implement a backward selection procedure based on the AIC criterion

```
step.backward <- step(glm.additif)
```

```
Start:  AIC=1579.54
default ~ student + balance + income

          Df Deviance    AIC
- income   1   1571.7 1577.7
<none>         1571.5 1579.5
- student  1   1579.0 1585.0
- balance  1   2907.5 2913.5

Step:  AIC=1577.68
default ~ student + balance

          Df Deviance    AIC
<none>         1571.7 1577.7
- student  1   1596.5 1600.5
- balance  1   2908.7 2912.7
```

Variable selection

- We can use *stepAIC* (*MASS* library) with AIC (option "p=2") or BIC (option "p=log(n)")

```
library(MASS)
stepAIC(glm.additif, direction=c("backward"),p=2,trace=0) # AIC
```

```
Call:  glm(formula = default ~ student + balance, family = binomial(link = "logit"),
    data = Default)

Coefficients:
(Intercept)   studentYes      balance
 -10.749496    -0.714878     0.005738

Degrees of Freedom: 9999 Total (i.e. Null);  9997 Residual
Null Deviance:      2921
Residual Deviance: 1572     AIC: 1578
```

```
stepAIC(glm.additif, direction=c("backward"),p=log(nrow(Default))) # BIC
```

```
Start:  AIC=1579.54
default ~ student + balance + income

          Df Deviance    AIC
- income   1   1571.7 1577.7
<none>         1571.5 1579.5
- student  1   1579.0 1585.0
- balance  1   2907.5 2913.5

Step:  AIC=1577.68
default ~ student + balance
```

Model with interaction

- Full model with all interactions (order 2):

$$\begin{aligned}\mathrm{logit}[\pi_\theta(\mathbf{x}_i)] &= \theta_0 + \theta_2 x_i^{(2)} + \theta_3 x_i^{(3)} + \theta_{23} x_i^{(2)} x_i^{(3)} \\ &\quad + (\beta_1 + \beta_2 x_i^{(2)} + \beta_3 x_i^{(3)}) \mathbb{1}_{x_i^{(1)}=1}\end{aligned}$$

- Then, use a variable selection procedure to simplify the model and valid with a testing procedure

Model with interaction

```
glm.full<-glm(default~.^2,data=Default,family="binomial")
summary(glm.full)
```

```
Call:
glm(formula = default ~ .^2, family = "binomial", data = Default)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.4848  -0.1417  -0.0554  -0.0202   3.7579

Coefficients:
                     Estimate Std. Error z value Pr(>|z|)
(Intercept)        -1.104e+01  1.866e+00  -5.914 3.33e-09 ***
studentYes         -5.201e-01  1.344e+00  -0.387    0.699
balance             5.882e-03  1.180e-03   4.983 6.27e-07 ***
income              4.050e-06  4.459e-05   0.091    0.928
studentYes:balance -2.551e-04  7.905e-04  -0.323    0.747
studentYes:income   1.447e-05  2.779e-05   0.521    0.602
balance:income     -1.579e-09  2.815e-08  -0.056    0.955
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 1571.1  on 9993  degrees of freedom
AIC: 1585.1

Number of Fisher Scoring iterations: 8
```

Model with interaction

```
stepAIC(glm.full, direction=c("backward"),p=log(nrow(Default)),trace=0)
```

```
Call:  glm(formula = default ~ student + balance, family = "binomial",
    data = Default)

Coefficients:
(Intercept)   studentYes      balance
 -10.749496    -0.714878     0.005738

Degrees of Freedom: 9999 Total (i.e. Null);  9997 Residual
Null Deviance:      2921
Residual Deviance: 1572     AIC: 1578
```

```
anova(glm.sansincome,glm.full)
```

```
Analysis of Deviance Table

Model 1: default ~ student + balance
Model 2: default ~ (student + balance + income)^2
  Resid. Df Resid. Dev Df Deviance
1      9997     1571.7
2      9993     1571.1  4  0.61588
```

$T = \mathcal{D}(M_1) - \mathcal{D}(M_2) \xrightarrow[n\to+\infty]{\mathcal{L}} \chi^2(4)$

on ne rejette pas H_0 au risque 5%

$\Rightarrow$ on ne conserve que les variables Student & Bal.

Focus on “default ∼ student+balance”

- Model: $Y_i \sim \mathcal{B}(\pi_\theta(\mathbf{x}_i))$ indep. with

$$\text{logit}[\pi_\theta(\mathbf{x}_i)] = \theta_0 + \theta_1 \mathbb{1}_{x_i^{(1)}=1} + \theta_2 x_i^{(2)} = X_i\theta \text{ where } X_i = (1, \mathbb{1}_{x_i^{(1)}=1}, x_i^{(2)})$$

```
glm.final = glm(default ~ student + balance, data=Default, family=binomial)
summary(glm.final)
```

```
Call:
glm(formula = default ~ student + balance, family = binomial,
    data = Default)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.4578  -0.1422  -0.0559  -0.0203   3.7435

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.075e+01  3.692e-01 -29.116  < 2e-16 ***
studentYes  -7.149e-01  1.475e-01  -4.846 1.26e-06 ***
balance      5.738e-03  2.318e-04  24.750  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2920.6  on 9999  degrees of freedom
Residual deviance: 1571.7  on 9997  degrees of freedom
AIC: 1577.7

Number of Fisher Scoring iterations: 8
```

$\hat{\theta}$

Focus on "default $\sim$ student+balance"

- Probability prediction: $= F(\hat{\eta}_i)$

$$\hat{\pi}_i = \frac{e^{\hat{\eta}_i}}{1 + e^{\hat{\eta}_i}} \text{ with } \hat{\eta}_i = X_i \hat{\theta}_{ML}$$

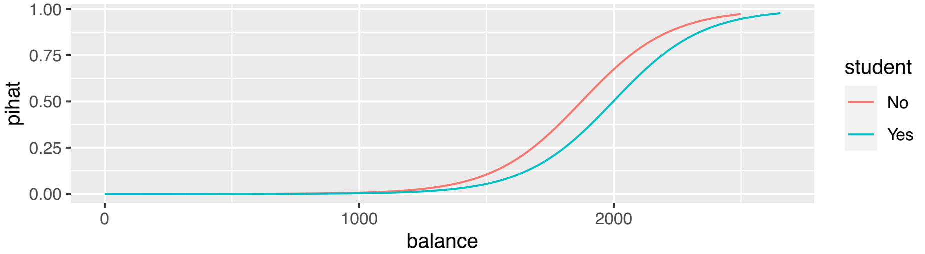
- Fitted values $\hat{Y}_i = \mathbb{1}_{\hat{\pi}_i > 0.5} = \delta$

```
hatpi <- glm.final$fitted.values
table(default,hatpi>0.5)
```

$\hat{y}_i$

y_i

```
default FALSE TRUE
     No  9628   39
    Yes   228  105
```



Parameter interpretation

- 2 individuals with the same balance value (*bal*), one student and one non-student $\mathbf{x} = (1, 1, bal)$ and $\tilde{\mathbf{x}} = (1, 0, bal)$ then

$$OR(\mathbf{x}, \tilde{\mathbf{x}}) = \frac{e^{\theta_0+\theta_1+\theta_2 bal}}{e^{\theta_0+\theta_2 bal}} = e^{\theta_1}$$

thus a student is $e^{-0.7149} = 0.489$ times likely to be in default than a non-student for the same value of *balance*.

- 2 individuals (both student or non-student) and balance($\mathbf{x}$)= balance($\tilde{\mathbf{x}}$)+1

$$OR(\mathbf{x}, \tilde{\mathbf{x}}) = e^{\theta_2 \overbrace{(balance(\mathbf{x})-balance(\tilde{\mathbf{x}}))}^{=1}}$$

thus when we increase the balance variable by one unit, the chance of being at default is multiplied by $e^{0.005738} = 1.005$.

Outline

1 Logistic regression

2 Log regression

Outline

2 Log regression

- Description of the example
- Log regression with one regressor
- Multiple log regression

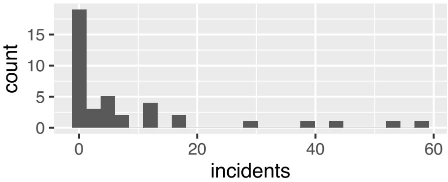
Example : Number of maritime accidents

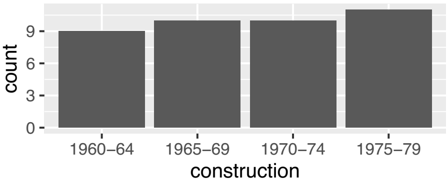
- We are interested in the variable *incidents* = number of damage incidents per month of commissioning of a ship
- The data frame contains 40 observations on 5 ship types in 4 vintages and 2 service periods:
  - *type*: factor with levels "A" to "E" for the different ship types
  - *construction*: factor with levels "1960-64", "1965-69", "1970-74", "1975-79" for the periods of construction
  - *operation*: factor with levels "1960-74", "1975-79" for the periods of operation
  - *service*: aggregate months of service

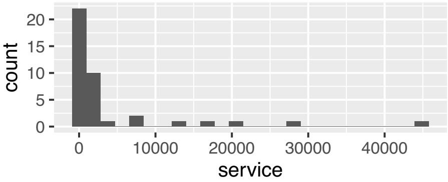
Example

```
data("ShipAccidents")
#str(ShipAccidents)
summary(ShipAccidents)
```

```
 type   construction   operation      service          incidents
 A:8    1960-64: 9    1960-74:19   Min.   :    0.0   Min.   : 0.0
 B:8    1965-69:10    1975-79:21   1st Qu.:  175.8   1st Qu.: 0.0
 C:8    1970-74:10                 Median :  782.0   Median : 2.0
 D:8    1975-79:11                 Mean   : 4089.3   Mean   : 8.9
 E:8                               3rd Qu.: 2078.5   3rd Qu.:11.0
                                   Max.   :44882.0   Max.   :58.0
```







Outline

2 **Log regression**

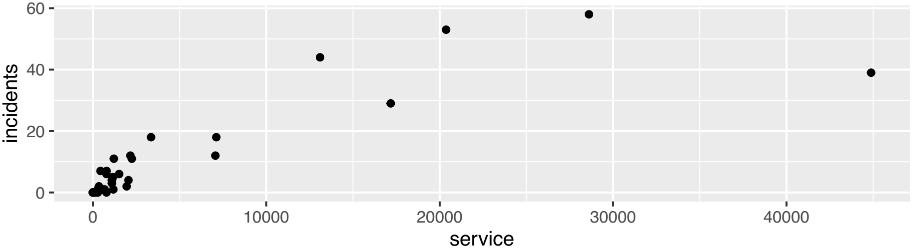
- Description of the example
- Log regression with one regressor
- Multiple log regression

With a quantitative explanatory variable

- Goal: explain the response variable *incidents* (Y) with the quantitative variable *service* (x)

- Model:

$$\begin{cases} Y_i \sim \mathcal{P}(\lambda(\mathbf{x}_i)), \ \forall i = 1, \ldots, n \\ \ln[\lambda(\mathbf{x}_i)] = \theta_0 + \theta_1 x_i \\ Y_1, \ldots, Y_n \text{ independent} \end{cases}$$



Example

```
fit.service <- glm(incidents ~ service, data=ShipAccidents, family=poisson)
summary(fit.service)
```

```
Call:
glm(formula = incidents ~ service, family = poisson, data = ShipAccidents)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-6.0040  -3.1674  -2.0055   0.9155   7.2372

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) 1.613e+00  7.150e-02   22.55   <2e-16 ***
service     6.417e-05  2.870e-06   22.36   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 730.25  on 39  degrees of freedom
Residual deviance: 374.55  on 38  degrees of freedom
AIC: 476.41

Number of Fisher Scoring iterations: 6
```

Example

```
import pandas as pd
import numpy as np
import statsmodels.api as sm
from statsmodels.formula.api import glm

Accidpy=r.ShipAccidents
fitservicepy=glm('incidents~service',data=Accidpy,family=sm.families.Poisson()).fit()
print(fitservicepy.summary())
```

```
                 Generalized Linear Model Regression Results
==============================================================================
Dep. Variable:              incidents   No. Observations:                   40
Model:                            GLM   Df Residuals:                       38
Model Family:                 Poisson   Df Model:                            1
Link Function:                    Log   Scale:                          1.0000
Method:                          IRLS   Log-Likelihood:                -236.21
Date:                Mar, 22 aoû 2023   Deviance:                       374.55
Time:                        09:45:58   Pearson chi2:                     368.
No. Iterations:                     6   Pseudo R-squ. (CS):             0.9999
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      1.6127      0.072     22.555      0.000       1.473       1.753
service     6.417e-05   2.87e-06     22.356      0.000    5.85e-05    6.98e-05
==============================================================================
```

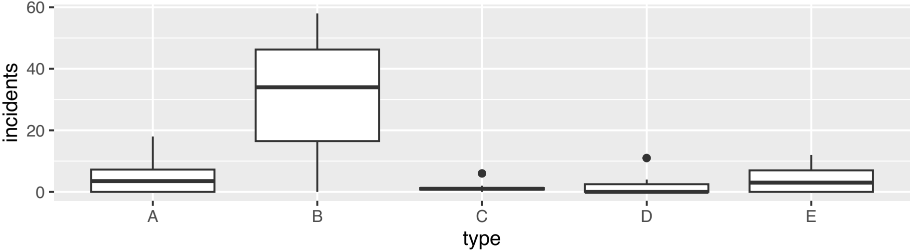
With a qualitative explanatory variable

- Goal: Explain *incidents* with the qualitative variable *type* having 5 levels.

- To make the model identifiable, we must choose a reference level (here, $type = A$).

- Model :

$$\begin{cases} Y_i \sim \mathcal{P}(\lambda(\mathbf{x}_i)), \ \forall i = 1, \ldots, n \\ \ln[\lambda(\mathbf{x}_i)] = \theta_0 + \theta_1 \mathbb{1}_{\texttt{type}_\texttt{i}=\texttt{B}} + \theta_2 \mathbb{1}_{\texttt{type}_\texttt{i}=\texttt{C}} + \theta_3 \mathbb{1}_{\texttt{type}_\texttt{i}=\texttt{D}} + \theta_4 \mathbb{1}_{\texttt{type}_\texttt{i}=\texttt{E}} \\ Y_1, \ldots, Y_n \text{ independent} \end{cases}$$



Example

```r
fit.type <- glm(incidents ~ type, data=ShipAccidents, family=poisson)
summary(fit.type)
```

```
Call:
glm(formula = incidents ~ type, family = poisson, data = ShipAccidents)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-7.9530  -2.0616  -0.4541   1.2873   4.3425

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   1.6582     0.1543  10.747  < 2e-16 ***
typeB         1.7957     0.1666  10.777  < 2e-16 ***
typeC        -1.2528     0.3273  -3.827  0.00013 ***
typeD        -0.9045     0.2875  -3.146  0.00165 **
typeE        -0.2719     0.2346  -1.159  0.24650
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 730.25  on 39  degrees of freedom
Residual deviance: 275.65  on 35  degrees of freedom
AIC: 383.52

Number of Fisher Scoring iterations: 6
```

Example

```python
fittypepy=glm('incidents~C(type)',data=Accidpy,family=sm.families.Poisson()).fit()
print(fittypepy.summary())
```

```
                 Generalized Linear Model Regression Results
==============================================================================
Dep. Variable:              incidents   No. Observations:                   40
Model:                            GLM   Df Residuals:                       35
Model Family:                 Poisson   Df Model:                            4
Link Function:                    Log   Scale:                          1.0000
Method:                          IRLS   Log-Likelihood:                -186.76
Date:                Mar, 22 aoû 2023   Deviance:                       275.65
Time:                        09:45:58   Pearson chi2:                     249.
No. Iterations:                     5   Pseudo R-squ. (CS):              1.000
Covariance Type:            nonrobust
================================================================================
                  coef    std err          z      P>|z|      [0.025      0.975]
--------------------------------------------------------------------------------
Intercept       1.6582      0.154     10.747      0.000       1.356       1.961
C(type)[T.B]    1.7957      0.167     10.777      0.000       1.469       2.122
C(type)[T.C]   -1.2528      0.327     -3.827      0.000      -1.894      -0.611
C(type)[T.D]   -0.9045      0.287     -3.146      0.002      -1.468      -0.341
C(type)[T.E]   -0.2719      0.235     -1.159      0.246      -0.732       0.188
================================================================================
```

Effect of variable *type*

- Since the variable *type* has 5 levels, a sub-model test is used to test the effect of this variable:
  - (M_0): $\ln[\lambda(\mathbf{x}_i)] = \theta_0$
  - (M_1): $\ln[\lambda(\mathbf{x}_i)] = \theta_0 + \theta_1 \mathbb{1}_{\mathtt{type_i=B}} + \theta_2 \mathbb{1}_{\mathtt{type_i=C}} + \theta_3 \mathbb{1}_{\mathtt{type_i=D}} + \theta_4 \mathbb{1}_{\mathtt{type_i=E}}$
- Test's statistics:

$$T = \mathcal{D}(M_0) - \mathcal{D}(M_1) \underset{n \to +\infty}{\overset{\mathcal{L}}{\to}} \chi^2(5-1)$$

- Reject zone: $\mathcal{R}_\alpha = \{T > v_{1-\alpha,4}\}$ where $v_{1-\alpha,4}$ is the $(1-\alpha)$ quantile of $\chi^2(4)$.
- P-value: $pval = \mathbb{P}_{\mathcal{H}_0}\left(T > T^{obs}\right) \underset{n \to +\infty}{\longrightarrow} \mathbb{P}(\chi^2(4) > T^{obs})$

Effect of variable *type*

```r
anova(glm(incidents ~ 1, data=ShipAccidents, family=poisson), fit.type, test="Chisq")
```

```
Analysis of Deviance Table

Model 1: incidents ~ 1
Model 2: incidents ~ type
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1        39     730.25
2        35     275.65  4    454.6 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```python
from scipy.stats import chi2
LR_stat=(-2)*(fittypepy.llnull - fittypepy.llf);
pvalue=1-chi2(4).cdf(LR_stat);
print(LR_stat)
```

```
454.60255359036773
```

```python
print(pvalue)
```

```
0.0
```

Outline

2 **Log regression**

- Description of the example
- Log regression with one regressor
- Multiple log regression

Multiple log regression

- Goal: Explain the response variable *incidents* with with all the available explanatory variables.

- Since a model with interactions (2nd order) has 37 parameters and the sample size is $n = 40$, we only consider an additive log regression model here.

- Model: $Y_i \sim \mathcal{P}(\lambda(\mathbf{x})_\mathbf{i})$ with

$$\begin{aligned}\ln[\lambda(\mathbf{x}_i)] = \quad & \theta_0 + \alpha_1 \mathbb{1}_{\mathtt{type_i=B}} + \alpha_2 \mathbb{1}_{\mathtt{type_i=C}} + \alpha_3 \mathbb{1}_{\mathtt{type_i=D}} + \alpha_4 \mathbb{1}_{\mathtt{type_i=E}} \\ & + \beta_1 \mathbb{1}_{const_i=\text{"65-69"}} + \beta_2 \mathbb{1}_{const_i=\text{"70-74"}} + \beta_3 \mathbb{1}_{const_i=\text{"75-79"}} \\ & + \gamma_1 \mathbb{1}_{op_i=\text{"75-79"}} + \theta_1 service_i\end{aligned}$$

Example

```r
fit.add <- glm(incidents ~ . , data=ShipAccidents, family=poisson)
summary(fit.add)
```

```

Call:
glm(formula = incidents ~ ., family = poisson, data = ShipAccidents)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.5810  -1.4773  -0.8972   0.5952   3.2154

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)          5.492e-04  2.787e-01   0.002 0.998427
typeB                5.933e-01  2.163e-01   2.743 0.006092 **
typeC               -1.190e+00  3.275e-01  -3.635 0.000278 ***
typeD               -8.210e-01  2.877e-01  -2.854 0.004321 **
typeE               -2.900e-01  2.351e-01  -1.233 0.217466
construction1965-69  1.148e+00  1.793e-01   6.403 1.53e-10 ***
construction1970-74  1.596e+00  2.242e-01   7.122 1.06e-12 ***
construction1975-79  5.670e-01  2.809e-01   2.018 0.043557 *
operation1975-79     8.619e-01  1.317e-01   6.546 5.92e-11 ***
service              7.270e-05  8.488e-06   8.565  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 730.253  on 39  degrees of freedom
Residual deviance:  99.793  on 30  degrees of freedom
AIC: 217.66

Number of Fisher Scoring iterations: 5
```

Example

```
fitaddpy =glm('incidents~C(type)+C(construction)+C(operation)+service',
              data=Accidpy,family=sm.families.Poisson()).fit()
print(fitaddpy.summary())
```

```
                 Generalized Linear Model Regression Results
==============================================================================
Dep. Variable:              incidents   No. Observations:                   40
Model:                            GLM   Df Residuals:                       30
Model Family:                 Poisson   Df Model:                            9
Link Function:                    Log   Scale:                          1.0000
Method:                          IRLS   Log-Likelihood:                -98.830
Date:                Mar, 22 aoû 2023   Deviance:                       99.793
Time:                        09:45:59   Pearson chi2:                     90.0
No. Iterations:                     6   Pseudo R-squ. (CS):              1.000
Covariance Type:            nonrobust
=============================================================================================
                                coef    std err          z      P>|z|      [0.025      0.975]
---------------------------------------------------------------------------------------------
Intercept                     0.0005      0.279      0.002      0.998      -0.546       0.547
C(type)[T.B]                  0.5933      0.216      2.743      0.006       0.169       1.017
C(type)[T.C]                 -1.1903      0.327     -3.635      0.000      -1.832      -0.548
C(type)[T.D]                 -0.8210      0.288     -2.854      0.004      -1.385      -0.257
C(type)[T.E]                 -0.2900      0.235     -1.233      0.217      -0.751       0.171
C(construction)[T.1965-69]    1.1479      0.179      6.403      0.000       0.796       1.499
C(construction)[T.1970-74]    1.5965      0.224      7.122      0.000       1.157       2.036
C(construction)[T.1975-79]    0.5670      0.281      2.018      0.044       0.016       1.118
C(operation)[T.1975-79]       0.8619      0.132      6.546      0.000       0.604       1.120
service                     7.27e-05   8.49e-06      8.565      0.000    5.61e-05    8.93e-05
=============================================================================================
```

Variable selection

- It is possible to implement a variable selection procedure using `step(fit.add)` (backward procedure with AIC criterion)

```
step(fit.add,trace=1)
```

```
Start:  AIC=217.66
incidents ~ type + construction + operation + service

               Df Deviance    AIC
<none>              99.793 217.66
- type          4  148.053 257.92
- operation     1  147.687 263.55
- service       1  182.605 298.47
- construction  3  191.419 303.29


Call:  glm(formula = incidents ~ type + construction + operation + service,
    family = poisson, data = ShipAccidents)

Coefficients:
        (Intercept)                typeB                typeC
          0.0005492            0.5932730           -1.1903189
              typeD                typeE  construction1965-69
         -0.8210370           -0.2899922            1.1478796
construction1970-74  construction1975-79      operation1975-79
          1.5964752            0.5669790            0.8618750
            service
          0.0000727

Degrees of Freedom: 39 Total (i.e. Null);  30 Residual
Null Deviance:      730.3
```

Test of sub-models

- We can for instance test the sub-model without the variables *contructions* and *operation*

```
fit.ssmod <- glm(incidents ~ type +service,data=ShipAccidents,family=poisson)
anova(fit.ssmod, fit.add,  test="Chisq")
```

```
Analysis of Deviance Table

Model 1: incidents ~ type + service
Model 2: incidents ~ type + construction + operation + service
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1        34    230.832
2        30     99.793  4   131.04 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
fitssmodpy = glm('incidents~C(type)+service',data=Accidpy,family=sm.families.Poisson()).fit()
LR_stat=(fitssmodpy.deviance - fitaddpy.deviance)
print(LR_stat)
```

```
131.0387423849605
```

```
print(1-chi2(4).cdf(LR_stat))
```

```
0.0
```

Prediction

- If we want to predict the average number of incidents for a ship with *type="A", construction= "65-69", operation="60-74", service=1000*

$$\hat{\lambda}_0 = e^{X_0 \hat{\theta}_{ML}} \text{ with } X_0 = (1, \underbrace{0, 0, 0, 0,}_{type} \underbrace{1, 0, 0}_{construction}, 0, 1000)$$

```
new.data = data.frame(type=factor("A"), construction=factor("1965-69"),operation=factor("1960-74"), service = 1
lambda_hat = exp(predict(fit.add,new.data))
lambda_hat
```

```
       1
3.391016
```

- Prediction of some probabilities: Let $A \sim \mathcal{P}(\hat{\lambda}_0)$. For instance,
  - ship has no incident: $\mathbb{P}(A = 0) = e^{-\hat{\lambda}_0}$
  - ship has at most one incident: $\mathbb{P}(A \leq 1) = (1 + \hat{\lambda}_0)e^{-\hat{\lambda}_0}$

```
c(exp(-lambda_hat),(1+lambda_hat) * exp(-lambda_hat))
```

```
         1          1
0.03367446 0.14786507
```

References I

[1] Jean-Marc Azais and Jean-Marc Bardet. *Le modèle linéaire par l'exemple-2e éd.: Régression, analyse de la variance et plans d'expérience illustrés avec R et SAS*. Dunod, 2012.

[2] Jean-Jacques Daudin. *Le modèle linéaire et ses extensions-Modèle linéaire général, modèle linéaire généralisé, modèle mixte, plans d'expériences (Niveau C)*. 2015.

[3] Peter McCullagh. *Generalized linear models*. Routledge, 2018.

[4] Nalini Ravishanker, Zhiyi Chi, and Dipak K Dey. *A first course in linear model theory*. CRC Press, 2021.

[5] Alvin C Rencher and G Bruce Schaalje. *Linear models in statistics*. John Wiley & Sons, 2008.