Optimal control of a (linear) elliptic equation: the optimality system. Exercise correction.

Exercise. Let us consider the toy example. The direct model (state equation) reads:

$$-div(\lambda\nabla y) + c\, y = u \text{ in } \Omega$$

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The functions λ and c belong to $L^\infty(\Omega)$ and are strictly positive.
The B.C. considered are mixed ones: $y = 0$ on Γ_0 and$-\lambda\partial_n y = \varphi$ on Γ_1, with $\partial\Omega = \Gamma_0 \cup \Gamma_1$.
The control u is the RHS of the model, $u \in H = L^2(\Omega)$.
We consider the following objective function:

$$J(u; y) = \int_\omega (y - y_d)^2 dx + \int_\Omega u^2 dx$$

where y_d is given and ω is a subset of Ω. $J(\cdot;\cdot)$ is of class C^1.
By classically defining $J_{obs}(y) = \int_\omega (y - y_d)^2 dx$ and $J_{reg}(u) = \int_\Omega u^2 dx$, we get: $J(u; y) = J_{obs}(y) + J_{reg}(u)$.

The cost function is defined as:

$$j(u) = J(u, y^u)$$

where y^u is the unique solution of the state equation.

We seek to solve the problem:

$$u^* = arg \min_{u\in U} j(u) \tag{0.1}$$

A) Write the straighforward expression of the gradient depending on $w^u = \frac{dy}{du}.\delta u$, w^u solution of the Linear Tangent Model (LTM).
You will detail the expression of the LTM.

B) Write the optimality system.

Correction

A) Let us derive the direct expression of the cost function "gradient" (actually the differential).
Given u_0 in H, for all δu in H,

$$\frac{dj}{du}(u_0) \cdot \delta u \;=\; \frac{\partial J}{\partial u}(u_0; y) \cdot \delta u + \frac{\partial J}{\partial y}(u_0; y) \cdot w^u \tag{0.2}$$

with $w^u = \frac{dy}{du} \cdot \delta u$ the solution of the LTM.
The latter reads:

$$\begin{cases} \text{Given } u \in H \text{ and } y^u \text{ the corresponding solution of the direct model,} \\ \text{given } \delta u \in H, \text{ find } w \text{ such that:} \\ -div(\lambda\nabla w) \;+\; c \;\; w \;=\; \delta u \text{ in } \Omega \end{cases} \tag{0.3}$$

with the linearized (wrt u) BCs: $w = 0$ on Γ_0 and $-\lambda\partial_n w = 0$ on Γ_1.

Observe that since the direct model is linear, only the RHS and the BCs of the LTM differ from the direct model.
The partial derivatives of the objective function reads:

$$\partial_u J(u_0; y) \cdot \delta u = J'_{reg}(u_0) \cdot \delta u = 2\int_\Omega u_0 \;\delta u\, dx$$

and

$$\partial_y J(u_0; y) \cdot \delta y = J'_{obs}(y) \cdot z = 2\int_\omega (y - y_d)\;\delta y\; dx$$

Then, given u and δu, the state y^u can be obtained by solving the state equation, next $w^u = (\frac{dy}{du}.\delta u)$ by solving the LTM. Next, the gradient expression (0.2) can be evaluated.

Weak forms

Considering the mixed BCs, the natural functional space for the state y (and w too) is: $V = \{z \in H^1(\Omega),\; z = 0 \text{ on } \Gamma_0\}$.
The weak formulation of the direct model reads as follows.

Find $y \in V$ satisfying:

$$\int_\Omega \lambda \; \nabla y \; \nabla z \; dx + \int_\Omega c \; y \; z \; dx = \int_\Omega u \; z \; dx - \int_{\Gamma_1} \varphi \; z \; ds \quad \forall z \in V$$

The weak formulation of the LTM reads as follows.

Find $w \in V$ satisfying:

$$\int_\Omega \lambda \; \nabla w \nabla z \; dx + \int_\Omega c \; w \; z \; dx = \int_\Omega \delta u \; z \; dx \quad \forall z \in V$$

B) The optimality system is the set of the following three equations: the state equation, the adjoint state equation and the necessary optimality condition $j'(u) \equiv \frac{dj}{du}(u) = 0$ (with the "gradient" expression derived from the adjoint equation).

Let us write the adjoint equation. The operator of the direct model is self-adjoint since linear symmetric.

Following the general expression derived in the course, the adjoint model reads:

$$(0.4) \qquad \left\{ \begin{array}{l} \text{Given a control value} u \text{ and } y^u \text{ the corresponding state of the system, find } p \in V \text{ satisfying:} \\ \int_\Omega \lambda \; \nabla p \nabla z \; dx + \int_\Omega c \; p \; z \; dx = \frac{\partial J}{\partial y}(u, y^u) \cdot z \qquad \forall z \in V \end{array} \right.$$

with: $\frac{\partial J}{\partial y}(u, y^u) \cdot z = 2 \int_\omega (y^u - y_d) \; z \; dx$.

Next, given u, y^u and p^u, the "gradient" expression (actually the differential) reads:

$$\forall \delta u, \quad j'(u) \cdot \delta u \equiv \frac{dj}{du}(u) \cdot \delta u \;\; = \;\; \frac{\partial J}{\partial u}(u; y^u) \cdot \delta u + \int_\Omega p^u \; \delta u \; dx$$

with $\frac{\partial J}{\partial u}(u, y^u) \cdot \delta u = J'_{reg}(u) \cdot \delta u$.

Hence the optimality system constituted by: the state equation, the adjoint equation and the 1st order optimality condition $j'(u) \cdot \delta u = 0$ for all δu.

Recall that after discretization, the "actual" gradient expression $\nabla j(u)$, $\nabla j(u) \in \mathbb{R}^m$, satisfies:

$$< \nabla j(u), \delta u >_{\mathbb{R}^m} = j'(u) \cdot \delta u \text{ for all } \delta u \in \mathbb{R}^m$$

Here, this provides the following gradient expression:

$$\nabla j(u) \;\; = \;\; (J'_{reg}(u) + p^u) \text{ in } \mathbb{R}^m$$

Note that since the RHS of the model is the control u, then the discrete state y, teh discrete adjoint state p and the discrete control u are necessarily of same dimension.