Exercises of Session 2: optimal control of ODE systems, Hamiltonian.

1 Exercise 0: the most simple LQ problem

Let $x(t)$ and $u(t)$ be two scalar fucntions defined on $[0, T]$, T given.

We consider the model equation

$$x'(t) = u(t) \text{ for } t \text{ in (0,T), with: } x(0) = x_0$$

One seeks to minimize the following cost function:

$$j(u) = \alpha_1 \int_0^T (x(u))^2(t)dt + \alpha_2 \int_0^T u^2(t)dt$$

For a sake of simplicity, we set: $\alpha_1 = \alpha_2 = \frac{1}{2}$.

Instructions

1) Write the optimality system of this inverse problem.

(Recall that these conditions provide the expression of the optimal control $u(t)$ in function of the adjoint variable, therefore implicitly in function of the state variable too).

2) a) Write the explicit expression of the optimal control u^*.

b) Write the explicit expression of the optimal trajectory x^*.

Correction

1) The optimality system

The Hamiltonian reads:

$$H(x, p, u)(t) = p(t) \cdot u(t) - \frac{1}{2}[\|x(t)\|^2 + \|u(t)\|^2]$$

Then the optimality system reads:

$$\begin{cases} x'(t) &= \partial_p H(x, p, u)(t) &= u(t) \\ -p'(t) &= \partial_x H(x, p, u)(t) &= -x(t) \\ 0 &= \partial_u H(x, p, u)(t) &\Leftrightarrow u(t) = p(t) \end{cases}$$

with the intial condition $x(0) = x_0$ and the final condition: $p(T) = 0$.

Therefore, we get:

$$\begin{cases} x'(t) &= u(t) \\ p"(t) &= u(t) \\ p(t) &= u(t) \end{cases}$$

The optimality system provides the optimal control u in function of $p(t)$. Here its expression is simply:

$$u(t) = p(t) \text{ for all } t \in [0, T] \quad (1)$$

with $p(T) = 0$.

The optimal control is nothing else than the adjoint.

In this very simple case, finally we have:

$$u(t) = p(t) = x'(t) \text{ for all } t \in [0, T] \quad (2)$$

with I.C. (at $t = 0$) for x and F.C. (at $t = T$) for p.

The golden objectif would be to have the expression of $u(t)$ in function of $x(t)$, that is a close loop law... This is what is investigated in next paragraph.

2) Expression of the optimal control $u(t)$

We have: $p''(t) = p(t)$ for all t. This is a linear homogeneous ODE order 2 in $p(t)$.

The characteristic equation of this 2nd order linear differential equation reads: $r^2 = 1$. Its roots are: $r = \pm 1$.

Therefore the general solution of the ODE is:

$$p(t) = c_1 \exp(t) + c_2 \exp(-t) \; \forall t \quad (3)$$

The final condition $p(T) = 0$ implies the condition: $c_1 + c_2 \exp(-2T) = 0$.

Since $p'(t) = c_1 \exp(t) - c_2 \exp(-t)$ and $p'(t) = x(t)$, the initial condition implies: $c_1 - c_2 = x_0$.

Therefore: $c_2 = -(\exp(-2T) + 1)^{-1} x_0$ and $c_1 = x_0[1 - (\exp(-2T) + 1)^{-1}]$.

Therefore the explicit expression of the adjoint solution reads:

$$p(t) = x_0[1 - (\exp(-2T) + 1)^{-1}] \exp(t) - (\exp(-2T) + 1)^{-1} x_0 \exp(-t) \quad (4)$$

This is the (explicit) expression of $u(t)$ too since $u(t) = p(t)$.

3) Resulting optimal trajectory $x(t)$ Next by deriving $p(t)$, one straighforwardly obtain $x(t)$ since $x(t) = p'(t)$. We have:

$$x(t) = x_0[1 - (\exp(-2T) + 1)^{-1}] \exp(t) + (\exp(-2T) + 1)^{-1} x_0 \exp(-t) \quad (5)$$

This is the optimal trajectory.

Remark. It would be easy to verify that the necessary optimality condition (Euler's equation) is satisfied: $j'(u) = 0$. (To be done ?...)

2 Exercise 1: How to optimally drive a vehicle ?

Let us consider the Newton law: $m\ddot{x}(t) = -\nabla V(t) + u(t)$ where V is the potential energy, $\nabla V = \dot{x}$.

We set: $m = 1$. Then the (direct) model reads:

$$x"(t) = -x'(t) + u(t) \text{ for } t \in (0, T)$$

with the I.C.: $x'(0) = x(0) = 0$.

The inverse problem.

The final time T is given. The question is how to act on u in order to maximize the distance $x(T)$ while minimizing the consumed « energy » ?

The answer can be modeled by minimizing one of the following cost functions: $j_i = J_i(x^u; u)$, $i = 1, 2$, with

$$J_1(x; u) = -\alpha_1 x(T) + \alpha_2 \int_0^T u^2(t)dt$$

or

$$J_2(x; u) = -\alpha_1 (x(T))^2 + \alpha_2 \int_0^T u^2(t)dt$$

with $x(T)$ solution of the state equation, with u given.

The scalars values α_* are given weight coefficients; they have to be a-priori set.

For a sake of simplicity, we consider: $\alpha_1 = \alpha_2 = \frac{1}{2}$.

Let us make a few remarks.

- The term $\int_0^T u^2(t)dt$ enables to minimize the consumed energy; moreover it acts as a regularization term (Tykonov's regularization term).

- This term $\int_0^T u^2(t)dt$ is stricly convex in u.

On the contrary the term is $-\alpha_1(x(T))^2$ is concave in x....

As a consequence, the problem is <u>not</u> a LQ problem. The uniqueness of the optimal contol is here not guaranted anymore... However, this does not prevent us to investigate the stationnary point(s) of the Hamiltonian. Indeed, in any case these conditions are necessary conditions.

This is what is done below: investigating the stationnary point(s) of the Hamiltonian.

Instructions 1) Write the optimality system in both cases.

Recall these conditions provide conditions of stationary points of the Hamiltonian therefore the optimality system.

The latter represent here necessary conditions of optimality, not sufficient ones.

The calculations should answer the question of existence-uniqueness too...

2) In the case of j_1,

a) write the explicit expression of the optimal control u.

b) write the explicit expression of the optimal trajectory x.

Correction Let us remark that the equation enables to introduce the natural change of variable $y(t) = x'(t)$ and to solve the 1st order ODE:

$$y'(t) = -y(t) + u(t) \text{ for } t \in (0, T)$$

Note that the 0-th order term $x(T)$ in $J_k(x; u)$ reads in function of $y(T)$ as :

$$x(T) = \int_0^T y(s)ds \equiv g(y(T)) \tag{6}$$

Therefore: $\nabla g(y(T)) \cdot \delta y = \int_0^T \delta y(s)ds$, soit $\nabla g(y(T)) : \cdot \mapsto \int_0^T \cdot ds$.
Soit $p(T)(.) = -\int_0^T \cdot ds$.
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Instead, we here choose to write the 2nd order equation as a 1st order system by setting:

$$X = (x, x')^T \ , \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \ , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & +1 \end{pmatrix} \ , \quad U = (*, u)^T$$

where * may denote any value, 0 for example.
(The command operator B is here not of maximal rank).

With these notations, the direct model reads:

$$X'(t) = AX(t) + BU(t) \text{ for } t \text{ in (0,T), with the I.C.: } X(0) = 0$$

(Note that $BU = U$).
Using the manuscript notations, the considered cost functions read:

$$j_k(U) = +\alpha_1 g_k(X(T)) + \alpha_2 \int_0^T \|U(t)\|^2 dt \quad , k = 1, 2$$

with: $g_1(X(T)) = -MX(T)$ and $g_2(X(T)) = -MX^2(T)$, $M = (1, 0)^T$.

Note that the matrix norm $\|U\|$ may deserve to be detailed. It denotes for example the Frobenius norm.

1) The optimality system.
The Hamiltonian reads the same in both cases:

$$H(X, P, U) = P(AX + BU) - \frac{1}{2}\|U\|^2$$

with $P = (p_1, p_2)(t)$.

Then the optimality system reads:

$$\begin{cases} X' &= AX + BU \\ -P' &= PA \text{ (recall, } P \text{ is a line vector)} \\ U^T &= PB \Leftrightarrow u(t) = p_2(t) \end{cases}$$

The adjoint equation re-reads as follows. For all t,

$$\begin{cases} p_1'(t) &= 0 \\ p_2'(t) &= (-p_1 + p_2)(t) \end{cases} \tag{7}$$

The final condition $p(T)$ depends on the case. In the general case (see course), it reads: $P(T) = -\alpha_1 \nabla_X g_k(X(T))$, $k = 1, 2$.

In the case of j_1, the final condition for P reads: $p_1(T) = \alpha_1$ and $p_2(T) = 0$.
In the case of j_2, the final condition reads: $p_1(T) = 2\alpha_1 x(T)$ and $p_2(T) = 0$.

Let us point out that the optimality system does not provide an explicit expression of the optimal control $u(t)$ but an expression in function of $p(t) = (p_1, p_2)(t)$.

However $\underline{\text{if}}$ one manages to explicitly solve the adjoint equation then one may obtain an *explicit* expression of $u(t)$...

2) Case j_1.

a) The optimal control expression.

We have: $p_1(t) = \alpha_1 = \frac{1}{2}$ for all t.

Let us try to solve the adjoint equation to obtain the expression of $p_2(t)$ $\forall t \in (0, T)$.

From the system above, we have:

$$p_2'(t) = p_2(t) - \frac{1}{2} \quad \forall t$$

The solution of the corresponding homogeneous equation reads: $c_1 \exp(t - T)$ with $c_1 \in R$.

Next, we guess the following form for the complete solution:

$$p_2(t) = c_1 \exp(t - T) + c_2$$

Since $p_2(T) = 0$, c_2 must satisfy: $c_2 = -c_1$.

Then, the guess expression leads to:

$$p_2'(t) - p_2(t) = c_1 = -\frac{1}{2}$$

Therefore the adjoint fonction $p = (p_1, p_2)$ reads: $\forall t \in (0, T)$,

$$p_1(t) = \frac{1}{2} \; ; \; p_2(t) = \frac{1}{2}(1 - \exp(t - T))$$

Finally, the optimal control reads:

$$u(t) = p_2(t) = \frac{1}{2}(1 - \exp(t - T)) \quad \forall t \in (0, T) \tag{8}$$

This is the expression of the optimal control.

This consists to start with a maximal acceleration equal to $(1 - \exp(-T))$, next exponentially decreasing to 0.

Remark . If the cost function expression would have contained a term in $\|x(t)\|_W^2$ then the adjoint equation would lead to a fully coupled first order system in (X, P) hence a-priori impossible to solve explicitly "by hand"... However since the present problem is a LQ problem then the Riccati equation would make possible to provide the explicit expression of the optimal control u.

b) The resulting optimal trajectory

Given the expression of the optimal control u, we can deduce the optimal trajectory by solving the state equation.

We have:

$x''(t) + x'(t) = u(t) +$ I.C. with $u(t)$ given by the expression above.

Let us set $y(t) = x'(t)$. We obtain:

$$y'(t) + y(t) = \frac{1}{2}(1 - \exp(t - T)) \text{ with } y(0) = 0$$

The solution of this first order non homogeneous linear ODE exists and is unique.
It can be computed numerically, for example by using a Runge Kutta scheme (eg. RK4).

Let us explicitly calculate the solution $y(t)$.
The solution of the homogeneous equation reads: $y_0(t) = c_1 \exp(-t)$.
Next, let us seek a particular solution $y_*(t)$ of the following form: $c_2 \exp(t - T) + c_3$.
By injecting in the equation, we get: $c_2 = -1/4$; $c_3 = 1/2$.
It follows that: $y(t) = y_0(t) + y_*(t) = c_1 \exp(-t) - \frac{1}{4}\exp(t - T) + \frac{1}{2}$.

The I.C. $y(0) = 0$ implies that: $c_1 = \frac{1}{4}\exp(-T) - \frac{1}{2}$.

Therefore:

$$y(t) = \frac{1}{4}\left(\exp(-t - T) - \exp(t - T) - 2\exp(-t) + 2\right) \tag{9}$$

Next, we have: $y(t) = x'(t)$. By integrating and by using the I.C. $x(0) = 0$, we obtain:

$$x(t) = -\frac{1}{4}\left(\exp(-t - T) - \exp(t - T) - 2\exp(-t) + 2t - 2\exp(-T) + 2\right) \tag{10}$$

This is the expression of the optimal trajectory. *(Calcul à verifier svp :)*

3 Exercise 2

The direct model reads:

$$x"(t) = +u(t) \text{ for } t \text{ in (0,T), with the I.C.: } x'(0) = x(0) = 0$$

The inverse problem.
The final time T is given. The problem consists to minimize the following cost function:

$$j(u) = -\alpha_1(x(T))^2 + \alpha_2 \int_0^T u^2(t)dt$$

with $x(T)$ solution of the state equation, with u given.
The scalar values α_* are given weight coefficients, to be tuned by the modeler.
For a sake of simplicity, we set: $\alpha_1 = \alpha_2 = \frac{1}{2}$.

Instructions
1) Write the optimality condition.
2) Write an explicit expression of the optimal control u, and an expression of the optimal trajectory x.

Correction First, let us write the second order scalar differential equation as a first order system. We set:

$$X = (x, x')^T \ , \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & +1 \end{pmatrix} \ , \quad U = (0, u)^T$$

Then, the direct model reads:

$$X'(t) = AX(t) + BU(t) \text{ for } t \text{ in (0,T), with the I.C.: } X(0) = 0$$

Using the manuscript notations, the cost function reads:

$$j(u) = -\alpha_1 g(x(T)) + \alpha_2 \int_0^T u^2(t)dt$$

with: $g(x(T)) = x^2(T)$.
This term can be re-rewritten as follows: $g(X(T)) = MX^2(T)$ with
$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The Hamiltonian reads:

$$H(X, P, U) = P(AX + BU) - \frac{1}{2}\|U\|^2$$

with $P = (p_1, p_2)(t)$.

The optimality system reads:

$$\left\{ \begin{array}{rcl} X' &=& AX + BU \\ -P' &=& PA \text{ (recall, } P \text{ is a line vector)} \\ U^T &=& PB \Leftrightarrow u(t) = p_2(t) \end{array} \right.$$

with the final condition depending on the case: $P(T) = +\alpha_1 \nabla_X g(X(T))$.

The adjoint equation re-reads as follows. For all t,

$$\begin{cases} p_1'(t) & = 0 \\ p_2'(t) & = -p_1(t) \end{cases} \tag{11}$$

Its final condition reads: $p_1(T) = 2\alpha_1 x(T)$, $p_2(T) = 0$.

Let us solve the adjoint equation.
We have: $p_1(t) = 2\alpha_1 x(T) = x(T)$ for all t.
Furthermore,

$$p_2(t) = x(T)(T - t) \ \ \forall t$$

Finally, the optimal control reads:

$$u(t) = p_2(t) = x(T)(T - t) \ \ \forall t \in (0, T)$$

It is the feedback law !

Given the expression of the optimal control u, we can deduce the optimal trajectory by solving the state equation.
We have:
$x"(t) = x(T)(T - t)$ + I.C..
Therefore:

$$x'(t) = -\frac{1}{2}x(T)(T-t)^2 + c_1$$

It follows:

$$x(t) = \frac{1}{6}x(T)(T-t)^3 + c_1 t + c_0$$

The I.C. implies that: $c_0 = -\frac{1}{6}x(T)T^3$ and $c_1 = \frac{1}{2}x(T)T^2$.

Calculs à vérifier svp.

In conclusion, *explicit expressions* for both the optimal control and the optimal trajectory have been obtained.

Recall that in the general LQ case, such explicit calculations are not possible.