Contact: Max Zhao
Email: jz3445@columbia.edu

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## ① conservation of momentum & conservation of mass

consider a bounded domain of any kind $\boxed{\alpha}$

### Conservation of mass :

we have $\frac{d}{dt}\int_{\alpha} \rho \, dV + \int_{\partial\alpha} (\rho \cdot \vec{v}) \cdot \vec{n} \, ds = 0$

$\rho$ is density , $\alpha$ is volume , $\partial V$ is boundary of
$\vec{v}$ is velocity $= (u, v, w)$ , $\vec{n}$ is the outward direction of boundary

Divergence theorem : if $v : \partial\Omega \to \mathbb{R}^n$ is outward pointing normal vector to $\partial\Omega$ , we have

$$\int_{\Omega} \nabla u \, dV = \int_{\partial\Omega} u \cdot v \, dS$$

So we have $\frac{d}{dt}\int_{\alpha} \rho \, dV + \int_{\partial\alpha} (\rho \cdot \vec{v}) \cdot \vec{n} \, ds$

$$= \frac{d}{dt}\int_{\alpha} \rho \, dV + \int_{\alpha} \nabla(\rho \cdot v) \, dV$$

$$= \int_{\alpha} \frac{\partial \rho}{\partial t} + \nabla(\rho \cdot v) \, dV$$

$$= 0$$

So $\frac{\partial \rho}{\partial t} + \nabla(\rho \cdot v) = 0$ since the equality holds for all $\alpha$

### Conservation of Linear Momentum:

Similar to conservation of mass , recall momentum $P = m \cdot v$
where $m$ is mass . $v$ is the velocity
And note that $\Delta P = F \cdot \Delta t$ , where $F$ are forces acting on the system
So we have

$$\frac{d}{dt}\int_{\alpha} \rho \cdot \vec{v}\, dV + \int_{\partial\alpha} (\rho \cdot \vec{v}) \cdot \vec{v} \cdot \vec{n}\, ds = \sum F_i$$

Force to consider: ① Body force (gravity, magnetic fields ... )

② contact force on surface (Cauchy stress tensor)

So $$\frac{d}{dt}\int_{\alpha} \rho \cdot \vec{v}\, dV + \int_{\partial\alpha} (\rho \cdot \vec{v}) \cdot \vec{v} \cdot \vec{n}\, ds = \int_{\alpha} \rho \cdot B\, dV + \int_{\partial\alpha} T \cdot \vec{n}\, ds$$

where $B$ is body force on the fluid system

$T$ is Cauchy stress tensor

Next. apply the Divergence Theorem again:

> Divergence theorem: if $v : \partial\Omega \to \mathbb{R}^n$ is outward pointing normal vector to $\partial\Omega$, we have
>
> $$\int_{\Omega} \nabla u\, dV = \int_{\partial\Omega} u \cdot v\, dS$$

we have

$$\frac{d}{dt}\int_{\alpha} \rho \cdot \vec{v}\, dV + \int_{\partial\alpha} (\rho \cdot \vec{v}) \cdot \vec{v} \cdot \vec{n}\, ds - \int_{\alpha} \rho \cdot B\, dV - \int_{\partial\alpha} T \cdot \vec{n}\, ds$$

$$= \int_{\alpha} \frac{\partial(\rho \cdot \vec{v})}{\partial t} dV + \int_{\alpha} \nabla \cdot (\rho \cdot \vec{v})\vec{v}\, dV - \int_{\alpha} \rho \cdot B\, dV - \int_{\alpha} \nabla(T)\, dV$$

$$= \int_{\alpha} \frac{\partial(\rho \cdot \vec{v})}{\partial t} + \nabla \cdot (\rho \cdot \vec{v})\vec{v} - \rho \cdot B - \nabla(T)\, dV$$

$$= 0$$

So $\frac{\partial(\rho \cdot \vec{v})}{\partial t} + \nabla \cdot (\rho \cdot \vec{v})\vec{v} - \rho \cdot B - \nabla(T) = 0$ since the equality holds for all $\alpha$

Now we are ready to put together the Navier-stokes equations with some assumptions !!

## ② Navier-stokes equations

$$\boxed{\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla(\rho \cdot v) = 0 \\ \dfrac{\partial(\rho \cdot \vec{v})}{\partial t} + \nabla(\rho \cdot \vec{v})\vec{v} - \rho \cdot B - \nabla(T) = 0 \end{cases}}$$

Now we need to make some assumptions about parameters:

① water density $\rho$ is consider to be constant

Effect: $\dfrac{\partial \rho}{\partial t} = 0$, can take out $\rho$ for all derivative terms

② $B = g + B'$, $g$ is earth gravity

$B'$ could be viscous force, coriolis force, ...

③ Since Newtonian fluid: $T = P \cdot I + T'$, $P$ is pressure

$I$ is identity matrix

$T'$ is stress matrix

so $\nabla(T) = -\nabla(P) + \nabla(T')$

$$\boxed{\text{so} \begin{cases} \rho \nabla(v) = 0 \\ \dfrac{\partial(\rho \cdot \vec{v})}{\partial t} + \nabla(\rho \cdot \vec{v})\vec{v} = \rho B + \nabla(T) \\ \qquad\qquad\qquad\qquad = \rho \cdot g + \rho B' - \nabla(P) + \nabla(T') \end{cases}}$$

let $T_{ij}$ denotes the element in the $T'$ matrix

we are ready to write out the final form of N-S equations

Navier-stokes equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = \rho g + \rho B' - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

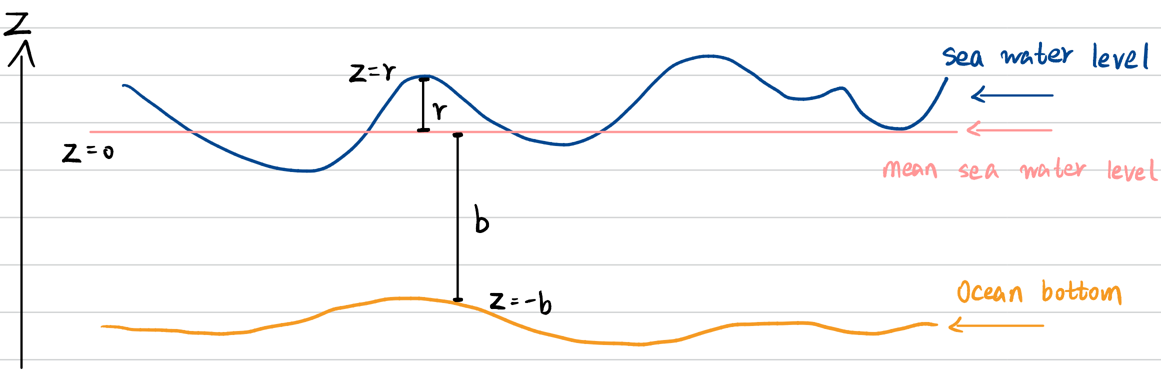
$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial \rho vw}{\partial z} = \rho g + \rho B' - \frac{\partial P}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho uw}{\partial x} + \frac{\partial \rho vw}{\partial y} + \frac{\partial \rho w^2}{\partial z} = \rho g + \rho B' - \frac{\partial P}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

We can assume $B' = 0$ for now for simplicity

3 Define B.C. for fluid flow (used to solve for unknown constant)

First, introduce some new parameters:




$$h(x, y, t) = r + b$$

which is the function we are interested in

**B.C:**

At $z = r$:

No relative normal flow:

$$\frac{Dh}{Dt} = w_r \quad , \quad D \text{ here is the material derivative}$$

so $$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\cancelto{0}{\frac{\partial h}{\partial z}} = w_r$$

At $z = -b$:

No slip: $w_{-b} = 0$

④ Depth-integrate Navier stokes equations

**Introduce Leibniz Integral Rule:**

$$\frac{\partial}{\partial z}\int_{a(x)}^{b(x)} f(x,z)\,dx = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial z}\,dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

$$\int_{-b}^{r} \nabla(v)\,dz = \int_{-b}^{r} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\,dz$$

$$= \underbrace{\int_{-b}^{r} \frac{\partial u}{\partial x}}_{} + \frac{\partial v}{\partial y}\,dz + w_r - \cancelto{0}{w_{-b}}$$

u is f

x, y is z

$$= \frac{\partial}{\partial h}\int_{-b}^{r} u\,dz + \frac{\partial}{\partial y}\int_{-b}^{r} v\,dz - u\Big|_{z=r}\frac{\partial h}{\partial x}$$

$$+ u\Big|_{z=-b}\frac{\partial(-b)}{\partial z} - v\Big|_{z=r}\frac{\partial h}{\partial y} + v\Big|_{z=-b}\frac{\partial(-b)}{\partial y} + w_r$$

**Define depth averaged velocity:**

$$\bar{u} = \frac{1}{h}\int_{-b}^{r} u\,dz$$

$$\bar{v} = \frac{1}{h}\int_{-b}^{r} v\,dz$$

$$\frac{\partial}{\partial h}\int_{-b}^{r} u\, dz + \frac{\partial}{\partial y}\int_{-b}^{r} v\, dz - u\big|_{z=r}\frac{\partial h}{\partial x}$$

$$+ u\big|_{z=-b}\frac{\partial(-b)}{\partial z} - v\big|_{z=r}\frac{\partial h}{\partial y} + v\big|_{z=-b}\frac{\partial(-b)}{\partial y} + w_r$$

$$= \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} - u\big|_{z=r}\frac{\partial h}{\partial x}$$

$$+ u\big|_{z=-b}\underbrace{\cancel{\frac{\partial(-b)}{\partial z}}}_{0} - v\big|_{z=r}\frac{\partial h}{\partial y} + v\big|_{z=-b}\underbrace{\cancel{\frac{\partial(-b)}{\partial y}}}_{0} + w_r$$

By Boundary conditions:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\cancel{\frac{\partial h}{\partial z}} = w_r$$

Plug in $w_r$ into the equation

$$= \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} - u\big|_{z=r}\frac{\partial h}{\partial x} - v\big|_{z=r}\frac{\partial h}{\partial y} + w_r$$

$$= \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} + \frac{\partial h}{\partial t}$$

$$= 0$$

So continuity equation:

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} = 0}$$

Now derive the two momentum equations:

By similar arguments: (left hand side)

$$\begin{cases} \int_{-b}^{r} \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z}\, dz = \frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u}^2}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} + F \\ \int_{-b}^{r} \frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial \rho vw}{\partial z}\, dz = \frac{\partial h\bar{v}}{\partial t} + \frac{\partial h\bar{v}^2}{\partial y} + \frac{\partial h\bar{u}\bar{v}}{\partial x} + G \end{cases}$$

F, G are differential advection functions since $\overline{f \cdot g} \neq \bar{f} \cdot \bar{g}$

Nothing really interesting happens for the last eq. in N-S equations

(Right hand side)

After integration, only terms $\begin{cases} -g(r+b)\dfrac{\partial r}{\partial x}\dfrac{\partial b}{\partial x} \\ -g(r+b)\dfrac{\partial r}{\partial y}\dfrac{\partial b}{\partial y} \end{cases}$ will be considered on the RHS

terms like surface friction, stree terms will be assume to be negligible

so
$$g(r+b)\frac{\partial r}{\partial x}\frac{\partial b}{\partial x} = gh\frac{\partial r}{\partial x}\frac{\partial b}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)$$
$$g(r+b)\frac{\partial r}{\partial y}\frac{\partial b}{\partial y} = gh\frac{\partial r}{\partial x}\frac{\partial b}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}gh^2\right)$$

Since $h(x,y,t) = r+b$

So
$$\frac{\partial h\bar{u}}{\partial t} + \frac{\partial h\bar{u}^2}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)$$
$$\frac{\partial h\bar{v}}{\partial t} + \frac{\partial h\bar{v}^2}{\partial y} + \frac{\partial h\bar{u}\bar{v}}{\partial x} = \frac{\partial}{\partial y}\left(\frac{1}{2}gh^2\right)$$

So
$$\boxed{\begin{aligned} \frac{\partial h\bar{u}}{\partial t} + \frac{\partial(h\bar{u}^2+\frac{1}{2}gh^2)}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} &= 0 \\ \frac{\partial h\bar{v}}{\partial t} + \frac{\partial(h\bar{v}^2+\frac{1}{2}gh^2)}{\partial y} + \frac{\partial h\bar{u}\bar{v}}{\partial x} &= 0 \end{aligned}}$$

Therefore: conservative form of 2D shallow Water Equations:
(without additional source terms)

$$\boxed{\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} &= 0 \\ \frac{\partial h\bar{u}}{\partial t} + \frac{\partial(h\bar{u}^2+\frac{1}{2}gh^2)}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} &= 0 \\ \frac{\partial h\bar{v}}{\partial t} + \frac{\partial(h\bar{v}^2+\frac{1}{2}gh^2)}{\partial y} + \frac{\partial h\bar{u}\bar{v}}{\partial x} &= 0 \end{aligned}}$$

## 5 Non-conservative form of shallow water equations

Given
$$\frac{\partial h}{\partial t}+\frac{\partial(\bar{u}h)}{\partial x}+\frac{\partial(\bar{v}h)}{\partial y}=0$$
$$\frac{\partial h\bar{u}}{\partial t}+\frac{\partial(h\bar{u}^2+\frac{1}{2}gh^2)}{\partial x}+\frac{\partial h\bar{u}\bar{v}}{\partial y}=0$$
$$\frac{\partial h\bar{v}}{\partial t}+\frac{\partial(h\bar{v}^2+\frac{1}{2}gh^2)}{\partial y}+\frac{\partial h\bar{u}\bar{v}}{\partial x}=0$$

$h(x,y,t)=r(x,y,t)+b$

since $h=r+b$

we have
$$\frac{\partial r}{\partial t}+\frac{\partial((r+b)\bar{u})}{\partial x}+\frac{\partial((r+b)\bar{v})}{\partial y}=0$$
$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+g\frac{\partial r}{\partial x}=0$$
$$\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+g\frac{\partial r}{\partial y}=0$$

### Assumptions and Observations:

1. suppose rise of sea water level $r$ is inconsequential comparing to $b$

So
$$\frac{\partial r}{\partial t}+\frac{\partial((r+b)\bar{u})}{\partial x}+\frac{\partial((r+b)\bar{v})}{\partial y}=0$$
$$\Downarrow$$
$$\frac{\partial r}{\partial t}+r\frac{\partial \bar{u}}{\partial x}+r\frac{\partial \bar{v}}{\partial y}=0$$

2. suppose geostrophic equilibrium $\Rightarrow$ non-linear terms $\to 0$

So
$$\frac{\partial \bar{u}}{\partial t}+g\frac{\partial r}{\partial x}=0$$
$$\frac{\partial \bar{v}}{\partial t}+g\frac{\partial r}{\partial y}=0$$

Denote $r$ as $\bar{h}$ meaning sea water level rise

$b$ is bathymetry depth, $\bar{u}$ is $x$-direction velocity, $\bar{v}$ is $y$-direction velocity

System of our interest is finally achieved

$$\boxed{\begin{aligned} &\frac{\partial \bar{h}}{\partial t} + b\frac{\partial \bar{u}}{\partial x} + b\frac{\partial \bar{v}}{\partial y} = 0 \\ &\frac{\partial \bar{u}}{\partial t} + g\frac{\partial \bar{h}}{\partial x} = 0 \\ &\frac{\partial \bar{v}}{\partial t} + g\frac{\partial \bar{h}}{\partial y} = 0 \end{aligned}}$$

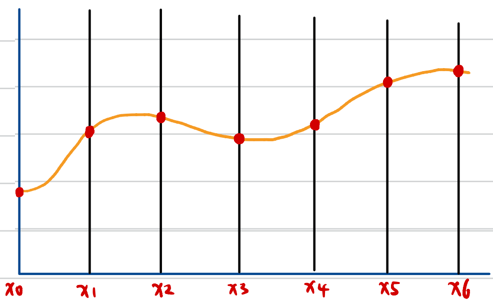
6 Finite Difference Method (forward difference scheme)

Compare to Finite Volume Method (Godunov 1st order scheme)

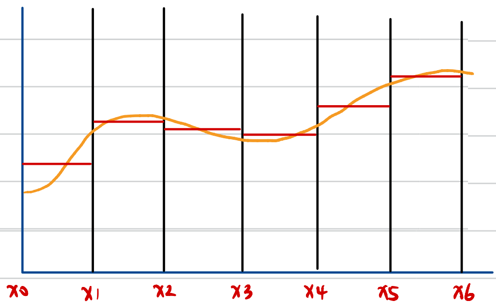
what is FDM, FVM ?

Given Initial Condition, this is how FDM, FVM discretize functions

FDM




FVM




Additional : Flux function at interfaces

FDM for SWE:

$$\frac{\partial h}{\partial t} = -b\frac{\partial u}{\partial x} - b\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial h}{\partial y}$$

$$\Downarrow$$

let superscript denote time advancement

let subscript denote grid advancement

Use Forward Difference Scheme:

$$\frac{d}{dx}f(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

Define $\Delta t = t_{i+1} - t_i$, $\Delta x = x_{i+1} - x_i$, $\Delta y = y_{i+1} - y_i$

$= t_i - t_{i-1}$ $\quad = x_i - x_{i-1}$ $\quad = y_i - y_{i-1}$

$$\frac{h_i^{i+1} - h_i^i}{\Delta t} = -b \cdot \frac{u_{i+1}^i - u_i^i}{\Delta x} - b \cdot \frac{v_{i+1}^i - v_i^i}{\Delta y}$$

$$\frac{u_i^{i+1} - u_i^i}{\Delta t} = -g \cdot \frac{h_{i+1}^i - h_i^i}{\Delta x}$$

$$\frac{v_i^{i+1} - v_i^i}{\Delta t} = -g \cdot \frac{h_{i+1}^i - h_i^i}{\Delta y}$$

Truncation Error ($\tau$)

$$u_{i+1} = u_i + \Delta x \frac{\partial u}{\partial x}\Big|_i + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i + \dots$$

$$u_i = u_i$$

$$\text{so} \quad \frac{u_{i+1} - u_i}{\Delta x} = \frac{\partial u}{\partial x}\Big|_i + O(\Delta x^2)$$

However, this is only for each grid

number of grid $= \frac{1}{\Delta x}$

$$\text{So} \quad \tau = O(\Delta x^2)\cdot\frac{1}{\Delta x} = O(\Delta x)$$

So this is a first order scheme.

comparison: FVM using Godunov's 1st order scheme:

Godunov's 1st order scheme is one of HRSC,
High Resolution Shock Capturing scheme for conservation law that compute boundary of sol. from sol. of Riemann Problems.

Riemann Problems: I.V.P with I.C

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x \geq 0 \end{cases}$$

Godunov's scheme define $u_L, u_R$ to to cell averages of two adjacent cells. Since FDM is just linear representation of the cell, the value will be the same.

Difference: FVM will conserve locally, FDM will only conserve globally

## 7 Necessary CFL condition for stability & convergence

To get the necessary condition for stability, we need the matrix form:

$$\frac{\partial u}{\partial t} = A \cdot u$$

$$A = \begin{pmatrix} 1 & -2i\frac{\Delta t}{\Delta x}\sin(a_x) & -2i\frac{\Delta t}{\Delta y}\sin(a_y) \\ -2i\frac{\Delta t}{\Delta x}\sin(a_x) & \left(1-4\frac{\Delta t^2}{\Delta x^2}\sin^2(a_x)\right) & -4\frac{\Delta t^2}{\Delta x\Delta y}\sin(a_x)\sin(a_y) \\ -2i\frac{\Delta t}{\Delta x}\sin(a_y) & -4\frac{\Delta t^2}{\Delta x\Delta y}\sin(a_x)\sin(a_y) & \left(1-4\frac{\Delta t^2}{\Delta y^2}\sin^2(a_y)\right) \end{pmatrix}$$

( behind the scene, convert discretization to discrete Fourier form )

with $a_x = 1/2 k_x \Delta x$

$a_y = 1/2 k_y \Delta y$

For simplicity: let's only consider 1D flow with only $x$ direction

$$A = \begin{pmatrix} 1 & -2i\frac{\Delta t}{\Delta x}\sin(a_x) \\ -2i\frac{\Delta t}{\Delta x}\sin(a_x) & \left(1-4\frac{\Delta t^2}{\Delta x^2}\sin^2(a_x)\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -ia \\ -ia & 1-a^2 \end{pmatrix} \qquad a = 2\frac{\Delta t}{\Delta x}\cdot \sin(1/2 k_x \Delta x)$$

By von Neumann stability: $\|A\|_2 \leq 1$

$\|A\|_2 = \max|\lambda|$ where $\lambda$ is eigenvalue of matrix $A$

$$\lambda_1 = 1 - \frac{a^2}{2} + \sqrt{(1-a^2/2)^2 - 1}$$

$$\lambda_2 = 1 - \frac{a^2}{2} - \sqrt{(1-a^2/2)^2 - 1}$$

if $1 - \frac{a^2}{2} \geqslant 1$ or $1 - \frac{a^2}{2} \leq -1$

$\max(\lambda_1, \lambda_2) > 1$

so $-1 \leq 1 - \frac{a^2}{2} \leq 1$

$4 \geqslant a^2 \geqslant 0$

$\rightarrow$ so $\left(2 \frac{\Delta t}{\Delta x} \cdot \sin(1/2 k_x \Delta x)\right)^2 \leq 4$

$4 \frac{\Delta t^2}{\Delta x^2} \cdot \sin^2(1/2 k_x \Delta x) \leq 4$

so $\frac{\Delta t^2}{\Delta x^2} \leq 1 \Rightarrow \frac{\Delta t}{\Delta x} \leq 1$

since phase speed is $\sqrt{g \cdot b}$

phase speed can not travel further than grid distance for a given time step $\leftarrow$ $\frac{\Delta t \cdot \sqrt{g \cdot b}}{\Delta x} \leq 1$

so $\Delta t \leq \Delta x \cdot (\sqrt{g \cdot b})^{-\frac{1}{2}}$ for 1D

so $\Delta t \leq \Delta x \cdot (\frac{1}{2})^{\frac{1}{2}} \cdot (\sqrt{g \cdot b})^{-\frac{1}{2}}$ for 2D