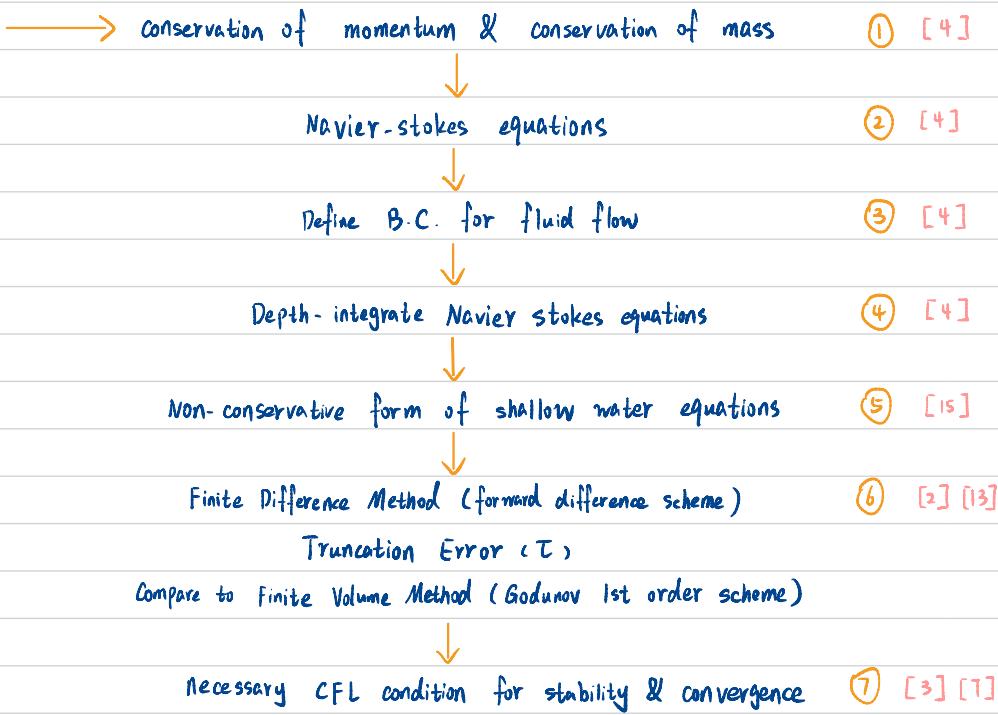


## Outline of structure



## ① Conservation of momentum & conservation of mass

consider a bounded domain of any kind



Conservation of mass:

$$\text{we have } \frac{d}{dt} \int_{\Omega} \rho dV + \int_{\partial\Omega} (\rho \vec{v}) \cdot \vec{n} dS = 0$$

$\rho$  is density,  $\Omega$  is volume,  $\partial\Omega$  is boundary of

$\vec{v}$  is velocity =  $(u, v, w)$ ,  $\vec{n}$  is the outward direction of boundary

Divergence theorem: if  $v: \partial\Omega \rightarrow \mathbb{R}^n$  is outward pointing normal vector to  $\partial\Omega$ , we have

$$\int_{\Omega} \nabla u \cdot dV = \int_{\partial\Omega} u \cdot v dS$$

$$\text{so we have } \frac{d}{dt} \int_{\Omega} \rho dV + \int_{\partial\Omega} (\rho \vec{v}) \cdot \vec{n} dS$$

$$= \frac{d}{dt} \int_{\Omega} \rho dV + \int_{\Omega} \nabla(\rho \cdot v) dV$$

$$= \int_{\Omega} \frac{\partial \rho}{\partial t} + \nabla(\rho \cdot v) dV$$

$$= 0$$

$$\text{So } \frac{\partial \rho}{\partial t} + \nabla(\rho \cdot v) = 0 \text{ since the equality holds for all } \Omega$$

Conservation of Linear Momentum:

Similar to conservation of mass, recall momentum  $P = m \cdot v$

where  $m$  is mass,  $v$  is the velocity

And note that  $\Delta P = F \cdot \Delta t$ , where  $F$  are forces acting on the system

so we have

$$\frac{d}{dt} \int_{\Omega} P \vec{v} dV + \int_{\partial\Omega} (P \vec{v}) \cdot \vec{n} ds = \sum F_i$$

Force to consider: ① Body force (gravity, magnetic fields...)  
 ② contact force on surface (Cauchy stress tensor)

$$\text{So } \frac{d}{dt} \int_{\Omega} P \vec{v} dV + \int_{\partial\Omega} (P \vec{v}) \cdot \vec{n} ds = \int_{\Omega} P \cdot B dV + \int_{\partial\Omega} T \cdot \vec{n} ds$$

where  $B$  is body force on the fluid system

$T$  is Cauchy stress tensor

Next, apply the Divergence Theorem again:

Divergence theorem: if  $v : \partial\Omega \rightarrow \mathbb{R}^n$  is outward pointing normal vector to  $\partial\Omega$ , we have

$$\int_{\Omega} \nabla u \cdot dV = \int_{\partial\Omega} u \cdot v ds$$

we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} P \vec{v} dV + \int_{\partial\Omega} (P \vec{v}) \cdot \vec{n} ds - \int_{\Omega} P \cdot B dV - \int_{\partial\Omega} T \cdot \vec{n} ds \\ &= \int_{\Omega} \frac{\partial(P \vec{v})}{\partial t} dV + \int_{\Omega} \nabla \cdot (P \vec{v}) \vec{v} dV - \int_{\Omega} P \cdot B dV - \int_{\Omega} \nabla(T) \cdot dV \\ &= \int_{\Omega} \frac{\partial(P \vec{v})}{\partial t} + \nabla \cdot (P \vec{v}) \vec{v} - P \cdot B - \nabla(T) \cdot dV \\ &= 0 \end{aligned}$$

$$\text{So } \frac{\partial(P \vec{v})}{\partial t} + \nabla \cdot (P \vec{v}) \vec{v} - P \cdot B - \nabla(T) = 0 \text{ since the equality holds for all } \alpha$$

Now we are ready to put together the Navier-Stokes equations with some assumptions !!

## ② Navier-Stokes equations

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} + \nabla(p \cdot v) = 0 \\ \frac{\partial(p \vec{v})}{\partial t} + \nabla(p \vec{v}) \vec{v} - p \cdot B - \nabla(T) = 0 \end{array} \right.$$

Now we need to make some assumptions about parameters:

① water density  $\rho$  is considered to be constant

Effect:  $\frac{\partial \rho}{\partial t} = 0$ , can take out  $\rho$  for all derivative terms

②  $B = g + B'$ ,  $g$  is earth gravity

$B'$  could be viscous force, coriolis force, ...

③ Since Newtonian fluid:  $T = P \cdot I + T'$ ,  $P$  is pressure

$I$  is identity matrix

$T'$  is stress matrix

$$\text{so } \nabla(T) = -\nabla(P) + \nabla(T')$$

$$\text{so } \int p \nabla(v) = 0$$

$$\begin{aligned} \frac{\partial(p \vec{v})}{\partial t} + \nabla(p \vec{v}) \vec{v} &= P B + \nabla(T) \\ &= P g + P B' - \nabla(P) + \nabla(T') \end{aligned}$$

let  $T_{ij}$  denotes the element in the  $T'$  matrix

we are ready to write out the final form of N-S equations

## Navier-Stokes equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial p u}{\partial t} + \frac{\partial p u^2}{\partial x} + \frac{\partial p u v}{\partial y} + \frac{\partial p u w}{\partial z} = \rho g + \rho B' - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

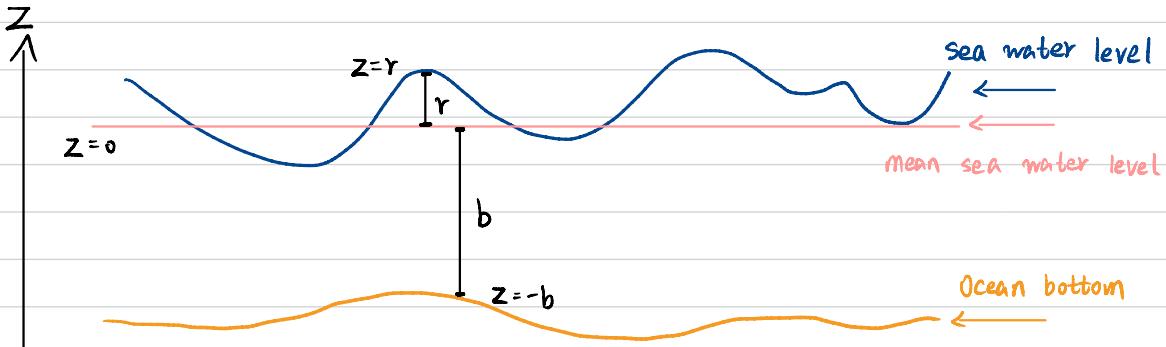
$$\frac{\partial p v}{\partial t} + \frac{\partial p u v}{\partial x} + \frac{\partial p v^2}{\partial y} + \frac{\partial p v w}{\partial z} = \rho g + \rho B' - \frac{\partial P}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$\frac{\partial p w}{\partial t} + \frac{\partial p u w}{\partial x} + \frac{\partial p v w}{\partial y} + \frac{\partial p w^2}{\partial z} = \rho g + \rho B' - \frac{\partial P}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

We can assume  $B' = 0$  for now for simplicity

③ Define B.C. for fluid flow (used to solve for unknown constant)

First, introduce some new parameters:



$$h(x, y, t) = r + b$$

which is the function we are interested in

B.C:

At  $z = r$ :

No relative normal flow:

$$\frac{Dh}{Dt} = W_r \quad , \quad D \text{ here is the material derivative}$$

$$\text{so } \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \frac{\partial h}{\partial z} = W_r$$

At  $z = -b$ :

No slip:  $w_{-b} = 0$

(4) Depth-integrate Navier-Stokes equations

Introduce Leibniz Integral Rule:

$$\frac{\partial}{\partial z} \int_{a(x)}^{b(x)} f(x, z) dx = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial z} dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z}$$

$$\int_{-b}^r \nabla(v) dz = \int_{-b}^r \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz$$

$$= \int_{-b}^r \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} dz + W_r - W_{-b}$$

$\begin{matrix} u \text{ is } f \\ x, y \text{ is } z \end{matrix}$

$$= \frac{\partial}{\partial x} \int_{-b}^r u dz + \frac{\partial}{\partial y} \int_{-b}^r v dz - u \Big|_{z=r} \frac{\partial h}{\partial x}$$

$$+ u \Big|_{z=-b} \frac{\partial(-b)}{\partial z} - v \Big|_{z=r} \frac{\partial h}{\partial y} + v \Big|_{z=-b} \frac{\partial(-b)}{\partial y} + W_r$$

Define depth averaged velocity:

$$\bar{u} = \frac{1}{h} \int_{-b}^r u dz$$

$$\bar{v} = \frac{1}{h} \int_{-b}^r v dz$$

$$\begin{aligned}
& \frac{\partial}{\partial x} \int_{-b}^r u dz + \frac{\partial}{\partial y} \int_{-b}^r v dz - u \Big|_{z=r} \frac{\partial h}{\partial x} \\
& + u \Big|_{z=-b} \frac{\partial(-b)}{\partial z} - v \Big|_{z=r} \frac{\partial h}{\partial y} + v \Big|_{z=-b} \frac{\partial(-b)}{\partial y} + wr \\
= & \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} - u \Big|_{z=r} \frac{\partial h}{\partial x} \\
& + u \Big|_{z=-b} \cancel{\frac{\partial(-b)}{\partial z}} - v \Big|_{z=r} \frac{\partial h}{\partial y} + v \Big|_{z=-b} \cancel{\frac{\partial(-b)}{\partial y}} + wr
\end{aligned}$$

By Boundary conditions:

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \cancel{\frac{\partial h}{\partial z}} = wr$$

Plug in wr into the equation

$$\begin{aligned}
& = \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} - u \Big|_{z=r} \frac{\partial h}{\partial x} - v \Big|_{z=r} \frac{\partial h}{\partial y} + wr \\
& = \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} + \frac{\partial h}{\partial t} \\
& = 0
\end{aligned}$$

So continuity equation:

$$\frac{\partial h}{\partial t} + \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} = 0$$

Now derive the two momentum equations:

By similar arguments: (left hand side)

$$\begin{cases} \int_{-b}^r \frac{\partial \bar{u}u}{\partial t} + \frac{\partial \bar{u}u^2}{\partial x} + \frac{\partial \bar{u}uv}{\partial y} + \frac{\partial \bar{u}uw}{\partial z} dz = \frac{\partial \bar{u}u}{\partial t} + \frac{\partial \bar{u}u^2}{\partial x} + \frac{\partial \bar{u}uv}{\partial y} + F \\ \int_{-b}^r \frac{\partial \bar{v}v}{\partial t} + \frac{\partial \bar{v}v^2}{\partial x} + \frac{\partial \bar{v}uv}{\partial y} + \frac{\partial \bar{v}vw}{\partial z} dz = \frac{\partial \bar{v}v}{\partial t} + \frac{\partial \bar{v}v^2}{\partial x} + \frac{\partial \bar{v}uv}{\partial y} + G \end{cases}$$

Nothing really interesting happens for the last eq. in N-S equations

F, G are differential advection functions  
since  $\bar{F}\bar{g} + \bar{F}\bar{g}$

(Right hand side)

After integration, only terms  $\int -g(r+b) \frac{\partial r}{\partial x} \frac{\partial b}{\partial x}$  will be considered on the RHS  
 $\int -g(r+b) \frac{\partial r}{\partial y} \frac{\partial b}{\partial y}$

terms like surface friction, stress terms will be assumed to be negligible

$$\text{so } g(r+b) \frac{\partial r}{\partial x} \frac{\partial b}{\partial x} = gh \frac{\partial r}{\partial x} \frac{\partial b}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{2} gh^2 \right)$$

$$g(r+b) \frac{\partial r}{\partial y} \frac{\partial b}{\partial y} = gh \frac{\partial r}{\partial y} \frac{\partial b}{\partial y} = \frac{\partial}{\partial y} \left( \frac{1}{2} gh^2 \right)$$

Since  $h(x, y, t) = r + b$

$$\text{so } \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = \frac{\partial}{\partial x} \left( \frac{1}{2} gh^2 \right)$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}^2}{\partial y} + \frac{\partial \bar{u}\bar{v}}{\partial x} = \frac{\partial}{\partial y} \left( \frac{1}{2} gh^2 \right)$$

$$\text{so } \frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{u}^2 + \frac{1}{2} gh^2)}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = 0$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial (\bar{v}^2 + \frac{1}{2} gh^2)}{\partial y} + \frac{\partial \bar{u}\bar{v}}{\partial x} = 0$$

Therefore: conservative form of 2D shallow Water Equations:  
 (without additional source terms)

$$\frac{\partial h}{\partial t} + \frac{\partial (\bar{u}h)}{\partial x} + \frac{\partial (\bar{v}h)}{\partial y} = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{u}^2 + \frac{1}{2} gh^2)}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = 0$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial (\bar{v}^2 + \frac{1}{2} gh^2)}{\partial y} + \frac{\partial \bar{u}\bar{v}}{\partial x} = 0$$

(5) Non-conservative form of shallow water equations

$$\text{Given } \frac{\partial h}{\partial t} + \frac{\partial(\bar{u}h)}{\partial x} + \frac{\partial(\bar{v}h)}{\partial y} = 0$$

$$\frac{\partial h\bar{u}}{\partial t} + \frac{\partial(h\bar{u}^2 + \frac{1}{2}gh^2)}{\partial x} + \frac{\partial h\bar{u}\bar{v}}{\partial y} = 0 \quad h(x,y,t) = r(x,y,t) + b$$

$$\frac{\partial h\bar{v}}{\partial t} + \frac{\partial(h\bar{v}^2 + \frac{1}{2}gh^2)}{\partial y} + \frac{\partial h\bar{u}\bar{v}}{\partial x} = 0$$

since  $h = r + b$

$$\text{we have } \frac{\partial r}{\partial t} + \frac{\partial((r+b)\bar{u})}{\partial x} + \frac{\partial((r+b)\bar{v})}{\partial y} = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + g \frac{\partial r}{\partial x} = 0$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + g \frac{\partial r}{\partial y} = 0$$

Assumptions and Observations:

① suppose rise of sea water level  $r$  is inconsequential comparing to  $b$

$$\text{So } \frac{\partial r}{\partial t} + \frac{\partial((r+b)\bar{u})}{\partial x} + \frac{\partial((r+b)\bar{v})}{\partial y} = 0$$

$$\downarrow$$

$$\frac{\partial r}{\partial t} + b \frac{\partial \bar{u}}{\partial x} + b \frac{\partial \bar{v}}{\partial y} = 0$$

② suppose geostrophic equilibrium  $\Rightarrow$  non-linear terms  $\rightarrow 0$

$$\text{So } \frac{\partial \bar{u}}{\partial t} + g \frac{\partial r}{\partial x} = 0$$

$$\frac{\partial \bar{v}}{\partial t} + g \frac{\partial r}{\partial y} = 0$$

Denote  $r$  as  $\bar{h}$  meaning sea water level rise  
 $b$  is bathymetry depth,  $\bar{u}$  is  $x$ -direction velocity,  $\bar{v}$  is  $y$ -direction velocity  
 System of our interest is finally achieved

$$\frac{\partial \bar{h}}{\partial t} + b \frac{\partial \bar{u}}{\partial x} + b \frac{\partial \bar{v}}{\partial y} = 0$$

$$\frac{\partial \bar{u}}{\partial t} + g \frac{\partial \bar{h}}{\partial x} = 0$$

$$\frac{\partial \bar{v}}{\partial t} + g \frac{\partial \bar{h}}{\partial y} = 0$$

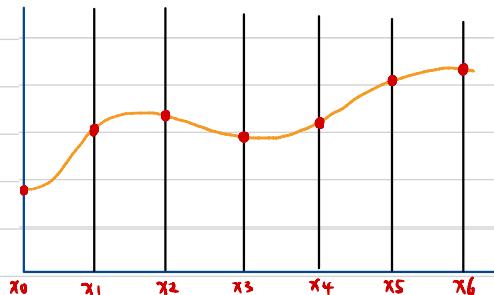
## ⑥ Finite Difference Method (forward difference scheme)

Compare to finite Volume Method (Godunov 1st order scheme)

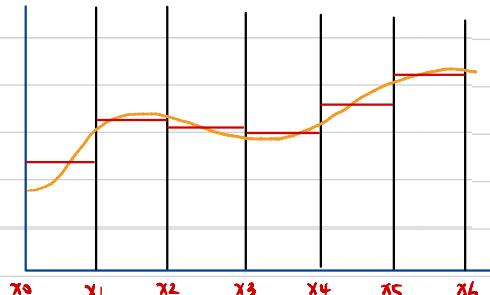
what is FDM, FVM ?

Given Initial condition, this is how FDM, FVM discretize functions

FDM



FVM



Additional : Flux function at interfaces

FDM for SWE:

$$\frac{\partial h}{\partial t} = -b \frac{\partial u}{\partial x} - b \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} = -g \frac{\partial h}{\partial y}$$



let superscript denote time advancement

let subscript denote grid advancement

Use Forward Difference Scheme:

$$\frac{d}{dx} f(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Define  $\Delta t = t_{i+1} - t_i$ ,  $\Delta x = x_{i+1} - x_i$ ,  $\Delta y = y_{i+1} - y_i$   
 $= t_{i+1} - t_i$        $= x_{i+1} - x_i$        $= y_{i+1} - y_i$

$$\frac{h^{i+1} - h^i}{\Delta t} = -b \cdot \frac{u^{i+1} - u^i}{\Delta x} - b \cdot \frac{v^{i+1} - v^i}{\Delta y}$$

$$\frac{u^{i+1} - u^i}{\Delta t} = -g \cdot \frac{h^{i+1} - h^i}{\Delta x}$$

$$\frac{v^{i+1} - v^i}{\Delta t} = -g \cdot \frac{h^{i+1} - h^i}{\Delta y}$$

Truncation Error ( $\tau$ )

$$U_{i+1} = U_i + \Delta x \frac{\partial u}{\partial x} \Big|_i + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} \Big|_i + \dots$$

$$U_i = U_i$$

$$\text{so } \frac{U_{i+1} - U_i}{\Delta x} = \frac{\partial u}{\partial x} \Big|_i + O(\Delta x^2)$$

However, this is only for each grid

$$\text{number of grid} = \frac{1}{\Delta x}$$

$$\begin{aligned} \text{so } \tau &= O(\Delta x^2) \cdot \frac{1}{\Delta x} \\ &= O(\Delta x) \end{aligned}$$

So this is a first order scheme.

Comparison: FVM using Godunov's 1st order scheme:

Godunov's 1st order scheme is one of HRSC,

High Resolution Shock Capturing scheme for conservation law that compute boundary of sol. from sol. of Riemann Problems.

Riemann Problems: IVP with IC

$$u(x, 0) = \begin{cases} U_L, & x < 0 \\ U_R, & x \geq 0 \end{cases}$$

Godunov's scheme define  $U_L, U_R$  to be cell averages of two adjacent cells. Since FDM is just linear representation of the cell, the value will be the same.

Difference: FVM will conserve locally, FDM will only conserve globally

## ⑦ Necessary CFL condition for stability & convergence

To get the necessary condition for stability, we need the matrix form:

$$\frac{\partial u}{\partial t} = A \cdot u$$

$$A = \begin{pmatrix} 1 & -2i\frac{\Delta t}{\Delta x} \sin(a_x) & -2i\frac{\Delta t}{\Delta y} \sin(a_y) \\ -2i\frac{\Delta t}{\Delta x} \sin(a_x) & \left(1 - 4\frac{\Delta t^2}{\Delta x^2} \sin^2(a_x)\right) & -4\frac{\Delta t^2}{\Delta x \Delta y} \sin(a_x) \sin(a_y) \\ -2i\frac{\Delta t}{\Delta x} \sin(a_y) & -4\frac{\Delta t^2}{\Delta x \Delta y} \sin(a_x) \sin(a_y) & \left(1 - 4\frac{\Delta t^2}{\Delta y^2} \sin^2(a_y)\right) \end{pmatrix}$$

(behind the scene, convert discretization to discrete Fourier form)

$$\text{with } a_x = 1/2k_x \Delta x$$

$$a_y = 1/2k_y \Delta y$$

For simplicity: let's only consider 1D flow with only  $x$  direction

$$A = \begin{pmatrix} 1 & -2i\frac{\Delta t}{\Delta x} \sin(a_x) \\ -2i\frac{\Delta t}{\Delta x} \sin(a_x) & \left(1 - 4\frac{\Delta t^2}{\Delta x^2} \sin^2(a_x)\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -ia \\ -ia & 1-a^2 \end{pmatrix} \quad a = 2 \frac{\Delta t}{\Delta x} \cdot \sin(1/2k_x \Delta x)$$

By von Neumann stability:  $\|A\|_2 \leq 1$

$\|A\|_2 = \max |\lambda|$  where  $\lambda$  is eigenvalue of matrix  $A$

$$\lambda_1 = 1 - \frac{a^2}{2} + \sqrt{(1-a^2/2)^2 - 1}$$

$$\lambda_2 = 1 - \frac{a^2}{2} - \sqrt{(1-a^2/2)^2 - 1}$$

if  $|1 - \frac{a^2}{2}| > 1$  or  $|1 - \frac{a^2}{2}| \leq -1$

$$\max(\lambda_1, \lambda_2) > 1$$

$$\text{so } -1 \leq 1 - \frac{a^2}{2} \leq 1$$

$$4 \geq a^2 \geq 0$$

$$\Rightarrow \text{so } \left( 2 \frac{\Delta t}{\Delta x} \cdot \sin(1/2 k_x \Delta x) \right)^2 \leq 4$$

$$4 \frac{\Delta t^2}{\Delta x^2} \cdot \sin^2(1/2 k_x \Delta x) \leq 4$$

Now change to dimensional criterion

phase speed can  $\leftarrow \frac{\Delta \tilde{t}}{\Delta \tilde{x}} \sqrt{g \cdot b} \leq \sqrt{1}$   
 not travel further

than grid distance

for a given time step

phase speed is  $\sqrt{g \cdot b}$

$$\text{so } \Delta \tilde{t} \leq \Delta \tilde{x} \cdot (\sqrt{g \cdot b})^{\frac{1}{2}} \text{ for 1D}$$

$$\text{For 2D: } \Delta \tilde{t} \leq \sqrt{\frac{g \cdot b}{\Delta \tilde{x}^2} + \frac{g \cdot b}{\Delta \tilde{y}^2}}$$

$$\text{so } \Delta \tilde{t} \leq \Delta \tilde{x} \cdot (\frac{1}{2})^{\frac{1}{2}} \cdot (\sqrt{g \cdot b})^{-\frac{1}{2}} \text{ for 2D}$$