

# TAF STAR – TAF OPE

## UE Cœur 1

### Course 5

## *Filtering of physical signals*

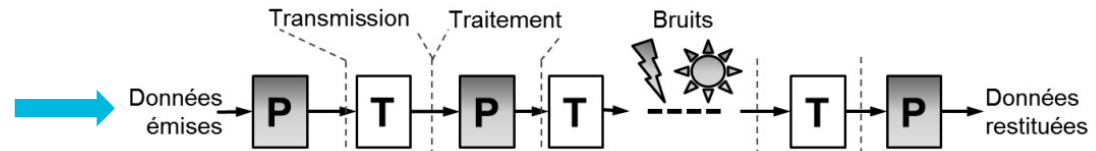
**Bruno Fracasso**  
Département Optique

**September 2023**

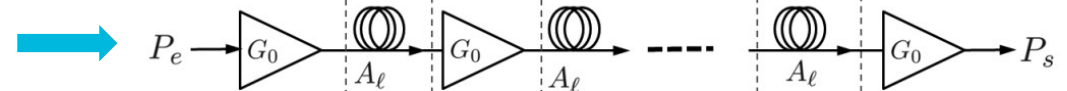
- 1) Physical signal transformations
- 2) Physical filter models
- 3) Filtering of random signals (noise)

## ■ Modular decompositions of a transmission system

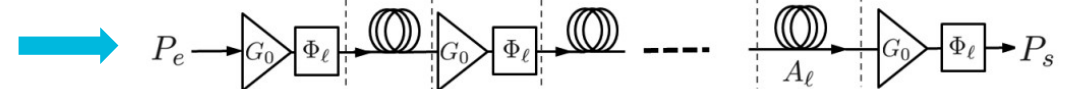
a) cascading of "P" (processing) and "T" (transmission) sections



b) chain of amplifiers and attenuators  $G_0 + A_\ell = 0$  dB



c) chain of amplifiers and filters: dispersive channel



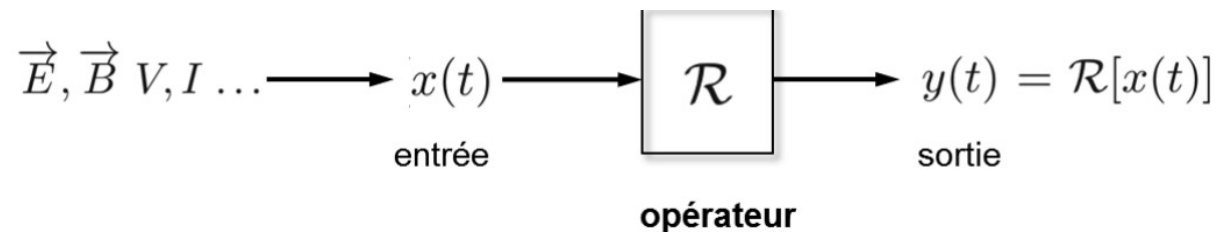
## ■ Case of a dispersive channel (c)

- Path dispersion
- Modal dispersion
- Chromatic dispersion

# Modelling of a physical (continuous) system

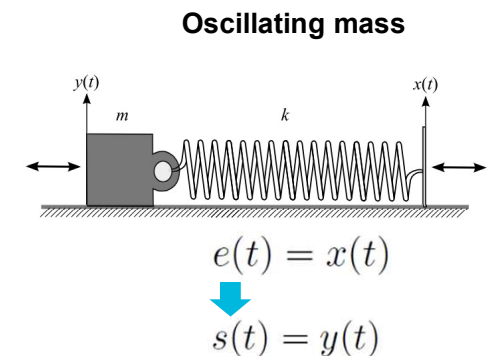
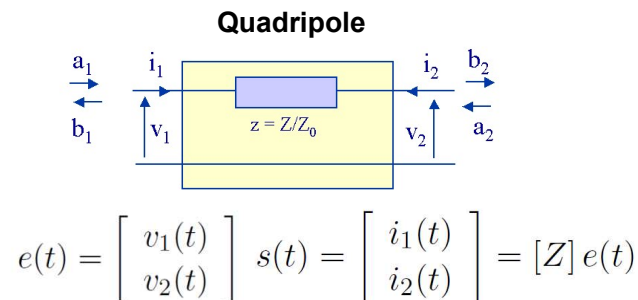
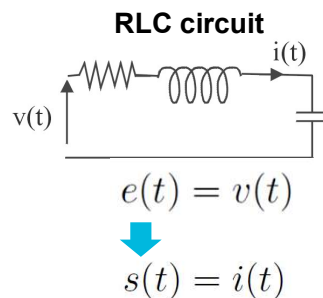
## ■ Black box model: input-output relationship

- **Electromagnetic input quantities:** current, voltage, electric field, magnetic field, voltage or current wave



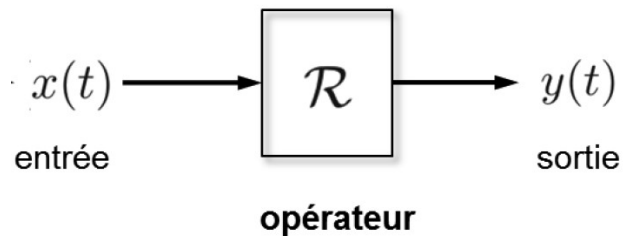
## ● Three main types of operators

- *Linear and invariant (filters)*
- *Parametric: multipliers, modulators, samplers*
- *Non-linear without memory: rectifiers, limiters, skimmers, etc...*



# Impulse response of the physical operator

## ■ General case (non-linear)

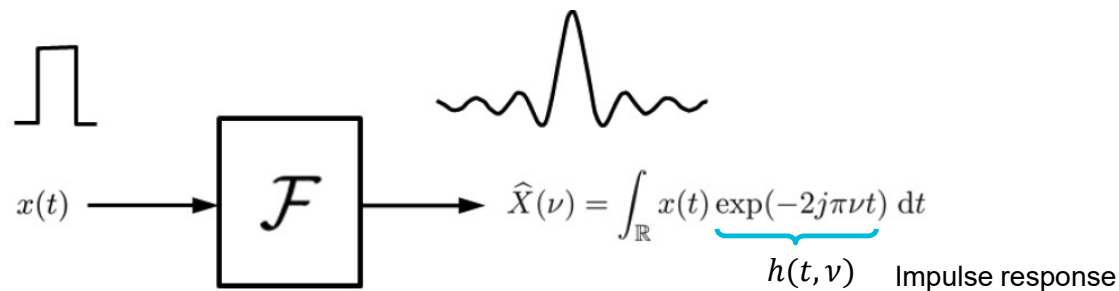


$$y(t) = \mathcal{R}[x(t)] = \int_{\mathbb{R}} \underbrace{h[t, \tau, x(\tau)]}_{\text{Impulse response (output at time } t \text{ to a localized Dirac impact at input } \tau)} d\tau$$

## ■ Linear case

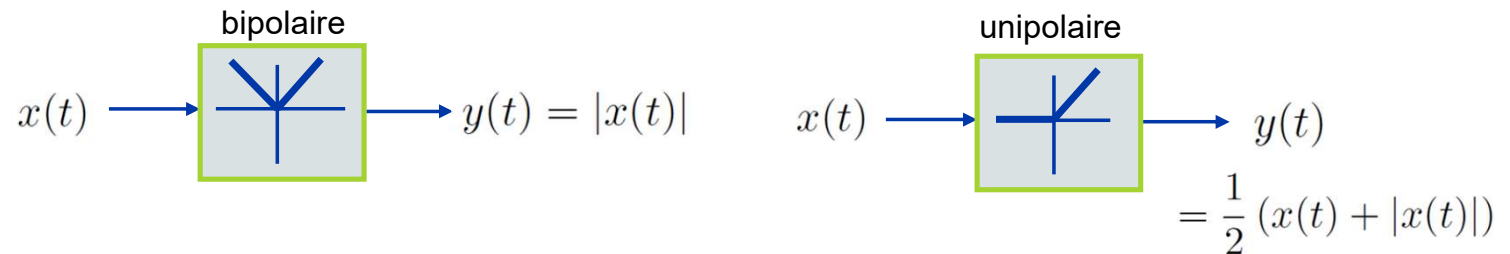
$$y(t) = \int_{\mathbb{R}} x(\tau) h(t, \tau) d\tau \quad \longrightarrow \quad \mathcal{R}[\lambda e_1(t) + \mu e_2(t)] = \lambda \mathcal{R}[e_1(t)] + \mu \mathcal{R}[e_2(t)]$$

- **Non-stationary case** :  $h(t_1, \tau) \neq h(t_2, \tau)$  pour  $t_1 \neq t_2$
- **Example**: the Fourier “transformer”

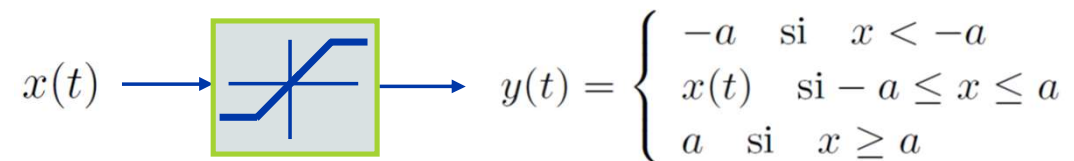


# Some non-linear operators (memoryless)

## ■ Rectifier



## ■ Skimmer

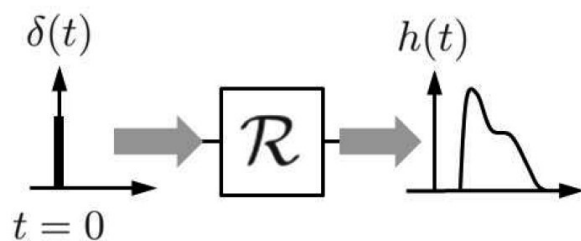


## ■ Quadratic detector

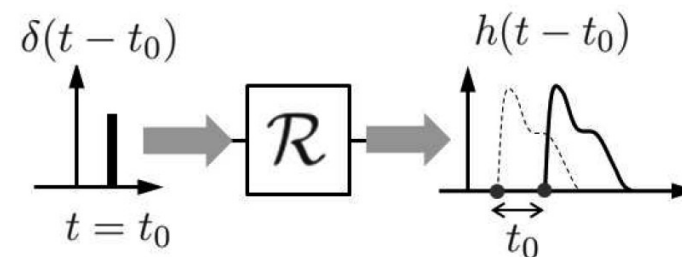


# Linear invariant operator (the filter !)

## ■ Temporal description



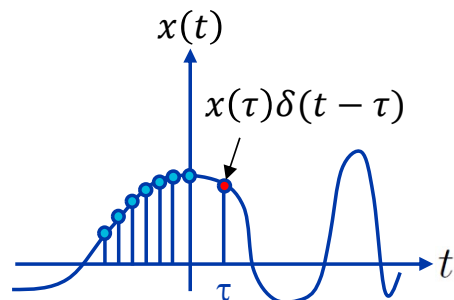
**Impulse response**  
Also called "Green" function



**Time invariance of the impulse response**

■ Two properties: linearity + invariance :  $x(t) = \alpha\delta(t - \tau) \implies y(t) = \alpha h(t - \tau)$

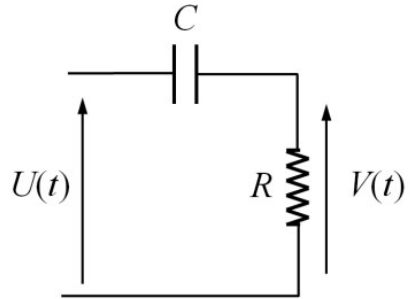
■ Response to any input  $x(t)$



$$y(t) = \int_{\mathbb{R}} x(\tau) h(t - \tau) d\tau = (h * x)(t)$$

**Filtering  $\leftrightarrow$  Convolution**

# Example of a filter: RC circuit (high-pass)

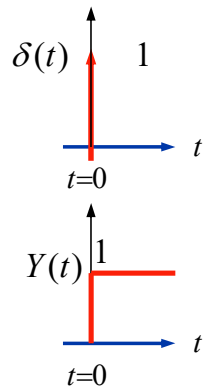


Input-output relationship :

$$U(t) = V(t) + \frac{1}{RC} \int_0^t V(\tau) d\tau \quad (1)$$

## ■ Representation by a linear filter

- ▶ Search for a filtering (or convolution) relationship
- ▶ Rewriting of the output-input relationship (1)



$$U(t) = \underbrace{\left[ \delta + \frac{1}{RC} Y \right]}_D * V(t) \quad \Rightarrow \quad \text{Impulse response : } h(t) = D^{-1} = Y(t) \exp\left(-\frac{t}{RC}\right)$$

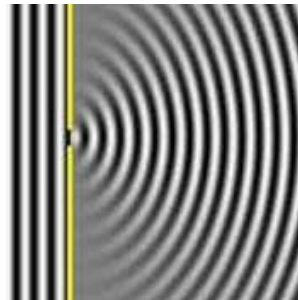
- ▶ Determination of the output voltage

$$V(t) = h(t) * U(t) = Y(t) \int_t^0 U'(\tau) \exp\left(-\frac{t-\tau}{RC}\right) d\tau + U(0)Y(t) \exp\left(-\frac{t}{RC}\right)$$



# Free-space wave propagation and diffraction (1/2)

## ■ Diffraction of an acoustic wave



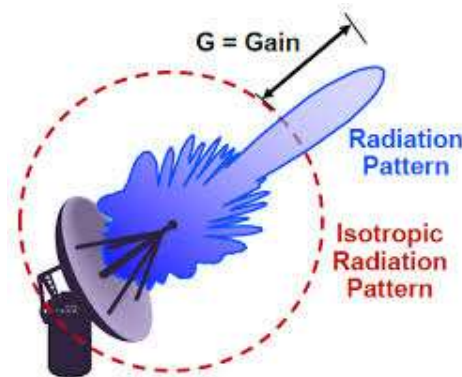
d'Alembert propagation equation

$$\Delta\phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

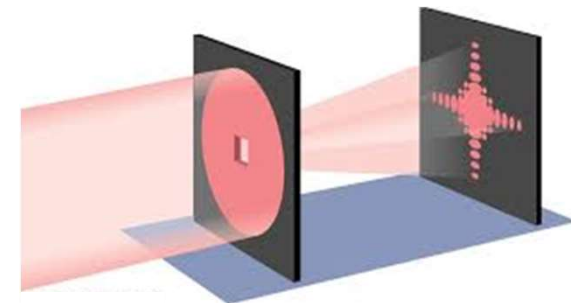
(pressure progressive wave)

## ■ Diffraction of an electromagnetic wave

- Electric field propagation : also governed by d'Alembert's equation
- Field radiated by a radio antenna
- Diffraction of an optical field through an aperture

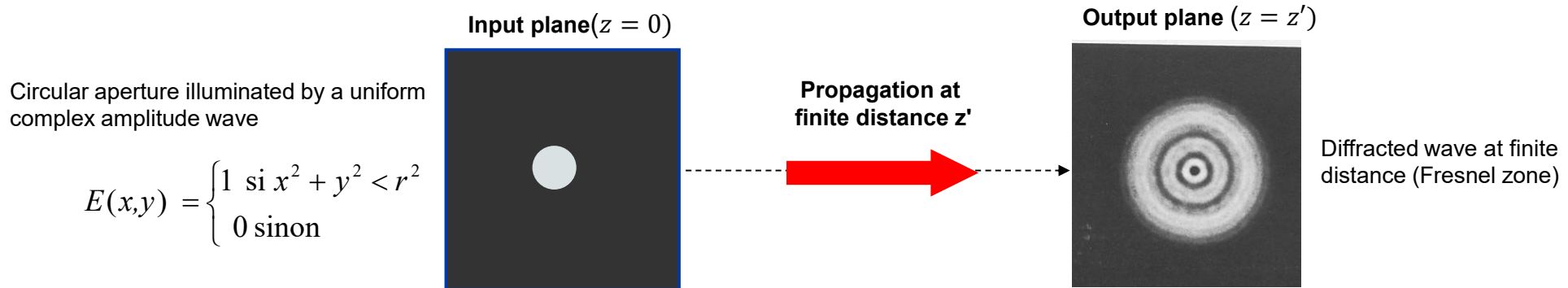


Radiation pattern of a parabolic antenna



Laser beam diffraction by a square aperture

# Free-space wave propagation and diffraction (2/2)



Solving the diffraction integral (Fresnel-Kirchoff)

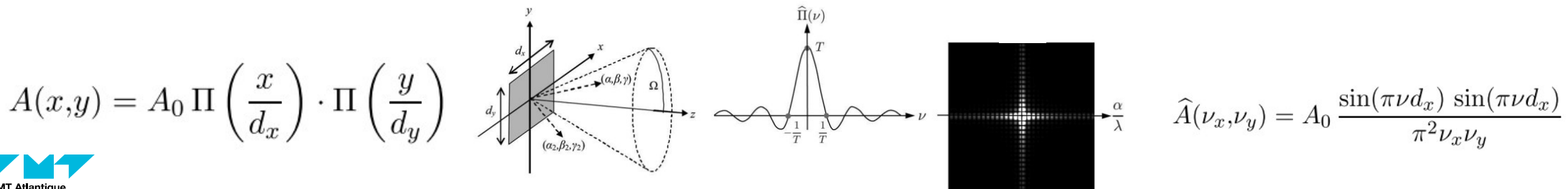


$$S(x', y') = (E * N_f)(x', y')$$

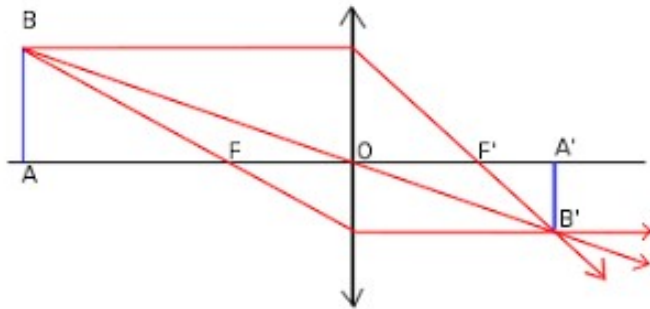
Spatial convolution

with  $N_f(x, y) = \exp \left[ -\frac{2i\pi}{\lambda z'} (x^2 + y^2) \right]$   
(Fresnel Kernel)

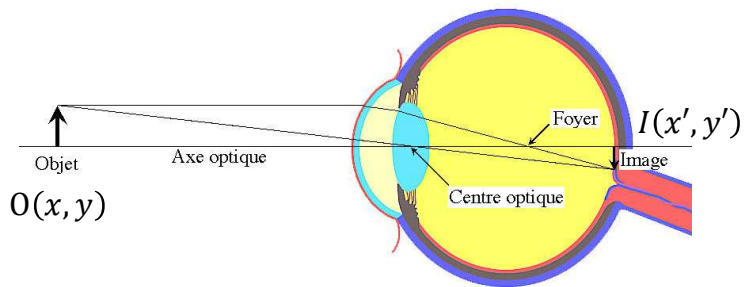
- If there is convolution, then free space acts as **a (spatial) filter** on the wave components.
- If  $(x^2 + y^2)/\lambda z' \ll 1$ , then  $S(x', y')$  is the Fourier transform (spatial) of  $E(x, y)$



# Convolution and imaging

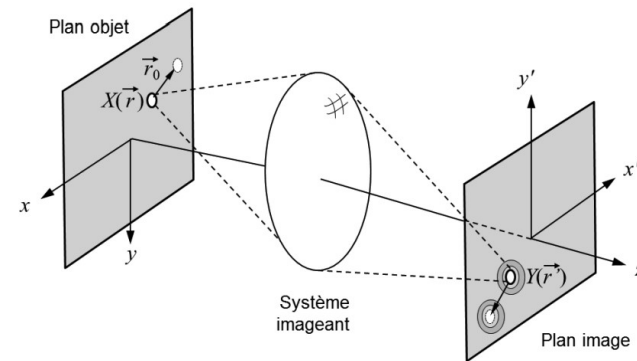


Formation of an image by a converging lens

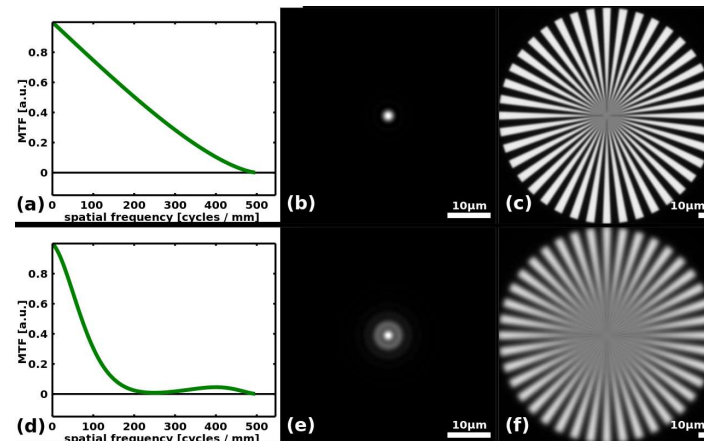


Formation of an image by the eye

$$I(x', y') = (O * h)(x', y')$$



Spatial impulse response of a converging lens  
-> linear and spatially invariant

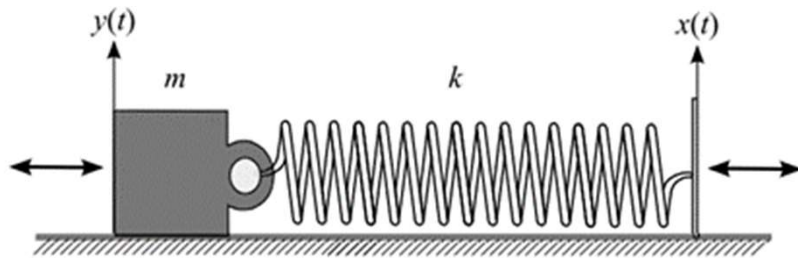


- Lens-Eye transfer function
- Spatial frequency

Modulation transfer function (MTF) of a converging lens

# Convolution (and filtering) in mechanics

- **Système masse-ressort** : on tire sur la tige et on observe le mouvement de la masse



Oscillateur mécanique amorti par frottement.

$$m \frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + k[y(t) - x(t)] = 0$$



$$\alpha = \gamma/2m \quad \omega_0 = \sqrt{k/m} \quad x(t) = \frac{1}{\omega_0^2} \underbrace{[\delta'' + 2\alpha\delta' + \omega_0^2\delta]}_M * y(t) \quad \Rightarrow \quad M(t) = [\delta' + \underbrace{(\alpha - j\omega_1)}_{\lambda_1} \delta] * [\delta' + \underbrace{(\alpha + j\omega_1)}_{\lambda_2} \delta]$$

En rappelant que  $Y e^{-\lambda t}$  est l'inverse de convolution de la distribution  $\delta' + \lambda\delta$  (justifié en page 101), la réponse impulsionnelle mécanique s'écrit :

$$h(t) = \omega_0^2 M^{-1}(t) = \omega_0^2 Y e^{-\lambda_1 t} * Y e^{-\lambda_2 t}$$

$$[YU * YV](t) = Y(t) \int_0^t U(\tau)V(t-\tau) d\tau$$



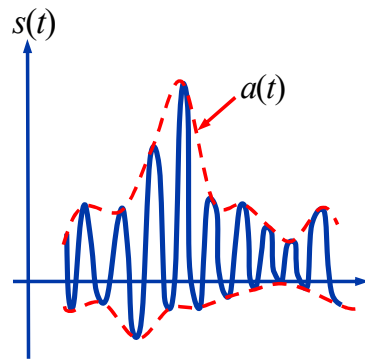
$$h(t) = \omega_0^2 Y(t) \int_0^t e^{-\lambda_1 \tau} e^{-\lambda_2 (t-\tau)} d\tau$$

$$= \frac{\omega_0^2 Y(t) e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} \left[ e^{-(\lambda_1 - \lambda_2)\tau} \right]_0^t$$

$$= \frac{\omega_0^2}{\omega_1} Y(t) \sin(\omega_1 t) e^{-\alpha t}$$

# Filtering a narrowband signal

## ■ Signal à "bande étroite" $s(t)$

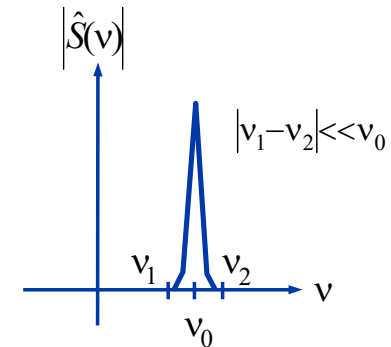


**Enveloppe**  
(variations "lentes")

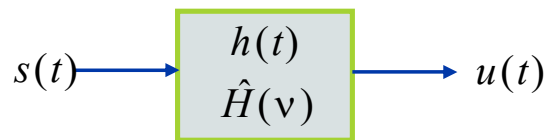
$$s(t) = a(t) e^{2i\pi\nu_0 t}$$



$$\hat{S}(\nu) = \hat{A}(\nu - \nu_0)$$



## ■ Que donne le filtrage de $s(t)$ ?



$$\hat{H}(\nu) = |\hat{H}(\nu)| \exp[i\phi(\nu)]$$

### Réponse en phase dans la bande filtrée

Approximation linéaire (1<sup>er</sup> ordre)

$$\phi(\nu) = \phi(\nu_0) + (\nu - \nu_0) \left( \frac{d\phi}{d\nu} \right)_{\nu=\nu_0}$$

# Phase and group delay of a filter

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## ■ On définit alors :

- Le temps de propagation de phase
- Le temps de propagation de groupe

$$\hat{H}(\nu) = |\hat{H}(\nu)| \exp[i\phi(\nu)]$$

$$\phi(\nu) = \phi(\nu_0) + (\nu - \nu_0) \left( \frac{d\phi}{d\nu} \right)_{\nu=\nu_0}$$

$$\hat{H}(\nu) = \underbrace{|\hat{H}(\nu)|}_{\simeq 1} \exp \{ -2i\pi [\nu_0(t_\varphi - t_g) + \nu t_g] \}$$

$t_\varphi = -\frac{\phi(\nu_0)}{2\pi\nu_0}$   
 $t_g = -\frac{1}{2\pi} \left( \frac{d\phi}{d\nu} \right)_{\nu=\nu_0}$

## ■ Le signal de sortie : $\hat{U}(\nu) = \hat{S}(\nu)\hat{H}(\nu)$

$$\downarrow$$

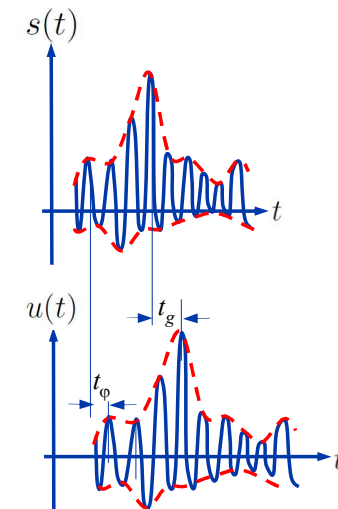
$$= \hat{S}(\nu) \exp(-2i\pi\nu t_g) \exp[-2i\pi\nu_0(t_\varphi - t_g)]$$

Transformée de  
Fourier inverse

$$u(t) = a(t - t_g) \exp[-2i\pi\nu_0(t - t_\varphi)]$$

L'enveloppe subit  
un retard  $t_g$

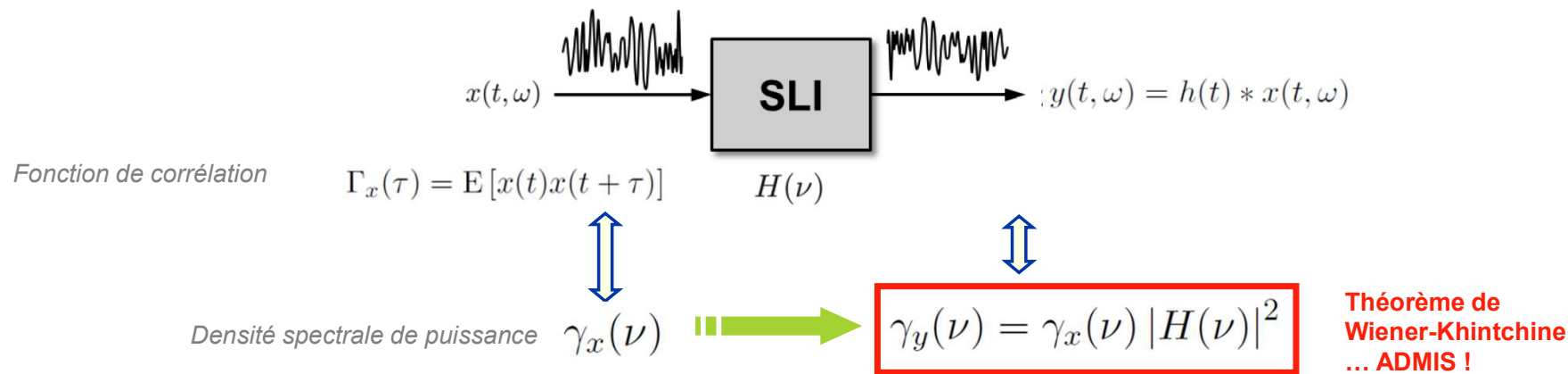
La composante sinusoïdale  
(porteuse) subit un retard  $t_\varphi$



## ■ Application : propagation d'impulsions dans une fibre optique, modélisée par un filtre dispersif

# Random signal (noise) filtering

- Signal aléatoire stationnaires au 2ème ordre, centré (voir cours 4).



- Filtrage d'un bruit blanc (thermique, grenaille)

$$\gamma_x(\nu) = B_0 \longrightarrow \gamma_y(\nu) = B_0 |H(\nu)|^2$$

La sortie n'est donc plus "blanche",  
sauf si  $|H(\nu)| = 1$  (filtre à retard)

- Puissance moyenne totale de bruit en sortie de filtre

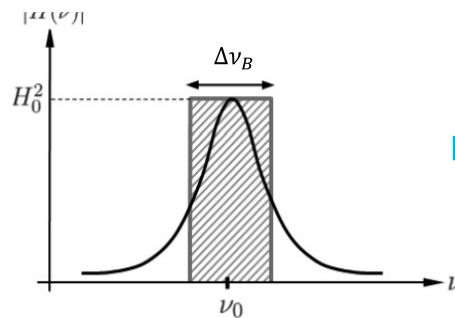
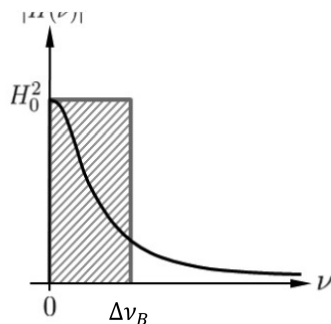
$$P_y^{\text{moy}} = \int_{-\infty}^{+\infty} \gamma_y(\nu) d\nu = \sigma_y^2 = B_0 \int_{-\infty}^{+\infty} |H(\nu)|^2 d\nu$$

# Equivalent noise band of a filter

## ■ Définition

- La **bande équivalente de bruit** d'un filtre est définie par la bande-passante d'un filtre rectangulaire qui laisserait passer la même puissance totale moyenne de bruit (blanc) que le filtre considéré :

$$B_0 \Delta\nu_B H_0^2 = \int_0^{+\infty} B_0 |H(\nu)|^2 d\nu$$



$$\Delta\nu_B = \frac{1}{H_0^2} \int_0^{+\infty} |H(\nu)|^2 d\nu$$

Bande équivalente de bruit du filtre

## ■ Puissance moyenne de bruit en sortie du filtre :

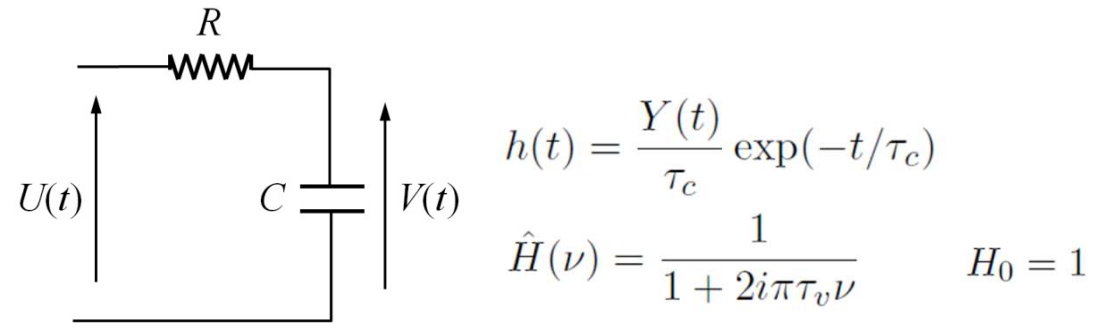
$$P_{\text{bruit}}^s = \sigma_y^2 = B_0 H_0^2 \Delta\nu_B$$

- **Conclusion** : pour un bruit blanc, le filtre possède un gain constant  $H_0$  dans la bande  $\Delta\nu_B$
- **Remarque** : pour certains filtres, la valeur de la bande équivalente de bruit est *proche de* la bande-passante ...mais il ne faut pas confondre les deux



# First-order electronic filter (low pass)

■ On pose  $RC = \tau_c$



► Bande équivalente de bruit :

$$\Delta\nu_B = \int_0^{+\infty} \frac{d\nu}{1 + 4\pi^2\tau_c^2\nu^2} = \frac{1}{2\pi\tau_c} [\text{artcg}(\nu)]_0^{+\infty} = \frac{1}{4\pi\tau_c} = \frac{\pi}{2} \Delta\nu_{-3\text{dB}}$$

► Puissance moyenne de la tension bruit en sortie (aux bornes du condensateur)

$$P_s^{\text{moy}} = B_0 H_0^2 \Delta\nu_B = 4kTR \cdot \frac{1}{4RC} = \frac{kT}{C}$$