TAF STAR – TAF OPE

UE Cœur 1

Course 5

Filtering of physical signals

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Outline

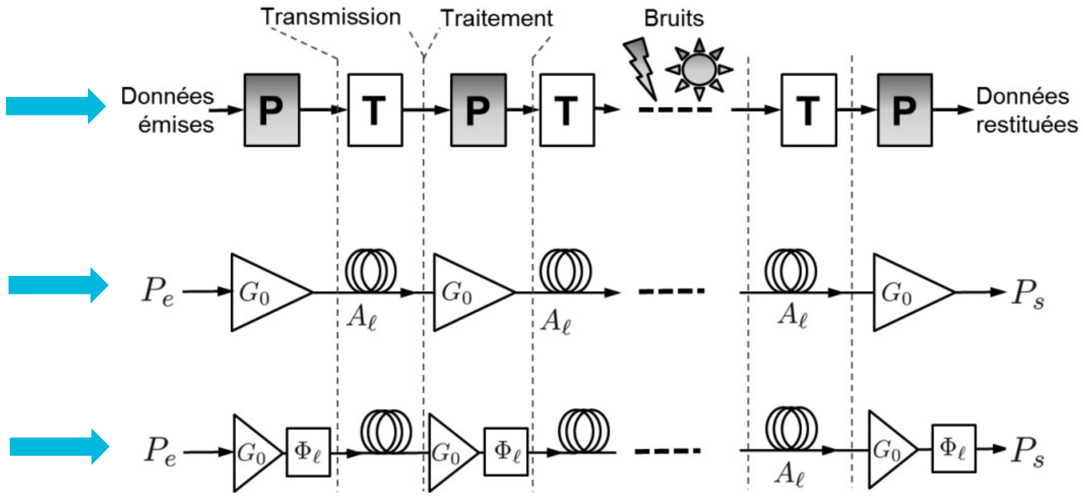
Introduction

Modular decompositions of a transmission system

a) cascading of "P" (processing) and "T" (transmission) sections

b) chain of amplifiers and attenuators $G_0 + A_\ell = 0$ dB

c) chain of amplifiers and filters: dispersive channel

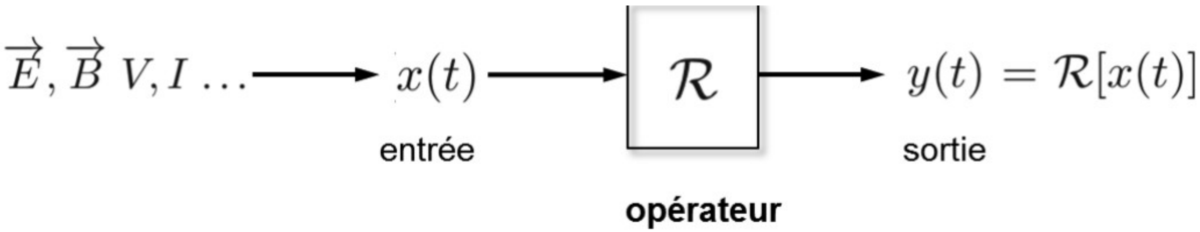


Case of a dispersive channel (c)

- Path dispersion
- Modal dispersion
- Chromatic dispersion

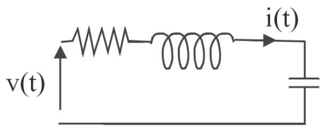
Modelling of a physical (continuous) system

- **Black box model**: input-output relationship
  - **Electromagnetic input quantities**: current, voltage, electric field, magnetic field, voltage or current wave



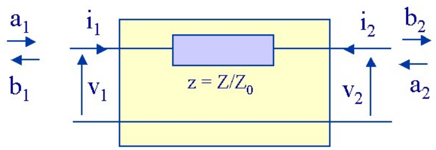
  - **Three main types of operators**
    - *Linear and invariant (filters)*
    - *Parametric: multipliers, modulators, samplers*
    - *Non-linear without memory: rectifiers, limiters, skimmers, etc…*

RLC circuit



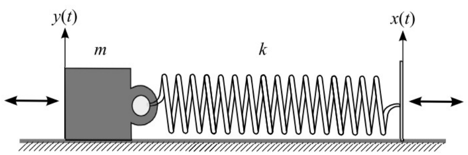
$$e(t) = v(t)$$
$$s(t) = i(t)$$

Quadripole



$$e(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \quad s(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = [Z]\, e(t)$$

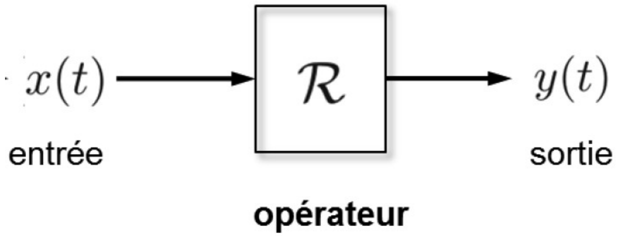
Oscillating mass



$$e(t) = x(t)$$
$$s(t) = y(t)$$

Impulse response of the physical operator

- General case (non-linear)

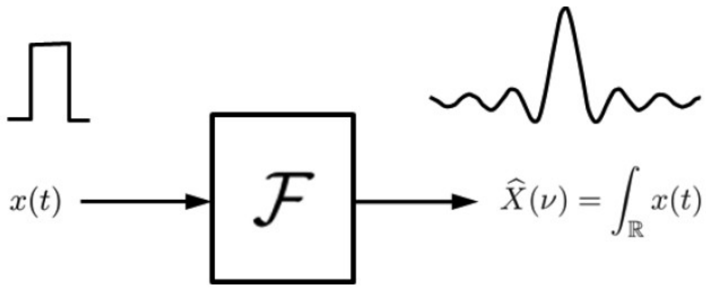


$$y(t) = \mathcal{R}\left[x(t)\right] = \int_{\mathbb{R}} \underbrace{h[t,\tau,x(\tau)]}_{\text{Impulse response (output at time } t \text{ to a localized Dirac impact at input } \tau)} \mathrm{d}\tau$$

- Linear case

$$y(t) = \int_{\mathbb{R}} x(\tau)\, h(t,\tau) \mathrm{d}\tau \quad \Longrightarrow \quad \mathcal{R}\left[\lambda e_1(t) + \mu e_2(t)\right] = \lambda \mathcal{R}\left[e_1(t)\right] + \mu \mathcal{R}\left[e_2(t)\right]$$

  - Non-stationary case : $h(t_1,\tau) \neq h(t_2,\tau)$ pour $t_1 \neq t_2$
  - **Example**: the Fourier “transformer”



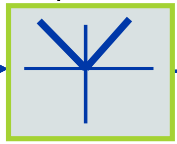
$$\widehat{X}(\nu) = \int_{\mathbb{R}} x(t) \underbrace{\exp(-2j\pi\nu t)}_{h(t,\nu)} \, \mathrm{d}t$$

Impulse response

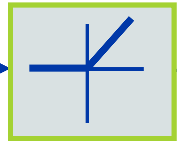
Some non-linear operators (memoryless)

- **Rectifier**

bipolaire

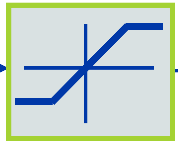
$x(t)$ →  → $y(t) = |x(t)|$

unipolaire

$x(t)$ →  → $y(t)$

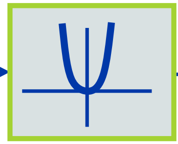
$$= \frac{1}{2}\left(x(t) + |x(t)|\right)$$

- **Skimmer**

$x(t)$ →  →

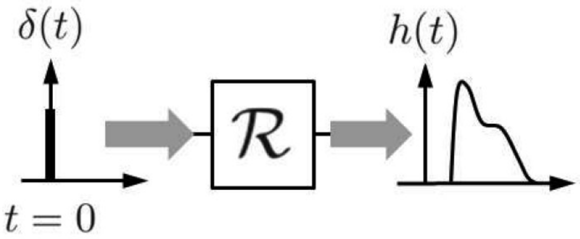
$$y(t) = \begin{cases} -a & \text{si} \quad x < -a \\ x(t) & \text{si} -a \leq x \leq a \\ a & \text{si} \quad x \geq a \end{cases}$$

- **Quadratic detector**

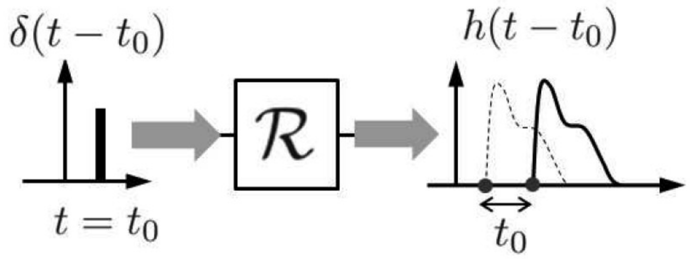
$x(t)$ →  → $y(t) = x(t)^2$

Linear invariant operator (the filter !)

- Temporal description



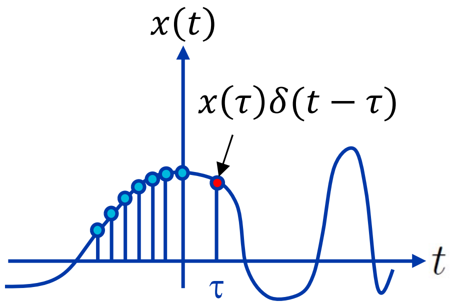
Impulse response
Also called "Green" function



Time invariance of the impulse response

- Two properties: linearity + invariance : $x(t) = \alpha\delta(t-\tau) \implies y(t) = \alpha h(t-\tau)$

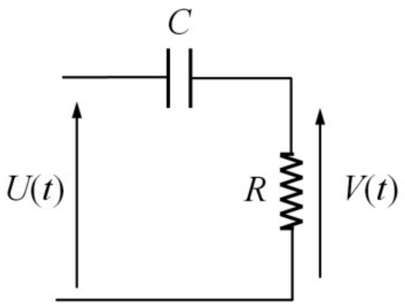
- Response to any input $x(t)$



$$y(t) = \int_{\mathbb{R}} x(\tau)\, h(t-\tau)\mathrm{d}\tau = (h * x)(t)$$

Filtering ↔ Convolution

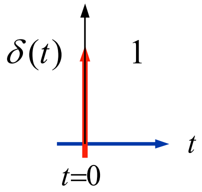
Example of a filter: RC circuit (high-pass)



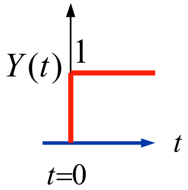
Input-output relationship :

$$U(t) = V(t) + \frac{1}{RC}\int_0^t V(\tau)\,\mathrm{d}\tau \qquad (1)$$

- **Representation by a linear filter**
  - Search for a filtering (or convolution) relationship
  - Rewriting of the output-input relationship (1)



$$U(t) = \underbrace{\left[\delta + \frac{1}{RC}Y\right]}_{D} * V(t) \;\Rightarrow\; \text{Impulse response : } h(t) = D^{-1} = Y(t)\exp\left(-\frac{t}{RC}\right)$$

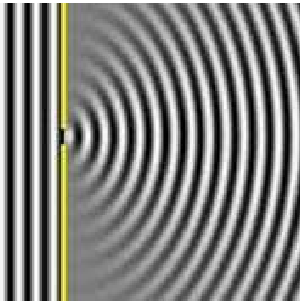


  - Determination of the output voltage

$$\boxed{V(t) = h(t) * U(t) = Y(t)\int_t^0 U'(\tau)\exp\left(-\frac{t-\tau}{RC}\right)\mathrm{d}\tau + U(0)Y(t)\exp\left(-\frac{t}{RC}\right)}$$

Free-space wave propagation and diffraction (1/2)

- **Diffraction of an acoustic wave**

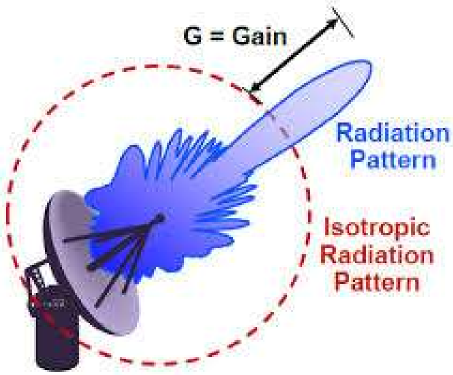


d'Alembert propagation equation

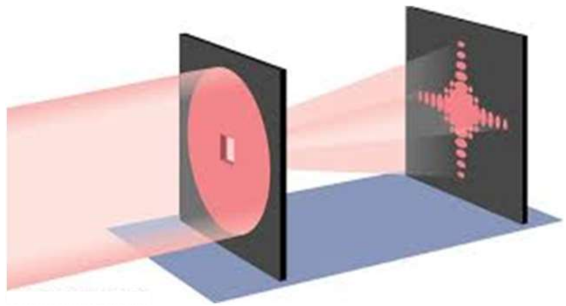
$$\boxed{\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0}$$

(pressure progressive wave)

- **Diffraction of an electromagnetic wave**
  - Electric field propagation : also governed by d'Alembert's equation
  - Field radiated by a radio antenna
  - Diffraction of an optical field through an aperture



Radiation pattern of a parabolic antenna



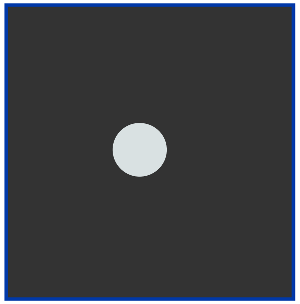
Laser beam diffraction by a square aperture

Free-space wave propagation and diffraction (2/2)

Circular aperture illuminated by a uniform complex amplitude wave

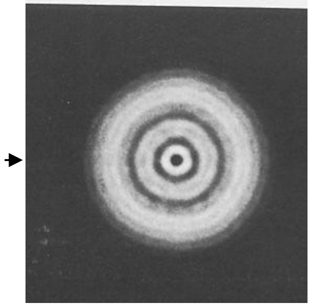
$$E(x,y) = \begin{cases} 1 \ \text{si}\ x^2 + y^2 < r^2 \\ 0 \ \text{sinon} \end{cases}$$

Input plane $(z = 0)$



Propagation at finite distance z'

Output plane $(z = z')$



Diffracted wave at finite distance (Fresnel zone)

Solving the diffraction integral (Fresnel-Kirchoff)

$$\boxed{S(x',y') = (E * N_f)(x',y')}$$

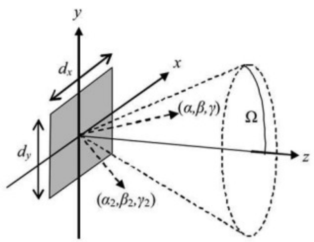
Spatial convolution

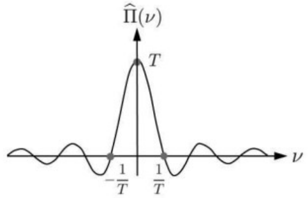
$$\text{with} \quad N_f(x,y) = \exp\left[-\frac{2i\pi}{\lambda z'}(x^2+y^2)\right]$$

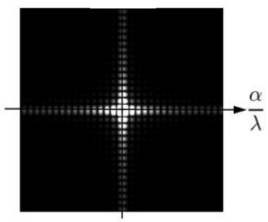
(Fresnel Kernel)

- If there is convolution, then free space acts as **a (spatial) filter** on the wave components.
- If $(x^2+y^2)/\lambda z' \ll 1$, then $S(x',y')$ is the Fourier transform (spatial) of $E(x,y)$

$$A(x,y) = A_0\,\Pi\left(\frac{x}{d_x}\right)\cdot\Pi\left(\frac{y}{d_y}\right)$$

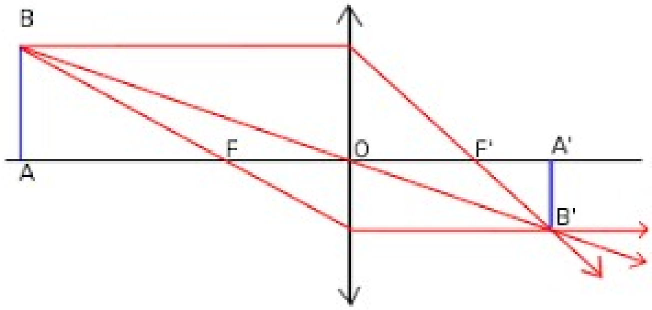




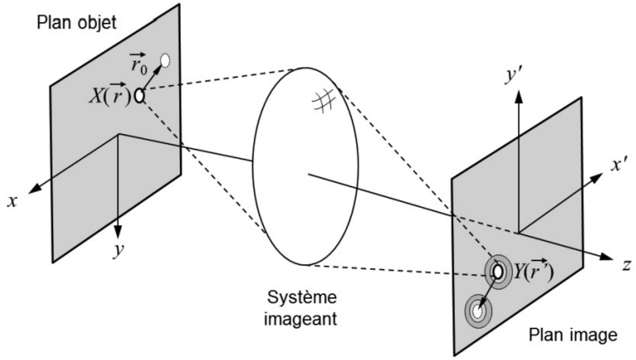


$$\hat{A}(\nu_x,\nu_y) = A_0\frac{\sin(\pi\nu d_x)\,\sin(\pi\nu d_x)}{\pi^2\nu_x\nu_y}$$

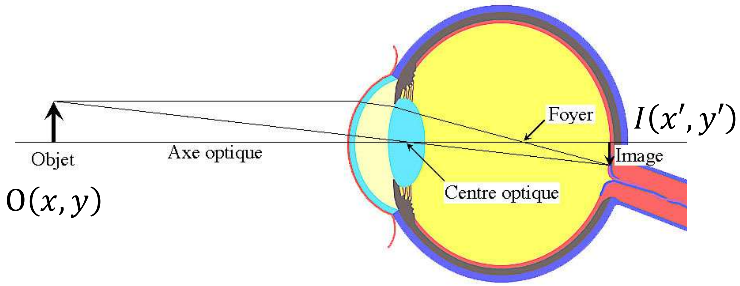
Convolution and imaging



Formation of an image by a converging lens

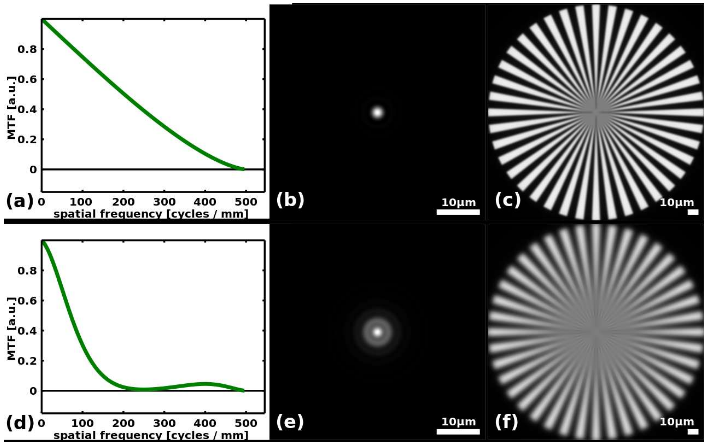


Spatial impulse response of a converging lens
-> linear and spatially invariant



Formation of an image by the eye

$$I(x', y') = (O * h)(x', y')$$

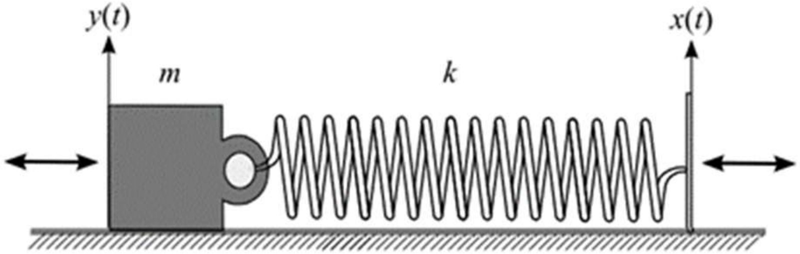


- Lens-Eye transfer function
- Spatial frequency

Modulation transfer function (MTF) of a converging lens

Convolution (and filtering) in mechanics

- **Système masse-ressort** : on tire sur la tige et on observe le mouvement de la masse



Oscillateur mécanique amorti par frottement.

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}y}{\mathrm{d}t} + k[y(t) - x(t)] = 0$$

$$\begin{aligned}\alpha &= \gamma/2m \\ \omega_0 &= \sqrt{k/m}\end{aligned} \qquad x(t) = \frac{1}{\omega_0^2}\underbrace{\left[\delta'' + 2\alpha\delta' + \omega_0^2\delta\right]}_{M} * y(t)$$

$$M(t) = [\delta' + \underbrace{(\alpha - j\omega_1)}_{\lambda_1}\delta] * [\delta' + \underbrace{(\alpha + j\omega_1)}_{\lambda_2}\delta]$$

En rappelant que $Y\,\mathrm{e}^{-\lambda t}$ est l'inverse de convolution de la distribution $\delta' + \lambda\delta$ (justifié en page 101), la réponse impulsionnelle mécanique s'écrit :

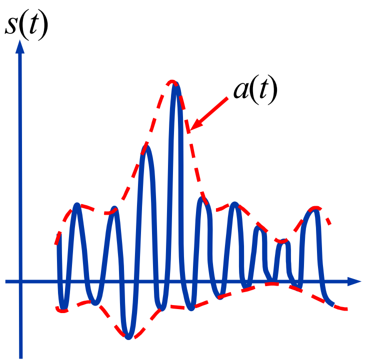
$$h(t) = \omega_0^2\, M^{-1}(t) = \omega_0^2\, Y\,\mathrm{e}^{-\lambda_1 t} * Y\,\mathrm{e}^{-\lambda_2 t}$$

$$[YU * YV](t) = Y(t)\int_0^t U(\tau)V(t-\tau)\,\mathrm{d}\tau$$

$$\begin{aligned} h(t) = {} & \omega_0^2\, Y(t)\int_0^t \mathrm{e}^{-\lambda_1\tau}\,\mathrm{e}^{-\lambda_2(t-\tau)}\,\mathrm{d}t \\ = {} & \frac{\omega_0^2\, Y(t)\,\mathrm{e}^{-\lambda_2 t}}{\lambda_2 - \lambda_1}\left[\mathrm{e}^{-(\lambda_1-\lambda_2)\tau}\right]_0^t \\ & \boxed{= \frac{\omega_0^2}{\omega_1}\, Y(t)\sin(\omega_1 t)\,\mathrm{e}^{-\alpha t}} \end{aligned}$$

Filtering a narrowband signal

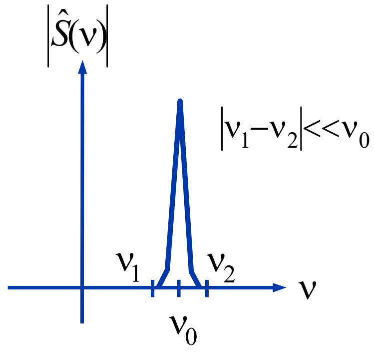
- **Signal à "bande étroite" $s(t)$**



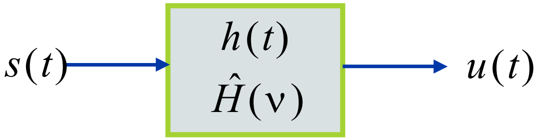
Enveloppe
(variations "lentes")

$$s(t) = a(t)\, e^{2i\pi\nu_0 t}$$

$$\hat{S}(\nu) = \hat{A}(\nu-\nu_0)$$



- **Que donne le filtrage de $s(t)$?**



$$\hat{H}(\nu) = \left|\hat{H}(\nu)\right| \exp\left[i\phi(\nu)\right]$$

Réponse en phase dans la bande filtrée

Approximation linéaire (1er ordre)

$$\phi(\nu) = \phi(\nu_0) + (\nu - \nu_0)\left(\frac{d\phi}{d\nu}\right)_{\nu=\nu_0}$$

Phase and group delay of a filter

- **On définit alors :**
  - Le temps de propagation de phase
  - Le temps de propagation de groupe

$$\hat{H}(\nu) = \left|\hat{H}(\nu)\right| \exp\left[i\phi(\nu)\right]$$

$$\phi(\nu) = \phi(\nu_0) + (\nu - \nu_0)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\nu}\right)_{\nu=\nu_0}$$

$$t_\varphi = -\frac{\phi(\nu_0)}{2\pi\nu_0}$$

$$t_g = -\frac{1}{2\pi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\nu}\right)_{\nu=\nu_0}$$

$$\boxed{\hat{H}(\nu) = \underbrace{\left|\hat{H}(\nu)\right|}_{\simeq 1} \exp\left\{-2i\pi\left[\nu_0(t_\varphi - t_g) + \nu t_g\right]\right\}}$$

- **Le signal de sortie :** $\hat{U}(\nu) = \hat{S}(\nu)\hat{H}(\nu)$

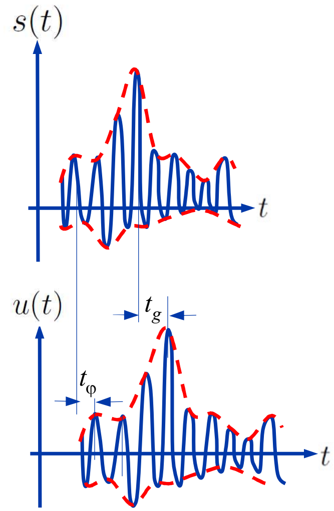
$$= \hat{S}(\nu)\exp(-2i\pi\nu t_g)\,\exp\left[-2i\pi\nu_0\,(t_\varphi - t_g)\right]$$

Transformée de Fourier inverse

$$u(t) = a(t - t_g)\,\exp\left[-2i\pi\nu_0(t - t_\varphi)\right]$$

L'enveloppe subit un retard t_g

La composante sinusoïdale (porteuse) subit un retard t_φ

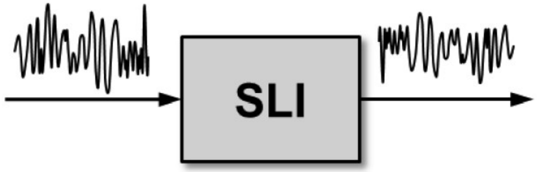


- **Application :** propagation d'impulsions dans une fibre optique, modélisée par un filtre dispersif

Random signal (noise) filtering

- Signal aléatoire stationnaires au 2ème ordre, centré (voir cours 4).

$x(t,\omega)$



$y(t,\omega) = h(t) * x(t,\omega)$

Fonction de corrélation $\Gamma_x(\tau) = \mathrm{E}\left[x(t)x(t+\tau)\right]$ $\quad H(\nu)$

$\Updownarrow \qquad \Updownarrow$

Densité spectrale de puissance $\gamma_x(\nu)$ $\Rightarrow$ $\boxed{\gamma_y(\nu) = \gamma_x(\nu)\left|H(\nu)\right|^2}$

Théorème de Wiener-Khintchine … ADMIS !

- Filtrage d'un bruit blanc (thermique, grenaille)

$$\gamma_x(\nu) = B_0 \Rightarrow \gamma_y(\nu) = B_0\left|H(\nu)\right|^2$$

La sortie n'est donc plus "blanche", sauf si $|H(\nu)| = 1$ (filtre à retard)

- Puissance moyenne totale de bruit en sortie de filtre

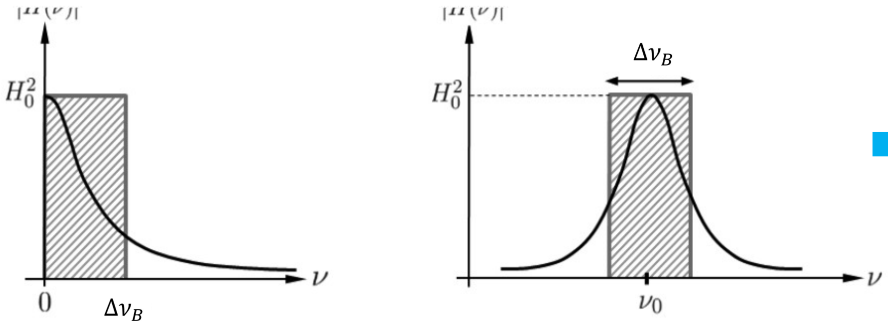
$$\boxed{P_y^{\mathrm{moy}} = \int_{-\infty}^{+\infty} \gamma_y(\nu)\,\mathrm{d}\nu = \sigma_y^2 = B_0 \int_{-\infty}^{+\infty} \left|H(\nu)\right|^2\,\mathrm{d}\nu}$$

Equivalent noise band of a filter

Définition

- La **bande équivalente de bruit** d'un filtre est définie par la bande-passante d'un filtre rectangulaire qui laisserait passer la même puissance totale moyenne de bruit (blanc) que le filtre considéré :

$$B_0 \Delta\nu_B H_0^2 = \int_0^{+\infty} B_0 |H(\nu)|^2 \mathrm{d}\nu$$



$$\boxed{\Delta\nu_B = \frac{1}{H_0^2} \int_0^{+\infty} |H(\nu)|^2 \mathrm{d}\nu}$$

Bande équivalente de bruit du filtre

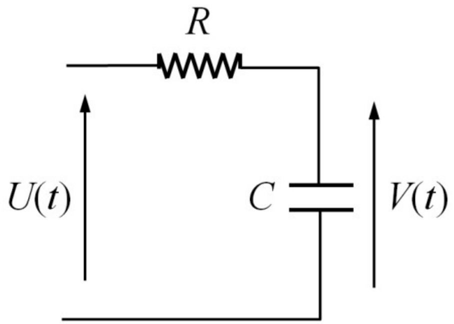
Puissance moyenne de bruit en sortie du filtre :

$$\boxed{P^s_{\text{bruit}} = \sigma_y^2 = B_0 H_0^2 \Delta\nu_B}$$

- **Conclusion** : pour un bruit blanc, le filtre possède un gain constant H_0 dans la bande $\Delta\nu_B$
- **Remarque** : pour certains filtres, la valeur de la bande équivalente de bruit est *proche de* la bande-passante …mais il ne faut pas confondre les deux

First-order electronic filter (low pass)

- **On pose** RC = τ_c



$$h(t) = \frac{Y(t)}{\tau_c} \exp(-t/\tau_c)$$

$$\hat{H}(\nu) = \frac{1}{1 + 2i\pi\tau_v\nu} \qquad H_0 = 1$$

- Bande équivalente de bruit :

$$\Delta\nu_B = \int_0^{+\infty} \frac{\mathrm{d}\nu}{1 + 4\pi^2\tau_c^2\nu^2} = \frac{1}{2\pi\tau_c}\left[\mathrm{artcg}(\nu)\right]_0^{+\infty} = \frac{1}{4\pi\tau_c} = \frac{\pi}{2}\Delta\nu_{-3\mathrm{dB}}$$

- Puissance moyenne de la tension bruit en sortie (aux bornes du condensateur)

$$\boxed{P_s^{\mathrm{moy}} = B_0 H_0^2 \Delta\nu_B = 4kTR \cdot \frac{1}{4RC} = \frac{kT}{C}}$$