Формулы

1. Гиперболические функции.

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$
$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

2. Производные.

1. $c' = 0, c = const$
2. $(x^n)' = nx^{n-1}$
3. $(a^x)' = a^x \cdot \ln a$
4. $(e^x)' = e^x$
5. $(\ln x)' = \frac{1}{x}$
6. $(\log_a x)' = \frac{1}{x \ln a}$
7. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$
8. $(\sin x)' = \cos x$
9. $(\cos x)' = -\sin x$
10. $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$
11. $(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$
12. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
13. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
14. $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$
15. $(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}$
16. $(\operatorname{sh} x)' = \operatorname{ch} x$
17. $(\operatorname{ch} x)' = \operatorname{sh} x$
18. $(\operatorname{th} x)' = \frac{1}{\operatorname{ch}^2 x}$
19. $(\operatorname{cth} x)' = -\frac{1}{\operatorname{sh}^2 x}$

3. Первая производная от параметрической функции.

$$y'_x = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'_t}{x'_t}$$

4. Вторая производная от параметрической функции.

$$y''_{xx} = \left(\frac{y'_t}{x'_t}\right)'_t \cdot t'_x = \frac{y''_{tt}x'_t - x''_{tt}y'_t}{(x'_t)^2} \cdot \frac{1}{x'_t} = \frac{y''_{tt}x'_t - x''_{tt}y'_t}{(x'_t)^3}$$

5. Производные высших порядков.

1. $f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$
2. $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)', n \in \mathbb{N}$
3. $(a^x)^{(n)} = a^x \ln^n a$
4. $(e^x)^{(n)} = e^x$
5. $(\sin ax)^{(n)} = a^n \sin(ax + \frac{\pi n}{2})$
6. $(\cos ax)^{(n)} = a^n \cos(ax + \frac{\pi n}{2})$
7. $((ax+b)^\alpha)^{(n)} = a^n \alpha(\alpha-1)\ldots(\alpha-n+1)(ax+b)^{\alpha-n} = a^n C_\alpha^n n!(ax+b)^{\alpha-n}$
8. $(\log_a |x|)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n \ln a}$
9. $(\ln |x|)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$

6. Формула Лейбница.

$$(uv)^n = \sum_{k=0}^{n} C_n^k u^{(n-k)} v^{(k)}$$

7. Формула Тейлора.

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \\ &+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o\left((x-x_0)^n\right), \quad x \to x_0 \\ f(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^n + o\left((x-x_0)^n\right), \quad x \to x_0 \end{aligned} \tag{1}$$

8. Формула Маклорена.

8.1 Общий вид.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + o(x^n), \quad x \to 0 \tag{2}$$

8.2 Для четной функции.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} + o(x^{2n+1}) \tag{3}$$

8.3 Для нечетной функции.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} + o(x^{2n+2}) \tag{4}$$

8.4 Частые функции.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + o(x^n) \\ e^x &= \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) \end{aligned} \tag{5}$$

$$\begin{aligned} \operatorname{sh} x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \\ \operatorname{sh} x &= \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \end{aligned} \tag{6}$$

$$\begin{aligned} \operatorname{ch} x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \\ \operatorname{ch} x &= \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) \end{aligned} \tag{7}$$

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \\ \sin x &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \end{aligned} \tag{8}$$

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}) \\ \cos x &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) \end{aligned} \tag{9}$$

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + o(x^n) \\ (1+x)^\alpha &= \sum_{k=0}^{n} C_\alpha^k x^k + o(x^n), \end{aligned} \tag{10}$$

где $C_\alpha^0 = 1$, $C_\alpha^k = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$

$$\frac{1}{1+x} = \sum_{k=0}^{n} (-1)^n x^k + o(x^n) \tag{11}$$

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + o(x^n) \tag{12}$$

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = \sum_{k=0}^{n} C_{-1/2}^k (-1)^k x^k + o(x^n) \tag{13}$$

где

$$C_{-1/2}^k = \frac{(-1/2)(-1/2-1)\cdots(-1/2-k+1)}{k!} = (-1)^k \frac{(2k-1)!!}{2^k k!}$$

$$(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1} x^n}{n} + o(x^n)$$

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{k} + o(x^n) \tag{14}$$

$$\ln(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} + o(x^n) \tag{15}$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + o(x^{2n+1}) \tag{16}$$

8.5 Формула из производной

Если

$$f' = \sum_{k=0}^{n} b_k (x-x_0)^k + o\left((x-x_0)^n\right)$$

, то

$$f(x) = f(x_0) + \sum_{k=0}^{n} \frac{b_k}{k+1} (x-x_0)^{k+1} + o\left((x-x_0)^{n+1}\right) \tag{17}$$

8.6 Еще функции.

$$\operatorname{tg} x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5) \tag{18}$$

$$\operatorname{ctg} x = x^{-1} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + o(x^5) \tag{19}$$

$$\begin{aligned}
\arcsin x &= x + \frac{x^3}{6} + \frac{3x^5}{40} + o(x^6) \\
(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{n} C_{-1/2}^{k} x^{2k} + o(x^{2n+1}) \\
\arcsin x &= x + \sum_{k=1}^{n} C_{-1/2}^{k} \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})
\end{aligned} \tag{20}$$

$$\begin{aligned}
\arccos x &= \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} + o(x^6) \\
\arccos x &= \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - x - \sum_{k=1}^{n} C_{-1/2}^{k} \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})
\end{aligned} \tag{21}$$

$$\begin{aligned}
\operatorname{arctg} x &= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6) \\
(\operatorname{arctg} x)' &= \frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + o(x^{2n+1}) \\
\operatorname{arctg} x &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})
\end{aligned} \tag{22}$$

$$\begin{aligned}
\operatorname{arcctg} x &= \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + o(x^6) \\
\operatorname{arcctg} x &= \frac{\pi}{2} - \operatorname{arctg} x = \frac{\pi}{2} - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})
\end{aligned} \tag{23}$$

$$\operatorname{th} x = x - \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6) \tag{24}$$

,

$$\operatorname{cth} x = x^{-1} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + o(x^6) \tag{25}$$

9. Кривые.

9.1 Кривизна

$$k = \frac{|[r', r'']|}{|r'|^3}$$

9.2 Кручение.

$$\chi = \frac{(r', r'', r''')}{k^2|r'|^6} = \frac{(r', r'', r''')}{|[r', r'']|^2}$$

9.3 Площадь.

$$S = \int_a^b y(x)dx$$

$$S = \int_a^b y(t)x'(t)dx$$

$$S = \frac{1}{2}\int_a^b r^2(\phi)d\phi$$

9.4 Длина.

$$s = \int_a^b \sqrt{1 + y'^2}dx$$

$$s = \int_a^b \sqrt{x' + y'^2}dt$$

$$s = \int_a^b \sqrt{r^2 + r'^2}d\phi$$

9.5 Объем.

$$V = \pi \int_a^b y^2(x)dx$$

$$V = \pi \int_a^b y^2(t)x'(t)dx$$

10. Эталонные интегралы.

Эталон	Сходится	Расходится
$\int_0^1 \frac{dx}{x^\alpha}$	$\alpha < 1$	$\alpha \geqslant 1$
$\int_1^{+\infty} \frac{dx}{x^\alpha}$	$\alpha > 1$	$\alpha \leqslant 1$
$\int_0^{+\infty} \frac{dx}{e^{\alpha x}}$	$\alpha > 0$	$\alpha \leqslant 0$
$\int_0^{\frac{1}{2}} \frac{dx}{x^\alpha \lvert \ln x \rvert^\beta}$	$\alpha < 1, \beta \in \mathbb{R}$ или $\alpha = 1, \beta > 1$	$\alpha > 1, \beta \in \mathbb{R}$ или $\alpha = 1, \beta \leqslant 1$
$\int_2^{+\infty} \frac{dx}{x^\alpha \ln^\beta x}$	$\alpha > 1, \beta \in \mathbb{R}$ или $\alpha = 1, \beta > 1$	$\alpha < 1, \beta \in \mathbb{R}$ или $\alpha = 1, \beta \leqslant 1$
$\int_1^{+\infty} \frac{dx}{e^\alpha x^\beta}$	$\alpha > 0, \beta \in \mathbb{R}$ или $\alpha = 0, \beta > 1$	$\alpha > 0, \beta \in \mathbb{R}$ или $\alpha = 0, \beta \leqslant 1$