Title

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Contents

Chapter 1

An Introduction to Representation Theory

Intro paragraph to lead into the definitions.

Definition 1.1 (Representation of a group)**.** Let G be a group. A *representation* of G is a homomorphism from G to a group of operators on a linear vector space V. The dimension of V is the *dimension* or *degree* of the representation.

If X is a representation of G on V, then X is a map

$$g \in G \xrightarrow{X} X(g) \tag{1.1}$$

in which $X(g)$ is an operator on the vector space V. For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \ldots, n\}$, we can realize each operator $X(g)$ as an $n \times n$ matrix $D(g)$.

$$X(g) \left| e_i \right\rangle = \sum_{j=1}^{n} \left| e_j \right\rangle D(g)^j{}_i = \left| e_j \right\rangle D(g)^j{}_i, \tag{1.2}$$

where the first index j is the row index and the second index i is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$X(g_1)X(g_2) = X(g_1 g_2), \tag{1.3}$$

which satisfies the group multiplication rules. Keep Dirac notation here? If so, reference appendix on Dirac notation.

Definition 1.2. If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

Example 1.1. The simplest representation of any group G is the *trivial* representation, in which every $g \in G$ is realized by $g \mapsto 1$. This representation is clearly degenerate.

Example 1.2. Consider the symmetric group S_n. The *defining* representation of S_n encodes each $\sigma \in S_n$ by placing a 1 in the j-th row and i-th column of the matrix $D(\sigma)$ if σ sends i to j, and 0 otherwise. For example, in S_3, the permutation (23) has the matrix representation

$$D\big((23)\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

whereas the permutation (123) is realized by the matrix

$$D\big((123)\big) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The above example involves a finite group. Infinite groups can also have representations, as demonstrated in the following example.

Example 1.3. Let G be the group of continuous rotations in the xy-plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space V_2. Then we define a representation of G on V_2 by the familiar rotation operation

$$\mathbf{e}_1' = X(\phi)\mathbf{e}_1 = \mathbf{e}_1 \cdot \cos\phi + \mathbf{e}_2 \cdot \sin\phi \tag{1.4}$$
$$\mathbf{e}_2' = X(\phi)\mathbf{e}_2 = -\mathbf{e}_1 \cdot \sin\phi + \mathbf{e}_2 \cdot \cos\phi, \tag{1.5}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthonormal basis vectors of V_2. This gives us the matrix representation

$$D(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{1.6}$$

To further illuminate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \mathbf{x} \in V_2$, then we have

$$\mathbf{x}' = X(\phi)\mathbf{x} = \mathbf{e}_j x'^j, \tag{1.7}$$

where $x'^j = D(\phi)^j{}_i x^i$. Can probably simplify the notation

Definition 1.3 (Equivalence of Representations)**.** For a group G, two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define the following.

Definition 1.4 (Characters of a Representation)**.** The *character* $\chi(g)$ of an element $g \in G$ in a representation $X(g)$ is defined as $\chi(g) = \text{Tr } D(g)$.

Since trace is independent of basis, the character serves as a class label.

1.1 Irreducibility and Invariant Subspaces

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

Definition 1.5 (Invariant Subspace)**.** Let $X(G)$ be a representation of G on a vector space V, and W a subspace of V such that $X(g)\,|x\rangle \in W$ for all $\mathbf{x} \in W$ and $g \in G$. Then W is an *invariant subspace* of V with respect to $X(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $X(G)$.

The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

Definition 1.6 (Irreducible Representation)**.** A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $X(G)$. Otherwise, it is *reducible*. If $X(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $X(G)$, then the representation is *fully reducible*.

Example 1.4. Different example! Under the group of 2-dimensional rotations, consider the 1-dimensional subspace spanned by $\hat{\mathbf{e}}_1$. This subspace is not invariant under 2-dimensional rotations, because a rotation of $\hat{\mathbf{e}}_1$ by $\pi/2$ results in the vector $\hat{\mathbf{e}}_2$ that is clearly not in the subspace spanned by $\hat{\mathbf{e}}_1$. A similar argument shows that the subspace spanned by $\hat{\mathbf{e}}_2$ is not invariant under 2-dimensional rotations.

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5**?

Schur's Lemmas?

Theorem 1.1. *Let G be a finite group. The number of irreducible representations of G is equal to the number of conjugacy classes in G. Moreover, the degree of each irreducible representation is equal to the size of the corresponding conjugacy class in G.*

Proof. DO IT!!! □

Corollary 1.1.1. *Let G be a finite abelian group. Then the irreducible representations of G are one-dimensional.*

Proof. Since G is abelian, the conjugacy classes of G are the elements of G themselves. By Theorem 1.1, the number of irreducible representations of G is equal to the number of conjugacy classes of G, which is equal to the number of elements of G. Furthermore, the degree of each irreducible representation is equal to the size of the corresponding conjugacy class in G, which is always 1. Therefore, the irreducible representations of G are one-dimensional. □

Chapter 2

Examples in Physics

Intro paragraph here?

2.1 Rotations in a plane and the group SO(2)

R vs U inconsistency from earlier notation

E vs I inconsistency with later on!

Resolve index notation at some point.

Intro paragraph here?

Reference appendix on Dirac notation somewhere in here.

2.1.1 The rotation group

Consider the rotations of a 2-dimensional Euclidean vector space about the origin. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be orthonormal basis vectors of this space. Using geometry, we can determine how a rotation by some angle ϕ, written in operator form as $R(\phi)$, acts on the basis vectors:

$$R(\phi)\mathbf{e}_1 = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi \tag{2.1}$$

$$R(\phi)\mathbf{e}_2 = -\mathbf{e}_1 \sin\phi + \mathbf{e}_2 \cos\phi. \tag{2.2}$$

In matrix form, we can write

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{2.3}$$

which allows us to write Eqns. 2.1 and 2.2 in a condensed form

$$R(\phi)\mathbf{e}_i = \mathbf{e}_j R(\phi)^j{}_i, \tag{2.4}$$

where we are summing over $j = 1, 2$.

Let $\mathbf{x}$ be an arbitrary vector in the plane. Then $\mathbf{x}$ has components x^i in the basis $\{\mathbf{e}_i\}$, where $i = 1, 2$. Equivalently, we can write $\mathbf{x} = \mathbf{e}_i x^i$. Then under rotations, $\mathbf{x}$ transforms in accordance to the basis vectors

$$\begin{aligned} R(\phi)\mathbf{x} &= R(\phi)\mathbf{e}_i x^i \\ &= \mathbf{e}_j R(\phi)^j{}_i x^i \\ &= \left(\mathbf{e}_1 R(\phi)^1{}_i + \mathbf{e}_2 R(\phi)^2{}_i\right) x^i \\ &= (\mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi)\, x^1 + (\mathbf{e}_1(-\sin\phi) + \mathbf{e}_2 \cos\phi)\, x^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right) \mathbf{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \mathbf{e}_2. \end{aligned} \tag{2.5}$$

Notice that $R(\phi)R^\top(\phi) = E$ where E is the identity matrix. This is precisely what defines *orthogonal matrices*. For 2-dimensional vectors in the plane, it is clear that these rotations do not change the length of said vectors. This can be verified by using Eqn. 2.5:

$$\begin{aligned} |R(\phi)\mathbf{x}|^2 &= |\mathbf{e}_j R(\phi)^j{}_i x^i|^2 \\ &= \left|\left(x^1 \cos\phi - x^2 \sin\phi\right) \mathbf{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \mathbf{e}_2\right|^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right)^2 + \left(x^1 \sin\phi + x^2 \cos\phi\right)^2 \\ &= \left(\cos^2\phi + \sin^2\phi\right) x^1 x_1 + \left(\sin^2\phi + \cos^2\phi\right) x^2 x_2 \\ &= x^1 x_1 + x^2 x_2 = |\mathbf{x}|^2. \end{aligned} \tag{2.6}$$

Similarly, notice that for any continuous rotation by angle ϕ, $\det R(\phi) = \cos^2\phi + \sin^2\phi = 1$. In general, orthogonal matrices have determinant equal to ± 1. However, the result of the above determinant of $R(\phi)$ implies that all continuous rotations in the 2-dimensional plane have determinant equal to

+1. These are the *special orthogonal matrices of rank 2*. This family of matrices is denoted SO(2). Furthermore, there is a one-to-one correspondence with SO(2) matrices and rotations in a plane.

We define the group of continuous rotations in a plane by letting $R(0) = E$ be the identity element corresponding to no rotation (i.e., a rotation by angle $\phi = 0$), and defining the inverse of a rotation as $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi)$. Lastly, we define group multiplication as $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$ and note that $R(\phi) = R(\phi \pm 2\pi)$, which can be verified geometrically. Although SO(2) is technically a 2-dimensional representation of a more abstract rotation group, it is often just referred to as the rotation group due to the nature of the construction. Thus, group elements of SO(2) can be labelled by the angle of rotation $\phi \in [0, 2\pi)$.

2.1.2 Infinitesimal rotations

Consider an infinitesimal rotation labelled by some infinitesimal angle $d\phi$. This is equivalent to the identity plus some small rotation, which can be written as

$$R(d\phi) = E - id\phi J \tag{2.7}$$

where the scalar quantity $-i$ is introduced for later convenience and J is some quantity independent of the rotation angle. If we consider the rotation $R(\phi + d\phi)$, then there are two equivalent ways to interpret this rotation

$$R(\phi + d\phi) = R(\phi)R(d\phi) = R(\phi)(E - id\phi J) = R(\phi) - id\phi R(\phi)J, \tag{2.8}$$

$$R(\phi + d\phi) = R(\phi) + dR(\phi) = R(\phi) + d\phi \frac{dR(\phi)}{d\phi}, \tag{2.9}$$

where the second equation can be thought of as a Taylor expansion of $R(\phi + d\phi)$ about ϕ. Equating the two expressions for $R(\phi + d\phi)$ yields

$$dR(\phi) = -id\phi R(\phi)J. \tag{2.10}$$

Solving this differential equation (with boundary condition $R(0) = E$) provides us with an equation for any group element involving J:

$$R(\phi) = e^{-i\phi J}, \tag{2.11}$$

where J is called the *generator* of the group.

The explicit form of J is found as follows. To first order in $d\phi$, we have

$$R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}.$$

Comparing to Eqn. 2.7,

$$E - id\phi J = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix} \implies J = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Notice that $J^2 = E$, which implies that even powers of J equal the identity matrix and odd powers of J equal J. Taylor expanding $e^{-iJ\phi}$ gives

$$\begin{aligned} R(\phi) = e^{-iJ\phi} &= E - iJ\phi - E\frac{\phi^2}{2!} - iJ\frac{\phi^3}{3!} + \cdots \\ &= E\left(\sum_{n=0}^{\infty}(-1)^n\frac{\phi^{2n}}{(2n)!}\right) - iJ\left(\sum_{n=0}^{\infty}(-1)^n\frac{\phi^{2n+1}}{(2n+1)!}\right) \\ &= E\cos\phi - iJ\sin\phi \\ &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \end{aligned}$$

Therefore, the generator J can be used to recover the rotation matrix for an arbitrary angle ϕ. Clearly, the map $R(\phi) \mapsto e^{-iJ\phi}$ is a valid homomorphism that respects the periodic nature of SO(2).

2.1.3 Irreducible representations of SO(2)

Equipped with the generator J, we can construct the irreducible representations of SO(2). First, consider a representation U of SO(2) defined on a finite dimensional vector space V. Then $U(\phi)$ is the corresponding representation of $R(\phi)$. The same argument as in Section 2.1.2 can be applied to an infinitesimal rotation to give

$$U(\phi) = e^{-iJ\phi},$$

which is an operator on V (for convenience, the same symbol J is used to denote the generator of the representation).

Since U is a representation of rotations that preserves the length of vectors, we have

$$\begin{aligned}
|a|^2 = |U(\phi)a|^2, \forall |a\rangle \in V \iff & \langle a|a\rangle = \langle U(\phi)a|U(\phi)a\rangle = \langle a|U(\phi)^\dagger U(\phi)|a\rangle \\
\iff & U(\phi)^\dagger U(\phi) = E \\
\iff & e^{iJ^\dagger\phi}e^{-iJ\phi} = e^{-i(J-J^\dagger)\phi} = 1 \\
\iff & J = J^\dagger.
\end{aligned}$$

Therefore, not only must U be unitary, but the generator J must be Hermitian. This fact becomes especially important in the physical interpretation of the representations of 3-dimensional rotations in Section 2.4.

According to Corollary 1.1.1, the abelian nature of SO(2) implies that all of its irreducible representations are one-dimensional. Then for any $|\alpha\rangle \in V$, the minimal subspace containing $|\alpha\rangle$ that is invariant under SO(2) is one-dimensional. Hence,

$$\begin{aligned}
J|\alpha\rangle &= \alpha|\alpha\rangle, \\
U(\phi)|\alpha\rangle &= e^{-iJ\phi}|\alpha\rangle = e^{-i\alpha\phi}|\alpha\rangle,
\end{aligned}$$

where the (real) number α is used as a label for the eigenvector of J with eigenvalue α. The periodicity conditions of SO(2) imply that $|\alpha\rangle = U(2\pi)|\alpha\rangle$, or equivalently, $e^{-i\alpha 2\pi} = 1$. This implies that α must be an integer, as $e^{i2\pi m} = 1$ for $m \in \mathbb{Z}$. Then U has a corresponding 1-dimensional representation for an integer m, defined by

$$\begin{aligned}
J|m\rangle &= m|m\rangle, \\
U^m(\phi)|m\rangle &= e^{-im\phi}|m\rangle.
\end{aligned}$$

Though already true by Corollary 1.1.1, these representations are clearly irreducible, as there is no way to reduce the dimension of a 1-dimensional representation.

In general, the *single-valued irreducible representations of SO(2)* are defined as

$$U^m(\phi) = e^{-im\phi}, \tag{2.12}$$

for $m \in \mathbb{Z}$.

If $m = 0$, then $R(\phi) \mapsto U^0(\phi) = 1$, which corresponds to the trivial representation. If instead $m = 1$, then $R(\phi) \mapsto U^1(\phi) = e^{-i\phi}$, which maps rotations in SO(2) to distinct points on the unit circle in the complex plane. The $m = 1$ representation is faithful because each rotation by ϕ has a unique image under $U^1(\phi)$, which is clear when interpreting rotations of unit vectors geometrically. As ϕ ranges from 0 to 2π, U^1 traces over the unit circle in $\mathbb{C}$ in the counterclockwise direction. Similarly, U^{-1} traces over the unit circle in the clockwise direction because $U^{-1}(\phi) = e^{i\phi}$. The $m = -1$ case is therefore faithful as well. In general, U^n covers the unit circle $|n|$ times as ϕ ranges from 0 to 2π, and is not faithful for $n \neq \pm 1$.

The irreducible representations of SO(2) are orthonormal in the sense that

$$\frac{1}{2\pi}\int_0^{2\pi} (U^m(\phi))^\dagger U^n(\phi)\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n-m)\phi}\, d\phi = \delta_{nm}.$$

2.1.4 Multivalued representations

If we relax the periodic condition on U to $U(2n\pi) = E$ for some $n \in \mathbb{Z}$, then the resulting 1-dimensional irreducible representations of SO(2) become multivalued. Consider the same construction of U^m in Section 2.1.3, but now with $m \in \mathbb{Q}$. For $m = \frac{1}{2}$, we have

$$U^{1/2}(2\pi + \phi) = e^{-i\pi - i\frac{\phi}{2}} = -e^{-i\frac{\phi}{2}} = -U^{1/2}(\phi).$$

Hence, the rotation $R(\phi)$ is assigned to both $\pm e^{i\phi/2}$ in the $U^{1/2}$ representation. For this reason, it can be said that $U^{1/2}$ is a *two-valued* representation of SO(2).

Despite this ambiguity in the realization of rotations in SO(2), the periodicity condition is still satisfied, as $U^{1/2}(4\pi) = e^{i2\pi} = 1$. In other words, the double-valued representation of SO(2) traverses the unit circle twice before returning to the identity. In general, $U^{n/m}$ is an m-valued representation of SO(2) for $\frac{n}{m} \in \mathbb{Q}$ and $\gcd(n, m) = 1$.

The physical implications of these irreducible representations will become clear when generalizing to rotations in 3-dimensional space in Section 2.4. Next, a similar construction will be done for the group of continuous translations in one dimension.

2.1.5 State vector decomposition

The concept of J generating 2-dimensional rotations is summarized in the following example. Consider a particle in a plane with polar coordinates (r, ϕ). The state vector of this particle is $|\phi\rangle$, where the coordinate r is suppressed in the vector notation, as the action of SO(2) preserves vector lengths. Note that the state vector $|\phi\rangle$ belongs to some Hilbert space V that is not necessarily the same as the physical space of the particle.

In general, we have $U(\theta)\,|\phi\rangle = |\theta + \phi\rangle$, as expected. Then $|\phi\rangle$ can be decomposed as

$$|\phi\rangle = U(\phi)\,|\mathcal{O}\rangle = e^{-iJ\phi}\,|\mathcal{O}\rangle\,,$$

where $|\mathcal{O}\rangle$ is a "standard" state vector aligned with a pre-selected x-axis. The triviality of this result must not be overlooked, for it is important to note that any arbitrary state vector can be decomposed into $e^{-iJ\phi}$ acting on $|\mathcal{O}\rangle$ [16]. This notion generalizes beyond the 2-dimensional case, and will be revisited in the more general case of rotations in 3 spatial dimensions in Section 2.4.

Since the set of eigenvectors of J form a basis for V, an arbitrary state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J:

$$|\phi\rangle = \left(\sum_m |m\rangle\,\langle m|\right)|\phi\rangle = \sum_m \langle m|\phi\rangle\,|m\rangle\,,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi) m|\mathcal{O}\rangle = e^{-im\phi}\,\langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J. Note that m is left unspecified, as the allowable values of m depend on the representation of SO(2) and thus the vector space V.

By construction, the eigenstates of J are invariant under rotations, so we are free to modify them up to a phase factor (i.e., pick different representatives from the eigenspaces). For example, we can choose the basis vector $|m\rangle$ to instead be $e^{ikm}\,|m\rangle$ for some $k \in \mathbb{R}$. With this strategy, all eigenvectors $|m\rangle$ can be oriented along the direction of $|\mathcal{O}\rangle$ so that $\langle m|\mathcal{O}\rangle = 1$. Again, note that the inner product $\langle m|\mathcal{O}\rangle$ is a projection of the *state* $|m\rangle$ onto the *state*

$|\mathcal{O}\rangle$, not to be confused with the projection of position vectors in the physical space of this system.

Thus, we can write the state vector $|\phi\rangle$ as

$$|\phi\rangle = \sum_m e^{-im\phi} |m\rangle . \tag{2.13}$$

As a result, the action of J on the state $|\phi\rangle$ can be written as

$$J|\phi\rangle = \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}|m\rangle\right) = i\frac{\partial}{\partial\phi}|\phi\rangle .$$

For fixed m, multiplying Eqn. 2.13 by $\frac{1}{2\pi}e^{im\phi}$ and integrating over ϕ, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{im\phi}|\phi\rangle\, d\phi &= \sum_n \left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(n-m)\phi}\, d\phi\right)|n\rangle \\ &= \sum_n \left(\frac{1}{2\pi}\int_0^{2\pi} (U^m(\phi))^\dagger U^n(\phi)\, d\phi\right)|n\rangle \\ &= \sum_n \delta_{mn}|n\rangle = |m\rangle .\end{aligned}$$

Then for an arbitrary state $|\psi\rangle \in V$, it follows that

$$\begin{aligned}|\psi\rangle = \sum_m \langle m|\psi\rangle |m\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_m e^{im\phi}\langle m|\psi\rangle\right)|\phi\rangle\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_m \langle\phi|m\rangle\langle m|\psi\rangle\right)|\phi\rangle\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\langle\phi|\underbrace{\left(\sum_m |m\rangle\langle m|\right)}_{\text{identity}}|\psi\rangle|\phi\rangle\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\langle\phi|\psi\rangle|\phi\rangle\, d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\psi(\phi)|\phi\rangle\, d\phi,\end{aligned}$$

where $\langle\phi|\psi\rangle$ is written as the *wavefunction* $\psi(\phi)$ since ϕ is a continuous parameter.

Add wavefunction description (to appendix?) for uncountably infinite-dim Hilbert space. How you can write $\langle x|\Psi\rangle$ as just the *wavefunction* $\Psi(x)$ since x is a continuous variable. Also, add QM citations. Textbooks, etc.

Then the action of J generalizes to

$$\langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = \frac{1}{i}\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi),$$

where we have projected $J|\psi\rangle$ onto the eigenbasis of J. If we let x and y be the Cartesian coordinates of the plane, then

$$\begin{aligned}\phi = \arctan\left(\frac{y}{x}\right) \implies \frac{\partial}{\partial\phi} &= \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y}\\ &= \frac{\partial}{\partial\phi}(r\cos\phi)\frac{\partial}{\partial x} + \frac{\partial}{\partial\phi}(r\sin\phi)\frac{\partial}{\partial y}\\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\\ &= (\mathbf{r}\times\nabla)\cdot\mathbf{e}_z,\end{aligned}$$

where $\mathbf{r} = (x, y, z)$ and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Therefore, the general form of J is

$$J = -i\frac{\partial}{\partial\phi} = -i(\mathbf{r}\times\nabla)\cdot\mathbf{e}_z. \tag{2.14}$$

This result is of particular significance to quantum mechanics, as J has the same form as the orbital *angular momentum operator* $\hat{L}_z$ (normalized to $\hbar$ here), where z is the axis of rotation of the plane[8].

2.2 Continuous 1-dimensional translations

Consider the group of continuous translations in one dimension, denoted by T_1, and let V be a 1-dimensional vector space with coordinate axis x. Then a vector $|x_0\rangle \in V$ is analogous to the point $x_0 \in \mathbb{R}$ on the real line. The translation of $|x_0\rangle$ by some amount x is described by the operator $T(x)$ in which

$$T(x)\,|x_0\rangle = |x + x_0\rangle\,.$$

The operator $T(x)$ has the expected group properties

$$T(0) = E, \tag{2.15}$$
$$T(x)^{-1} = T(-x), \tag{2.16}$$
$$T(x_1)T(x_2) = T(x_1 + x_2). \tag{2.17}$$

Consider an infinitesimal translation $T(dx)$. This derivation is identical to finding the generator J for SO(2) in Section 2.1.2. Thus, we rewrite

$$T(dx) = E - idxP,$$

where, for the moment, P is an arbitrary quantity. Eqns. 2.8 and 2.9 apply to $T(x)$ with P replacing J, T replacing R, and x replacing ϕ. This yields the familiar differential equation

$$\frac{dT(x)}{T(x)} = -iPdx, \tag{2.18}$$

along with the boundary condition Eqn. 2.15, which implies

$$T(x) = e^{-iPx}. \tag{2.19}$$

The exponential form of $T(x)$ satisfies the group properties of T_1 and is a valid representation of the group. Therefore, P generates T_1. A similar decomposition of state vectors as in Section 2.1.5 can be done for T_1. Specifically, for $|x\rangle \in V$, we have

$$|x\rangle = T(x)\,|\mathcal{O}\rangle = e^{-iPx}\,|\mathcal{O}\rangle\,,$$

where $|\mathcal{O}\rangle$ is the standard state in V.

2.2.1 Irreducible representations of T_1

Consider a unitary representation U of T_1 on a finite dimensional vector space V. As before, U can be reduced to $U(x) = e^{-iPx}$, where P is the generator of the representation. The unitarity of U requires that P be Hermitian, as in the case of J for SO(2). It follows that the eigenvalues of P, labeled by p, are real. Since T_1 is abelian, Corollary 1.1.1 implies that the irreducible representations of T_1 are all 1-dimensional. Similar to Section 2.1.3, the irreducible representation $U^p(x)$ of $T(x)$ is given by

$$\begin{aligned} P\,|p\rangle &= p\,|p\rangle\,, \\ U^p(x)\,|p\rangle &= e^{-iPx}\,|p\rangle = e^{-ipx}\,|p\rangle\,. \end{aligned}$$

The above description satisfies Eqns. 2.15–2.17 with no further restrictions on p.

Notice that the eigenvalues of P are continuous, in contrast to the discrete eigenvalues of J for SO(2) which were a result of the periodicity condition. The resulting orthonomality of the irreducible representations of T_1 are given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (U^p(x))^\dagger U^{p'}(x)\,dx = \int_{-\infty}^{\infty} e^{-i(p'-p)x}\,dx = \delta(p'-p)$$

where the normalization by 2π is chosen by convention.

2.2.2 Explicit form of P

Performing the same arguments as in Section 2.1.5 for T_1, we can expand a localized state $|x\rangle$ in terms of the eigenvectors of P:

$$|x\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle p|x\rangle\,|p\rangle\;dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\,|p\rangle\;dp,$$

where the sums from Section 2.1.5 are replaced by integrals due to the continuous and unbounded nature of p. Multiplying by e^{ipx} for some fixed p and integrating over x, we obtain an expression of $|p\rangle$ in terms of $|x\rangle$:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ipx}\,|x\rangle\;dx &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(p'-p)x}\,|p'\rangle\;dx\right)dp' \\ &= \int_{-\infty}^{\infty} \delta(p'-p)\,|p'\rangle\;dp' = |p\rangle\,. \end{aligned}$$

The relationship between $|p\rangle$ and $|x\rangle$ is the familiar Fourier transform, where the state $|p\rangle$ is the momentum space representation of the state $|x\rangle$, which corresponds to position space.

The action of P on $|x\rangle$ can then be written as

$$P\,|x\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} P\,|p\rangle\; dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} p e^{-ipx}\,|p\rangle\; dp = i\frac{\partial}{\partial x}\,|x\rangle\,.$$

Therefore, an arbitrary state $|\psi\rangle$ can be expressed in either the position or momentum basis:

$$|\psi\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x)\,|x\rangle\; dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(p)\,|p\rangle\; dp,$$

where again $\psi(\cdot) = \langle\cdot|\psi\rangle$ is the wavefunction of the state $|\psi\rangle$ projected onto the relevant basis.

Lastly, we obtain the explicit form of P by viewing its action on $|\psi\rangle$ with respect to the position basis:

$$\langle x|P|\psi\rangle = \langle P^{\dagger}x|\psi\rangle = \frac{1}{i}\frac{\partial}{\partial x}\,\langle x|\psi\rangle = -i\frac{\partial}{\partial x}\psi(x).$$

The above form of P agrees with the (normalized) quantum mechanical linear momentum operator $\hat{p}$ [8].

2.2.3 Generalization to 3-dimensional space

The derivation in Section 2.2.2 generalizes to 3-dimensional space, where the group of 3-dimensional translations T_3 is defined by

$$\begin{aligned} T(\mathbf{r})\,|\mathbf{r}_0\rangle &= T(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)\,|x_0\mathbf{e}_x + y_0\mathbf{e}_y + z_0\mathbf{e}_z\rangle \\ &= |(x+x_0)\mathbf{e}_x + (y+y_0)\mathbf{e}_y + (z+z_0)\mathbf{e}_z\rangle \\ &= |\mathbf{r}+\mathbf{r}_0\rangle\,, \end{aligned}$$

subject to the equivalent group properties of T_1 in Eqns. 2.15–2.17 with $\mathbf{r}$ replacing x.

Notice that $T_3 \simeq T_1 \oplus T_1 \oplus T_1$, where the group operation is defined as $T(x_1, y_1, z_1)T(x_2, y_2, z_2) = T(x_1+x_2, y_1+y_2, z_1+z_2)$. In other words, T_3 can be decomposed into independent 1-dimensional translations along each axis (or more generally along the span of each basis vector in 3-space). Thus, following the same procedure as in Section 2.1.2, an infinitesimal translation

$$T(d\mathbf{r}) = E - idxP_x\mathbf{e}_x - idyP_y\mathbf{e}_y - idzP_z\mathbf{e}_z$$

produces the following relations:

$$dT(x_j) = -idx_jT(x_j)P_j,$$

for $j = 1, 2, 3$ and $(x_1, x_2, x_3) = (x, y, z)$. This gives the expected result, namely

$$T(\mathbf{r}) = e^{-iP_xx}e^{-iP_yy}e^{-iP_zz} = e^{-i\mathbf{P}\cdot\mathbf{r}},$$

where the generator of 3-dimensional translations is the vector $\mathbf{P} = (P_x, P_y, P_z)$. The consequences of the separability of T_3 allows the results from Section 2.2.2 to be applied independently to each axis of translation. The intuitive generalization of T_1 to T_3 lets us immediately write down the explicit form of the generator for 3-dimensional translations. Since

$$P_j = -i\frac{\partial}{\partial x_j}, \tag{2.20}$$

we have

$$\mathbf{P} = -i\nabla. \tag{2.21}$$

Again, up to $\hbar$, Eqn. 2.21 is precisely the quantum mechanical linear momentum operator in 3 dimensions, often denoted $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$.

2.3 Symmetry, invariance, and conserved quantities

Physically, the generators $\mathbf{P}$ and J alter a (quantum) system by translation and rotation. These transformations correspond to the Hermitian operators $\hat{\mathbf{p}}$ and $\hat{L}_z$ that act on the state vectors belonging to the Hilbert space describing the system. Hence, the (real) eigenvalues of P and J correspond to the physical observables (measurable quantities) of linear and angular momentum, respectively. Armed with the explicit forms of these operators, the physical ramifications of the irreducible representations of T_3 and SO(2) can now be demonstrated.

According to Ehrenfest's theorem (see Appendix A.4), if a physical system represented by a Hamiltonian $\hat{H}$ is invariant under a transformation generated by an operator $\hat{A}$, then the physical observable corresponding to $\hat{A}$ is conserved. In other words, the expectation value of $\hat{A}$ is constant in time if the commutator $[\hat{H}, \hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H} = 0$.

The generators obtained in previous sections fit this framework. If a Hamiltonian $\hat{H}$ is invariant under translations or rotations, then $[\hat{H}, \mathbf{P}] = [\hat{H}, \hat{\mathbf{p}}] = 0$ or $[\hat{H}, J] = [\hat{H}, \hat{L}_z] = 0$, respectively. Therefore, the physical observables of linear and angular momentum are conserved in systems with translational and rotational symmetry, respectively. The following are examples of physical systems that exhibit these symmetries and the conserved quantities that result from them.

2.3.1 Conservation of linear momentum

Consider a free particle in three spatial dimensions. The Hamiltonian of this system is given by

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m},$$

which gives the quantum operator

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m}.$$

Notice that

$$\left[\hat{\mathbf{p}}^2, \hat{\mathbf{p}}\right] = \left[(-i\hbar\nabla)^2, -i\hbar\nabla\right] = i\hbar^3\left[\nabla^2, \nabla\right] = i\hbar^3\left(\nabla^3 - \nabla^3\right) = 0,$$

where $\nabla^3 = \nabla \cdot \nabla \cdot \nabla$. It follows that

$$[\hat{H}, \hat{\mathbf{p}}] = \left[\frac{\hat{\mathbf{p}}^2}{2m}, \hat{\mathbf{p}}\right] = \frac{1}{2m}\left[\hat{\mathbf{p}}^2, \hat{\mathbf{p}}\right] = 0$$

Therefore, linear momentum is conserved in this system. This result is expected, as the Hamiltonian of a free particle is invariant under translations in space, which are generated by $\mathbf{P}$. The conservation of linear momentum is a direct consequence of the translational symmetry of the system.

2.3.2 Conservation of angular momentum

Now, consider the Hamiltonian describing a free particle in confined to a radially symmetric scalar potential $V(\mathbf{r})$. The quantum analog of the Hamiltonian is the operator

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}),$$

where $\hat{V}(\mathbf{r})$ is the potential operator defined by $\hat{V}(\mathbf{r})\,|\mathbf{r}\rangle = V(\mathbf{r})\,|\mathbf{r}\rangle$.

Intuitively, a potential that depends solely on the radial coordinate should be invariant under rotations, as there is no angular dependence. According to Noether's theorem, the rotational symmetry of the system implies that angular momentum is conserved. This claim is equivalent to showing that $[\hat{H}, \hat{L}_z] = 0$, where $\hat{L}_z$ is the operator corresponding to the generator of rotations in the xy-plane, derived as J in Section 2.1.

The angular momentum operator $\hat{L}_z$ is given by

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi},$$

where ϕ is the polar angle in the xy-plane. The following result is immediate:

$$[V(\mathbf{r}), \hat{L}_z] = 0,$$

since $V(\mathbf{r})$ does not have ϕ-dependence.

Recall that we can express $\hat{L}_z$ in Cartesian coordinates as

$$\hat{L}_z = -i\hbar\,(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \cdot \mathbf{e}_z = -i\hbar\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = -y\hat{p}_x + x\hat{p}_y, \tag{2.22}$$

where the *position operator* is defined as $\hat{\mathbf{r}}\,|\mathbf{r}\rangle = \mathbf{r}\,|\mathbf{r}\rangle$. To reduce clutter, the components $\hat{x}, \hat{y}, \hat{z}$ of $\hat{\mathbf{r}}$ are written without the hats.

First, we can reduce the commutator $[\hat{\mathbf{p}}^2, \hat{L}_z]$ to a simpler form:

$$\begin{aligned}
[\hat{\mathbf{p}}^2, \hat{L}_z] &= \left[\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, -y\hat{p}_x + x\hat{p}_y\right] \\
&= \left[\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, -y\hat{p}_x\right] + \left[\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, x\hat{p}_y\right] \\
&= \left[\hat{p}_y^2, -y\hat{p}_x\right] + \left[\hat{p}_x^2, x\hat{p}_y\right],
\end{aligned}$$

since the components of $\hat{\mathbf{p}}$ commute with each other. Further simplification is done using Eqns. A.1, A.2, A.4 and A.5:

$$\begin{aligned}
\left[\hat{p}_y^2, -y\hat{p}_x\right] &= \hat{p}_y[\hat{p}_y, -y\hat{p}_x] + [\hat{p}_y, -y\hat{p}_x]\hat{p}_y \\
&= \hat{p}_y \left(-y\cancelto{0}{[\hat{p}_y, \hat{p}_x]} - [\hat{p}_y, y]\hat{p}_x\right) + \left(-y\cancelto{0}{[\hat{p}_y, \hat{p}_x]} - [\hat{p}_y, y]\hat{p}_x\right) \hat{p}_y \\
&= \hat{p}_y \left(i\hbar - \cancel{y\hat{p}_y} + \cancel{y\hat{p}_y}\right) \hat{p}_x + \left(i\hbar - \cancel{y\hat{p}_y} + \cancel{y\hat{p}_y}\right) \hat{p}_x\hat{p}_y \\
&= 2i\hbar\hat{p}_y\hat{p}_x,
\end{aligned}$$

and by a relabeling of the variables, we also have

$$\left[\hat{p}_x^2, x\hat{p}_y\right] = -\left[\hat{p}_x^2, -x\hat{p}_y\right] = -2i\hbar\hat{p}_x\hat{p}_y = -\left[\hat{p}_y^2, -y\hat{p}_x\right].$$

Therefore, $[\hat{\mathbf{p}}^2, \hat{L}_z] = 0$. It follows from Eqn. A.3 that

$$[\hat{H}, \hat{L}_z] = \left[\frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \hat{L}_z\right] = 0.$$

This result agrees with the previous claim that the emergence of conservation of angular momentum is due to the rotational symmetry of the system.

The profound connection between symmetries and conserved quantities is a fundamental principle in physics, and the results obtained in this section highlight the significance of representation theory in physics. The irreducible representations of T_3 and SO(2) provide the necessary mathematical framework to derive conservation laws without a preconceived notion of the physical universe.

2.4 3D rotations and the group SO(3)

As was done for translations in Section 2.2.3, we can generalize SO(2) to rotations in 3-dimensional space, albeit with less triviality. The group of rotations in Euclidean 3-space, which is synonymous with 3-dimensional linear operators that fix the length of vectors, is a *special orthogonal group in 3D*, denoted by SO(3).

Consider a rotation in three dimensions about an axis (vector) $\mathbf{n}$ by an angle θ. The rotation $R_{\mathbf{n}}(\theta)$ is a linear transformation that maps a vector $\mathbf{v}$ to a new vector $\mathbf{v}'$ such that $|\mathbf{v}| = |\mathbf{v}'|$. The rotation angle $\theta \in [0, 2\pi)$ is a continuous parameter, and every one-parameter subgroup of SO(3) can be written as $\{R_{\mathbf{n}}(\theta) \mid \theta \in [0, 2\pi)\}$ for fixed $\mathbf{n}$.

The set of rotations in a plane perpendicular $\mathbf{n}$ is clearly isomorphic to SO(2). Thus, for a fixed axis of rotation $\mathbf{n}$, an infinitesimal rotation $R_{\mathbf{n}}(d\theta)$ can be used to obtain a generator of rotations about $\mathbf{n}$. The derivation is identical to that of J for SO(2) in Section 2.1.2 since the group of rotations about $\mathbf{n}$ is isomorphic to SO(2). Hence, we can label the generator of rotations about $\mathbf{n}$ as $J_{\mathbf{n}}$, and the corresponding results from Section 2.1 can be applied to $J_{\mathbf{n}}$. Most notably, for arbitrary θ, we can write

$$R_{\mathbf{n}}(\theta) = e^{-i\theta J_{\mathbf{n}}}.$$

Consider the standard basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ in 3-dimensional Euclidean space. The generators of rotations about the x, y, z axes are denoted by J_x, J_y, J_z, respectively. With some work [16], it can be shown that the generator $J_{\mathbf{n}}$ is decomposable into J_x, J_y, J_z for any $\mathbf{n}$ by projection onto the standard basis:

$$J_{\mathbf{n}} = n_x J_x + n_y J_y + n_z J_z, \tag{2.23}$$

where $n_\mu = \mathbf{n} \cdot \mathbf{e}_\mu$ for $\mu = x, y, z$. The general rotation operator about $\mathbf{n}$ becomes

$$R_{\mathbf{n}}(\theta) = e^{-i\theta(n_x J_x + n_y J_y + n_z J_z)}.$$

As in Section 2.1.3, the unitarity of the rotation operator requires that the generators J_x, J_y, J_z be Hermitian and therefore have real eigenvalues.

Therefore, the set $\{J_x, J_y, J_z\}$ forms a basis for the generators of the one-parameter abelian subgroups of SO(3). As a result, we have a representation of SO(3) defined by the generator $\mathbf{J} = (J_x, J_y, J_z)$. Namely, for an arbitrary rotation $R_{\mathbf{n}}(\theta)$, we can write

$$R_{\mathbf{n}}(\theta) = e^{-i\theta\, \mathbf{n}\cdot\mathbf{J}}.$$

2.4.1 Explicit form of J

Since the subspace generated by each component of $\mathbf{J}$ is isomorphic to SO(2), we can use the same arguments made in Section 2.1.5 to obtain the explicit forms of the generators J_x, J_y, J_z. For $\mu = x, y, z$, Eqn. 2.14 generalizes to rotations about $\mathbf{e}_\mu$ as follows:

$$J_\mu = -i\frac{\partial}{\partial \phi_\mu} = -i\left(\mathbf{r} \times \nabla\right) \cdot \mathbf{e}_\mu,$$

where ϕ_μ is the polar angle in the plane perpendicular to $\mathbf{e}_\mu$. This allows an explicit expression for $\mathbf{J}$:

$$\mathbf{J} = -i\left(\mathbf{r} \times \nabla\right), \tag{2.24}$$

which, up to $\hbar$, is the quantum mechanical angular momentum operator in 3 dimensions [8], often written as $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$.

2.4.2 Commutation relations of SO(3) generators

The algebraic structure of the representation of SO(3) in terms of $\mathbf{J}$ is defined by the commutation relations of the basis generators J_x, J_y, J_z. By studying the underlying algebraic relationships between the generators, we can gain insight into the irreducible representations of SO(3) and the physical implications of these representations.

A consequence of the correspondence found in Section 2.4.1 is that the commutation relations of the generators J_x, J_y, J_z are identical to those of the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ up to $\hbar$. First, note that Eqn. 2.22 can be generalized to each component of $\mathbf{L}$:

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y, \tag{2.25}$$
$$\hat{L}_y = z\hat{p}_x - x\hat{p}_z, \tag{2.26}$$
$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{2.27}$$

Thus, we can write

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}. \tag{2.28}$$

The commutation relations of the angular momentum operators can then be found by direct computation:

$$
\begin{aligned}
[\hat{L}_z, \hat{L}_x] &= [\hat{L}_z, y\hat{p}_z - z\hat{p}_y] \\
&= [\hat{L}_z, y\hat{p}_z] - [\hat{L}_z, z\hat{p}_y] \\
&= y[\hat{L}_z, \hat{p}_z] + [\hat{L}_z, y]\hat{p}_z - z[\hat{L}_z, \hat{p}_y] - [\hat{L}_z, z]\hat{p}_y \\
&= 0 - i\hbar x\hat{p}_z + i\hbar z\hat{p}_x + 0 \\
&= i\hbar(z\hat{p}_x - x\hat{p}_z) = i\hbar\hat{L}_y,
\end{aligned}
$$

where the remaining details can be found in Appendix A.3. The appropriate permutation of the indices gives the other commutation relations. Altogether, the commutation relations of the angular momentum operators are

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \tag{2.29}$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \tag{2.30}$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y. \tag{2.31}$$

These commutation relations are identical to those of the generators J_x, J_y, J_z up to $\hbar$.

2.4.3 Irreducible representations of SO(3)

Due to the nontrivial interaction between J_x, J_y, J_z, the irreducible representations of SO(3) are not as straightforward to determine as those of SO(2). However, the commutation relations in Eqns. 2.29–2.31 provide the necessary foundation to proceed with the following analysis.

Let V be a finite-dimensional vector space corresponding to a representation of SO(3). The generators J_x, J_y, J_z act on V as linear operators, and the J-analogue of Eqns. 2.29–2.31 must be satisfied. To obtain an irreducible representation of SO(3), we seek a subspace of V that is invariant under SO(3) rotations. Equivalently, a subspace of V that is invariant under the action of J_x, J_y, J_z will be invariant under SO(3).

The most straightforward procedure for constructing an invariant subspace of V is by choosing a "standard" vector that is an eigenvector of one of the generators, and then applying SO(3) operations to generate the rest of the basis [16]. As is customary in physics, we choose the z-axis as the standard axis of rotation.

Let $|m\rangle$ be a normalized eigenvector of J_z, in which $J_z |m\rangle = m |m\rangle$ for some real number m. For reasons presently unknown, define a new operator

$$J^2 = \mathbf{J} \cdot \mathbf{J} = J_x^2 + J_y^2 + J_z^2.$$

It follows that

$$\begin{aligned}
[J^2, J_z] &= [J_x^2 + J_y^2 + J_z^2, J_z] \\
&= [J_x^2, J_z] + [J_y^2, J_z] \\
&= J_x[J_x, J_z] + [J_x, J_z]J_x + J_y[J_y, J_z] + [J_y, J_z]J_y \\
&= -iJ_xJ_y - iJ_yJ_x + iJ_yJ_x + iJ_xJ_y = 0.
\end{aligned}$$

A relabeling of z with x and y yields an identical result. Since J^2 commutes with J_x, J_y, J_z, it follows that J^2 commutes with any linear combination of J_x, J_y, J_z. Therefore, J^2 commutes with all SO(3) rotations.

As shown in Appendix A.4, commuting operators have a common set of eigenvectors. In this case, we choose the common eigenvectors of J^2 and J_z as the basis vectors of the invariant subspace of V. At this point, we have one eigenvector $|m\rangle$ of J_z and J^2. To obtain the other eigenvectors that span the invariant subspace, we first define two more operators. Let

$$J_\pm = J_x \pm iJ_y. \tag{2.32}$$

These operators have the following properties [16]:

$$\begin{aligned}
[J_z, J_\pm] &= [J_z, J_x] \pm i[J_z, J_y] = iJ_y \pm i(-iJ_x) = J_\pm, \\
[J_+, J_-] &= [J_x + iJ_y, J_x - iJ_y] = i[J_x, J_y] - i[J_y, J_x] = 2iJ_z, \\
J^2 &= J_z^2 + J_x^2 + J_y^2 \\
&= J_z^2 + J_+J_- + i(J_yJ_x - J_xJ_y) \\
&= J_z^2 + J_+J_- + i(-iJ_z) \\
&= J_z^2 + J_-J_+ + J_z \\
&= J_z^2 + J_+J_- - J_z \\
J_\pm^\dagger &= J_\mp.
\end{aligned}$$

Notice that

$$\begin{aligned}
J_zJ_\pm |m\rangle &= [J_z, J_\pm] |m\rangle + J_\pm J_z |m\rangle \\
&= \pm J_\pm |m\rangle + mJ_\pm |m\rangle \\
&= (m \pm 1)J_\pm |m\rangle .
\end{aligned}$$

Therefore, $J_\pm |m\rangle$ are either eigenstates of J_z with eigenvalue $m \pm 1$ or zero. The name of $J_\pm$ as the *ladder operators* is justified by the fact that they raise or lower the eigenvalue of J_z by one unit as if climbing the rungs of a ladder. Assume that $J_+ |m\rangle \neq 0$. Then the eigenvector $J_+ |m\rangle$ can be normalized and written as $|m+1\rangle$. Similarly, $|m-1\rangle$ can be defined as $J_- |m\rangle$, assuming it is nonzero.

With the ladder operators, we can generate a set of eigenvectors of J_z (and J^2) by repeated application on the standard eigenvector $|m\rangle$. Since V is assumed to be finite, this process must terminate at some point. Let j denote the largest eigenvalue of J_z in the invariant subspace, and similarly let l denote the lowest. In other words, we have

$$J_+ |j\rangle = 0, \quad J_- |l\rangle = 0,$$

so that any further application of the corresponding ladder operator returns zero.

The span of eigenvectors $\{|l\rangle, |l+1\rangle, \ldots, |j-1\rangle, |j\rangle\}$ is indeed an invariant subspace of V under SO(3) rotations. Since $J_x = \frac{1}{2}(J_+ + J_-)$ and $J_y = \frac{1}{2i}(J_+ - J_-)$, it follows that their action on $\{|l\rangle, |l+1\rangle, \ldots, |j-1\rangle, |j\rangle\}$ is closed within the subspace.

Additionally, for any eigenvector $|m\rangle$, we know specifically that

$$\begin{aligned}
J^2 |m\rangle &= (J_z^2 - J_z + J_+ J_-) |m\rangle \\
&= (m^2 - m + m(m-1)) |m\rangle \\
&= m(m+1) |m\rangle .
\end{aligned}$$

We gain further insight into this invariant subspace by noting that

2.4.4 Physical implications of SO(3) representations

As discussed in Section 2.3, the eigenvalues of the components of $\mathbf{J}$ and thus $\hat{\mathbf{L}}$ correspond to physical observables. In particular, the eigenvalues of $\hat{L}_z$...

Make a note that it's really the irreducible representation of $\mathfrak{so}(3)$, but it seamlessly translates to the group representations?

2.4.5 Further applications

This is a section about what else you can do with the irreducible representations of SO(3) that I won't get into in this thesis.

- As with SO(2), if a system is invariant under 3D rotations (radially symmetric), then angular momentum is conserved. For each axis of rotation, we have $[\hat{H}, \hat{L}_\mu] = 0$. Can make the same arguments as in Section 2.3.2 with all three angular momentum generators.
- Show that radially symmetric systems with basis eigenvectors $|E, l, m\rangle$ of $\hat{H}, \mathbf{J}^2, J_3$ are separable into a radial function and a spherical harmonic (Tung 7.5.1).
- Direct product, multiple particle states, Clebsch-Gordan coefficients, etc.
- Talk about $U(2)$?

Notes below:

- **Discuss Lie groups/algebras specifically?**
- The real generalization is to 3 spatial dimensions, SO(3), which then has the Lie algebra $\mathfrak{so}(3)$ with generators J_i and familiar commutation relations.
- The eigenvalues of J are real since it is Hermitian, and so they correspond to physical observables. In particular, the eigenvalues m of J correspond to the angular momentum of a quantum system (really it's a projection of the total angular momentum onto the axis of rotation normalized to $\hbar$). When m is an integer, the representation U^m corresponds to integer-spin particles, such as bosons or gravitons. When m is a half-integer, the representation U^m corresponds to half-integer-spin particles, such as fermions.
- The **quantization of angular momentum in quantum mechanics** is a direct consequence of the representation theory of SO(2) (really SO(3))!!! The allowable values of angular momentum are quantized because the eigenvalues of the generator J are quantized. Moreover, for eigenvalue m, the possible spin states with angular momentum m correspond to the multiple values (U^m's satisfying the periodicity condition

for the eigenvector $|m\rangle$): $-m, -m+1, \dots, m-1, m$ (normalized to $\hbar$). Jumping between spin states is done by the ladder operators $J_\pm$ in SO(3).

- Regarding the above bullet point, need to do SO(3) Lie algebra stuff in order to discuss ladder operators and hence the connection to the discretized angular momentum values!

- Not until SO(3). Example for $U^{1/2}$, or in physics $j = \frac{1}{2}$. The spin state of an electron can either be up $+U^{1/2}$ or down $-U^{1/2}$, which corresponds to the two-valued-ness of the representation. A rotation of 2π results in a change of sign (a change in spin state). Moreover, the spin state of a spin-$\frac{1}{2}$ particle is described by a *spinor* (a two-component complex-valued vector). The purely mathematical consequences of double-valued representations of SO(2) explains the emergent behavior of spinors under coordinate rotations.

2.5 Outline

1. ~~Explicit form of P.~~

2. ~~Explicit Hamiltonians, conservation, symmetry, invariance section.~~

3. SO(3) construction. How much new stuff do I need to do vs just generalize the other components of J like what was done for T_3?

4. Lie group/algebra definitions and contextualized in the above.

5. Finally, the physical applications of the above. Meaning the connection of irreducible reps to spin states, quantization of observables (angular momentum), etc.

6. Then if roughly finished with the main goals/content of this chapter, **need** to go back to chapter 1 and do Schur's lemmas (if needed?) and prove the correspondence between irreducible representations and conjugacy classes.

Chapter 4

To-Do List

Potential committee members:

- **Anton Kaul**
- **Patrick Orson**
- **Eric Brussel**
- *Rob Easton*

- Redo the Chapter 1 with nicer notation and stray away from Tung's notation when possible.
- More straightforward examples of representations
- At least briefly discuss $U(n)$ either here or in braid rep chapter.
- Finish/modify irreducible rep. example in Chapter 1.
- Fix out equation numbers
- **Sections to add:** SO(3) and related applications to QM
- Do a little more context on the physics: describe what the heck a quantum Hilbert space is and braket notation.
- Also note why we care about unitary matrices, in appendix as well.

- Show $\psi_n(\sigma_i)$ invertible? Yes, eventually
- derive $\psi_n^{\mathbf{r}}(\sigma_i)$ matrices or state?
- Show $\psi_n^{\mathbf{r}}(\sigma_i)$ invertible? Yes, eventually
- ~~Explicitly show why Burau isn't able to be made unitary? [3]~~
- Separate chapters into braid group and braid group reps.?

- Concluding paragraph on first section to lead into the more physics-y stuff.
- ~~Show the additional cross terms from $N = 2$ to $N = 3$ and beyond.~~
- Add paragraph on gauge theory/motivation. (Appendix)
- Anyon fusion rules
- τ anyon/Fibonacci anyon example. Relate to singlet/triplet states in spin-1/2 system.
- ~~Move anyon calculations to appendix?~~
- Spend some time on `MATLAB` thing

- Conclusion/future of anyons/braid group in physics.
- Abstract
- Title
- Acknowledgements

Format!!

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Appendix A

Physics Background

A.1 Dirac notation

Bra-ket notation, “Hilbert space”, inner product, etc.

A.2 Commutator Identities

$$[A, B] = -[B, A] \tag{A.1}$$
$$[A, -B] = -AB + BA = -[A, B]. \tag{A.2}$$

$$\begin{aligned}
[A, B + C] &= A(B + C) - (B + C)A \\
&= AB + AC - BA - CA \\
&= AB - BA + AC - CA \\
&= [A, B] + [A, C].
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
[A^2, B] &= [AA, B] \\
&= AAB - BAA \\
&= AAB - ABA + ABA - BAA \\
&= A(AB - BA) + (AB - BA)A \\
&= A[A, B] + [A, B]A.
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
[A, BC] &= ABC - BCA \\
&= ABC - BAC + BAC - BCA \\
&= (AB - BA)C + B(AC - CA).
\end{aligned} \tag{A.5}$$

A.3 Commutation relations for SO(3)

$$[y,\hat{p}_y] = y\hat{p}_y - \hat{p}_y y = \cancel{y\hat{p}_y} - \overbrace{\left(-i\hbar + \cancel{y\hat{p}_y}\right)}^{\text{product rule}} = i\hbar,$$

$$[\hat{L}_z,\hat{p}_z] = [x\hat{p}_y - y\hat{p}_x,\hat{p}_z] = [x\hat{p}_y,\hat{p}_z] - [y\hat{p}_x,\hat{p}_z] = 0.$$

$$[\hat{L}_z,z] = [x\hat{p}_y - y\hat{p}_x,z] = [x\hat{p}_y,z] - [y\hat{p}_x,z] = 0.$$

$$[\hat{L}_z,\hat{p}_y] = [x\hat{p}_y - y\hat{p}_x,\hat{p}_y] = \cancelto{0}{[x\hat{p}_y,\hat{p}_y]} - [y\hat{p}_x,\hat{p}_y] = -y\cancelto{0}{[\hat{p}_x,\hat{p}_y]} - [y,\hat{p}_y]\hat{p}_x = -i\hbar\hat{p}_x.$$

$$[\hat{L}_z,y] = [x\hat{p}_y - y\hat{p}_x,y] = [x\hat{p}_y,y] - \cancelto{0}{[y\hat{p}_x,y]} = x[\hat{p}_y,y] + \cancelto{0}{[x,y]}\hat{p}_y = -i\hbar x.$$

A.4 Ehrenfest's theorem and conserved quantities

Possible reference here [8]!

Suppose G is an operator on a quantum Hilbert space of states. The quantity $\langle G\rangle$ is conserved if

$$\frac{d\langle G\rangle}{dt} = 0.$$

Recall the time-dependent Schrödinger equation

$$\hat{H}\psi = i\hbar\frac{d\psi}{dt} \implies \frac{d\psi}{dt} = \frac{1}{i\hbar}\hat{H}\psi.$$

Then if G is time-independent we have

$$
\begin{aligned}
\frac{d\langle G\rangle}{dt} &= \frac{d}{dt}\langle\psi|G|\psi\rangle \\
&= \left\langle \frac{d\psi}{dt}\middle|G\middle|\psi\right\rangle + \left\langle\psi\middle|G\middle|\frac{d\psi}{dt}\right\rangle + \cancelto{0}{\left\langle\psi\middle|\frac{\partial G}{\partial t}\middle|\psi\right\rangle} \\
&= \left\langle \frac{1}{i\hbar}\hat{H}\psi\middle|G\middle|\psi\right\rangle + \left\langle\psi\middle|G\middle|\frac{1}{i\hbar}\hat{H}\psi\right\rangle \\
&= \frac{i}{\hbar}\Big(\langle\hat{H}\psi|G|\psi\rangle - \langle\psi|G|\hat{H}\psi\rangle\Big) \\
&= \frac{i}{\hbar}\Big(\langle\psi|\hat{H}^\dagger G|\psi\rangle - \langle\psi|G\hat{H}|\psi\rangle\Big) \\
&= \frac{i}{\hbar}\Big(\langle\psi|\hat{H}G|\psi\rangle - \langle\psi|G\hat{H}|\psi\rangle\Big) \text{ because } \hat{H} \text{ is Hermitian} \\
&= \frac{i}{\hbar}\langle\psi|\,(\hat{H}G - G\hat{H})\,|\psi\rangle \\
&= \frac{i}{\hbar}\langle\psi|\,[\hat{H},G]\,|\psi\rangle = 0 \iff [\hat{H},G] = 0.
\end{aligned}
$$

(linear in the second argument). (See Ehrenfest's theorem).

Thus, if $[\hat{H},G]=0$, it follows that

$$
\begin{aligned}
\hat{H}G - G\hat{H} = 0 &\iff \hat{H}G = G\hat{H} \\
&\iff G^{-1}\hat{H}G = \hat{H}.
\end{aligned}
$$

Therefore, $G^{-1}\hat{H}G$ and $\hat{H}$ share the same eigenvalues (observables), which is only true if $\hat{H}$ is invariant under G. If G generates a group of transformations, then $\hat{H}$ is invariant under the group of transformations generated by G. If G is unitary, this invariance is often expressed as

$$G^\dagger\hat{H}G = \hat{H}.$$

Running the argument in reverse, if $\hat{H}$ is invariant under the transformations generated by G, then $[\hat{H},G]=0$, which, by the Ehrenfest theorem, implies that $\langle G\rangle$ is conserved.

Appendix B

Multi-anyon system with harmonic potential

B.1 Gauge Theory and the Hamiltonian

B.2 Hamiltonian Terms

The last term in Eqn. 4.22 is the result of squaring the canonical momentum in Eqn. 4.21. To see this, let's isolate one of the terms. Fix i. Then,

$$\big(\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i)\big)^2 = p_i^2 - 2\mathbf{p}_i \cdot \mathbf{A}_i(\mathbf{r}_i) + A_i^2(\mathbf{r}_i).$$

By Eqn. 4.19, we have

$$\mathbf{A}_i{}^2(\mathbf{r}_i) = \left(\alpha \sum_{j \neq i} \frac{-y_{ij}\hat{x} + x_{ij}\hat{y}}{r_{ij}^2}\right)^2 = \alpha^2 \sum_{j,k \neq i} \frac{y_{ij}y_{ik} + x_{ij}x_{ik}}{r_{ij}^2 r_{ik}^2} = \alpha^2 \sum_{j,k \neq i} \frac{\mathbf{r}_{ij} \cdot \mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2},$$

which is the last term in Eqn. 4.22.

Moreover, the cross term in the expansion of $\big(\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i)\big)^2$ is

$$
\begin{aligned}
-2\mathbf{p}_i \cdot \mathbf{A}_i(\mathbf{r}_i) &= -2\mathbf{p}_i \cdot \left(\alpha \sum_{j \neq i} \frac{-y_{ij}\hat{x} + x_{ij}\hat{y}}{r_{ij}^2} \right) \\
&= -2\alpha \sum_{j \neq i} \frac{\mathbf{p}_i \cdot (-y_{ij}\hat{x} + x_{ij}\hat{y})}{r_{ij}^2} \\
&= -2\alpha \sum_{j \neq i} \frac{-p_{ix} y_{ij} + p_{iy} x_{ij}}{r_{ij}^2} \\
&= -2\alpha \sum_{j \neq i} \frac{(\mathbf{r}_{ij} \times \mathbf{p}_i) \cdot \hat{z}}{r_{ij}^2}.
\end{aligned}
$$

For each j, there is a corresponding term in Eqn. 4.22 with

$$
-2\alpha \frac{(\mathbf{r}_{ji} \times \mathbf{p}_j) \cdot \hat{z}}{r_{ji}^2} = -\alpha \frac{(\mathbf{r}_{ji} \times \mathbf{p}_j) \cdot \hat{z}}{r_{ij}^2} + \alpha \frac{(\mathbf{r}_{ij} \times \mathbf{p}_j) \cdot \hat{z}}{r_{ij}^2},
$$

where we rewrote one of the two terms to have $\mathbf{r}_{ij}$ instead of $\mathbf{r}_{ji}$. Then, for fixed i and j, the ij- and ji-term can be combined in the following manner:

$$
\begin{aligned}
-2\alpha \frac{(\mathbf{r}_{ij} \times \mathbf{p}_i) \cdot \hat{z}}{r_{ji}^2} - 2\alpha \frac{(\mathbf{r}_{ji} \times \mathbf{p}_j) \cdot \hat{z}}{r_{ji}^2} &= -\alpha \frac{(\mathbf{r}_{ij} \times \mathbf{p}_i) \cdot \hat{z}}{r_{ij}^2} + \alpha \frac{(\mathbf{r}_{ji} \times \mathbf{p}_i) \cdot \hat{z}}{r_{ji}^2} \\
&\quad + \alpha \frac{(\mathbf{r}_{ij} \times \mathbf{p}_j) \cdot \hat{z}}{r_{ij}^2} - \alpha \frac{(\mathbf{r}_{ji} \times \mathbf{p}_j) \cdot \hat{z}}{r_{ji}^2} \\
&= -\alpha \frac{(\mathbf{r}_{ij} \times (\mathbf{p}_i - \mathbf{p}_j)) \cdot \hat{z}}{r_{ij}^2} \\
&\quad - \alpha \frac{(\mathbf{r}_{ji} \times (\mathbf{p}_j - \mathbf{p}_i)) \cdot \hat{z}}{r_{ji}^2} \\
&= -\alpha \frac{(\mathbf{r}_{ij} \times \mathbf{p}_{ij}) \cdot \hat{z}}{r_{ij}^2} + \alpha \frac{(\mathbf{r}_{ji} \times \mathbf{p}_{ji}) \cdot \hat{z}}{r_{ji}^2} \\
&= -\alpha \frac{\ell_{ij}}{r_{ij}^2} - \alpha \frac{\ell_{ji}}{r_{ji}^2}.
\end{aligned}
$$

Then, summing over all $i \neq j$ yields the second-to-last term in Eqn. 4.22.