

Representation Theory and its Applications in Physics

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Abstract

Representation theory, which encodes the elements of a group as linear operators on a vector space, has far-reaching implications in physics. Fundamental results in quantum physics emerge directly from the representations describing physical symmetries. We first examine the connections between specific representations and the principles of quantum mechanics. Then, we shift our focus to the braid group, which describes the algebraic structure of braids. We apply representations of the braid group to physical systems in order to investigate quasiparticles known as anyons. Finally, we obtain governing equations of anyonic systems to highlight the physical differences between braiding statistics and Bose-Einstein/Fermi-Dirac statistics.

Chapter 6

To-Do List

Potential committee members:

- Anton Kaul
- Patrick Orson
- Eric Brussel
- *Rob Easton*

Questions for grad ed formatting

- Short figure captions.

Wednesday questions

- Chapter 2 Schur proof reference 531/Mendes?
- Beefy captions?
- Okay to just state $\psi_n^{\mathbf{r}}(\sigma_i)$ matrices?
- Show $\psi_n^{\mathbf{r}}(\sigma_i)$ invertible?
- Note the Lie algebra vs Lie group distinction in Chapter 3?

- **Do?** Anyon fusion rules. τ anyon/Fibonacci anyon example. Relate to singlet/triplet states in spin-1/2 system.
- Broken inline math?
- Spend some time on MATLAB thing?

-
- Add sigma inverse to 5.1 rot on Hilbert spaces
 - Discuss fusion rules and non-abelian-ness of them. Can cite paper without going into too much detail about it.
-

- ✓ Gauge theory background, QM background.
- ✓ Go over Appendix A and see if it needs more examples, maybe push to appendix.
- ✓ Do the stuff above listed for Chapter 4.
- ✓ Concluding paragraph on first section of Chapter 5 to lead into the more physics-y stuff.
- ✓ Conclusion/future of anyons/braid group in physics.
- ✓ Go over Chapter 2 with the relevant bullets above in mind.
- X Note the Lie algebra vs Lie group distinction in Chapter 3.
- ✓ Intro paragraphs for Chapter 3 and sections.
- ✓ Comment on faithfulness of the Burau representation.
- ✓ Introduction “chapter”.
- ✓ Abstract.
- ✓ Acknowledgements.

Bibliography

- [1] E. Artin. Theory of braids. *The Annals of Mathematics*, 48(1):101, January 1947.
- [2] Kirill I. Bolotin, Fereshte Ghahari, Michael D. Shulman, Horst L. Stormer, and Philip Kim. Observation of the fractional quantum hall effect in graphene. *Nature*, 462(7270):196–199, November 2009.
- [3] G. Date, M. V. N. Murthy, and Radhika Vathsan. Classical and quantum mechanics of anyons, 2003.
- [4] Amitesh Datta. A strong characterization of the entries of the burau matrices of 4-braids: The burau representation of the braid group b_4 is faithful almost everywhere, 2022.
- [5] Colleen Delaney, Eric C. Rowell, and Zhenghan Wang. Local unitary representations of the braid group and their applications to quantum computing, 2016.
- [6] Avinash Deshmukh. An introduction to anyons.
- [7] Bernard Field and Tapio Simula. Introduction to topological quantum computation with non-abelian anyons. 2018.
- [8] W. Fulton. *Algebraic Topology: A First Course*. Graduate Texts in Mathematics. Springer New York, 1997.
- [9] Juan Gonzalez-Meneses. Basic results on braid groups, 2010.
- [10] David J. Griffiths. *Introduction to Electrodynamics*. Cambridge University Press, June 2017.

- [11] David J. Griffiths and Darrell F. Schroeter. *Introduction to Quantum Mechanics*. Cambridge University Press, August 2018.
- [12] Brian C. Hall. *Quantum Theory for Mathematicians*. Springer New York, 2013.
- [13] Christian Kassel and Vladimir Turaev. *Homological Representations of the Braid Groups*, page 93–150. Springer New York, 2008.
- [14] Avinash Khare. *Fractional Statistics and Quantum Theory*. WORLD SCIENTIFIC, February 2005.
- [15] K Moriyasu. *An Elementary Primer for Gauge Theory*. WORLD SCIENTIFIC, October 1983.
- [16] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma. Non-abelian anyons and topological quantum computation. *Reviews of Modern Physics*, 80(3):1083–1159, September 2008.
- [17] Martin Palmer and Arthur Soulié. The bureau representations of loop braid groups. 2021.
- [18] Dale Rolfsen. Tutorial on the braid groups, 2010.
- [19] Craig C. Squier. The bureau representation is unitary. *Proceedings of the American Mathematical Society*, 90(2):199–202, 1984.
- [20] John R. Taylor. *Classical Mechanics*. University Science Books, board book edition, 1 2005.
- [21] Jean-Luc Thiffeault. The bureau representation of the braid group and its application to dynamics. Presentation given at Topological Methods in Mathematical Physics 2022, Seminar GEOTOP-A, September 2022.
- [22] Wu-Ki Tung. *Group theory in physics: An introduction to symmetry principles, group representations, and special functions in classical and quantum physics*. World Scientific Publishing, Singapore, Singapore, January 1985.
- [23] Frank Wilczek. Quantum mechanics of fractional-spin particles. *Physical Review Letters*, 49(14):957–959, October 1982.