Representation Theory and its Applications in Physics

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Outline:

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

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Invertibility

If X is a representation of G, then $X(g)^{-1} = X(g^{-1})$, $\forall g \in G$.

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Consequences:

1. $X(e) = I$, where e is the identity element of the group and I is the identity operator.
2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

Example: The Trivial Representation

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- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to equivalence.

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Example: Irreducible Representation of 2D Rotations

Note: The subspace spanned by $\mathbf{e}_1$ (or $\mathbf{e}_2$) is *not* invariant under rotations!

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Invariance of $\mathbf{e}_\pm$

Let $\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mp\mathbf{e}_1 + i\mathbf{e}_2)$. Then, $X(\phi)\mathbf{e}_\pm = e^{\pm i\phi}\mathbf{e}_\pm$.

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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6. One-dimensional representation are irreducible.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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2 Examples in Physics

Preliminaries

Outline:

1. Dirac notation.
2. Basic quantum mechanics.
3. Quantum Hilbert space.
4. The commutator.

Preliminaries: Dirac Notation

Preliminaries: Basic Quantum Mechanics

2D Rotations and SO(2)

1. The group of 2D rotations is SO(2).
2. General properties of SO(2).

Infinitesimal Rotations

- Go through the derivation of the generator of SO(2) in an appropriate level of detail.

Recovering the Rotation Matrix from J

- Do Taylor expansion thing to get the rotation matrix from J (looks familiar phys majors?)

Irreducible Representations of SO(2)

- Rep generated by J is unitary, J is Hermitian.
- SO(2) abelian implies 1D irreps (reference previous thm's).
- Construct 1D invariant subspaces, obtain 1D irreps.
- Get result about $m \in \mathbb{Z}$ for irrep label.
- Mention ortho/completeness relations?
- State vector decomposition. Probably don't have time to delve into detailed derivations but would be great to show part of the argument for getting explicit differential form of J.

Conservation of Angular Momentum

- Do commutator example with Hamiltonian and J.
- Discuss implications.
- We can do the same thing for translation group which gives us the familiar $\hat{p}$ operator and conservation of linear momentum!

Generalization to 3 Spatial Dimensions

- Show but don't derive $R_{\mathbf{n}}(\theta)$ decomposition into **J** components.
- We have basis from the components of **J**.
- Ladies and gentlemen, we got SO(3)...
- **J** component differential forms?
- Commutation relations, in some form talk about $J_\pm, J^2$ and final eigenvalue results.

Connection to Quantum Mechanics

- Discuss connection between generators and quantum operators, eigenvalues and classical observables, discretization (!), etc.
- This is the kicker. I will get very excited here probably.

Multi-valued Irreducible Representations and Spinors

Not sure where to put this...

- Let's come back to SO(2) for a second...
- Show $m = 1/2$ irreps.
- Discuss implications, spinors, etc...

3 The Braid Group

Basic Definitions

- Formal definitions.
- Physical/intuitive visualization and interpretation.
- Standard generators.
- Automorphisms of $\pi_1(\mathbb{D}_n)$.
- Braid relations in this picture.
- 1D Reps.
- Burau representation.
- Note on faithfulness.
- Unitary representation from reduced Burau.

4 Physical Applications of the Braid Group

Rotations of Quantum Hilbert Space

- 1D action on Hilbert space, permuting particles, compare/contrast to bosons/fermions.
- Talk about nontrivial braiding effects.
- Example of unitary braid rep acting on Hilbert space.

Anyons: A Consequence of Braiding

- Introduce anyons.
- Discuss how anyons are described by the braid group.
- Fusion rules, abelian vs nonabelian anyons.
- Non-interacting anyons.
- Non-interacting anyons in harmonic potential.
- Nontrivial braiding effects anyone?
- Applications of anyons! (quantum computing, topological quantum field theory, FQHE, etc.)

Summary/Conclusion

Acknowledgements, questions, references (?)