Title

Max Varverakis

January 14, 2024

Chapter 1

Representation Theory Background

Definition 1.1 (Representations of a Group)**.** If there is a homomorphism from a group G to a group of operators $U(G)$ on a linear vector space V, we say that $U(G)$ forms a *representation* of G with dimension $\dim V$.

The representation is a map

$$g \in G \xrightarrow{U} U(g) \tag{1.1}$$

in which $U(g)$ is an operator on the vector space V. For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \ldots, n\}$, we can realize each operator $U(g)$ as an $n \times n$ matrix $D(g)$.

$$U(g) \left| e_i \right\rangle = \sum_{j=1}^{n} \left| e_j \right\rangle D(g)^j{}_i = \left| e_j \right\rangle D(g)^j{}_i, \tag{1.2}$$

where the first index j is the row index and the second index i is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$U(g_1)U(g_2) = U(g_1 g_2), \tag{1.3}$$

which satisfies the group multiplication rules.

Definition 1.2. If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

Example 1.1. Let G be the group of continuous rotations in the xy-plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space V_2. Then we define a representation of G on V_2 by the familiar rotation operation

$$\hat{e}'_1 = U(\phi)\hat{e}_1 = \hat{e}_1 \cdot \cos\phi + \hat{e}_2 \cdot \sin\phi \tag{1.4}$$
$$\hat{e}'_2 = U(\phi)\hat{e}_2 = -\hat{e}_1 \cdot \sin\phi + \hat{e}_2 \cdot \cos\phi. \tag{1.5}$$

This gives us the matrix representation

$$D(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1.6}$$

To further illustrate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \vec{x} \in V_2$, then we have

$$\vec{x}' = U(\phi)\vec{x} = \hat{e}_j x'^j, \tag{1.7}$$

where $x'^j = D(\phi)^j{}_i x^i$.

Definition 1.3 (Equivalence of Representations)**.** For a group G, two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define

Definition 1.4 (Characters of a Representation)**.** The *character* $\chi(g)$ of an element $g \in G$ in a representation $U(g)$ is defined as $\chi(g) = \mathrm{Tr}\ D(g)$.

Since trace is independent of basis, the character serves as a class label.

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

Definition 1.5 (Invariant Subspace)**.** Let $U(G)$ be a representation of G on a vector space V, and W a subspace of V such that $U(g)\,|x\rangle \in W$ for all $\vec{x} \in W$ and $g \in G$. Then W is an *invariant subspace* of V with respect to $U(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $U(G)$.

The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

Definition 1.6 (Irreducible Representation)**.** A representation $U(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $U(G)$. Otherwise, it is *reducible.* If $U(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $U(G)$, then the representation is *fully reducible.*

Example 1.2. Under the group of 2-dimensional rotations, consider the 1-dimensional subspace spanned by $\hat{e}_1$. This subspace is not invariant under 2-dimensional rotations, because a rotation of $\hat{e}_1$ by $\pi/2$ results in the vector $\hat{e}_2$ that is clearly not in the subspace spanned by $\hat{e}_1$. A similar argument shows that the subspace spanned by $\hat{e}_2$ is not invariant under 2-dimensional rotations.

However, consider the linear combination of basis vectors

$$\hat{e}_\pm = \frac{1}{\sqrt{2}}\left(\mp\hat{e}_1 + i\hat{e}_2\right), \tag{1.8}$$

where $i = \sqrt{-1}$. Then a rotation by angle ϕ, denoted in operator form as $U(\phi)$, acts on $\hat{e}_\pm$ by

$$\begin{aligned}
U(\phi)\left|\hat{e}_+\right\rangle &= U(\phi)\frac{1}{\sqrt{2}}(-\hat{e}_1 + i\hat{e}_2) && (1.9)\\
&= \frac{1}{\sqrt{2}}(-U(\phi)\left|\hat{e}_1\right\rangle + iU(\phi)\left|\hat{e}_2\right\rangle)\\
&= \frac{1}{\sqrt{2}}\left(-\hat{e}_1\cos\phi - \hat{e}_2\sin\phi - i\hat{e}_1\sin\phi + i\hat{e}_2\cos\phi\right)\\
&= \frac{1}{\sqrt{2}}\left(-\hat{e}_1(\cos\phi + i\sin\phi) + i\hat{e}_2(\cos\phi - i\sin\phi)\right)\\
&\textbf{Not Done}\\
&= \hat{e}_+(\cos\phi - i\sin\phi)\\
&= \hat{e}_+e^{-i\phi}, && (1.10)\\
\text{and } U(\phi)\left|\hat{e}_-\right\rangle &= \hat{e}_-e^{i\phi}. && (1.11)
\end{aligned}$$

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5**?

Example 1.3 (Generator of SO(2))**.** Consider the rotations of a 2-dimensional Euclidean vector space about the origin. Let $\hat{e}_1$ and $\hat{e}_2$ be orthonormal basis vectors of this space. Using geometry, we can determine how a rotation by some angle ϕ, written in operator form as $R(\phi)$, acts on the basis vectors:

$$R(\phi)\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi \tag{1.12}$$
$$R(\phi)\hat{e}_2 = -\hat{e}_1 \sin\phi + \hat{e}_2 \cos\phi. \tag{1.13}$$

In matrix form, we can write

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{1.14}$$

which allows us to write Eqn. 1.12 and Eqn. 1.13 in a condensed form

$$R(\phi)\hat{e}_i = \hat{e}_j {R(\phi)^j}_i, \tag{1.15}$$

where we are summing over $j = 1, 2$.

Now, let $\vec{x}$ be an arbitrary vector in the plane. Then $\vec{x}$ has components x^i in the basis $\{\hat{e}_i\}$, where $i = 1, 2$. Equivalently, we can write $\vec{x} = \hat{e}_i x^i$. Then under rotations, $\vec{x}$ transforms in accordance to the basis vectors

$$\begin{aligned} R(\phi)\vec{x} &= R(\phi)\hat{e}_i x^i \\ &= \hat{e}_j {R(\phi)^j}_i x^i \\ &= \left(\hat{e}_1 {R(\phi)^1}_i + \hat{e}_2 {R(\phi)^2}_i\right) x^i \\ &= (\hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi)\, x^1 + (\hat{e}_1(-\sin\phi) + \hat{e}_2 \cos\phi)\, x^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right) \hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \hat{e}_2. \end{aligned} \tag{1.16}$$

Observe that $R(\phi)R^\top(\phi) = E$ where E is the identity matrix. This is precisely what defines *orthogonal matrices.* For 2-dimensional vectors in the plane, it is clear that these rotations do not change the length of said vectors. This can be verified by using Eqn. 1.16:

$$\begin{aligned} |R(\phi)\vec{x}|^2 &= |\hat{e}_j {R(\phi)^j}_i x^i|^2 \\ &= \left|\left(x^1 \cos\phi - x^2 \sin\phi\right) \hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \hat{e}_2\right|^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right)^2 + \left(x^1 \sin\phi + x^2 \cos\phi\right)^2 \\ &= \left(\cos^2\phi + \sin^2\phi\right) x^1 x_1 + \left(\sin^2\phi + \cos^2\phi\right) x^2 x_2 \\ &= x^1 x_1 + x^2 x_2 = |\vec{x}|^2. \end{aligned} \tag{1.17}$$

Similarly, notice that for any continuous rotation by angle ϕ, $\det R(\phi) = \cos^2\phi + \sin^2\phi = 1$. In general, orthogonal matrices have determinant equal to ± 1. However, the result of the above determinant of $R(\phi)$ implies that all continuous rotations in the 2-dimensional plane have determinant equal to $+1$. These are the *special orthogonal matrices of rank 2*. This family of matrices is denoted SO(2). Furthermore, there is a one-to-one correspondence with SO(2) matrices and rotations in a plane.

We define the group of continuous rotations in a plane by letting $R(0) = E$ be the identity element corresponding to no rotation (i.e., a rotation by angle $\phi = 0$), and defining the inverse of a rotation as $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi)$. This group can be called the SO(2) group. Lastly, we define group multiplication as $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$ and note that $R(\phi) = R(\phi \pm 2\pi)$, which can be verified geometrically. Thus, group elements of SO(2) can be labelled by the angle of rotation $\phi \in [0, 2\pi)$.

Now we can find a generator of *sotwo* by considering an infinitesimal rotation, labelled by some infinitesimal angle $d\phi$. Then this is equivalent to the identity plus some small rotation, which we can write as

$$R(\mathrm{d}\phi) = E - i\mathrm{d}\phi J \tag{1.18}$$

where the scalar quantity $-i$ is introduced for later convenience and J is some quantity independent of the rotation angle. If we consider the rotation $R(\phi + \mathrm{d}\phi)$, then there are two equivalent ways to interpret this rotation

$$R(\phi + \mathrm{d}\phi) = R(\phi)R(\mathrm{d}\phi) = R(\phi)(E - i\mathrm{d}\phi J) = R(\phi) - i\mathrm{d}\phi R(\phi)J \tag{1.19}$$

$$R(\phi + \mathrm{d}\phi) = R(\phi) + \mathrm{d}R(\phi) = R(\phi) + \mathrm{d}\phi\frac{\mathrm{d}R(\phi)}{\mathrm{d}\phi} \tag{1.20}$$

where the second equation can be thought of as a Taylor expansion of $R(\phi + \mathrm{d}\phi)$ about ϕ. Equating the two expressions for $R(\phi + \mathrm{d}\phi)$ yields

$$\mathrm{d}R(\phi) = -i\mathrm{d}\phi R(\phi)J. \tag{1.21}$$

Solving this differential equation (with boundary condition $R(0) = E$) provides us with an equation for any group element involving J:

$$R(\phi) = e^{-i\phi J}, \tag{1.22}$$

where J is called the *generator* of the group.

Chapter 2

Topological Definitions

The braid group is formally defined in terms of topology. In order to understand the braid group, we must first understand the underlying topological properties that are used to define the braid group. Similar to an isomorphism in algebra, we can define a notion of equivalence in topology.

Definition 2.1. Consider X and Y to be two topological spaces. A *homotopy* between two continuous functions $f, g : X \to Y$ is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. If such a homotopy exists, we say that f and g are *homotopic*.

The homotopy H can be thought of as a continuous deformation of f into g. The interval $[0, 1]$ represents the "time" parameter of the deformation. At time equal to 0, the function H is equal to f, and at time equal to 1, the function H is equal to g. If two functions are homotopic, then they belong to the same homotopy class, which is an equivalence class of functions under the relation of homotopy.

Definition 2.2. A *loop* on a topological space X is a continuous function $\ell : [0, 1] \to X$ such that $\ell(0) = \ell(1)$. In other words, the path of ℓ starts and ends at the same point in X. Often, this point is called the *base point* of the loop.

Definition 2.3. A *homotopy class of loops* on a topological space X is an equivalence class of loops under the relation of homotopy. More plainly, a homotopy of loops is a continuous transformation of one loop into another.

If two loops $\ell_1, \ell_2 : [0,1] \to X$ with base point $\xi \in X$ are homotopic, then there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that:

1. $H(0,t) = \xi = H(1,t)$ for all $t \in [0,1]$, and
2. $H(s,0) = \ell_1(s)$ and $H(s,1) = \ell_2(s)$ for all $s \in [0,1]$.

Property 1 ensures that the starting/ending point of the loop remains fixed throughout the deformation from ℓ_1 to ℓ_2, and property 2 follows from the definition of a homotopy.

Definition 2.4. The *fundamental group* of a topological space X with base point ξ is defined as the collection of loops on X with base point ξ modulo homotopy. In other words, the fundamental group is the collection of equivalence classes of loops under homotopy. This is written as

$$\pi(X, \xi) := \{\text{loops } \ell \text{ on } X \text{ with base point } \xi\} / \text{homotopy}.$$

Often times, the base point of a loop is arbitrary, so we can write $\pi(X)$ instead of $\pi(X, \xi)$ to denote the fundamental group of X.

The group structure of the fundamental group is defined as operations on the loops themselves. Consider two loops $\ell_1, \ell_2 : [0,1] \to X$ with base point ξ. Then the product $\ell_1 \cdot \ell_2$ is defined in terms of *concatenation* of the two loops. Specifically, this defines a new loop $(\ell_1 \cdot \ell_2)(t) = \mathcal{L}(t) : [0,1] \to X$ where $\mathcal{L}(t) = \ell_1(2t)$ on $\left[0, \frac{1}{2}\right]$ and $\mathcal{L}(t) = \ell_2(2t-1)$ on $\left[\frac{1}{2}, 1\right]$. Loop concatenation can be thought of as stitching the loops together at the shared base point. As t ranges from 0 to 1, we can think of the first half of the deformation as traversing the first loop at twice the original speed, and then traveling along the second loop at twice the original speed in the second half of the deformation.

In the above description, recall that each loop ℓ is actually an equivalence class $[\ell]$ under the relation of homotopy. So the concatenation of two loops ℓ_1 and ℓ_2 is actually the concatenation of any two loops belonging to the equivalence classes $[\ell_1]$ and $[\ell_2]$, which becomes the equivalence class $[\ell_1 \cdot \ell_2]$.

In the fundamental group, the inverse of an element is the identical topological path traversed in the opposite direction. Hence, if $\gamma : [0,1] \to X$ is a loop on X, then $\gamma^{-1}(t) := \gamma(1-t)$.

Chapter 3

The Braid Group

Definition 3.1. The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}; z_i \neq z_j, \forall i \neq j\}$. Alternatively, consider $\mathcal{D}$ to be the collection of all hyperplanes $H_{i,j} = \{z_i = z_j\} \in \mathbb{C}^n$ for $1 \leq i < j \leq n$. Then we can define $M_n = \mathbb{C}^n \setminus \mathcal{D}$.

Note that $(z_1, z_2, z_3, \ldots, z_n)$ and $(z_2, z_1, z_3, \ldots, z_n)$ are different points in the configuration space M_n. Before studying the various interpretations of the braid group, we first define the braid group itself.

Definition 3.2. The *pure braid group* on n strands, denoted PB_n, is the fundamental group of M_n. One can write $PB_n = \pi_1(M_n)$.

3.1 Visualization of the Braid Group

We can think of a pure braid as a loop in M_n:

$$\begin{aligned}\beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_n(t)),\end{aligned}$$

with some base point. Conventionally, we define the base point as the n-tuple of integers $(1, 2, 3, \ldots, n) \in \mathbb{C}^n$. Then a pure braid can be though of the motion of these points in the complex plane as t ranges from 0 to 1 in which $\beta_i(t)$ is defined and $\beta_i(t) \neq \beta_j(t)$ for every $t \in [0, 1]$ and $i \neq j \in \{1, 2, \ldots, n\}$. Because each β_i is a loop, it must start and end at the point

i (e.g., $\beta_i(0) = \beta_i(1) = i$). Recall that the loops are actually equivalence classes of loops under homotopy. As a result, we can continuously deform the motion of the n points while maintaining the same pure braid (up to equivalence) so long as we preserve the pairwise distinction of the points for all time $t \in [0, 1]$.

A common visualization of pure braids is as strands in 3-dimensional space.