Representation Theory and its Applications in Physics

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Presented by
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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then $X(g)$ can be realized as an $n \times n$ matrix.

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2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial X-invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to matrix similarity.

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Schur's Lemmas

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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Corollary

If G is a finite abelian group, then the irreducible representations of G are one-dimensional.

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How does this help in physics?

The groups corresponding to physical transformations have irreducible representations that lead to fundamental insights in physics.

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2 Examples in Physics

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

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- We call J the *generator* of SO(2) rotations.

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4. Each invariant subspace is spanned by an eigenvector of J:

$$J|m\rangle = m|m\rangle,$$
$$U(\phi)|m\rangle = e^{-iJ\phi}|m\rangle = e^{-im\phi}|m\rangle.$$

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Irreducible Representations of SO(2)

Process to obtaining irreducibles:

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Theorem

The single-valued irreducible representations of SO(2) are defined as

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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Definition

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all 3×3 orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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Theorem

The irreducible representations of SO(3) are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ and the $2j+1$ eigenvectors spanning an invariant subspace are labelled by their eigenvalues: $m = -j, -j+1, \ldots, j-1, j$.

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- *Quantum spin* is labeled by j and has possible spin states $|m\rangle$.

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Discretization of Angular Momentum for Free

Discretization (quantization) of angular momentum follows directly from the irreducible representations of SO(3)!

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But that's not all folks!

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3. Lorentz invariance $\implies$ conservation of energy and momentum!

3 The Braid Group

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Definition

The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

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- A *braid* β is a *loop*[6] in M_n and can be thought of as a configuration that evolves over time:

$$\begin{aligned} \beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\big), \end{aligned}$$

[6] The topological formalisms that define the braid group are omitted for times sake.

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Definition

The *braid group* B_n is the fundamental group of M_n/S_n, where S_n is the symmetric group on n elements.

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Visualization of Braids

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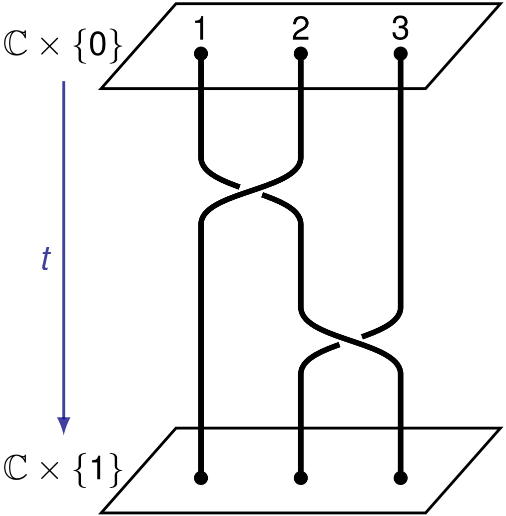
Braid on 3 strands.



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- Each path traced out by a point in the configuration space is a *strand*.

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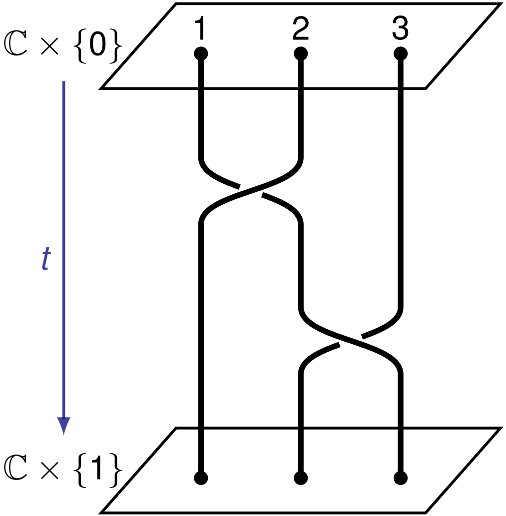

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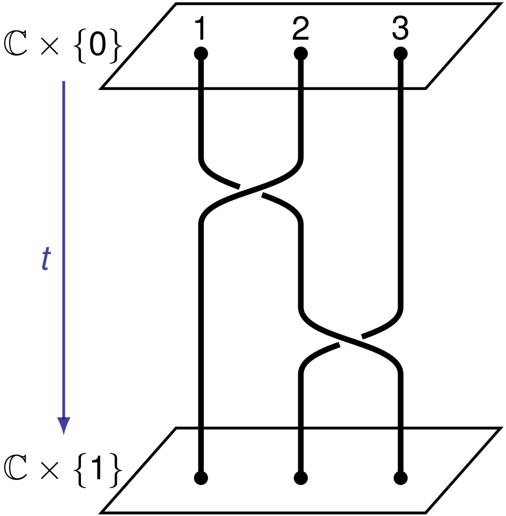

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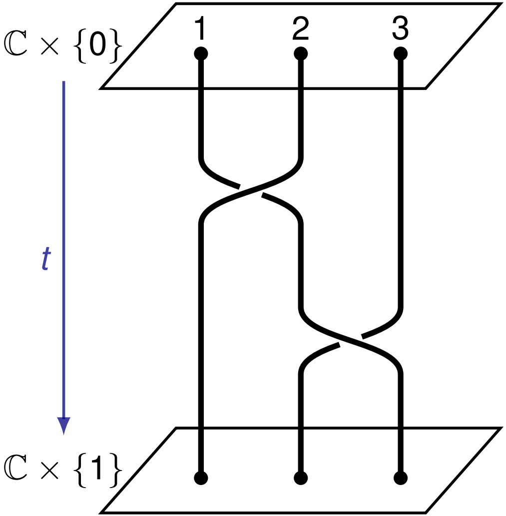

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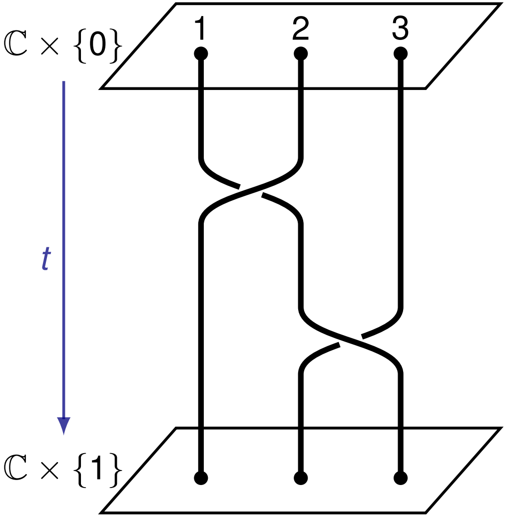

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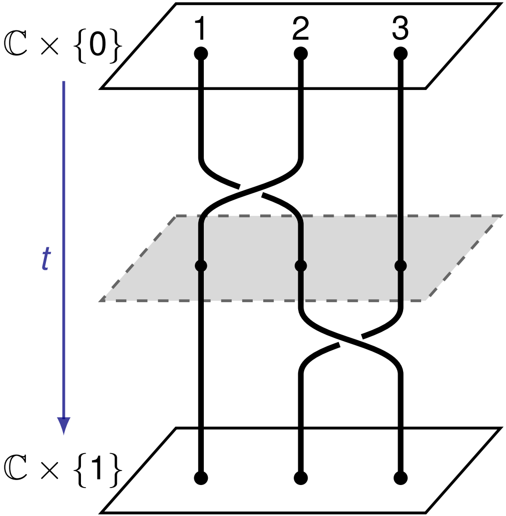

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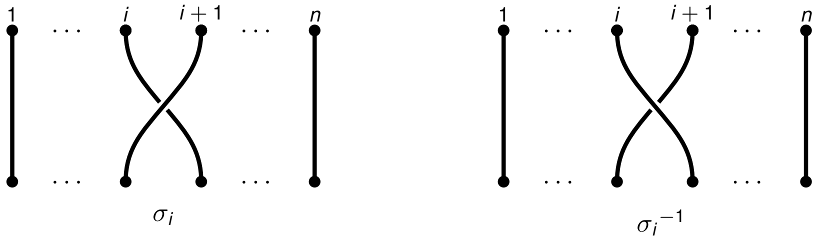

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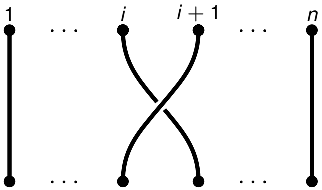
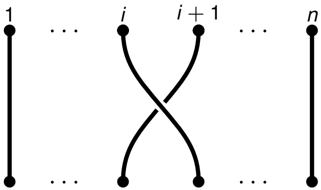


σ_i



$\sigma_i{}^{-1}$

- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β.

Alternative Description of B_n

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Definition

The braid group on n strands, denoted B_n, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, \;\; |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle .$$

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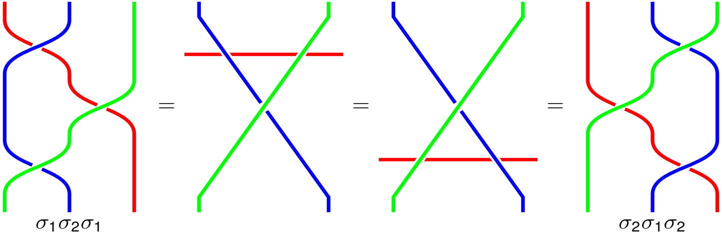
Comment: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



One-Dimensional Representations of the Braid Group

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- *Unitary representations* of B_n can be constructed from the reduced Burau representation.

Unitary Representation of the Braid Group

Definition

A matrix $M \in \mathrm{GL}_n(\mathbb{C})$ is *unitary* if $M^\dagger = M^{-1}$.

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Definition

Define the unitary representation $\mathcal{U} : B_3 \to U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

$$\mathcal{U}(\sigma_2) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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What are the physical implications of this nonabelian unitary representation?

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What are the physical implications of this nonabelian unitary representation?

Answer: Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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Braiding action: For any degree-k braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\, \psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}} \psi(r_1, r_2, \ldots, r_n),$$

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Braiding action: The transformed basis states due to the action of σ_1 are

$$|1'\rangle = \mathcal{U}(\sigma_1)_{1,1}\,|1\rangle + \mathcal{U}(\sigma_1)_{1,2}\,|2\rangle = \frac{1}{2}e^{-i\frac{\pi}{6}}\left(\sqrt{3}\,e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|1\rangle + |2\rangle\right),$$

$$|2'\rangle = \mathcal{U}(\sigma_1)_{2,1}\,|1\rangle + \mathcal{U}(\sigma_1)_{2,2}\,|2\rangle = \frac{1}{2}e^{-i\frac{\pi}{6}}\left(|1\rangle - \sqrt{3}\,e^{-i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|2\rangle\right).$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system[7].

[7] Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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- Edge cases: *bosons* and *fermions*.

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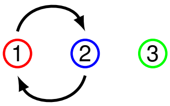

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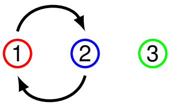
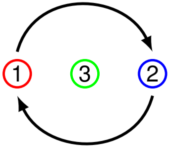


Trajectory B



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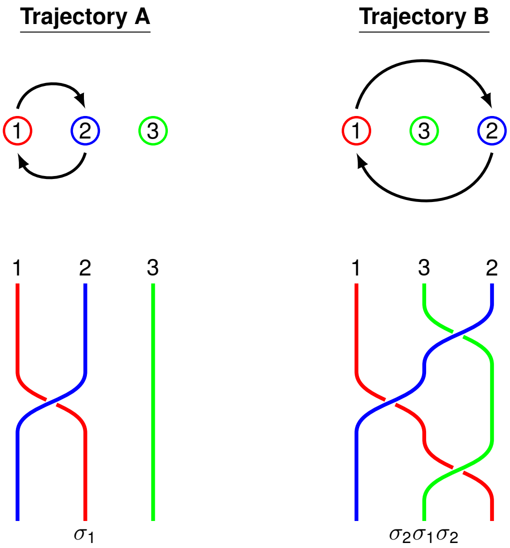

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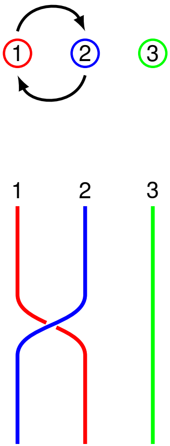

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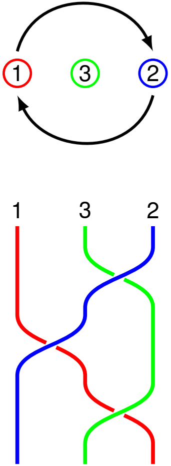

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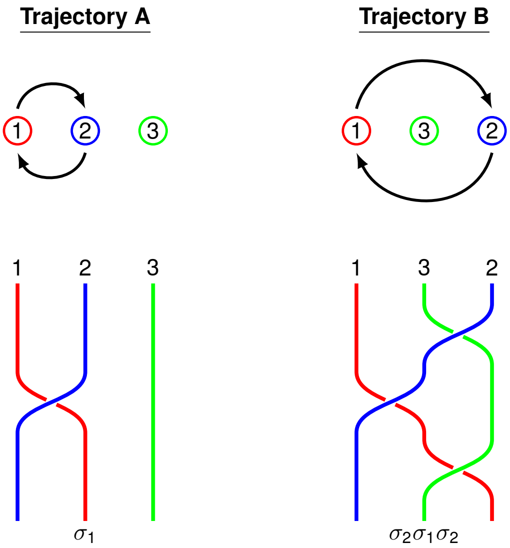

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- The *fractional quantum Hall effect* is a physical manifestation of anyonic braiding in 2D electron systems (fractional charge, fractional statistics).
- Anyons can have different topological flavors, leading to special *fusion rules* that can be used to describe the behavior of anyonic systems.
- Specific fusion rules $+$ nonabelian anyons $=$ fault-tolerant topological *quantum computer*. This is an ongoing area of research.

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Thank you for your attention!

A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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Generalize to N anyons: Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i<j}^{N} \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^{N} \mathbf{r}_i{}^2$$

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$$\mathcal{H} = \frac{1}{2m}\sum_{i=1}^{N}\left(\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i)\right)^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N}r_i^2$$

Expand:
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Question

Why is this useful?

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For any group G, the trivial representation takes $g \mapsto 1$ for all $g \in G$.

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- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[8]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[8] 1-dimensional representations are always irreducible!

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Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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Let X be an irreducible representation of a group G and T a linear operator that commutes with all $X(g)$ for $g \in G$. Then T is a scalar multiple of the identity operator.

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Corollary

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Definition

The *Burau representation* of the braid group B_n is defined on the standard generators:

$$\psi_n : B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$
$$\sigma_i \mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

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The Burau representation satisfies the braid relations:

$$\psi_n(\sigma_i)\psi_n(\sigma_j) = \psi_n(\sigma_j)\psi_n(\sigma_i) \text{ for } |i-j| > 1,$$
$$\psi_n(\sigma_i)\psi_n(\sigma_{i+1})\psi_n(\sigma_i) = \psi_n(\sigma_{i+1})\psi_n(\sigma_i)\psi_n(\sigma_{i+1}) \text{ for } i \in \{1, \ldots, n-2\}.$$

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Notice: $U \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-t+t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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Block structure of $\psi_n(\sigma_i) \implies \mathbf{1} = \begin{bmatrix}1\\1\\ \vdots\\ 1\end{bmatrix}$ is invariant under $\psi_n(\sigma_i)\ \forall\, i = 1, 2, \ldots, n-1$

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$\implies$ **Burau representation is reducible**!

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^{\dagger}\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle\,,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

SO(2) Explicit form of J

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J:

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m|\right) |\phi\rangle = \sum_m \langle m|\phi\rangle |m\rangle ,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi)m|\mathcal{O}\rangle = \langle e^{im\phi}m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J.
Thus,

$$J|\phi\rangle = \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}|m\rangle\right) = i\frac{\partial}{\partial\phi}|\phi\rangle$$
$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

SO(3) Invariance $\implies$ Commute with Hamiltonian

$$
\begin{aligned}
&\phi = \arctan\left(\frac{y}{x}\right) \\
&\quad \implies \frac{\partial}{\partial \phi} = (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z \implies J = -i\frac{\partial}{\partial \phi} = -i\,(\mathbf{r} \times \nabla) \cdot \mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \\
&\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x \\
&\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,
\end{aligned}
$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

From Invariant Subspace to the Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle \,,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle \,,$$

$$J_z |j, m\rangle = m |j, m\rangle \,,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \, |j, m \pm 1\rangle \,,$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J^2, J_i] = 0.$$