# Representation Theory and its Applications in Physics

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**Presented by**
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**Outline:**

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

# 1 Introduction to Representation Theory

# Definition of a Representation

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**Remark**

If $V$ is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then $X$ can be realized as an $n \times n$ matrix.

# Properties of Representations

## Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

$$X(gh) = X(g)X(h), \quad \forall g, h \in G.$$

## Invertibility

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1. $X(e) = I$, where $e$ is the identity element of the group and $I$ is the identity operator.
2. In the matrix presentation of $X$, $X(g)$ is invertible for all $g \in G$.

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- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

# Example: A Faithful Representation of $S_n$

**Defining representation of $S_n$**

The defining representation $D$ of $S_n$ encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends $i$ to $j$, then place a 1 the $i$-th column and $j$-th row of the representation matrix.

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E.g., in $S_3$:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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- This representation is faithful.

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**Representation:** Let $X$ be a representation of $G$ on $V_2$ with[1]

$$X(\phi)\mathbf{e}_1 = \mathbf{e}_1 \cdot \cos\phi + \mathbf{e}_2 \cdot \sin\phi$$
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**Question**

How do we classify representations of a group?

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- We can use characters to classify representations.

# Decomposing Representations

**Definition**

A representation $X(G)$ on $V$ is *irreducible* if there is no non-trivial invariant subspace[2] in $V$ with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to equivalence.

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**Decomposition of $X$**

The span of each $\mathbf{e}_\pm$ is an $X$-invariant subspace of $V_2$. In this basis, we rewrite $X$ as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

# Schur's Lemmas (pt. 1)

**Lemma**

*Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group $G$. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then $T$ is either the zero map or invertible.*

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4. By the rank-nullity theorem, conclude that $T$ is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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**Corollary**

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4. The element $h$ was arbitrary, so $X(g) = \lambda_g I$ for all $g \in G$.
5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.

# Consequence of Schur's Lemmas

**Corollary**

*If G is a finite abelian group, then the irreducible representations of G are one-dimensional.*

**Proof (sketch)**

1. Fix $h \in G$.
2. Since $G$ is abelian, $X(h)X(g) = X(g)X(h)$ for all $g \in G$.
3. Schur's second lemma implies $X(h) = \lambda_h I$ for some scalar $\lambda_h$.
4. The element $h$ was arbitrary, so $X(g) = \lambda_g I$ for all $g \in G$.
5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.
6. One-dimensional representations are irreducible.

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**How does this help in physics?**

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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# 2 Examples in Physics

# Preliminaries: Physics Conventions

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator $A$ on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:

$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

# Orthonormality, Completeness, and Wavefunctions

## Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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## Definition

For a continuous basis labelled by $|x\rangle$ where $x$ is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

# Properties of 2D Rotations

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

# The SO(2) Group

**Definition**

The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all $2 \times 2$ orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_{\pm}$).

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- We call $J$ the *generator* of SO(2) rotations.

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# Recovering the Rotation Matrix from $J$

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$$\begin{aligned} J\,|m\rangle &= m\,|m\rangle\,, \\ U(\phi)\,|m\rangle &= e^{-iJ\phi}\,|m\rangle = e^{-im\phi}\,|m\rangle\,. \end{aligned}$$

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**Theorem**

*The single-valued irreducible representations of SO(2) are defined as*

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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# Generalization to 3 Spatial Dimensions

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**Definition**

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all $3 \times 3$ orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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Arguably the most defining characteristic of quantum mechanics is that classically measurable quantities become discretized (quantized) when observed on the quantum scale. Without any physical motivation, the irreducible representations of SO(3) gave it to us for free!

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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*This is the tip of the iceberg!*

## 3 The Braid Group

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The *configuration space* of $n$ ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

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- A *braid* $\beta$ is a *loop*[8] in $M_n$ and can be thought of as a configuration that evolves over time:

$$\begin{aligned}\beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\big),\end{aligned}$$

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**Definition**

The *braid group* $B_n$ is the (fundamental) group of all complex-valued $n$-tuples $(M_n)$ up to *homotopy*.

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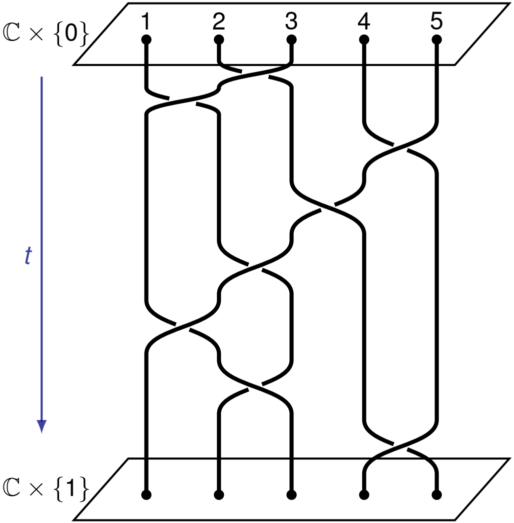
*Braid on 5 strands.*



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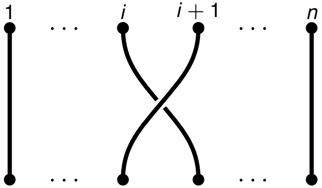
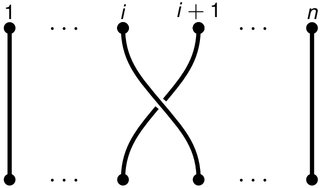



$\sigma_i$

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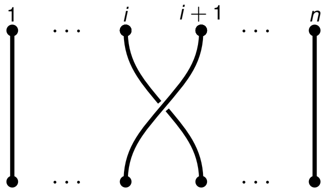


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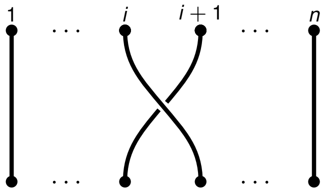


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- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of $\beta$.

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## Definition

The braid group on $n$ strands, denoted $B_n$, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, \; |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle.$$

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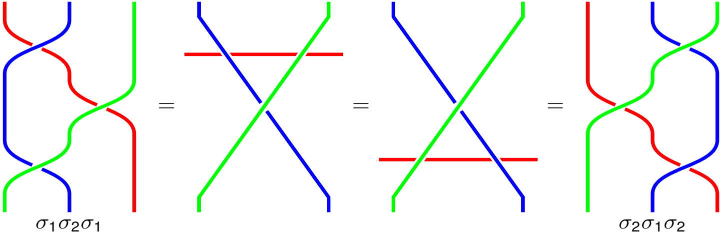
**Comment:** $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



# One-Dimensional Representations of the Braid Group

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Hence, for any $\beta \in B_n$ with degree $k$:

$$p_\theta(\beta) = p_\theta(\sigma_1^{m_1}\sigma_2^{m_2}\cdots\sigma_{n-1}^{m_{n-1}}) = e^{i\theta(m_1+m_2+\cdots+m_{n-1})} = e^{ik\theta}.$$

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## Definition

The *Burau representation* of the braid group $B_n$ is defined on the standard generators:

$$\psi_n : B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$
$$\sigma_i \mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

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The Burau representation satisfies the braid relations:

$$\psi_n(\sigma_i)\psi_n(\sigma_j) = \psi_n(\sigma_j)\psi_n(\sigma_i) \text{ for } |i-j| > 1,$$
$$\psi_n(\sigma_i)\psi_n(\sigma_{i+1})\psi_n(\sigma_i) = \psi_n(\sigma_{i+1})\psi_n(\sigma_i)\psi_n(\sigma_{i+1}) \text{ for } i \in \{1, \ldots, n-2\}.$$

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**Notice:** $$U\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1-t & t\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1-t+t\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

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$\implies$ **Burau representation is reducible**!

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## Definition

Define the unitary representation $\mathcal{U} : B_3 \to U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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**Answer:** Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

# 4 Physical Applications of the Braid Group

# (Abelian) Braiding Action on a Quantum System

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**Braiding action:** For any degree-$k$ braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \ldots, r_n),$$

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$$|1'\rangle = \mathcal{U}(\sigma_1)_{1,1}\,|1\rangle + \mathcal{U}(\sigma_1)_{1,2}\,|2\rangle = \frac{1}{2} e^{-i\frac{\pi}{6}} \left( \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)}\, |1\rangle + |2\rangle \right),$$

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**Remark**

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system[9].

[9] Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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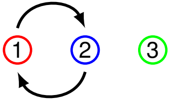

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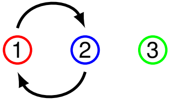
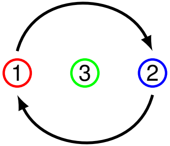



**Trajectory B**



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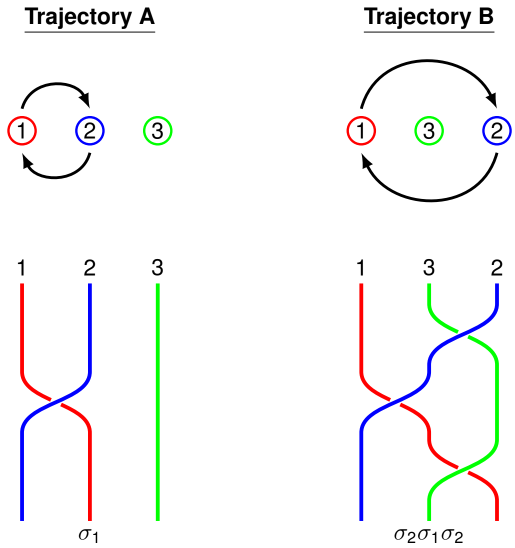

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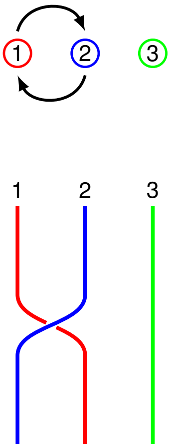


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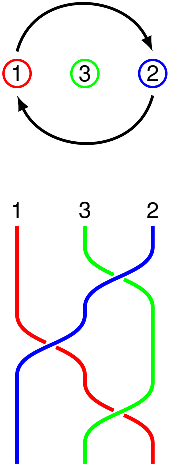


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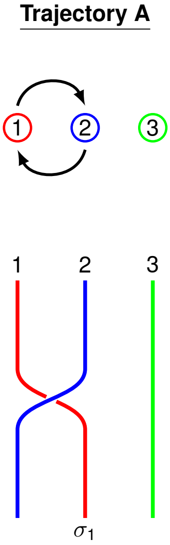


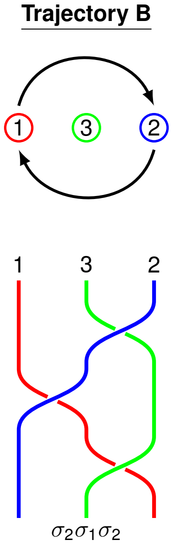

# A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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**Generalize to $N$ anyons:** Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i<j}^{N} \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^{N} \mathbf{r}_i{}^2$$

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***i*-th anyon Hamiltonian:**
$$\mathcal{H}_i = \frac{1}{2m}\underbrace{\big(\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i)\big)}_{\text{canonical momentum}}{}^2 + \frac{m\omega^2}{2}r_i^2$$

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$$\mathcal{L} = \frac{m}{2}\sum_{i=1}^{N}\left[\dot{\mathbf{r}}^2 - \omega^2 \mathbf{r}_i^2\right] + \alpha \sum_{i<j}^{N} \dot{\mathbf{r}}_{ij} \cdot \frac{(-y_{ij}\hat{x} + x_{ij}\hat{y})}{r_{ij}^2}$$

**Gauge potential:**

$$\mathbf{A}_i(\mathbf{r}_i) = \alpha \sum_{j\neq i} \frac{\hat{z} \times \mathbf{r}_{ij}}{r_{ij}^2} = \alpha \sum_{j \neq i} \frac{-y_{ij}\hat{x} + x_{ij}\hat{y}}{r_{ij}^2}$$

**$i$-th anyon Hamiltonian:**

$$\mathcal{H}_i = \frac{1}{2m}\underbrace{\left(\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i)\right)}_{\text{canonical momentum}}^2 + \frac{m\omega^2}{2} r_i^2$$

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**Expand:**

$$\mathcal{H} = \boxed{\frac{1}{2m}\sum_{i=1}^{N}p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N}r_i^2 - \frac{\alpha}{2m}\sum_{\substack{i=1\\ j\neq i}}^{N}\frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{\substack{i=1\\ j,k\neq i}}^{N}\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}}$$

# Interpreting the $N$-anyon Hamiltonian

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**Question**

Why is this useful?

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# Physical Implications of Nontrivial Braiding Effects

- The fractional quantum Hall effect is a physical manifestation of anyonic braiding in 2D electron systems (fractional charge, fractional statistics).
- Depending on the specific representation of the braid group, one can define topological properties of different flavors of anyons. The corresponding combination (fusion) rules can be used to describe the behavior of anyonic systems.
- Certain nonabelian anyons (defined by specific fusion rules) are alleged to enable fault-tolerant quantum computers, and is an ongoing area of research.

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1. Representation theory is a powerful tool that can be used to obtain fundamental results in quantum mechanics and beyond.
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**Thank you for your attention!**

# SO(3) Calculations (pt. 1)

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of $J$:

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m|\right) |\phi\rangle = \sum_m \langle m|\phi\rangle \, |m\rangle \, ,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi) m|\mathcal{O}\rangle = \langle e^{im\phi} m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of $J$.
Thus,

$$J\,|\phi\rangle = \sum_m e^{-im\phi} J\,|m\rangle = \sum_m m e^{-im\phi}\,|m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}\,|m\rangle\right) = i\frac{\partial}{\partial\phi}\,|\phi\rangle$$

$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

# SO(3) Calculations (pt. 2)

$$\phi = \arctan\left(\frac{y}{x}\right)$$
$$\implies \frac{\partial}{\partial\phi} = (\mathbf{r}\times\nabla)\cdot\mathbf{e}_z \implies J = -i\frac{\partial}{\partial\phi} = -i\,(\mathbf{r}\times\nabla)\cdot\mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$$
$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$
$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}),\hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2,\hat{L}_z] = 0 \implies [\hat{H},\hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

# Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle .$$