Representation Theory and its Applications in Physics

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Presented by
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Outline:

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

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1. $X(e) = I$, where e is the identity element of the group and I is the identity operator.
2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

Example: The Trivial Representation

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- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- The decomposition of a representation into irreducibles is unique up to equivalence.

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Example: Irreducible Representation of 2D Rotations

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.
6. One-dimensional representations are irreducible.

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How does this help in physics?

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2 Examples in Physics

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

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Definition

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- We call J the *generator* of SO(2) rotations.

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Recovering the Rotation Matrix from J

To first order in $d\phi$: $R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}$

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Theorem

The single-valued irreducible representations of SO(2) are defined as

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- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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Discretization of Angular Momentum for Free

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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This is the tip of the iceberg!

3 The Braid Group

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- A *braid* β is a *loop*[8] in M_n and can be thought of as a configuration that evolves over time:

$$\beta : [0,1] \to M_n$$
$$t \mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \dots, \beta_n(t)\big),$$

[8]The topological formalisms that define the braid group are omitted for times sake.

The Braid Group

Definition

The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

- Note that $(z_1, z_2, z_3, \ldots, z_n)$ and $(z_2, z_1, z_3, \ldots, z_n)$ are distinct configurations in M_n.
- A *braid* β is a *loop*[8] in M_n and can be thought of as a configuration that evolves over time:

$$\begin{aligned} \beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\big), \end{aligned}$$

Definition

The *braid group* B_n is the (fundamental) group of all complex-valued n-tuples (M_n) up to *homotopy*.

[8] The topological formalisms that define the braid group are omitted for times sake.

Visualization of Braids

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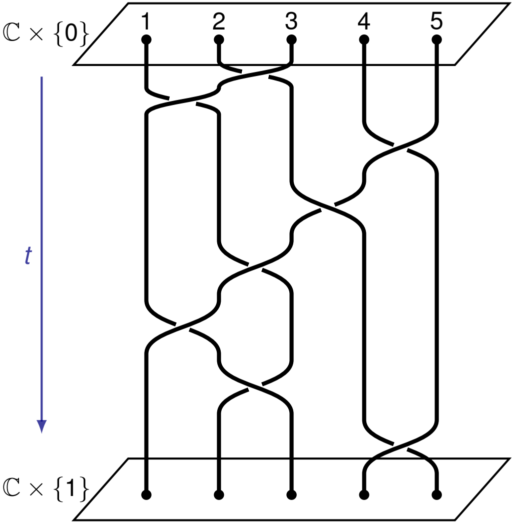
Braid on 5 strands.



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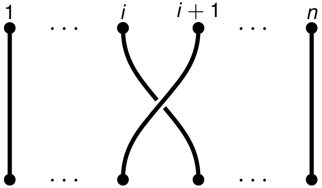

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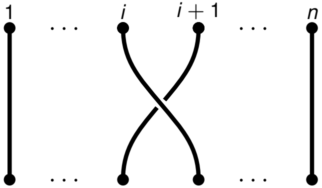

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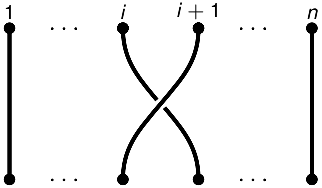

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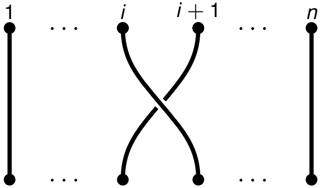

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- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β.

Alternative Description of B_n

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The braid group on n strands, denoted B_n, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, \;\; |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle .$$

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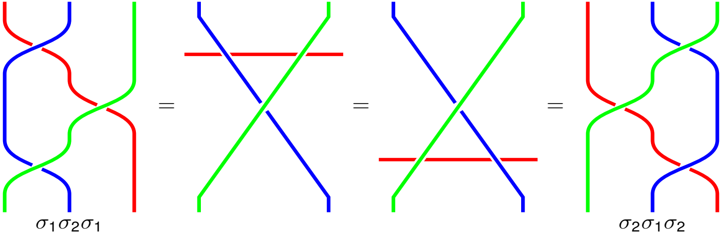
Comment: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



One-Dimensional Representations of the Braid Group

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Hence, for any $\beta \in B_n$ with degree k:

$$p_\theta(\beta) = p_\theta(\sigma_1^{m_1}\sigma_2^{m_2}\cdots\sigma_{n-1}^{m_{n-1}}) = e^{i\theta(m_1+m_2+\cdots+m_{n-1})} = e^{ik\theta}.$$

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Define the unitary representation $\mathcal{U} : B_3 \to U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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Answer: Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \ldots, r_n),$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system[9].

[9] Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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- Edge cases: *bosons* and *fermions*.

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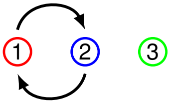

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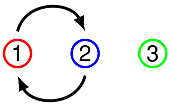

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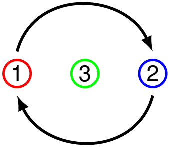

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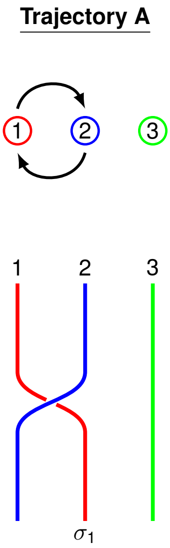

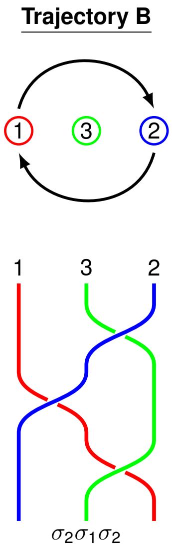

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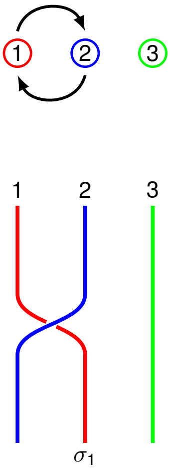
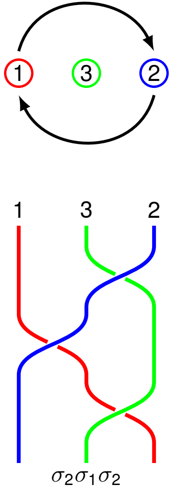


Trajectory B



Nontrivial Braiding Effects in 1D Representations

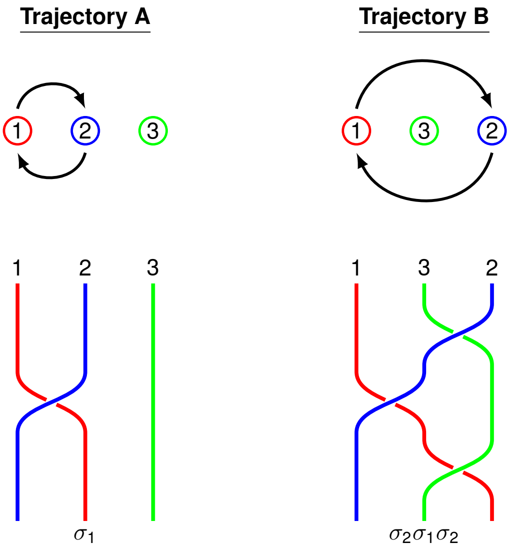
Recall: A braid is only well-defined if all particle trajectories are known.

Consequences:

1. A permutation of two anyons requires the knowledge of the positions of all other anyons in the system.
2. This is a consequence of the so-called *nontrivial braiding effects* of the braid group.

1D representation:

$$\left.\begin{aligned}\sigma_1 &\mapsto e^{i\theta}\\ \sigma_2\sigma_1\sigma_2 &\mapsto e^{3i\theta}\end{aligned}\right\} \neq \text{ if } \theta \notin \pi\mathbb{Z}$$



A Physicists Approach to Anyons

Consider N identical non-interacting anyons with positions $\mathbf{r}_i = (x_i, y_i)$ in a harmonic potential. Let $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$.

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Harmonic Potential: $V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \frac{1}{2} m\omega^2 \left(\mathbf{r}_1{}^2 + \mathbf{r}_2{}^2 + \cdots + \mathbf{r}_N{}^2\right)$

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Interpreting the N-anyon Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N} r_i^2 - \frac{\alpha}{2m}\sum_{\substack{i=1\\ j\neq i}}^{N} \frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{\substack{i=1\\ j,k\neq i}}^{N} \frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}$$

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Question

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- Depending on the specific representation of the braid group, one can define topological properties of different flavors of anyons. The corresponding combination (fusion) rules can be used to describe the behavior of anyonic systems.
- Certain nonabelian anyons (defined by specific fusion rules) are alleged to enable fault-tolerant quantum computers, and is an ongoing area of research.

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Thank you for your attention!

SO(3) Calculations (pt. 1)

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J:

$$|\phi\rangle = \left(\sum_m |m\rangle\langle m|\right)|\phi\rangle = \sum_m \langle m|\phi\rangle\,|m\rangle\,,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi)m|\mathcal{O}\rangle = \langle e^{im\phi}m|\mathcal{O}\rangle = e^{-im\phi}\langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J.
Thus,

$$J|\phi\rangle = \sum_m e^{-im\phi}J|m\rangle = \sum_m me^{-im\phi}|m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}|m\rangle\right) = i\frac{\partial}{\partial\phi}|\phi\rangle$$

$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

SO(3) Calculations (pt. 2)

$$\phi = \text{arctan}\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial \phi} = (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z \implies J = -i\frac{\partial}{\partial \phi} = -i\,(\mathbf{r} \times \nabla) \cdot \mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle .$$