# Thesis Title

## Cal Poly

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# Chapter 1

# Background Info

**Definition 1.1** (Representations of a Group). If there is a homomorphism from a group $G$ to a group of operators $U(G)$ on a linear vector space $V$, we say that $U(G)$ forms a *representation* of $G$ with dimension $\dim V$.

The representation is a map

$$g \in G \xrightarrow{U} U(g) \tag{1.1}$$

in which $U(g)$ is an operator on the vector space $V$. For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \ldots, n\}$, we can realize each operator $U(g)$ as an $n \times n$ matrix $D(g)$.

$$U(g) \left| e_i \right\rangle = \sum_{j=1}^{n} \left| e_j \right\rangle D(g)^j{}_i = \left| e_j \right\rangle D(g)^j{}_i, \tag{1.2}$$

where the first index $j$ is the row index and the second index $i$ is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$U(g_1)U(g_2) = U(g_1 g_2), \tag{1.3}$$

which satisfies the group multiplication rules.

**Definition 1.2.** If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

**Example 1.1.** Let $G$ be the group of continuous rotations in the $xy$-plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space $V_2$. Then we define a representation of $G$ on $V_2$ by the familiar rotation operation

$$\hat{e}_1' = U(\phi)\hat{e}_1 = \hat{e}_1 \cdot \cos\phi + \hat{e}_2 \cdot \sin\phi \tag{1.4}$$

$$\hat{e}_2' = U(\phi)\hat{e}_2 = -\hat{e}_1 \cdot \sin\phi + \hat{e}_2 \cdot \cos\phi. \tag{1.5}$$

This gives us the matrix representation

$$D(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1.6}$$

To further illustrate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \vec{x} \in V_2$, then we have

$$\vec{x}' = U(\phi)\vec{x} = \hat{e}_j x'^j, \tag{1.7}$$

where $x'^j = D(\phi)^j{}_i x^i$.

**Definition 1.3** (Equivalence of Representations)**.** For a group $G$, two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define

**Definition 1.4** (Characters of a Representation)**.** The *character* $\chi(g)$ of an element $g \in G$ in a representation $U(g)$ is defined as $\chi(g) = \text{Tr } D(g)$.

Since trace is independent of basis, the character serves as a class label.

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

**Definition 1.5** (Invariant Subspace)**.** Let $U(G)$ be a representation of $G$ on a vector space $V$, and $W$ a subspace of $V$ such that $U(g)|x\rangle \in W$ for all $\vec{x} \in W$ and $g \in G$. Then $W$ is an *invariant subspace* of $V$ with respect to $U(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $U(G)$.

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The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

**Definition 1.6** (Irreducible Representation). A representation $U(G)$ on $V$ *irreducible* if there is no non-trivial invariant subspace in $V$ with respect to $U(G)$. Otherwise, it is *reducible*. If $U(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $U(G)$, then the representation is *fully reducible*.

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5**?

**Example 1.2** (Generator of SO(2)). Consider the rotations of a 2-dimensional Euclidean vector space about the origin. Let $\hat{e}_1$ and $\hat{e}_2$ be orthonormal basis vectors of this space. Using geometry, we can determine how a rotation by some angle $\phi$, written in operator form as $R(\phi)$, acts on the basis vectors:

$$R(\phi)\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi \tag{1.8}$$
$$R(\phi)\hat{e}_2 = -\hat{e}_1 \sin\phi + \hat{e}_2 \cos\phi. \tag{1.9}$$

In matrix form, we can write

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{1.10}$$

which allows us to write Eqn. 1.8 and Eqn. 1.9 in a condensed form

$$R(\phi)\hat{e}_i = \hat{e}_j R(\phi)^j{}_i, \tag{1.11}$$

where we are summing over $j = 1, 2$.

Now, let $\vec{x}$ be an arbitrary vector in the plane. Then $\vec{x}$ has components $x^i$ in the basis $\{\hat{e}_i\}$, where $i = 1, 2$. Equivalently, we can write $\vec{x} = \hat{e}_i x^i$. Then under rotations, $\vec{x}$ transforms in accordance to the basis vectors

$$\begin{aligned} R(\phi)\vec{x} &= R(\phi)\hat{e}_i x^i \\ &= \hat{e}_j R(\phi)^j{}_i x^i \\ &= \left(\hat{e}_1 R(\phi)^1{}_i + \hat{e}_2 R(\phi)^2{}_i\right) x^i \\ &= \left(\hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi\right) x^1 + \left(\hat{e}_1(-\sin\phi) + \hat{e}_2 \cos\phi\right) x^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right) \hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \hat{e}_2. \end{aligned} \tag{1.12}$$

Observe that $R(\phi)R^\top(\phi) = E$ where $E$ is the identity matrix. This is precisely what defines *orthogonal matrices*. For 2-dimensional vectors in the plane, it is clear that these rotations do not change the length of said vectors. This can be verified by using Eqn. 1.12:

$$
\begin{aligned}
|R(\phi)\vec{x}|^2 &= |\hat{e}_j R(\phi)^j{}_i x^i|^2 \qquad\qquad\qquad\qquad\qquad (1.13)\\
&= \left|\left(x^1 \cos\phi - x^2 \sin\phi\right)\hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right)\hat{e}_2\right|^2\\
&= \left(x^1 \cos\phi - x^2 \sin\phi\right)^2 + \left(x^1 \sin\phi + x^2 \cos\phi\right)^2\\
&= \left(\cos^2\phi + \sin^2\phi\right)x^1 x_1 + \left(\sin^2\phi + \cos^2\phi\right)x^2 x_2\\
&= x^1 x_1 + x^2 x_2 = |\vec{x}|^2.
\end{aligned}
$$

Similarly, notice that for any continuous rotation by angle $\phi$, $\det R(\phi) = \cos^2\phi + \sin^2\phi = 1$. In general, orthogonal matrices have determinant equal to $\pm 1$. However, the result of the above determinant of $R(\phi)$ implies that all continuous rotations in the 2-dimensional plane have determinant equal to $+1$. These are the *special orthogonal matrices of rank 2*. This family of matrices is denoted SO(2). Furthermore, there is a one-to-one correspondence with SO(2) matrices and rotations in a plane.

We define the group of continuous rotations in a plane by letting $R(0) = E$ be the identity element corresponding to no rotation (i.e., a rotation by angle $\phi = 0$), and defining the inverse of a rotation as $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi)$. This group can be called the SO(2) group. Lastly, we define group multiplication as $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$ and note that $R(\phi) = R(\phi \pm 2\pi)$, which can be verified geometrically. Thus, group elements of SO(2) can be labelled by the angle of rotation $\phi \in [0, 2\pi)$.

Now we can find a generator of *sotwo* by considering an infinitesimal rotation, labelled by some infinitesimal angle $d\phi$. Then this is equivalent to the identity plus some small rotation, which we can write as

$$
R(\mathrm{d}\phi) = E - i\mathrm{d}\phi J \qquad\qquad\qquad\qquad (1.14)
$$

where the scalar quantity $-i$ is introduced for later convenience and $J$ is some quantity independent of the rotation angle. If we consider the rotation

$R(\phi + \mathrm{d}\phi)$, then there are two equivalent ways to interpret this rotation

$$R(\phi + \mathrm{d}\phi) = R(\phi)R(\mathrm{d}\phi) = R(\phi)(E - i\mathrm{d}\phi J) = R(\phi) - i\mathrm{d}\phi R(\phi)J \quad (1.15)$$

$$R(\phi + \mathrm{d}\phi) = R(\phi) + \mathrm{d}R(\phi) = R(\phi) + \mathrm{d}\phi\frac{\mathrm{d}R(\phi)}{\mathrm{d}\phi} \quad (1.16)$$

where the second equation can be thought of as a Taylor expansion of $R(\phi + \mathrm{d}\phi)$ about $\phi$. Equating the two expressions for $R(\phi + \mathrm{d}\phi)$ yields

$$\mathrm{d}R(\phi) = -i\mathrm{d}\phi R(\phi)J. \quad (1.17)$$

Solving this differential equation (with boundary condition $R(0) = E$) provides us with an equation for any group element involving $J$:

$$R(\phi) = e^{-i\phi J}, \quad (1.18)$$

where $J$ is called the *generator* of the group.