

Title

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Chapter 1

An Introduction to Representation Theory

1.1 Introduction

Intro paragraph to lead into the definitions.

Definition 1.1 (Representation of a group). Let G be a group. A *representation* of G is a homomorphism from G to a group of operators on a linear vector space V . The dimension of V is the *dimension* or *degree* of the representation.

If X is a representation of G on V , then X is a map

$$g \in G \xrightarrow{X} X(g) \tag{1.1}$$

in which $X(g)$ is an operator on the vector space V . For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \dots, n\}$, we can realize each operator $X(g)$ as an $n \times n$ matrix $D(g)$.

$$X(g) |e_i\rangle = \sum_{j=1}^n |e_j\rangle D(g)^j_i = |e_j\rangle D(g)^j_i, \tag{1.2}$$

where the first index j is the row index and the second index i is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$X(g_1)X(g_2) = X(g_1g_2), \tag{1.3}$$

which satisfies the group multiplication rules.

Definition 1.2. If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

Example 1.1. The simplest representation of any group G is the *trivial* representation, in which every $g \in G$ is realized by $g \mapsto 1$. This representation is clearly degenerate.

Example 1.2. Consider the symmetric group S_n . The *defining* representation of S_n encodes each $\sigma \in S_n$ by placing a 1 in the j -th row and i -th column of the matrix $D(\sigma)$ if σ sends i to j , and 0 otherwise. For example, in S_3 , the permutation (23) has the matrix representation

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

whereas the permutation (123) is realized by the matrix

$$D((123)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The above example involves a finite group. Infinite groups can also have representations, as demonstrated in the following example.

Example 1.3. Let G be the group of continuous rotations in the xy -plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space V_2 . Then we define a representation of G on V_2 by the familiar rotation operation

$$\hat{e}'_1 = X(\phi)\hat{e}_1 = \hat{e}_1 \cdot \cos \phi + \hat{e}_2 \cdot \sin \phi \quad (1.4)$$

$$\hat{e}'_2 = X(\phi)\hat{e}_2 = -\hat{e}_1 \cdot \sin \phi + \hat{e}_2 \cdot \cos \phi, \quad (1.5)$$

where \hat{e}_1 and \hat{e}_2 are orthonormal basis vectors of V_2 . This gives us the matrix representation

$$D(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}. \quad (1.6)$$

To further illuminate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \vec{x} \in V_2$, then we have

$$\vec{x}' = X(\phi)\vec{x} = \hat{e}_j x'^j, \quad (1.7)$$

where $x'^j = D(\phi)^j_i x^i$. **Can probably simplify the notation**

Definition 1.3 (Equivalence of Representations). For a group G , two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define the following.

Definition 1.4 (Characters of a Representation). The *character* $\chi(g)$ of an element $g \in G$ in a representation $X(g)$ is defined as $\chi(g) = \text{Tr } D(g)$.

Since trace is independent of basis, the character serves as a class label.

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

Definition 1.5 (Invariant Subspace). Let $X(G)$ be a representation of G on a vector space V , and W a subspace of V such that $X(g)|x\rangle \in W$ for all $\vec{x} \in W$ and $g \in G$. Then W is an *invariant subspace* of V with respect to $X(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $X(G)$.

The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

Definition 1.6 (Irreducible Representation). A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $X(G)$. Otherwise, it is *reducible*. If $X(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $X(G)$, then the representation is *fully reducible*.

Example 1.4. Under the group of 2-dimensional rotations, consider the 1-dimensional subspace spanned by \hat{e}_1 . This subspace is not invariant under 2-dimensional rotations, because a rotation of \hat{e}_1 by $\pi/2$ results in the vector \hat{e}_2 that is clearly not in the subspace spanned by \hat{e}_1 . A similar argument shows that the subspace spanned by \hat{e}_2 is not invariant under 2-dimensional rotations.

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5?**

1.2 Rotations in a plane and the group SO(2)

R vs U inconsistency from earlier notation

1.2.1 The rotation group

Consider the rotations of a 2-dimensional Euclidean vector space about the origin. Let \hat{e}_1 and \hat{e}_2 be orthonormal basis vectors of this space. Using geometry, we can determine how a rotation by some angle ϕ , written in operator form as $R(\phi)$, acts on the basis vectors:

$$R(\phi)\hat{e}_1 = \hat{e}_1 \cos \phi + \hat{e}_2 \sin \phi \quad (1.8)$$

$$R(\phi)\hat{e}_2 = -\hat{e}_1 \sin \phi + \hat{e}_2 \cos \phi. \quad (1.9)$$

In matrix form, we can write

$$R(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \quad (1.10)$$

which allows us to write Eqns. 1.8 and 1.9 in a condensed form

$$R(\phi)\hat{e}_i = \hat{e}_j R(\phi)^j_i, \quad (1.11)$$

where we are summing over $j = 1, 2$.

Let \vec{x} be an arbitrary vector in the plane. Then \vec{x} has components x^i in the basis $\{\hat{e}_i\}$, where $i = 1, 2$. Equivalently, we can write $\vec{x} = \hat{e}_i x^i$. Then under rotations, \vec{x} transforms in accordance to the basis vectors

$$\begin{aligned} R(\phi)\vec{x} &= R(\phi)\hat{e}_i x^i \\ &= \hat{e}_j R(\phi)^j_i x^i \\ &= (\hat{e}_1 R(\phi)^1_i + \hat{e}_2 R(\phi)^2_i) x^i \\ &= (\hat{e}_1 \cos \phi + \hat{e}_2 \sin \phi) x^1 + (\hat{e}_1 (-\sin \phi) + \hat{e}_2 \cos \phi) x^2 \\ &= (x^1 \cos \phi - x^2 \sin \phi) \hat{e}_1 + (x^1 \sin \phi + x^2 \cos \phi) \hat{e}_2. \end{aligned} \quad (1.12)$$

Notice that $R(\phi)R^\top(\phi) = E$ where E is the identity matrix. This is precisely what defines *orthogonal matrices*. For 2-dimensional vectors in the plane, it is clear that these rotations do not change the length of said vectors. This can be verified by using Eqn. 1.12:

$$\begin{aligned}
|R(\phi)\vec{x}|^2 &= |\hat{e}_j R(\phi)^j_i x^i|^2 & (1.13) \\
&= |(x^1 \cos \phi - x^2 \sin \phi) \hat{e}_1 + (x^1 \sin \phi + x^2 \cos \phi) \hat{e}_2|^2 \\
&= (x^1 \cos \phi - x^2 \sin \phi)^2 + (x^1 \sin \phi + x^2 \cos \phi)^2 \\
&= (\cos^2 \phi + \sin^2 \phi) x^1 x_1 + (\sin^2 \phi + \cos^2 \phi) x^2 x_2 \\
&= x^1 x_1 + x^2 x_2 = |\vec{x}|^2.
\end{aligned}$$

Similarly, notice that for any continuous rotation by angle ϕ , $\det R(\phi) = \cos^2 \phi + \sin^2 \phi = 1$. In general, orthogonal matrices have determinant equal to ± 1 . However, the result of the above determinant of $R(\phi)$ implies that all continuous rotations in the 2-dimensional plane have determinant equal to $+1$. These are the *special orthogonal matrices of rank 2*. This family of matrices is denoted $SO(2)$. Furthermore, there is a one-to-one correspondence with $SO(2)$ matrices and rotations in a plane.

We define the group of continuous rotations in a plane by letting $R(0) = E$ be the identity element corresponding to no rotation (i.e., a rotation by angle $\phi = 0$), and defining the inverse of a rotation as $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi)$. Lastly, we define group multiplication as $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$ and note that $R(\phi) = R(\phi \pm 2\pi)$, which can be verified geometrically. Although $SO(2)$ is technically a 2-dimensional representation of a more abstract rotation group, it is often just referred to as the rotation group due to the nature of the construction. Thus, group elements of $SO(2)$ can be labelled by the angle of rotation $\phi \in [0, 2\pi)$.

1.2.2 Infinitesimal rotations

Consider an infinitesimal rotation labelled by some infinitesimal angle $d\phi$. This is equivalent to the identity plus some small rotation, which can be written as

$$R(d\phi) = E - id\phi J \quad (1.14)$$

where the scalar quantity $-i$ is introduced for later convenience and J is some quantity independent of the rotation angle. If we consider the rotation

$R(\phi + d\phi)$, then there are two equivalent ways to interpret this rotation

$$R(\phi + d\phi) = R(\phi)R(d\phi) = R(\phi)(E - id\phi J) = R(\phi) - id\phi R(\phi)J, \quad (1.15)$$

$$R(\phi + d\phi) = R(\phi) + dR(\phi) = R(\phi) + d\phi \frac{dR(\phi)}{d\phi}, \quad (1.16)$$

where the second equation can be thought of as a Taylor expansion of $R(\phi + d\phi)$ about ϕ . Equating the two expressions for $R(\phi + d\phi)$ yields

$$dR(\phi) = -id\phi R(\phi)J. \quad (1.17)$$

Solving this differential equation (with boundary condition $R(0) = E$) provides us with an equation for any group element involving J :

$$R(\phi) = e^{-i\phi J}, \quad (1.18)$$

where J is called the *generator* of the group.

The explicit form of J is found as follows. To first order in $d\phi$, we have

$$R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}.$$

Comparing to Eqn. 1.14,

$$E - id\phi J = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix} \implies J = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Notice that $J^2 = E$, which implies that even powers of J equal the identity matrix and odd powers of J equal J . Taylor expanding $e^{-iJ\phi}$ gives

$$\begin{aligned} R(\phi) &= e^{-iJ\phi} = E - iJ\phi - E\frac{\phi^2}{2!} - iJ\frac{\phi^3}{3!} + \dots \\ &= E \left(\sum_{n=0}^{\infty} (-1)^n \frac{\phi^{2n}}{(2n)!} \right) - iJ \left(\sum_{n=0}^{\infty} (-1)^n \frac{\phi^{2n+1}}{(2n+1)!} \right) \\ &= E \cos \phi - iJ \sin \phi \\ &= \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}. \end{aligned}$$

Therefore, the generator J can be used to recover the rotation matrix for an arbitrary angle ϕ . Clearly, the map $R(\phi) \mapsto e^{-iJ\phi}$ is a valid homomorphism that respects the periodic nature of $\text{SO}(2)$.

1.2.3 Irreducible representations of SO(2)

Equipped with the generator J , we can construct the irreducible representations of SO(2). First, consider a representation U of SO(2) defined on a finite dimensional vector space V . Then $U(\phi)$ is the corresponding representation of $R(\phi)$. The same argument as in Section 1.2.2 can be applied to an infinitesimal rotation to give

$$U(\phi) = e^{-iJ\phi},$$

which is an operator on V (for convenience, the same symbol J is used to denote the generator of the representation).

Since U is a representation of rotations that preserves the length of vectors, we have

$$\begin{aligned} |a|^2 = |U(\phi)a|^2, \forall |a\rangle \in V &\iff \langle a|a\rangle = \langle U(\phi)a|U(\phi)a\rangle = \langle a|U(\phi)^\dagger U(\phi)|a\rangle \\ &\iff U(\phi)^\dagger U(\phi) = E \\ &\iff e^{iJ^\dagger\phi} e^{-iJ\phi} = e^{-i(J-J^\dagger)\phi} = 1 \\ &\iff J = J^\dagger. \end{aligned}$$

Therefore, not only must U be unitary, but the generator J must be Hermitian. This fact becomes especially important in the physical interpretation of the representations of SO(2) in Section 1.2.5.

Then the minimal invariant subspace of any $|\alpha\rangle \in V$ under SO(2) must satisfy

$$\begin{aligned} J|\alpha\rangle &= \alpha|\alpha\rangle, \\ U(\phi)|\alpha\rangle &= e^{-iJ\phi}|\alpha\rangle = e^{-i\alpha\phi}|\alpha\rangle, \end{aligned}$$

where the eigenvalue α of J is used as a label for the eigenvectors of J . The periodicity conditions of SO(2) imply that $|\alpha\rangle = U(2\pi)|\alpha\rangle$, or equivalently, $e^{-i\alpha 2\pi} = 1$. This implies that α must be an integer, as $e^{i2\pi m} = 1$ for $m \in \mathbb{Z}$. Then U has a corresponding 1-dimensional representation for each integer m , defined by

$$\begin{aligned} J|m\rangle &= m|m\rangle, \\ U^m(\phi)|m\rangle &= e^{-im\phi}|m\rangle. \end{aligned}$$

Since these representations correspond to 1-dimensional invariant subspaces, they are irreducible.

If $m = 0$, then $R(\phi) \rightarrow U^0(\phi) = 1$, which corresponds to the trivial representation. If instead $m = 1$, then $R(\phi) \rightarrow U^1(\phi) = e^{-i\phi}$, which maps rotations in $\text{SO}(2)$ to distinct points on the unit circle in the complex plane. The $m = 1$ representation is faithful because each rotation by ϕ has a unique image under $U^1(\phi)$, which is clear when interpreting rotations of unit vectors geometrically. As ϕ ranges from 0 to 2π , U^1 traces over the unit circle in \mathbb{C} in the counterclockwise direction. Similarly, U^{-1} traces over the unit circle in the clockwise direction because $U^{-1}(\phi) = e^{i\phi}$. In general, U^n covers the unit circle n times as ϕ ranges from 0 to 2π .

Get eigenvalues and eigenvectors of J to write $\text{SO}(2)$ in block diagonal form?

1.2.4 Multivalued representations

If we restrict the periodic condition on U to $U(2n\pi) = E$ for some $n \in \mathbb{Z}$, then the resulting 1-dimensional irreducible representations of $\text{SO}(2)$ become multivalued. Consider the same construction of U^m in Section 1.2.3, but now with $m \in \mathbb{Q}$. For $m = \frac{1}{2}$, we have

$$U^{1/2}(2\pi + \phi) = e^{-i\pi - i\frac{\phi}{2}} = -e^{-i\frac{\phi}{2}} = -U^{1/2}(\phi).$$

Hence, the rotation $R(\phi)$ is assigned to both $\pm e^{i\phi/2}$ in the $U^{1/2}$ representation. For this reason, it can be said that $U^{1/2}$ is a *two-valued* representation of $\text{SO}(2)$.

Despite this ambiguity in the realization of rotations in $\text{SO}(2)$, the periodicity condition is still satisfied, as $U^{1/2}(4\pi) = e^{i2\pi} = 1$. In other words, the double-valued representation of $\text{SO}(2)$ traverses the unit circle twice before returning to the identity. In general, $U^{n/m}$ is an m -valued representation of $\text{SO}(2)$ for $\frac{n}{m} \in \mathbb{Q}$ and $\gcd(n, m) = 1$.

1.2.5 Application of $\text{SO}(2)$ to quantum mechanics

The generalization of the $\text{SO}(2)$ group to quantum mechanics is the unitary group $\text{U}(1)$, which is the group of continuous phase transformations. The group elements of $\text{U}(1)$ are complex numbers of unit modulus, similar to

that of the irreducible representations of $SO(2)$ found in Section 1.2.3. The requirement of unitary matrices in quantum mechanics is a consequence of the fact that physical transformations, such as rotations and translations, must preserve probabilities. In quantum mechanics, probabilities are encoded in the norm of the state vectors. By definition, unitary transformations preserve the norm of vectors, which is why they represent physical transformations in quantum mechanics. **Move that second half to appendix and combine with discussion of bracket notation et al.?**

- **Discuss Lie groups/algebras specifically?**
- The real generalization is to 3 spatial dimensions, $SO(3)$, which then has the Lie algebra $\mathfrak{so}(3)$ with generators J_i and familiar commutation relations.
- The eigenvalues of J are real since it is Hermitian, and so they correspond to physical observables. In particular, the eigenvalues m of J correspond to the angular momentum of a quantum system (really it's a projection of the total angular momentum onto the axis of rotation normalized to \hbar). When m is an integer, the representation U^m corresponds to integer-spin particles, such as bosons or gravitons. When m is a half-integer, the representation U^m corresponds to half-integer-spin particles, such as fermions.
- The **quantization of angular momentum in quantum mechanics** is a direct consequence of the representation theory of $SO(2)$ (really $SO(3)$)!!! The allowable values of angular momentum are quantized because the eigenvalues of the generator J are quantized. **Moreover, for eigenvalue m , the possible spin states with angular momentum m correspond to the multiple values (U^m 's satisfying the periodicity condition for the eigenvector $|m\rangle$): $-m, -m + 1, \dots, m - 1, m$ (normalized to \hbar). Jumping between spin states is done by the ladder operators J_{\pm} in $SO(3)$.**
- Example for $U^{1/2}$, or in physics $j = \frac{1}{2}$. The spin state of an electron can either be up $+U^{1/2}$ or down $-U^{1/2}$, which **corresponds to the two-valued-ness** of the representation. A rotation of 2π results in a change of sign (a change in spin state). Moreover, the spin state of a spin- $\frac{1}{2}$ particle is described by a *spinor* (a two-component complex-valued vector). The purely mathematical consequences of double-valued rep-

representations of $SO(2)$ explains the emergent behavior of spinors under coordinate rotations.

1.3 Continuous translations

- Look at the 1D translation group T and its Lie algebra $\mathfrak{t}(1)$.
- Go through same process as $SO(2)$ for T in terms of infinitesimal translations and the generator P .
- Point out differences between observables (discretization vs continuous).
- Pg. 106 in Tung, show/derive the $|x\rangle \leftrightarrow |p\rangle$ transformations?

1.4 Ehrenfest's theorem and conserved quantities

Move most of this to appendix? Just keep part about J and P specifically here?

- Get specific \hat{H} that commutes with J and P . I'm thinking $\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{X})$?

Possible reference here [7]!

Suppose G is an operator on a quantum Hilbert space of states. The quantity $\langle G \rangle$ is conserved if

$$\frac{d\langle G \rangle}{dt} = 0.$$

Recall the time-dependent Schrödinger equation

$$\hat{H}\psi = i\hbar \frac{d\psi}{dt} \implies \frac{d\psi}{dt} = \frac{1}{i\hbar} \hat{H}\psi.$$

Then if G is time-independent we have

$$\begin{aligned}
\frac{d\langle G \rangle}{dt} &= \frac{d}{dt} \langle \psi | G | \psi \rangle \\
&= \left\langle \frac{d\psi}{dt} \middle| G \middle| \psi \right\rangle + \left\langle \psi \middle| G \middle| \frac{d\psi}{dt} \right\rangle + \left\langle \psi \middle| \frac{\partial G}{\partial t} \middle| \psi \right\rangle \xrightarrow{0} \\
&= \left\langle \frac{1}{i\hbar} \hat{H} \psi \middle| G \middle| \psi \right\rangle + \left\langle \psi \middle| G \middle| \frac{1}{i\hbar} \hat{H} \psi \right\rangle \\
&= \frac{i}{\hbar} \left(\langle \hat{H} \psi | G | \psi \rangle - \langle \psi | G | \hat{H} \psi \rangle \right) \\
&= \frac{i}{\hbar} \left(\langle \psi | \hat{H}^\dagger G | \psi \rangle - \langle \psi | G \hat{H} | \psi \rangle \right) \\
&= \frac{i}{\hbar} \left(\langle \psi | \hat{H} G | \psi \rangle - \langle \psi | G \hat{H} | \psi \rangle \right) \text{ because } \hat{H} \text{ is Hermitian} \\
&= \frac{i}{\hbar} \langle \psi | (\hat{H} G - G \hat{H}) | \psi \rangle \\
&= \frac{i}{\hbar} \langle \psi | [\hat{H}, G] | \psi \rangle = 0 \iff [\hat{H}, G] = 0.
\end{aligned}$$

(linear in the second argument). (See Ehrenfest's theorem).

Thus, if $[\hat{H}, G] = 0$ and G is unitary, it follows that

$$\begin{aligned}
\hat{H}G - G\hat{H} = 0 &\iff \hat{H}G = G\hat{H} \\
&\iff G^{-1}\hat{H}G = \hat{H}.
\end{aligned}$$

Thus, $G^{-1}\hat{H}G$ and \hat{H} share the same eigenvalues (observables), which is only true if \hat{H} is invariant under G . If G generates a group of transformations, then \hat{H} is invariant under the group of transformations generated by G . Often times, this invariance is expressed as

$$G^\dagger \hat{H} G = \hat{H}$$

since G is unitary.

Running the argument in reverse, if \hat{H} is invariant under the transformations generated by G , then $[\hat{H}, G] = 0$, which, by the Ehrenfest theorem, implies that $\langle G \rangle$ is conserved.

Therefore, if a physical system represented by a Hamiltonian H is invariant under rotations, then $[H, R(\phi)] = 0$ for all $\phi \in \mathbb{R}$ and thus $[H, J] = 0$, so angular momentum is conserved.

Chapter 4

To-Do List

Potential committee members:

- Anton Kaul
- Patrick Orson
- Eric Brussel
- *Rob Easton*

-
- **First:** Do Lorentz group example, finish off content that you want in there. Then go back and fix notation once you know that you won't need it for Lorentz group. Alternatively, you can just invent the notation solely for that example!
 - Redo the Chapter 1 with nicer notation and stray away from Tung's notation when possible.
 - More straightforward examples of representations?
 - At least briefly discuss $U(n)$ either here or in braid rep chapter.
 - Finish/modify irreducible rep. example in Chapter 1.
 - Do more for $SO(2)$ example?
 - Fix out equation numbers

- **Examples to add:** 1D conservation of momentum from T , angular momentum connection to $SO(2)$?, Lorentz group
- Do a little more context on the physics. Maybe put in appendix, but describe what the heck a quantum Hilbert space is and bra-ket notation.
- Also note why we care about unitary matrices, probs in appendix as well.

-
- Show $\psi_n(\sigma_i)$ invertible? Yes, eventually
 - derive $\psi_n^r(\sigma_i)$ matrices or state?
 - Show $\psi_n^r(\sigma_i)$ invertible? Yes, eventually
 - Explicitly show why Burau isn't able to be made unitary? [3]
 - Separate chapters into braid group and braid group reps.?

-
- Concluding paragraph on first section to lead into the more physics-y stuff.
 - ~~Show the additional cross terms from $N=2$ to $N=3$ and beyond.~~
 - Add paragraph on gauge theory/motivation.
 - Anyon fusion rules
 - τ anyon/Fibonacci anyon example. Relate to singlet/triplet states in spin-1/2 system.
 - ~~Move anyon calculations to appendix?~~
 - Spend some time on MATLAB thing

-
- Conclusion/future of anyons/braid group in physics.
 - Abstract
 - Title
 - Acknowledgements

Format!!

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