

# Representation Theory and its Applications in Physics

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**Presented by**

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## Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group



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## **1 Introduction to Representation Theory**

# Definition of a Representation

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Let  $G$  be a group. A *representation* of  $G$  is a homomorphism from  $G$  to a group of operators on a linear vector space  $V$ . The dimension of  $V$  is the *dimension* or *degree* of the representation.

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If  $X$  is a representation of  $G$  on a vector space  $V$ , then  $X$  is a map

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## Remark

If  $V$  is finite-dimensional with basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ , then  $X$  can be realized as an  $n \times n$  matrix.

# Properties of Representations

## Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

$$X(gh) = X(g)X(h), \quad \forall g, h \in G.$$

## Invertibility

If  $X$  is a representation of  $G$ , then  $X(g)^{-1} = X(g^{-1})$ ,  $\forall g \in G$ .

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2. In the matrix presentation of  $X$ ,  $X(g)$  is invertible for all  $g \in G$ .

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- ▶ For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- ▶ If a representation is injective, then it is a *faithful representation*.



# Example: A Faithful Representation of $S_n$

## Defining representation of $S_n$

The defining representation  $D$  of  $S_n$  encodes the action of the symmetric group on the standard basis of  $\mathbb{R}^n$ . If a permutation sends  $i$  to  $j$ , then place a 1 the  $i$ -th column and  $j$ -th row of the representation matrix.

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E.g., in  $S_3$ :

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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- ▶ The defining representation of  $S_n$  is  $n$ -dimensional.
- ▶ This representation is faithful.

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**Representation:** Let  $X$  be a representation of  $G$  on  $V_2$  with<sup>1</sup>

$$X(\phi)\mathbf{e}_1 = \mathbf{e}_1 \cdot \cos \phi + \mathbf{e}_2 \cdot \sin \phi$$

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## Question

How do we classify representations of a group?

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- ▶ If two representations have the same character for all  $g \in G$ , then they are equivalent.
- ▶ We can use characters to classify representations.

# Decomposing Representations

## Definition

A representation  $X(G)$  on  $V$  is *irreducible* if there is no non-trivial invariant subspace<sup>2</sup> in  $V$  with respect to  $X(G)$ . Otherwise,  $X(G)$  is *reducible*.

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- ▶ Irreducible representations are the building blocks of all representations.
- ▶ A reducible representation can be decomposed into a direct sum of irreducible representations.
- ▶ The decomposition of a representation into irreducibles is unique up to equivalence.

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## Invariance of $\mathbf{e}_{\pm}$

Let  $\mathbf{e}_{\pm} = \frac{1}{\sqrt{2}} (\mp \mathbf{e}_1 + i \mathbf{e}_2)$ . Then,  $X(\phi) \mathbf{e}_{\pm} = e^{\pm i\phi} \mathbf{e}_{\pm}$ .

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## Decomposition of $X$

The span of each  $\mathbf{e}_{\pm}$  is an  $X$ -invariant subspace of  $V_2$ . In this basis, we rewrite  $X$  as a direct sum of the 1D irreducible representations<sup>3</sup>:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

---

<sup>3</sup>1-dimensional representations are always irreducible!

# Schur's Lemmas (pt. 1)

## Lemma

*Let  $X : G \rightarrow V$  and  $Y : G \rightarrow W$  be irreducible representations of a group  $G$ . If there exists a fixed linear transformation  $T : V \rightarrow W$  such that  $TX(g) = Y(g)T$  for all  $g \in G$ , then  $T$  is either the zero map or invertible.*

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1. The kernel of  $T$  is invariant under  $X(G)$ .
2. The image of  $T$  is invariant under  $Y(G)$ .
3. Since  $X$  and  $Y$  are irreducible,  $\ker(T) = \{\mathbf{0}\}$  and  $\text{im}(T) = V$  or  $\ker(T) = V$  and  $\text{im}(T) = \{0\}$ .

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1. The kernel of  $T$  is invariant under  $X(G)$ .
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4. By the rank-nullity theorem, conclude that  $T$  is either the zero map or invertible.

# Schur's Lemma's (pt. 2)

## Lemma

*Let  $X$  be an irreducible representation of a group  $G$  and  $T$  a linear operator that commutes with all  $X(g)$  for  $g \in G$ . Then  $T$  is a scalar multiple of the identity operator.*

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2. Then  $T - \lambda I$  is not invertible.
3. By assumption,  $(T - \lambda I)X(g) = X(g)(T - \lambda I)$  for all  $g \in G$ .

# Schur's Lemma's (pt. 2)

## Lemma

*Let  $X$  be an irreducible representation of a group  $G$  and  $T$  a linear operator that commutes with all  $X(g)$  for  $g \in G$ . Then  $T$  is a scalar multiple of the identity operator.*

## Proof (sketch)

1. Consider  $\lambda$  to be an eigenvalue of  $T$ .
2. Then  $T - \lambda I$  is not invertible.
3. By assumption,  $(T - \lambda I)X(g) = X(g)(T - \lambda I)$  for all  $g \in G$ .
4. By previous lemma,  $T - \lambda I = 0 \implies T = \lambda I$ .



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5.  $X(G)$  is equivalent to the representation  $g \mapsto \lambda_g$  for all  $g \in G$ .
6. One-dimensional representations are irreducible.



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## How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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## **2 Examples in Physics**

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This *special* property is summarized by noting  $\det R(\phi) = 1$  for all  $\phi \in [0, 2\pi)$ .

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- ▶  $SO(2)$  is *reducible* (earlier example with  $\mathbf{e}_{\pm}$ ).

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- ▶ With  $R(0) = I$  boundary condition:  $R(\phi) = e^{-i\phi J}$ .
- ▶ We call  $J$  the *generator* of SO(2) rotations.

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## Theorem

*The single-valued irreducible representations of  $SO(2)$  are defined as*

$$U^m(\phi) = e^{-im\phi}, \quad \forall m \in \mathbb{Z}.$$

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## Definition

The *special orthogonal group* in three dimensions, denoted  $\text{SO}(3)$ , is the group of all  $3 \times 3$  orthogonal matrices with determinant equal to  $+1$ .  $\text{SO}(3)$  rotations are generated by the components of the Hermitian generator  $\mathbf{J} = [J_x, J_y, J_z]^T$ .

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- ▶ This generalizes to other types of angular momentum, such as *spin angular momentum*!

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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*This is the tip of the iceberg!*



**CAL POLY**

### **3 The Braid Group**

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- ▶ A *braid*  $\beta$  is a *loop*<sup>8</sup> in  $M_n$  and can be thought of as a configuration that evolves over time:

$$\begin{aligned}\beta &: [0, 1] \rightarrow M_n \\ t &\mapsto \beta(t) = (\beta_1(t), \beta_2(t), \dots, \beta_n(t)),\end{aligned}$$

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The *braid group*  $B_n$  is the (fundamental) group of all complex-valued  $n$ -tuples  $(M_n)$  up to *homotopy*.

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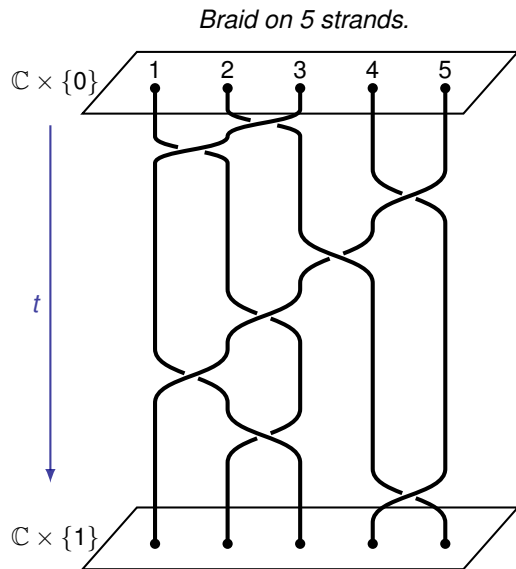
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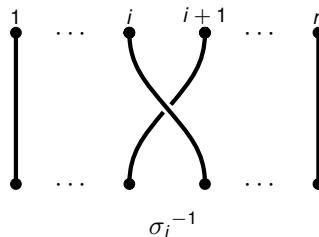
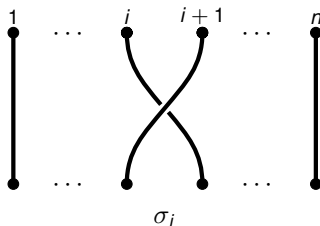
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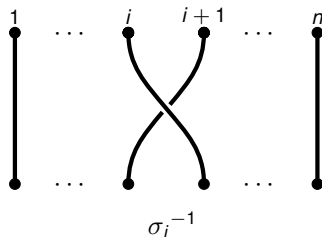
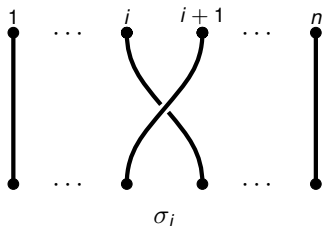
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- ▶ The *degree* of a braid  $\beta \in B_n$  is the sum of the powers of the standard generators in the decomposition of  $\beta$ .

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## Definition

The braid group on  $n$  strands, denoted  $B_n$ , is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right. \right\rangle.$$



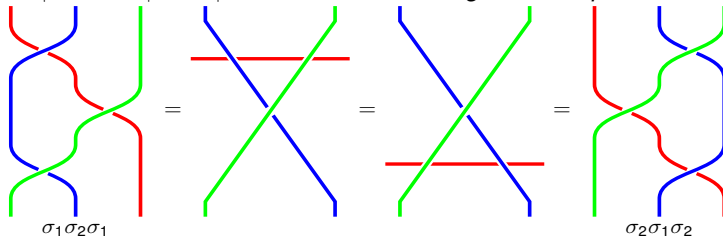
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**Comment:**  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  is known as the *Yang-Baxter equation*, visualized below:



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$$\psi_n : B_n \rightarrow \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$
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The Burau representation satisfies the braid relations:

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## Definition

Define the unitary representation  $\mathcal{U} : B_3 \rightarrow U(2)$  by

$$\mathcal{U}(\sigma_1) = \frac{1}{2}e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}e^{-i\arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$
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**Answer:** Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!



**CAL POLY**

## **4 Physical Applications of the Braid Group**



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**Braiding action:** For any degree- $k$  braid  $\beta \in B_n$ , we have

$$\psi(r_{1'}, r_{2'}, \dots, r_{n'}) = p_\theta(\beta) \psi(r_1, r_2, \dots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase} \\ \text{shift}}} \psi(r_1, r_2, \dots, r_n),$$

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## Remark

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system<sup>9</sup>.

<sup>9</sup>Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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- ▶ Edge cases: *bosons* and *fermions*.



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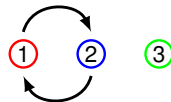
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Trajectory A



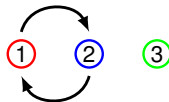
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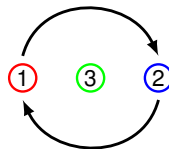
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Trajectory B



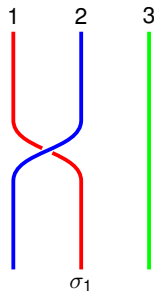
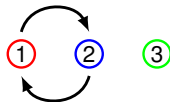
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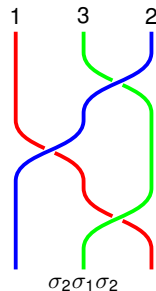
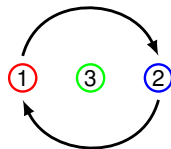
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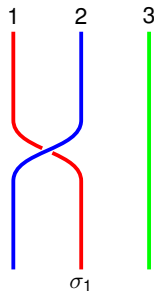
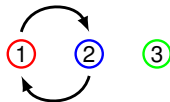
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## 1D representation:

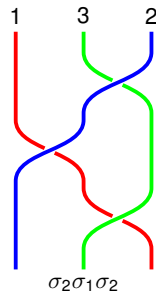
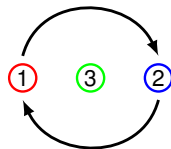
$$\sigma_1 \mapsto e^{i\theta}$$

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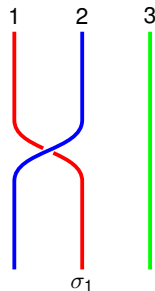
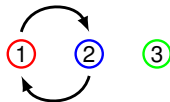
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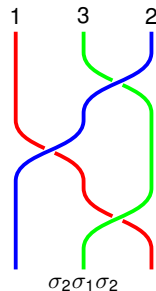
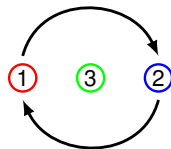
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$$\left. \begin{array}{l} \sigma_1 \mapsto e^{i\theta} \\ \sigma_2 \sigma_1 \sigma_2 \mapsto e^{3i\theta} \end{array} \right\} \neq \text{if } \theta \notin \pi\mathbb{Z}$$

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# A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions  $\mathbf{r}_1 = (x_1, y_1)$  and  $\mathbf{r}_2 = (x_2, y_2)$  in a harmonic potential. Let  $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$  be the relative angle between the two anyons and  $\dot{\phi} = \frac{d\phi}{dt}$ .

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**Generalize to  $N$  anyons:** Let  $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$ ,

$$\mathcal{L} = \sum_{i=1}^N \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i < j}^N \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^N \mathbf{r}_i^2$$

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Rewrite  $N$ -anyon  $\mathcal{L}$ :

$$\mathcal{L} = \frac{m}{2} \sum_{i=1}^N [\dot{\mathbf{r}}^2 - \omega^2 \mathbf{r}_i^2] + \alpha \sum_{i < j} \dot{\mathbf{r}}_{ij} \cdot \frac{(-y_{ij} \hat{x} + x_{ij} \hat{y})}{r_{ij}^2}$$

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$$\mathcal{L} = \frac{m}{2} \sum_{i=1}^N [\dot{\mathbf{r}}^2 - \omega^2 \mathbf{r}_i^2] + \alpha \sum_{i < j} \dot{\mathbf{r}}_{ij} \cdot \frac{(-y_{ij} \hat{x} + x_{ij} \hat{y})}{r_{ij}^2}$$

Gauge potential:

$$\mathbf{A}_i(\mathbf{r}_i) = \alpha \sum_{j \neq i} \frac{\hat{z} \times \mathbf{r}_{ij}}{r_{ij}^2} = \alpha \sum_{j \neq i} \frac{-y_{ij} \hat{x} + x_{ij} \hat{y}}{r_{ij}^2}$$

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$i$ -th anyon Hamiltonian:

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Expand:

$$\mathcal{H} = \frac{1}{2m} \sum_{i=1}^N p_i^2 + \frac{m\omega^2}{2} \sum_{i=1}^N r_i^2 - \frac{\alpha}{2m} \sum_{\substack{i=1 \\ j \neq i}}^N \frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m} \sum_{\substack{i=1 \\ j, k \neq i}}^N \frac{\mathbf{r}_{ij} \cdot \mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}$$

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## Question

Why is this useful?

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- ▶ Depending on the specific representation of the braid group, one can define topological properties of different flavors of anyons. The corresponding combination (fusion) rules can be used to describe the behavior of anyonic systems.
- ▶ Certain nonabelian anyons (defined by specific fusion rules) are alleged to enable fault-tolerant quantum computers, and is an ongoing area of research.

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**Thank you for your attention!**

# SO(3) Calculations (pt. 1)

The state  $|\phi\rangle$  can be decomposed into a linear combination of the eigenvectors of  $J$ :

$$|\phi\rangle = \left( \sum_m |m\rangle \langle m| \right) |\phi\rangle = \sum_m \langle m|\phi\rangle |m\rangle,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi)m|\mathcal{O}\rangle = \langle e^{im\phi}m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of  $|\phi\rangle$  onto the eigenvector  $|m\rangle$  of  $J$ .

Thus,

$$\begin{aligned} J|\phi\rangle &= \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i \frac{\partial}{\partial \phi} (e^{-im\phi} |m\rangle) = i \frac{\partial}{\partial \phi} |\phi\rangle \\ \implies \langle \phi|J|\psi\rangle &= \langle J^\dagger \phi|\psi\rangle = -i \frac{\partial}{\partial \phi} \langle \phi|\psi\rangle = -i \frac{\partial}{\partial \phi} \psi(\phi). \end{aligned}$$

## SO(3) Calculations (pt. 2)

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial \phi} = (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z \implies J = -i \frac{\partial}{\partial \phi} = -i (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z = \frac{1}{\hbar} \hat{L}_z \implies \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,$$

where the last line easily generalizes to  $\hat{\mathbf{L}}$ .

# Lie Algebra

$$\mathcal{J}^2 |j\rangle = (J_- J_+ + J_z + \mathcal{J}_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$\mathcal{J}^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle .$$