# Representation Theory and its Applications in Physics

June 5, 2024

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**Presented by**
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**Outline:**

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

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# Definition of a Representation

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**Remark**

If $V$ is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then $X$ can be realized as an $n \times n$ matrix.

# Properties of Representations

## Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

$$X(gh) = X(g)X(h), \quad \forall g, h \in G.$$

## Invertibility

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2. In the matrix presentation of $X$, $X(g)$ is invertible for all $g \in G$.

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- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

# Example: A Faithful Representation of $S_n$

## Defining representation of $S_n$

The defining representation $D$ of $S_n$ encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends $i$ to $j$, then place a 1 the $i$-th column and $j$-th row of the representation matrix.

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E.g., in $S_3$:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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- This representation is faithful.

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**Representation:** Let $X$ be a representation of $R$ on $V_2$ with[1]

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**Question**

How do we classify representations of a group?

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- We can use characters to classify representations.

# Decomposing Representations

**Definition**

A representation $X(G)$ on $V$ is *irreducible* if there is no non-trivial invariant subspace[2] in $V$ with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to equivalence.

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**Decomposition of $X$**

The span of each $\mathbf{e}_\pm$ is an $X$-invariant subspace of $V_2$. In this basis, we rewrite $X$ as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

# Schur's Lemmas (pt. 1)

**Lemma**

*Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group $G$. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then $T$ is either the zero map or invertible.*

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4. By the rank-nullity theorem, conclude that $T$ is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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**Corollary**

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4. The element $h$ was arbitrary, so $X(g) = \lambda_g I$ for all $g \in G$.
5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.

# Consequence of Schur's Lemmas

## Corollary

*If $G$ is a finite abelian group, then the irreducible representations of $G$ are one-dimensional.*

## Proof (sketch)

1. Fix $h \in G$.
2. Since $G$ is abelian, $X(h)X(g) = X(g)X(h)$ for all $g \in G$.
3. Schur's second lemma implies $X(h) = \lambda_h I$ for some scalar $\lambda_h$.
4. The element $h$ was arbitrary, so $X(g) = \lambda_g I$ for all $g \in G$.
5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.
6. One-dimensional representations are irreducible.

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**How does this help in physics?**

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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# Preliminaries

**Skip preliminaries?**

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator $A$ on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

# Orthonormality, Completeness, and Wavefunctions

**Definition**

Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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**Definition**

For a continuous basis labelled by $|x\rangle$ where $x$ is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

# Preliminaries: Basic Quantum Mechanics

- Talk about probabilities and whatnot? Eigenvalues = observables? Or just mention when connecting later stuff to physics?

# Properties of 2D Rotations

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

# The SO(2) Group

**Definition**

The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all $2 \times 2$ orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- With $R(0) = I$ boundary condition: $\boxed{R(\phi) = e^{-i\phi J}}$.
- We call $J$ the *generator* of SO(2) rotations.

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# Recovering the Rotation Matrix from $J$

To first order in $d\phi$: $R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}$

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J\,|m\rangle &= m\,|m\rangle\,, \\
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**Theorem**

*The single-valued irreducible representations of SO(2) are defined as*

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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- The standard generators along each axis $\{J_x, J_y, J_z\}$ form a basis for all rotation generators: $J_{\mathbf{n}} = n_x J_x + n_y J_y + n_z J_z$.

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**Consequence:** Any rotation in Euclidean 3-space can be written in terms of the generators:

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**Definition**

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all $3 \times 3$ orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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# Connection to Quantum Mechanics: Punchline

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**Discretization of Angular Momentum for Free**

Arguably the most defining characteristic of quantum mechanics is that classically measurable quantities become discretized (quantized) when observed on the quantum scale. Without any physical motivation, the irreducible representations of SO(3) gave it to us for free!

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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*This is the tip of the iceberg!*

CPSectionPage169

# The Braid Group

- Definition: config space and standard visualization

# Standard Generators

- $\sigma_i$ generators.
- Define *degree*?
- Braid relations.
- Skip YBE verification?

# Automorphisms of the Free Group

- Automorphisms of $\pi_1(\mathbb{D}_n)$.
- Braid relations in this picture.

# One-Dimensional Representations of the Braid Group

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Hence, for any $\beta \in B_n$ with degree $k$:

$$p_\theta(\beta) = p_\theta(\sigma_1^{m_1}\sigma_2^{m_2}\cdots\sigma_{n-1}^{m_{n-1}}) = e^{i\theta(m_1+m_2+\cdots+m_{n-1})} = e^{ik\theta}.$$

# The Burau Representation

- Go through arguments/motivation for Burau?
- Show covering space picture/diagrams?
- Define Burau representation.
- Note on faithfulness!
- Quickly show it's reducible with the **1** eigenvector?

# Unitary Representation of the Braid Group

- Define reduced Burau representation.
- Obtain unitary representation from reduced Burau. (Not sure how much detail to go into here.)
- Maybe just jump right to defining the unitary reps in the $2 \times 2$ case?
- Comment on why we want a unitary rep!

# Nonabelian Characteristics of the Unitary Representation

- Compare and contrast $\mathcal{U}(\sigma_{1,2})$ to their inverses.
- Note that $[\mathcal{U}(\sigma_{1,2}), \mathcal{U}(\sigma_{2,1})] \neq 0$ to highlight nonabelian-ness.

**Question**

What are the physical implications of this nonabelian representation?

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# (Abelian) Braiding Action on a Quantum System

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**Braiding action:** For any degree-$k$ braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \dots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \dots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \dots, r_n),$$

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**Braiding action:** The transformed basis states due to the action of $\sigma_1$ are

$$|1'\rangle = \mathcal{U}(\sigma_1)_{1,1}\,|1\rangle + \mathcal{U}(\sigma_1)_{1,2}\,|2\rangle = \frac{1}{2}e^{-i\frac{\pi}{6}}\left(\sqrt{3}\,e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|1\rangle + |2\rangle\right),$$

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**Remark**

The action of a nonabelian braid group representation on a quantum system leads to nontrivial rotations in the many-particle Hilbert space that describes the quantum system.

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# Nontrivial Braiding Effects in 1D Representations

- Introduce the idea (define it).
- Show diagram to illustrate nontrivial braiding effects qualitatively.
- Nontrivial braiding in 1D rep corresponding to diagram.
- Hint at a greater conclusion but first need to look into the physics perspective...

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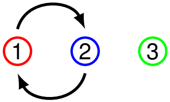
**Trajectory A**

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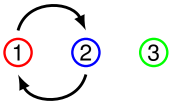
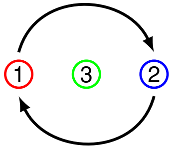



**Trajectory B**

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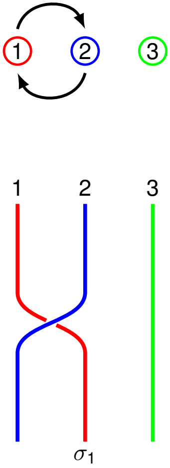


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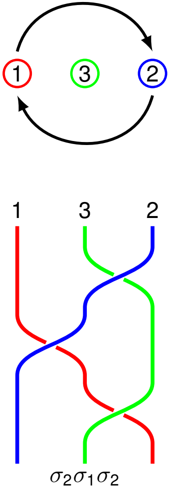

# A Physicists Approach to Anyons (pt. 1)

- Non-interacting anyons.

## A Physicists Approach to Anyons (pt. 2)

- Non-interacting anyons in harmonic potential.
- Arrive at $N$-anyon Hamiltonian.

# Nontrivial Braiding Effects in the Hamiltonian

- Compare $N = 2$ to $N = 3$ Hamiltonian from previous slide.
- Highlight nontrivial braiding effects in the $N = 3$ case.
- How does this compare to bosons/fermions? (maybe redundant depending on the depth of the previous discussion)
- Question the physical implications.

# Physical Implications of Nontrivial Braiding Effects

- FQHE
- Fault-tolerant quantum computing

# Summary/Conclusion

- Outline the talk: what did we talk about?
- What are the takeaways?
- Acknowledgements, questions, references (?)