Title

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April 13, 2024

Contents

Chapter 4

Anyons

4.1 Braiding action on a quantum system

As discussed in [4], each of the standard generators of the braid group can be realized as unitary operators on a quantum system. For the one-dimensional representations of the braid group (Section 3.5), the action on a quantum system is merely a phase shift by $e^{ik\theta}$ for some $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}$ the degree of the braid. We can see this explicitly by considering the action of $\beta \in B_n$ on a wavefunction $\psi(r_1, \ldots, r_n)$ by permuting the identical particles fixed at positions $r_1, r_2, \ldots, r_n$:

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\psi(r_1, r_2, \ldots, r_n) = e^{ik\theta}\psi(r_1, r_2, \ldots, r_n),$$

where $r_{i'}$ denotes particle i's position after the braid β has been applied.

Physically, we can think of each σ_i as a clockwise exchange of identical particles i and $i+1$ that live in 2 spatial dimensions. In general, particles that obey the braid group permutation rules are known as *anyons*, and their statistics are determined by the braid group representation that describes the system. The analysis of anyons will continue in the later sections of this chapter.

Higher degree representations of the braid group are generally nonabelian. Thus, the action of the braid group on a quantum system will not always be as straightforward as a phase shift. This is a direct consequence of the *nontrivial braiding effects*, or nonabelian nature, of the braid group.

In Example 3.4, a 2-dimensional, nonabelian, unitary representation of B_3 was constructed. Consider a degenerate set of two quantum states with orthonormal basis $\psi_1(r_1, r_2, r_3)$ and $\psi_2(r_1, r_2, r_3)$. Each basis state can be thought of a column vector, written as $|1\rangle$ and $|2\rangle$, for ψ_1 and ψ_2 respectively.

For the unitary representation $\mathcal{U}$ constructed in Example 3.4, the action of σ_1 on the basis states is given by

$$\begin{aligned}\mathcal{U}(\sigma_1)\,|1\rangle &= \frac{1}{2}e^{-i\frac{\pi}{6}}\left(\sqrt{3}\,e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|1\rangle + |2\rangle\right),\\ \mathcal{U}(\sigma_1)\,|2\rangle &= \frac{1}{2}e^{-i\frac{\pi}{6}}\left(|1\rangle - \sqrt{3}\,e^{-i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|2\rangle\right).\end{aligned}$$

Similarly, the action of σ_2 on the basis states yields

$$\begin{aligned}\mathcal{U}(\sigma_2)\,|1\rangle &= \frac{1}{2}e^{-i\frac{\pi}{6}}\left(-\sqrt{3}\,e^{-i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|1\rangle + |2\rangle\right),\\ \mathcal{U}(\sigma_2)\,|2\rangle &= \frac{1}{2}e^{-i\frac{\pi}{6}}\left(|1\rangle + \sqrt{3}\,e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)}\,|2\rangle\right).\end{aligned}$$

As a result of these nonabelian braiding effects, we can see that the action of the braid group on this system corresponds to nontrivial rotations in the many-particle Hilbert space that describes the quantum system [10, 4].

More generally, if we have a set of $m \geq 2$ degenerate states in terms of $r_1, \dots, r_n$ with orthonormal basis $\psi_1, \psi_2, \dots, \psi_m$, then the basis states transform under the action of $\sigma_k \in B_n$ as

$$\psi_i' = \sum_j [\Xi(\sigma_k)]_{ij}\psi_j,$$

where $[\Xi(\sigma_k)]_{ij}$ is the (i, j)-th entry of the unitary matrix $\Xi(\sigma_k)$ for some representation $\Xi : B_n \to U(m)$.

Insert concluding paragraph here! *Comment on why we care about getting the Lagrangian and Hamiltonian for anyons? In general?*

4.2 Two Non-Interacting Anyons

The interaction term in the Lagrangian for two anyons due to the braiding of the anyons is given by

$$\mathcal{L}_{\text{int}} = \hbar\alpha\dot{\phi}, \tag{4.1}$$

where a dot indicates a total time derivative $\frac{d}{dt}$ and $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ is the relative angle between the two anyons with positions $\vec{r}_1 = (x_1, y_1)$ and $\vec{r}_2 = (x_2, y_2)$. As in the previous section, $\alpha \in [0, 1]$ is the *winding angle* or braiding statistic of the anyons. The parameter α can also be thought of as an angle modulo π. Though the relative angle ϕ is ambiguous for identical particles, the derivative $\frac{d\phi}{dt} = \dot{\phi}$ is well-defined.

Notice that if we take $\alpha \to 0$, the interaction term vanishes as expected for bosons. Similarly, for $\alpha > 0$, ϕ becomes singular if $\vec{r}_1 = \vec{r}_2$, which motivates the Pauli exclusion principle for fermions. In fact, this means that for any $\alpha > 0$, the corresponding anyons exhibit some form of the Pauli exclusion principle.

The classical kinetic energy of this system is

$$T = \frac{1}{2}m\left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}\right), \tag{4.2}$$

as expected. Then the Lagrangian for this system is

$$\mathcal{L}\left(\dot{\vec{r}}_1, \dot{\vec{r}}_2, \dot{\phi}\right) = T + \mathcal{L}_{\text{int}} = \frac{1}{2}m\left(\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}\right) + \hbar\alpha\dot{\phi}, \tag{4.3}$$

which can also be viewed as the Lagrangian for 2 interacting bosons/fermions.

We can redefine the Lagrangian in terms of the relative and center-of-mass coordinates

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}, \tag{4.4}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \tag{4.5}$$

where $\vec{r}$ is the relative position vector and $\vec{R}$ is the center-of-mass position vector of the two anyons. Note that we are assuming that the mass of the two particles are equal ($m_1 = m_2$). Classically, the momentum of a particle is given by the product of its mass and velocity. Then the corresponding center-of-mass and relative momenta:

$$\vec{P} = 2m\dot{\vec{R}} = 2m\frac{\dot{\vec{r}}_1 + \dot{\vec{r}}_2}{2} = m\dot{\vec{r}}_1 + m\dot{\vec{r}}_2 = \vec{p}_1 + \vec{p}_2, \tag{4.6}$$

$$\vec{p} = \mu\dot{\vec{r}} = \frac{m}{2}\left(\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right) = \frac{\vec{p}_1 - \vec{p}_2}{2}, \tag{4.7}$$

where m is the mass of each anyon and μ is the reduced mass of the system.

With this in mind, we derive the following identity:

$$\dot{\vec{R}} + \frac{1}{4}\dot{\vec{r}} = \frac{\left(\dot{\vec{r}}_1 + \dot{\vec{r}}_2\right)^2}{4} + \frac{\left(\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right)^2}{4} = \frac{\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2}}{2}. \tag{4.8}$$

Thus, the Lagrangian decomposes into relative and center-of-mass components upon making the substitution from the above identity:

$$\mathcal{L} = \underbrace{m\dot{\vec{R}}^2}_{\mathcal{L}_R} + \underbrace{\frac{m\dot{\vec{r}}^{\,2}}{4} + \hbar\alpha\dot{\phi}}_{\mathcal{L}_r}, \tag{4.9}$$

where the squared velocities indicate magnitude squared. Observe that the center-of-mass component of the Lagrangian, $\mathcal{L}_R$, is independent of the braiding parameter α. We can further simplify the relative component of the Lagrangian, $\mathcal{L}_r$, by noting that we can briefly write the coordinate $\vec{r}$ in polar form by representing it as a complex number $\vec{r} = re^{i\phi}$. It follows that

$$\left|\dot{\vec{r}}(r,\phi)\right|^2 = \left|\frac{d}{dt}\vec{r}(r,\phi)\right|^2 = \left|\left(\dot{r} + ir\dot{\phi}\right)e^{i\phi}\right|^2 = \dot{r}^2 + r^2\dot{\phi}^2. \tag{4.10}$$

Hence, we rewrite the relative component of the Lagrangian as

$$\mathcal{L}_r = \frac{m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)}{4} + \hbar\alpha\dot{\phi}. \tag{4.11}$$

Recall that the classical relative angular momentum can be described by:

$$p_\phi = \frac{d\mathcal{L}}{d\dot{\phi}} = \frac{mr^2}{2}\dot{\phi} + \hbar\alpha. \tag{4.12}$$

Now, the Hamiltonian for this system can be constructed:

$$\begin{aligned}\mathcal{H} &= P\dot{R} + p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L} \\ &= \frac{P^2}{4m} + \frac{p_r^2}{m} + \frac{mr^2}{4}{p_\phi}^2 \\ &= \frac{P^2}{4m} + \frac{p_r^2}{m} + \frac{\left(p_\phi - \hbar\alpha\right)^2}{mr^2}.\end{aligned} \tag{4.13}$$

Once again, the center-of-mass component of the Hamiltonian is independent of α, and so we can focus on the relative component of the Hamiltonian, which is

$$\mathcal{H}_r = \frac{p_r^2}{m} + \frac{\left(p_\phi - \hbar\alpha\right)^2}{mr^2}. \tag{4.14}$$

For the purposes of this work, we need not carry out to find the energy eigenstates corresponding to the quantum operator of this relative Hamiltonian. More about this is found in [8].

4.3 Anyons in Harmonic Potential

Placing anyons in a harmonic potential alters the Hamiltonian obtained in Section 4.2. The potential energy of a 2-anyon system is given by

$$V(\vec{r}_1, \vec{r}_2) = \frac{1}{2}m\omega^2\left(\vec{r}_1^{\,2} + \vec{r}_2^{\,2}\right) = m\omega^2\left(\vec{R}^2 + \frac{1}{4}\vec{r}^{\,2}\right), \tag{4.15}$$

where ω is the angular frequency of the harmonic potential. We can make the same substitution as in Section 4.2 to write the potential in terms of the relative and center-of-mass coordinates. As is a recurring theme, the center-of-mass component of the potential has no dependence on the braiding parameter α, and corresponds to a 2-dimensional quantum harmonic oscillator problem for a particle of mass $2m$.

Note that we can generalize Eqn. 4.11 (now omitting the subscript r) to an N-anyon system in a harmonic potential by writing

$$\mathcal{L} = \sum_{i=1}^{N}\frac{m}{2}\dot{\vec{r}}_i^{\,2} + \hbar\alpha\sum_{\substack{i=1\\ j\neq i}}^{N}\dot{\phi}_{ij} - \frac{m\omega^2}{2}\sum_{i=1}^{N}\vec{r}_i^{\,2}, \tag{4.16}$$

where $\phi_{ij} = \arctan\left(\frac{y_i - y_j}{x_i - x_j}\right)$ is the relative angle between anyons i and j. For brevity, we write $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ to denote the relative coordinates between the anyons.

More generally, let $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ be the relative coordinate between anyons i and j, and define $r_{ij}^2 = |\vec{r}_{ij}|^2$. Then we can solve directly for $\dot{\phi}_{ij}$ as follows:

$$
\begin{aligned}
\dot{\phi}_{ij} = \frac{d\phi_{ij}}{dt} = \frac{d}{dt}\arctan\left(\frac{y_{ij}}{x_{ij}}\right) &= \frac{\frac{d}{dt}\left(\frac{y_{ij}}{x_{ij}}\right)}{1 + \left(\frac{y_{ij}}{x_{ij}}\right)^2} \\
&= \frac{x_{ij}\dot{y}_{ij} - \dot{x}_{ij}y_{ij}}{x_{ij}^2\left[1 + \left(\frac{y_{ij}}{x_{ij}}\right)^2\right]} \\
&= \frac{x_{ij}\dot{y}_{ij} - \dot{x}_{ij}y_{ij}}{x_{ij}^2 + y_{ij}^2} \\
&= \frac{\vec{r}_{ij} \times \dot{\vec{r}}_{ij}}{r_{ij}^2}.
\end{aligned}
$$

Setting $\hbar = 1$, we can rewrite the Lagrangian as [2]

$$
\mathcal{L} = \frac{m}{2}\sum_{i=1}^{N}\left[\dot{\vec{r}}^2 - \omega^2\vec{r}_i^2\right] + \alpha\sum_{i<j}^{N}\frac{\vec{r}_{ij} \times \dot{\vec{r}}_{ij}}{r_{ij}^2}, \tag{4.17}
$$

which can be expanded as

$$
\mathcal{L} = \frac{m}{2}\sum_{i=1}^{N}\left[\dot{\vec{r}}^2 - \omega^2\vec{r}_i^2\right] + \alpha\sum_{i<j}^{N}\dot{\vec{r}}_{ij} \cdot \frac{(-y_{ij}\hat{x} + x_{ij}\hat{y})}{r_{ij}^2}. \tag{4.18}
$$

The last term in Eqn. 4.18 is of similar form to the vector (gauge) potential associated with the i-th anyon [8, 2, 9]:

$$
\vec{A}_i(\vec{r}_i) = \alpha\sum_{j\neq i}\frac{\hat{z} \times \vec{r}_{ij}}{r_{ij}^2} = \alpha\sum_{j\neq i}\frac{-y_{ij}\hat{x} + x_{ij}\hat{y}}{r_{ij}^2}, \tag{4.19}
$$

where $\hat{z}$ is the unit vector perpendicular to the r_{ij}-plane. Here, α serves as the coupling constant, which dictates the strength of the interaction between anyons in the system.

For the i-th anyon, the contribution to the Hamiltonian can be written as

$$\mathcal{H}_i = \frac{1}{2m}\left(\vec{p}_i - \vec{A}_i(\vec{r}_i)\right)^2 + \frac{m\omega^2}{2} r_i^2, \tag{4.20}$$

where $p_i - A_i(\vec{r}_i)$ is known as the *canonical momentum* of the system. This is a required modification since we must account for the motion of the anyons in the presence of the gauge potential in addition to their mechanical momentum.

Then, with only essential coupling between the anyons (due to the gauge potential), the Hamiltonian for the N-anyon system in a harmonic potential is given by

$$\mathcal{H} = \frac{1}{2m}\sum_{i=1}^{N}\left(\vec{p}_i - \vec{A}_i(\vec{r}_i)\right)^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N} r_i^2. \tag{4.21}$$

Substituting Eqn. 4.19 into Eqn. 4.21, we have

$$\mathcal{H} = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N} r_i^2 - \frac{\alpha}{2m}\sum_{\substack{i=1\\ j\neq i}}^{N} \frac{\vec{\ell}_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{\substack{i=1\\ j,k\neq i}}^{N} \frac{\vec{r}_{ij}\cdot\vec{r}_{ik}}{r_{ij}^2 r_{ik}^2}, \tag{4.22}$$

where $\vec{\ell}_{ij} = (\vec{r}_i - \vec{r}_j)\times(\vec{p}_i - \vec{p}_j)$ is the relative angular momentum of anyon i and j.

4.3.1 Nontrivial braiding effects

The last term in Eqn. 4.22, which can be thought of as a long-range 2- and 3-particle interaction term, highlights the key difference between the braid group and the symmetric group when studying the physics of a multi-particle system. As previously established, anyons obeying braiding statistics exhibit nontrivial braiding effects.

To better understand these nontrivial braiding effects, first consider two different trajectories of anyons, as illustrated in Figure 4.1 [8]. From a top-down perspective, trajectory A involves a clockwise exchange of anyons 1 and 2 while anyon 3 does not take part in the exchange. In trajectory B, anyon 1 is again swapped with anyon 2 in a clockwise fashion, but anyon 3 is now involved in the exchange by being in between anyons 1 and 2. This is where the differences between the symmetric group and the braid group become apparent. In the symmetric group, both trajectories would be equivalent, but in the braid group, the two trajectories are distinct. This claim is verified by comparing the braid representation of each trajectory in the bottom row of Figure 4.1.

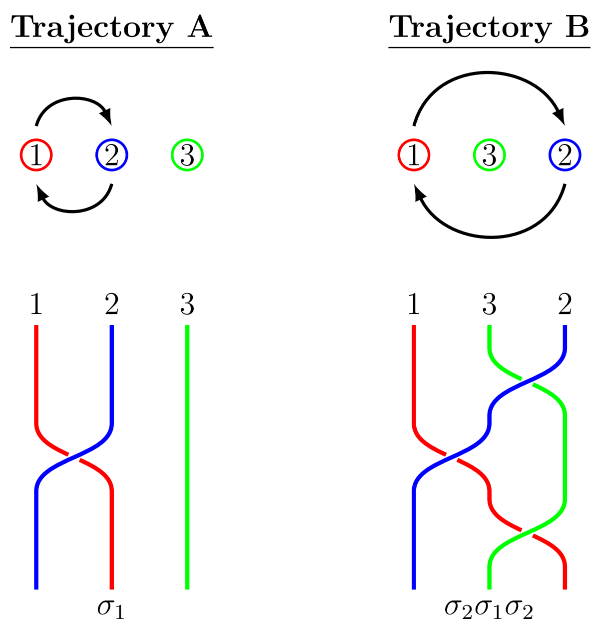

Figure 4.1: Two possible trajectories of three anyons viewed from above (top) and represented as braids (bottom).

Even in the one-dimensional (abelian) representation of the braid group, the nontrivial braiding effects emerge for the trajectories depicted in Figure 4.1. The braid representations for trajectory A and trajectory B are $\sigma_1 \mapsto e^{i\theta}$ and $\sigma_2\sigma_1\sigma_2 \mapsto e^{3i\theta}$, respectively. As long as the choice of θ is not an integer multiple of π, it follows that $e^{i\theta} \neq e^{3i\theta}$. This is precisely the nontrivial braiding effect. If there were some wavefunction $\psi(r_1, r_2, r_3)$ that described the system, then the action of the braid group on ψ would yield different phase changes for the two trajectories. Despite the fact that both trajectories involve the same exchange of anyons 1 and 2, the relative position of anyon 3 played a significant role in the resultant effect of particle exchange on the system. It is then evident that for a system of N anyons, every particle exchange must also simultaneously consider the relative positions of all other anyons in the system in order to properly encode the braiding action.

Clearly, nontrivial braiding affects must be accounted for when studying a system that obeys braiding statistics. In fact, nontrivial braiding is already encoded into the Hamiltonian found earlier in this section. For a 2-anyon system, the long-range interaction term manifests in a familiar form. Isolating the last term in Eqn. 4.22 for $N = 2$ anyons, we see that

$$\frac{\alpha^2}{2m}\left(\frac{\vec{r}_{12}\cdot\vec{r}_{12}}{r_{12}^2 r_{12}^2}+\frac{\vec{r}_{21}\cdot\vec{r}_{21}}{r_{21}^2 r_{21}^2}\right)=\frac{\alpha^2}{m r_{12}^2},$$

which is analogous to the Coulombic interaction between two charged particles. The nontrivial braiding effects are not present in the 2-anyon system.

Only when we have $N \geq 3$ anyons will the nontrivial braiding effects truly present themselves in the long-range interaction term of the Hamiltonian. For example, if we take $N = 3$, the last term in Eqn. 4.22 becomes

$$\frac{\alpha^2}{m}\left(\underbrace{\frac{1}{r_{12}^2}+\frac{1}{r_{13}^2}+\frac{1}{r_{23}^2}}_{\text{Coulomb-like interaction}}+\underbrace{\frac{\vec{r}_{12}\cdot\vec{r}_{13}}{r_{12}^2 r_{13}^2}+\frac{\vec{r}_{21}\cdot\vec{r}_{23}}{r_{21}^2 r_{23}^2}+\frac{\vec{r}_{31}\cdot\vec{r}_{32}}{r_{31}^2 r_{32}^2}}_{\text{Nontrivial braiding}}\right).$$

With three anyons, the nontriviality of the long-range interaction becomes clear, as there are now cross terms that involve the relative positions of all three anyons. This result is similar to the observations made in Figure 4.1. The consequence of these three-body interaction terms is that the system is no longer separable into independent 2-anyon systems, which greatly complicates the quantum mechanics and statistical analysis of the system [8]. The nontrivial braiding effects of anyonic systems highlights the rich complexity of the braid group and its physical implications.

Appendix A

Multi-anyon system with harmonic potential

A.1 Gauge Theory and the Hamiltonian

A.2 Hamiltonian Terms

The last term in Eqn. 4.22 is the result of squaring the canonical momentum in Eqn. 4.21. To see this, let's isolate one of the terms. Fix i. Then,

$$\left(\vec{p}_i - \vec{A}_i(\vec{r}_i)\right)^2 = p_i^2 - 2\vec{p}_i \cdot \vec{A}_i(\vec{r}_i) + A_i^2(\vec{r}_i).$$

By Eqn. 4.19, we have

$$\vec{A}_i^2(\vec{r}_i) = \left(\alpha \sum_{j\neq i} \frac{-y_{ij}\hat{x} + x_{ij}\hat{y}}{r_{ij}^2}\right)^2 = \alpha^2 \sum_{j,k\neq i} \frac{y_{ij}y_{ik} + x_{ij}x_{ik}}{r_{ij}^2 r_{ik}^2} = \alpha^2 \sum_{j,k\neq i} \frac{\vec{r}_{ij} \cdot \vec{r}_{ik}}{r_{ij}^2 r_{ik}^2},$$

which is the last term in Eqn. 4.22.

Moreover, the cross term in the expansion of $\left(\vec{p}_i - \vec{A}_i(\vec{r}_i)\right)^2$ is

$$\begin{aligned}
-2\vec{p}_i \cdot \vec{A}_i(\vec{r}_i) &= -2\vec{p}_i \cdot \left(\alpha \sum_{j\neq i} \frac{-y_{ij}\hat{x} + x_{ij}\hat{y}}{r_{ij}^2}\right) \\
&= -2\alpha \sum_{j\neq i} \frac{\vec{p}_i \cdot (-y_{ij}\hat{x} + x_{ij}\hat{y})}{r_{ij}^2} \\
&= -2\alpha \sum_{j\neq i} \frac{-p_{ix}y_{ij} + p_{iy}x_{ij}}{r_{ij}^2} \\
&= -2\alpha \sum_{j\neq i} \frac{(\vec{r}_{ij} \times \vec{p}_i) \cdot \hat{z}}{r_{ij}^2}.
\end{aligned}$$

For each j, there is a corresponding term in Eqn. 4.22 with

$$-2\alpha \frac{(\vec{r}_{ji} \times \vec{p}_j) \cdot \hat{z}}{r_{ji}^2} = -\alpha \frac{(\vec{r}_{ji} \times \vec{p}_j) \cdot \hat{z}}{r_{ij}^2} + \alpha \frac{(\vec{r}_{ij} \times \vec{p}_j) \cdot \hat{z}}{r_{ij}^2},$$

where we rewrote one of the two terms to have $\vec{r}_{ij}$ instead of $\vec{r}_{ji}$. Then, for fixed i and j, the ij- and ji-term can be combined in the following manner:

$$\begin{aligned}
-2\alpha \frac{(\vec{r}_{ij} \times \vec{p}_i) \cdot \hat{z}}{r_{ji}^2} - 2\alpha \frac{(\vec{r}_{ji} \times \vec{p}_j) \cdot \hat{z}}{r_{ji}^2} &= -\alpha \frac{(\vec{r}_{ij} \times \vec{p}_i) \cdot \hat{z}}{r_{ij}^2} + \alpha \frac{(\vec{r}_{ji} \times \vec{p}_i) \cdot \hat{z}}{r_{ji}^2} \\
&\quad + \alpha \frac{(\vec{r}_{ij} \times \vec{p}_j) \cdot \hat{z}}{r_{ij}^2} - \alpha \frac{(\vec{r}_{ji} \times \vec{p}_j) \cdot \hat{z}}{r_{ji}^2} \\
&= -\alpha \frac{(\vec{r}_{ij} \times (\vec{p}_i - \vec{p}_j)) \cdot \hat{z}}{r_{ij}^2} \\
&\quad - \alpha \frac{(\vec{r}_{ji} \times (\vec{p}_j - \vec{p}_i)) \cdot \hat{z}}{r_{ji}^2} \\
&= -\alpha \frac{(\vec{r}_{ij} \times \vec{p}_{ij}) \cdot \hat{z}}{r_{ij}^2} + \alpha \frac{(\vec{r}_{ji} \times \vec{p}_{ji}) \cdot \hat{z}}{r_{ji}^2} \\
&= -\alpha \frac{\vec{\ell}_{ij}}{r_{ij}^2} - \alpha \frac{\vec{\ell}_{ji}}{r_{ji}^2}.
\end{aligned}$$

Then, summing over all $i \neq j$ yields the second-to-last term in Eqn. 4.22.

Chapter 4

To-Do List

- Redo the Chapter 1 with nicer notation and stray away from Tung's notation when possible.
- Finish/modify irreducible rep. example in Chapter 1.
- Lorentz group example

- Show $\psi_n(\sigma_i)$ invertible? Yes, eventually
- derive $\psi_n^{\mathbf{r}}(\sigma_i)$ matrices or state?
- Show $\psi_n^{\mathbf{r}}(\sigma_i)$ invertible? Yes, eventually
- ~~Explicitly show why Burau isn't able to be made unitary? [3]~~
- Separate chapters into braid group and braid group reps.?

- Concluding paragraph on first section to lead into the more physics-y stuff.
- ~~Show the additional cross terms from $N = 2$ to $N = 3$ and beyond.~~
- Add paragraph on gauge theory/motivation.
- Anyon fusion rules

- τ anyon/Fibonacci anyon example. Relate to singlet/triplet states in spin-1/2 system.
- ~~Move anyon calculations to appendix?~~

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