Title

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Chapter 1

Background Info

Definition 1.1 (Representations of a Group)**.** If there is a homomorphism from a group G to a group of operators $U(G)$ on a linear vector space V, we say that $U(G)$ forms a *representation* of G with dimension $\dim V$.

The representation is a map

$$g \in G \xrightarrow{U} U(g) \tag{1.1}$$

in which $U(g)$ is an operator on the vector space V. For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \ldots, n\}$, we can realize each operator $U(g)$ as an $n \times n$ matrix $D(g)$.

$$U(g) \left| e_i \right\rangle = \sum_{j=1}^{n} \left| e_j \right\rangle D(g)^j{}_i = \left| e_j \right\rangle D(g)^j{}_i, \tag{1.2}$$

where the first index j is the row index and the second index i is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$U(g_1)U(g_2) = U(g_1 g_2), \tag{1.3}$$

which satisfies the group multiplication rules.

Definition 1.2. If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

Example 1.1. Let G be the group of continuous rotations in the xy-plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space V_2. Then we define a representation of G on V_2 by the familiar rotation operation

$$\hat{e}_1' = U(\phi)\hat{e}_1 = \hat{e}_1 \cdot \cos\phi + \hat{e}_2 \cdot \sin\phi \tag{1.4}$$
$$\hat{e}_2' = U(\phi)\hat{e}_2 = -\hat{e}_1 \cdot \sin\phi + \hat{e}_2 \cdot \cos\phi. \tag{1.5}$$

This gives us the matrix representation

$$D(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1.6}$$

To further illustrate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \vec{x} \in V_2$, then we have

$$\vec{x}' = U(\phi)\vec{x} = \hat{e}_j x'^j, \tag{1.7}$$

where $x'^j = D(\phi)^j{}_i x^i$.

Definition 1.3 (Equivalence of Representations)**.** For a group G, two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define

Definition 1.4 (Characters of a Representation)**.** The *character* $\chi(g)$ of an element $g \in G$ in a representation $U(g)$ is defined as $\chi(g) = \text{Tr } D(g)$.

Since trace is independent of basis, the character serves as a class label.

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

Definition 1.5 (Invariant Subspace)**.** Let $U(G)$ be a representation of G on a vector space V, and W a subspace of V such that $U(g)\,|x\rangle \in W$ for all $\vec{x} \in W$ and $g \in G$. Then W is an *invariant subspace* of V with respect to $U(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $U(G)$.

The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

Definition 1.6 (Irreducible Representation)**.** A representation $U(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $U(G)$. Otherwise, it is *reducible.* If $U(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $U(G)$, then the representation is *fully reducible.*

Example 1.2. Under the group of 2-dimensional rotations, consider the 1-dimensional subspace spanned by $\hat{e}_1$. This subspace is not invariant under 2-dimensional rotations, because a rotation of $\hat{e}_1$ by $\pi/2$ results in the vector $\hat{e}_2$ that is clearly not in the subspace spanned by $\hat{e}_1$. A similar argument shows that the subspace spanned by $\hat{e}_2$ is not invariant under 2-dimensional rotations.

However, consider the linear combination of basis vectors

$$\hat{e}_{\pm} = \frac{1}{\sqrt{2}} \left(\mp \hat{e}_1 + i \hat{e}_2 \right), \tag{1.8}$$

where $i = \sqrt{-1}$. Then a rotation by angle ϕ, denoted in operator form as $U(\phi)$, acts on $\hat{e}_{\pm}$ by

$$\begin{aligned}
U(\phi) \left| \hat{e}_+ \right\rangle &= U(\phi) \frac{1}{\sqrt{2}} (-\hat{e}_1 + i\hat{e}_2) &(1.9)\\
&= \frac{1}{\sqrt{2}} (\hat{e}_1(-\cos\phi - i\sin\phi) + \hat{e}_2(\sin\phi + i\cos\phi))\\
&= \frac{1}{\sqrt{2}} (-\hat{e}_1 + i\hat{e}_2)(\cos\phi - i\sin\phi)\\
&= \hat{e}_+(\cos\phi - i\sin\phi)\\
&= \hat{e}_+ e^{-i\phi}, &(1.10)\\
\text{and } U(\phi) \left| \hat{e}_- \right\rangle &= \hat{e}_- e^{i\phi}. &(1.11)
\end{aligned}$$

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5**?

Example 1.3 (Generator of SO(2))**.** Consider the rotations of a 2-dimensional Euclidean vector space about the origin. Let $\hat{e}_1$ and $\hat{e}_2$ be orthonormal basis vectors of this space. Using geometry, we can determine how a rotation by some angle ϕ, written in operator form as $R(\phi)$, acts on the basis vectors:

$$R(\phi)\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi \tag{1.12}$$

$$R(\phi)\hat{e}_2 = -\hat{e}_1 \sin\phi + \hat{e}_2 \cos\phi. \tag{1.13}$$

In matrix form, we can write

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{1.14}$$

which allows us to write Eqn. 1.12 and Eqn. 1.13 in a condensed form

$$R(\phi)\hat{e}_i = \hat{e}_j R(\phi)^j{}_i, \tag{1.15}$$

where we are summing over $j = 1, 2$.

Now, let $\vec{x}$ be an arbitrary vector in the plane. Then $\vec{x}$ has components x^i in the basis $\{\hat{e}_i\}$, where $i = 1, 2$. Equivalently, we can write $\vec{x} = \hat{e}_i x^i$. Then under rotations, $\vec{x}$ transforms in accordance to the basis vectors

$$\begin{aligned} R(\phi)\vec{x} &= R(\phi)\hat{e}_i x^i \\ &= \hat{e}_j R(\phi)^j{}_i x^i \\ &= \left(\hat{e}_1 R(\phi)^1{}_i + \hat{e}_2 R(\phi)^2{}_i\right) x^i \\ &= \left(\hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi\right) x^1 + \left(\hat{e}_1(-\sin\phi) + \hat{e}_2 \cos\phi\right) x^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right) \hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \hat{e}_2. \end{aligned} \tag{1.16}$$

Observe that $R(\phi)R^\top(\phi) = E$ where E is the identity matrix. This is precisely what defines *orthogonal matrices*. For 2-dimensional vectors in the plane, it is clear that these rotations do not change the length of said vectors. This can be verified by using Eqn. 1.16:

$$\begin{aligned} |R(\phi)\vec{x}|^2 &= |\hat{e}_j R(\phi)^j{}_i x^i|^2 \\ &= \left|\left(x^1 \cos\phi - x^2 \sin\phi\right) \hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \hat{e}_2\right|^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right)^2 + \left(x^1 \sin\phi + x^2 \cos\phi\right)^2 \\ &= \left(\cos^2\phi + \sin^2\phi\right) x^1 x_1 + \left(\sin^2\phi + \cos^2\phi\right) x^2 x_2 \\ &= x^1 x_1 + x^2 x_2 = |\vec{x}|^2. \end{aligned} \tag{1.17}$$

Similarly, notice that for any continuous rotation by angle ϕ, $\det R(\phi) = \cos^2\phi + \sin^2\phi = 1$. In general, orthogonal matrices have determinant equal to ± 1. However, the result of the above determinant of $R(\phi)$ implies that all continuous rotations in the 2-dimensional plane have determinant equal to $+1$. These are the *special orthogonal matrices of rank 2*. This family of matrices is denoted SO(2). Furthermore, there is a one-to-one correspondence with SO(2) matrices and rotations in a plane.

We define the group of continuous rotations in a plane by letting $R(0) = E$ be the identity element corresponding to no rotation (i.e., a rotation by angle $\phi = 0$), and defining the inverse of a rotation as $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi)$. This group can be called the SO(2) group. Lastly, we define group multiplication as $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$ and note that $R(\phi) = R(\phi \pm 2\pi)$, which can be verified geometrically. Thus, group elements of SO(2) can be labelled by the angle of rotation $\phi \in [0, 2\pi)$.

Now we can find a generator of *sotwo* by considering an infinitesimal rotation, labelled by some infinitesimal angle $d\phi$. Then this is equivalent to the identity plus some small rotation, which we can write as

$$R(\mathrm{d}\phi) = E - i\mathrm{d}\phi J \tag{1.18}$$

where the scalar quantity $-i$ is introduced for later convenience and J is some quantity independent of the rotation angle. If we consider the rotation $R(\phi + \mathrm{d}\phi)$, then there are two equivalent ways to interpret this rotation

$$R(\phi + \mathrm{d}\phi) = R(\phi)R(\mathrm{d}\phi) = R(\phi)(E - i\mathrm{d}\phi J) = R(\phi) - i\mathrm{d}\phi R(\phi)J \tag{1.19}$$

$$R(\phi + \mathrm{d}\phi) = R(\phi) + \mathrm{d}R(\phi) = R(\phi) + \mathrm{d}\phi\frac{\mathrm{d}R(\phi)}{\mathrm{d}\phi} \tag{1.20}$$

where the second equation can be thought of as a Taylor expansion of $R(\phi + \mathrm{d}\phi)$ about ϕ. Equating the two expressions for $R(\phi + \mathrm{d}\phi)$ yields

$$\mathrm{d}R(\phi) = -i\mathrm{d}\phi R(\phi)J. \tag{1.21}$$

Solving this differential equation (with boundary condition $R(0) = E$) provides us with an equation for any group element involving J:

$$R(\phi) = e^{-i\phi J}, \tag{1.22}$$

where J is called the *generator* of the group.