Representation Theory and its Applications in Physics

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Presented by
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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then $X(g)$ can be realized as an $n \times n$ matrix.

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2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to equivalence.

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Invariance of $\mathbf{e}_\pm$

Let $\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mp\mathbf{e}_1 + i\mathbf{e}_2)$. Then, $X(\phi)\mathbf{e}_\pm = e^{\pm i\phi}\mathbf{e}_\pm$.

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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6. One-dimensional representations are irreducible.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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2 Examples in Physics

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

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Properties of SO(2):

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- We call J the *generator* of SO(2) rotations.

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Theorem

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- Quantum spin is a property that is labeled by j and has possible spin states $|m\rangle$.

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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This is the tip of the iceberg!

3 The Braid Group

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The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

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- A *braid* β is a *loop*[7] in M_n and can be thought of as a configuration that evolves over time:

$$\beta : [0,1] \to M_n$$
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Definition

The *braid group* B_n is the (fundamental) group of all complex-valued n-tuples (M_n) up to *homotopy*.

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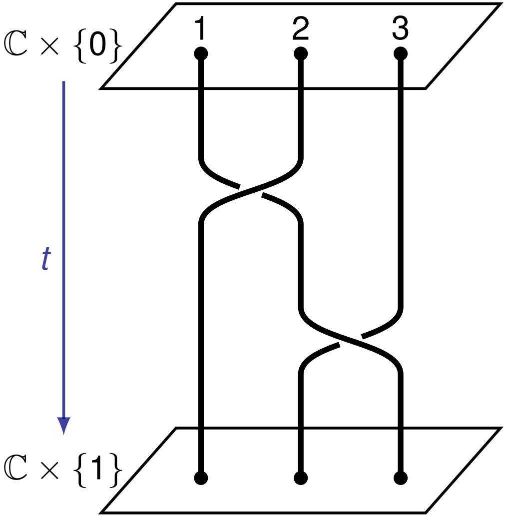

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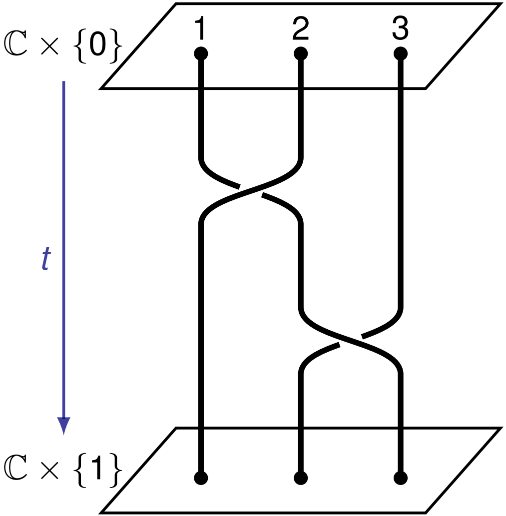

Visualization of Braids

- Each path traced out by a point in the configuration space is a *strand*.
- We can think of a braid on n strands as the motion of n distinct points in the complex plane over a normalized time interval.
- Visualized in $\mathbb{C} \times [0, 1]$.
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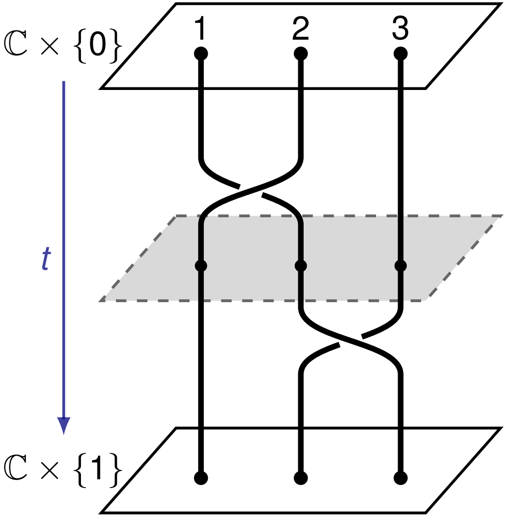

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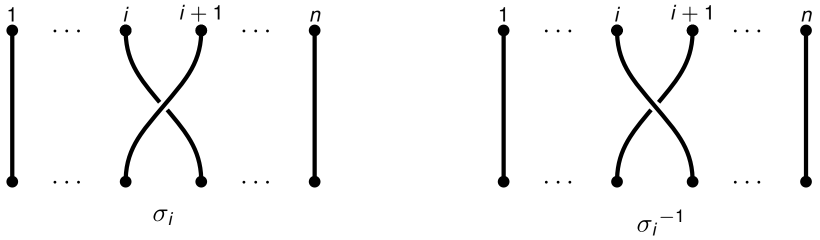

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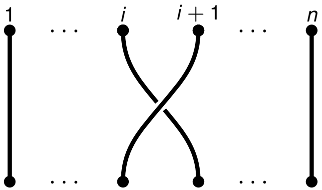
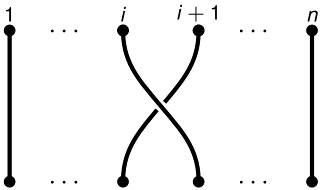


σ_i



${\sigma_i}^{-1}$

- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β.

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Definition

The braid group on n strands, denoted B_n, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, \; |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle .$$

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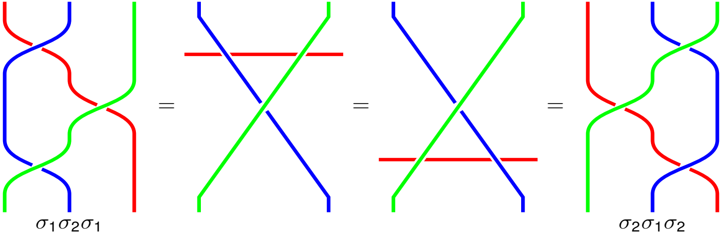
Comment: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



One-Dimensional Representations of the Braid Group

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For $\theta \in \mathbb{R}$ and $j = 1, 2, \ldots, n-1$, we define some *one-dimensional representations* of B_n:

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Hence, for any $\beta \in B_n$ with degree k:

$$p_\theta(\beta) = p_\theta(\sigma_1^{m_1}\sigma_2^{m_2}\cdots\sigma_{n-1}^{m_{n-1}}) = e^{i\theta(m_1+m_2+\cdots+m_{n-1})} = e^{ik\theta}.$$

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Define the unitary representation $\mathcal{U} : B_3 \to U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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What are the physical implications of this nonabelian unitary representation?

Answer: Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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Braiding action: For any degree-k braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \ldots, r_n),$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system[8].

[8] Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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- Edge cases: *bosons* and *fermions*.

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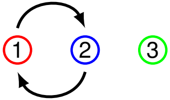

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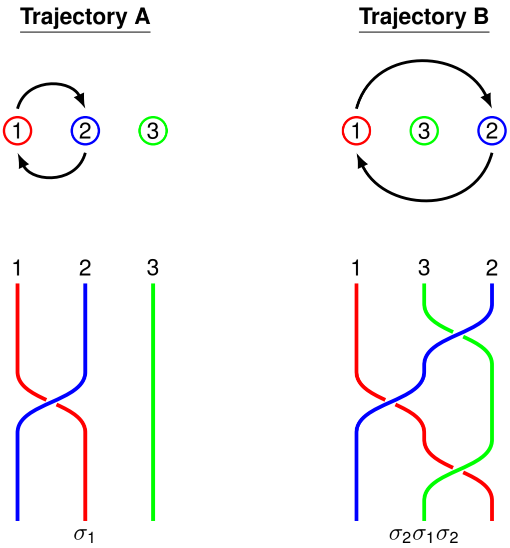

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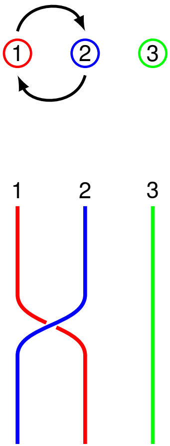
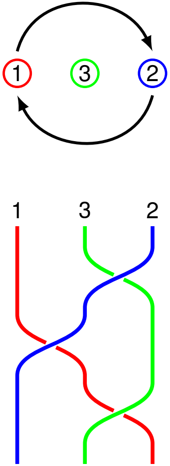


σ_1

Trajectory B



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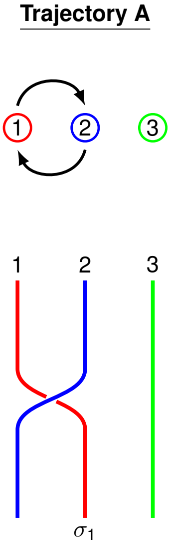

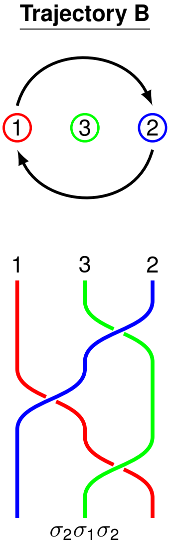

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- Anyons can have different topological flavors, leading to special *fusion rules* that can be used to describe the behavior of anyonic systems.
- Specific fusion rules $+$ nonabelian anyons $=$ fault-tolerant topological *quantum computer*. This is an ongoing area of research.

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Summary

Main Takeaways:

1. Representation theory is a powerful tool that can be used to obtain fundamental results in quantum mechanics and beyond.
2. Unitary representations of the braid group can act on $(2+1)$-dimensional quantum systems, resulting in anyons.
3. Anyons exhibit fractional statistics in contrast to the boson/fermion dichotomy.
4. The nontrivial braiding effects of anyons results in useful physical properties that can be exploited for various physical applications.

Thank you for your attention!

A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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Generalize to N anyons: Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i<j}^{N} \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^{N} \mathbf{r}_i{}^2$$

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$$\mathcal{L} = \frac{m}{2}\sum_{i=1}^{N}\left[\dot{\mathbf{r}}^2 - \omega^2\mathbf{r}_i^2\right] + \alpha\sum_{i<j}^{N}\dot{\mathbf{r}}_{ij}\cdot\frac{(-y_{ij}\hat{x} + x_{ij}\hat{y})}{r_{ij}^2}$$

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Expand:
$$\mathcal{H} = \boxed{\frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N} r_i^2 - \frac{\alpha}{2m}\sum_{\substack{i=1\\ j\neq i}}^{N} \frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{\substack{i=1\\ j,k\neq i}}^{N} \frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}}$$

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Question

Why is this useful?

The Burau Representation

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Definition

The *Burau representation* of the braid group B_n is defined on the standard generators:

$$\psi_n : B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$
$$\sigma_i \mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

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The Burau representation satisfies the braid relations:

$$\psi_n(\sigma_i)\psi_n(\sigma_j) = \psi_n(\sigma_j)\psi_n(\sigma_i) \text{ for } |i-j| > 1,$$
$$\psi_n(\sigma_i)\psi_n(\sigma_{i+1})\psi_n(\sigma_i) = \psi_n(\sigma_{i+1})\psi_n(\sigma_i)\psi_n(\sigma_{i+1}) \text{ for } i \in \{1, \ldots, n-2\}.$$

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$\implies$ **Burau representation is reducible**!

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- Equivalent ways to write the same thing:

$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

SO(2) Explicit form of J

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J:

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m|\right) |\phi\rangle = \sum_m \langle m|\phi\rangle \, |m\rangle \, ,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi) m|\mathcal{O}\rangle = \langle e^{im\phi} m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J.
Thus,

$$J\,|\phi\rangle = \sum_m e^{-im\phi} J\,|m\rangle = \sum_m m e^{-im\phi}\,|m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}\,|m\rangle\right) = i\frac{\partial}{\partial\phi}\,|\phi\rangle$$

$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger \phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\,\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

SO(3) Invariance $\implies$ Commute with Hamiltonian

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial\phi} = (\mathbf{r}\times\nabla)\cdot\mathbf{e}_z \implies J = -i\frac{\partial}{\partial\phi} = -i\,(\mathbf{r}\times\nabla)\cdot\mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}),\hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2,\hat{L}_z] = 0 \implies [\hat{H},\hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

From Invariant Subspace to the Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle ,$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J^2, J_i] = 0.$$