Representation Theory and its Applications in Physics

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

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1. $X(e) = I$, where e is the identity element of the group and I is the identity operator.
2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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Comments:

- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

[2] Invariant subspace $W \subset V$: $X(g)\mathbf{w} \in W, \forall \mathbf{w} \in W$

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- The decomposition of a representation into irreducibles is unique up to equivalence.

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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How does this help in physics?

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2 Examples in Physics

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- We call J the *generator* of SO(2) rotations.

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Theorem

The single-valued irreducible representations of SO(2) are defined as

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- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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Discretization of Angular Momentum for Free

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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This is the tip of the iceberg!

3 The Braid Group

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- A *braid* β is a *loop*[8] in M_n and can be thought of as a configuration that evolves over time:

$$\begin{aligned} \beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \dots, \beta_n(t)\big), \end{aligned}$$

[8]The topological formalisms that define the braid group are omitted for times sake.

The Braid Group

Definition

The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

- Note that $(z_1, z_2, z_3, \ldots, z_n)$ and $(z_2, z_1, z_3, \ldots, z_n)$ are distinct configurations in M_n.
- A *braid* β is a *loop*[8] in M_n and can be thought of as a configuration that evolves over time:

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Definition

The *braid group* B_n is the (fundamental) group of all complex-valued n-tuples (M_n) up to *homotopy*.

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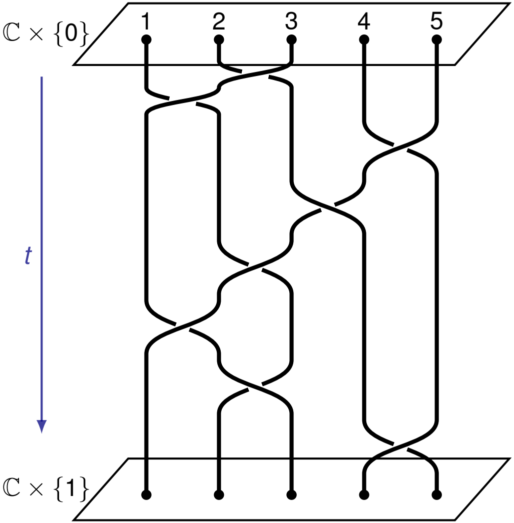
Braid on 5 strands.



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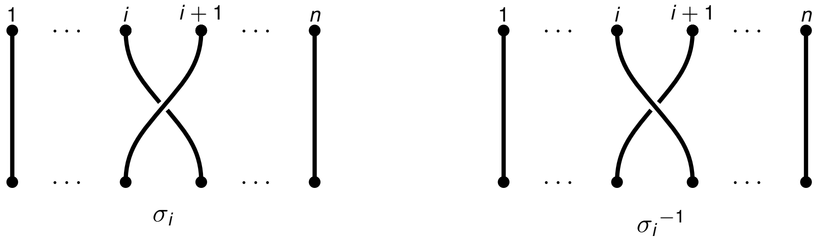

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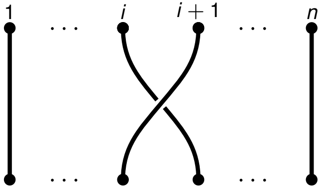
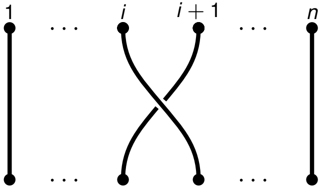


σ_i



${\sigma_i}^{-1}$

- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β.

Alternative Description of B_n

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Definition

The braid group on n strands, denoted B_n, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle.$$

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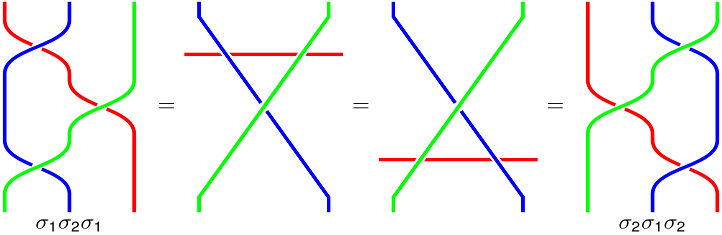
Comment: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



One-Dimensional Representations of the Braid Group

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Hence, for any $\beta \in B_n$ with degree k:

$$p_\theta(\beta) = p_\theta(\sigma_1^{m_1}\sigma_2^{m_2}\cdots\sigma_{n-1}^{m_{n-1}}) = e^{i\theta(m_1+m_2+\cdots+m_{n-1})} = e^{ik\theta}.$$

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Define the unitary representation $\mathcal{U} : B_3 \to U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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Answer: Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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Braiding action: For any degree-k braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \ldots, r_n),$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system[9].

[9] Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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- Edge cases: *bosons* and *fermions*.

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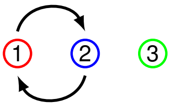

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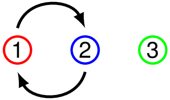
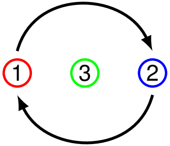


Trajectory B



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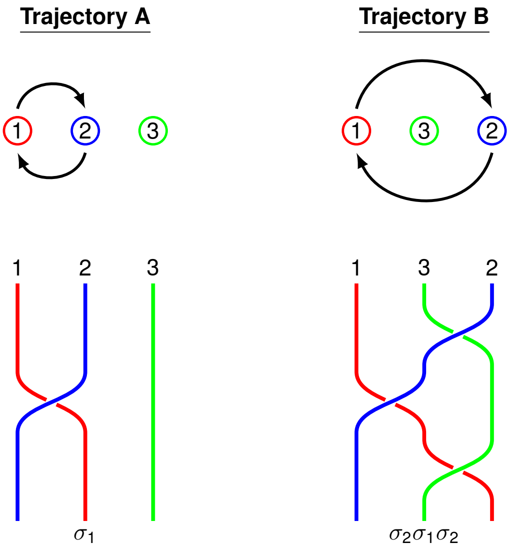

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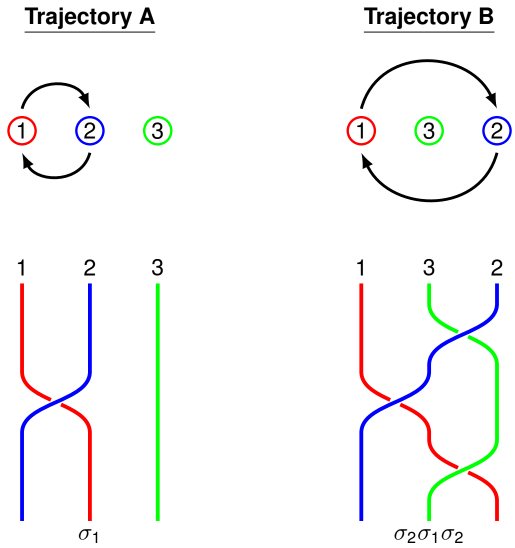

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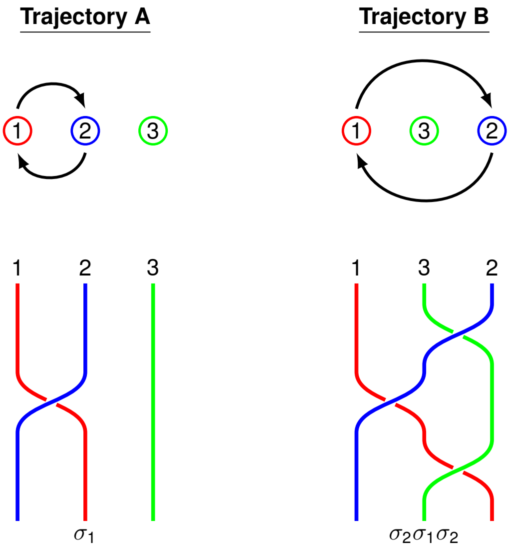
$$\left.\begin{aligned} \sigma_1 &\mapsto e^{i\theta} \\ \sigma_2\sigma_1\sigma_2 &\mapsto e^{3i\theta} \end{aligned}\right\} \neq \text{ if } \theta \notin \pi\mathbb{Z}$$



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- Anyons can have different topological flavors, leading to special *fusion rules* that can be used to describe the behavior of anyonic systems.
- Specific fusion rules $+$ nonabelian anyons $=$ fault-tolerant topological *quantum computer*. This is an ongoing area of research.

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Thank you for your attention!

SO(3) Calculations (pt. 1)

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J:

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m|\right) |\phi\rangle = \sum_m \langle m|\phi\rangle |m\rangle ,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi)m|\mathcal{O}\rangle = \langle e^{im\phi}m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J.
Thus,

$$J|\phi\rangle = \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}|m\rangle\right) = i\frac{\partial}{\partial\phi}|\phi\rangle$$

$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

SO(3) Calculations (pt. 2)

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial\phi} = (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z \implies J = -i\frac{\partial}{\partial\phi} = -i\,(\mathbf{r} \times \nabla) \cdot \mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle .$$