Representation Theory and its Applications in Physics

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Chapter 1

Introduction

The intimate connection between abstract mathematics and the physical world highlights the beautiful complexity of nature. In our efforts to understand the fundamental processes that govern the universe, we grow increasingly reliant on the principles of mathematics. One such tool that bridges the abstract and physical regimes is known as representation theory.

In this thesis, we examine representation theory and observe the physical consequences that emerge from the mathematics. First, we begin in Chapter 2 with a brief overview of representation theory. Then in Chapter 3, we elucidate specific applications of representation theory in the context of quantum physics, arriving at remarkably fundamental results. In Chapter 4, we introduce the braid group and explore some of its representations. Finally, we highlight physical applications of the braid group and its representations in Chapter 5.

It is assumed that the reader has a basic understanding of group theory and linear algebra. A familiarity with topology is also required, and the relevant definitions can be found in Appendix A. Additionally, notational conventions are chosen to align with the physics literature which at times differs from the standard mathematical notation. Some discussions in the later chapters also assume a general understanding of classical and quantum mechanics. To those less familiar with those topics and the notation, a supplementary overview is given in Appendix B.

Chapter 6

To-Do List

Potential committee members:

- **Anton Kaul**
- **Patrick Orson**
- **Eric Brussel**
- *Rob Easton*

Questions for grad ed formatting

- Bold figure captions.
- Short figure captions.
- Compressed citations.
- Make sure every citation is used in the document?
- Weird matrices.

- Beefy captions?
- Okay to just state $\psi_n^{\mathbf{r}}(\sigma_i)$ matrices?
- Show $\psi_n^{\mathbf{r}}(\sigma_i)$ invertible?

- **Do?** Anyon fusion rules. τ anyon/Fibonacci anyon example. Relate to singlet/triplet states in spin-1/2 system.
- Spend some time on `MATLAB` thing?

✓ Gauge theory background, QM background.

✓ Go over Appendix A and see if it needs more examples, maybe push to appendix.

✓ Do the stuff above listed for Chapter 4.

✓ Concluding paragraph on first section of Chapter 5 to lead into the more physics-y stuff.

✓ Conclusion/future of anyons/braid group in physics.

✓ Go over Chapter 2 with the relevant bullets above in mind.

X Note the Lie algebra vs Lie group distinction in Chapter 3.

✓ Intro paragraphs for Chapter 3 and sections.

✓ Comment on faithfulness of the Burau representation.

✓ Introduction "chapter".

☐ Abstract.

☐ Acknowledgements.

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