# Representation Theory and its Applications in Physics

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**Presented by**
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**Outline:**

# 1 Introduction to Representation Theory

# Definition of a Representation

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**Remark**

If $V$ is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then $X(g)$ can be realized as an $n \times n$ matrix.

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2. In the matrix presentation of $X$, $X(g)$ is invertible for all $g \in G$.

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- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

# Example: A Faithful Representation of $S_n$

**Defining representation of $S_n$**

The defining representation $D$ of $S_n$ encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends $i$ to $j$, then place a 1 the $i$-th column and $j$-th row of the representation matrix.

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E.g., in $S_3$:

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**Question**

How do we classify representations of a group?

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- We can use characters to classify representations.

# Decomposing Representations

**Definition**

A representation $X(G)$ on $V$ is *irreducible* if there is no non-trivial invariant subspace[2] in $V$ with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- The decomposition of a representation into irreducibles is unique up to equivalence.

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**Decomposition of $X$**

The span of each $\mathbf{e}_\pm$ is an $X$-invariant subspace of $V_2$. In this basis, we rewrite $X$ as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

# Schur's Lemmas (pt. 1)

**Lemma**

*Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group $G$. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then $T$ is either the zero map or invertible.*

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4. By the rank-nullity theorem, conclude that $T$ is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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**Corollary**

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5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.

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5. $X(G)$ is equivalent to the representation $g \mapsto \lambda_g$ for all $g \in G$.
6. One-dimensional representations are irreducible.

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**How does this help in physics?**

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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# 2 Examples in Physics

# Properties of 2D Rotations

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**Rotations preserve vector lengths:**

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

# The SO(2) Group

**Definition**

The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all $2 \times 2$ orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- With $R(0) = I$ boundary condition: $\boxed{R(\phi) = e^{-i\phi J}}$.
- We call $J$ the *generator* of SO(2) rotations.

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# Recovering the Rotation Matrix from $J$

To first order in $d\phi$: $R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}$

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**Theorem**

*The single-valued irreducible representations of SO(2) are defined as*

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**Definition**

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all $3 \times 3$ orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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- The eigenvalues of these operators correspond to the measurable angular momenta of the quantum system.
- Quantum spin is a property that is labeled by $j$ and has possible spin states $|m\rangle$.

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But that's not all folks!

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1. Any system with radial symmetry is invariant under SO(3) rotations, so $[\hat{H}, \mathbf{J}] = 0$.

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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*This is the tip of the iceberg!*

# 3 The Braid Group

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- A *braid* $\beta$ is a *loop*[7] in $M_n$ and can be thought of as a configuration that evolves over time:

$$\begin{aligned} \beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\big), \end{aligned}$$

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The *braid group* $B_n$ is the (fundamental) group of all complex-valued $n$-tuples ($M_n$) up to *homotopy*.

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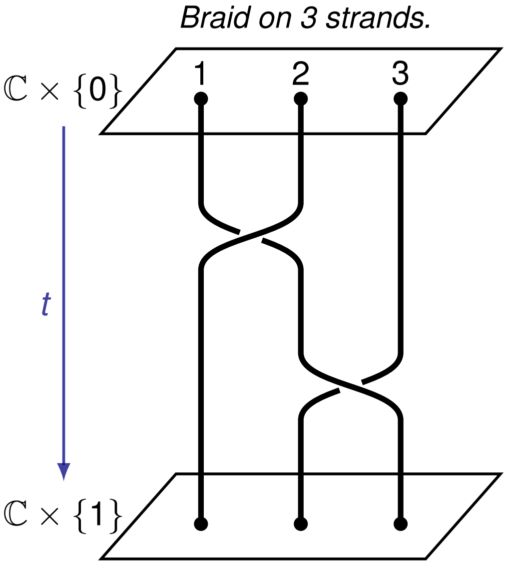

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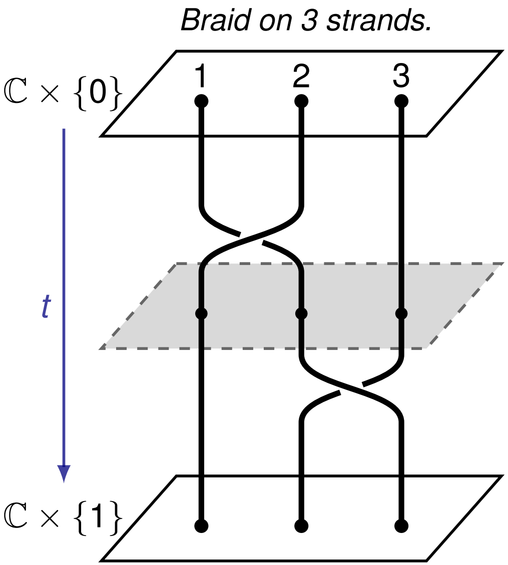

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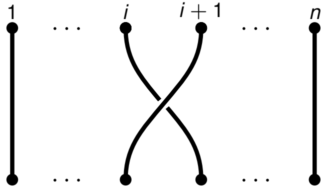
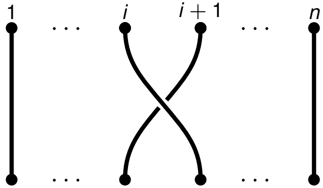



$\sigma_i$

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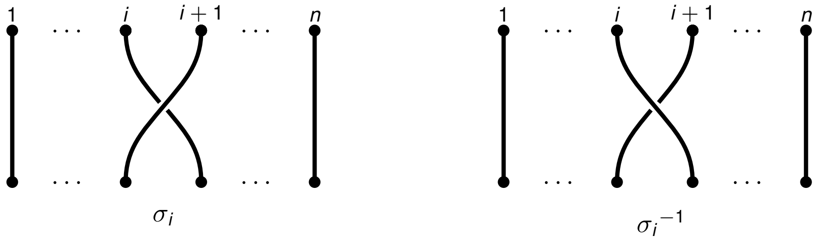


- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of $\beta$.

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The braid group on $n$ strands, denoted $B_n$, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle.$$

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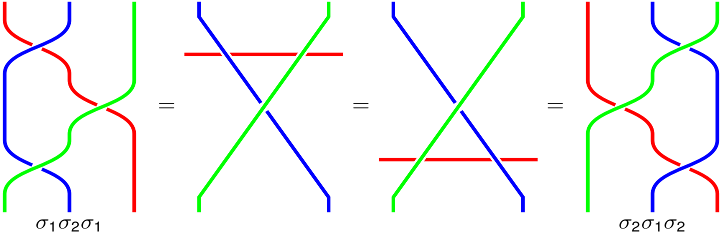
**Comment:** $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



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Hence, for any $\beta \in B_n$ with degree $k$:

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**Definition**

Define the unitary representation $\mathcal{U} : B_3 \to U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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**Answer:** Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

# 4 Physical Applications of the Braid Group

# (Abelian) Braiding Action on a Quantum System

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**Braiding action:** For any degree-$k$ braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \ldots, r_n),$$

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**Remark**

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system[8].

[8] Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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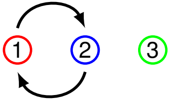

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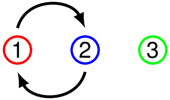
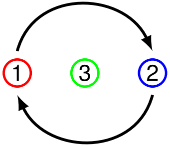



**Trajectory B**



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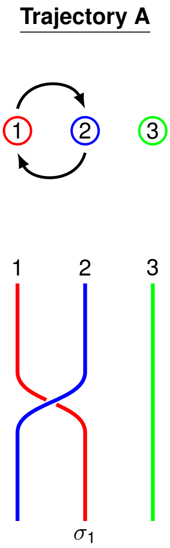


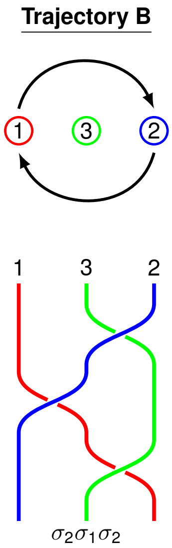

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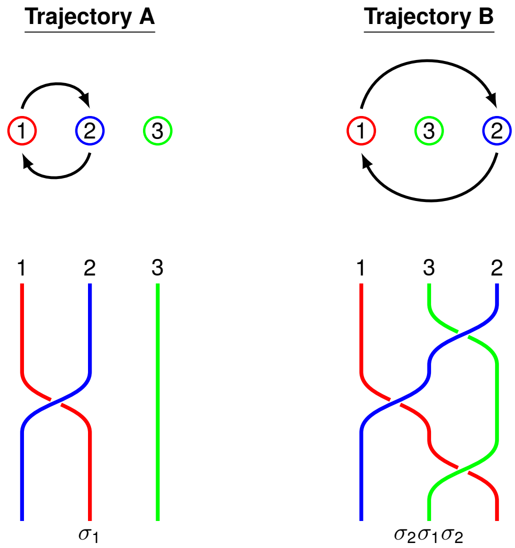

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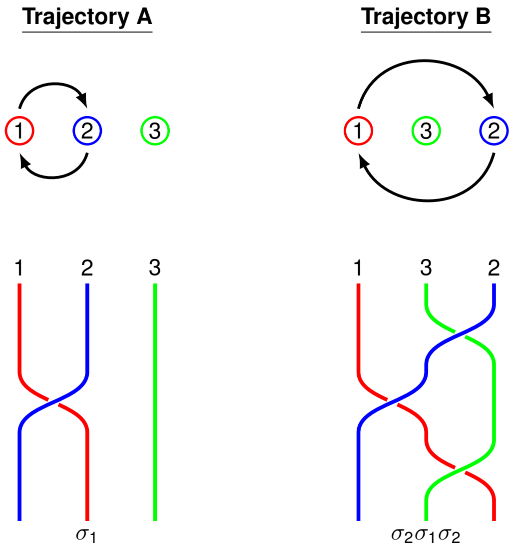

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- Specific fusion rules $+$ nonabelian anyons $=$ fault-tolerant topological *quantum computer*. This is an ongoing area of research.

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**Thank you for your attention!**

# A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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$$\mathcal{L}\left(r_1, r_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \dot{\phi}\right) = T + \mathcal{L}_{\text{int}} - V(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2} m \left(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2\right) + \hbar\alpha\dot{\phi} - \frac{1}{2} m\omega^2 \left(\mathbf{r}_1{}^2 + \mathbf{r}_2{}^2\right)$$

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**Generalize to $N$ anyons:** Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i<j}^{N} \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^{N} {\mathbf{r}_i}^2$$

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**Expand:**
$$\mathcal{H} = \boxed{\frac{1}{2m}\sum_{i=1}^{N}p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N}r_i^2 - \frac{\alpha}{2m}\sum_{\substack{i=1\\ j\neq i}}^{N}\frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{\substack{i=1\\ j,k\neq i}}^{N}\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}}$$

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# Nontrivial Braiding Effects in the Hamiltonian

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**Question**

Why is this useful?

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## Definition

The *Burau representation* of the braid group $B_n$ is defined on the standard generators:

$$\psi_n : B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$

$$\sigma_i \mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

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The Burau representation satisfies the braid relations:

$$\psi_n(\sigma_i)\psi_n(\sigma_j) = \psi_n(\sigma_j)\psi_n(\sigma_i) \text{ for } |i-j| > 1,$$
$$\psi_n(\sigma_i)\psi_n(\sigma_{i+1})\psi_n(\sigma_i) = \psi_n(\sigma_{i+1})\psi_n(\sigma_i)\psi_n(\sigma_{i+1}) \text{ for } i \in \{1, \ldots, n-2\}.$$

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$\implies$ **Burau representation is reducible**!

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- Equivalent ways to write the same thing:

$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

# Orthonormality, Completeness, and Wavefunctions

**Definition**

Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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**Definition**

For a continuous basis labelled by $|x\rangle$ where $x$ is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

# SO(2) Explicit form of $J$

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of $J$:

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m|\right) |\phi\rangle = \sum_m \langle m|\phi\rangle |m\rangle ,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi) m|\mathcal{O}\rangle = \langle e^{im\phi} m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of $J$.
Thus,

$$J|\phi\rangle = \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}|m\rangle\right) = i\frac{\partial}{\partial\phi}|\phi\rangle$$
$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

# SO(3) Invariance $\implies$ Commute with Hamiltonian

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial\phi} = (\mathbf{r}\times\nabla)\cdot\mathbf{e}_z \implies J = -i\frac{\partial}{\partial\phi} = -i\,(\mathbf{r}\times\nabla)\cdot\mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

# From Invariant Subspace to the Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle ,$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J^2, J_i] = 0.$$