Representation Theory and its Applications in Physics

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

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Invertibility

If X is a representation of G, then $X(g)^{-1} = X(g^{-1})$, $\forall g \in G$.

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2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Representation: Let X be a representation of R on V_2 with[1]

$$X(\phi)\mathbf{e}_1 = \mathbf{e}_1 \cdot \cos\phi + \mathbf{e}_2 \cdot \sin\phi$$
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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to equivalence.

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

[4] If the representation matrices have entries in $\mathbb{C}$.

2 Examples in Physics

Preliminaries

Skip preliminaries?

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5. Operators that are self-adjoint are called *Hermitian*.

Preliminaries: Dirac notation

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- The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

Preliminaries: Basic Quantum Mechanics

- Talk about probabilities and whatnot? Eigenvalues = observables? Or just mention when connecting later stuff to physics?

Properties of 2D Rotations

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

The SO(2) Group

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The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all 2×2 orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- We call J the *generator* of SO(2) rotations.

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Recovering the Rotation Matrix from J

To first order in $d\phi$: $R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}$

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$$J\left|m\right\rangle = m\left|m\right\rangle,$$
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Theorem

The single-valued irreducible representations of SO(2) are defined as

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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Definition

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all 3×3 orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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- One can obtain the explicit form of $\mathbf{J}$ and subsequently its components J_x, J_y, J_z. These are precisely the angular momentum operators in quantum mechanics.

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Using a similar process to generate SO(3) invariant subspaces that correspond to irreducible representations, we summarize the results in a theorem:

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- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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But that's not all folks!

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1. Any system with radial symmetry is invariant under SO(3) rotations, so $[\hat{H}, \mathbf{J}] = 0$.
2. Conservation of angular momentum is a direct result of the radial symmetry of the system.
3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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   - the Pauli exclusion principle

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3 The Braid Group

Basic Definitions

- Formal definitions.
- Physical/intuitive visualization and interpretation.
- Standard generators.
- Automorphisms of $\pi_1(\mathbb{D}_n)$.
- Braid relations in this picture.
- 1D Reps.
- Burau representation.
- Note on faithfulness.
- Unitary representation from reduced Burau.

4 Physical Applications of the Braid Group

Rotations of Quantum Hilbert Space

- 1D action on Hilbert space, permuting particles, compare/contrast to bosons/fermions.
- Talk about nontrivial braiding effects.
- Example of unitary braid rep acting on Hilbert space.

Anyons: A Consequence of Braiding

- Introduce anyons.
- Discuss how anyons are described by the braid group.
- Fusion rules, abelian vs nonabelian anyons.
- Non-interacting anyons.
- Non-interacting anyons in harmonic potential.
- Nontrivial braiding effects anyone?
- Applications of anyons! (quantum computing, topological quantum field theory, FQHE, etc.)

Summary/Conclusion

Acknowledgements, questions, references (?)