Representation Theory and its Applications in Physics

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Presented by
Max Varverakis (mvarvera@calpoly.edu)

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Outline:

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1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

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Invertibility

If X is a representation of G, then $X(g)^{-1} = X(g^{-1})$, $\forall g \in G$.

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1. $X(e) = I$, where e is the identity element of the group and I is the identity operator.
2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

Example: The Trivial Representation

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Comments:

- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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- This representation is faithful.

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Representation: Let X be a representation of R on V_2 with[1]

$$X(\phi)\mathbf{e}_1 = \mathbf{e}_1 \cdot \cos\phi + \mathbf{e}_2 \cdot \sin\phi$$
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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

[2] Invariant subspace $W \subset V$: $X(g)\mathbf{w} \in W$, $\forall\, \mathbf{w} \in W$

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- A reducible representation can be decomposed into a direct sum of irreducible representations.
- The decomposition of a representation into irreducibles is unique up to equivalence.

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Example: Irreducible Representation of 2D Rotations

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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3. Since X and Y are irreducible, $\ker(T) = \{\mathbf{0}\}$ and $\operatorname{im}(T) = V$ or $\ker(T) = V$ and $\operatorname{im}(T) = \{0\}$.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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2. Then $T - \lambda I$ is not invertible.
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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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4. The element h was arbitrary, so $X(g) = \lambda_g I$ for all $g \in G$.
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6. One-dimensional representations are irreducible.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

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2 Examples in Physics

Preliminaries: Physics Conventions

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

Preliminaries: Basic Quantum Mechanics

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$$R(\phi)R^\top(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

The SO(2) Group

Definition

The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all 2×2 orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_{\pm}$).

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- We call J the *generator* of SO(2) rotations.

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Recovering the Rotation Matrix from J

To first order in $d\phi$: $R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}$

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4. Each invariant subspace is spanned by an eigenvector of J:

$$\begin{aligned} J\,|m\rangle &= m\,|m\rangle\,, \\ U(\phi)\,|m\rangle &= e^{-iJ\phi}\,|m\rangle = e^{-im\phi}\,|m\rangle\,. \end{aligned}$$

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5. Periodicity of SO(2) $\implies$ $e^{-i2\pi m} = 1 \implies m \in \mathbb{Z}$.

Theorem

The single-valued irreducible representations of SO(2) are defined as

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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Consequence: Any rotation in Euclidean 3-space can be written in terms of the generators:

$$R_{\mathbf{n}}(\theta) = e^{-i\theta J_{\mathbf{n}}} = e^{-i\theta(n_x J_x + n_y J_y + n_z J_z)} = e^{-i\theta\, \mathbf{n}\cdot\mathbf{J}}.$$

Definition

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all 3×3 orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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The irreducible representations of SO(3) are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ *and the* $2j + 1$ *eigenvectors spanning an invariant subspace are labelled by their eigenvalues:* $m = -j, -j + 1, \ldots, j - 1, j.$

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- One can obtain the explicit form of $\mathbf{J}$ and subsequently its components J_x, J_y, J_z. These are precisely the angular momentum operators in quantum mechanics.

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Consequences:

- One can obtain the explicit form of $\mathbf{J}$ and subsequently its components J_x, J_y, J_z. These are precisely the angular momentum operators in quantum mechanics.
- The eigenvalues of $\mathbf{J}^2$ and J_z are $j(j + 1)$ and m, respectively[7]. In quantum physics, these eigenvalues correspond to the observable total angular momentum and its z-component.

[7] Typically, the z-axis is chosen as the standard axis.

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The irreducible representations of SO(3) are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ *and the* $2j + 1$ *eigenvectors spanning an invariant subspace are labelled by their eigenvalues:* $m = -j, -j + 1, \ldots, j - 1, j.$

Consequences:

- One can obtain the explicit form of **J** and subsequently its components J_x, J_y, J_z. These are precisely the angular momentum operators in quantum mechanics.
- The eigenvalues of $\mathbf{J}^2$ and J_z are $j(j+1)$ and m, respectively[7]. In quantum physics, these eigenvalues correspond to the observable total angular momentum and its z-component.
- This generalizes to other types of angular momentum, such as *spin angular momentum*!

[7] Typically, the z-axis is chosen as the standard axis.

Connection to Quantum Mechanics: Punchline

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Discretization of Angular Momentum for Free

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But that's not all folks!

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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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This is the tip of the iceberg!

3 The Braid Group

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Definition

The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

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- A *braid* β is a *loop*[8] in M_n and can be thought of as a configuration that evolves over time:

$$\begin{aligned} \beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\big), \end{aligned}$$

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The *braid group* B_n is the (fundamental) group of all complex-valued n-tuples (M_n) up to *homotopy*.

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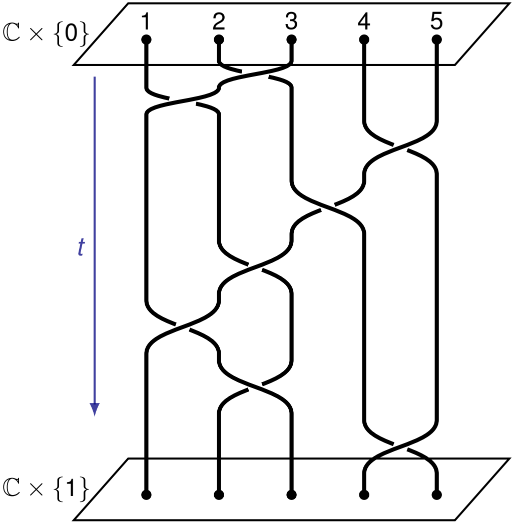
Braid on 5 strands.



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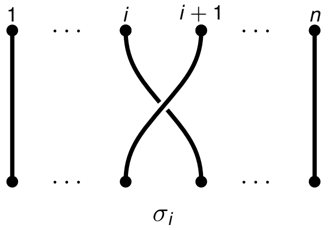

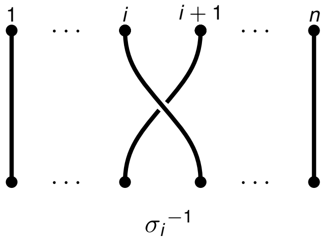

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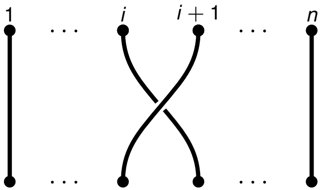
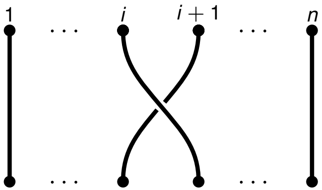


σ_i



${\sigma_i}^{-1}$

- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β.

Alternative Description of B_n

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The braid group on n strands, denoted B_n, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle .$$

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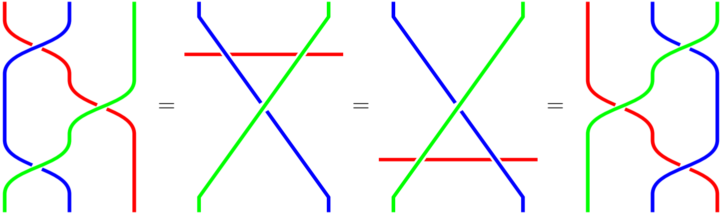
Comment: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$

One-Dimensional Representations of the Braid Group

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For $\theta \in \mathbb{R}$ and $j = 1, 2, \ldots, n-1$, we define some *one-dimensional representations* of B_n:

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The *Burau representation* of the braid group B_n is defined on the standard generators:

$$\begin{aligned} \psi_n : B_n &\to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}]) \\ \sigma_i &\mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}. \end{aligned}$$

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The Burau representation satisfies the braid relations:

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$\implies$ **Burau representation is reducible**!

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$$\mathcal{U}(\sigma_1) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

$$\mathcal{U}(\sigma_2) = \frac{1}{2} e^{-i\frac{\pi}{6}} \begin{bmatrix} -\sqrt{3}\, e^{-i \arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & \sqrt{3}\, e^{i \arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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Answer: Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!

4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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Braiding action: For any degree-k braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\,\psi(r_1, r_2, \ldots, r_n),$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to nontrivial rotations in the many-particle Hilbert space that describes the quantum system.

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- Edge cases: *bosons* and *fermions*.

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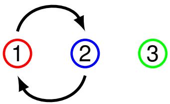

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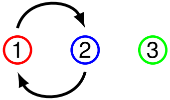
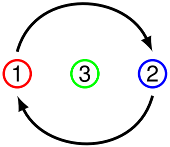


Trajectory B



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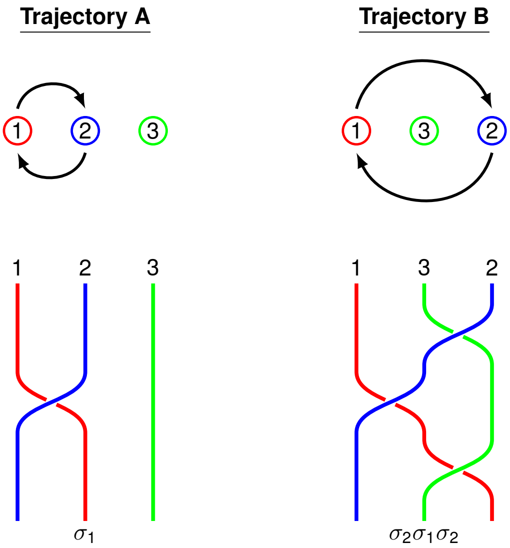

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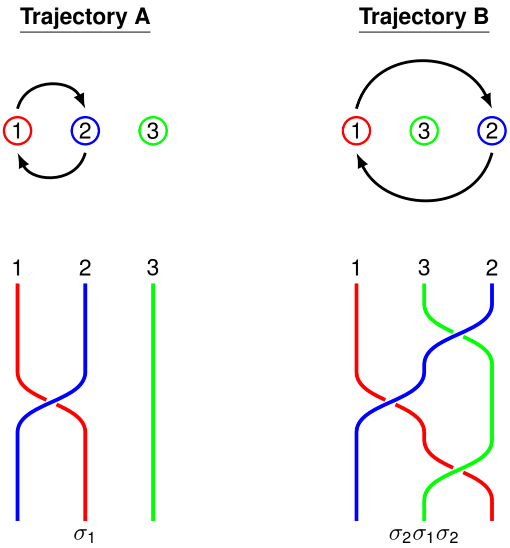

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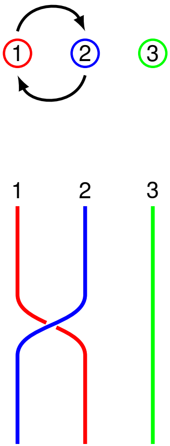

σ_1

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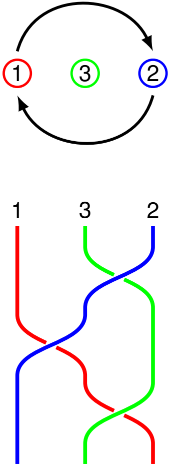

$\sigma_2\sigma_1\sigma_2$

A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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Generalize to N anyons: Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i<j}^{N} \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^{N} {\mathbf{r}_i}^2$$

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Expand:
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Nontrivial Braiding Effects in the Hamiltonian

Nontrivial braiding effects emerge from the *long-range interaction* term when $N \geq 3$.

Nontrivial Braiding Effects in the Hamiltonian

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Question

Why is this useful?

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- The fractional quantum Hall effect is a physical manifestation of anyonic braiding in 2D electron systems (fractional charge, fractional statistics).
- Depending on the specific representation of the braid group, one can define topological properties of different flavors of anyons. The corresponding combination (fusion) rules can be used to describe the behavior of anyonic systems.
- Certain nonabelian anyons (defined by specific fusion rules) are alleged to enable fault-tolerant quantum computers, and is an ongoing area of research.

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Thank you for your attention!

SO(3) Calculations (pt. 1)

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J:

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m|\right) |\phi\rangle = \sum_m \langle m|\phi\rangle |m\rangle ,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi)m|\mathcal{O}\rangle = \langle e^{im\phi}m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J.
Thus,

$$J|\phi\rangle = \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i\frac{\partial}{\partial\phi}\left(e^{-im\phi}|m\rangle\right) = i\frac{\partial}{\partial\phi}|\phi\rangle$$

$$\implies \langle\phi|J|\psi\rangle = \langle J^\dagger\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\langle\phi|\psi\rangle = -i\frac{\partial}{\partial\phi}\psi(\phi).$$

SO(3) Calculations (pt. 2)

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial\phi} = (\mathbf{r}\times\nabla)\cdot\mathbf{e}_z \implies J = -i\frac{\partial}{\partial\phi} = -i\,(\mathbf{r}\times\nabla)\cdot\mathbf{e}_z = \frac{1}{\hbar}\hat{L_z} \implies \hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle .$$