

Representation Theory and its Applications in Physics

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Presented by

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group



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1 Introduction to Representation Theory

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, then $X(g)$ can be realized as an $n \times n$ matrix.

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2. In the matrix presentation of X , $X(g)$ is invertible for all $g \in G$.

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Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial X -invariant subspace² in V . Otherwise, $X(G)$ is *reducible*.

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- ▶ A reducible representation can be decomposed into a direct sum of irreducible representations.
- ▶ The decomposition of a representation into irreducibles is unique up to matrix similarity.

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Schur's Lemmas

Lemma

Let $X : G \rightarrow V$ and $Y : G \rightarrow W$ be irreducible representations of a group G . If there exists a fixed linear transformation $T : V \rightarrow W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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Corollary

If G is a finite abelian group, then the irreducible representations of G are one-dimensional.

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How does this help in physics?

The groups corresponding to physical transformations have irreducible representations that lead to fundamental insights in physics.

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2 Examples in Physics

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Rotation matrices are orthogonal:

$$R(\phi)R^\top(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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This *special* property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

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The *special orthogonal group* in two dimensions, denoted $SO(2)$, is the group of all 2×2 orthogonal matrices with determinant equal to $+1$.⁴

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- ▶ With $R(0) = I$ boundary condition: $R(\phi) = e^{-i\phi J}$.
- ▶ We call J the *generator* of $SO(2)$ rotations.

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3. $SO(2)$ is abelian: Schur's Lemmas \implies all irreducible representations are 1D.
4. Each invariant subspace is spanned by an eigenvector of J :

$$J|m\rangle = m|m\rangle ,$$
$$U(\phi)|m\rangle = e^{-iJ\phi}|m\rangle = e^{-im\phi}|m\rangle .$$

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Theorem

The single-valued irreducible representations of $SO(2)$ are defined as

$$U^m(\phi) = e^{-im\phi}, \forall m \in \mathbb{Z}.$$

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Definition

The *special orthogonal group* in three dimensions, denoted $\text{SO}(3)$, is the group of all 3×3 orthogonal matrices with determinant equal to $+1$. $\text{SO}(3)$ rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^T$.

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*The irreducible representations of $SO(3)$ are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, and the $2j + 1$ eigenvectors spanning an invariant subspace are labelled by their eigenvalues:
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- ▶ *Quantum spin* is labeled by j and has possible spin states $|m\rangle$.

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Discretization of Angular Momentum for Free

Discretization (quantization) of angular momentum follows directly from the irreducible representations of $SO(3)$!

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But that's not all folks!

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3. Lorentz invariance \implies conservation of energy and momentum!



CAL POLY

3 The Braid Group

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The *configuration space* of n ordered distinct points in the complex plane \mathbb{C} is defined as $M_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

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- ▶ A *braid* β is a *loop*⁶ in M_n and can be thought of as a configuration that evolves over time:

$$\begin{aligned}\beta : [0, 1] &\rightarrow M_n \\ t &\mapsto \beta(t) = (\beta_1(t), \beta_2(t), \dots, \beta_n(t)),\end{aligned}$$

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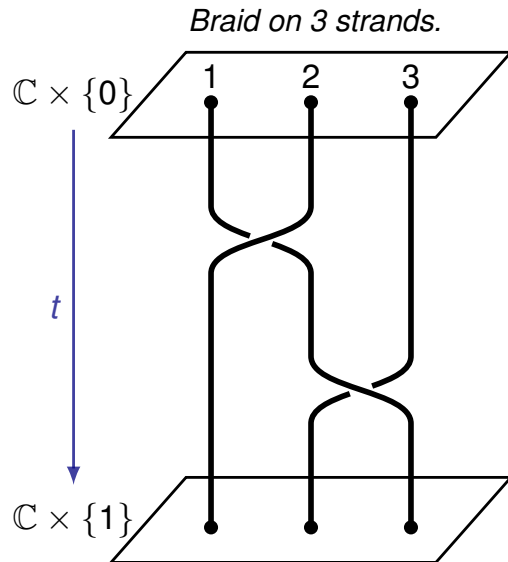
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The *braid group* B_n is the fundamental group of M_n/S_n , where S_n is the symmetric group on n elements.

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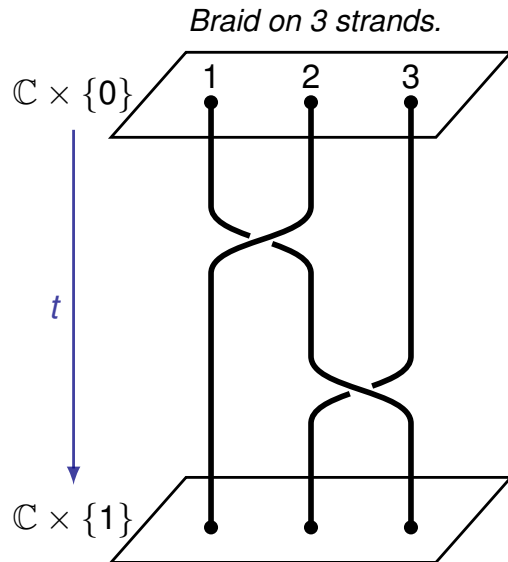
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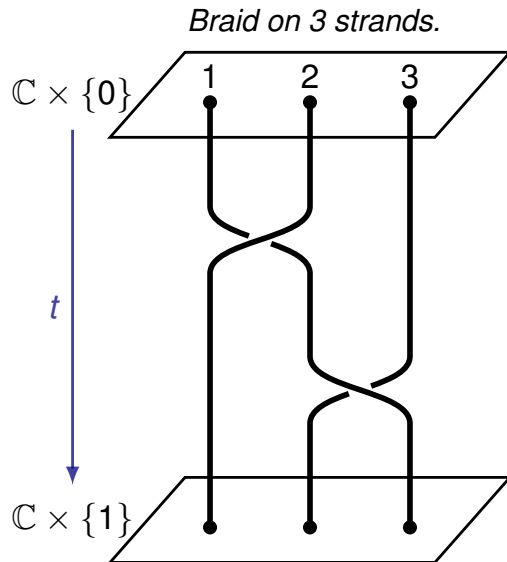
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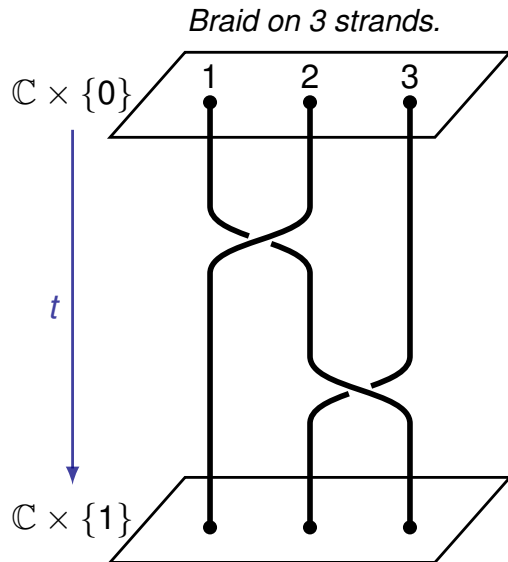
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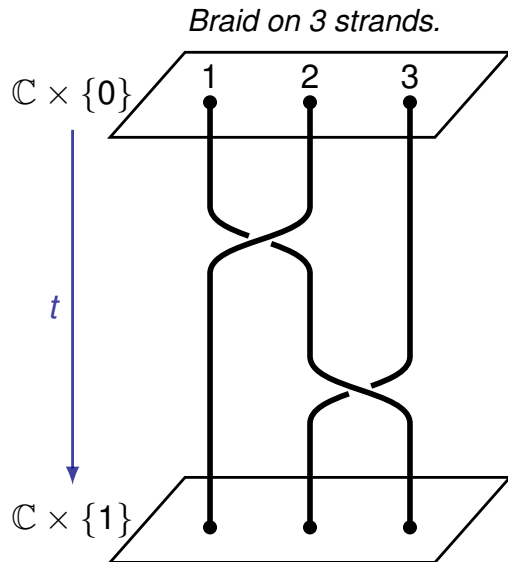
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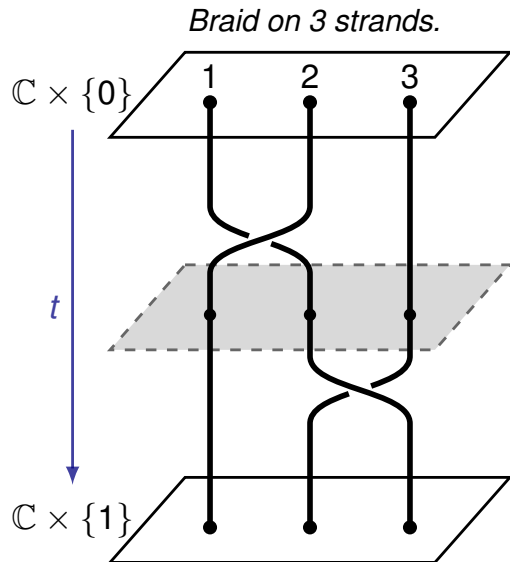
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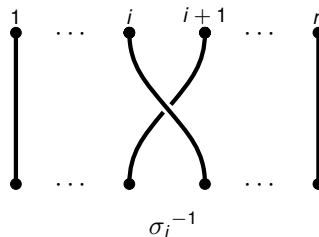
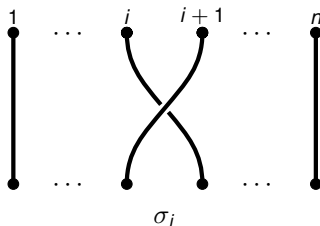
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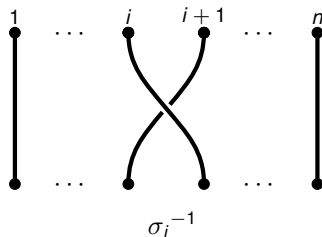
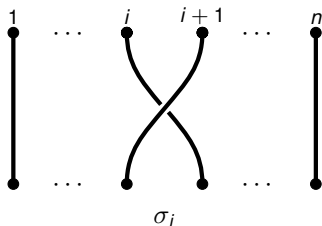
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- ▶ The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β .

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The braid group on n strands, denoted B_n , is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right. \right\rangle.$$

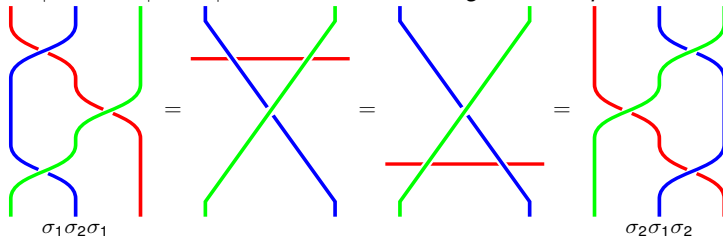
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Comment: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



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Unitary Representation of the Braid Group

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Define the unitary representation $\mathcal{U} : B_3 \rightarrow U(2)$ by

$$\mathcal{U}(\sigma_1) = \frac{1}{2}e^{-i\frac{\pi}{6}} \begin{bmatrix} \sqrt{3}e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & -\sqrt{3}e^{-i\arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$
$$\mathcal{U}(\sigma_2) = \frac{1}{2}e^{-i\frac{\pi}{6}} \begin{bmatrix} -\sqrt{3}e^{-i\arctan\left(\frac{1}{\sqrt{2}}\right)} & 1 \\ 1 & \sqrt{3}e^{i\arctan\left(\frac{1}{\sqrt{2}}\right)} \end{bmatrix}$$

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What are the physical implications of this nonabelian unitary representation?

Answer: Unitary matrices preserve inner products, so the unitary representations of the braid group can act on a quantum system by braiding particles!



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4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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Braiding action: For any degree- k braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \dots, r_{n'}) = p_\theta(\beta) \psi(r_1, r_2, \dots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase} \\ \text{shift}}} \psi(r_1, r_2, \dots, r_n),$$

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Braiding action: The transformed basis states due to the action of σ_1 are

$$\begin{aligned} |1'\rangle &= \mathcal{U}(\sigma_1)_{1,1} |1\rangle + \mathcal{U}(\sigma_1)_{1,2} |2\rangle = \frac{1}{2} e^{-i\frac{\pi}{6}} \left(\sqrt{3} e^{i\arctan(\frac{1}{\sqrt{2}})} |1\rangle + |2\rangle \right), \\ |2'\rangle &= \mathcal{U}(\sigma_1)_{2,1} |1\rangle + \mathcal{U}(\sigma_1)_{2,2} |2\rangle = \frac{1}{2} e^{-i\frac{\pi}{6}} \left(|1\rangle - \sqrt{3} e^{-i\arctan(\frac{1}{\sqrt{2}})} |2\rangle \right). \end{aligned}$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to *nontrivial rotations* in the many-particle Hilbert space that describes the quantum system⁷.

⁷Nayak et al., 2008, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics*

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- ▶ Edge cases: *bosons* and *fermions*.

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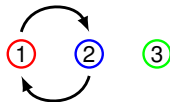
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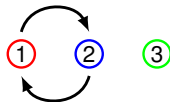
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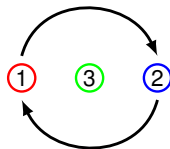
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Trajectory B



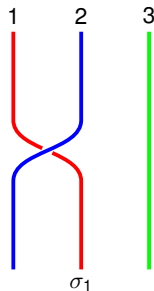
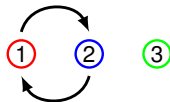
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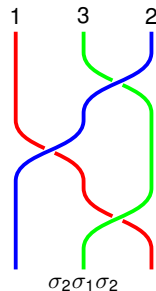
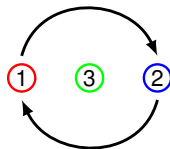
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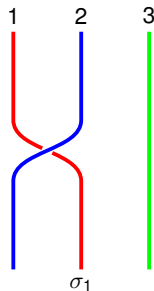
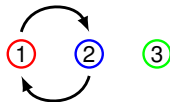
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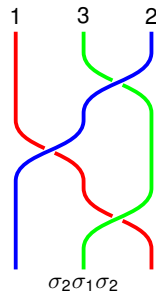
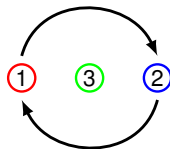
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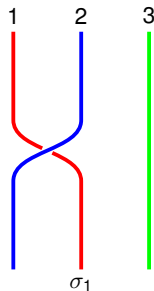
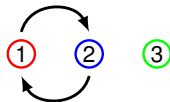
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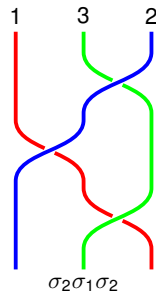
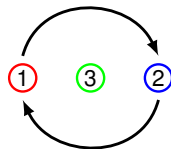
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- ▶ Anyons can have different topological flavors, leading to special *fusion rules* that can be used to describe the behavior of anyonic systems.
- ▶ Specific fusion rules + nonabelian anyons = fault-tolerant topological *quantum computer*. This is an ongoing area of research.

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Thank you for your attention!

A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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Generalize to N anyons: Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^N \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i < j}^N \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^N \mathbf{r}_i^2$$

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i -th anyon Hamiltonian:

$$\mathcal{H}_i = \frac{1}{2m} \underbrace{(\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i))^2}_{\text{canonical momentum}} + \frac{m\omega^2}{2} r_i^2$$

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$$\mathcal{L} = \frac{m}{2} \sum_{i=1}^N [\dot{\mathbf{r}}_i^2 - \omega^2 \mathbf{r}_i^2] + \alpha \sum_{i < j} \dot{\mathbf{r}}_{ij} \cdot \frac{(-y_{ij} \hat{x} + x_{ij} \hat{y})}{r_{ij}^2}$$

Gauge potential:

$$\mathbf{A}_i(\mathbf{r}_i) = \alpha \sum_{j \neq i} \frac{\hat{z} \times \mathbf{r}_{ij}}{r_{ij}^2} = \alpha \sum_{j \neq i} \frac{-y_{ij} \hat{x} + x_{ij} \hat{y}}{r_{ij}^2}$$

i -th anyon Hamiltonian:

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Expand:

$$\mathcal{H} = \frac{1}{2m} \sum_{i=1}^N p_i^2 + \frac{m\omega^2}{2} \sum_{i=1}^N r_i^2 - \frac{\alpha}{2m} \sum_{\substack{i=1 \\ j \neq i}}^N \frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m} \sum_{\substack{i=1 \\ j, k \neq i}}^N \frac{\mathbf{r}_{ij} \cdot \mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}$$

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Question

Why is this useful?

Example: The Trivial Representation

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Comments:

- ▶ The trivial representation is always one-dimensional.
- ▶ For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- ▶ If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of \mathbb{R}^n . If a permutation sends i to j , then place a 1 the i -th column and j -th row of the representation matrix.

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- ▶ The defining representation of S_n is n -dimensional.
- ▶ This representation is faithful.

Example: Irreducible Representation of 2D Rotations

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Let $\mathbf{e}_{\pm} = \frac{1}{\sqrt{2}} (\mp \mathbf{e}_1 + i \mathbf{e}_2)$. Then, $X(\phi) \mathbf{e}_{\pm} = e^{\pm i \phi} \mathbf{e}_{\pm}$.

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Decomposition of X

The span of each \mathbf{e}_{\pm} is an X -invariant subspace of V_2 . In this basis, we rewrite X as a direct sum of the 1D irreducible representations⁸:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

⁸1-dimensional representations are always irreducible!

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Lemma

Let $X : G \rightarrow V$ and $Y : G \rightarrow W$ be irreducible representations of a group G . If there exists a fixed linear transformation $T : V \rightarrow W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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Let X be an irreducible representation of a group G and T a linear operator that commutes with all $X(g)$ for $g \in G$. Then T is a scalar multiple of the identity operator.

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Corollary

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Definition

The *Burau representation* of the braid group B_n is defined on the standard generators:

$$\psi_n : B_n \rightarrow \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$$

$$\sigma_i \mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

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Definition

The *Burau representation* of the braid group B_n is defined on the standard generators:

$$\begin{aligned} \psi_n : B_n &\rightarrow \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}]) \\ \sigma_i &\mapsto \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{bmatrix}. \end{aligned}$$

The Burau representation satisfies the braid relations:

$$\begin{aligned} \psi_n(\sigma_i)\psi_n(\sigma_j) &= \psi_n(\sigma_j)\psi_n(\sigma_i) \text{ for } |i-j| > 1, \\ \psi_n(\sigma_i)\psi_n(\sigma_{i+1})\psi_n(\sigma_i) &= \psi_n(\sigma_{i+1})\psi_n(\sigma_i)\psi_n(\sigma_{i+1}) \text{ for } i \in \{1, \dots, n-2\}. \end{aligned}$$

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\implies **Burau representation is reducible!**

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5. Operators that are self-adjoint are called *Hermitian*.

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- ▶ The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A|\psi\rangle$.
- ▶ Equivalent ways to write the same thing:

$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|A|\psi\rangle = \langle\phi|A\psi\rangle.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the **orthonormality** and **completeness** relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n| \right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the **wavefunction** $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

SO(2) Explicit form of J

The state $|\phi\rangle$ can be decomposed into a linear combination of the eigenvectors of J :

$$|\phi\rangle = \left(\sum_m |m\rangle \langle m| \right) |\phi\rangle = \sum_m \langle m|\phi\rangle |m\rangle,$$

where

$$\langle m|\phi\rangle = \langle m|U(\phi)|\mathcal{O}\rangle = \langle U^\dagger(\phi)m|\mathcal{O}\rangle = \langle e^{im\phi}m|\mathcal{O}\rangle = e^{-im\phi} \langle m|\mathcal{O}\rangle$$

is the projection of $|\phi\rangle$ onto the eigenvector $|m\rangle$ of J .

Thus,

$$\begin{aligned} J|\phi\rangle &= \sum_m e^{-im\phi} J|m\rangle = \sum_m m e^{-im\phi} |m\rangle = \sum_m i \frac{\partial}{\partial \phi} (e^{-im\phi} |m\rangle) = i \frac{\partial}{\partial \phi} |\phi\rangle \\ \implies \langle \phi|J|\psi\rangle &= \langle J^\dagger \phi|\psi\rangle = -i \frac{\partial}{\partial \phi} \langle \phi|\psi\rangle = -i \frac{\partial}{\partial \phi} \psi(\phi). \end{aligned}$$

SO(3) Invariance \implies Commute with Hamiltonian

$$\phi = \arctan\left(\frac{y}{x}\right)$$

$$\implies \frac{\partial}{\partial \phi} = (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z \implies J = -i \frac{\partial}{\partial \phi} = -i (\mathbf{r} \times \nabla) \cdot \mathbf{e}_z = \frac{1}{\hbar} \hat{L}_z \implies \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$$

$$\hat{\mathbf{p}} = -i\nabla \implies \hat{L}_z = x\hat{p}_y - y\hat{p}_x$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}), \quad [V(\mathbf{r}), \hat{L}_z] = 0, \quad [\hat{\mathbf{p}}^2, \hat{L}_z] = 0 \implies [\hat{H}, \hat{L}_z] = 0,$$

where the last line easily generalizes to $\hat{\mathbf{L}}$.

From Invariant Subspace to the Lie Algebra

$$J^2 |j\rangle = (J_- J_+ + J_z + J_z^2) |j\rangle = (0 + j + j^2) |j\rangle = j(j+1) |j\rangle ,$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle ,$$

$$J_z |j, m\rangle = m |j, m\rangle ,$$

$$J_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle ,$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J^2, J_i] = 0.$$