Representation Theory and its Applications in Physics

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

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Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

$$X(gh) = X(g)X(h), \quad \forall g, h \in G.$$

Invertibility

If X is a representation of G, then $X(g)^{-1} = X(g^{-1})$, $\forall g \in G$.

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2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

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Question

How do we classify representations of a group?

Equivalent Representations

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- We can use characters to classify representations.

Decomposing Representations

Definition

A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

[2] Invariant subspace $W \subset V$: $X(g)\mathbf{w} \in W, \forall\, \mathbf{w} \in W$

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- The decomposition of a representation into irreducibles is unique up to equivalence.

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Example: Irreducible Representation of 2D Rotations

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Invariance of $\mathbf{e}_\pm$

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

[4] If the representation matrices have entries in $\mathbb{C}$.

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Preliminaries

Skip preliminaries?

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle \, ,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

Preliminaries: Basic Quantum Mechanics

- Talk about probabilities and whatnot? Eigenvalues = observables? Or just mention when connecting later stuff to physics?

Properties of 2D Rotations

Let R denote the familiar rotation matrix representation from before.

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

The SO(2) Group

Definition

The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all 2×2 orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- We call J the *generator* of SO(2) rotations.

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Recovering the Rotation Matrix from J

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Process to obtaining irreducibles:

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4. Each invariant subspace is spanned by an eigenvector of J:

$$\begin{aligned} J\,|m\rangle &= m\,|m\rangle\,, \\ U(\phi)\,|m\rangle &= e^{-iJ\phi}\,|m\rangle = e^{-im\phi}\,|m\rangle\,. \end{aligned}$$

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Theorem

The single-valued irreducible representations of SO(2) are defined as

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

Generalization to 3 Spatial Dimensions

- Show but don't derive $R_{\mathbf{n}}(\theta)$ decomposition into **J** components.
- We have basis from the components of **J**.
- Ladies and gentlemen, we got SO(3)...
- **J** component differential forms?
- Commutation relations, in some form talk about $J_{\pm}, J^2$ and final eigenvalue results.

Conservation of Angular Momentum

I think it makes most sense to do this after generalizing to 3D...

Let V be the vector space that U^m acts on.

- The Hermiticity of J allows us to obtain an eigenbasis of V

Connection to Quantum Mechanics

- Discuss connection between generators and quantum operators, eigenvalues and classical observables, discretization (!), etc.
- This is the kicker. I will get very excited here probably.

Multi-valued Irreducible Representations and Spinors

Not sure where to put this. . .

- Let's come back to SO(2) for a second. . .
- Show $m = 1/2$ irreps.
- Discuss implications, spinors, etc. . .

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Basic Definitions

- Formal definitions.
- Physical/intuitive visualization and interpretation.
- Standard generators.
- Automorphisms of $\pi_1(\mathbb{D}_n)$.
- Braid relations in this picture.
- 1D Reps.
- Burau representation.
- Note on faithfulness.
- Unitary representation from reduced Burau.

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Rotations of Quantum Hilbert Space

- 1D action on Hilbert space, permuting particles, compare/contrast to bosons/fermions.
- Talk about nontrivial braiding effects.
- Example of unitary braid rep acting on Hilbert space.

Anyons: A Consequence of Braiding

- Introduce anyons.
- Discuss how anyons are described by the braid group.
- Fusion rules, abelian vs nonabelian anyons.
- Non-interacting anyons.
- Non-interacting anyons in harmonic potential.
- Nontrivial braiding effects anyone?
- Applications of anyons! (quantum computing, topological quantum field theory, FQHE, etc.)

Summary/Conclusion

Acknowledgements, questions, references (?)