Representation Theory and its Applications in Physics

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

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Invertibility

If X is a representation of G, then $X(g)^{-1} = X(g^{-1})$, $\forall g \in G$.

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2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

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A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

[2] Invariant subspace $W \subset V$: $X(g)\mathbf{w} \in W$, $\forall\, \mathbf{w} \in W$

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- The decomposition of a representation into irreducibles is unique up to equivalence.

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

[4] If the representation matrices have entries in $\mathbb{C}$.

2 Examples in Physics

Preliminaries

Skip preliminaries?

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle\langle n|\right)|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

Preliminaries: Basic Quantum Mechanics

- Talk about probabilities and whatnot? Eigenvalues = observables? Or just mention when connecting later stuff to physics?

Properties of 2D Rotations

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

The SO(2) Group

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The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all 2×2 orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_{\pm}$).

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- We call J the *generator* of SO(2) rotations.

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Recovering the Rotation Matrix from J

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Theorem

The single-valued irreducible representations of SO(2) are defined as

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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Definition

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all 3×3 orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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Consequences:

- One can obtain the explicit form of $\mathbf{J}$ and subsequently its components J_x, J_y, J_z. These are precisely the angular momentum operators in quantum mechanics.

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Using a similar process to generate SO(3) invariant subspaces that correspond to irreducible representations, we summarize the results in a theorem:

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The irreducible representations of SO(3) are labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ and the $2j+1$ eigenvectors spanning an invariant subspace are labelled by their eigenvalues: $m = -j, -j+1, \ldots, j-1, j$.

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- The eigenvalues of $\mathbf{J}^2$ and J_z are $j(j + 1)$ and m, respectively[7]. In quantum physics, these eigenvalues correspond to the observable total angular momentum and its z-component.
- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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Connection to Quantum Mechanics: Punchline

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But that's not all folks!

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This is the tip of the iceberg!

3 The Braid Group

The Braid Group

Definition

The *configuration space* of n ordered distinct points in the complex plane $\mathbb{C}$ is defined as $M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, \forall i \neq j\}$.

- Note that $(z_1, z_2, z_3, \ldots, z_n)$ and $(z_2, z_1, z_3, \ldots, z_n)$ are distinct configurations in M_n.
- A *braid* β is a *loop*[8] in M_n and can be thought of as a configuration that evolves over time:

$$\begin{aligned} \beta : [0,1] &\to M_n \\ t &\mapsto \beta(t) = \big(\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\big), \end{aligned}$$

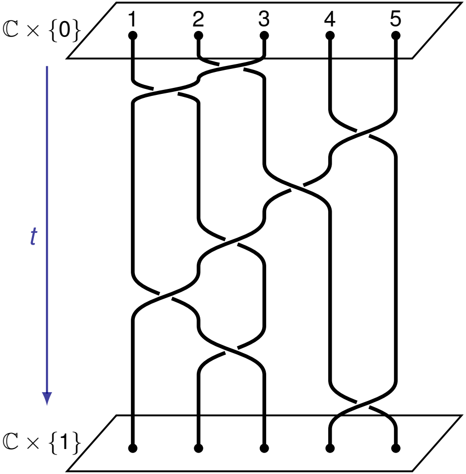
Definition

The *braid group* B_n is the (fundamental) group of all complex-valued n-tuples (M_n) up to *homotopy*.

[8] The topological formalisms that define the braid group are omitted for times sake.

Visualization of Braids

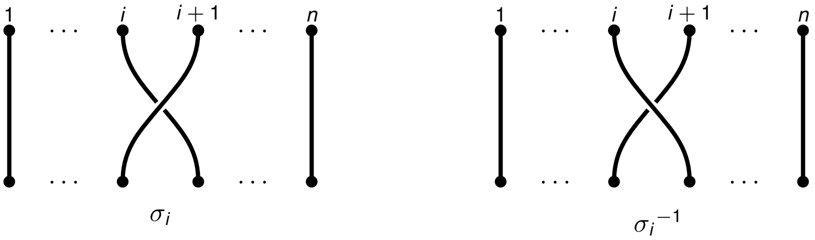
- Each path traced out by a point in the configuration space is a *strand*.
- The number of strands of a braid is equal to the number of points in the configuration space tuples.
- We can think of a braid on n strands as the motion of n distinct points in the complex plane over a normalized time interval.
- Each trajectory is a strand, and the braid is the collection of all strands.
- A braid is defined up to *homotopy*.
- Visualized in $\mathbb{C} \times [0, 1]$.



Braid on 5 strands.

Standard Generators

- Every braid can be decomposed into a finite product of *standard generators* that permute adjacent points.
- The standard generators of B_n are defined as $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$, in which:



- The *degree* of a braid $\beta \in B_n$ is the sum of the powers of the standard generators in the decomposition of β.

Alternative Description of B_n

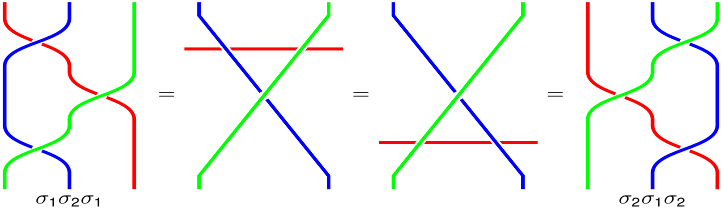
Definition

The braid group on n strands, denoted B_n, is generated by the standard generators that follow the *braid relations*, summarized below:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle .$$

Comment:

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is known as the *Yang-Baxter equation*, visualized below:



One-Dimensional Representations of the Braid Group

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For $\theta \in \mathbb{R}$ and $j = 1, 2, \ldots, n-1$, we define some *one-dimensional representations* of B_n:

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Hence, for any $\beta \in B_n$ with degree k:

$$p_\theta(\beta) = p_\theta(\sigma_1^{m_1}\sigma_2^{m_2}\cdots\sigma_{n-1}^{m_{n-1}}) = e^{i\theta(m_1+m_2+\cdots+m_{n-1})} = e^{ik\theta}.$$

The Burau Representation

- Go through arguments/motivation for Burau?
- Show covering space picture/diagrams?
- Define Burau representation.
- Note on faithfulness!
- Quickly show it's reducible with the **1** eigenvector?

Unitary Representation of the Braid Group

- Define reduced Burau representation.
- Obtain unitary representation from reduced Burau. (Not sure how much detail to go into here.)
- Maybe just jump right to defining the unitary reps in the 2×2 case?
- Comment on why we want a unitary rep!

Nonabelian Characteristics of the Unitary Representation

- Compare and contrast $\mathcal{U}(\sigma_{1,2})$ to their inverses.
- Note that $[\mathcal{U}(\sigma_{1,2}), \mathcal{U}(\sigma_{2,1})] \neq 0$ to highlight nonabelian-ness.

Question

What are the physical implications of this nonabelian representation?

4 Physical Applications of the Braid Group

(Abelian) Braiding Action on a Quantum System

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Braiding action: For any degree-k braid $\beta \in B_n$, we have

$$\psi(r_{1'}, r_{2'}, \ldots, r_{n'}) = p_\theta(\beta)\,\psi(r_1, r_2, \ldots, r_n) = \underbrace{e^{ik\theta}}_{\substack{\text{phase}\\ \text{shift}}}\psi(r_1, r_2, \ldots, r_n),$$

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Remark

The action of a nonabelian braid group representation on a quantum system leads to nontrivial rotations in the many-particle Hilbert space that describes the quantum system.

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- Edge cases: *bosons* and *fermions*.

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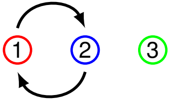

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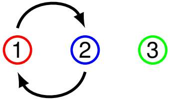
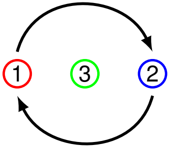


Trajectory B



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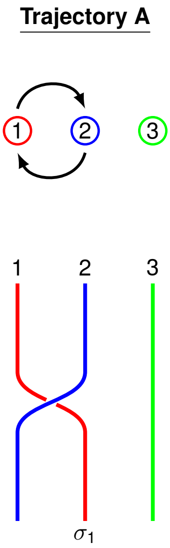

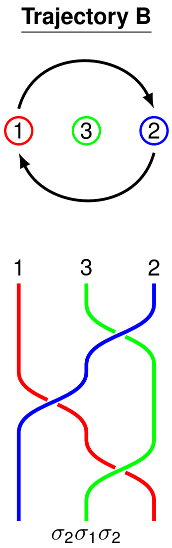

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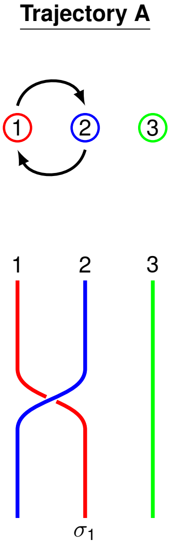
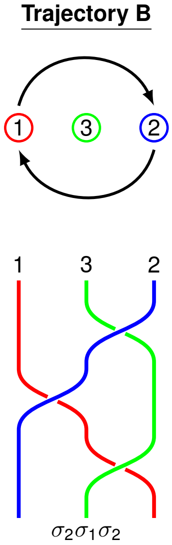
$$\sigma_2\sigma_1\sigma_2 \mapsto e^{3i\theta}$$





Nontrivial Braiding Effects in 1D Representations

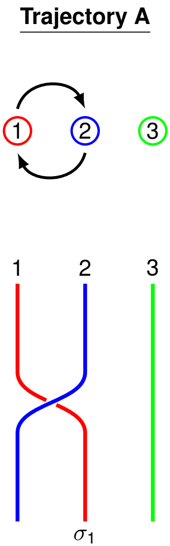
Recall: A braid is only well-defined if all particle trajectories are known.

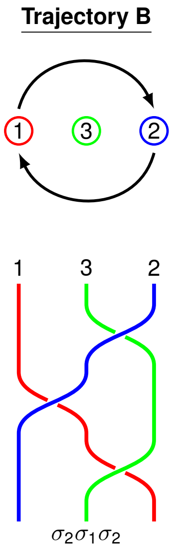
Consequences:

1. A permutation of two anyons requires the knowledge of the positions of all other anyons in the system.
2. This is a consequence of the so-called *nontrivial braiding effects* of the braid group.

1D representation:

$$\left.\begin{aligned} \sigma_1 &\mapsto e^{i\theta} \\ \sigma_2\sigma_1\sigma_2 &\mapsto e^{3i\theta} \end{aligned}\right\} \neq \ \text{if } \theta \notin \pi\mathbb{Z}$$





A Physicists Approach to Anyons (Lagrangian)

Consider two identical non-interacting anyons with positions $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in a harmonic potential. Let $\phi = \arctan\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ be the relative angle between the two anyons and $\dot{\phi} = \frac{d\phi}{dt}$.

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Generalize to N anyons: Let $\phi_{ij} = \arctan\left(\frac{y_j - y_i}{x_j - x_i}\right)$,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{m}{2} \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{i<j}^{N} \dot{\phi}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^{N} \mathbf{r}_i{}^2$$

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$$\mathcal{H}_i = \frac{1}{2m}\big(\underbrace{\mathbf{p}_i - \mathbf{A}_i(\mathbf{r}_i)}_{\text{canonical momentum}}\big)^2 + \frac{m\omega^2}{2} r_i^2$$

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Expand:
$$\mathcal{H} = \boxed{\frac{1}{2m}\sum_{i=1}^{N}p_i^2 + \frac{m\omega^2}{2}\sum_{i=1}^{N}r_i^2 - \frac{\alpha}{2m}\sum_{\substack{i=1\\ j\neq i}}^{N}\frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m}\sum_{\substack{i=1\\ j,k\neq i}}^{N}\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}^2 r_{ik}^2}}$$

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Question

Why is this useful?

Physical Implications of Nontrivial Braiding Effects

- FQHE
- Fusion rules?
- Fault-tolerant quantum computing

Summary/Conclusion

- Summary: what did we talk about?
- What are the main takeaways?
- Acknowledgements, questions, references (?)