# Thesis Title

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# Chapter 1

# Background Info

**Definition 1.1** (Representations of a Group)**.** If there is a homomorphism from a group $G$ to a group of operators $U(G)$ on a linear vector space $V$, we say that $U(G)$ forms a *representation* of $G$ with dimension $\dim V$.

The representation is a map

$$g \in G \xrightarrow{U} U(g) \tag{1.1}$$

in which $U(g)$ is an operator on the vector space $V$. For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \ldots, n\}$, we can realize each operator $U(g)$ as an $n \times n$ matrix $D(g)$.

$$U(g)\left|e_i\right\rangle = \sum_{j=1}^{n} \left|e_j\right\rangle D(g)^j{}_i = \left|e_j\right\rangle D(g)^j{}_i, \tag{1.2}$$

where the first index $j$ is the row index and the second index $i$ is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$U(g_1)U(g_2) = U(g_1 g_2), \tag{1.3}$$

which satisfies the group multiplication rules.

**Definition 1.2.** If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

**Example 1.1.** Let $G$ be the group of continuous rotations in the $xy$-plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space $V_2$. Then we define a representation of $G$ on $V_2$ by the familiar rotation operation

$$\hat{e}_1' = U(\phi)\hat{e}_1 = \hat{e}_1 \cdot \cos\phi + \hat{e}_2 \cdot \sin\phi \tag{1.4}$$

$$\hat{e}_2' = U(\phi)\hat{e}_2 = -\hat{e}_1 \cdot \sin\phi + \hat{e}_2 \cdot \cos\phi. \tag{1.5}$$

This gives us the matrix representation

$$D(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{1.6}$$

To further illustrate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \vec{x} \in V_2$, then we have

$$\vec{x}' = U(\phi)\vec{x} = \hat{e}_j x'^j, \tag{1.7}$$

where $x'^j = D(\phi)^j{}_i x^i$.

**Definition 1.3** (Equivalence of Representations)**.** For a group $G$, two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define

**Definition 1.4** (Characters of a Representation)**.** The *character* $\chi(g)$ of an element $g \in G$ in a representation $U(g)$ is defined as $\chi(g) = \mathrm{Tr}\ D(g)$.

Since trace is independent of basis, the character serves as a class label.

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

**Definition 1.5** (Invariant Subspace)**.** Let $U(G)$ be a representation of $G$ on a vector space $V$, and $W$ a subspace of $V$ such that $U(g)\,|x\rangle \in W$ for all $\vec{x} \in W$ and $g \in G$. Then $W$ is an *invariant subspace* of $V$ with respect to $U(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $U(G)$.

The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

**Definition 1.6** (Irreducible Representation)**.** A representation $U(G)$ on $V$ *irreducible* if there is no non-trivial invariant subspace in $V$ with respect to $U(G)$. Otherwise, it is *reducible.* If $U(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $U(G)$, then the representation is *fully reducible.*

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5**?