Representation Theory and its Applications in Physics

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

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1 Introduction to Representation Theory

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

$$X(gh) = X(g)X(h), \quad \forall g, h \in G.$$

Invertibility

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Consequences:

1. $X(e) = I$, where e is the identity element of the group and I is the identity operator.
2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

Example: The Trivial Representation

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- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not invertible, so we call it a *degenerate representation*.
- If a representation is an isomorphism, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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- The defining representation of S_n is n-dimensional.
- This representation is faithful.

Example: An Intuitive Representation of 2D Rotations

1. Note that representations work for continuous groups too!
2. Define rotation group.
3. Obtain familiar 2D rotation matrix.
4. Can you think of other ways to represent 2D rotations? What about $e^{i\phi}$ parameterization? How many ways to do this? How many ways are unique? What does it mean to be unique?

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Question

How do we classify representations of a group?

Character of a Representation

1. Basics of characters.
2. Similar representations have the same character.
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Answer: Something to lead into irreducibility.

Decomposing Representations

1. Define irreducibility.
2. Relate to invariant subspaces.
3. Schur's Lemmas?
4. Note the consequence of abelian groups and one-dimensional representations. Will be useful later. . .

Example: Irreducible Representation of 2D Rotations

1. Back to our rotation group example...
2. Span of $\mathbf{e}_1$ or $\mathbf{e}_2$ not invariant under rotations.
3. Define $\mathbf{e}_\pm$.
4. Show each is invariant.
5. Decompose previous representation into direct sum of these two.

A Note About Irreducibility

1. Irreducible representations are the building blocks of all representations.
2. Different ways to construct representations from the irreducibles.
3. The irreducibles offer insight into the structure of the group.
4. As for applications in physics, we can use irreps to describe the symmetries of physical systems and understand their consequences.

2 Examples in Physics

Preliminaries

Outline:

1. Dirac notation.
2. Basic quantum mechanics.
3. Quantum Hilbert space.
4. The commutator.

Preliminaries: Dirac Notation

Preliminaries: Basic Quantum Mechanics

2D Rotations and SO(2)

1. The group of 2D rotations is SO(2).
2. General properties of SO(2).

Infinitesimal Rotations

- Go through the derivation of the generator of SO(2) in an appropriate level of detail.

Recovering the Rotation Matrix from J

- Do Taylor expansion thing to get the rotation matrix from J (looks familiar phys majors?)

Irreducible Representations of SO(2)

- Rep generated by J is unitary, J is Hermitian.
- SO(2) abelian implies 1D irreps (reference previous thm's).
- Construct 1D invariant subspaces, obtain 1D irreps.
- Get result about $m \in \mathbb{Z}$ for irrep label.
- Mention ortho/completeness relations?
- State vector decomposition. Probably don't have time to delve into detailed derivations but would be great to show part of the argument for getting explicit differential form of J.

Conservation of Angular Momentum

- Do commutator example with Hamiltonian and J.
- Discuss implications.
- We can do the same thing for translation group which gives us the familiar $\hat{p}$ operator and conservation of linear momentum!

Generalization to 3 Spatial Dimensions

- Show but don't derive $R_{\mathbf{n}}(\theta)$ decomposition into **J** components.
- We have basis from the components of **J**.
- Ladies and gentlemen, we got SO(3)...
- **J** component differential forms?
- Commutation relations, in some form talk about $J_\pm, J^2$ and final eigenvalue results.

Connection to Quantum Mechanics

- Discuss connection between generators and quantum operators, eigenvalues and classical observables, discretization (!), etc.
- This is the kicker. I will get very excited here probably.

Multi-valued Irreducible Representations and Spinors

Not sure where to put this...

- Let's come back to SO(2) for a second...
- Show $m = 1/2$ irreps.
- Discuss implications, spinors, etc...

3 The Braid Group

Basic Definitions

- Formal definitions.
- Physical/intuitive visualization and interpretation.
- Standard generators.
- Automorphisms of $\pi_1(\mathbb{D}_n)$.
- Braid relations in this picture.
- 1D Reps.
- Burau representation.
- Note on faithfulness.
- Unitary representation from reduced Burau.

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4 Physical Applications of the Braid Group

Rotations of Quantum Hilbert Space

- 1D action on Hilbert space, permuting particles, compare/contrast to bosons/fermions.
- Talk about nontrivial braiding effects.
- Example of unitary braid rep acting on Hilbert space.

Anyons: A Consequence of Braiding

- Introduce anyons.
- Discuss how anyons are described by the braid group.
- Fusion rules, abelian vs nonabelian anyons.
- Non-interacting anyons.
- Non-interacting anyons in harmonic potential.
- Nontrivial braiding effects anyone?
- Applications of anyons! (quantum computing, topological quantum field theory, FQHE, etc.)

Summary/Conclusion

Acknowledgements, questions, references (?)