Representation Theory and its Applications in Physics

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CPTitlePage169

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Outline:

1. Introduction to Representation Theory
2. Examples in Physics
3. The Braid Group
4. Physical Applications of the Braid Group

CPSectionPage169

Definition of a Representation

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Remark

If V is finite-dimensional with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, then X can be realized as an $n \times n$ matrix.

Properties of Representations

Group Multiplication

Representations are group morphisms, so they satisfy the group multiplication rule:

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Invertibility

If X is a representation of G, then $X(g)^{-1} = X(g^{-1})$, $\forall g \in G$.

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1. $X(e) = I$, where e is the identity element of the group and I is the identity operator.
2. In the matrix presentation of X, $X(g)$ is invertible for all $g \in G$.

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- The trivial representation is always one-dimensional.
- For groups with more than one element, the trivial representation is not injective, so we call it a *degenerate representation*.
- If a representation is injective, then it is a *faithful representation*.

Example: A Faithful Representation of S_n

Defining representation of S_n

The defining representation D of S_n encodes the action of the symmetric group on the standard basis of $\mathbb{R}^n$. If a permutation sends i to j, then place a 1 the i-th column and j-th row of the representation matrix.

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E.g., in S_3:

$$D((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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Representation: Let X be a representation of R on V_2 with[1]

$$\begin{aligned} X(\phi)\mathbf{e}_1 &= \mathbf{e}_1 \cdot \cos\phi + \mathbf{e}_2 \cdot \sin\phi \\ X(\phi)\mathbf{e}_2 &= -\mathbf{e}_1 \cdot \sin\phi + \mathbf{e}_2 \cdot \cos\phi \end{aligned}$$

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Question

How do we classify representations of a group?

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- We can use characters to classify representations.

Decomposing Representations

Definition

A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace[2] in V with respect to $X(G)$. Otherwise, $X(G)$ is *reducible*.

[2] Invariant subspace $W \subset V$: $X(g)\mathbf{w} \in W, \forall \mathbf{w} \in W$

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- The decomposition of a representation into irreducibles is unique up to equivalence.

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Example: Irreducible Representation of 2D Rotations

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Invariance of $\mathbf{e}_\pm$

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Decomposition of X

The span of each $\mathbf{e}_\pm$ is an X-invariant subspace of V_2. In this basis, we rewrite X as a direct sum of the 1D irreducible representations[3]:

$$X(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix}.$$

[3] 1-dimensional representations are always irreducible!

Schur's Lemmas (pt. 1)

Lemma

Let $X : G \to V$ and $Y : G \to W$ be irreducible representations of a group G. If there exists a fixed linear transformation $T : V \to W$ such that $TX(g) = Y(g)T$ for all $g \in G$, then T is either the zero map or invertible.

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4. By the rank-nullity theorem, conclude that T is either the zero map or invertible.

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4. By previous lemma, $T - \lambda I = 0 \implies T = \lambda I$.

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How does this help in physics?

Irreducible representations can describe symmetries of physical systems with remarkably fundamental implications.

[4] If the representation matrices have entries in $\mathbb{C}$.

CPSectionPage169

Preliminaries

Skip preliminaries?

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5. Operators that are self-adjoint are called *Hermitian*.

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- The action of an operator A on a vector $|\psi\rangle$ is written as $|A\psi\rangle = A\,|\psi\rangle$.
- Equivalent ways to write the same thing:
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|\,A\,|\psi\rangle = \langle\phi|A\psi\rangle\,.$$

Orthonormality, Completeness, and Wavefunctions

Definition

Let $\{|1\rangle, |2\rangle, |3\rangle, \dots\}$ be an orthonormal basis for some quantum Hilbert space. In the context of physics, the orthonormality and completeness relations of the basis vectors allow any state vector $|\psi\rangle$ to be written as a linear combination of the basis vectors:

$$|\psi\rangle = \left(\sum_n |n\rangle \langle n|\right) |\psi\rangle = \sum_n |n\rangle \langle n|\psi\rangle ,$$

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Definition

For a continuous basis labelled by $|x\rangle$ where x is a continuous parameter, the *wavefunction* $\psi(x)$ is the projection: $\langle x|\psi\rangle = \psi(x)$.

Preliminaries: Basic Quantum Mechanics

- Talk about probabilities and whatnot? Eigenvalues = observables? Or just mention when connecting later stuff to physics?

Properties of 2D Rotations

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This special property is summarized by noting $\det R(\phi) = 1$ for all $\phi \in [0, 2\pi)$.

The SO(2) Group

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The *special orthogonal group* in two dimensions, denoted SO(2), is the group of all 2×2 orthogonal matrices with determinant equal to $+1$.[5]

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- SO(2) is *reducible* (earlier example with $\mathbf{e}_\pm$).

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- We call J the *generator* of SO(2) rotations.

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Recovering the Rotation Matrix from J

To first order in $d\phi$: $R(d\phi) = \begin{bmatrix} 1 & -d\phi \\ d\phi & 1 \end{bmatrix}$

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Theorem

The single-valued irreducible representations of SO(2) are defined as

$$U^m(\phi) = e^{-im\phi},\ \forall\, m \in \mathbb{Z}.$$

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Definition

The *special orthogonal group* in three dimensions, denoted SO(3), is the group of all 3×3 orthogonal matrices with determinant equal to $+1$. SO(3) rotations are generated by the components of the Hermitian generator $\mathbf{J} = [J_x, J_y, J_z]^\top$.

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- The eigenvalues of $\mathbf{J}^2$ and J_z are $j(j+1)$ and m, respectively[7]. In quantum physics, these eigenvalues correspond to the observable total angular momentum and its z-component.
- This generalizes to other types of angular momentum, such as *spin angular momentum*!

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Connection to Quantum Mechanics: Punchline

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Discretization of Angular Momentum for Free

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But that's not all folks!

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1. Any system with radial symmetry is invariant under SO(3) rotations, so $[\hat{H}, \mathbf{J}] = 0$.
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3. Similar arguments can be made for the continuous group of translations in space, leading to the conservation of linear momentum for translationally invariant systems.

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This is the tip of the iceberg!

CPSectionPage169

The Braid Group

- Definition: config space and standard visualization

Standard Generators

- σ_i generators.
- Define *degree*?
- Braid relations.
- Skip YBE verification?

Automorphisms of the Free Group

- Automorphisms of $\pi_1(\mathbb{D}_n)$.
- Braid relations in this picture.

One-Dimensional Representations of the Braid Group

- Define 1D reps.
- Show the abelian-ness of these reps

For $\theta \in \mathbb{R}$:

$$\begin{aligned} p_\theta : B_n &\to \mathbb{C}_{|z|=1} \\ \sigma_i &\mapsto e^{i\theta}, \end{aligned}$$

Clearly, p_θ is a homomorphism, and it is unitary because

$$p_\theta(\sigma_i)^\dagger = \left(e^{i\theta}\right)^\dagger = e^{-i\theta} = \left(e^{i\theta}\right)^{-1} = p_\theta(\sigma_i)^{-1}.$$

The Burau Representation

- Go through arguments/motivation for Burau?
- Show covering space picture/diagrams?
- Define Burau representation.
- Note on faithfulness!
- Quickly show it’s reducible with the $\mathbf{1}$ eigenvector?

Unitary Representation of the Braid Group

- Define reduced Burau representation.
- Obtain unitary representation from reduced Burau. (Not sure how much detail to go into here.)
- Maybe just jump right to defining the unitary reps in the 2×2 case?
- Comment on why we want a unitary rep!

Nonabelian Characteristics of the Unitary Representation

- Compare and contrast $\mathcal{U}(\sigma_{1,2})$ to their inverses.
- Note that $[\mathcal{U}(\sigma_{1,2}), \mathcal{U}(\sigma_{2,1})] \neq 0$ to highlight nonabelian-ness.

Question

What are the physical implications of this nonabelian representation?

CPSectionPage169

Braiding Action on a Quantum System

- 1D action on Hilbert space, permuting particles, compare/contrast to bosons/fermions. Note the abelian characteristics of this rep.
- Talk about nontrivial braiding effects.
- Example of unitary braid rep acting on Hilbert space to highlight nonabelian-ness.

Anyons: A Consequence of Braiding

- Introduce anyons.
- Discuss how anyons are described by the braid group.
- Abelian vs nonabelian anyons.
- Mention fusion rules.

Nontrivial Braiding Effects in 1D Representations

- Introduce the idea (define it).
- Show diagram to illustrate nontrivial braiding effects qualitatively.
- Nontrivial braiding in 1D rep corresponding to diagram.
- Hint at a greater conclusion but first need to look into the physics perspective...

A Physicists Approach to Anyons (pt. 1)

- Non-interacting anyons.

A Physicists Approach to Anyons (pt. 2)

- Non-interacting anyons in harmonic potential.
- Arrive at N-anyon Hamiltonian.

Nontrivial Braiding Effects in the Hamiltonian

- Compare $N = 2$ to $N = 3$ Hamiltonian from previous slide.
- Highlight nontrivial braiding effects in the $N = 3$ case.
- How does this compare to bosons/fermions? (maybe redundant depending on the depth of the previous discussion)
- Question the physical implications.

Physical Implications of Nontrivial Braiding Effects

- FQHE
- Fault-tolerant quantum computing

Summary/Conclusion

- Outline the talk: what did we talk about?
- What are the takeaways?
- Acknowledgements, questions, references (?)