Title

Max Varverakis

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Contents

Chapter 1

An Introduction to Representation Theory

1.1 Introduction

Intro paragraph to lead into the definitions.

Definition 1.1 (Representation of a group)**.** Let G be a group. A *representation* of G is a homomorphism from G to a group of operators on a linear vector space V. The dimension of V is the *dimension* or *degree* of the representation.

If X is a representation of G on V, then X is a map

$$g \in G \xrightarrow{X} X(g) \tag{1.1}$$

in which $X(g)$ is an operator on the vector space V. For a set of basis vectors $\{\hat{e}_i, i = 1, 2, \ldots, n\}$, we can realize each operator $X(g)$ as an $n \times n$ matrix $D(g)$.

$$X(g) \left| e_i \right\rangle = \sum_{j=1}^{n} \left| e_j \right\rangle D(g)^j{}_i = \left| e_j \right\rangle D(g)^j{}_i, \tag{1.2}$$

where the first index j is the row index and the second index i is the column index. We use the Einstein summation convention, so repeated indices are summed over. Note that the operator multiplication is defined as

$$X(g_1)X(g_2) = X(g_1 g_2), \tag{1.3}$$

which satisfies the group multiplication rules.

Definition 1.2. If the homomorphism defining the representation is an isomorphism, then the representation is *faithful*. Otherwise, it is *degenerate*.

Example 1.1. Consider the symmetric group S_n. The *defining* representation of S_n encodes each $\sigma \in S_n$ by placing a 1 in the j-th row and i-th column of the matrix $D(\sigma)$ if σ sends i to j, and 0 otherwise. For example, in S_3, the permutation (23) has the matrix representation

$$D\big((23)\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

whereas the permutation (123) is realized by the matrix

$$D\big((123)\big) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The above example involves a finite group. Infinite groups can also have representations, as demonstrated in the following example.

Example 1.2. Let G be the group of continuous rotations in the xy-plane about the origin. We can write $G = \{R(\phi), 0 \leq \phi \leq 2\pi\}$ with group operation $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$. Consider the 2-dimensional Euclidean vector space V_2. Then we define a representation of G on V_2 by the familiar rotation operation

$$\hat{e}_1' = X(\phi)\hat{e}_1 = \hat{e}_1 \cdot \cos\phi + \hat{e}_2 \cdot \sin\phi \tag{1.4}$$

$$\hat{e}_2' = X(\phi)\hat{e}_2 = -\hat{e}_1 \cdot \sin\phi + \hat{e}_2 \cdot \cos\phi, \tag{1.5}$$

where $\hat{e}_1$ and $\hat{e}_2$ are orthonormal basis vectors of V_2. This gives us the matrix representation

$$D(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{1.6}$$

To further illuminate this representation, if we consider an arbitrary vector $\hat{e}_i x^i = \vec{x} \in V_2$, then we have

$$\vec{x}\,' = X(\phi)\vec{x} = \hat{e}_j x'^j, \tag{1.7}$$

where $x'^j = D(\phi)^j{}_i x^i$. Can probably simplify the notation

Definition 1.3 (Equivalence of Representations)**.** For a group G, two representations are *equivalent* if they are related by a similarity transformation. Equivalent representations form an equivalence class.

To determine whether two representations belong to the same equivalence class, we define the following.

Definition 1.4 (Characters of a Representation)**.** The *character* $\chi(g)$ of an element $g \in G$ in a representation $X(g)$ is defined as $\chi(g) = \text{Tr } D(g)$.

Since trace is independent of basis, the character serves as a class label.

Vector space representations of a group have familiar substructures, which are useful in constructing representations of the group.

Definition 1.5 (Invariant Subspace)**.** Let $X(G)$ be a representation of G on a vector space V, and W a subspace of V such that $X(g)\,|x\rangle \in W$ for all $\vec{x} \in W$ and $g \in G$. Then W is an *invariant subspace* of V with respect to $X(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $X(G)$.

The identification of invariant subspaces on vector space representations leads to the following distinction of the representations.

Definition 1.6 (Irreducible Representation)**.** A representation $X(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $X(G)$. Otherwise, it is *reducible.* If $X(G)$ is reducible and its orthogonal complement to the invariant subspace is also invariant with respect to $X(G)$, then the representation is *fully reducible.*

Example 1.3. Under the group of 2-dimensional rotations, consider the 1-dimensional subspace spanned by $\hat{e}_1$. This subspace is not invariant under 2-dimensional rotations, because a rotation of $\hat{e}_1$ by $\pi/2$ results in the vector $\hat{e}_2$ that is clearly not in the subspace spanned by $\hat{e}_1$. A similar argument shows that the subspace spanned by $\hat{e}_2$ is not invariant under 2-dimensional rotations.

The irreducible representation matrices satisfy orthonormality and completeness relations. **Thm. 3.5**?

1.2 Rotations in a plane and the group SO(2)

R vs U inconsistency from earlier notation

1.2.1 The rotation group

Consider the rotations of a 2-dimensional Euclidean vector space about the origin. Let $\hat{e}_1$ and $\hat{e}_2$ be orthonormal basis vectors of this space. Using geometry, we can determine how a rotation by some angle ϕ, written in operator form as $R(\phi)$, acts on the basis vectors:

$$R(\phi)\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi \tag{1.8}$$
$$R(\phi)\hat{e}_2 = -\hat{e}_1 \sin\phi + \hat{e}_2 \cos\phi. \tag{1.9}$$

In matrix form, we can write

$$R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{1.10}$$

which allows us to write Eqns. 1.8 and 1.9 in a condensed form

$$R(\phi)\hat{e}_i = \hat{e}_j R(\phi)^j{}_i, \tag{1.11}$$

where we are summing over $j = 1, 2$.

Let $\vec{x}$ be an arbitrary vector in the plane. Then $\vec{x}$ has components x^i in the basis $\{\hat{e}_i\}$, where $i = 1, 2$. Equivalently, we can write $\vec{x} = \hat{e}_i x^i$. Then under rotations, $\vec{x}$ transforms in accordance to the basis vectors

$$\begin{aligned} R(\phi)\vec{x} &= R(\phi)\hat{e}_i x^i \qquad (1.12) \\ &= \hat{e}_j R(\phi)^j{}_i x^i \\ &= \left(\hat{e}_1 R(\phi)^1{}_i + \hat{e}_2 R(\phi)^2{}_i\right) x^i \\ &= \left(\hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi\right) x^1 + \left(\hat{e}_1(-\sin\phi) + \hat{e}_2 \cos\phi\right) x^2 \\ &= \left(x^1 \cos\phi - x^2 \sin\phi\right) \hat{e}_1 + \left(x^1 \sin\phi + x^2 \cos\phi\right) \hat{e}_2. \end{aligned}$$

Notice that $R(\phi)R^\top(\phi) = E$ where E is the identity matrix. This is precisely what defines *orthogonal matrices*. For 2-dimensional vectors in the plane, it is clear that these rotations do not change the length of said vectors. This can be verified by using Eqn. 1.12:

$$\begin{aligned} |R(\phi)\vec{x}|^2 &= |\hat{e}_j R(\phi)^j{}_i x^i|^2 \\ &= \left|\left(x^1\cos\phi - x^2\sin\phi\right)\hat{e}_1 + \left(x^1\sin\phi + x^2\cos\phi\right)\hat{e}_2\right|^2 \\ &= \left(x^1\cos\phi - x^2\sin\phi\right)^2 + \left(x^1\sin\phi + x^2\cos\phi\right)^2 \\ &= \left(\cos^2\phi + \sin^2\phi\right)x^1x_1 + \left(\sin^2\phi + \cos^2\phi\right)x^2x_2 \\ &= x^1x_1 + x^2x_2 = |\vec{x}|^2. \end{aligned} \tag{1.13}$$

Similarly, notice that for any continuous rotation by angle ϕ, $\det R(\phi) = \cos^2\phi + \sin^2\phi = 1$. In general, orthogonal matrices have determinant equal to ± 1. However, the result of the above determinant of $R(\phi)$ implies that all continuous rotations in the 2-dimensional plane have determinant equal to $+1$. These are the *special orthogonal matrices of rank 2*. This family of matrices is denoted SO(2). Furthermore, there is a one-to-one correspondence with SO(2) matrices and rotations in a plane.

We define the group of continuous rotations in a plane by letting $R(0) = E$ be the identity element corresponding to no rotation (i.e., a rotation by angle $\phi = 0$), and defining the inverse of a rotation as $R^{-1}(\phi) = R(-\phi) = R(2\pi - \phi)$. This group can be called the SO(2) group. Lastly, we define group multiplication as $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2)$ and note that $R(\phi) = R(\phi \pm 2\pi)$, which can be verified geometrically. Thus, group elements of SO(2) can be labelled by the angle of rotation $\phi \in [0, 2\pi)$.

1.2.2 Infinitesimal rotations

Consider an infinitesimal rotation labelled by some infinitesimal angle $d\phi$. This is equivalent to the identity plus some small rotation, which can be written as

$$R(\mathrm{d}\phi) = E - i\mathrm{d}\phi J \tag{1.14}$$

where the scalar quantity $-i$ is introduced for later convenience and J is some quantity independent of the rotation angle. If we consider the rotation $R(\phi + \mathrm{d}\phi)$, then there are two equivalent ways to interpret this rotation

$$R(\phi + \mathrm{d}\phi) = R(\phi)R(\mathrm{d}\phi) = R(\phi)(E - i\mathrm{d}\phi J) = R(\phi) - i\mathrm{d}\phi R(\phi)J, \tag{1.15}$$

$$R(\phi + \mathrm{d}\phi) = R(\phi) + \mathrm{d}R(\phi) = R(\phi) + \mathrm{d}\phi\frac{\mathrm{d}R(\phi)}{\mathrm{d}\phi}, \tag{1.16}$$

where the second equation can be thought of as a Taylor expansion of $R(\phi + \mathrm{d}\phi)$ about ϕ. Equating the two expressions for $R(\phi + \mathrm{d}\phi)$ yields

$$\mathrm{d}R(\phi) = -i\mathrm{d}\phi R(\phi) J. \tag{1.17}$$

Solving this differential equation (with boundary condition $R(0) = E$) provides us with an equation for any group element involving J:

$$R(\phi) = e^{-i\phi J}, \tag{1.18}$$

where J is called the *generator* of the group.

The explicit form of J is found as follows. To first order in $\mathrm{d}\phi$, we have

$$R(\mathrm{d}\phi) = \begin{bmatrix} 1 & -\mathrm{d}\phi \\ \mathrm{d}\phi & 1 \end{bmatrix}.$$

Comparing to Eqn. 1.14,

$$E - i\mathrm{d}\phi J = \begin{bmatrix} 1 & -\mathrm{d}\phi \\ \mathrm{d}\phi & 1 \end{bmatrix} \implies J = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

In fact, notice that $J^2 = E$, which implies that even powers of J equal the identity matrix and odd powers of J equal J. Taylor expanding $e^{-iJ\phi}$ gives

$$\begin{aligned} R(\phi) = e^{-iJ\phi} &= E - iJ\phi - E\frac{\phi^2}{2!} - iJ\frac{\phi^3}{3!} + \cdots \\ &= E\left(\sum_{n=0}^{\infty}(-1)^n\frac{\phi^{2n}}{(2n)!}\right) - iJ\left(\sum_{n=0}^{\infty}(-1)^n\frac{\phi^{2n+1}}{(2n+1)!}\right) \\ &= E\cos\phi - iJ\sin\phi \\ &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \end{aligned}$$

Therefore, the generator J can be used to recover the rotation matrix for an arbitrary angle ϕ. Clearly, the map $R(\phi) \mapsto e^{-iJ\phi}$ is a valid homomorphism that respects the periodic nature of SO(2).

1.3 Irreducible representations of SO(2)

With the generator J in hand, we can now construct the irreducible representations of SO(2). First, consider a representation U of SO(2) defined on a finite dimensional vector space V. Then $U(\phi)$ is the corresponding matrix representation of $R(\phi)$. The same argument as in Section 1.2.2 can be applied to an infinitesimal rotation to give

$$U(\phi) = e^{-iJ\phi},$$

which is an operator on V.

1.4 Note

Possible reference here [7]!

Suppose G is an operator on a quantum Hilbert space of states. The quantity $\langle G \rangle$ is conserved if

$$\frac{d \langle G \rangle}{dt} = 0.$$

Recall the time-dependent Schrödinger equation

$$\hat{H}\psi = i\hbar \frac{d\psi}{dt} \implies \frac{d\psi}{dt} = \frac{1}{i\hbar}\hat{H}\psi.$$

Then if G is time-independent we have

$$\begin{aligned}
\frac{d\langle G\rangle}{dt} &= \frac{d}{dt}\langle\psi|G|\psi\rangle \\
&= \left\langle \frac{d\psi}{dt}\middle| G \middle| \psi\right\rangle + \left\langle \psi \middle| G \middle| \frac{d\psi}{dt}\right\rangle + \cancelto{0}{\left\langle \psi \middle| \frac{\partial G}{\partial t} \middle| \psi\right\rangle} \\
&= \left\langle \frac{1}{i\hbar}\hat{H}\psi \middle| G \middle| \psi\right\rangle + \left\langle \psi \middle| G \middle| \frac{1}{i\hbar}\hat{H}\psi\right\rangle \\
&= \frac{i}{\hbar}\left(\langle\hat{H}\psi|G|\psi\rangle - \langle\psi|G|\hat{H}\psi\rangle\right) \\
&= \frac{i}{\hbar}\left(\langle\psi|\hat{H}^{\dagger}G|\psi\rangle - \langle\psi|G\hat{H}|\psi\rangle\right) \\
&= \frac{i}{\hbar}\left(\langle\psi|\hat{H}G|\psi\rangle - \langle\psi|G\hat{H}|\psi\rangle\right) \text{ because } \hat{H} \text{ is Hermitian} \\
&= \frac{i}{\hbar}\langle\psi|\,(\hat{H}G - G\hat{H})\,|\psi\rangle \\
&= \frac{i}{\hbar}\langle\psi|\,[\hat{H},G]\,|\psi\rangle = 0 \iff [\hat{H},G] = 0.
\end{aligned}$$

(linear in the second argument). (See Ehrenfest's theorem).

Thus, if $[\hat{H},G]=0$ and G is unitary, it follows that

$$\begin{aligned}
\hat{H}G - G\hat{H} = 0 &\iff \hat{H}G = G\hat{H} \\
&\iff G^{-1}\hat{H}G = \hat{H}.
\end{aligned}$$

Thus, $G^{-1}\hat{H}G$ and $\hat{H}$ share the same eigenvalues (observables), which is only true if $\hat{H}$ is invariant under G. If G generates a group of transformations, then $\hat{H}$ is invariant under the group of transformations generated by G. Often times, this invariance is expressed as

$$G^{\dagger}\hat{H}G = \hat{H}$$

since G is unitary.

Running the argument in reverse, if $\hat{H}$ is invariant under the transformations generated by G, then $[\hat{H},G]=0$, which, by the Ehrenfest theorem, implies that $\langle G\rangle$ is conserved.

Therefore, if a physical system represented by a Hamiltonian H is invariant under rotations, then $[H,R(\phi)]=0$ for all $\phi\in\mathbb{R}$ and thus $[H,J]=0$, so angular momentum is conserved.

Chapter 4

To-Do List

Potential committee members:

- **Anton Kaul**
- **Patrick Orson**
- **Eric Brussel**
- *Rob Easton*

- **First:** Do Lorentz group example, finish off content that you want in there. Then go back and fix notation once you know that you won't need it for Lorentz group. Alternatively, you can just invent the notation solely for that example!
- Redo the Chapter 1 with nicer notation and stray away from Tung's notation when possible.
- More straightforward examples of representations?
- At least briefly discuss $U(n)$ either here or in braid rep chapter.
- Finish/modify irreducible rep. example in Chapter 1.
- Do more for SO(2) example?
- Fix out equation numbers

- **Examples to add:** 1D conservation of momentum from T, ang-momentum connection to SO(2)?, ~~Lorentz group~~
- Do a little more context on the physics. Maybe put in appendix, but describe what the heck a quantum Hilbert space is and braket notation.

- Show $\psi_n(\sigma_i)$ invertible? Yes, eventually
- derive $\psi_n^{\mathbf{r}}(\sigma_i)$ matrices or state?
- Show $\psi_n^{\mathbf{r}}(\sigma_i)$ invertible? Yes, eventually
- ~~Explicitly show why Burau isn't able to be made unitary? [3]~~
- Separate chapters into braid group and braid group reps.?

- Concluding paragraph on first section to lead into the more physics-y stuff.
- ~~Show the additional cross terms from $N = 2$ to $N = 3$ and beyond.~~
- Add paragraph on gauge theory/motivation.
- Anyon fusion rules
- τ anyon/Fibonacci anyon example. Relate to singlet/triplet states in spin-1/2 system.
- ~~Move anyon calculations to appendix?~~
- Spend some time on `MATLAB` thing

- Conclusion/future of anyons/braid group in physics.
- Abstract
- Title
- Acknowledgements

Format!!

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