Physics 408

Build Your Own Problems

Max Varverakis

September 30, 2023

Griffiths 1.34

This problem is a modification of Griffiths 1.34. The original problem asks us to verify Stokes' theorem for the vector function $\vec{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$ using the triangular surface made by connecting vertices $(0,0,0)$, $(0,2,0)$, and $(0,0,2)$. In the original problem, the integrals are shown to be

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_C \vec{v} \cdot d\vec{l} = -\frac{8}{3}. \tag{1}$$

Adaptation

I want to modify the surface that we use to test Stokes' theorem. In particular, I want to connect the points $(0,2,0)$ and $(0,0,2)$ by a circular arc.

Solution

The figure below depicts the modified surface. Each path segment is labeled with a number.

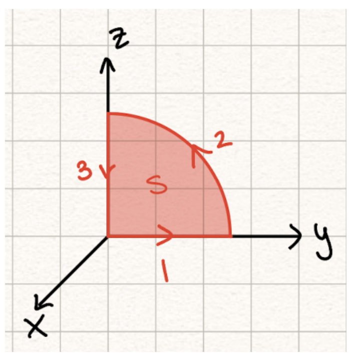

Figure 1: Modified surface for Stokes' theorem.

First, let's calculate the surface integral. We have the following:

- $d\vec{a} = dydz\hat{\mathbf{x}}$ as by the right-hand rule convention.
- $\vec{\nabla} \times \vec{v} = (-2y)\hat{\mathbf{x}} + (-3z)\hat{\mathbf{y}} + (-x)\hat{\mathbf{z}} = (-2y)\hat{\mathbf{x}} + (-3z)\hat{\mathbf{y}} + 0 \cdot \hat{\mathbf{z}}$ since $x = 0$ for this surface. Therefore, $(\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = (-2y)dydz$.
- Note that the upper integration bound for y is $\sqrt{4 - z^2}$ since path 2 is an arc from a circle of radius 2 centered at $(0,0,0)$ with corresponding equation $y^2 + z^2 = 4$.

The surface integral is then

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \int_0^2 dz \int_0^{\sqrt{4-z^2}} (-2y)dy = -\frac{16}{3}. \tag{2}$$

Next, we calculate the integral over the boundary of the surface. Since $d\vec{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$, it follows that

$$\vec{v} \cdot d\vec{l} = (xy)dx + (2yz)dy + (3zx)dz = 2yzdy \tag{3}$$

with the simplification due to $x = 0$.

Let's calculate the line integral over each path segment:

1. For path (1), $x = z = dx = dz = 0$ where $y : 0 \to 2$. Plugging in, this gives $\vec{v} \cdot d\vec{l} = 0$. Clearly, this path segment does not contribute to the line integral. In other words,
$$\int \vec{v} \cdot d\vec{l} = 0. \tag{4}$$

2. For path (2), $x = dx = 0$. We parameterize y in terms of z by $y = \sqrt{4 - z^2}$ where $z : 0 \to 2$. Thus, $dy = -\frac{z}{\sqrt{4-z^2}}dz$. Now, in terms of our parameterization, we can rewrite $\vec{v} \cdot d\vec{l} = 2yzdy = -2z^2dz$. Therefore, the line integral over path (2) is
$$\int \vec{v} \cdot d\vec{l} = -2\int_0^2 z^2dz = -\frac{16}{3}. \tag{5}$$

3. Lastly, we have to integrate over path (3). We have $x = y = dx = dy = 0$ and $z : 2 \to 0$. Thus, $\vec{v} \cdot d\vec{l} = 0$ after plugging in for y and dy. Similar to path (1), $\vec{v} \cdot d\vec{l} = 0 \implies \int \vec{v} \cdot d\vec{l} = 0$.

To find the integral around the closed loop of (1) $\to$ (2) $\to$ (3), we add the line integrals over each path segment to obtain

$$\oint \vec{v} \cdot d\vec{l} = 0 - \frac{16}{3} + 0 = -\frac{16}{3}. \tag{6}$$

Both the surface and line integrals are equal to $-\frac{16}{3}$, so Stokes' theorem is verified for this modified surface!