Engineering Dynamics Assignment

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Figure 1: A worker using a hand-held vibratory tool (drill) exposed to hand-arm vibration.

Introduction

Resonance exists not only in mechanical systems but also in biological systems, such as the human body, which can experience resonant motion. For example, when using a power drill, as shown in Figure 1, the hand and arm vibrate. With continuous use of vibratory tools over many years, laborers are exposed to the risk of developing degenerative hand-arm vibration syndrome (HAVS). Therefore, studying the vibration dynamics of the hand-arm system is essential to design future tools that suppress these oscillations. In this report, we derive a dynamical model to determine the eigenfrequencies, eigenmodes, and damping ratio of this hand-arm system.

The report begins with an overview of how the model is constructed, followed by a discussion of the kinematics, energies, and equations of motion. The system is then linearized around the equilibrium point to determine the eigenfrequencies and eigenmodes for both undamped and damped conditions. Finally, we visualize the system and present a conclusion.

The Model

If modelled with all its complexities, the hand-arm vibratory system is a three-dimensional problem consisting of many degrees of freedom. The muscles are continuum elements that show viscoelastic behavior upon loading, and bones are stiff parts that undergo elastic deformations when subjected to external forces. In the assignment though, we use a simple model comprising discrete mechanical elements to explain the dynamics of this system (see Figure 2). Our goal is to model the motion only in the x - y plane. For this we use three masses (m_1, m_2, and m_3), five translational springs (k_1 - k_5), one rotational spring ($k_(t3)$), five translational dampers (c_1 - c_5), and one rotational damper ($c_(t3)$). Masses m_1, m_2, and m_3 represent the palm, lower arm, and upper arm, respectively. J_3g is the mass moment of inertia for m_3 with respect to its center of mass. The upper arm (m_3) is attached to the lower arm (m_2) with the elbow joint and to the body (environment) with the shoulder joint. Lg

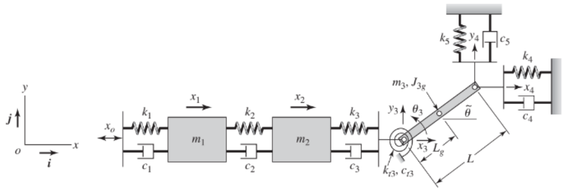

Figure 2: Mass-spring-damper model for the hand-arm vibration system.

shows the center of mass of the upper arm with respect to the elbow joint whose total length is L. On the other hand, $\tilde{\theta}$ is the orientation of the upper arm while working with the hand-held vibratory tool. For the worker's comfort, and ease of usage of the vibrating tool, it is desired that this angle remains constant while using the tool. In our simple model, all springs and dampers are assumed to have zero mass and only move linearly in their defined direction. The system is excited externally with a prescribed displacement x_0, and $\dot{x_0}$ applied to m_1. It is worth noting that masses m_1 and m_2 can only move along the x-direction of our local coordinate; however, elbow and shoulder joints can move both vertically and horizontally.

Vibration Analysis

Kinematics

To model the motion of the hand-arm vibration system, we need to define the generalized coordinates. These coordinates describe the system's movement and allow us to capture its behavior in a concise mathematical form. For this system, the generalized coordinates were chosen to be $x_1 = q_1$, $x_2 = q_2$, $x_3 = q_3$, $y_3 = q_4$ and $\theta_3 = q_5$. Therefore resulting in the vector:

$$q^T = \begin{bmatrix} x_1 & x_2 & x_3 & y_3 & \theta_3 \end{bmatrix}.$$

Where x_1 represents the horizontal displacement of the palm m_1, x_2 represents the horizontal displacement of the lower arm m_2, x_3 represents the horizontal displacement of the upper arm m_3, y_3 represents the vertical displacement of the upper arm m_3 and θ_3 represents the rotational angle of the upper arm m_3.

Using these coordinates, the displacement of each segment of the arm in the x- and y-directions can be expressed. For the upper part of the upper arm (represented by mass m_3), the horizontal and vertical positions are given by x_4 and y_4 but the following hereunder are also the holonomic constraints:

$$\begin{aligned} y_1 &= 0 \\ y_2 &= 0 \\ x_4 &= x_3 + L\cos(\theta_3) \\ y_4 &= y_3 + L\sin(\theta_3) \end{aligned}$$

Where L is the length of the upper arm, and θ_3 is the angle of rotation about the elbow joint. The degrees of freedom the system has been determined using the table below:

Table 1: System Identification: N is number of nodes, Coordinates is the number of free movement without constraints, Rigid is the number of rigid links in the system and DOF is the degrees of freedom

N	Coordinates	Rigid	DOF
4	8	3	5

The system therefore has five degrees of freedom (DOF), which correspond to the five independent motions described by the generalized coordinates.

Energies

In systems like the hand-arm system, the vibrational energy transmitted through a tool (such as a power drill) interacts with the body, potentially leading to Hand-Arm Vibration Syndrome (HAVS) over time. The analysis of the system's energies helps us understand how much vibration energy is absorbed by the body and where it gets dissipated.

So, the next step is to calculate the energies of the system, which will be essential for formulating the Lagrangian equations of motion later on. The system's total energy consists of the kinetic (T), potential (V) and dissipative (D) energy components. We use the following formulas to calculate the energies as seen for translation and rotation.

$$T = T_{transl} + T_{rot} = \frac{1}{2}\sum_i m_i \dot{q}_i^2 + \frac{1}{2}\sum_j I_j \dot{\theta}_j^2$$

$$V = \frac{1}{2}\sum_i k_i (q_i - q_{i,0})^2 + \frac{1}{2}\sum_j k_{\theta_j}(\theta_j - \theta_{j,0})^2$$

$$D = \frac{1}{2}\sum_i c_i \dot{q}_i^2 + \frac{1}{2}\sum_j c_{\theta_j}\dot{\theta}_j^2$$

with $i = 1, 2, 3, 4$ and $j = 3$ in this report's case.
Applying these formulas to the hand-arm vibration system, the following energies can be derived:

$$T_{\text{total}} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\left(\dot{x}_3 - L_g\dot{\theta}_3\sin\theta_3\right)^2 + \frac{1}{2}m_3\left(\dot{y}_3 + L_g\dot{\theta}_3\cos\theta_3\right)^2 + \frac{1}{2}J_{3g}\dot{\theta}_3^2$$

$$V_{\text{total}} = \frac{1}{2}k_1(x_0 - x_1)^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3(x_2 - x_3)^2 + \frac{1}{2}k_{t3}(\theta_3 - \hat{\theta}_3)^2 + \frac{1}{2}k_4\left(x_3 + L\cos\theta_3\right)^2 + \frac{1}{2}k_5\left(y_3 + L\sin\theta_3\right)^2$$

$$D_{\text{total}} = \frac{1}{2}c_1(\dot{x}_0 - \dot{x}_1)^2 + \frac{1}{2}c_2(\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{2}c_3(\dot{x}_2 - \dot{x}_3)^2 + \frac{1}{2}c_{t3}\dot{\theta}_3^2 + \frac{1}{2}c_4\left(\dot{x}_3 - L\dot{\theta}_3\sin\theta_3\right)^2 + \frac{1}{2}c_5\left(\dot{y}_3 + L\dot{\theta}_3\cos\theta_3\right)^2$$

Equations of motion

Now that we have the expressions for the kinetic energy T, potential energy V, and dissipation function D, we can apply the Lagrange equation to derive the equations of motion for the hand-arm vibration system. The Lagrange equation is formulated as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0$$

Since we determined that the system has 5 degrees of freedom, the result is 5 equations of motion.
Equation for x_1:

$$\begin{aligned} &m_1\frac{d^2x_1(t)}{dt^2} - \frac{c_1}{2}\left(2\frac{dx_0(t)}{dt} - 2\frac{dx_1(t)}{dt}\right) + \frac{c_2}{2}\left(2\frac{dx_1(t)}{dt} - 2\frac{dx_2(t)}{dt}\right) \\ &- \frac{k_1}{2}\left(2x_0(t) - 2x_1(t)\right) + \frac{k_2}{2}\left(2x_1(t) - 2x_2(t)\right) \end{aligned} \tag{1}$$

Equation for x_2:

$$\begin{aligned} &m_2\frac{d^2x_2(t)}{dt^2} - \frac{c_2}{2}\left(2\frac{dx_1(t)}{dt} - 2\frac{dx_2(t)}{dt}\right) + \frac{c_3}{2}\left(2\frac{dx_2(t)}{dt} - 2\frac{dx_3(t)}{dt}\right) \\ &- \frac{k_2}{2}\left(2x_1(t) - 2x_2(t)\right) + \frac{k_3}{2}\left(2x_2(t) - 2x_3(t)\right) \end{aligned} \tag{2}$$

Equation for x_3:

$$\begin{aligned} &\frac{c_4}{2}\left(2\frac{dx_3(t)}{dt} - 2L\sin(\theta_3(t))\frac{d\theta_3(t)}{dt}\right) \\ &- \frac{m_3}{2}\left(2L_g\cos(\theta_3(t))\left(\frac{d\theta_3(t)}{dt}\right)^2 - 2\frac{d^2x_3(t)}{dt^2} + 2L_g\sin(\theta_3(t))\frac{d^2\theta_3(t)}{dt^2}\right) \\ &- \frac{c_3}{2}\left(2\frac{dx_2(t)}{dt} - 2\frac{dx_3(t)}{dt}\right) + \frac{k_4}{2}\left(2x_3(t) + 2L\cos(\theta_3(t))\right) - \frac{k_3}{2}\left(2x_2(t) - 2x_3(t)\right) \end{aligned} \tag{3}$$

Equation for y_3:

$$\begin{aligned}\frac{m_3}{2}&\left(2L_g\cos(\theta_3(t)rl)\frac{d^2\theta_3(t)}{dt^2}-2L_g\sin(\theta_3(t))\left(\frac{d\theta_3(t)}{dt}\right)^2+2\frac{d^2y_3(t)}{dt^2}\right)\\&+\frac{k_5}{2}\left(2y_3(t)+2L\sin(\theta_3(t))\right)+\frac{c_5}{2}\left(2\frac{dy_3(t)}{dt}+2L\cos(\theta_3(t))\frac{d\theta_3(t)}{dt}\right)\end{aligned}\tag{4}$$

Equation for θ_3:

$$\begin{aligned}&c_{t3}\frac{d\theta_3(t)}{dt}-\frac{k_{t3}}{2}\left(2\tilde{\theta}-2\theta_3(t)\right)+J_{3g}\frac{d^2\theta_3(t)}{dt^2}\\&+Lc_5\cos(\theta_3(t))\left(\frac{dy_3(t)}{dt}+L\cos(\theta_3(t))\frac{d\theta_3(t)}{dt}\right)\\&+L_gm_3\cos(\theta_3(t))\left(L_g\cos(\theta_3(t))\frac{d^2\theta_3(t)}{dt^2}-L_g\sin(\theta_3(t))\left(\frac{d\theta_3(t)}{dt}\right)^2+\frac{d^2y_3(t)}{dt^2}\right)\\&-Lc_4\sin(\theta_3(t))\left(\frac{dx_3(t)}{dt}-L\sin(\theta_3(t))\frac{d\theta_3(t)}{dt}\right)\\&-Lk_4\sin(\theta_3(t))\left(x_3(t)+L\cos(\theta_3(t))\right)+Lk_5\cos(\theta_3(t))\left(y_3(t)+L\sin(\theta_3(t))\right)\\&+L_gm_3\sin(\theta_3(t))\left(L_g\cos(\theta_3(t))\left(\frac{d\theta_3(t)}{dt}\right)^2-\frac{d^2x_3(t)}{dt^2}+L_g\sin(\theta_3(t))\frac{d^2\theta_3(t)}{dt^2}\right)\end{aligned}\tag{5}$$

Linearization

After obtaining the the non-linear Lagrangian dynamics it is useful to linearize the system around its operating (equilibrium) point $\tilde{\theta}$. Where the dynamics around this operating point are given by section .

$$M\ddot{q}+C\dot{q}+Kq=Q$$

Matrices M, C, K represent the mass, damping, and stiffness matrix respectively. These matrices are defined by the Hessian matrix of the multivariate energy functions T, D, V. The equilibrium point is described by the vector $q_{eq}=\begin{bmatrix}0&0&0&0&\pi/2\end{bmatrix}^T$

Mass Matrix:

$$M=\left.\frac{\partial^2T}{\partial\dot{q}\partial\dot{q}}\right|_{q=q_{\mathrm{eq}}}=\begin{bmatrix}m_1&0&0&0&0\\0&m_2&0&0&0\\0&0&m_3&0&-L_gm_3\\0&0&0&m_3&0\\0&0&-L_gm_3&0&m_3L_g^2+J_{3g}\end{bmatrix}$$

Damping Matrix:

$$C=\left.\frac{\partial^2D}{\partial\dot{q}\partial\dot{q}}\right|_{q=q_{\mathrm{eq}}}=\begin{bmatrix}c_1+c_2&-c_2&0&0&0\\-c_2&c_2+c_3&-c_3&0&0\\0&-c_3&c_3+c_4&0&0\\0&0&0&c_5&-Lc_4\\0&0&0&-Lc_4&c_4L^2+c_{t3}\end{bmatrix}$$

Stiffness Matrix:

$$K=\left.\frac{\partial^2V}{\partial q\partial q}\right|_{q=q_{\mathrm{eq}}}=\begin{bmatrix}k_1+k_2&-k_2&0&0&0\\-k_2&k_2+k_3&-k_3&0&0\\0&-k_3&k_3+k_4&0&-Lk_4\\0&0&0&k_5&0\\0&0&-Lk_4&0&k_{t3}+L^2k_4-L^2k_5\end{bmatrix}$$

Because the system is subject to an imposed motion described by $\dot{x_0}$ and x_0 there will be a non zero Q vector affecting the dynamics of the system. This motion is imposed on mass 1, meaning that Q can be formulated as in Equation 6.

$$Q = C \begin{bmatrix} \dot{x_0} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + K \begin{bmatrix} x_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (c_1 + c_2)\dot{x_0} + (k_1 + k_2)x_0 \\ -c_2\dot{x_0} - k_2 x_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Eigenfrequencies and eigenmodes without damping

In order to simulate the system, the eigenfrequencies and eigenmodes have to be found. Solving the problem in Equation 7 results in the eigenfrequencies ω_r.

$$\det(K - \omega^2 M) = 0 \tag{7}$$

Using this relation in combination with the value substituted K and M matrices, the eigenfrequencies can be found for the undamped system.

$$\begin{bmatrix} \omega_{1,u} & \omega_{2,u} & ... & \omega_{5,u} \end{bmatrix} = 10^3 \times \begin{bmatrix} 0.1043 & 0.1384 & 0.6627 & 0.7703 & 1.3913 \end{bmatrix}^T$$

By substituting ω and solving $(K - \omega^2 M)q = 0$ the eigenmodes are obtained and put in matrix form X_u.

$$X_u = \begin{bmatrix} x_{1,u} & x_{2,u} & ... & x_{5,u} \end{bmatrix} = \begin{bmatrix} 0.0790 & -0.0000 & 1.4078 & -0.1082 & 0.0101 \\ 0.7159 & -0.0000 & -0.0106 & 0.3542 & -0.3205 \\ 0.7252 & -0.0000 & -0.0667 & -0.2088 & 1.3915 \\ 0.0000 & 0.6901 & 0.0000 & 0.0000 & 0 \\ 2.1934 & -0.0000 & -0.0097 & 2.1562 & 7.2828 \end{bmatrix}$$

All eigenfrequencies found are non-zero, so there are no rigid-body modes.

Eigenfrequencies and eigenmodes with damping

In the following case we will introduce damping to our system. Solving the eigenvalue problem in Equation 8 results in the eigenvalues below. The assumption that the system is lightly damped is made. This means that the found eigenvalues have the form $\lambda_r = -\frac{\beta_r}{\mu_r} + i\omega_r$

$$\det(\lambda^2 M + \lambda C + K) = 0 \tag{8}$$

$$\lambda = -\frac{\beta}{\mu} + i\omega = 10^3 \times \begin{bmatrix} -1.0902 + 0.7818i \\ -1.0902 - 0.7818i \\ -0.2029 + 0.6218i \\ -0.2029 - 0.6218i \\ -0.1493 + 0.7706i \\ -0.1493 - 0.7706i \\ -0.0466 + 0.0969i \\ -0.0466 - 0.0969i \\ -0.0119 + 0.1379i \\ -0.0119 - 0.1379i \end{bmatrix}$$

The damped eigenmodes are obtained by solving $(\lambda^2 M + \lambda C + K)q = 0$ and are shown in matrix form $X_d =$

$$\begin{bmatrix} 0.0125 + 0.0092i & 0.0125 - 0.0092i & -0.3323 - 0.7694i & -0.3323 + 0.7694i & -0.0195 + 0.1071i \\ -0.0593 - 0.0096i & -0.0593 + 0.0096i & -0.0311 + 0.0739i & -0.0311 - 0.0739i & -0.0876 + 0.0146i \\ 0.2133 + 0.0134i & 0.2133 - 0.0134i & -0.0971 + 0.1371i & -0.0971 - 0.1371i & 0.0123 - 0.0818i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.9749 - 0.0036i & 0.9749 + 0.0036i & 0.0789 + 0.5067i & 0.0789 - 0.5067i & -0.9730 - 0.1633i \end{bmatrix}$$

$$\begin{matrix} -0.0195 - 0.1071i & 0.0140 + 0.0285i & 0.0140 - 0.0285i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ -0.0876 + 0.0146i & 0.2694 + 0.1181i & 0.2694 - 0.1181i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0123 - 0.0818i & 0.2730 + 0.1198i & 0.2730 - 0.1198i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i & -0.0863 - 0.9963i & -0.0863 + 0.9963i \\ -0.9730 - 0.1633i & 0.8364 + 0.3521i & 0.8364 - 0.3521i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \end{matrix}$$

By introducing damping into the model we can also take the dampingratio in considiration. The damping ratio for each eigenfrequency is defined by $\epsilon_r = \frac{\beta_r}{2\omega_r \mu_r}$. Using the definition for the lightly damped eigenvalue this can be rewriten to $\epsilon_r = \frac{-\mathrm{Re}(\lambda_r)}{2\mathrm{Im}(\lambda_r)}$. Which results in the damping ratios:

$$\epsilon = \begin{bmatrix} 0.1608 & 0.3628 & 0.4124 & 0.3142 & 0.4587 \end{bmatrix}^T$$

Visualization

We have made some animations to visualise the eigenmodes corresponding to the eigenfrequencies of the system. These animations can be found in the same ZIPfile as this report along with the MatLab-code that we have used for all the calculations.

Conclusion

This report investigates the resonance phenomenon within a hand-arm vibratory system, specifically during operations such as drilling. Utilizing a simplified five degrees of freedom model constrained to the x-y plane, the study derives the equations of motion through Lagrange's method. Linearization around a system equilibrium facilitates an in-depth analysis of the eigenfrequencies and eigenmodes, revealing key insights into vibrational behaviors.

In conclusion, this report presents a mathematical framework for modeling hand-arm vibrations, underscoring the significance of incorporating vibrational dynamics into tool design to mitigate health risks associated with extended exposure. Prospective work could refine the current model by adding further complexities, validating the equations experimentally, and developing targeted strategies to dampen unwanted oscillations in practical scenarios.