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Assignment 4 Theory Questions

COMP3670: Introduction to Machine Learning

Errata: All corrections are in red.

Note: For the purposes of this assignment, if X is a random variable we let p_X denote the probability density function (pdf) of X, F_X to denote it's cumulative distribution function, and P to denote probabilities. These can all be related as follows:

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} p_X(z)dz$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_{a}^{b} p_X(z)dz$$

Often, we will simply write p_X as p, where it's clear what random variable the distribution refers to. You should show your derivations, but **you may use a computer algebra system (CAS)** to assist with integration or differentiation. We are not assessing your ability to integrate/differentiate here.[1].

Question 1 Continuous Bayesian Inference 5+5+2+4+4+6+6+5=37 credits

Let X be a random variable representing the outcome of a biased coin with possible outcomes $\mathcal{X} = \{0, 1\}$, $x \in \mathcal{X}$. The bias of the coin is itself controlled by a random variable Θ, with outcomes[2] $\theta \in \boldsymbol{\theta}$, where

$$\boldsymbol{\theta} = \{\theta \in \mathbb{R} : 0 \leq \theta \leq 1\}$$

The two random variables are related by the following conditional probability distribution function of X given Θ.

$$p(X = 1 \mid \Theta = \theta) = \theta$$

$$p(X = 0 \mid \Theta = \theta) = 1 - \theta$$

We can use $p(X = 1 \mid \theta)$ as a shorthand for $p(X = 1 \mid \Theta = \theta)$.

We wish to learn what θ is, based on experiments by flipping the coin.

We flip the coin a number of times.[3] After each coin flip, we update the probability distribution for θ to reflect our new belief of the distribution on θ, based on evidence.

Suppose we flip the coin n times, and obtain the sequence of coin flips [4] $x_{1:n}$.

a) Compute the new PDF for θ after having observed n consecutive **ones** (that is, $x_{1:n}$ is a sequence where $\forall i. x_i = 1$), for an arbitrary prior pdf $p(\theta)$. Simplify your answer as much as possible.

b) Compute the new PDF for θ after having observed n consecutive **zeros**, (that is, $x_{1:n}$ is a sequence where $\forall i. x_i = 0$) for an arbitrary prior pdf $p(\theta)$. Simplify your answer as much as possible.

c) Compute $p(\theta | x_{1:n} = 1^n)$ for the uniform prior $p(\theta) = 1$.

d) Compute the expected value μ_n of θ after observing n consecutive ones, with a uniform prior $p(\theta) = 1$. Provide intuition explaining the behaviour of μ_n as $n \to \infty$.

[1] For example, asserting that $\int_0^1 x^2 \left(x^3 + 2x\right) dx = 2/3$ with no working out is adequate, as you could just plug the integral into Wolfram Alpha using the command `Integrate[x^2(x^3 + 2x),{x,0,1}]`

[2] For example, a value of $\theta = 1$ represents a coin with 1 on both sides. A value of $\theta = 0$ represnts a coin with 0 on both sides, and $\theta = 1/2$ represents a fair, unbaised coin.

[3] The coin flips are independent and identically distributed (i.i.d).

[4] We write $x_{1:n}$ as shorthand for the sequence $x_1 x_2 \dots x_n$.

e) Compute the variance σ_n^2 of the distribution of θ after observing n consecutive ones, with a uniform prior $p(\theta) = 1$. Provide intuition explaining the behaviour of σ_n^2 as $n \to \infty$.

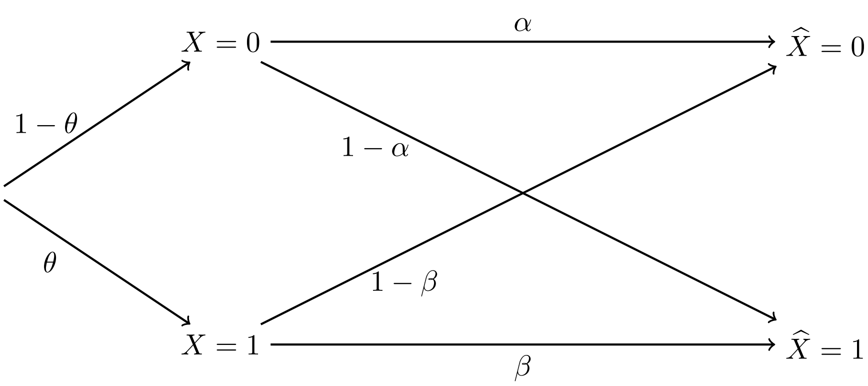
f) Compute the *maximum a posteriori* estimation θ_{MAP_n} of the distribution on θ after observing n consecutive ones, with a uniform prior $p(\theta) = 1$. Provide intuition explaining how θ_{MAP_n} varies with n.

g) Given we have observed n consecutive coin flips of ones in a row, what do you think would be a better choice for the best guess of the true value of θ? μ_n or θ_{MAP}? Justify your answer. (Assume $p(\theta) = 1$.)

h) Plot the probability distributions $p(\theta | x_{1:n} = 1)$ over the interval $0 \leq \theta \leq 1$ for $n \in \{0, 1, 2, 3, 4\}$ to compare them. Assume $p(\theta) = 1$.

Question 2 Bayesian Inference on Imperfect Information (4+5+8+4+4=25 credits)

We have a Bayesian agent running on a computer, trying to learn information about what the parameter θ could be in the coin flip problem, based on observations through a noisy camera. The noisy camera takes a photo of each coin flip and reports back if the result was a 0 or a 1. Unfortunately, the camera is not perfect, and sometimes reports the wrong value.[5] The probability that the camera makes mistakes is controlled by two parameters α and β, that control the likelihood of correctly reporting a zero, and a one, respectively. Letting X denote the true outcome of the coin, and $\widehat{X}$ denoting what the camera reported back, we can draw the relationship between X and $\widehat{X}$ as shown.



So, we have

$$
\begin{aligned}
p(\widehat{X} = 0 \mid X = 0) &= \alpha \\
p(\widehat{X} = 0 \mid X = 1) &= 1 - \beta \\
p(\widehat{X} = 1 \mid X = 1) &= \beta \\
p(\widehat{X} = 1 \mid X = 0) &= 1 - \alpha
\end{aligned}
$$

We would now like to investigate what posterior distributions are obtained, as a function of the parameters α and β.

a) (5 credits) Briefly comment about how the camera behaves for $\alpha = \beta = 1$, for $\alpha = \beta = 1/2$, and for $\alpha = \beta = 0$. For each of these cases, how would you expect this would change how the agent updates it's prior to a posterior on θ, given an observation of $\widehat{X}$? (No equations required.) You shouldn't need any assumptions about $p(\theta)$ for this question.

b) (10 credits) Compute $p(\widehat{X} = x | \theta)$ for all $x \in \{0, 1\}$.

[5] The errors made by the camera are i.i.d, in that past camera outputs do not affect future camera outputs.

c) (15 credits) The coin is flipped, and the camera reports seeing a one. (i.e. that $\hat{X} = 1$.) Given an arbitrary prior $p(\theta)$, compute the posterior $p(\theta|\hat{X} = 1)$. What does $p(\theta|\hat{X} = 1)$ simplify to when $\alpha = \beta = 1$? When $\alpha = \beta = 1/2$? When $\alpha = \beta = 0$? Explain your observations.

d) Compute $p(\theta|\hat{X} = 1)$ for the uniform prior $p(\theta) = 1$. Simplify it under the assumption that $\beta := \alpha$.

e) (10 credits) Let $\beta = \alpha$. Plot $p(\theta|\hat{X} = 1)$ as a function of θ, for all $\alpha \in \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ on the same graph to compare them. Comment on how the shape of the distribution changes with α. Explain your observations. (Assume $p(\theta) = 1$.)

Question 3 **Relating Random Variables** (10+7+5+16=38 credits)

A casino offers a new game. Let $X \sim f_X$ be a random variable on $(0, 1]$ with pdf p_X. Let Y be a random variable on $[1, \infty)$ such that $Y = 1/X$. A random number c is sampled from Y, and the player guesses a number $m \in [1, \infty)$. If the player's guess m was lower than c, then the player wins $m - 1$ dollars from the casino (which means higher guesses pay out more money). But if the player guessed too high, $(m \geq c)$, they go bust, and have to pay the casino 1 dollar.

a) Show that the probability density function p_Y for Y is given by

$$p_Y(y) = \frac{1}{y^2} p_X(\frac{1}{y})$$

b) Hence, or otherwise, compute the expected profit for the player under this game. Your answer will be in terms of m and p_X, and should be as simplified as possible.

c) Suppose the casino chooses a uniform distribution over $(0, 1]$ for X, that is,

$$p_X(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What strategy should the player use to maximise their expected profit?

d) Find a pdf $p_X : (0, 1] \to \mathbb{R}$ such that for any $B > 0$, there exists a corresponding player guess m such that the expected profit for the player is at least B. (That is, prove that the expected profit for p_X, as a function of m, is unbounded.)

Make sure that your choice for p_X is a valid pdf, i.e. it should satisfy

$$\int_0^1 p_X(x)dx = 1 \text{ and } p_X(x) \geq 0$$

You should also briefly mention how you came up with your choice for p_X.

Hint: We want X to be extremely biased towards small values, so that Y is likely to be large, and the player can choose higher values of m without going bust.