

COMP3670/6670: Introduction to Machine Learning

Release Date. Aug 4th, 2021

Due Date. 11:59pm, Aug 22th, 2021

Maximum credit. 100

Exercise 1

Solving Linear Systems

(4+4 credits)

Find the set \mathcal{S} of all solutions \mathbf{x} of the following inhomogenous linear systems $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} and \mathbf{b} are defined as follows. Write the solution space \mathcal{S} in parametric form.

(a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 4 & 3 \\ 2 & 7 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

Exercise 2

Inverses

(4 credits)

For what values of $[a, b, c]^T \in \mathbb{R}^3$ does the inverse of the following matrix exist?

$$\begin{bmatrix} 1 & a & b \\ 1 & 1 & c \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise 3

Subspaces

(3+3+3+3 credits)

Which of the following sets are subspaces of \mathbb{R}^3 ? Prove your answer. (That is, if it is a subspace, you must demonstrate the subspace axioms are satisfied, and if it is not a subspace, you must show which axiom fails.)

(a) $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

(b) $B = \{(x, y, z) : x + y + z = 0\}$.

(c) $C = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$

(d) $D =$ The set of all solutions \mathbf{x} to the matrix equation $\mathbf{Ax} = \mathbf{b}$, for some matrix \mathbf{A} and some vector \mathbf{b} . (Hint: Your answer may depend on \mathbf{A} and \mathbf{b} .)

Exercise 4

Linear Independence

(4+8+8 credits)

Let V and W be vector spaces. Let $T : V \rightarrow W$ be a linear transformation.

(a) Prove that $T(\mathbf{0}) = \mathbf{0}$.

- (b) For any integer $n \geq 1$, prove that given a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in V and a set of coefficients $\{c_1, \dots, c_n\}$ in \mathbb{R} , that

$$T(c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \dots + c_nT(\mathbf{v}_n)$$

- (c) Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of linearly **dependent** vectors in V .
 Define $\mathbf{w}_1 := T(\mathbf{v}_1), \dots, \mathbf{w}_n := T(\mathbf{v}_n)$.
 Prove that $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a set of linearly **dependent** vectors in W .

Exercise 5

Inner Products

(4+8 credits)

- (a) Show that if an inner product $\langle \cdot, \cdot \rangle$ is symmetric and linear in the first argument, then it is bilinear.
 (b) Define $\langle \cdot, \cdot \rangle$ for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + 2(x_1y_2 + x_2y_1)$$

Which of the three inner product axioms does $\langle \cdot, \cdot \rangle$ satisfy?

Exercise 6

Orthogonality

(8+6 credits)

Let V denote a vector space together with an inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$.

Let \mathbf{x}, \mathbf{y} be **non-zero** vectors in V .

- (a) Prove or disprove that if \mathbf{x} and \mathbf{y} are orthogonal, then they are linearly independent.
 (b) Prove or disprove that if \mathbf{x} and \mathbf{y} are linearly independent, then they are orthogonal.

Exercise 7

Properties of Norms

(4+4+10 credits)

Given a vector space V with two norms $\|\cdot\|_a : V \rightarrow \mathbb{R}_{\geq 0}$ and $\|\cdot\|_b : V \rightarrow \mathbb{R}_{\geq 0}$, we say that the two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are ε -equivalent if for any $\mathbf{v} \in V$, we have that

$$\varepsilon\|\mathbf{v}\|_a \leq \|\mathbf{v}\|_b \leq \frac{1}{\varepsilon}\|\mathbf{v}\|_a.$$

where $\varepsilon \in (0, 1]$.

If $\|\cdot\|_a$ is ε -equivalent to $\|\cdot\|_b$, we denote this as $\|\cdot\|_a \stackrel{\varepsilon}{\sim} \|\cdot\|_b$.

- (a) Is ε -equivalence reflexive for all $\varepsilon \in (0, 1]$?
 (Is it true that $\|\cdot\|_a \stackrel{\varepsilon}{\sim} \|\cdot\|_a$?)
 (b) Is ε -equivalence symmetric for all $\varepsilon \in (0, 1]$?
 (Does $\|\cdot\|_a \stackrel{\varepsilon}{\sim} \|\cdot\|_b$ imply $\|\cdot\|_b \stackrel{\varepsilon}{\sim} \|\cdot\|_a$?)
 (c) Assuming that $V = \mathbb{R}^2$, prove that $\|\cdot\|_1 \stackrel{\varepsilon}{\sim} \|\cdot\|_2$ for the largest ε possible.

Exercise 8

Projections

(3+3+3+3 credits)

Consider the Euclidean vector space \mathbb{R}^3 with the dot product. A subspace $U \subset \mathbb{R}^3$ and vector $\mathbf{x} \in \mathbb{R}^3$ are given by

$$U = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 12 \\ 12 \\ 18 \end{bmatrix}$$

- (a) Show that $\mathbf{x} \notin U$.
 (b) Determine the orthogonal projection of \mathbf{x} onto U , denoted $\pi_U(\mathbf{x})$.
 (c) Show that $\pi_U(\mathbf{x})$ can be written as a linear combination of $[1, 1, 1]^T$ and $[2, 1, 0]^T$.
 (d) Determine the distance $d(\mathbf{x}, U) := \min_{\mathbf{y} \in U} \|\mathbf{x} - \mathbf{y}\|_2$.