Q1.

(1).

Assumes $\lambda=0$ is an eigenvalue of A.

$$Ax=\lambda x=0 \qquad (x\neq 0)$$

But A is an invertible matrix.

$$\Rightarrow A\cdot A^{-1}\cdot x=A^{-1}0=0$$

$$\Rightarrow I\cdot x=0$$

$\because$ I is a indentity matrix.

$\therefore$ $x=0$

However. $x=0$ does not satisfy $Ax=\lambda x$ $(x\neq 0)$

Therefore, all the eigenvalues of A are non-zero.

(2).

Prove that $Ax=\lambda x \Rightarrow A^{-1}x=\lambda^{-1}x$ $(x\neq 0)$

$$Ax=\lambda x. \quad (\lambda\neq 0)$$

$$A^{-1}\cdot A\cdot x=A^{-1}\cdot\lambda\cdot x$$

$$I \cdot x = A^{-1} \cdot \lambda \cdot x$$

$$x = A^{-1} \cdot \lambda \cdot x$$

$$\frac{1}{\lambda} \cdot x = A^{-1} \cdot x$$

$$\Rightarrow \lambda^{-1} \cdot x = A^{-1} \cdot x$$

Therefore, any eigenvalues λ. of A.

λ^{-1} is a eigenvalue of A^{-1}!

Q2.

when $n=1$:

x is an eigenvector of B with eigenvalue λ

when $n=n$, Suppose:

x is an eigenvector of B^n with eigenvalue λ^n.

when $n=n+1$, Prove:

x is an eigenvector of B^{n+1} with eigenvalue λ^{n+1}

$$B^{n+1} x = (BB^n) x$$

$$= (B \lambda^n) x$$

$= \lambda^n (Bx)$

$= \lambda^n (\lambda x)$

$= \lambda^{n+1} x$

Therefore, $B^{n+1} x = \lambda^{n+1} x$

Q3.

(1).

Suppose:

$\{x_1 \ldots\ldots x_m\}$ is linear correlation.

Then:

$x_{p+1} \in \{x_1, x_2 \ldots\ldots x_p\}$

$\{x_1 \ldots\ldots x_p\}$ is linear independent.

$(1 \le p < m)$

$\Rightarrow \{x_1 \ldots\ldots x_p, x_{p+1}\}$ is linear correlation.

$\Rightarrow a_1 x_1 + \ldots\ldots + a_p x_p + a_{p+1} x_{p+1} = 0 \ldots\ldots ①$

we have:

$$A x_1 = \lambda_1 x_1$$

$$A x_{p+1} = \lambda_{p+1} x_{p+1} \quad \cdots\cdots ②$$

$\because$ ① $= 0$

$\therefore$ $A \cdot$ ① $= 0$

we have:

$$A \cdot (a_1 x_1 + \cdots + a_p x_p + a_{p+1} x_{p+1}) = 0$$

$$A a_1 x_1 + \cdots\cdots + A a_p x_p + A a_{p+1} x_{p+1} = 0$$

$$a_1 A x_1 + \cdots\cdots a_p A x_p + a_{p+1} A x_{p+1} = 0 \quad \cdots\cdots ③$$

$$\Rightarrow a_1 \lambda_1 x_1 + \cdots\cdots a_p \lambda_p x_p + a_{p+1} \lambda_{p+1} x_{p+1} = 0$$

$$\cdots\cdots ④$$

④ $- (\lambda_{p+1})$ ③ :

$$a_1(\lambda_1 - \lambda_{p+1}) x_1 + \cdots + a_p(\lambda_p - \lambda_{p+1}) x_p + a_{p+1} \cdot 0 \cdot x_{p+1} = 0$$

we have:

$\{x_1, \ldots, x_n\}$ is linear independent.

$$\Rightarrow a_p(\lambda_p - \lambda_{p+1}) = 0 \qquad p \in [1, p]$$

$\{\lambda_1 \dots \lambda_n\}$ are distinct eigenvalues.

$\Rightarrow \quad a_p = 0 \quad p \in [1, p] \cdots ⑤$

$a_1 x_1 + \dots + a_p x_p + a_{p+1} x_{p+1} = 0 \quad \cdots ①$

$\Rightarrow \quad a_{p+1} x_{p+1} = 0 \quad \cdots ①$ and ⑤

However.

$\{x_1 \dots x_p, x_{p+1}\}$ is linear correlation.

$a_1 x_1 + \dots + a_p x_p + a_{p+1} x_{p+1} = 0 \quad \cdots ①$

$\Rightarrow a_{p+1} \neq 0$

Hence. $\quad x_{p+1} = 0$

But $\quad x_{p+1} \neq 0 \quad \cdots$ eigenvalue.

Summarizing:

$\{x_1, \dots x_n\}$ is linearly independent.

(2).

For any matrix $B \in R^{n \times n}$:

The number of eigenvalues of A is same as the number of roots of characteristic equation:

$$|A - \lambda I| = 0.$$

And. the highest term of λ:

$$\lambda_{\text{higest term}} = n$$

Therefore, the equation at most has n roots (includes roots with same values)

Summarizing:

for. any matrix $B \in \mathbb{R}^{n \times n}$, at most has n distinct eigenvalues for B.

Q4.

(1)

For any $n \times n$ matrix A:

$$\det(A) = \sum_{k=1}^{n} a_{ik} A_{ik} \quad \cdots ①$$

$$= \sum_{k=1}^{n} a_{kj} A_{kj} \quad \cdots ②$$

$(i, j \in [1, n])$

when $n=1$:

$$A^T = A$$

then: $\det(A^T) = \det(A)$

when $n=k$:

Assumes that $\det(A^T) = \det(A)$

when $n=k+1$:

Compute $\det(A)$ by row 1:

$$\det(A) = \sum_{i=1}^{k+1} (-1)^{1+i} a_{1i} \det(M_{1i})$$
$$= \sum_{i=1}^{k+1} (-1)^{1+i} a_{1i} \det(M_{1i}^T) \quad \cdots\cdots ③$$

compute $\det(A^T)$ by column 1:

$$\det(A^T) = \sum_{j=1}^{k+1} (-1)^{1+i} a_{j1} \det(M_{j1}^T) \quad \cdots\cdots ④$$
$$= \sum_{j=1}^{k+1} (-1)^{1+i} a_{j1} \det(M_{j1})$$

we have: ③ = ④ ······ ① and ②

③ is the expansion of $\det(A)$ by row 1, which is same as the expansion of $\det(A^T)$ by column 1.

Therefore, $\det(A) = \det(A^T)$

(2)

when $n=1$:

$$I_n = [1]$$

then: $\det(I_n) = 1$

when $n=k$:

$\because$ I_n is the $n \times n$ identity matrix

$\therefore$ $a_{ij} = 1$ when $i=j$

$a_{ij} = 0$ when $i \neq j$ $\quad (i, j \in [1, k])$

$$\det(I_n) = \sum_{j_1 j_2 \cdots j_k} (-1)^{\tau(j_1, \cdots j_k)} a_{1j_1} a_{2j_2} \cdots a_{kj_k}$$

$$= a_{11} a_{22} \cdots a_{nn} + 0 + 0 + \cdots + 0$$

$$= 1$$

Q5.

To prove that:

$$\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2.$$

$$\lambda_1 v_1^T v_2 = v_2^T (\lambda_1 v_1).$$
$$= v_2^T (A v_1)$$
$$= (A v_1)^T v_2$$
$$= v_1^T A^T v_2$$
$$= v_1^T A v_2 \quad \cdots \text{A is a symmetric matrix}$$
$$= v_1^T (\lambda_2 v_2)$$
$$= \lambda_2 v_1^T v_2.$$

Therefore. $\lambda_1 v_1^T v_2 - \lambda_2 v_1^T v_2 = 0$

$$(\lambda_1 - \lambda_2) v_1^T v_2 = 0$$

$$v_1^T v_2 = 0 \quad \cdots \quad \lambda_1 \neq \lambda_2$$

Summarizing:

v_1, v_2 are orthogonal.

Q6

(1).

$\det(A-\lambda I)$

$= \det\left(\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$

$= \begin{vmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$

$= -(1+\lambda)(4-\lambda) - 2\cdot 3$

$= \lambda^2 - 3\lambda - 10$

$= (\lambda+2)(\lambda-5)$

$(\lambda+2)(\lambda-5)=0$

Therefore, $\lambda_1 = -2$, $\lambda_2 = 5$

(2)

$\begin{bmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} \vec{x} = \vec{0}$

For $\lambda=5$ we obtain:

$$\begin{bmatrix} -1-5 & 2 \\ 3 & 4-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E_5=\text{Span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

For $\lambda=-2$ we obtain:

$$\begin{bmatrix} -1-(-2) & 2 \\ 3 & 4-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E_{-2}=\text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

(3).

The set of all eigen vectors of A:

span $\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right)$

Assumes $\vec{v_1}, \vec{v_2}$ are linear dependence:

$$\sum_{i=1}^{2} a_i v_i = 0 \qquad (a_i \neq 0)$$

$$\begin{bmatrix}a_1\\3a_1\end{bmatrix} + \begin{bmatrix}-2a_2\\a_2\end{bmatrix} = \begin{bmatrix}a_1-2a_2\\3a_1+a_2\end{bmatrix}$$

$$\begin{cases} a_1 - 2a_2 = 0 & \cdots\cdots ① \\ 3a_1 + a_2 = 0 & \cdots\cdots ② \end{cases}$$

② $\times 2$ + ①: $\quad 7 \cdot a_1 = 0$

However, $a_1 \neq 0$

Therefore., $\vec{v_1}, \vec{v_2}$ are linear independent.

$\Rightarrow \quad \vec{v_1}, \vec{v_2}$ spans R^2.

(4).

let. $P = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 3 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

$$P \cdot D = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 15 \end{bmatrix}$$

$$P^{-1} = -\frac{2}{7} \begin{bmatrix} 3 & -1 \\ -\frac{1}{2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

$$P \cdot D \cdot P^{-1} = \begin{bmatrix} -7 & 14 \\ 21 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= A$$

(5)

$$A^n = P \cdot D^n P^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & 3 \end{bmatrix} \cdot \begin{bmatrix} -2^n & 0 \\ 0 & 5^n \end{bmatrix} \cdot \begin{bmatrix} -\frac{6}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

$$= \begin{bmatrix} (-1)\cdot(-2)^n + 1\cdot 0 & -1\cdot 0 + 1\cdot 5^n \\ 2^{-1}\cdot(-2)^n + 3\cdot 0 & \frac{1}{2}\cdot 0 + 3\cdot 5^n \end{bmatrix} \begin{bmatrix} -6 & 2 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-2)^n & 5^n \\ (-1)(-2)^{n-1} & 3\cdot 5^n \end{bmatrix} \begin{bmatrix} -6 & 2 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6\cdot(-2)^n + 5^n & (-2)^{n+1} + 2\cdot 5^n \\ 6(-2)^{n-1} + 3\cdot 5^n & (-2)^n + 6\cdot 5^n \end{bmatrix}$$