Question 1

(a)

When $n=1$:

$$P(\theta | x_1 = 1)$$
$$= P(\theta)\frac{P(x_1=1|\theta)}{P(x_1=1)}$$
$$= P(\theta)\frac{P(x_1=1|\theta)}{\int_0^1 P(x_1=1|\theta)P(\theta)d\theta}$$
$$= P(\theta)\frac{\theta}{\int_0^1 \theta P(\theta)d\theta} \quad \cdots\cdots\cdots P(x=1|\theta=0)=\theta$$
$$= P(\theta)\frac{\theta}{E[\theta]} \quad \cdots\cdots E[f(x)] = \int f(x)P(x)\,dx$$

when $n=n$:

Assumes that $P(\theta | x_{1:n} = 1^n) = P(\theta)\dfrac{\theta^n}{E[\theta^n]}$

To prove:

$$P(\theta | x_{1:n+1} = 1^{n+1}) = P(\theta)\frac{\theta^{n+1}}{E[\theta^{n+1}]}$$

$$p(\theta \mid x_{1:n+1} = 1^{n+1})$$

$$= p(\theta \mid x_{1:n} = 1^n) \frac{p(x_{n+1} = 1 \mid \theta, x_{1:n} = 1^n)}{p(x_{n+1} = 1 \mid x_{1:n} = 1^n)}$$

$$= p(\theta \mid x_{1:n} = 1^n) \frac{p(n+1 = 1 \mid \theta, x_{1:n} = 1^n)}{\int_0^1 p(x_{n+1} = 1, \theta \mid x_{1:n} = 1^n)\, d\theta}$$

$$= p(\theta \mid x_{1:n} = 1^n) \frac{p(n+1 = 1 \mid \theta, x_{1:n} = 1^n)}{\int_0^1 p(x_{n+1} = 1 \mid \theta, x_{1:n} = 1^n)\, p(\theta \mid x_{1:n} = 1^n)\, d\theta}$$

$$= p(\theta \mid x_{1:n} = 1^n) \frac{\theta}{\int_0^1 \theta \cdot p(\theta \mid x_{1:n} = 1^n)\, d\theta}$$

$$\cdots\cdots p(x_{n+1} = 1 \mid \theta, x_{1:n} = 1^n) = 1$$

$$= \frac{\theta^n p(\theta)}{E[\theta^n]} \frac{\theta}{\int_0^1 \theta \frac{\theta^n p(\theta)}{E[\theta^n]}\, d\theta}$$

$$= p(\theta) \frac{\theta^{n+1}}{\int_0^1 \theta^{n+1} p(\theta)\, d\theta}$$

$$= p(\theta) \frac{\theta^{n+1}}{E[\theta^{n+1}]}$$

Therefore, $p(\theta \mid x_{1:n} = 1^n) = p(\theta) \dfrac{\theta^n}{E[\theta^n]}$

(b)

When $n=0$:

$P(\theta|x_1=1)$

$= P(\theta)\dfrac{P(x_1=0|\theta)}{P(x_1=0)}$

$= P(\theta)\dfrac{P(x_1=0|\theta)}{\int_0^1 P(x_1=0|\theta)P(\theta)d\theta}$

$= P(\theta)\dfrac{1-\theta}{\int_0^1 (1-\theta)P(\theta)d\theta}$ .... $P(X=1|\theta=0)=\theta$

$= P(\theta)\dfrac{1-\theta}{E[1-\theta]}$ ...... $E[f(x)]=\int f(x)p(x)\,dx.$

when $n=n$:

Assumes that $P(\theta|x_{1:n}=0^n) = P(\theta)\dfrac{(1-\theta)^n}{E[(1-\theta)^n]}$

To prove:

$$P(\theta|x_{1:n+1}=0^{n+1}) = P(\theta)\frac{(1-\theta)^{n+1}}{E[(1-\theta)^{n+1}]}$$

$P(\theta|x_{1:n+1}=0^{n+1})$

$$= P(\theta | x_{1:n} = 0^k) \frac{P(x_{n+1} = 0 | \theta, x_{1:n} = 0^n)}{P(x_{n+1} = 0 | x_{1:n} = 0^n)}$$

$$= P(\theta | x_{1:n} = 0^n) \frac{P(_{n+1} = 0 | \theta, x_{1:n} = 0^n)}{\int_0^1 P(x_{n+1} = 0, \theta | x_{1:n} = 0^n)\, d\theta}$$

$$= P(\theta | x_{1:n} = 0^n) \frac{P(_{n+1} = 0 | \theta, x_{1:n} = 0^n)}{\int_0^1 P(x_{n+1} = 0 | \theta, x_{1:n} = 0^n)\, P(\theta | x_{1:n} = 0^n)\, d\theta}$$

$$= P(\theta | x_{1:n} = 0^n) \frac{1-\theta}{\int_0^1 (1-\theta) P(\theta | x_{1:n} = 0^n)\, d\theta}$$

$$\cdots\cdots P(x_{n+1} = 1 | \theta, x_{1:n} = 1^n) = 1$$

$$= \frac{(1-\theta)^n p(\theta)}{E[(1-\theta)^n]} \frac{1-\theta}{\int_0^1 (1-\theta) \frac{\theta^n p(\theta)}{E[(1-\theta)^n]}\, d\theta}$$

$$= p(\theta) \frac{(1-\theta)^{n+1}}{E[(1-\theta)^{n+1}]}$$

Therefore, $P(\theta | x_{1:n} = 0^n) = p(\theta) \dfrac{(1-\theta)^n}{E[(1-\theta)^n]}$

(c).

$$P(\theta | X_{1:n} = 1^n) = P(\theta) \frac{\theta^n}{\int_0^1 \theta^n p(\theta)\, d\theta}$$

For $\int_0^1 \theta^n p(\theta)\, dn$:

$$\int_0^1 \theta^n \, d\theta = \frac{\theta^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$$

Therefore, $P(\theta | X_{1:n} = 1^n) = (n+1)\, \theta^n$.

(d)

$P(\theta | X_{1:n} = 1^n) = (n+1)\, \theta^n$. ...... From Q1.C

$$\begin{aligned}
M_n &= \int_0^1 \theta \cdot P(\theta | X_{1:n} = 1^n)\, d\theta \\
&= \int_0^1 (n+1)\, \theta^{n+1}\, d\theta \\
&= (n+1) \cdot \frac{\theta^{n+2}}{\theta^{n+1}} \Big|_0^1 \\
&= \frac{n+1}{n+2}
\end{aligned}$$

when $n \to \infty$:

$$\lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \frac{1+\lim_{n\to\infty}(\frac{1}{n})}{1+\lim_{n\to\infty}(\frac{2}{n})} = 1$$

Therefore. when $n \to \infty$, $M_n \to 1$.

(e).

$P(\theta \mid x_{1:n} = 1^{n}) = (n+1)\,\theta^{n}$. ..... From Q1.c

$$\sigma_n^2 = \int_0^1 (\theta - M_n)^2 \cdot P(\theta \mid x_{1:n} = 1^{n})\, d\theta$$

$$= \int_0^1 (n+1)\left(\theta - \frac{n+1}{n+2}\right)^2 \theta^{n}\, d\theta$$

$$= \int_0^1 (n+1)\left(\theta^2 + \left(\frac{n+1}{n+2}\right)^2 - \frac{2\theta(n+1)}{n+2}\right)\theta^{n}\, d\theta$$

$$= \int_0^1 \left((n+1)\theta^2 + \frac{(n+1)^3}{(n+2)^2} - \frac{2\theta(n+1)^2}{n+2}\right)\theta^{n}\, d\theta$$

$$= \int_0^1 \left((n+1)\theta^{n+2} + \frac{(n+1)^3}{(n+2)^2}\theta^{n} - \frac{2(n+1)^2}{n+2}\theta^{n+1}\right) d\theta$$

$$= \left[\frac{n+1}{n+3}\theta^{n+3} + \frac{(n+1)^2}{(n+2)^2}\theta^{n+1} - \frac{2(n+1)^2}{(n+2)^2}\theta^{n+2}\right]\Bigg|_0^1$$

$$= \frac{n+1}{n+3} + \frac{(n+1)^2}{(n+2)^2} - \frac{2(n+1)^2}{(n+2)^2}$$

when $n \to \infty$:

$$\sigma^2_n = \frac{(n+1)(n+2)^2}{(n+3)(n+2)^2} - \frac{(n+1)^2(n+3)}{(n+2)^2(n+3)}$$

$$= \frac{(n+1)(n+2)^2 - (n+1)^2(n+3)}{(n+3)(n+2)^2}.$$

$$= \frac{\frac{n+1}{n+3} \cdot \frac{(n+2)^2}{n+3} - \frac{(n+1)^2}{n+3}}{(n+2)^2}.$$

$\because$ $(n+2)^2 > \frac{n+1}{n+3} \cdot \frac{(n+2)^2}{n+3}$, $(n+2)^2 > \frac{(n+1)^2}{n+3}$

$\therefore$ when $n \to \infty$, $\sigma^2_n \to 0$

(f)

$P(\theta | x_{1:n} = 1^n) = (n+1)\,\theta^n.$ .... From Q1.c

For $\theta \in (0,1]$:

$P'(\theta | x_{1:n} = 1^n) = n(n+1)\,\theta^{n-1} > 0$

Then the maximum θ_{MAP_n} is 1.

For any $\theta \in [0,1]$ has the same probability.

Therefore. $\theta_{MAP_n} = 1$ for any $\theta \in [0,1]$.

(g)

Because. for any $\theta \in [0,1]$ has the same probability. The best. guess of θ, M_n and θ_{map} is 1.

(h)

$P(\theta | X_{1:n} = 1^n) = (n+1)\,\theta^n$. ...... From Q1.c

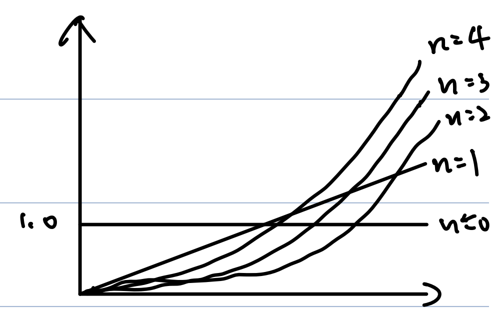
$n=0$: $P(\theta | X_{1:0} = 1^0) = 1$

$n=1$: $P(\theta | X_{1:1} = 1^1) = 2\theta$

$n=2$: $P(\theta | X_{1:2} = 1^2) = 3\theta^2$

$n=3$: $P(\theta | X_{1:3} = 1^3) = 4\theta^3$

$n=4$: $P(\theta | X_{1:4} = 1^4) = 5\theta^4$



Therefore, $p(\theta)$. approaching 1 with increase of n.

Question2

(a)

For $\alpha=\beta=1$:

when the coin is 0, camera's result is 0.

Therefore, the agent updates it's posterior approaching 0.

when the coin is 1, camera's result is 1.

Therefore, the agent updates it's posterior approaching 1.

For $\alpha=\beta=1/2$:

Whatever the coin is 0 or 1, camera's result has 50% to gets correct answer, which means the posterior will not change.

Therefore, the agent will not update it's posterior.

For $\alpha=\beta=0$:

when the coin is 0, camera's result is 1.

Therefore, the agent updates it's posterior approaching 0.

when the coin is 1, camera's result is 0.

Therefore, the agent updates it's posterior approaching 1.

(b)

$$P(\hat{x}=x \mid \theta)$$
$$=\sum_{n=0}^{1} P(\hat{x}=x,\ X=n \mid \theta)$$
$$=\sum_{n=0}^{1} P(\hat{x}=x \mid X=n, \theta)\cdot P(X=n \mid \theta)$$
$$=P(\hat{x}=x \mid X=0, \theta)P(X=0 \mid \theta)+P(\hat{x}=x \mid X=1, \theta)P(X=1 \mid \theta)$$

we have:

$$P(\hat{x}=0 \mid X=0)=\alpha \quad \cdots\cdots ①$$
$$P(\hat{x}=0 \mid X=1)=1-\beta \quad \cdots ②$$
$$P(\hat{x}=1 \mid X=1)=\beta \quad \cdots\cdots ③$$
$$P(\hat{x}=1 \mid X=0)=1-\alpha \quad \cdots ④$$

Base on ② and ③:

Because $x=0$ is the resule of θ for $P(\hat{x}=x|x=0,\theta)$,

and $x=1$ is the resule of θ for $P(\hat{x}=x|x=1,\theta)$.

$$
\begin{aligned}
&P(\hat{x}=x|\theta)\\
&=P(\hat{x}=x|x=0,\theta)(1-\theta)+P(\hat{x}=x|x=1,\theta)\theta\\
&=P(\hat{x}=x|x=0)(1-\theta)+P(\hat{x}=x|x=1)\theta
\end{aligned}
$$

when $x=0$:

$$
\begin{aligned}
&P(\hat{x}=0|\theta)\\
&=P(\hat{x}=0|x=0)(1-\theta)+P(\hat{x}=0|x=1)\theta\\
&=\alpha(1-\theta)+(1-\beta)\theta \quad \text{------ ① and ②}
\end{aligned}
$$

when $x=1$:

$$
\begin{aligned}
&P(\hat{x}=1|\theta)\\
&=P(\hat{x}=1|x=0)(1-\theta)+P(\hat{x}=1|x=1)\theta\\
&=(1-\alpha)(1-\theta)+\beta\theta \quad \text{------ ③ and ④}
\end{aligned}
$$

(c)

Base on Bayes theorem:

$$P(\theta \mid \hat{x}=1) = P(\theta)\cdot\frac{P(\hat{x}=1\mid\theta)}{P(\hat{x}=1)}$$

$$= P(\theta)\cdot\frac{P(\hat{x}=1\mid\theta)}{\int_0^1 P(\hat{x}=1\mid\theta)P(\theta)\,d\theta}$$

we have:

$$P(\hat{x}=1\mid\theta) = (1-\alpha)(1-\theta)+\beta\theta \quad \text{------ From Q2.b}$$

$$P(\theta\mid\hat{x}=1) = P(\theta)\cdot\frac{(1-\alpha)(1-\theta)+\beta\theta}{\int_0^1((1-\alpha)(1-\theta)+\beta\theta)P(\theta)\,d(\theta)}$$

when $\alpha=\beta=1$

$$P(\theta\mid\hat{x}=1) = P(\theta)\frac{\theta}{\int_0^1 \theta P(\theta)\,d(\theta)}$$

$$= P(\theta)\frac{\theta}{E[\theta]} \quad \text{------ } E[f(x)] = \int f(x)P(x)\,dx$$

when $\alpha=\beta=1/2$

$$p(\theta|\hat{x}=1) = p(\theta)\,\frac{\frac{1}{2}\cdot(1-\theta)+\beta\theta}{\int_0^1\left(\frac{1}{2}(1-\theta)+\frac{1}{2}\theta\right)p(\theta)\,d(\theta)}$$

$$= p(\theta)\,\frac{\frac{1}{2}}{\frac{1}{2}\int_0^1 p(\theta)\,d(\theta)}$$

$$= p(\theta) \qquad \cdots\cdots \int_0^1 p(\theta)\,d(\theta)=1$$

when $\alpha=\beta=0$:

$$p(\theta|\hat{x}=1) = p(\theta)\,\frac{1-\theta}{\int_0^1 (1-\theta)\,p(\theta)\,d(\theta)}$$

$$= p(\theta)\,\frac{1-\theta}{E[1-\theta]} \qquad \cdots\cdots E[f(x)]=\int f(x)p(x)\,dx.$$

(d)

We have $p(\theta|\hat{x}=1) = p(\theta)\cdot\dfrac{(1-\alpha)(1-\theta)+\beta\theta}{\int_0^1((1-\alpha)(1-\theta)+\beta\theta)\,p(\theta)\,d(\theta)}$

$\cdots\cdots$ From Q2.c

when $P(\theta)=1$:

$$P(\theta|\hat{x}=1) = P(\theta)\cdot\frac{(1-\alpha)(1-\theta)+\beta\theta}{\int_0^1((1-\alpha)(1-\theta)+\beta\theta)P(\theta)d(\theta)}$$

$$= \frac{(1-\alpha)(1-\theta)+\beta\theta}{\int_0^1((1-\alpha)(1-\theta)+\beta\theta)d(\theta)}$$

For $\int_0^1((1-\alpha)(1-\theta)+\beta\theta)$:

$$\int_0^1((1-\alpha)(1-\theta)+\beta\theta) = \int_0^1((1-\theta-\alpha+\alpha\theta)+\beta\theta)d\theta$$

$$= (\theta-\frac{1}{2}\theta^2-\alpha\theta+\alpha\frac{\theta^2}{2}+\beta\frac{\theta^2}{2})\Big|_0^1$$

$$= -\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}$$

Base on $\alpha := \beta$

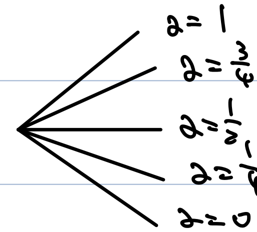
$$P(\theta|\hat{x}=1) = \frac{2((1-\alpha)(1-\theta)+\beta\theta)}{-\alpha+\beta+1}$$

$$= \frac{2((1-\theta-\alpha+\alpha\theta)+\beta\theta)}{1}$$

$$= 2(1-\theta-\alpha+2\alpha\theta)$$

$$= 2-2\theta-2\alpha+4\alpha\theta$$

(e)



$\alpha = 0$: $P(\theta \mid \hat{x} = 1) = 2 - 2\theta$

$\alpha = \frac{1}{4}$: $P(\theta \mid \hat{x} = 1) = 2 - 2\theta - \frac{1}{2} + \theta = \frac{3}{2} - \theta$

$\alpha = \frac{2}{4}$: $P(\theta \mid \hat{x} = 1) = 2 - 2\theta - 1 + 2\theta = 1$

$\alpha = \frac{3}{4}$: $P(\theta \mid \hat{x} = 1) = 2 - 2\theta - \frac{3}{2} + 3\theta = \frac{1}{2} + \theta$

$\alpha = 1$: $P(\theta \mid \hat{x} = 1) = 2 - 2\theta - 2 + 4\theta = 2\theta$

Therefore:

1. when $0 < \alpha < 0.5$, the agent resule is negative correlation of the coin resule.
2. when $\alpha = 0.5$, we can not judge.
3. when $0.5 < \alpha < 1$, the agent resule is positive correlation of the coin resule.
4. when $\alpha = 1$, the agent result same as coin.
5. when $\alpha = 0$, the agent res opposite of coin.

Question 3

(a)

cdf of X:

$F_X(x) = P(X \leq x)$

$= \int_{-\infty}^{x} p_X(u)\,du$

$= \int_{0}^{x} p_X(u)\,du$ ...... $x \in (0,1]$

Because X, Y are continuous random variable,

and $X > 0$, $Y > 0$.

cdf of Y:

$F_Y(y) = P(Y \leq y)$

$= P(\frac{1}{X} \leq y)$

$= P(\frac{1}{y} \leq X)$

$= 1 - P(X < \frac{1}{y})$ ...... $y \in [1, +\infty)$

$P(X < \frac{1}{y}) = \int_0^1 P_X(n)\,dn$

Therefore, $F_Y(y) = 1 - \int_0^1 P_X(n)\,dn$

$P_Y(y)$

$= F'_Y(y)$

$= \frac{d}{dy}\left(1 - \int_0^{\frac{1}{y}} P_X(n)\,dn\right)$

$= -\frac{d}{dy}\left(\int_0^{\frac{1}{y}} P_X(n)\,dn\right)$

$= -\left(\frac{1}{y}\right)' P_X\left(\frac{1}{y}\right)$

$= \frac{1}{y^2} P_X\left(\frac{1}{y}\right)$

..... $\frac{d}{dx}\int_a^{u(x)} f(n)\,dn = u'(x) f(u(x))$.

(b)

when player wins:

The player wins : $m-1$

The probability : $\int_m^{\infty} P_Y(y)\,dy$

when player lose:

The player loss: 1

The probability: $\int_1^m P_Y(y)\,dy$.

Then,

$$E = (m-1)\int_m^{\infty} P_Y(y)\,dy - \int_1^m P_Y(y)\,dy$$

$$= m\int_m^{\infty} P_Y(y)\,dy - \int_1^m P_Y(y)\,dy - \int_m^{\infty} P_Y(y)\,dy$$

$$= m\int_m^{\infty} P_Y(y)\,dy - \int_1^{\infty} P_Y(y)\,dy$$