Exercise 1

1. Triangle inequality:

For $x, y \in V$:

$\|x+y\| = \sqrt{\langle x+y, y+x \rangle}$

$\|x+y\|^2 = \langle x+y, y+x \rangle$

$= \langle x, y \rangle + \langle x, x \rangle + \langle y, y \rangle + \langle y, x \rangle$

$= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$

$= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \quad \cdots\cdots ①$

$(\|x\| + \|y\|)^2 = \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 . \quad \cdots\cdots ②$

① and ②: $\quad \|x+y\| \leq \|x\| + \|y\|$

Hence, $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$

2. Absolute homogeneity:

For $x \in V$, $s \in \mathbb{R}$:

$\|sx\| = \sqrt{\langle sx, sx \rangle} = \sqrt{s^2 \langle x, x \rangle} = |s| \, \|x\|$

Hence, $\|sx\| = |s|\,\|x\|$ for all $x \in V$ and all scalars s.

3. Positive definiteness:

For $x \in V$:

$\|x\| = \sqrt{\langle x, x\rangle} \geq 0$

Hence, $\|x\| \geq 0$ for all $x \in V$. Then $\|\cdot\|$ is a norm.

Execrise 2

Q1

For $x, a, b \in R^n$:

$$\nabla_x (x^T a b^T x) = \frac{d}{dx}(x^T a b^T x)$$

Let $f(x) = x^T a b^T x$:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^T a b^T (x+\Delta x) - x^T a b^T x}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{x^T a b^T x + x^T a b^T \Delta x + \Delta x^T a b^T x + \Delta x^T a b^T \Delta x - x^T a b^T x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^T a b^T \Delta x + \Delta x^T a b^T x + \Delta x^T a b^T \Delta x}{\Delta x}$$

$$= (a b^T x)^T + x^T a b^T$$

$$= x^T b a^T + x^T a b^T$$

$$= a^T x b^T + b^T x a^T$$

Q2.

For $x \in \mathbb{R}^n$:

$$\nabla_x (x^T B x) = \frac{d}{dx} (x^T B x)$$

Let $f(x) = x^T B x$:

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^T B (x+\Delta x) - x^T B x}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{x^T B x + x^T B \Delta x + \Delta x^T B x + \Delta x^T B \Delta x - x^T B x}{\Delta x}$$

$= x^T B + (Bx)^T$

$= x^T B + B^T x^T$

$= x^T (B + B^T)$

Execrise 3

$\because$ A, B are symmetric, positive definite matrixes.

$\therefore$ $x^T A x$, $x^T B x > 0$ ...... ①

For $\forall x \in V \setminus \{0\}$, $p, q > 0$; ..... ②

$$x^T (pA + qB) x = x^T pA x + x^T qB x$$

$$= p x^T A x + q x^T B x$$

$$p x^T A x + q x^T B x > 0 \quad \text{......} \quad ① \text{ and } ②$$

Hence, $pA + qB$ is symmetric and positive definite.

Execrise 4

Q1

$$L(\theta, c) = (y - X\theta - c)^T A (y - X\theta - c) + \theta^T B \theta + c^T A c$$

$$= y^T A y - y^T A X\theta - y^T A c - (X\theta)^T A y + (X\theta)^T A X\theta$$

$$+ (X\theta)^T A c - c^T A y + c^T A X\theta + c^T A c + \theta^T B \theta + c^T A c$$

$$= y^T A y - 2y^T A X\theta - 2y^T A c + 2(X\theta)^T A c + 2c^T A c$$

$$+ (X\theta)^T A X\theta + \theta^T B \theta$$

$$= y^T A y - 2y^T A X\theta - 2y^T A c + 2\theta^T X^T A c + 2c^T A c$$

$$+ \theta^T X^T A X \theta + \theta^T B \theta$$

$$\nabla L_\theta(\theta, c) = -2y^T A X + 2(X^T A c)^T + \theta^T (X^T A X + (X^T A X)^T)$$

$$+ \theta^T (B + B^T)$$

$$= -2y^T A X + 2c^T A X + 2\theta^T X^T A X + 2\theta^T B$$

Q2

$$-2y^T A X + 2c^T A X + 2\theta^T X^T A X + 2\theta^T B = 0$$

$2\theta^T(x^TAx+B)=2(y^TAx-C^TAx)$ ...... ①

$\theta(x^TA^Tx+B^T)=x^TA^Ty-x^TA^TC$ ... ①/2 and Transpose

$\theta(x^TAx+B)=x^TAy-x^TAC$ ..... A, B are positive definite matrices.

If x^TAx+B is a positive definite matrix,

then x^TAx+B is invertible.

For $\forall w\in V\setminus\{0\}$:

$$w^T(x^TAx+B)w$$

$$=w^T(x^TAx)w+w^TBw$$

$\because$ $\|x\|^2:=\langle x,x\rangle_A$ $\quad$ $\langle x,x\rangle_A:=x^TAx$

let x^TAx as Z: $\quad\langle y,y\rangle_Z:=\|y\|^2>0$ .... ①

$\because$ B is positive definite matrix, $y^TBy>0$.—. ②

① and ②: $\quad x^TAx+B$ is a is positive definite matrix

Hence, x^TAx+B is invertible.

Therefore, $\theta=(x^TAy-x^TAC)(x^TAx+B)^{-1}$

Q3

$$L(\theta,c) = y^T A y - 2y^T A X\theta - 2y^T A c + 2\theta^T X^T A c + 2c^T A c$$
$$+ \theta^T X^T A X\theta + \theta^T B\theta \quad \cdots \text{From Q1}$$

$$\nabla_c L(\theta,c) = -2y^T A + 2\theta^T X^T A + 2c^T(A + A^T)$$
$$= -2y^T A + 2\theta^T X^T A + 4c^T A$$

Q4

$\because \nabla_c L(\theta,c) = 0$

$$\therefore -2y^T A + 2\theta^T X^T A + 4c^T A = 0$$
$$2c^T A = y^T A - \theta^T X^T A$$
$$2cA^T = yA^T - A^T X\theta \quad \cdots \text{Transpose.}$$
$$2cA = Ay - AX\theta$$
$$c = A^{-1}\left(\tfrac{1}{2}Ay - \tfrac{1}{2}AX\theta\right)$$
$$= \tfrac{1}{2}(y - X\theta) \quad \cdots\cdots \text{A is positive definite Matrix.}$$

Q5

$$\theta = (X^T A y - X^T A C)(X^T A X + B)^{-1} \quad \cdots\cdots \text{From Q2}$$

$$\left.\begin{aligned} A &= I \\ C &= 0 \\ B &= \lambda I \end{aligned}\right\} \quad \theta = (X^T I y - 0) \cdot (X^T I X + \lambda I)^{-1}$$

$$= (X^T I y)(X^T I X + \lambda I)^{-1}$$

$$= (X^T X + \lambda I)^{-1} X^T y$$