Exercise 1

(a)

when $Ax=b$:

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{array}\right] \xrightarrow{\text{Swap } R_1 \text{ and } R_3} \begin{bmatrix} 2 & 7 & 1 & -2 \\ 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \end{bmatrix} \xrightarrow[R_1/2]{R_1-2R_2} \begin{bmatrix} 1 & 7/2 & 1/2 & -1 \\ 0 & 1 & 5 & -2 \\ 0 & 1 & 5 & -4 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 7/2 & 1/2 & -1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Because $0 \neq -2$, then No solution.

(b)

when $Ax=b$:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 4 & 0 & 3 & 12 \end{array}\right] \xrightarrow{R_2/2-R_1} \begin{bmatrix} 2 & 3 & 1 & 6 \\ 0 & -3 & \frac{1}{2} & 0 \end{bmatrix} \xrightarrow[R_2/3]{R_1/2} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 3 \\ 0 & -1 & \frac{1}{6} & 0 \end{bmatrix}$$

It's already in the REF, we let $z=0$

$$\begin{cases} x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 = 3 \\ -x_2 + \frac{1}{6}x_3 = 0 \end{cases}$$

A particular solution: $S_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

when $Ax = 0$:

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{6} & 0 \end{bmatrix}$$

let $z=0$, we get $S_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

let $z=1$, we get $S_3 = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{6} \\ 1 \end{bmatrix}$

General solution:

$$S = \left\{ x \in \mathbb{R}^3 : x = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{6} \\ 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \lambda_1 \in \mathbb{R} \right\}$$

Exercise 2

$$A^{-1} = \frac{1}{|A|} \cdot A^*$$

$M_{11} = 1-c$ $\quad$ $M_{12} = 1-c$ $\quad$ $M_{13} = 0$

$M_{21} = a-b$ $\quad$ $M_{22} = 1-b$ $\quad$ $M_{23} = 1-a$

$M_{31} = ac-b$ $\quad$ $M_{32} = c-b$ $\quad$ $M_{33} = 1-a$

$$A^* = \begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1-c & -a+b & ac-b \\ c-1 & 1-b & b-c \\ 0 & a-1 & a-1 \end{bmatrix}$$

$$|A| = (1\times1\times1 + 1\times1\times b + 1\times a\times c) - (1\times1\times b + 1\times c\times1 + 1\times1\times a)$$

$$= ac - a - c + 1$$

$$A^{-1} = \frac{1}{|A|} \cdot A^* = \frac{1}{ac-a-c+1} \cdot A^*$$

when $(a-1)(c-1) \neq 0$, A^{-1} exist.

Exercise 3

(a)

A is not a subspace of R^3.

Because $\forall x \in A, \lambda \in R, \lambda x \in A$ is NOT satisfied.

e.g. $x = (1, 0)$, $\lambda = -1$, $\lambda x = (-1, 0)$, but $\lambda x \notin A$.

(b)

B is a subspace of R^3.

Because. $B \subseteq R^3$, $B \neq 0$, in particular $0 \in B$

Closure of B: $\forall x, y \in B$, $x + y \in B$

$\forall x \in B, \lambda \in R, \lambda x \in B$.

(c)

C is not a subspace of R^3.

Because $\forall x, y \in C$, $x + y \in C$ is NOT satisfield.

e.g.

$x = (i, 0)$ $\quad y = (0, j)$ $\quad (i \neq j \neq 0)$

$x + y = (i, j)$ $\quad$ but $x + y \notin C$

(d)

when $b \neq 0$:

D is not a subspace of R^3.

Because $\forall x, y \in D$, $x + y \in D$ is NOT satisfield.

$\forall x \in D, \lambda \in R, \lambda x \in D$ is NOT satisfield.

And. $0 \notin B$

e.g.

$\lambda = -1$, $x \in D$, $A(\lambda x) = \lambda(Ax) = -b \neq b$

$x, y \in D$, $Ax = b$, $Ay = b$, $A(x+y) = Ax + Ay = 2b \neq b$

When $b = 0$:

D is a subspace of R^3:

Because $D \subseteq R^3$, $D \neq \emptyset$; in particular $0 \in D$

Closure of D: $\forall x, y \in D$, $x + y \in D$

$\forall x \in D$ $\lambda \in R$, $\lambda x \in D$

Exercise 4

(a)

Because $T: V \rightarrow W$ is a linear transformation.

Then $T(a+b) = T(a) + T(b)$

when $a = b = 0$

$$T(0+0) = 2T(0)$$

$$T(0) = 2T0$$

Therefore, $T(0) = 0$

(b)

Base on the definition of linear transformation:

$\forall x, y \in V$, $\forall \lambda \in R$, $\varphi \in R$:

$$(\lambda x + \varphi y) = \lambda(x) + \varphi(y)$$

when $n=1$, $y=0$:

$$T(c_1 v_1 + y) = c_1 T(v_1) + T(y) = T(c_1 v_1)$$

when $n=n$:

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 T(v_1) + \dots c_n T(v_n)$$

when $n = n+1$:

$$T(c_1 v_1 + \dots c_n v_n + c_{n+1} v_{n+1})$$

$$= T(c_1 v_1 + \dots c_n v_n) + c_{n+1} v_{n+1}$$

$= c_1T(v_1) + \cdots c_nT(v_n) + c_{n+1}T(v_{n+1})$

Therefore, For $n \geq 1$, we have that:

$$T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n)$$

(c)

Base on the definition of linearly dependent:

$$c_1v_1 + c_2v_2 + \cdots c_nv_n = 0 \quad (c_1 \cdots c_n \text{ not all zero})$$

As definition of W:

$$\{w_1 \cdots w_n\} = \{T(v_1) \cdots T(v_n)\}$$

To prove that:

$$c_1T(v_1) + \cdots + c_nT(v_n) = 0$$

Because $\{v_1 \cdots v_n\}$ is a set of linearly dependent. vectors in V. Then we have:

$$c_1v_1 + c_2v_2 + \cdots c_nv_n = 0$$

And $T: V \to W$ is a linear transformation

Therefore, $c_1T(v_1) + \cdots + c_nT(v_n) = 0$

Exercise 5

(a)

It shows that $<\cdot,\cdot>$ is symmetric and linear in first arguement, to prove that $<\cdot,\cdot>$ is linear in both arguements, as know as bilinear.

$$<x,y> = <y,x> \quad \cdots\cdots ①$$

$$<\lambda x + \psi y, z> = \lambda(x,z) + \psi(y,z) \quad \cdots\cdots ②$$

to prove that $<x, \lambda y + \psi z> = \lambda<x,y> + \psi<x,z>$

$$\begin{aligned}<x, \lambda y + \psi z> &= <\lambda y + \psi z, x> \quad \cdots\cdots ① \\ &= \lambda<y,x> + \psi<z,x> \quad \cdots\cdots ② \\ &= \lambda<x,y> + \psi<x,z> \quad \cdots\cdots ①\end{aligned}$$

$$<x, \lambda y + \psi z> = \lambda<x,y> + \psi<x,z>$$

Therefore. it is bilinear.

(b)

Let $x = [x_1, x_2]^T$, $y = [y_1, y_2]^T$. Then,

$$\begin{aligned}<x,y> &= x_1y_1 + x_2y_2 + 2(x_1y_2 + x_2y_1) \\ &= y_1x_1 + y_2x_2 + 2(y_2x_1 + y_1x_2) = <y,x>\end{aligned}$$

Therefore, $<\cdot,\cdot>$ is symmetric.

Let $x = [-1, -1]$ $y = [1, 0]$

$\langle x, y \rangle = (-1) \times (-1) + (-1) \times 0 + 2 \times ((-1) \times 0 + (-1) \times 1)$

$= 1 + 0 - 2 = -1 < 0$

Therefore, $\langle \cdot, \cdot \rangle$ is NOT positive definite.

Let $z = [z_1, z_2]^T \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

$\langle x+y, z \rangle$

$= (x_1 + y_1) z_1 + (x_2 + y_2) z_2 + 2((x_1 + y_1) z_2 + (x_2 + y_2) z_1)$

$= x_1 z_1 + y_1 z_1 + x_2 z_2 + y_2 z_2 + 2(x_1 z_2 + y_1 z_2 + x_2 z_1 + y_2 z_1)$

$= \langle x, z \rangle + \langle y, z \rangle$

$\langle x, y+z \rangle$

$= x_1 (y_1 + z_1) + x_2 (y_2 + z_2) + 2(x_1 (y_2 + z_2) + x_2 (y_1 + z_1))$

$= x_1 y_1 + x_1 z_1 + x_2 y_2 + x_2 z_2 + 2(x_1 y_2 + x_1 z_2 + x_2 y_1 + x_2 z_1)$

$= \langle x, z \rangle + \langle y, z \rangle$

$\langle \lambda x, y \rangle = \lambda x_1 y_1 + \lambda x_2 y_2 + 2(\lambda x_1 y_2 + \lambda x_2 y_1)$

$= \lambda x_1 y_1 + \lambda x_2 y_2 + 2\lambda x_1 y_2 + 2\lambda x_2 y_1$

$= \lambda (x_1 y_1 + x_2 y_2 + 2(x_1 y_2 + x_2 y_1))$

$= \lambda \langle x, y \rangle$

Therefore, $\langle \cdot, \cdot \rangle$ is bilinear.

Exercise 6

(a)

If x and y are orthologonal, then:

$$\langle x, y\rangle = 0 \quad \cdots\cdots (1)$$

Assumes that. x and y are linearl independent:

$$ax + by = 0 \quad (a, b \text{ not all zero}) \cdots (2)$$

Because $\langle x, 0\rangle = 0$, then:

$$\begin{aligned}\langle x, ax+by\rangle &= a\langle x, x\rangle + b\langle x, y\rangle \\ &= a\langle x, x\rangle + 0 \quad \cdots\cdots (1) \\ &= 0 \quad \cdots\cdots (2)\end{aligned}$$

$$a\langle x, x\rangle = 0$$

In the same way, $\langle 0, y\rangle = 0$

$$\begin{aligned}\langle ax+by, y\rangle &= a\langle x, y\rangle + b\langle y, y\rangle \\ &= 0 + b\langle y, y\rangle \quad \cdots (1) \\ &= 0 \quad \cdots\cdots (2)\end{aligned}$$

Because we have $x \neq 0$, $y \neq 0$, then:

$$\begin{cases}\langle x, x\rangle > 0 \\ \langle y, y\rangle > 0 \\ a\langle x, x\rangle = 0 \\ b\langle y, y\rangle = 0\end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0\end{cases}$$

But if x and y are linear independent:

a, b are not all zero.

Therefore, x and y are linear dependent.

(b)

let i and j are orthogonal, then:

$$\langle i, j \rangle = 0$$

let $x = i$, $y = i + j$, then x and y are linearly independent.

$$ax + by = 0 \quad (a, b \text{ not all zero})$$

$$ai + b(i+j) = 0$$

$$(a+b)i + bj = 0$$

then: $\begin{cases} a+b=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$

But a, b should not all zero.

Therefore, if x and y are linearly independent, they are not always orthogonal.

Exercise 7

(a)

To prove that $\varepsilon\|V\|_a \le \|V\|_a \le \frac{1}{\varepsilon}\|V\|_a$.

Because $\varepsilon \in (0,1]$, $\|V\|_a \ge 0$

$$\begin{cases} \varepsilon\|V\|_a \le \|V\|_a & \cdots\cdots ① \\ \|V\|_a \le \frac{1}{\varepsilon}\|V\|_a & \cdots\cdots ② \end{cases}$$

Therefore:

$$\varepsilon\|V\|_a \le \|V\|_a \le \frac{1}{\varepsilon}\|V\|_a. \quad \cdots\cdots ①+②$$

(b)

To prove that $\varepsilon\|V\|_b \le \|V\|_a \le \frac{1}{\varepsilon}\|V\|_b$.

Because $\varepsilon\|V\|_a \le \|V\|_b \le \frac{1}{\varepsilon}\|V\|_a$, $\varepsilon \in (0,1]$:

$$\begin{cases} \varepsilon\|V\|_a \le \|V\|_b & \Rightarrow \|V\|_a \le \frac{1}{\varepsilon}\|V\|_b \cdots\cdots ③ \\ \|V\|_b \le \frac{1}{\varepsilon}\|V\|_a & \Rightarrow \varepsilon\|V\|_b \le \|V\|_a \cdots\cdots ④ \end{cases}$$

Therefore:

$$\varepsilon\|V\|_b \le \|V\|_a \le \frac{1}{\varepsilon}\|V\|_b \quad \cdots\cdots ③+④$$

(c)

$\|V\|_1 = |V_1| + |V_2|$ $\qquad$ $\|V\|_1^2 = |V_1|^2 + |V_2|^2 + 2|V_1|\cdot|V_2| \cdots\cdots ①$

$\|V\|_2 = \sqrt{V_1^2 + V_2^2}$ $\qquad$ $(\|V\|_1 - \|V\|_2)^2 > 0 \cdots\cdots ②$

$$2|V_1|\cdot|V_2| + |V_1|^2 + |V_2|^2 \le 2|V_1|^2 + 2|V_2|^2$$

Then: $\qquad \|V\|_1^2 \le 2\|V\|_2^2 \qquad (\|V\|_1 \ge 0,\ \|V\|_2 \ge 0)$

Therefore:

$$\frac{\sqrt{2}}{2}\|V\|_1 \le \|V\|_2 \text{, largest } \varepsilon \text{ is } \frac{\sqrt{2}}{2} \quad ---- 3$$

To prove that: $\frac{\sqrt{2}}{2}\|V\|_1 \le \|V\|_2 \le \sqrt{2}\|V\|_1$

As 3 shows: $\frac{\sqrt{2}}{2}\|V\|_1 \le \|V\|_2$

$$\|V\|_2^2 = V_1^2 + V_2^2 \quad ----- 4$$

$$(\sqrt{2}\|V\|_1)^2 = 2\|V\|_1^2 = 2\times(|V_1|^2 + |V_2|^2 + 2|V_1|\cdot|V_2|$$

$$= 2V_1^2 + 2V_2^2 + 2|V_1|\cdot|V_2| \quad --- 5$$

Base on 4 and 5:

$$\|V\|_2 \le \sqrt{2}\|V\|_1 \quad ---- 6$$

As 3 and 6 shows:

$$\frac{\sqrt{2}}{2}\|V\|_1 \le \|V\|_2 \le \sqrt{2}\|V\|_1$$

Exercise 8

(a)

If $x \in U$, then:

$$a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}12\\12\\18\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 12\\1 & 1 & 12\\1 & 0 & 18\end{bmatrix} \xrightarrow[R_1-R_3]{R_1-R_2} \begin{bmatrix}1 & 2 & 12\\0 & 1 & 0\\0 & 2 & -6\end{bmatrix} \xrightarrow{R_2\times 2-R_3}$$

$$\begin{bmatrix}1 & 2 & 12\\0 & 1 & 0\\0 & 0 & 6\end{bmatrix}$$

Because $0 \neq 6$,
Then no solution.

(b)

$P_\pi = B(B^TB)^{-1}B^T$

$\pi_U(x) = B(B^TB)^{-1}B^Tx$

$$B = \begin{bmatrix}1 & 2\\1 & 1\\1 & 0\end{bmatrix} \qquad B^TB = \begin{bmatrix}1 & 1 & 1\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}3 & 3\\3 & 5\end{bmatrix}$$

$$B^Tx = \begin{bmatrix}1 & 1 & 1\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}12\\12\\18\end{bmatrix} = \begin{bmatrix}42\\36\end{bmatrix}$$

we solve the normal equation $B^T B \lambda = B^T x$ to find λ

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix}$$

$$\Rightarrow \lambda = \begin{bmatrix} 17 \\ -3 \end{bmatrix}$$

$$\pi_U(x) = B\lambda = \begin{bmatrix} 11 \\ 14 \\ 17 \end{bmatrix}$$

(c)

To find solution of:

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \pi_U(x) = \begin{bmatrix} 11 \\ 14 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 11 \\ 1 & 1 & 14 \\ 1 & 0 & 17 \end{bmatrix} \xrightarrow[R_1 - R_2]{R_1 - R_3} \begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{bmatrix}$$

$$\xrightarrow{2 \cdot R_2 - R_3} \begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} a = 17 \\ b = -3 \end{cases}$$

(d)

If $x \in U$: $d(x, U) = 0$

If $x \notin U$:

Assumes that y is the orthogonal projection of x onto U, then $|z|$ is $d(x, U)$. And x, y, z consist of a triangle. Then. $z = x - y$.

Therefore: $d(x, U) = \min_{y \in U} \|x - y\|_2$

e.g.