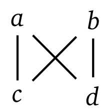
**ASSIGNMENT 5 (DUE ON 17 SEPTEMBER 2021 AT 11:59PM)**

**MATH2301, SEMESTER 2, 2021**

INSTRUCTOR: ASILATA BAPAT

(1) Compute all values of the Möbius function $\mu$ for the "bow-tie" poset shown below, using either the recursive formula discussed in class or the matrix representation.




(2) Compute the following values of the Möbius function $\mu$ for the divisor poset order relation on the positive natural numbers. (Note that even though this is not a finite poset, it is locally finite, and so the Möbius function makes perfect sense, and the calculations are the same.)
   (a) $\mu([5,75])$
   (b) $\mu([1,2021])$
   (c) $\mu([1,2020])$

(3) Let $P$ be the subset poset of $\{a,b,c\}$. Fix an ordering on $P$ and set up the matrix $M_\zeta$. (It will be a large matrix!) Let $p\colon P \to \mathbb{R}$ be the function $p(X) = 1$. Use matrix-vector product to compute $\zeta * p$.

(4) Let $P$ be the divisor poset of a fixed positive natural number $n$. Let $\epsilon$ be the function on $P$ defined as

$$\epsilon(m) = \begin{cases} 1 & m = 1 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Prove that $\epsilon * \zeta$ is the constant function 1. That is, show that

$$(\epsilon * \zeta)(m) = 1$$

   for every $m \in P$.

   (b) Show using the formulas we found in class, that

$$\mu([d,m]) = \mu([1,m/d])$$

   for every $d \mid m$ with $d, m \in P$.

   (c) By applying appropriate convolution with $\mu$ to the formula from the first subpart, show that for every $m > 1$ in $P$, we have

$$\sum_{d \mid m} \mu([1,m/d]) = 0.$$

   Hint: Recall that if $p\colon P \to \mathbb{R}$ a function on a poset $P$, and if $f, g \in \mathscr{A}(P)$, then

$$(p * f) * g = p * (f * g).$$

(5) Consider the number 888. In this problem we will use the inclusion-exclusion principle to compute the number of positive integers $m$ such that $1 \le m \le 888$, and $\gcd(m, 888) = 1$.
   (a) We have $888 = 2^3 \times 3 \times 37$. This is the prime factorisation. Briefly justify that if $1 \le m \le 888$ and $\gcd(m, 888) = 1$, then none of the primes $\{2,3,37\}$ divides $m$.
   (b) Consider the subset poset of $\{2,3,37\}$. For $X \subseteq \{2,3,37\}$, set $p(X)$ to be the number of positive integers $1 \le m \le 888$ that are divisible by the primes in $X$ (and also perhaps other primes). Find formulas for $p(X)$ for all values of $X$. (Hint: use a technique similar to the one we used in class for numbers from 1 to 100.)

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(c) For $X \subseteq \{2,3,37\}$, set $q(X)$ to be the number of positive integers $1 \le m \le 888$ that are divisible by exactly the primes appearing in $X$ and no other primes from $\{2,3,37\}$. Briefly justify that

$$p(X) = (q * \zeta)(X).$$

(d) Using the fact that

$$q(X) = (p * \mu)(X)$$

for every $X \subseteq \{2,3,37\}$, find the value of $q(\emptyset)$. Conclude that this is your answer!

(6) (Bonus, not for credit!) Let $n$ be any positive natural number. We say that $\phi(n)$ is the number of positive integers $m$ such that $1 \le m \le n$, and $\gcd(m,n) = 1$. This is called *Euler's $\phi$ function*, or *Euler's totient function*. Suppose that $n$ has the prime factorisation

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

with $p_1 < p_2 < \cdots < p_k$ distinct primes, and $a_i \ge 1$ for every $i$.

(a) Find a formula for $\phi(n)$ using the technique of the previous problem.

(b) Show by calculation that in the divisor poset of $n$, we have

$$(\phi * \zeta)(m) = n$$

for every $m \mid n$.

(c) Deduce the following identity in the divisor poset of $n$:

$$\phi(n) = \sum_{d \mid n} d \cdot \mu([d,n]).$$