WORKSHEET 6

MATH2301, SEMESTER 2, 2021

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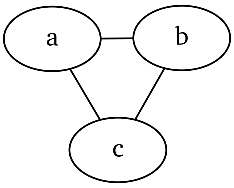
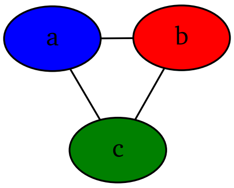
Let $G = (V, E)$ be an undirected graph without multiple edges or self-loops. Recall that the edge relation in an undirected graph is symmetric. So we will draw edges in undirected graphs as single undirected connections (not two arrows) between vertices. For example, here is an undirected graph.



All graphs in this worksheet will be of this form, so I will stop writing "undirected graph without multiple edges or self-loops".

For some k, fix a set of k "colours", which is $C = \{c_1, \ldots, c_k\}$. A *k-colouring* of G is an assignment $V \to C$. We think of each vertex of G as being *coloured* by one of the k possible colours. A *proper* k-colouring of G is a k-colouring, such that the two endpoints of any given edge receive different colours. For example, here is a 3-colouring of the graph above, where I think of my colours as being red, blue, and green.



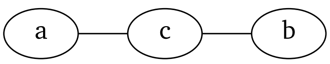
In this worksheet we will discover a formula for the number of proper k-colourings of any undirected graph (without multiple edges and self-loops). This is called the *chromatic polynomial* of the graph G, and we will denote it by $p_k(G)$. Part of the challenge in this worksheet is reading and understanding all the definitions, and you should do this with the help of your groupmates and demonstrator.

1. Graph colourings, using deletion-contraction recursion

(1) (Warm-up) Find the smallest k for which the graph shown above has a proper k-colouring, and describe all proper k-colourings for such a k.

(2) (Warm-up) Together with your group, draw one or two bigger examples of undirected graphs, and find the smallest k for which those examples have proper k-colourings. Can you calculate how many different proper k-colourings there are?

(3) Let $e \in E$ be an edge of G. We can "delete" the edge e as well as its reverse, to obtain a new undirected graph, which we call $G \backslash e$. For example, deleting the undirected edge between a and b yields the following.



Together with your group, use the examples you had written down earlier and choose one or two edges to delete in each example. Then write down $G \backslash e$ for each example and convince yourself and your groupmates that it is correct.

(4) Let $e \in E$ be an edge. We can also "contract" the edge e to obtain a new undirected graph, which we will call G/e. You should think of this procedure as "shrinking" the edge e and its reverse down to a point, so that you merge its two endpoints. Keep all other edges as they are, and collapse multiple

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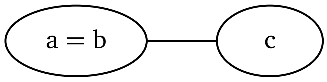
edges down to single edges. For example, if you contract the edge between a and b in the original example, you obtain the following. Note that since there was an edge $a - c$ and also an edge $b - c$, just merging a and b gives a duplicate edge from $a = b$ to c, but then we collapse it down to a single edge.



Together with your group, use the examples you had written down earlier and choose one or two edges to contract in each example. Then write down G/e for each example and convince yourself and your groupmates that it is correct.

(5) Now consider $G = (V, E)$. Let $p_G(k)$ be the number of proper k-colourings of G. Convince yourself and your groupmates that

$$p_G(k) = p_{G\setminus e}(k) - p_{G/e}(k).$$

(Hint: when is a proper k-colouring of $G\setminus e$ also a proper k-colouring of G?)

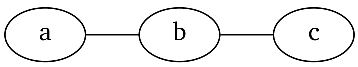
(6) Whether or not you proved the formula above, use it to construct formulas for $p_G(k)$ in some or all of the examples you constructed earlier. In particular, find $p_G(k)$ of the graph drawn in the original example.

2. (Bonus) Graph colourings, using bond posets and Möbius inversion

Recall the bond poset from the previous worksheet. For convenience, I recall the definition here. A *bond* of G is a partition of the vertices of G into disjoint subsets

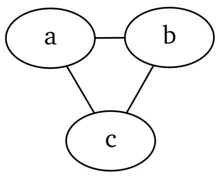
$$V = \bigsqcup_{i=1}^{n} V_i,$$

such that each subgraph of G whose edges have endpoints in a single subset V_i is a connected subgraph. For example, suppose that G is the following graph.



In this case, the partition $\{\{a, b\}, \{c\}\}$ is a bond, because the subgraph determined by $\{a, b\}$ is connected (has one edge between a and b), and the subgraph determined by $\{c\}$ is trivially connected. On the other hand, the partition $\{\{a, c\}, \{b\}\}$ is not a bond, because the subgraph determined by $\{a, c\}$ has no paths from a to c, and is not connected.

(1) (Warm-up) Draw the bond poset for the following graph.



(2) Let G be a graph, and let $V_1 \sqcup \cdots \sqcup V_m$ be a bond. A k-colouring of this bond is a k-colouring of the vertices of G, such that all vertices in V_1, get the same colour, all vertices in V_2 get the same colour, and so on. A *proper* k-colouring of this bond is a k-colouring of the bond, that additionally satisfies the following condition: if some vertex of V_i is connected to some vertex of V_j, then V_i and V_j receive different colours.

For the bond poset you drew in the previous example, describe/draw some proper bond colourings.

(3) Let B be the bond poset of a graph G. Write functions $f_k, g_k : B \to \mathbb{R}$ as follows. If $b \in B$, then

$$f_k(b) = \text{number of not necessarily proper } k\text{-colourings of the bond } b,$$
$$g_k(b) = \text{number of proper } k\text{-colourings of the bond } b.$$

Do some examples to get acquainted with these definitions.

(4) Find an explicit formula for $f_k(b)$ in terms of the number of parts in the bond.

(5) Convince yourself and your groupmates that in any bond poset B, we have

$$f_k(b) = \sum_{b \preceq c} g_k(c).$$

(6) Whether or not you proved the formula above, use an appropriate convolution with the Möbius function to find a formula for $g_k(b)$ in terms of $f_k(b)$.

(7) Check that the number of proper k-colourings of G is simply the number of proper bond colourings of the unique minimum element of the bond poset. Now in your original example, read off an explicit formula for $p_k(G)$.