WORKSHEET 1

MATH2301, SEMESTER 2, 2021

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1. SETS AND SET NOTATION

(1) Some of the following sets are the same, and some are different. Decide and discuss.

(a) $\emptyset$

(b) $P(\emptyset)$

(c) $\emptyset \cap P(\emptyset)$

(d) $\emptyset \cup P(\emptyset)$

(e) $\emptyset \times P(\emptyset)$

(f) $P(\emptyset) \times P(\emptyset)$

Solution. Here are all the sets in plainer language.

(a) The empty set, $\emptyset$.

(b) The set containing the empty set, $\{\emptyset\}$.

(c) The empty set, $\emptyset$.

(d) The set containing the empty set, $\{\emptyset\}$

(e) The empty set, $\emptyset$.

(f) The set $\{(\emptyset, \emptyset)\}$.

$\square$

(2) The symbol $\exists$ reads as "there exists". The symbol $\forall$ reads as "for all". The symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of natural numbers (including zero), rational numbers, integers, and reals respectively. Consider the following set:

$$\left\{x \in \mathbb{Q} \mid \exists y \in \mathbb{Z} \text{ such that } x = \frac{y}{2} + 1\right\}.$$

Some of the following sets are the same as the set described above, and some are not. Decide and discuss.

(a) $\left\{x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} \text{ such that } x = \frac{y}{2} + 1\right\}$.

(b) $\{x \in \mathbb{Q} \mid \exists y \in \mathbb{Z} \text{ such that } 2x = y\}$.

(c) $\left\{x \in \mathbb{Z} \mid \forall y \in \mathbb{Q} \text{ we have } x = \frac{y}{2} + 1\right\}$.

(d) $\left\{x \in \mathbb{R} \mid \exists y \in \mathbb{Z} \text{ such that } x = \frac{y}{2} + 1\right\}$.

Solution. Here are all the sets described in plainer language. The original set is the set of all x that can be described as half an integer, plus one. All integers have this form; for example, $3 = 4/2 + 1$. Additionally, all half-integers also have this form. For example, $5/2 = 3/2 + 1$. Nothing else has this form: we can conclude this by solving for y in the equation given to get

$$y = 2(x - 1)$$

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where $x \in \mathbb{Q}$ and $y \in \mathbb{Z}$. Since y is an integer, the quantity $x - 1$ can have a denominator of either 1 or 2.

So the original set consists of the integers and the half-integers.

(a) This set simply describes all the integers, because all integers can be expressed in the form shown. It is not the same as the original set.

(b) This set describes all rational whose denominator is either 1 or 2. This is the same as the original set.

(c) This set is the empty set: it is never true for a rational x that $x = y/2 + 1$ *for all* $y \in \mathbb{Q}$.

(d) This is the same as the original set: if the given equation holds true, then x must already belong to the rationals, so the constraint is the same as the constraint of the original set. □

2. Relations

(1) Give examples of relations that have the listed properties, in each case. Remember that you can always give an "abstract" example: that is, any appropriate subset of $S \times S$ for a well-chosen S.

(a) Symmetric but not reflexive.

Solution. Many possible examples. For instance, $\{(0,1),(1,0)\}$ on the set $\{0,1\}$. □

(b) Both symmetric and anti-symmetric.

Solution. Symmetric means that if $(x,y) \in R$ then $(y,x) \in R$. Anti-symmetric means that if $(x,y) \in R$ and if $x \neq y$, then $(y,x) \notin R$. The only possibility is that only elements of the form (x,x) are in R. (But not all such pairs have to be in the relation.) So for example, we can take the relation $\{(0,0),(2,2)\}$ on the set $\{0,1,2\}$. □

(c) Neither symmetric nor anti-symmetric.

Solution. Many possible examples. For instance, $\{(0,1),(1,0),(0,2)\}$ on the set $\{0,1,2\}$. □

(2) For each kind of relation mentioned, discuss what special property the graph of the relation must satisfy, and also what special property the adjacency matrix for the graph of that relation must satisfy.

(a) Reflexivity

Solution. Every node must have a self-loop. All diagonal entries of the adjacency matrix are equal to 1. □

(b) Symmetry

Solution. For every arrow (x,y), we also have an arrow (y,x). The adjacency matrix is a symmetric matrix: equal to its own transpose. □

(c) Anti-symmetry

Solution. If there is an edge (x,y) with $y \neq x$, then there is no backwards edge. If the (i,j)th entry of the adjacency matrix is non-zero for some $i \neq j$, then the (j,i)th entry must be zero. □

(d) Being a function

Solution. Every node has exactly one outgoing edge. There is exactly one non-zero entry in each row of the adjacency matrix. $\square$

(e) $(*)$ Transitivity

Solution. This is a slightly harder one! Think about it for now, but we will answer this together in class in a few weeks. $\square$

(3) Let S be the set of all possible orderings of the tuple $(1, 2, 3, 4)$. For example, $(2, 4, 1, 3)$ and $(4, 3, 2, 1)$ are elements of S.

(a) How many elements does S have?

Solution. S has $4! = 24$ elements. $\square$

(b) Suppose that s is an element of S. A *swap* on s swaps two of the numbers in s. For example, swapping 1 and 2 in $s = (1, 2, 3, 4)$ results in $(2, 1, 3, 4)$. Define a relation R on S by saying that $(s, t) \in R$ if we can get to t from s via an even number of swaps. Check whether R is an equivalence relation on S. If it is, find its equivalence classes.

Solution. Yes, R is an equivalence relation (I'll leave the checks to you but ask me if you have questions). For now you are meant to compute the equivalence classes by hand. It turns out that there are two of them. I'll write the tuple (a, b, c, d) as $abcd$ for short. The first one is

$$\{1234, 3124, 2314, 4132, 2431, 4213, 3241, 1423, 1324, 2143, 4321, 3412\}.$$

These are all the orderings that are reachable from 1234 by an even number of swaps. The remaining equivalence class consists of the other 12 possibilities, which I will not write out (you can do that on your own if you wish). $\square$