MATH 2301

* Nim strategy

** Example

$(1, 2, 5, 7)$
$\Leftarrow$
N-position, according to the theorem.

$$\begin{array}{r} 1_2 \\ 10_2 \\ 101_2 \\ \oplus\ 111_2 \\ \hline 001_2 \end{array} \neq 0$$

** Theorem

A position in nim is an "N" position iff its nim-sum is non-zero. It is a "P" position iff its nim-sum is zero.

** Proof sketch

We need to show the following:

1) Any move from a position with sum $= 0$ lands us in a position with nonzero sum.

✓ 2) From a position with sum $\neq 0$, there is at least one move to a position with sum $= 0$.

$$\begin{array}{rl} 1_2 & \\ 10_2 & \\ 101_2 & n \\ \oplus\ 111_2 & \\ \hline 001_2 & s \end{array}$$

Steps to achieve (2):

1) Look at first column from the left with an odd number of 1s.
2) Choose a pile that has a 1 in that column.
3) Let n be chosen pile, and s be the nim-sum.
4) Take $n \oplus s$, and replace n with $n \oplus s$

$$\left.\begin{array}{r} 101_2 \\ \oplus\ 001_2 \\ \hline 100_2 \end{array}\right\} \text{ the move } (1, 2, 4, 7) \text{ has nim-sum } 0.$$

More generally: $(x_1, \dots, x_k)$ be a nim config.

$S = x_1 \oplus \cdots \oplus x_k$

Suppose $s > 0$.

Follow steps ① & ②. Suppose that x_m is a pile size that has a 1 in the leftmost column with an odd number of 1s.

Make the following move: Change x_m to $(x_m \oplus s)$

Why is this a valid move? Is the new nim-sum 0?

$$\begin{array}{r} 1_2 \\ 10_2 \\ 101_2 \\ \oplus\ 111_2 \\ \hline 001_2 \end{array} \quad \rightsquigarrow \quad \begin{array}{r} 1_2 \\ 10_2 \\ 100_2 \\ \oplus\ 111_2 \\ \hline 000_2 \end{array}$$

** Prop: The new nim-sum is zero.

Pf: $x_1 \oplus \cdots \oplus x_m \oplus \cdots \oplus x_k = s$ (old eqn)

$x_1 \oplus \cdots \oplus (x_m \oplus s) \oplus \cdots \oplus x_k = ?$ (new eqn)

commutativity of $\oplus$ $\;= s \oplus (x_1 \oplus \cdots \oplus x_k) = s \oplus s = 0$

** Prop Under the previous assumptions, $(x_m \oplus s) < x_m$.

Pf:

$$\begin{array}{r} 1_2 \\ 10_2 \\ 101_2 \\ \oplus\ 111_2 \\ \hline 001_2 \end{array}$$

Note: x_m has a 1 in the first column from the left in which s has a 1.

$x_m \oplus s$ in binary looks like:

$$\begin{array}{rl} * * \cdots * 1 * * * \cdots * & \leftarrow x_m \\ \oplus\ 0\,0\,0 \cdots 0\, 1 * * * \cdots * & \leftarrow s \\ \hline \underbrace{* * \cdots *}_{\text{same as } x_m} 0 \underbrace{* * \cdots *}_{\text{something}} & \end{array} \quad \Big\}\ * = \text{unknown.}$$

$$\begin{array}{l} q \cdots p{+}1\ p \cdots 1\, 0 \\ * * \cdots * 1 * * * \cdots * \\ 0\,0\,0 \cdots 0\, 1 * * * \cdots * \\ \hline * * \cdots * 0 * * \cdots * \end{array}$$

$\rightsquigarrow x_m$ contains 2^p

$\rightsquigarrow (x_m \oplus s)$ does not contain 2^p, but all higher places equal x_m.

This implies that $(x_m \oplus s) < x_m$, because the largest possible number in the second part of $(x_m \oplus s)$ (after the 0) can be $111\cdots1_2 = (1+2+2^2+\cdots+2^{p-1})$. This sum equals $(2^p - 1) < 2^p$.

$\Rightarrow$ Changing x_m to $x_m \oplus s$ is a valid nim move.

Part (1) of the proof:

Suppose $(x_1, \dots, x_k)$ is a game position with $x_1 \oplus \cdots \oplus x_k = 0$.

Suppose we make a move in x_m, changing it to x_m'.

New nim-sum:

$x_1 \oplus x_2 \oplus \cdots x_m' \oplus \cdots \oplus x_k = s$.

Consider $s \oplus 0 = s = (x_1 \oplus \cdots \oplus x_m' \oplus \cdots \oplus x_k) \oplus (x_1 \oplus \cdots \oplus x_m \oplus \cdots \oplus x_k)$

$s = x_m' \oplus x_m$. (everything else cancels).

$$\begin{array}{rl} x_m = & * * \cdots * * \\ x_m' = & * * \cdots * * \ \oplus \\ \hline & * * \cdots * * = s \end{array}$$

$s = 0$ if and only if $x_m' = x_m$ binary digit by digit.

So $s = 0$ iff $x_m' = x_m$.

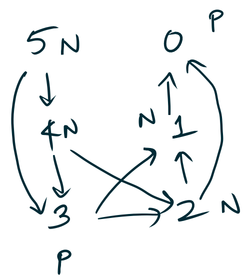
But $x_m' < x_m$ (we made a move)

$\Rightarrow s \neq 0$

** Grundy labelling

Let G be any impartial combinatorial game.

Eg. : $n = 5$, subtraction game with $S = \{1, 2\}$



Grundy labelling = more sophisticated labelling of the game graph.