

WORKSHEET 2
MATH2301, SEMESTER 2, 2021

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1. EQUIVALENCE RELATIONS AND MODULAR ARITHMETIC

- (1) Find all possible equivalence relations on the following set $S = \{a, b, c\}$.

Solution. We group by equivalence classes. There can be at most three equivalence classes, since each element lives in some equivalence class. We have the following possibilities:

- (a) No two distinct elements are equivalent. The equivalence classes are $\{\{a\}, \{b\}, \{c\}\}$.
- (b) $a \sim b$ but c is not in the same class. The equivalence classes are $\{\{a, b\}, \{c\}\}$.
- (c) $c \sim b$ but a is not in the same class. The equivalence classes are $\{\{c, b\}, \{a\}\}$.
- (d) $a \sim c$ but b is not in the same class. The equivalence classes are $\{\{a, c\}, \{b\}\}$.
- (e) All three elements are equivalent. The equivalence classes are $\{\{a, b, c\}\}$.

□

- (2) Is the following relation an equivalence relation?

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a positive integer multiple of } 3\}.$$

Solution. No, it is not; it is not symmetric.

□

- (3) Consider the relation on \mathbb{Z} described by

$$\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid (a^2 - b^2) \text{ is an integer multiple of } 5\}.$$

Show that it is an equivalence relation, and find the **distinct** equivalence classes.

Solution. Showing that it is an equivalence is easy. Note that $5 \mid (a^2 - b^2)$ means that $5 \mid (a+b)(a-b)$. This is true if and only if either $5 \mid (a+b)$ or $5 \mid (a-b)$. Saying that $5 \mid (a+b)$ is the same as saying that $a \equiv -b$ modulo 5, and saying that $5 \mid (a-b)$ is the same as saying that $a \equiv b$ modulo 5. For any number a , we find all numbers b such that either $a+b$ or $a-b$ is divisible by 5. This only depends on the remainder of a modulo 5. There are 5 possibilities for this **remainder**: 0, 1, 2, 3, 4. We note that under the new relation, we additionally have $1 \sim 4$ and $2 \sim 3$. The set of numbers divisible by 5 forms its own equivalence class. So there are three equivalence classes, which can be represented as $[0]$, $[1]$, and $[2]$.

□

- (4) Find the smallest non-negative integer b that satisfies the following equalities, or justify why it does not exist. The number d is the modulus.

- (a) $[17] + [b] = [2]$ with $d = 7$.

Solution. $[17] = [3]$ so we need $[b] = [-1] = [6]$. The answer is 6.

□

- (b) $[3b] = [0]$ with $d = 6$.

Solution. $b = 2$ by direct check. □

(c) $[3b] = [1]$ with $d = 6$.

Solution. Such a b does not exist. Note that if $[3b] = [1]$ then $3b - 1 = 6k$ for some integer k . Rearranging, we have $3b - 6k = 1$. The left hand side is a multiple of 3 but the right hand side is not. □

(d) $[b^2] = [4]$ with $d = 6$.

Solution. $b = 2$ by direct check. □

(e) $[b^2] = [-1]$ with $d = 15$.

Solution. First note that for any integer a , we have $[a^2] = [(a+15)^2]$, so it suffices to check the integers from 0 to 14 inclusive. We can check explicitly that we never get a remainder of 14, which is what we want ($[14] = [-1]$).

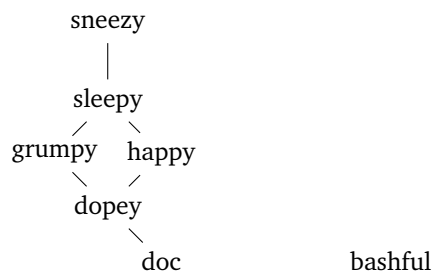
Number	Square	Remainder modulo 15
0	0	0
1	1	1
2	4	4
3	9	9
4	16	1
5	25	10
6	36	6
7	49	4
8	64	4
9	81	6
10	100	10
11	121	1
12	144	9
13	169	4
14	196	1

Notice that the pattern is symmetric. This is because we also have $[b^2] = [(15 - b)^2]$, so we only really needed to check the first half of the table. □

2. PARTIAL ORDERS

- (1) Draw the Hasse diagram of the partial order from Thursday's class on the set of dwarfs from Snow White. (Sleepy, Grumpy, etc; see the notes for the full definition.)

Solution. The Hasse diagram looks as follows.



□

- (2) Write down some topological sortings of the relation from the previous problem. Can you figure out how many there are?

Solution. The element **bashful** can go anywhere since it is unrelated to everything. So let us ignore it for now. The elements from **doc** to **sneezy** have to be in order, aside from the order of the elements **grumpy** and **happy**. We have two options there:

(doc, dopey, grumpy, happy, sleepy, sneezy),

and

(doc, dopey, happy, grumpy, sleepy, sneezy).

Once we select one of these, we have seven options to insert the element **bashful**: either in front, or in between two successive elements, or at the end. The choices for where to put **happy** are independent from the two choices we had before, so the total answer is $2 \times 7 = 14$. \square

- (3) Draw the adjacency matrices of one or two (or more...) of the topological sortings that you found above. Do you see a pattern?

Solution. With respect to any topological sorting, the adjacency matrix is always upper-triangular (the only non-zero entries are above the **diagonal**). \square

- (4) Draw all possible shapes of Hasse diagrams on three elements without labelling.

Solution. There are five possibilities. I'll leave you to find them all. \square

- (5) Draw all possible shapes of Hasse diagrams on four elements without labelling.

Solution. There are 16 possibilities. I'll leave you to find them all. \square