WORKSHEET 2

MATH2301, SEMESTER 2, 2021

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1. Equivalence relations and modular arithmetic

(1) Find all possible equivalence relations on the following set $S = \{a, b, c\}$.

Solution. We group by equivalence classes. There can be at most three equivalence classes, since each element lives in some equivalence class. We have the following possibilities:

(a) No two distinct elements are equivalent. The equivalence classes are $\{\{a\}, \{b\}, \{c\}\}$.

(b) $a \sim b$ but c is not in the same class. The equivalence classes are $\{\{a, b\}, \{c\}\}$.

(c) $c \sim b$ but a is not in the same class. The equivalence classes are $\{\{c, b\}, \{a\}\}$.

(d) $a \sim c$ but b is not in the same class. The equivalence classes are $\{\{a, c\}, \{b\}\}$.

(e) All three elements are equivalent. The equivalence classes are $\{\{a, b, c\}\}$.

□

(2) Is the following relation an equivalence relation?

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a positive integer multiple of } 3\}.$$

Solution. No, it is not; it is not symmetric. □

(3) Consider the relation on $\mathbb{Z}$ described by

$$\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid (a^2 - b^2) \text{ is an integer multiple of } 5\}.$$

Show that it is an equivalence relation, and find the distinct equivalence classes.

Solution. Showing that it is an equivalence is easy. Note that $5|(a^2-b^2)$ means that $5|(a+b)(a-b)$. This is true if and only if either $5|(a+b)$ or $5|(a-b)$. Saying that $5|(a+b)$ is the same as saying that $a \equiv -b$ modulo 5, and saying that $5|(a-b)$ is the same as saying that $a \equiv b$ modulo 5. For any number a, we find all numbers b such that either $a+b$ or $a-b$ is divisible by 5. This only depends on the remainder of a modulo 5. There are 5 possibilities for this remainder: 0, 1, 2, 3, 4. We note that under the new relation, we additionally have $1 \sim 4$ and $2 \sim 3$. The set of numbers divisible by 5 forms its own equivalence class. So there are three equivalence classes, which can be represented as $[0]$, $[1]$, and $[2]$. □

(4) Find the smallest non-negative integer b that satisfies the following equalities, or justify why it does not exist. The number d is the modulus.

(a) $[17] + [b] = [2]$ with $d = 7$.

Solution. $[17] = [3]$ so we need $[b] = [-1] = [6]$. The answer is 6. □

(b) $[3b] = [0]$ with $d = 6$.

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Solution. $b = 2$ by direct check. □

(c) $[3b] = [1]$ with $d = 6$.

Solution. Such a b does not exist. Note that if $[3b] = [1]$ then $3b - 1 = 6k$for some integer k. Rearranging, we have $3b - 6k = 1$. The left hand side is a multiple of 3 but the right hand side is not. □

(d) $[b^2] = [4]$ with $d = 6$.

Solution. $b = 2$ by direct check. □

(e) $[b^2] = [-1]$ with $d = 15$.

Solution. First note that for any integer a, we have $[a^2] = [(a+15)^2]$, so it suffices to check the integers from 0 to 14 inclusive. We can check explicitly that we never get a remainder of 14, which is what we want ($[14] = [-1]$).

Number	Square	Remainder modulo 15
0	0	0
1	1	1
2	4	4
3	9	9
4	16	1
5	25	10
6	36	6
7	49	4
8	64	4
9	81	6
10	100	10
11	121	1
12	144	9
13	169	4
14	196	1

Notice that the pattern is symmetric. This is because we also have $[b^2] = [(15 - b)^2]$, so we only really needed to check the first half of the table. □

2. Partial orders

(1) Draw the Hasse diagram of the partial order from Thursday's class on the set of dwarfs from Snow White. (Sleepy, Grumpy, etc; see the notes for the full definition.)

Solution. The Hasse diagram looks as follows.

$$
\begin{tikzcd}
 & \text{sneezy} \arrow[d, no head] & & \\
 & \text{sleepy} \arrow[dl, no head] \arrow[dr, no head] & & \\
\text{grumpy} \arrow[dr, no head] & & \text{happy} \arrow[dl, no head] & \\
 & \text{dopey} \arrow[dr, no head] & & \\
 & & \text{doc} & \text{bashful}
\end{tikzcd}
$$

□

(2) Write down some topological sortings of the relation from the previous problem. Can you figure out how many there are?

Solution. The element `bashful` can go anywhere since it is unrelated to everything. So let us ignore it for now. The elements from `doc` to `sneezy` have to be in order, aside from the order of the elements `grumpy` and `happy`. We have two options there:

$$(\text{doc}, \text{dopey}, \text{grumpy}, \text{happy}, \text{sleepy}, \text{sneezy}),$$

and

$$(\text{doc}, \text{dopey}, \text{happy}, \text{grumpy}, \text{sleepy}, \text{sneezy}).$$

Once we select one of these, we have seven options to insert the element `bashful`: either in front, or in between two successive elements, or at the end. The choices for where to put `happy` are independent from the two choices we had before, so the total answer is $2 \times 7 = 14$. □

(3) Draw the adjacency matrices of one or two (or more...) of the topological sortings that you found above. Do you see a pattern?

Solution. With respect to any topological sorting, the adjacency matrix is always upper-triangular (the only non-zero entries are above the diagonal). □

(4) Draw all possible shapes of Hasse diagrams on three elements without labelling.

Solution. There are five possibilities. I'll leave you to find them all. □

(5) Draw all possible shapes of Hasse diagrams on four elements without labelling.

Solution. There are 16 possibilities. I'll leave you to find them all. □