ASSIGNMENT 3 (DUE ON 20 AUGUST 2021 AT 11:59PM)

MATH2301, SEMESTER 2, 2021

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(1) Consider a graph whose adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the number of paths of length 4 from 1 to 3.

Solution. We solve this by taking A^4 and computing the entry in the spot $(1,3)$.

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^4 = (A^2)^2 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

So the answer is 10. □

(2) (a) Find (without explicit calculation) an example of a 4×4 nonzero adjacency matrix such that all powers of this matrix beyond the 10th power are zero. Justify briefly.

(b) Show that the 8th power of any such matrix must also be zero.

(c) Is it true that the cube of any such matrix also has to be zero?

Solution. We can do this by drawing a graph where there are no paths of length 10 or higher between any pair of vertices. For example, we can consider

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that if the 8th power is not zero, then some entry (i,j) is nonzero, so there is a length nine path from i to j. But there are only four vertices, so there must be some loop in this path! By repeating the loop several times, we can get longer and longer paths from i to j, so there must certainly be paths of length beyond 10. But all powers beyond the 10th power are zero, so this can only happen if the 8th power of the matrix was zero to begin with.

The cube need not be zero. In our example the cube of our matrix is non-zero:

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

(3) Draw the graph of the relation $R = \{(a,b) \mid 0 \le b - a \le 2\}$ on the set $S = \{1,2,3,4\}$. Draw the graph of this relation, and also the transitive closure of the graph of this relation. Also write down the adjacency matrices of both graphs (using the drawing, not using Boolean product).

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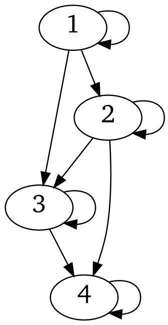
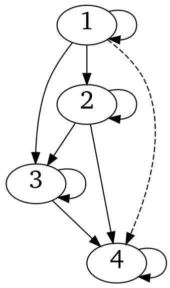
Solution. The graph is as follows (first figure). We need to add one extra edge from 1 to 4 to make it transitive. This is shown with a dashed edge in the second figure.





Let A be the adjacency matrix of the first graph and B the adjacency matrix of the second graph. Then we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

(4) Find the transitive closure of the relation $R = \{(a,b) \mid a + b > 3\}$ on the set $\{1,2,3\}$ using Boolean powers of the adjacency matrix.

Solution. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

To take the transitive closure, we have to take Boolean powers up to 3 and "add" (that is, "or") them. Note that

$$A^{*2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^{*3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

So the adjacency matrix of the transitive closure is

$$A \vee A^{*2} \vee A^{*3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

□

(5) (a) Recall from the last assignment what it means to be the maximal element of a poset. Using that knowledge, come up with a definition of what it means to be a minimal element of a poset.

Solution. Let $(P, \preceq)$ be a poset. An element $a \in P$ is called *minimal* if there is no $b \in P$ such that $b \neq a$ and $b \preceq a$. □

(b) In class we defined the transitive closure of a relation R on a set S as the minimal relation R' with the property that $R \subseteq R'$. This means that if R'' is any other transitive relation on S such that $R \subseteq R''$, then $R' \subseteq R''$. Fill in the details of the proof of the following statement: any relation R on a set S has a unique transitive closure.

- Suppose that T_1 and T_2 are both transitive closures of a relation R on a set S.
- $\cdots$ $\boxed{\text{Fill this in}}$ $\cdots$

Solution. First note that both being transitive closures means that $R \subseteq T_1$ and $R \subseteq T_2$, and that T_1 and T_2 are both transitive relations. Further, since T_1 is a transitive closure, we have that $T_1 \subseteq T_2$. Similarly, since T_2 is a transitive closure, we have that $T_2 \subseteq T_1$. $\square$

- Therefore $T_1 = T_2$.