WORKSHEET 4

MATH2301, SEMESTER 2, 2021

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1. General questions about posets

(1) Find an example of a poset in which every two element subset has a glb, but not every two element subset has an lub.

Solution. Many options. For example, a $\vee$ shaped poset with three elements. □

(2) Let $(P, \preceq)$ be a poset. If $x \preceq y$, we say that y *covers* x if $y \neq x$ and there is no z such that $x \prec z \prec y$. Find an example of a poset P and an element $x \in P$ such that x is not a maximal element, but such that no element of P covers x.

Solution. Many options. For example, $\mathbb{Q}$ with the standard $\leq$ relation. For any given x, we cannot find a covering element y, because the element $(x+y)/2$ will always be strictly between the two! □

(3) We say that a poset is *locally finite* if every interval $[x, y]$ has a finite size. Find an example of a poset that is not locally finite.

Solution. Many options. For example, $\mathbb{Q}$ with the standard $\leq$ relation. □

(4) Find an example of an infinite poset that is locally finite.

Solution. Many options. For example, $\mathbb{N}$ with the standard $\leq$ relation. □

2. The incidence algebra

Let P be the divisor poset whose underlying set is $\mathbb{N}$, and where $a \preceq b$ if $a \mid b$. Note that this poset is not finite, but it is *locally finite*. The incidence algebra also makes sense in this setup. Let $L, R \in \mathscr{A}_P$ be the "left endpoint" and "right endpoint" functions, defined as follows:

$$L([x, y]) = x, \quad R([x, y]) = y.$$

(1) Compute a few values of $\zeta * L$ and $L * \zeta$ on intervals of type $[1, n]$. For example, you could take $n = 6, 15, 24$, or any other positive integers of your choice. What are these new functions doing?

Solution. We write down the answers for $n = 15$ and leave you to work out the others. Note that 15 has four divisors: 1, 3, 5, and 15, and these are exactly the values of d so that $1 \preceq d \preceq 15$. So we see that

$$(\zeta * L)([1, 15]) = \sum_{1 \preceq d \preceq 15} \zeta([1, d]) L([d, 15]) = \sum_{1 \preceq d \preceq 15} d.$$

In this case, the answer is

$$1 + 3 + 5 + 15 = 24.$$

This function just adds up all divisors of n when you apply it to $[1, n]$.

Date: August 22, 2021.

Similarly we have

$$(L * \zeta)([1,15]) = \sum_{1 \preceq d \preceq 15} L([1,d])\zeta([d,15]) = \sum_{1 \preceq d \preceq 15} 1.$$

In this case, the answer is $1+1+1+1=4$. This function just counts the number of divisors of n when you apply it to $[1,n]$. □

(2) What will $\zeta * L$ and $L * \zeta$ do to a general interval of type $[a,b]$? (Answer in words.)

Solution. Arguing as above, we see that when evaluated on $[a,b]$, the function $\zeta * L$ adds up all divisors of b that are multiples of a, that is, adds up all d such that $a \preceq d \preceq b$. Similarly, when evaluated on $[a,b]$, the function $L * \zeta$ counts all divisors of b that are multiples of a. □

(3) Compute a few values of $\zeta * R$ and $R * \zeta$ on intervals of type $[1,n]$. What are these new functions doing?

Solution. We write down the answers for $n = 15$ and leave you to work out the others. Note that 15 has four divisors: 1, 3, 5, and 15, and these are exactly the values of d so that $1 \preceq d \preceq 15$. So we see that

$$(\zeta * R)([1,15]) = \sum_{1 \preceq d \preceq 15} \zeta([1,d])R([d,15]) = \sum_{1 \preceq d \preceq 15} 15.$$

In this case, the answer is

$$15+15+15+15=60.$$

This function just counts up all divisors of n and multiplies the result by n, when you apply it to $[1,n]$.

Similarly we have

$$(R * \zeta)([1,15]) = \sum_{1 \preceq d \preceq 15} R([1,d])\zeta([d,15]) = \sum_{1 \preceq d \preceq 15} d.$$

In this case, the answer is $1+3+5+15=24$. This function just adds up the divisors of n when you apply it to $[1,n]$. □

(4) What will $\zeta * R$ and $R * \zeta$ do to a general interval of type $[a,b]$? (Answer in words.)

Solution. Arguing as above, we see that when evaluated on $[a,b]$, the function $\zeta * R$ evaluates to b times the number of divisors of b that are multiples of a. Similarly, when evaluated on $[a,b]$, the function $R * \zeta$ adds up all divisors of b that are multiples of a. □

(5) Define a function ϕ in the incidence algebra of P as follows. First, for any interval of the form $[1,n]$, set $\phi(n)$ to be the number of positive integers m such that $1 \le m \le n$, such that the gcd of m and n equals 1. For any other interval of the form $[a,b]$, note that we must have $a \preceq b$ which means that $a \mid b$. In this case, set $\phi([a,b])$ to be the number of positive integers m such that $1 \le m \le (b/a)$, such that the gcd of m and b/a equals 1.

(a) Compute a few values of ϕ.

Solution. Try a few different values. Can you find formulas for $\phi([1,p])$ where p is a prime? What about $\phi([1,p^k])$?
(Challenge) What about a general formula? □

(b) (Bonus) Can you show that $(\phi * \zeta)([a,b]) = b/a$?

Solution. This problem is slightly beyond the scope of this class, because it uses techniques that we haven't really seen here. You will probably see the solution if you take MATH3301, so you are welcome to wait until then. If you're interested in trying to solve it now, come and talk to me and I would be happy to guide you through the solution. □

3. The partitions poset

Let n be a positive integer. A *partition* of n is an ordered tuple $(a_1, \dots, a_k)$ of positive integers, such that $a_1 \geq a_2 \cdots \geq a_k$ and $n = a_1 + \cdots + a_k$. For example, the possible partitions of 3 are (3), $(2,1)$, and $(1,1,1)$.

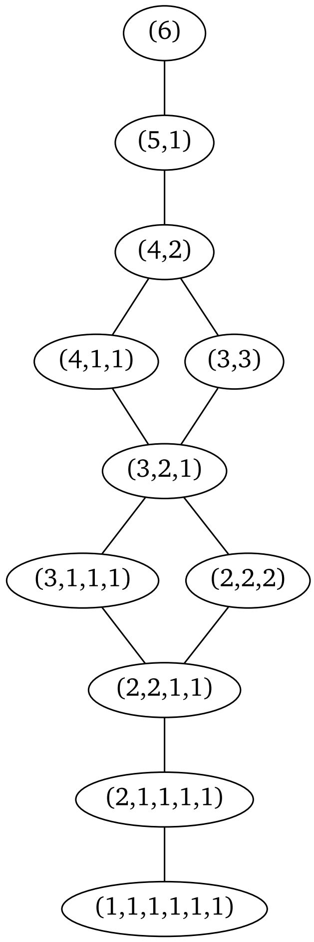
The set of all partitions of a fixed n has a partial order, defined as follows. Suppose that $\bar{a} = (a_1, \dots, a_k)$ and $\bar{b} = (b_1, \dots, b_l)$ are two partitions of n. (Note that k and l can be different.) Set $a_m = 0$ for all $m > k$, and similarly $b_m = 0$ for all $m > l$. We say that $\bar{a} \preceq \bar{b}$ if all of the following are true:

$$a_1 \leq b_1, (a_1 + a_2) \leq (b_1 + b_2), \dots, (a_1 + \cdots + a_m) \leq (b_1 + \cdots + b_m),$$

where $m = \max\{k, l\}$. For example, the partitions $(4,1,1)$ and $(3,3)$ of the number 6 are incomparable under this ordering, but we have $(3,2,1) \preceq (4,1,1)$

(1) Find all partitions of the number 6 and draw the corresponding Hasse diagram.

Solution. There are 11 different partitions, with the following Hasse diagram.



□

(2) Check that $n = 6$ is the first time that this poset is not a total order.

Solution. This is easy to check. For up to five elements, the Hasse diagrams are linear. □