WORKSHEET 3

MATH2301, SEMESTER 2, 2021

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(1) The *transpose* of an $m \times n$ matrix A, denoted A^t, is the matrix such that

$$(A^t)_{(i,j)} = A_{(j,i)}.$$

If A is the adjacency matrix of a graph, then so is A^t. What can you say about the graph corresponding to the transposed matrix?

Solution. The transpose will "flip" all connections. So to get the graph corresponding to A^t, we take the original graph and reverse all the edges. ☐

(2) Suppose that for an adjacency matrix of a graph, all the entries in the ith row and the ith column are zero. What can you conclude about the graph?

Solution. This means that the ith vertex has no outgoing edges, and also no incoming edges. In other words, it is an "isolated" vertex, and in particular the graph is not connected. ☐

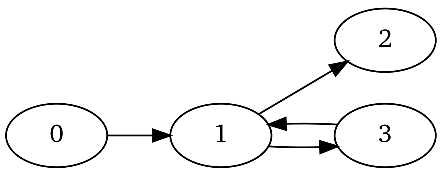
(3) Convince yourself (using an example or two) that the adjacency matrix of a graph really changes if you relabel the vertices in a different order.

Solution. Convince yourself. ☐

(4) Find a non-zero 5×5 matrix whose square is zero.

Solution. Make this the adjacency matrix of any graph that visibly has no length two paths. For example, a graph on five vertices where the only edges are $1 \to 2$, $3 \to 4$, and $3 \to 5$. ☐

(5) Find the transitive closure of the following graph using Boolean multiplication.



Solution. The adjacency matrix is the following 4×4 matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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The Boolean powers are as follows:

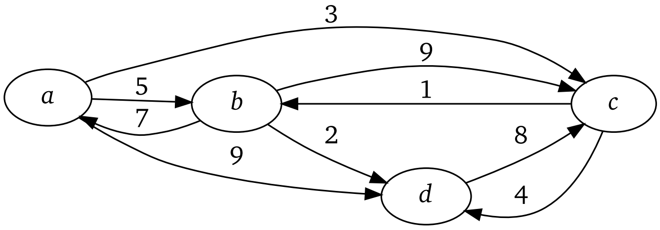
$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The transitive closure is the (Boolean) sum of all of these, which gives the following matrix:

$$A \vee A^{*2} \vee A^{*3} \vee A^{*4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

□

(6) Consider the following graph, with edge weights as listed.



(a) Use the $\{\min, +\}$ matrix product to compute the minimum-weight paths of any length between any two vertices.

Solution. The weighted adjacency matrix is as follows:

$$W = \begin{pmatrix} 0 & 5 & 3 & 9 \\ 7 & 0 & 9 & 2 \\ \infty & 1 & 0 & 4 \\ \infty & \infty & 8 & 0 \end{pmatrix}.$$

The min-plus powers are as follows:

$$W^{\odot 2} = \begin{pmatrix} 0 & 4 & 3 & 7 \\ 7 & 0 & 9 & 2 \\ 8 & 1 & 0 & 3 \\ \infty & 9 & 8 & 0 \end{pmatrix}, \quad W^{\odot 3} = \begin{pmatrix} 0 & 4 & 3 & 6 \\ 7 & 0 & 9 & 2 \\ 8 & 1 & 0 & 3 \\ 16 & 9 & 8 & 0 \end{pmatrix}.$$

The last (third) power tells us the minimum-cost paths of length at most three (and hence the minimum-cost paths of any length) between any two vertices in the graph. □

(b) (*) Can you use a variant of a matrix product to compute the *maximum* weight paths of any length between any two vertices? How would you need to modify the adjacency matrix of a graph to do this? Does a maximum weight path always exist between any pair of vertices?

Solution. Yes, the appropriate variant would be a "max-plus" product, where the "min" operation is replaced by a "max" operation. Let us assume again that all weights are non-negative. Note that instead of putting "∞", we should put "-∞", and whenever there is a loop from a vertex to itself, we should put the value of that loop rather than 0. We won't always have a max-weight path. If there are cycles in the graph, then going around a cycle always increases the weight, but we can certainly use this method to compute the maximum-weight path of at most a given number of edges between any two vertices: just take the appropriate power under the "max-plus" product. □

(7) Recall that a graph $G = (V, E)$ is called *undirected* when the edge relation E is symmetric. The adjacency matrices of undirected graphs are symmetric, namely equal to their own transpose. An undirected graph is called *bipartite* if the vertex set can be written as a disjoint union $V = V_1 \sqcup V_2$,

such that there are no edges between elements of V_1, and no edges between elements of V_2. In other words, whenever $(a, b) \in E$, we either have $a \in V_1$ and $b \in V_2$, or $a \in V_2$ and $b \in V_1$. Show that if a graph is bipartite, you can order the vertices so that the adjacency matrix has the form

$$\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix},$$

where the numbers "0" represent square matrices of all zeroes, B is some (not necessarily square) matrix.

Solution. Label the vertices of the graph so that all elements of V_1 come before all elements of V_2. Suppose that V_1 has k elements and V_2 has m elements. Note that $k+m$ is the total number of vertices. Then if we take a pair i, j where $1 \le i \le k$ and $1 \le j \le k$, then there is no edge from i to j and no edge from j to i. This says that the top left $k \times k$ block of the adjacency matrix is zero. Similarly, the bottom right $m \times m$ block is zero.

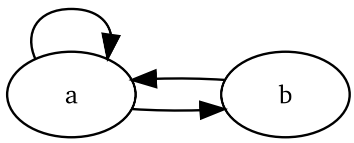
So we know that the matrix has the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

for some B and C. Let us show that $C = B^t$.

Suppose now that $1 \le i \le k$, and $1 \le j \le m$. The (i, j)th entry of B is exactly the $(i, k + j)$th entry of the adjacency matrix. That is the same as the $(k + j, i)$th entry by symmetry. On the other hand, it is easy to check that this is the same as the (j, i)th entry of C. This proves that $C = B^t$. □

(8) (Thanks to Lekh Bhatia for suggesting this exercise!) Consider the following graph.



Calculate for a few values of k the number of length k paths from a to itself. Can you find (and perhaps prove!) a pattern?

Solution. The adjacency matrix is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and taking powers gives the Fibonacci sequence in the top-left spot. (Recall that the Fibonacci sequence starts with 1, 1, such that the next number is always the sum of the previous two). I won't write out the proof here, but happy to discuss if you are interested! □