

ETH Zurich

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# Advanced Graph Algorithms and Optimization

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These notes will be updated throughout the course. They are likely to contain typos, and they may have mistakes or lack clarity in places. Feedback and comments are welcome. Please send to [kyng@inf.ethz.ch](mailto:kyng@inf.ethz.ch) or, even better, submit a pull request at

[https://github.com/rjkyng/agao24\\_script](https://github.com/rjkyng/agao24_script).

We want to thank scribes from the 2020 edition of the course who contributed to these notes: Hongjie Chen, Meher Chaitanya, Timon Knigge, and Tim Taubner – and we're grateful to all the readers who've submitted corrections, including Martin Kucera, Alejandro Cassis, and Luke Volpatti.

*A important note:* If you're a student browsing these notes to decide whether to take this course, please note that the current notes are incomplete. We will release parts later in the semester. You can take a look at last year's notes for an impression of the what the rest of the course will look like. Find them here:

[https://github.com/rjkyng/agao23\\_script/raw/main/agao23\\_script.pdf](https://github.com/rjkyng/agao23_script/raw/main/agao23_script.pdf)

There will, however, be some changes to the content compared to last year.

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# Chapter 1

## Course Introduction

### 1.1 Overview

This course will take us quite deep into modern approaches to graph algorithms using convex optimization techniques. By studying convex optimization through the lens of graph algorithms, we'll try to develop an understanding of fundamental phenomena in optimization. Much of our time will be devoted to flow problems on graphs. We will not only be studying these problems for their own sake, but also because they often provide a useful setting for thinking more broadly about optimization.

The course will cover some traditional discrete approaches to various graph problems, especially flow problems, and then contrast these approaches with modern, asymptotically faster methods based on combining convex optimization with spectral and combinatorial graph theory.

### 1.2 Electrical Flows and Voltages - a Graph Problem from Middle School?

We will dive right into graph problems by considering how electrical current moves through a network of resistors.

First, let us recall some middle school physics. If some of these things don't make sense to you, don't worry, in less than a paragraph from here, we'll be back to safely doing math.

Recall that a typical battery that one buys from Migros has two endpoints, and produces what is called a *voltage difference* between these endpoints.

One end of the battery will have a positive charge (I think that means an excess of positrons<sup>1</sup>), and the other a negative charge. If we connect the two endpoints with a wire, then a current will flow from one end of the battery to the other in an attempt to even out this imbalance of charge.

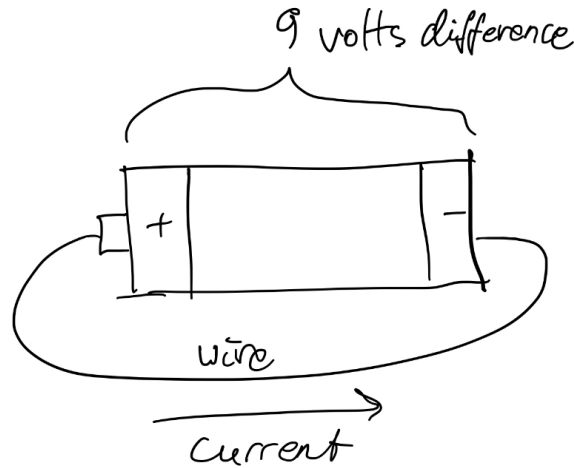


Figure 1.1: A 9 volts battery with a wire attached.

We can also imagine a kind of battery that tries to send a certain amount of current through the wires between its endpoints, e.g. 1 unit of charge per unit of time. This will be a little more convenient to work with, so let us focus on that case.

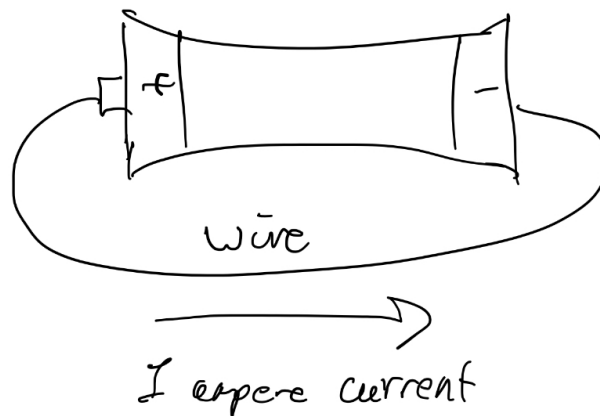


Figure 1.2: A 1 ampere battery with a wire attached.

A *resistor* is a piece of wire that connects two points  $u$  and  $v$ , and is completely described by a single number  $r$  called its *resistance*.

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<sup>1</sup>I'm joking, of course! Try Wikipedia if you want to know more. However, you will not need it for this class.



If the voltage difference between the endpoints of the resistor is  $x$ , and the resistance is  $r$  then this will create a flow of charge per unit of time of  $f = x/r$ . This is called Ohm's Law.

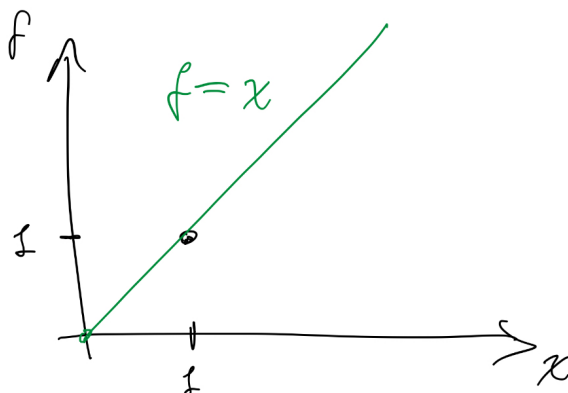


Figure 1.3: Ohm's Law for a resistor with resistance  $r = 1$ .

Suppose we set up a bunch of wires that route electricity from our current source  $s$  to our current sink  $t$  in some pattern:

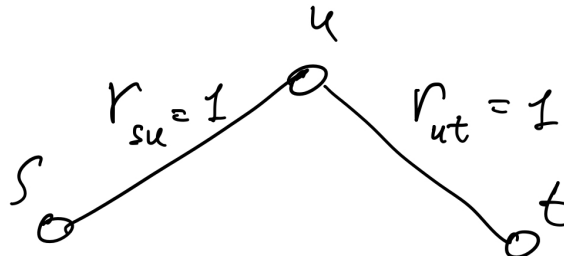


Figure 1.4: A path of two resistors.

We have one unit of charge flowing out of  $s$  per unit of time, and one unit coming into  $t$ . Because charge is conserved, the current flowing into any other point  $u$  must equal the amount flowing out of it. This is called Kirchhoff's Current Law.

To send one unit of current from  $s$  to  $t$ , we must be sending it first from  $s$  to  $u$  and then from  $u$  to  $t$ . So the current on edge  $(s, u)$  is 1 and the current on  $(u, t)$  is 1. By Ohm's Law, the voltage difference must also be 1 across each of the two wires. Thus, if the voltage is  $x$  at  $s$ , it must be  $x + 1$  at  $u$  and  $x + 2$  at  $t$ . What is  $x$ ? It turns out it doesn't matter: We only care about the differences. So let us set  $x = 0$ .

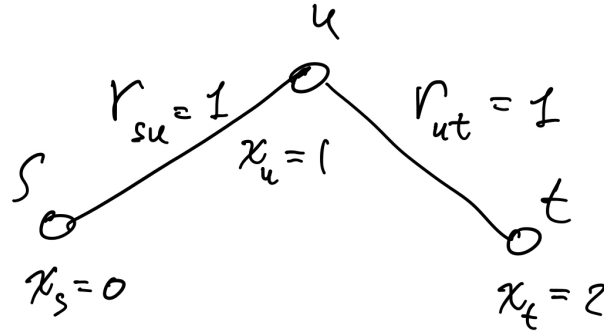


Figure 1.5: A path of two resistors.

Let us try one more example:

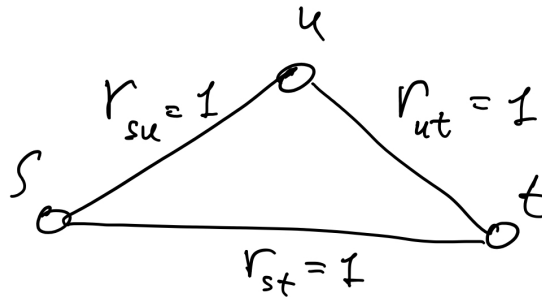


Figure 1.6: A network with three resistors.

How much flow will go directly from  $s$  to  $t$  and how much via  $u$ ?

Well, we know what the net current flowing into and out of each vertex must be, and we can use that to set up some equations. Let us say the voltage at  $s$  is  $x_s$ , at  $u$  is  $x_u$  and at  $t$  is  $x_t$ .

- Net current at  $s$ :  $-1 = (x_s - x_t) + (x_s - x_u)$
- Net current at  $u$ :  $0 = (x_u - x_s) + (x_u - x_t)$
- Net current at  $t$ :  $1 = (x_t - x_s) + (x_t - x_u)$

The following is a solution:  $x_s = 0$ ,  $x_u = \frac{1}{3}$ ,  $x_t = \frac{2}{3}$ . And as before, we can shift all the voltages by some constant  $x$  and get another solution  $x_s = x + 0$ ,  $x_u = x + \frac{1}{3}$ ,  $x_t = x + \frac{2}{3}$ . You might want to convince yourself that these are the only solutions.

**Electrical flows in general graphs.** Do we know enough to calculate the electrical flow in some other network of resistors? To answer this, let us think about the network as a graph. Consider an undirected graph  $G = (V, E)$  with  $|V| = n$  vertices and  $|E| = m$  edges, and let us assume  $G$  is connected. Let's associate a resistance  $r(e) > 0$  with every edge  $e \in E$ .

To keep track of the direction of the flow on each edge, it will be useful to assign an arbitrary direction to every edge. So let's do that, but remember that this is just a bookkeeping tool that helps us track where flow is going.

A *flow* in the graph is a vector  $\mathbf{f} : \mathbb{R}^E$ . The *net flow* of  $\mathbf{f}$  at a vertex  $u \in V$  is defined as  $\sum_{v \rightarrow u} \mathbf{f}(v, u) - \sum_{u \rightarrow v} \mathbf{f}(u, v)$ .

We say a flow routes the demands  $\mathbf{d} \in \mathbb{R}^V$  if the net flow at every vertex  $v$  is  $\mathbf{d}(v)$ .

We can assign a voltage to every vertex  $\mathbf{x} \in \mathbb{R}^V$ . Ohm's Law says that the electrical flow induced by these voltages will be  $\mathbf{f}(u, v) = \frac{1}{r(u, v)}(\mathbf{x}(v) - \mathbf{x}(u))$ .

Say we want to route one unit of current from vertex  $s \in V$  to vertex  $t \in V$ . As before, we can write an equation for every vertex saying that the voltage differences must produce the desired net current:

- Net current at  $s$ :  $-1 = \sum_{(s, v)} \frac{1}{r(s, v)}(\mathbf{x}(v) - \mathbf{x}(s))$
- Net current at  $u \in V \setminus \{s, t\}$ :  $0 = \sum_{(u, v)} \frac{1}{r(u, v)}(\mathbf{x}(v) - \mathbf{x}(u))$
- Net current at  $t$ :  $1 = \sum_{(t, v)} \frac{1}{r(t, v)}(\mathbf{x}(v) - \mathbf{x}(t))$

This gives us  $n$  constraints, exactly as many as we have voltage variables. However we have to be a little careful when trying to conclude that a solution exists, yielding voltages  $\mathbf{x}$  that gives induce an electrical flow routing the desired demand.

You will prove in the exercises (Week 1, Exercise 3) that a solution  $\mathbf{x}$  exists. The proof requires two important observations: Firstly that the graph is connected, and secondly that summed over all vertices, the net demand is zero, i.e. as much flow is coming into the network as is leaving it.

**The incidence matrix and the Laplacian matrix.** To have a more compact notation for net flow constraints, we also introduce the *edge-vertex incidence matrix* of the graph,  $\mathbf{B} \in \mathbb{R}^{V \times E}$ .

$$\mathbf{B}(v, e) = \begin{cases} 1 & \text{if } e = (u, v) \\ -1 & \text{if } e = (v, u) \\ 0 & \text{o.w.} \end{cases}$$

Now we can express the net flow constraint that  $\mathbf{f}$  routes  $\mathbf{d}$  by

$$\mathbf{B}\mathbf{f} = \mathbf{d}.$$

This is also called a conservation constraint. In our examples so far, we have  $\mathbf{d}(s) = -1$ ,  $\mathbf{d}(t) = 1$  and  $\mathbf{d}(u) = 0$  for all  $u \in V \setminus \{s, t\}$ .

If we let  $\mathbf{R} = \text{diag}_{e \in E} r(e)$  then Ohm's law tells us that  $\mathbf{f} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{x}$ . Putting these observations together, we have  $\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{x} = \mathbf{d}$ . The voltages  $\mathbf{x}$  that induce  $\mathbf{f}$  must solve this system of linear equations, and we can use that to compute both  $\mathbf{x}$  and  $\mathbf{f}$ . It is exactly

the same linear equation as the one we considered earlier. We can show that for a connected graph, a solution  $\mathbf{x}$  exists if and only if the flow into the graph equals the net flow out, which we can express as  $\sum_v \mathbf{d}(v) = 0$  or  $\mathbf{1}^\top \mathbf{d} = 0$ . You will show this as part of Exercise 3. This also implies that an electrical flow routing  $\mathbf{d}$  exists if and only if the net flow into the graph equals the net flow out, which we can express as  $\mathbf{1}^\top \mathbf{d} = 0$ .

The matrix  $\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top$  is called the *Laplacian* of the graph and is usually denoted by  $\mathbf{L}$ .

**An optimization problem in disguise.** So far, we have looked at electrical voltages and flows as arising from a set of linear equations – and it might not be apparent that this has anything to do with optimization. But transporting current through a resistor requires energy, which will be dissipated as heat by the resistor (i.e. it will get hot!). If we send a current of  $f$  across a resistor with a potential drop of  $x$ , then the amount of energy spent per unit of time by the resistor will be  $f \cdot x$ . This is called Joule’s Law. Applying Ohm’s law to a resistor with resistance  $r$ , we can also express this energy per unit of time as  $f \cdot x = x^2/r = r \cdot f^2$ . Since we aren’t bothering with units, we will even forget about time, and refer to these quantities as “energy”, even though a physicist would call them “power”.

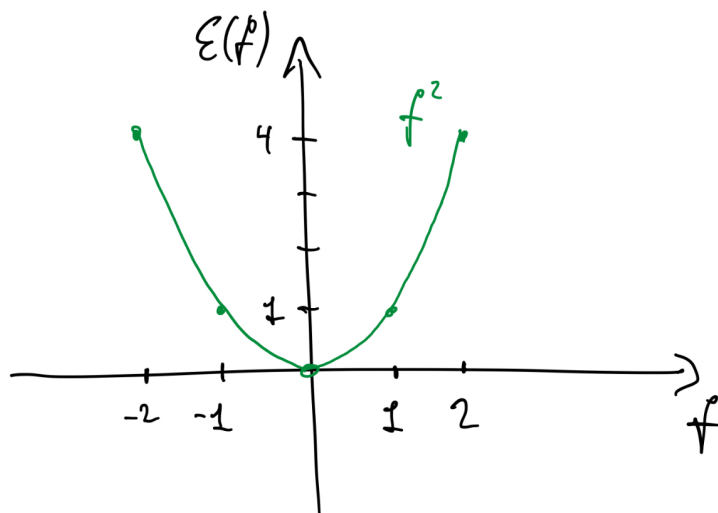


Figure 1.7: Energy as a function of flow in a resistor with resistance  $r = 1$ .

Now, another interesting question would seem to be: If we want to find a flow routing a certain demand  $\mathbf{d}$ , how should the flow behave in order to minimize the electrical energy spent routing the flow? The electrical energy of a flow vector  $\mathbf{f}$  is  $\mathcal{E}(\mathbf{f}) \stackrel{\text{def}}{=} \sum_e \mathbf{r}(e) \mathbf{f}(e)^2$ . We can phrase this as an optimization problem:

$$\begin{aligned} \min_{\mathbf{f} \in \mathbb{R}^E} \quad & \mathcal{E}(\mathbf{f}) \\ \text{s.t.} \quad & \mathbf{B}\mathbf{f} = \mathbf{d}. \end{aligned}$$

We call this problem *electrical energy-minimizing flow*. As we will prove later, the flow  $\mathbf{f}^*$  that minimizes the electrical energy among all flows that satisfy  $\mathbf{B}\mathbf{f} = \mathbf{d}$  is precisely the electrical flow.

**A pair of problems.** What about our voltages, can we also get them from some optimization problem? Well, we can work backwards from the fact that our voltages solve the equation  $\mathbf{L}\mathbf{x} = \mathbf{d}$ . Consider the function  $c(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{L}\mathbf{x} - \mathbf{x}^\top \mathbf{d}$ . We should ask ourselves some questions about this function  $c : \mathbb{R}^V \rightarrow \mathbb{R}$ . Is it continuous and continuously differentiable? The answer to this is yes, and that is not hard to see. Does the function have a minimum? This is maybe not immediately clear, but the minimum does indeed exist.

When this is minimized, the derivative of  $c(\mathbf{x})$  with respect to each coordinate of  $\mathbf{x}$  must be zero. This condition yields exactly the system of linear equations  $\mathbf{L}\mathbf{x} = \mathbf{d}$ . You will confirm this in Exercise 4 of the first exercise sheet.

Based on our derivative condition for the optimum, we can also express the electrical voltages as the solution to an optimization problem, namely

$$\min_{\mathbf{x} \in \mathbb{R}^V} c(\mathbf{x})$$

As you are probably aware, having the derivative of each coordinate equal zero is not a sufficient condition for being at the optimum of a function<sup>2</sup>.

It is also interesting to know whether *all* solutions to  $\mathbf{L}\mathbf{x} = \mathbf{d}$  are in fact minimizers of  $c$ . The answer is yes, and we will see some very general tools for proving statements like this in Chapter 2.

Altogether, we can see that routing electrical current through a network of resistors leads to a *pair* of optimization problems, let's call them  $\mathbf{f}^*$  and  $\mathbf{x}^*$ , and that the solutions to the two problems are related, in our case through the equation  $\mathbf{f}^* = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{x}^*$  (Ohm's Law). But why and how are these two optimization problems related?

Instead of minimizing  $c(\mathbf{x})$ , we can equivalently think about maximizing  $-c(\mathbf{x})$ , which gives the following optimization problem:  $\max_{\mathbf{x} \in \mathbb{R}^V} -c(\mathbf{x})$ . In fact, as you will show in the exercises for Week 1, we have  $\mathcal{E}(\mathbf{f}^*) = -c(\mathbf{x}^*)$ , so the minimum electrical energy is exactly the maximum value of  $-c(\mathbf{x})$ . More generally for *any* flow that routes  $\mathbf{d}$  and *any* voltages  $\mathbf{x}$ , we have  $\mathcal{E}(\mathbf{f}) \geq -c(\mathbf{x})$ . So, for any  $\mathbf{x}$ , the value of  $-c(\mathbf{x})$  is a lower bound on the minimum energy  $\mathcal{E}(\mathbf{f}^*)$ .

This turns out to be an instance of a much broader phenomenon, known as Lagrangian duality, which allows us to learn a lot about many optimization problems by studying two related pairs of problems, a minimization problem, and a related maximization problem that gives lower bounds on the optimal value of the minimization problem.

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<sup>2</sup>Consider the function in one variable  $c(x) = x^3$ .

**Solving  $\mathbf{L}\mathbf{x} = \mathbf{d}$ .** Given a graph  $G$  with resistances for the edges, and some net flow vector  $\mathbf{d}$ , how quickly can we compute  $\mathbf{x}$ ? Broadly speaking, there are two very different families of algorithms we could use to try to solve this problem.

We could solve the linear equation using something like *Gaussian Elimination* to compute an exact solution.

Alternatively, we could start with a guess at a solution, e.g.  $\mathbf{x}_0 = \mathbf{0}$ , and then we could try to make a change to  $\mathbf{x}_0$  to reach a new point  $\mathbf{x}_1$  with a lower value of  $c(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{L}\mathbf{x} - \mathbf{x}^\top \mathbf{d}$ , i.e.  $c(\mathbf{x}_1) < c(\mathbf{x}_0)$ . If we repeat a process like that for enough steps, say  $t$ , hopefully we eventually reach  $\mathbf{x}_t$  with  $c(\mathbf{x}_t)$  close to  $c(\mathbf{x}^*)$ , where  $\mathbf{x}^*$  is a minimizer of  $c(\mathbf{x})$  and hence  $\mathbf{L}\mathbf{x}^* = \mathbf{d}$ . Now, we also need to make sure that  $c(\mathbf{x}_t) \approx c(\mathbf{x}^*)$  implies that  $\mathbf{L}\mathbf{x}_t \approx \mathbf{d}$  in some useful sense.

One of the most basic algorithms in this framework of “guess and adjust” is called *Gradient Descent*, which we will study in Week 2. The rough idea is the following: if we make a very small step from  $\mathbf{x}$  to  $\mathbf{x} + \boldsymbol{\delta}$ , then a multivariate Taylor expansion suggests that  $c(\mathbf{x} + \boldsymbol{\delta}) - c(\mathbf{x}) \approx \sum_{v \in V} \delta(v) \frac{\partial c(\mathbf{x})}{\partial x(v)}$ .

If we are dealing with smooth convex function, this quantity is negative if we let  $\boldsymbol{\delta}(v) = -\epsilon \cdot \frac{\partial c(\mathbf{x})}{\partial x(v)}$  for some small enough  $\epsilon$  so the approximation holds well. So we should be able to make progress by taking a small step in this direction. That’s Gradient Descent! The name comes from the vector of partial derivatives, which is called the gradient.

As we will see later in this course, understanding electrical problems from an optimization perspective is crucial to develop fast algorithms for computing electrical flows and voltages, but to do very well, we also need to borrow some ideas from Gaussian Elimination.

What running times do different approaches get?

1. Using Gaussian Elimination, we can find  $\mathbf{x}$  s.t.  $\mathbf{L}\mathbf{x} = \mathbf{d}$  in  $O(n^3)$  time and with asymptotically faster algorithms based on matrix multiplication, we can bring this down to roughly  $O(n^{2.372})$ .
2. Meanwhile Gradient Descent will get a running time of  $O(n^3 m)$  or so – at least this is what a simple analysis suggests.
3. However, we can do much better: By combining ideas from both algorithms, and a bit more, we can get  $\mathbf{x}$  up to very high accuracy in time  $O(m \log^c n)$  where  $c$  is some small constant.

## 1.3 Convex Optimization

Recall our plot in Figure 1.7 of the energy required to route a flow  $f$  across a resistor with resistance  $r$ , which was  $\mathcal{E}(f) = r \cdot f^2$ . We see that the function has a special structure: the

graph of the function sits below the line joining any two points  $(f, \mathcal{E}(f))$  and  $(g, \mathcal{E}(g))$ . A function  $\mathcal{E} : \mathbb{R} \rightarrow \mathbb{R}$  that has this property is said to be convex.

Figure 1.8 shows the energy as a function of flow, along with two points  $(f, \mathcal{E}(f))$  and  $(g, \mathcal{E}(g))$ . We see the function sits below the line segment between these points.

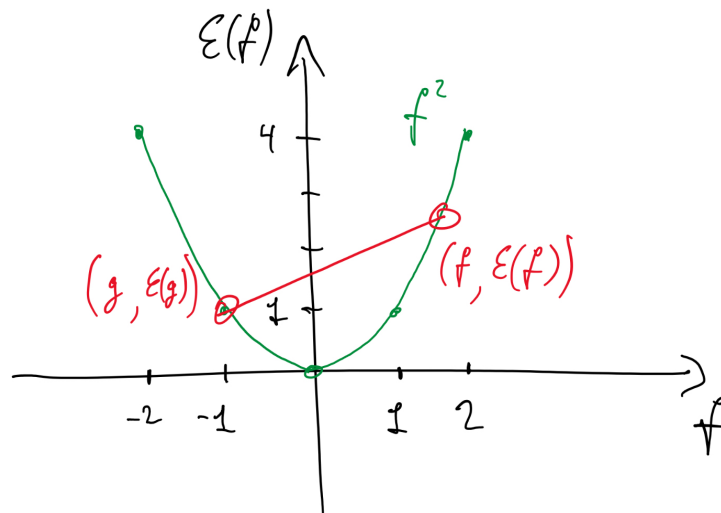


Figure 1.8: Energy as a function of flow in a resistor with resistance  $r = 1$ . The function is convex.

We can also interpret this condition as saying that for all  $\theta \in [0, 1]$

$$\mathcal{E}(\theta f + (1 - \theta)g) \leq \theta \mathcal{E}(f) + (1 - \theta)\mathcal{E}(g).$$

This immediately generalizes to functions  $\mathcal{E} : \mathbb{R}^m \rightarrow \mathbb{R}$ .

A *convex set* is a subset of  $S \subseteq \mathbb{R}^m$  s.t. if  $\mathbf{f}, \mathbf{g} \in S$  then for all  $\theta \in [0, 1]$  we have  $\theta \mathbf{f} + (1 - \theta)\mathbf{g} \in S$ .

Figure 1.9 shows some examples of sets that are and aren't convex.

Convex functions and convex sets are central to optimization, because for most problems of minimization of a convex function over a convex set, we can develop fast algorithms <sup>3</sup>.

So why convex functions and convex sets? One important reason is that for a convex function defined over a convex feasible set, any local minimum is also a global minimum, and this fact makes searching for an optimal solution computationally easier. In fact, this is closely related to why Gradient Descent works well on many convex functions.

Notice that the set  $\{\mathbf{f} : \mathbf{B}\mathbf{f} = \mathbf{d}\}$  is convex, i.e. the set of all flows that route a fixed demand

<sup>3</sup>There are some convex optimization problems that are NP-hard. That said, polynomial time algorithms exist for almost any convex problem you can come up with. The most general polynomial time algorithm for convex optimization is probably the Ellipsoid Method.

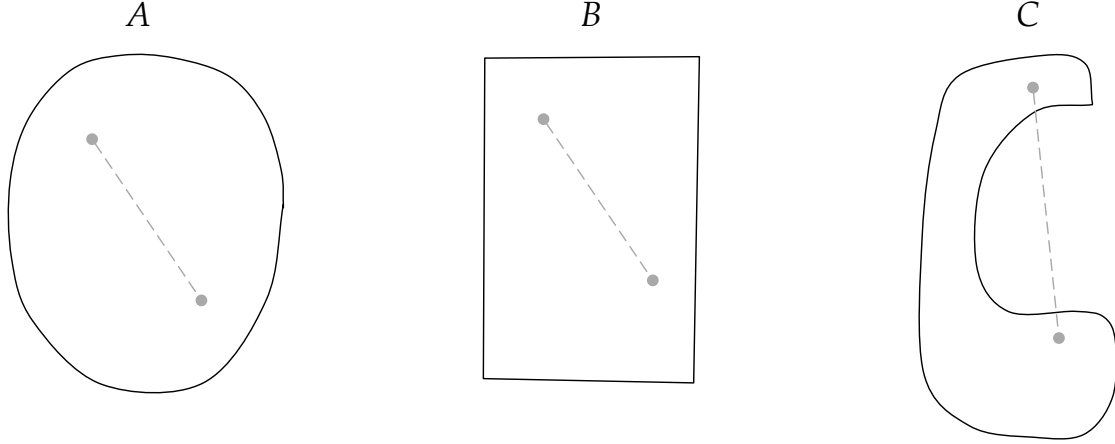


Figure 1.9: A depiction of convex and non-convex sets. The sets  $A$  and  $B$  are convex since the straight line between any two points inside them is also in the set. The set  $C$  is not convex.

$\mathbf{d}$  is convex. It is also easy to verify that  $\mathcal{E}(\mathbf{f}) = \sum_e \mathbf{r}(e) \mathbf{f}(e)^2$  is a convex function, and hence finding an electrical flow is an instance of convex minimization:

## 1.4 More Graph Optimization Problems

**Maximum flow.** Again, let  $G = (V, E)$  be an undirected, connected graph with  $n$  vertices and  $m$  edges. Suppose we want to find a flow  $\mathbf{f} \in \mathbb{R}^E$  that routes  $\mathbf{d}$ , but instead of trying to minimize electrical energy, we try to pick an  $\mathbf{f}$  that minimizes the largest amount of flow on any edge, i.e.  $\max_e |\mathbf{f}_e|$  – which we also denote by  $\|\mathbf{f}\|_\infty$ . We can write this problem as

$$\begin{aligned} \min_{\mathbf{f} \in \mathbb{R}^E} \quad & \|\mathbf{f}\|_\infty \\ \text{s.t.} \quad & \mathbf{B}\mathbf{f} = \mathbf{d} \end{aligned}$$

This problem is known as the Minimum Congested Flow Problem<sup>4</sup>. It is equivalent to the more famous Maximum Flow Problem.

The behavior of this kind of flow is very different than electrical flow. Consider the question of whether a certain demand can be routed  $\|\mathbf{f}\|_\infty \leq 1$ . Imagine sending goods from a source  $s$  to a destination  $t$  using a network of train lines that all have the same capacity and asking whether the network is able to route the goods at the rate you want: This boils down to whether routing exists with  $\|\mathbf{f}\|_\infty \leq 1$ , if we set it up right.

We have a very fast, convex optimization-based algorithm for Minimum Congested Flow: In  $m\epsilon^{-1} \log^{O(1)} n$  time, we can find a flow  $\tilde{\mathbf{f}}$  s.t.  $\mathbf{B}\tilde{\mathbf{f}} = \mathbf{d}$  and  $\|\tilde{\mathbf{f}}\|_\infty \leq (1 + \epsilon) \|\mathbf{f}^*\|_\infty$ , where  $\mathbf{f}^*$

<sup>4</sup>This version is called undirected, because the graph is undirected, and *uncapacitated* because we are aiming for the same bound on the flow on all edges.



is an optimal solution, i.e. an actual minimum congestion flow routing  $\mathbf{d}$ .

But what if we want  $\epsilon$  to be very small, e.g.  $1/m$ ? Then this running time isn't so good anymore. But, in this case, we can use other algorithms that find flow  $\mathbf{f}^*$  *exactly*. Unfortunately, these algorithms take time roughly  $m^{4/3+o(1)}$ .

Just as the electrical flow problem had a dual voltage problem, so maximum flow has a dual voltage problem, which is known as the  $s$ - $t$  minimum cut problem.

**Maximum flow, with directions and capacities.** We can make the maximum flow problem harder by introducing directed edges: To do so, we allow edges to exist in both directions between vertices, and we require that the flow on a directed edge is always non-negative. So now  $G = (V, E)$  is a directed graph. We can also make the problem harder by introducing capacities. We define a capacity vector  $\mathbf{c} \in \mathbb{R}^E \geq \mathbf{0}$  and try to minimize  $\|\mathbf{C}^{-1}\mathbf{f}\|_\infty$ , where  $\mathbf{C} = \text{diag}_{e \in E} \mathbf{c}(e)$ . Then our problem becomes

$$\begin{aligned} \min_{\mathbf{f} \in \mathbb{R}^E} & \quad \|\mathbf{C}^{-1}\mathbf{f}\|_\infty \\ \text{s.t.} & \quad \mathbf{B}\mathbf{f} = \mathbf{d} \\ & \quad \mathbf{f} \geq \mathbf{0}. \end{aligned}$$

For this capacitated, directed maximum flow problem, our best algorithms run in about  $O(m\sqrt{n})$  time in sparse graphs and  $O(m^{1.483})$  in dense graphs<sup>5</sup>, even if we are willing to accept fairly low accuracy solution. If the capacities are allowed to be exponentially large, the best running time we can get is  $O(mn)$ . For this problem, we do not yet know how to improve over classical combinatorial algorithms using convex optimization.

**Multi-commodity flow.** We can make the problem even harder still, by simultaneously trying to route two types of flow (imagine pipes with Coke and Pepsi). Our problem now looks like

$$\begin{aligned} \min_{\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^E} & \quad \|\mathbf{C}^{-1}(\mathbf{f}_1 + \mathbf{f}_2)\|_\infty \\ \text{s.t.} & \quad \mathbf{B}\mathbf{f}_1 = \mathbf{d}_1 \\ & \quad \mathbf{B}\mathbf{f}_2 = \mathbf{d}_2 \\ & \quad \mathbf{f}_1, \mathbf{f}_2 \geq \mathbf{0}. \end{aligned}$$

Solving this problem to high accuracy is essentially as hard as solving a general linear program! We should see later in the course how to make this statement precise.

If we in the above problem additionally require that our flows must be integer valued, i.e.  $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{N}_0$ , then the problem becomes NP-complete.

---

<sup>5</sup>Provided the capacities are integers satisfying a condition like  $\mathbf{c} \leq n^{100}\mathbf{1}$ .

**Random walks in a graph.** Google famously uses<sup>6</sup> the PageRank problem to help decide how to rank their search results. This problem essentially boils down to computing the *stable distribution* of a random walk on a graph. Suppose  $G = (V, E)$  is a directed graph where each outgoing edge  $(v, u)$ , which we will define as going from  $u$  to  $v$ , has a transition probability  $p_{(v,u)} > 0$  s.t.  $\sum_{z \leftarrow u} p_{(z,u)} = 1$ . We can take a step of a random walk on the vertex set by starting at some vertex  $u_0 = u$ , and then randomly pick one of the outgoing edges  $(v, u)$  with probability  $p_{(v,u)}$  and move to the chosen vertex  $u_1 = v$ . Repeating this procedure, to take a step from the next vertex  $u_1$ , gives us a *random walk* in the graph, a sequence of vertices  $u_0, u_1, u_2 \dots, u_k$ .

We let  $\mathbf{P} \in \mathbb{R}^{V \times V}$  be the matrix of transition probabilities given by

$$\mathbf{P}_{vu} = \begin{cases} p_{(v,u)} & \text{for } (u, v) \in E \\ 0 & \text{o.w.} \end{cases}$$

Any probability distribution over the vertices can be specified by a vector  $\mathbf{p} \in \mathbb{R}^V$  where  $\mathbf{p} \geq \mathbf{0}$  and  $\sum_v \mathbf{p}(v) = 1$ . We say that probability distribution  $\boldsymbol{\pi}$  on the vertices is a *stable distribution* of the random walk if  $\boldsymbol{\pi} = \mathbf{P}\boldsymbol{\pi}$ . A strongly connected graph always has exactly one stable distribution.

How quickly can we compute the stable distribution of a general random walk? Under some mild conditions on the stable distribution<sup>7</sup>, we can find a high accuracy approximation of  $\boldsymbol{\pi}$  in time  $O(m \log^c n)$  for some constant  $c$ .

This problem does not easily fit in a framework of convex optimization, but nonetheless, our fastest algorithms for it use ideas from convex optimization.

## Topics in this Course

In this course, we will try to address the following questions.

1. What are the fundamental tools of fast convex optimization?
2. What are some problems we can solve quickly on graphs using optimization?
3. What can graphs teach us about convex optimization?
4. What algorithm design techniques are good for getting algorithms that quickly find a crude approximate solution? And what techniques are best when we need to get a highly accurate answer?
5. What is special about flow problems?

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<sup>6</sup>At least they did at some point.

<sup>7</sup>Roughly something like  $\max_v 1/\boldsymbol{\pi}(v) \leq n^{100}$ .

# Part I

## Introduction to Convex Optimization

# Chapter 2

## Some Basic Optimization, Convex Geometry, and Linear Algebra

### 2.1 Overview

In this chapter, we will

1. Start with an overview (i.e. this list).
2. Learn some basic terminology and facts about optimization.
3. Recall our definition of convex functions and see how convex functions can also be understood in terms of a characterization based on first derivatives.
4. See how the first derivatives of a convex function can certify that we are at a global minimum.

### 2.2 Optimization Problems

Focusing for now on optimization over  $\mathbf{x} \in \mathbb{R}^n$ , we usually write optimization problems as:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & (\text{or } \max) \quad f(\mathbf{x}) \\ \text{s.t.} \quad & g_1(\mathbf{x}) \leq b_1 \\ & \cdot \\ & \cdot \\ & \cdot \\ & g_m(\mathbf{x}) \leq b_m \end{aligned}$$

where  $\{g_i(\mathbf{x})\}_{i=1}^m$  encode the constraints. For example, in the following optimization problem from the previous chapter

$$\begin{aligned} \min_{\mathbf{f} \in \mathbb{R}^E} \quad & \sum_e r(e) \mathbf{f}(e)^2 \\ \text{s.t.} \quad & \mathbf{B}\mathbf{f} = \mathbf{d} \end{aligned}$$

we have the constraint  $\mathbf{B}\mathbf{f} = \mathbf{d}$ . Notice that we can rewrite this constraint as  $\mathbf{B}\mathbf{f} \leq \mathbf{d}$  and  $-\mathbf{B}\mathbf{f} \leq -\mathbf{d}$  to match the above setting. The set of points which respect the constraints is called the *feasible set*.

**Definition 2.2.1.** For a given optimization problem the set  $\mathcal{F} = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i, \forall i \in [m]\}$  is called the **feasible set**. A point  $\mathbf{x} \in \mathcal{F}$  is called a **feasible point**, and a point  $\mathbf{x}' \notin \mathcal{F}$  is called an **infeasible point**.

Ideally, we would like to find optimal solutions for the optimization problems we consider. Let's define what we mean exactly.

**Definition 2.2.2.** For a *maximization* problem  $\mathbf{x}^*$  is called an **optimal solution** if  $f(\mathbf{x}^*) \geq f(\mathbf{x}), \forall \mathbf{x} \in \mathcal{F}$ . Similarly, for a *minimization* problem  $\mathbf{x}^*$  is an optimal solution if  $f(\mathbf{x}^*) \leq f(\mathbf{x}), \forall \mathbf{x} \in \mathcal{F}$ .

What happens if there are *no feasible points*? In this case, an optimal solution cannot exist, and we say the problem is infeasible.

**Definition 2.2.3.** If  $\mathcal{F} = \emptyset$  we say that the optimization problem is **infeasible**. If  $\mathcal{F} \neq \emptyset$  we say the optimization problem is **feasible**.

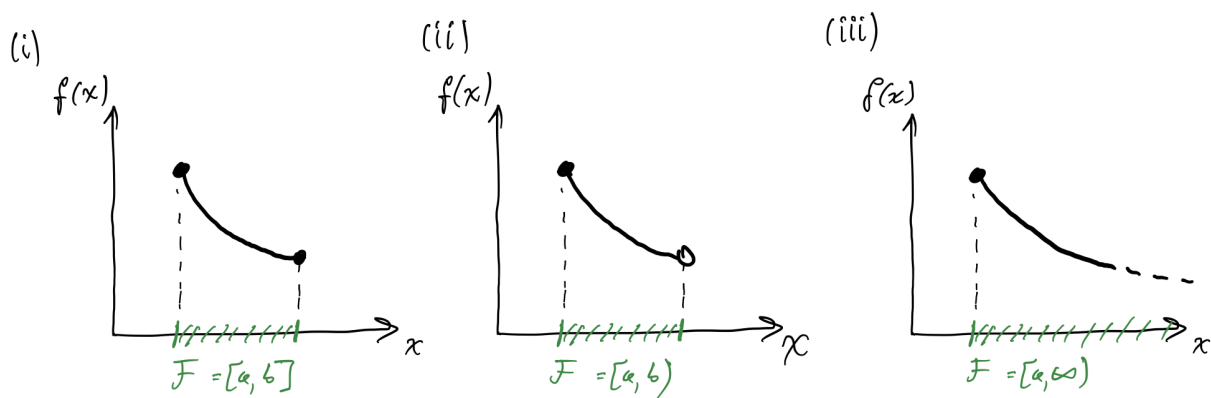


Figure 2.1

Consider three examples depicted in Figure 2.1:

(i)  $\mathcal{F} = [a, b]$

(ii)  $\mathcal{F} = [a, b)$

(iii)  $\mathcal{F} = [a, \infty)$

In the first example, the minimum of the function is attained at  $b$ . In the second case the region is open and therefore there is no minimum function value, since for every point we will choose, there will always be another point with a smaller function value. Lastly, in the third example, the region is unbounded and the function decreasing, thus again there will always be another point with a smaller function value.

**Sufficient Condition for Optimality.** The following theorem, which is a fundamental theorem in real analysis, gives us a sufficient (though not necessary) condition for optimality.

**Theorem** (Extreme Value Theorem). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function and  $\mathcal{F} \subseteq \mathbb{R}^n$  be nonempty, bounded, and closed. Then, the optimization problem  $\min f(\mathbf{x}) : \mathbf{x} \in \mathcal{F}$  has an optimal solution.

## 2.3 A Characterization of Convex Functions

Recall the definitions of convex sets and convex functions that we introduced in Chapter 1:

**Definition 2.3.1.** A set  $S \subseteq \mathbb{R}^n$  is called a **convex set** if any two points in  $S$  contain their line, i.e. for any  $\mathbf{x}, \mathbf{y} \in S$  we have that  $\theta\mathbf{x} + (1 - \theta)\mathbf{y} \in S$  for any  $\theta \in [0, 1]$ .

**Definition 2.3.2.** For a convex set  $S \subseteq \mathbb{R}^n$ , we say that a function  $f : S \rightarrow \mathbb{R}$  is **convex on  $S$**  if for any two points  $\mathbf{x}, \mathbf{y} \in S$  and any  $\theta \in [0, 1]$  we have that:

$$f(\theta\mathbf{x} + (1 - \theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta)f(\mathbf{y}).$$

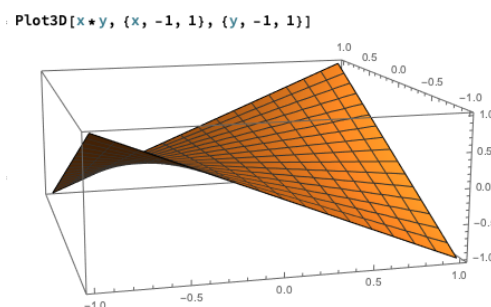


Figure 2.2: This plot shows the function  $f(x, y) = xy$ . For any fixed  $y_0$ , the function  $h(x) = f(x, y_0) = xy_0$  is linear in  $x$ , and so is a convex function in  $x$ . But is  $f$  convex?

We will first give an important characterization of convex function. To do so, we need to characterize multivariate functions via their Taylor expansion.

**Notation for this section.** In the rest of this section, we frequently consider a multivariate function  $f$  whose domain is a set  $S \subseteq \mathbb{R}^n$ , which we will require to be open. When we additionally require that  $S$  is convex, we will specify this. Note that  $S = \mathbb{R}^n$  is both open and convex and it suffices to keep this case in mind. Things sometimes get more complicated if  $S$  is not open, e.g. when the domain of  $f$  has a boundary. We will leave those complications for another time.

### 2.3.1 First-order Taylor Approximation

**Definition 2.3.3.** The **gradient** of a function  $f : S \rightarrow \mathbb{R}$  at point  $\mathbf{x} \in S$  is denoted  $\nabla f(\mathbf{x})$  and is defined as

$$\nabla f(\mathbf{x}) = \left[ \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}(1)}, \dots, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}(n)} \right]^\top$$

**First-order Taylor expansion.** For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  of a single variable, differentiable at  $x \in \mathbb{R}$

$$f(x + \delta) = f(x) + f'(x)\delta + o(|\delta|)$$

where by definition:

$$\lim_{\delta \rightarrow 0} \frac{o(|\delta|)}{|\delta|} = 0.$$

Similarly, a multivariate function  $f : S \rightarrow \mathbb{R}$  is said to be (*Fréchet*) *differentiable* at  $\mathbf{x} \in S$  when there exists  $\nabla f(\mathbf{x}) \in \mathbb{R}^n$  s.t.

$$\lim_{\delta \rightarrow 0} \frac{\|f(\mathbf{x} + \delta) - f(\mathbf{x}) - \nabla f(\mathbf{x})^\top \delta\|_2}{\|\delta\|_2} = 0.$$

Note that this is equivalent to saying that  $f(\mathbf{x} + \delta) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top \delta + o(\|\delta\|_2)$ .

We say that  $f$  is *continuously differentiable* on a set  $S \subseteq \mathbb{R}^n$  if it is differentiable and in addition the gradient is continuous on  $S$ . A differentiable convex function whose domain is an open convex set  $S \subseteq \mathbb{R}^n$  is always continuously differentiable<sup>1</sup>.

**Remark.** In this course, we will generally err on the side of being informal about functional analysis when we can afford to, and we will not worry too much about the details of different notions of differentiability (e.g. Fréchet and Gateaux differentiability), except when it turns out to be important.

**Theorem 2.3.4** (Taylor's Theorem, multivariate first-order remainder form). *If  $f : S \rightarrow \mathbb{R}$  is continuously differentiable over  $[\mathbf{x}, \mathbf{y}]$ , then for some  $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$ ,*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{z})^\top (\mathbf{y} - \mathbf{x}).$$

---

<sup>1</sup>See p. 248, Corollary 25.5.1 in *Convex Analysis* by Rockafellar (my version is the Second print, 1972). Rockafellar's corollary concerns finite convex functions, because he otherwise allows convex functions that may take on the values  $\pm\infty$ .

This theorem is useful for showing that the function  $f$  can be approximated by the affine function  $\mathbf{y} \rightarrow f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$  when  $\mathbf{y}$  is “close to”  $\mathbf{x}$  in some sense.

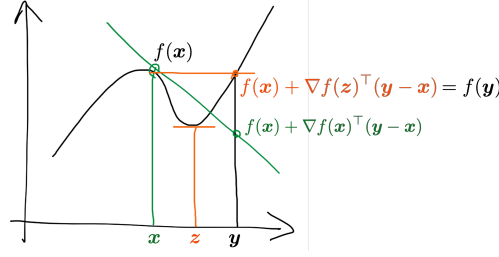


Figure 2.3: The function  $f(\mathbf{y})$  equals its linear approximation based at  $\mathbf{x}$ , with gradient coming from an intermediate point  $\mathbf{z}$ , i.e.  $f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{z})^\top (\mathbf{y} - \mathbf{x})$ .

### 2.3.2 Directional Derivatives

**Definition 2.3.5.** Let  $f : S \rightarrow \mathbb{R}$  be a function differentiable at  $\mathbf{x} \in S$  and let us consider  $\mathbf{d} \in \mathbb{R}^n$ . We define the **derivative of  $f$  at  $\mathbf{x}$  in direction  $\mathbf{d}$**  as:

$$Df(\mathbf{x})[\mathbf{d}] = \lim_{\lambda \rightarrow 0} \frac{f(\mathbf{x} + \lambda \mathbf{d}) - f(\mathbf{x})}{\lambda}$$

**Proposition 2.3.6.**  $Df(\mathbf{x})[\mathbf{d}] = \nabla f(\mathbf{x})^\top \mathbf{d}$ .

*Proof.* Using the first order expansion of  $f$  at  $\mathbf{x}$ :

$$f(\mathbf{x} + \lambda \mathbf{d}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\lambda \mathbf{d}) + o(\|\lambda \mathbf{d}\|_2)$$

hence, dividing by  $\lambda$  (and noticing that  $\|\lambda \mathbf{d}\|_2 = \lambda \|\mathbf{d}\|_2$ ):

$$\frac{f(\mathbf{x} + \lambda \mathbf{d}) - f(\mathbf{x})}{\lambda} = \nabla f(\mathbf{x})^\top \mathbf{d} + \frac{o(\lambda \|\mathbf{d}\|_2)}{\lambda}$$

letting  $\lambda$  go to 0 concludes the proof.  $\square$

### 2.3.3 Lower Bounding Convex Functions with Affine Functions

In order to prove the characterization of convex functions in the next section we will need the following lemma. This lemma says that any differentiable convex function can be lower bounded by an affine function.

**Theorem 2.3.7.** Let  $S$  be an open convex subset of  $\mathbb{R}^n$ , and let  $f : S \rightarrow \mathbb{R}$  be a differentiable function. Then,  $f$  is convex if and only if for any  $\mathbf{x}, \mathbf{y} \in S$  we have that  $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$ .



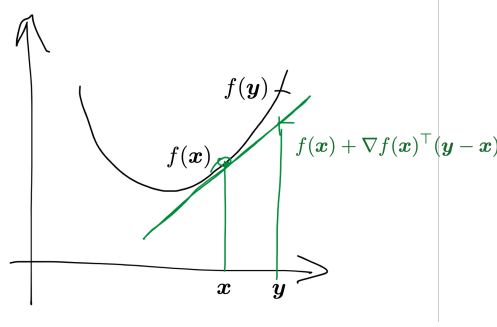


Figure 2.4: The convex function  $f(\mathbf{y})$  sits above the linear function in  $\mathbf{y}$  given by  $f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x})$ .

*Proof.* [  $\implies$  ] Assume  $f$  is convex, then for all  $\mathbf{x}, \mathbf{y} \in S$  and  $\theta \in [0, 1]$ , if we let  $\mathbf{z} = \theta\mathbf{y} + (1 - \theta)\mathbf{x}$ , we have that

$$f(\mathbf{z}) = f((1 - \theta)\mathbf{x} + \theta\mathbf{y}) \leq (1 - \theta)f(\mathbf{x}) + \theta f(\mathbf{y})$$

and therefore by subtracting  $f(\mathbf{x})$  from both sides we get:

$$\begin{aligned} f(\mathbf{x} + \theta(\mathbf{y} - \mathbf{x})) - f(\mathbf{x}) &\leq \theta f(\mathbf{y}) + (1 - \theta)f(\mathbf{x}) - f(\mathbf{x}) \\ &= \theta f(\mathbf{y}) - \theta f(\mathbf{x}). \end{aligned}$$

Thus we get that (for  $\theta > 0$ ):

$$\frac{f(\mathbf{x} + \theta(\mathbf{y} - \mathbf{x})) - f(\mathbf{x})}{\theta} \leq f(\mathbf{y}) - f(\mathbf{x})$$

Applying Proposition 2.3.6 with  $\mathbf{d} = \mathbf{x} - \mathbf{y}$  we have that:

$$\nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x}) = \lim_{\theta \rightarrow 0^+} \frac{f(\mathbf{x} + \theta(\mathbf{y} - \mathbf{x})) - f(\mathbf{x})}{\theta} \leq f(\mathbf{y}) - f(\mathbf{x}).$$

[  $\Leftarrow$  ] Assume that  $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x})$  for all  $\mathbf{x}, \mathbf{y} \in S$  and show that  $f$  is convex. Let  $\mathbf{x}, \mathbf{y} \in S$  and  $\mathbf{z} = \theta\mathbf{y} + (1 - \theta)\mathbf{x}$ . By our assumption we have that:

$$f(\mathbf{y}) \geq f(\mathbf{z}) + \nabla f(\mathbf{z})^\top(\mathbf{y} - \mathbf{z}) \quad (2.1)$$

$$f(\mathbf{x}) \geq f(\mathbf{z}) + \nabla f(\mathbf{z})^\top(\mathbf{x} - \mathbf{z}) \quad (2.2)$$

Observe that  $\mathbf{y} - \mathbf{z} = (1 - \theta)(\mathbf{y} - \mathbf{x})$  and  $\mathbf{x} - \mathbf{z} = \theta(\mathbf{x} - \mathbf{y})$ . Thus adding  $\theta$  times (2.1) to  $(1 - \theta)$  times (2.2) gives cancellation of the vectors multiplying the gradient, yielding

$$\begin{aligned} \theta f(\mathbf{y}) + (1 - \theta)f(\mathbf{x}) &\geq f(\mathbf{z}) + \nabla f(\mathbf{z})^\top \mathbf{0} \\ &= f(\theta\mathbf{y} + (1 - \theta)\mathbf{x}) \end{aligned}$$

This is exactly the definition of convexity. □

## 2.4 Conditions for Optimality

We now want to find necessary and sufficient conditions for local optimality.

**Definition 2.4.1.** Consider a differentiable function  $f : S \rightarrow \mathbb{R}$ . A point  $\mathbf{x} \in S$  at which  $\nabla f(\mathbf{x}) = \mathbf{0}$  is called a **stationary point**.

**Proposition 2.4.2.** *If  $\mathbf{x}$  is a local extremum of a differentiable function  $f : S \rightarrow \mathbb{R}$  then  $\nabla f(\mathbf{x}) = \mathbf{0}$ .*

*Proof.* Let us assume that  $\mathbf{x}$  is a local minimum for  $f$ . Then for all  $\mathbf{d} \in \mathbb{R}^n$ ,  $f(\mathbf{x}) \leq f(\mathbf{x} + \lambda \mathbf{d})$  for  $\lambda$  small enough. Hence:

$$0 \leq f(\mathbf{x} + \lambda \mathbf{d}) - f(\mathbf{x}) = \lambda \nabla f(\mathbf{x})^\top \mathbf{d} + o(\|\lambda \mathbf{d}\|)$$

dividing by  $\lambda > 0$  and letting  $\lambda \rightarrow 0^+$ , we obtain  $0 \leq \nabla f(\mathbf{x})^\top \mathbf{d}$ . But, taking  $\mathbf{d} = -\nabla f(\mathbf{x})$ , we get  $0 \leq -\|\nabla f(\mathbf{x})\|_2^2$ . This implies that  $\nabla f(\mathbf{x}) = \mathbf{0}$ .

The case where  $\mathbf{x}$  is a local maximum can be dealt with similarly. □

**Remark 2.4.3.** For this proposition to hold, it is important that  $S$  is open.

For convex functions however it turns out that a stationary point necessarily implies that the function is at its minimum. Together with the proposition above, this says that for a convex function on  $\mathbb{R}^n$  a point is optimal if and only if it is stationary.

**Proposition 2.4.4.** *Let  $S \subseteq \mathbb{R}^n$  be an open convex set and let  $f : S \rightarrow \mathbb{R}$  be a differentiable and convex function. If  $\mathbf{x}$  is a stationary point then  $\mathbf{x}$  is a global minimum.*

*Proof.* From Theorem 2.3.7 we know that for all  $\mathbf{x}, \mathbf{y} \in S$ :  $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$ . Since  $\nabla f(\mathbf{x}) = \mathbf{0}$  this implies that  $f(\mathbf{y}) \geq f(\mathbf{x})$ . As this holds for any  $\mathbf{y} \in S$ ,  $\mathbf{x}$  is a global minimum. □

# Chapter 3

## Convexity, Second Derivatives and Gradient Descent

**Notation for this chapter.** In this chapter, we sometimes consider a multivariate function  $f$  whose domain is a set  $S \subseteq \mathbb{R}^n$ , which we will require to be open. When we additionally require that  $S$  is convex, we will specify this. Note that  $S = \mathbb{R}^n$  is both open and convex and it suffices to keep this case in mind. Things sometimes get more complicated if  $S$  is not open, e.g. when the domain of  $f$  has a boundary. We will leave those complications for another time.

### 3.1 A Review of Linear Algebra

**Semi-definiteness of a matrix.** The following classification of symmetric matrices will be useful.

**Definition 3.1.1.** Let  $\mathbf{A}$  be a symmetric matrix in  $\mathbb{R}^{n \times n}$ . We say that  $\mathbf{A}$  is:

1. *positive definite* iff  $\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$  for all  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ ;
2. *positive semidefinite* iff  $\mathbf{x}^\top \mathbf{A} \mathbf{x} \geq 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ ;
3. If neither  $\mathbf{A}$  nor  $-\mathbf{A}$  is positive semi-definite, we say that  $\mathbf{A}$  is *indefinite*.

**Example: indefinite matrix.** Consider the following matrix  $\mathbf{A}$ :

$$\mathbf{A} := \begin{bmatrix} +4 & -1 \\ -1 & -2 \end{bmatrix}$$

For  $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , we have  $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 4 > 0$ . For  $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  we have  $\mathbf{x}^\top \mathbf{A} \mathbf{x} = -2 < 0$ .  $\mathbf{A}$  is therefore indefinite.

The following theorem gives a useful characterization of (semi)definite matrices.

**Theorem 3.1.2.** *Let  $\mathbf{A}$  be a symmetric matrix in  $\mathbb{R}^{n \times n}$ .*

1.  *$\mathbf{A}$  is positive definite iff all its eigenvalues are positive;*
2.  *$\mathbf{A}$  is positive semidefinite iff all its eigenvalues are non-negative;*

In order to prove this theorem, let us first recall the Spectral Theorem for symmetric matrices.

**Theorem 3.1.3** (The Spectral Theorem for Symmetric Matrices). *For all symmetric  $\mathbf{A} \in \mathbb{R}^{n \times n}$  there exist  $\mathbf{V} \in \mathbb{R}^{n \times n}$  and a diagonal matrix  $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$  s.t.*

1.  $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top$ .
2.  $\mathbf{V}^\top \mathbf{V} = \mathbf{I}$  (the  $n \times n$  identity matrix). I.e. the columns of  $\mathbf{V}$  form an orthonormal basis. Furthermore,  $\mathbf{v}_i$  is an eigenvector of  $\lambda_i(\mathbf{A})$ , the  $i$ th eigenvalue of  $\mathbf{A}$ .
3.  $\Lambda_{ii} = \lambda_i(\mathbf{A})$ .

Using the Spectral Theorem, we can show the following result:

**Theorem 3.1.4** (The Courant-Fischer Theorem). *Let  $\mathbf{A}$  be a symmetric matrix in  $\mathbb{R}^{n \times n}$ , with eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . Then*

1.

$$\lambda_i = \min_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=i}} \max_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

2.

$$\lambda_i = \max_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=n+1-i}} \min_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

Theorem 3.1.2 is an immediate corollary of Theorem 3.1.4, since we can see that the minimum value of the quadratic form  $\mathbf{x}^\top \mathbf{A} \mathbf{x}$  over  $\mathbf{x} \in W = \mathbb{R}^n$  is  $\lambda_1(\mathbf{A}) \|\mathbf{x}\|_2^2$ .

*Proof of Theorem 3.1.4.* We start by showing Part 1.

Consider letting  $W = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ , and normalize  $\mathbf{x} \in W$  so that  $\|\mathbf{x}\|_2 = 1$ . Then  $\mathbf{x} = \sum_{j=1}^i \mathbf{c}(j) \mathbf{v}_j$  for some vector  $\mathbf{c} \in \mathbb{R}^i$  with  $\|\mathbf{c}\|_2 = 1$ .

Using the decomposition from Theorem 3.1.3  $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top$  where  $\mathbf{\Lambda}$  is a diagonal matrix of eigenvalues of  $\mathbf{A}$ , which we take to be sorted in increasing order. Then  $\mathbf{x}^\top \mathbf{A} \mathbf{x} = \mathbf{x}^\top \mathbf{V}^\top \mathbf{\Lambda} \mathbf{V} \mathbf{x} = (\mathbf{V} \mathbf{x})^\top \mathbf{\Lambda} (\mathbf{V} \mathbf{x}) = \sum_{j=1}^i \lambda_j \mathbf{c}(j)^2 \leq \lambda_i \|\mathbf{c}\|_2^2 = \lambda_i$ . So this choice of  $W$  ensures the maximizer cannot achieve a value above  $\lambda_i$ .

But is it possible that the “minimizer” can do better by choosing a different  $W$ ? Let  $T = \text{span}\{\mathbf{v}_i, \dots, \mathbf{v}_n\}$ . As  $\dim(T) = n+1-i$  and  $\dim(W) = i$ , we must have  $\dim(W \cap T) \geq 1$ , by a standard property of subspaces. Hence for any  $W$  of this dimension,

$$\begin{aligned} \max_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} &\geq \max_{\mathbf{x} \in W \cap T, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \\ &\geq \min_{\substack{\text{subspace } V \subseteq T \\ \dim(V)=1}} \max_{\mathbf{x} \in V, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \lambda_i, \end{aligned}$$

where the last equality follows from a similar calculation to our first one. Thus,  $\lambda_i$  can always be achieved by the “maximizer” for all  $W$  of this dimension.

Part 2 can be dealt with similarly.

□

**Example: a positive semidefinite matrix.** Consider the following matrix  $\mathbf{A}$ :

$$\mathbf{A} := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For  $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , we have  $\mathbf{A}\mathbf{x} = \mathbf{0}$ , so  $\lambda = 0$  is an eigenvalue of  $\mathbf{A}$ . For  $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , we have  $\mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2\mathbf{x}$ , so  $\lambda = 2$  is the other eigenvalue of  $\mathbf{A}$ . As both are non-negative, by the theorem above,  $\mathbf{A}$  is positive semidefinite.

Since we are learning about symmetric matrices, there is one more fact that everyone should know about them. We’ll use  $\lambda_{\max}(\mathbf{A})$  denote maximum eigenvalue of a matrix  $\mathbf{A}$ , and  $\lambda_{\min}(\mathbf{A})$  the minimum.

**Claim 3.1.5.** For a symmetric matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\|\mathbf{A}\| = \max(|\lambda_{\max}(\mathbf{A})|, |\lambda_{\min}(\mathbf{A})|)$ .

## 3.2 Characterizations of Convexity and Optimality via Second Derivatives

We will now use the second derivatives of a function to obtain characterizations of convexity and optimality. We will begin by introducing the *Hessian*, the matrix of pairwise second derivatives of a function. We will see that it plays a role in approximating a function via a second-order Taylor expansion. We will then use *semi-definiteness* of the Hessian matrix to characterize both conditions of optimality as well as the convexity of a function.

**Definition 3.2.1.** Given a function  $f : S \rightarrow \mathbb{R}$  its **Hessian** matrix at point  $\mathbf{x} \in S$  denoted  $\mathbf{H}_f(\mathbf{x})$  (also sometimes denoted  $\nabla^2 f(\mathbf{x})$ ) is:

$$\mathbf{H}_f(\mathbf{x}) := \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(1)^2} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(1)\partial \mathbf{x}(2)} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(1)\partial \mathbf{x}(n)} \\ \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(2)\partial \mathbf{x}(1)} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(2)^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(2)\partial \mathbf{x}(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(n)\partial \mathbf{x}(1)} & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(n)\partial \mathbf{x}(2)} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}(n)^2} \end{bmatrix}$$

**Second-order Taylor expansion.** When  $f$  is twice differentiable it is possible to obtain an approximation of  $f$  by quadratic functions. Our definition of  $f : S \rightarrow \mathbb{R}$  being twice (Fréchet) differentiable at  $\mathbf{x} \in S$  is that there exists  $\nabla f(\mathbf{x}) \in \mathbb{R}^n$  and  $\mathbf{H}_f(\mathbf{x}) \in \mathbb{R}^{n \times n}$  s.t.

$$\lim_{\delta \rightarrow 0} \frac{\|f(\mathbf{x} + \delta) - f(\mathbf{x}) - (\nabla f(\mathbf{x})^\top \delta + \frac{1}{2} \delta^\top \mathbf{H}_f(\mathbf{x}) \delta)\|_2}{\|\delta\|_2^2} = 0.$$

This is equivalent to saying that for all  $\delta$

$$f(\mathbf{x} + \delta) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top \delta + \frac{1}{2} \delta^\top \mathbf{H}_f(\mathbf{x}) \delta + o(\|\delta\|_2^2).$$

where by definition:

$$\lim_{\delta \rightarrow 0} \frac{o(\|\delta\|_2^2)}{\|\delta\|_2^2} = 0$$

We say that  $f$  is *twice continuously differentiable* on a set  $S \subseteq \mathbb{R}^n$  if it is twice differentiable and in addition the gradient and Hessian are continuous on  $S$ .

As for first order expansions, we have a Taylor's Theorem, which we state in the so-called remainder form.

**Theorem 3.2.2** (Taylor's Theorem, multivariate second-order remainder form). *If  $f : S \rightarrow \mathbb{R}$  is twice continuously differentiable over  $[\mathbf{x}, \mathbf{y}]$ , then for some  $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$ ,*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{1}{2} (\mathbf{y} - \mathbf{x})^\top \mathbf{H}_f(\mathbf{z}) (\mathbf{y} - \mathbf{x})$$

### 3.2.1 A Necessary Condition for Local Extrema

Recall that in the previous chapter, we show the following proposition.

**Proposition 3.2.3.** *If  $\mathbf{x}$  is a local extremum of a differentiable function  $f : S \rightarrow \mathbb{R}$  then  $\nabla f(\mathbf{x}) = \mathbf{0}$ .*

We can now give the second-order necessary conditions for local extrema via the Hessian.

**Theorem 3.2.4.** Let  $f : S \rightarrow \mathbb{R}$  be a function twice differentiable at  $\mathbf{x} \in S$ . If  $\mathbf{x}$  is a local minimum, then  $\mathbf{H}_f(\mathbf{x})$  is positive semidefinite.

*Proof.* Let us assume that  $\mathbf{x}$  is a local minimum. We know from Proposition 3.2.3 that  $\nabla f(\mathbf{x}) = \mathbf{0}$ , hence the second-order expansion at  $\mathbf{x}$  takes the form:

$$f(\mathbf{x} + \lambda \mathbf{d}) = f(\mathbf{x}) + \lambda^2 \frac{1}{2} \mathbf{d}^\top \mathbf{H}_f(\mathbf{x}) \mathbf{d} + o(\lambda^2 \|\mathbf{d}\|_2^2)$$

Because  $\mathbf{x}$  is a local minimum, we can then derive

$$0 \leq \lim_{\lambda \rightarrow 0^+} \frac{f(\mathbf{x} + \lambda \mathbf{d}) - f(\mathbf{x})}{\lambda^2} = \frac{1}{2} \mathbf{d}^\top \mathbf{H}_f(\mathbf{x}) \mathbf{d}$$

This is true for any  $\mathbf{d}$ , hence  $\mathbf{H}_f(\mathbf{x})$  is positive semidefinite.  $\square$

**Remark 3.2.5.** Again, for this proposition to hold, it is important that  $S$  is open.

### 3.2.2 A sufficient condition for local extrema

A local minimum thus is a stationary point and has a positive semi-definite Hessian. The converse is almost true, but we need to strengthen the Hessian condition slightly.

**Theorem 3.2.6.** Let  $f : S \rightarrow \mathbb{R}$  be a function twice differentiable at a stationary point  $\mathbf{x} \in S$ . If  $\mathbf{H}_f(\mathbf{x})$  is positive definite then  $\mathbf{x}$  is a local minimum.

*Proof.* Let us assume that  $\mathbf{H}_f(\mathbf{x})$  is positive definite. We know that  $\mathbf{x}$  is a stationary point. We can write the second-order expansion at  $\mathbf{x}$ :

$$f(\mathbf{x} + \boldsymbol{\delta}) = f(\mathbf{x}) + \frac{1}{2} \boldsymbol{\delta}^\top \mathbf{H}_f(\mathbf{x}) \boldsymbol{\delta} + o(\|\boldsymbol{\delta}\|_2^2)$$

Because the Hessian is positive definite, it has a strictly positive minimum eigenvalue  $\lambda_{\min}$ , we can conclude that  $\boldsymbol{\delta}^\top \mathbf{H}_f(\mathbf{x}) \boldsymbol{\delta} \geq \lambda_{\min} \|\boldsymbol{\delta}\|_2^2$ . From this, we conclude that when  $\|\boldsymbol{\delta}\|_2^2$  is small enough,  $f(\mathbf{x} + \boldsymbol{\delta}) - f(\mathbf{x}) \geq \frac{1}{4} \lambda_{\min} \|\boldsymbol{\delta}\|_2^2 > 0$ . This proves that  $\mathbf{x}$  is a local minimum.  $\square$

**Remark 3.2.7.** When  $\mathbf{H}_f(\mathbf{x})$  is indefinite at a stationary point  $\mathbf{x}$ , we have what is known as a *saddle point*:  $\mathbf{x}$  will be a minimum along the eigenvectors of  $\mathbf{H}_f(\mathbf{x})$  for which the eigenvalues are positive and a maximum along the eigenvectors of  $\mathbf{H}_f(\mathbf{x})$  for which the eigenvalues are negative.

### 3.2.3 Characterization of convexity

**Definition 3.2.8.** For a convex set  $S \subseteq \mathbb{R}^n$ , we say that a function  $f : S \rightarrow \mathbb{R}$  is **strictly convex on  $S$**  if for any two points  $\mathbf{x}_1, \mathbf{x}_2 \in S$  and any  $\theta \in (0, 1)$  we have that:

$$f(\theta \mathbf{x}_1 + (1 - \theta) \mathbf{x}_2) < \theta f(\mathbf{x}_1) + (1 - \theta) f(\mathbf{x}_2).$$

**Theorem 3.2.9.** Let  $S \subseteq \mathbb{R}^n$  be open and convex, and let  $f : S \rightarrow \mathbb{R}$  be twice continuously differentiable.

1. If  $H_f(\mathbf{x})$  is positive semi-definite for any  $\mathbf{x} \in S$  then  $f$  is convex on  $S$ .
2. If  $H_f(\mathbf{x})$  is positive definite for any  $\mathbf{x} \in S$  then  $f$  is **strictly** convex on  $S$ .
3. If  $f$  is convex, then  $H_f(\mathbf{x})$  is positive semi-definite  $\forall \mathbf{x} \in S$ .

*Proof.*

1. By applying Theorem 3.2.2, we find that for some  $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$ :

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{1}{2} \left( (\mathbf{y} - \mathbf{x})^\top H_f(\mathbf{z}) (\mathbf{y} - \mathbf{x}) \right)$$

If  $H_f(\mathbf{z})$  is positive semi-definite, this necessarily implies that:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$$

and from Theorem 2.3.7 we get that  $f$  is convex.

2. if  $H_f(\mathbf{x})$  is positive definite, we have that:

$$f(\mathbf{y}) > f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}).$$

Applying the same idea as in Theorem 2.3.7 we can show that in this case  $f$  is **strictly** convex.

3. Let  $f$  be a convex function. For  $\mathbf{x} \in S$ , and some small  $\lambda > 0$ , for any  $\mathbf{d} \in \mathbb{R}^n$  we have that  $\mathbf{x} + \lambda \mathbf{d} \in S$ . From the Taylor expansion of  $f$  we get:

$$f(\mathbf{x} + \lambda \mathbf{d}) = f(\mathbf{x}) + \lambda \nabla f(\mathbf{x})^\top \mathbf{d} + \frac{\lambda^2}{2} \mathbf{d}^\top H_f(\mathbf{x}) \mathbf{d} + o(\lambda^2 \|\mathbf{d}\|_2^2).$$

From Lemma 2.3.7 we get that if  $f$  is convex then:

$$f(\mathbf{x} + \lambda \mathbf{d}) \geq f(\mathbf{x}) + \lambda \nabla f(\mathbf{x})^\top \mathbf{d}.$$

Therefore, we have that for any  $\mathbf{d} \in \mathbb{R}^n$ :

$$\frac{\lambda^2}{2} \mathbf{d}^\top H_f(\mathbf{x}) \mathbf{d} + o(\|\lambda \mathbf{d}\|^2) \geq 0$$

Dividing by  $\lambda^2$  and taking  $\lambda \rightarrow 0^+$  gives us that for any  $\mathbf{d} \in \mathbb{R}^n$ :  $\mathbf{d}^\top H_f(\mathbf{x}) \mathbf{d} \geq 0$ .  $\square$

**Remark 3.2.10.** It is important to note that if  $S$  is open and  $f$  is strictly convex, then  $H_f(\mathbf{x})$  may still (only) be positive semi-definite  $\forall \mathbf{x} \in S$ . Consider  $f(x) = x^4$  which is strictly convex, then the Hessian is  $H_f(x) = 12x^2$  which equals 0 at  $x = 0$ .



### 3.3 Gradient Descent - An Approach to Optimization?

We have begun to develop an understanding of convex functions, and what we have learned already suggests a way for us to try to find an approximate minimizer of a given convex function.

Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex and differentiable, and we want to solve

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

We would like to find  $\mathbf{x}^*$ , a global minimizer of  $f$ . Suppose we start with some initial guess  $\mathbf{x}_0$ , and we want to update it to  $\mathbf{x}_1$  with  $f(\mathbf{x}_1) < f(\mathbf{x}_0)$ . If we can repeatedly make updates like this, maybe we eventually find a point with nearly minimum function value, i.e. some  $\tilde{\mathbf{x}}$  with  $f(\tilde{\mathbf{x}}) \approx f(\mathbf{x}^*)$ ?

Recall that  $f(\mathbf{x}_0 + \boldsymbol{\delta}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^\top \boldsymbol{\delta} + o(\|\boldsymbol{\delta}\|_2)$ . This means that if we choose  $\mathbf{x}_1 = \mathbf{x}_0 - \lambda \nabla f(\mathbf{x}_0)$ , we get

$$f(\mathbf{x}_0 - \lambda \nabla f(\mathbf{x}_0)) = f(\mathbf{x}_0) - \lambda \|\nabla f(\mathbf{x}_0)\|_2^2 + o(\lambda \|\nabla f(\mathbf{x}_0)\|_2)$$

And because  $f$  is convex, we know that  $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$  unless we are already at a global minimum. So, for some small enough  $\lambda > 0$ , we should get  $f(\mathbf{x}_1) < f(\mathbf{x}_0)$  unless we're already at a global minimizer. This idea of taking a step in the direction of  $-\nabla f(\mathbf{x}_0)$  is what is called *Gradient Descent*. But how do we choose  $\lambda$  each time? And does this lead to an algorithm that quickly reaches a point with close to minimal function value? To get good answers to these questions, we need to assume more about the function  $f$  that we are trying to minimize.

In the following subsection, we will see some conditions that suffice. But there are also many other settings where one can show that some form of gradient descent converges.

#### 3.3.1 A Quantitative Bound on Changes in the Gradient

**Definition 3.3.1.** Let  $f : S \rightarrow \mathbb{R}$  be a differentiable function, where  $S \subseteq \mathbb{R}^n$  is convex and open. We say that  $f$  is  $\beta$ -gradient Lipschitz iff for all  $\mathbf{x}, \mathbf{y} \in S$

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2 \leq \beta \|\mathbf{x} - \mathbf{y}\|_2.$$

We also refer to this as  $f$  being  $\beta$ -smooth.

**Proposition 3.3.2.** Consider a twice continuously differentiable  $f : S \rightarrow \mathbb{R}$ . Then  $f$  is  $\beta$ -gradient Lipschitz if and only if for all  $\mathbf{x} \in S$ ,  $\|\mathbf{H}_f(\mathbf{x})\| \leq \beta$ .

You will prove this in Exercise 13 (Week 2) of the first exercise set.

**Proposition 3.3.3.** *Let  $f : S \rightarrow \mathbb{R}$  be a  $\beta$ -gradient Lipschitz function. Then for all  $\mathbf{x}, \mathbf{y} \in S$ ,*

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{\beta}{2} \|\mathbf{x} - \mathbf{y}\|_2^2$$

To prove this proposition, we need the following result from multi-variate calculus. This is a restricted form of the fundamental theorem of calculus for line integrals.

**Proposition 3.3.4.** *Let  $f : S \rightarrow \mathbb{R}$  be a differentiable function, and consider  $\mathbf{x}, \mathbf{y}$  such that  $[\mathbf{x}, \mathbf{y}] \in S$ . Let  $\mathbf{x}_\theta = \mathbf{x} + \theta(\mathbf{y} - \mathbf{x})$ . Then*

$$f(\mathbf{y}) = f(\mathbf{x}) + \int_{\theta=0}^1 \nabla f(\mathbf{x}_\theta)^\top (\mathbf{y} - \mathbf{x}) d\theta$$

Now, we're in a position to show Proposition 3.3.3

*Proof of Proposition 3.3.3.* Let  $f : S \rightarrow \mathbb{R}$  be a  $\beta$ -gradient Lipschitz function. Consider arbitrary  $\mathbf{x}, \mathbf{y} \in S$  such that  $[\mathbf{x}, \mathbf{y}] \in S$

$$\begin{aligned} f(\mathbf{y}) &= f(\mathbf{x}) + \int_{\theta=0}^1 \nabla f(\mathbf{x}_\theta)^\top (\mathbf{y} - \mathbf{x}) d\theta \\ &= f(\mathbf{x}) + \int_{\theta=0}^1 \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) d\theta + \int_{\theta=0}^1 (\nabla f(\mathbf{x}_\theta) - \nabla f(\mathbf{x}))^\top (\mathbf{y} - \mathbf{x}) d\theta \end{aligned}$$

Next we use Cauchy-Schwarz pointwise.

We also evaluate the first integral.

$$\leq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \int_{\theta=0}^1 \|\nabla f(\mathbf{x}_\theta) - \nabla f(\mathbf{x})\| \|\mathbf{y} - \mathbf{x}\| d\theta$$

Then we apply  $\beta$ -gradient Lipschitz and note  $\mathbf{x}_\theta - \mathbf{x} = \theta(\mathbf{y} - \mathbf{x})$ .

$$\leq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \int_{\theta=0}^1 \beta \theta \|\mathbf{y} - \mathbf{x}\|^2 d\theta.$$

$$= f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{\beta}{2} \|\mathbf{y} - \mathbf{x}\|^2.$$

□

### 3.3.2 Analyzing Gradient Descent

It turns out that just knowing a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex and  $\beta$ -gradient Lipschitz is enough to let us figure out a reasonable step size for Gradient Descent and let us analyze its convergence.

We start at a point  $\mathbf{x}_0 \in \mathbb{R}^n$ , and we try to find a point  $\mathbf{x}_1 = \mathbf{x}_0 + \boldsymbol{\delta}$  with lower function value. We will let our upper bound from Proposition 3.3.3 guide us, in fact, we could ask, what is the *best* update for minimizing this upper bound, i.e. a  $\boldsymbol{\delta}$  solving

$$\min_{\boldsymbol{\delta} \in \mathbb{R}^n} f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^\top \boldsymbol{\delta} + \frac{\beta}{2} \|\boldsymbol{\delta}\|^2$$

We can compute the best according to this upper bound by noting first that function is convex and continuously differentiable, and hence will be minimized at any point where the gradient is zero. Thus we want  $\mathbf{0} = \nabla_{\boldsymbol{\delta}} (f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^\top \boldsymbol{\delta} + \frac{\beta}{2} \|\boldsymbol{\delta}\|^2) = \nabla f(\mathbf{x}_0) + \beta \boldsymbol{\delta}$ , which occurs at  $\boldsymbol{\delta} = -\frac{1}{\beta} \nabla f(\mathbf{x}_0)$ .

Plugging in this value into the upper bound, we get that  $f(\mathbf{x}_1) \leq f(\mathbf{x}_0) - \|\nabla f(\mathbf{x}_0)\|_2^2 / 2\beta$ .

Now, as our algorithm, we will start with some guess  $\mathbf{x}_0$ , and then at every step we will update our guess using the best step based on our Proposition 3.3.3 upper bound on  $f$  at  $\mathbf{x}_i$ , and so we get

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \frac{1}{\beta} \nabla f(\mathbf{x}_i) \text{ and } f(\mathbf{x}_{i+1}) \leq f(\mathbf{x}_i) - \frac{\|\nabla f(\mathbf{x}_i)\|_2^2}{2\beta}. \quad (3.1)$$

Let us try to prove that our algorithm converges toward an  $\mathbf{x}$  with low function value.

We will measure this by looking at

$$\text{gap}_i = f(\mathbf{x}_i) - f(\mathbf{x}^*)$$

where  $\mathbf{x}^*$  is a global minimizer of  $f$  (note that there may not be a unique minimizer of  $f$ ). We will try to show that this function value gap grows small. Using  $f(\mathbf{x}_{i+1}) - f(\mathbf{x}_i) = \text{gap}_{i+1} - \text{gap}_i$ , we get

$$\text{gap}_{i+1} - \text{gap}_i \leq -\frac{\|\nabla f(\mathbf{x}_i)\|_2^2}{2\beta} \quad (3.2)$$

If the  $\text{gap}_i$  value is never too much bigger than  $\frac{\|\nabla f(\mathbf{x}_i)\|_2^2}{2\beta}$ , then this should help us show we are making progress. But how much can they differ? We will now try to show a limit on this.

Recall that in the previous chapter we showed the following theorem.

**Theorem 3.3.5.** *Let  $S$  be an open convex subset of  $\mathbb{R}^n$ , and let  $f : S \rightarrow \mathbb{R}$  be a differentiable function. Then,  $f$  is convex if and only if for any  $\mathbf{x}, \mathbf{y} \in S$  we have that  $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x})$ .*

Using the convexity of  $f$  and the lower bound on convex functions given by Theorem 3.3.5, we have that

$$f(\mathbf{x}^*) \geq f(\mathbf{x}_i) + \nabla f(\mathbf{x}_i)^\top (\mathbf{x}^* - \mathbf{x}_i) \quad (3.3)$$

Rearranging gets us

$$\begin{aligned} \text{gap}_i &\leq \nabla f(\mathbf{x}_i)^\top (\mathbf{x}_i - \mathbf{x}^*) \\ &\leq \|\nabla f(\mathbf{x}_i)\|_2 \|\mathbf{x}_i - \mathbf{x}^*\|_2 \end{aligned} \tag{3.4}$$

by Cauchy-Schwarz.

At this point, we are essentially ready to connect Equation (3.2) with Equation (3.4) and analyze the convergence rate of our algorithm.

However, at the moment, we see that the change  $\text{gap}_{i+1} - \text{gap}_i$  in how close we are to the optimum function value is governed by the norm of the gradient  $\|\nabla f(\mathbf{x}_i)\|_2$ , while the size of the gap is related to *both* this quantity and the distance  $\|\mathbf{x}_i - \mathbf{x}^*\|_2$  between the current solution  $\mathbf{x}_i$  and an optimum  $\mathbf{x}^*$ . Do we need both or can we get rid of, say, the distance? Unfortunately, with this algorithm and for this class of functions, a dependence on the distance is necessary. However, we can simplify things considerably using the following observation, which you will prove in the exercises (Exercise 2):

**Claim 3.3.6.** *When running Gradient Descent as given by the step in Equation (3.1), for all  $i$   $\|\mathbf{x}_i - \mathbf{x}^*\|_2 \leq \|\mathbf{x}_0 - \mathbf{x}^*\|_2$ .*

Combining this Claim with Equation (3.2) and Equation (3.4),

$$\text{gap}_{i+1} - \text{gap}_i \leq -\frac{1}{2\beta} \cdot \left( \frac{\text{gap}_i}{\|\mathbf{x}_0 - \mathbf{x}^*\|_2} \right)^2 \tag{3.5}$$

At this point, a simple induction will complete the proof of following result.

**Theorem 3.3.7.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $\beta$ -gradient Lipschitz, convex function. Let  $\mathbf{x}_0$  be a given starting point, and let  $\mathbf{x}^* \in \arg \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$  be a minimizer of  $f$ . The Gradient Descent algorithm given by*

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \frac{1}{\beta} \nabla f(\mathbf{x}_i)$$

*ensures that the  $k$ th iterate satisfies*

$$f(\mathbf{x}_k) - f(\mathbf{x}^*) \leq \frac{2\beta \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2}{k+1}.$$

Carrying out this induction is one of the exercises in problem set 2.

# Part II

## Spectral Graph Theory

# Chapter 4

## Introduction to Spectral Graph Theory

In this chapter, we will study graphs through linear algebra. This approach is known as Spectral Graph Theory and turns out to be surprisingly powerful. An in-depth treatment of many topics in this area can be found in [Spi19].

### 4.1 Recap: Incidence and Adjacency Matrices, the Laplacian Matrix and Electrical Energy

To exploit tools from linear algebra, we first have to introduce some interesting matrices that capture important properties of the graph  $G$ . In Chapter 1, we have already seen some of these matrices. Let us recall them and also revisit some additional material from the lectures before. For the rest of the lecture, we let  $G = (V, E, \mathbf{w})$  be an undirected graph, with  $n = |V|$  vertices and  $m = |E|$  edges, where  $\mathbf{w} \in \mathbb{R}_+^E$  assigns a positive weight for every edge. We also assume that  $G$  is connected!

**Fundamental Matrices.** We start by re-introducing the *edge-vertex incidence matrix*  $\mathbf{B}$  of the graph  $G$ . Therefore, we first have to associate an arbitrary direction to every edge. We then let  $\mathbf{B} \in \mathbb{R}^{V \times E}$ .

$$\mathbf{B}(v, e) = \begin{cases} 1 & \text{if } e = (u, v) \\ -1 & \text{if } e = (v, u) \\ 0 & \text{o.w.} \end{cases}$$

The edge directions are only there to help us track the meaning of signs of quantities defined on edges: The math we do should not depend on the choice of sign.

We define the *weight matrix*  $\mathbf{W} \in \mathbb{R}^{E \times E}$  be the diagonal matrix given by  $\mathbf{W} = \text{diag}(\mathbf{w})$ , i.e.  $\mathbf{W}(e, e) = \mathbf{w}(e)$ . The weighted degree of a vertex is defined as  $\mathbf{d}(v) = \sum_{\{u, v\} \in E} \mathbf{w}(u, v)$ .

Again we treat the edges as undirected. Let  $\mathbf{D} = \text{diag}(\mathbf{d})$  be the *degree matrix* that is a diagonal matrix in  $\mathbb{R}^{V \times V}$  with weighted degrees on the diagonal.

We define the *weighted adjacency matrix*  $\mathbf{A} \in \mathbb{R}^{V \times V}$  of a graph by

$$\mathbf{A}(u, v) = \begin{cases} \mathbf{w}(u, v) & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Note that we treat the edges as undirected here, so  $\mathbf{A}^\top = \mathbf{A}$ .

Finally, we define the *Laplacian* of the graph as  $\mathbf{L} = \mathbf{B} \mathbf{W} \mathbf{B}^\top$ . Note that in the first chapter, we defined the Laplacian as  $\mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top$ , where  $\mathbf{R}$  is the diagonal matrix with edge resistances on the diagonal. We want to think of high *weight* on an edge as expressing that two vertices are highly connected, whereas we think of high resistance on an edge as expressing that the two vertices are poorly connected, so we let  $\mathbf{w}(e) = 1/\mathbf{R}(e, e)$ . In Problem Set 1, you showed that  $\mathbf{L} = \mathbf{D} - \mathbf{A}$ , and that for  $\mathbf{x} \in \mathbb{R}^V$ ,

$$\mathbf{x}^\top \mathbf{L} \mathbf{x} = \sum_{\{a, b\} \in E} \mathbf{w}(a, b) (\mathbf{x}(a) - \mathbf{x}(b))^2.$$

**Flows and Voltages.** Let us next briefly review flows and voltages. Given the edge-vertex incidence matrix  $\mathbf{B}$ , we can now express the net flow constraint that  $\mathbf{f}$  routes a demand  $\mathbf{d}$  by

$$\mathbf{B} \mathbf{f} = \mathbf{d}.$$

This is also called a conservation constraint. In our examples so far, we have  $\mathbf{d}(s) = -1$ ,  $\mathbf{d}(t) = 1$  and  $\mathbf{d}(u) = 0$  for all  $u \in V \setminus \{s, t\}$ . Note that, unfortunately, the demand vector is typically denoted by  $\mathbf{d}$ , but so is the degree vector. Since it should always be clear from context which is which, we will also overload  $\mathbf{d}$  in this way!

We then define for any flow  $\mathbf{f} \in \mathbb{R}^E$  the electrical energy by

$$\mathcal{E}(\mathbf{f}) = \sum_e \mathbf{r}(e) \mathbf{f}(e)^2 = \mathbf{f}^\top \mathbf{R} \mathbf{f}$$

and for any electrical voltages  $\mathbf{x} \in \mathbb{R}^E$

$$\mathcal{E}(\mathbf{f}) = \mathbf{f}^\top \mathbf{R} \mathbf{f} = \mathbf{x}^\top \mathbf{L} \mathbf{x}.$$

While the former definition is standard, the second definition is motivated by the fact that by Ohm's law, where we use  $\mathbf{R} = \text{diag}_{e \in E} \mathbf{r}(e)$ , we have that electrical voltages  $\mathbf{x}$  will induce an electrical flow  $\mathbf{f} = \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{x}$ . And for such a pair  $\mathbf{f}$  and  $\mathbf{x}$ , we then have that the energy associated with  $\mathbf{x}$  is equal to the energy consumed by the flow  $\mathbf{f}$  induced by  $\mathbf{x}$ :

$$\mathcal{E}(\mathbf{f}) = \mathbf{f}^\top \mathbf{R} \mathbf{f} = \mathbf{x}^\top \mathbf{L} \mathbf{x} = \mathcal{E}(\mathbf{x}).$$

**The Courant-Fisher Theorem.** Let us also recall the Courant-Fisher theorem, which we proved in Chapter 3 (Theorem 3.1.4). Here we add an additional characterization of  $\lambda_i$ . Note that the second equality for both versions to obtain  $\lambda_i$  can be extracted from our proof of the Courant-Fisher theorem in Chapter 3.

**Theorem 4.1.1** (The Courant-Fisher Theorem, eigenbasis version). *Let  $\mathbf{A}$  be a symmetric matrix in  $\mathbb{R}^{n \times n}$ , with eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , and corresponding eigenvectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  which form an orthonormal basis. Then*

1.

$$\lambda_i = \min_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=i}} \max_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \min_{\substack{\mathbf{x} \perp \mathbf{x}_1, \dots, \mathbf{x}_{i-1} \\ \mathbf{x} \neq \mathbf{0}}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

2.

$$\lambda_i = \max_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=n+1-i}} \min_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \max_{\substack{\mathbf{x} \perp \mathbf{x}_{i+1}, \dots, \mathbf{x}_n \\ \mathbf{x} \neq \mathbf{0}}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

Of course, we also have  $\lambda_i(\mathbf{A}) = \frac{\mathbf{x}_i^\top \mathbf{A} \mathbf{x}_i}{\mathbf{x}_i^\top \mathbf{x}_i}$ .

## 4.2 Understanding Eigenvalues of the Laplacian

We would like to understand the eigenvalues of the Laplacian matrix of a graph.

But first, why should we care? It turns out that Laplacian eigenvalues can help us understand many properties of a graph. But we are going to start with a simple motivating observation: Electrical voltages  $\mathbf{x} \in \mathbb{R}^V$  consume electrical energy  $\mathcal{E}(\mathbf{x}) = \mathbf{x}^\top \mathbf{L} \mathbf{x}$ . This means that by the Courant-Fisher Theorem

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^\top \mathbf{L} \mathbf{x} \leq \lambda_n(L) \mathbf{x}^\top \mathbf{x}$$

And, for any voltages  $\mathbf{x} \perp \mathbf{1}$ ,

$$\mathcal{E}(\mathbf{x}) = \mathbf{x}^\top \mathbf{L} \mathbf{x} \geq \lambda_2(L) \mathbf{x}^\top \mathbf{x}.$$

Thus, we can use the eigenvalues to give upper and lower bounds on how much electrical energy will be consumed by the flow induced by  $\mathbf{x}$ , in terms compared to  $\mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|_2^2$ .

In a couple of chapters, we will also prove the following claim, which shows that the Laplacian eigenvalues can directly tell us about the electrical energy that is required to route a given demand.

**Claim 4.2.1.** *Given a demand vector  $\mathbf{d} \in \mathbb{R}^V$  such that  $\mathbf{d} \perp \mathbf{1}$ , the electrical voltages  $\mathbf{x}$  that route  $\mathbf{d}$  satisfy  $\mathbf{L} \mathbf{x} = \mathbf{d}$  and the electrical energy of these voltages satisfies*

$$\frac{\|\mathbf{d}\|_2^2}{\lambda_n} \leq \mathcal{E}(\mathbf{x}) \leq \frac{\|\mathbf{d}\|_2^2}{\lambda_2}$$



While the techniques in this section can be used to analyze various Laplacian matrices, we focus on understanding the eigenvalues of certain graph families that are particularly instructive. In this section, we obtain the following results.

Graph Family	$\lambda_2$ lower bound	$\lambda_2$ upper bound	$\lambda_n$ lower bound	$\lambda_n$ upper bound
Complete Graph $K_n$	$n$	$n$	$n$	$n$
Path Graph $P_n$	$\frac{1}{n^2}$	$\frac{12}{n^2}$	1	4
Binary Tree $T_n$	$\frac{1}{2n \log_2(n)}$	$\frac{2}{n-1}$	1	6

### 4.3 The Complete Graph

To get a sense of how Laplacian eigenvalues behave, let us start by considering the  $n$ -vertex complete graph with unit weights, which we denote by  $K_n$ . The adjacency matrix of  $K_n$  is  $\mathbf{A} = \mathbf{1}\mathbf{1}^\top - \mathbf{I}$ , since it has ones everywhere, except for the diagonal, where entries are zero. The degree matrix  $\mathbf{D} = (n-1)\mathbf{I}$ . Thus the Laplacian is  $\mathbf{L} = \mathbf{D} - \mathbf{A} = n\mathbf{I} - \mathbf{1}\mathbf{1}^\top$ .

Thus for any  $\mathbf{y} \perp \mathbf{1}$ , we have  $\mathbf{y}^\top \mathbf{L} \mathbf{y} = n\mathbf{y}^\top \mathbf{y} - (\mathbf{1}^\top \mathbf{y})^2 = n\mathbf{y}^\top \mathbf{y}$ .

From this, we can conclude that any  $\mathbf{y} \perp \mathbf{1}$  is an eigenvector of eigenvalue  $n$ , and that all  $\lambda_2 = \lambda_3 = \dots = \lambda_n = n$ .

### 4.4 The Path Graph

Next, let us try to understand  $\lambda_2$  and  $\lambda_n$  for  $P_n$ , the  $n$  vertex path graph with unit weight edges. I.e. the graph has edges  $E = \{\{i, i+1\} \text{ for } i = 1 \text{ to } (n-1)\}$ . This is in a sense the least well-connected unit weight graph on  $n$  vertices, whereas  $K_n$  is the most well-connected.

#### 4.4.1 Upper Bounding $\lambda_2(P_n)$ via Test Vectors

We can use the eigenbasis version of the Courant-Fisher theorem to observe that the second-smallest eigenvalue of the Laplacian is given by

$$\lambda_2(\mathbf{L}) = \min_{\substack{\mathbf{x} \neq \mathbf{0} \\ \mathbf{x}^\top \mathbf{1} = 0}} \frac{\mathbf{x}^\top \mathbf{L} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}. \quad (4.1)$$

We can get a better understanding of this particular case through a couple of simple observations. Suppose  $\mathbf{x} = \mathbf{y} + \alpha \mathbf{1}$ , where  $\mathbf{y} \perp \mathbf{1}$ . Then  $\mathbf{x}^\top \mathbf{L} \mathbf{x} = \mathbf{y}^\top \mathbf{L} \mathbf{y}$ , and  $\|\mathbf{x}\|_2^2 = \|\mathbf{y}\|_2^2 + \alpha^2 \|\mathbf{1}\|_2^2$ .

So for any given vector, you can increase the value of  $\frac{\mathbf{x}^\top \mathbf{L} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$ , by replacing  $\mathbf{x}$  with its component orthogonal to  $\mathbf{1}$ , which we denoted by  $\mathbf{y}$ .

We can conclude from Equation (4.1) that for *any* vector  $\mathbf{y} \perp \mathbf{1}$ ,

$$\lambda_2 \leq \frac{\mathbf{y}^\top \mathbf{L} \mathbf{y}}{\mathbf{y}^\top \mathbf{y}}$$

When we use a vector  $\mathbf{y}$  in this way to prove a bound on an eigenvalue, we call it a *test vector*.

Now, we'll use a test vector to give an upper bound on  $\lambda_2(\mathbf{L}_{P_n})$ . Let  $\mathbf{x} \in \mathbb{R}^V$  be given by  $\mathbf{x}(i) = (n+1) - 2i$ , for  $i \in [n]$ . This vector satisfies  $\mathbf{x} \perp \mathbf{1}$ . We picked this because we wanted a sequence of values growing linearly along the path, while also making sure that the vector is orthogonal to  $\mathbf{1}$ . Now

$$\begin{aligned} \lambda_2(\mathbf{L}_{P_n}) &\leq \frac{\sum_{i \in [n-1]} (\mathbf{x}(i) - \mathbf{x}(i+1))^2}{\sum_{i=1}^n \mathbf{x}(i)^2} \\ &= \frac{\sum_{i=1}^{n-1} 2^2}{\sum_{i=1}^n (n+1-2i)^2} \\ &= \frac{4(n-1)}{(n+1)n(n-1)/3} \\ &= \frac{12}{n(n+1)} \leq \frac{12}{n^2}. \end{aligned}$$

Later, we will prove a lower bound that shows this value is right up to a constant factor. But the test vector approach based on the Courant-Fischer theorem doesn't immediately work when we want to prove lower bounds on  $\lambda_2(\mathbf{L})$ .

#### 4.4.2 Lower Bounding $\lambda_n(P_n)$ via Test Vectors

We can see from either version of the Courant-Fischer theorem that

$$\lambda_n(\mathbf{L}) = \max_{\mathbf{v} \neq \mathbf{0}} \frac{\mathbf{v}^\top \mathbf{L} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}}. \quad (4.2)$$

Thus for *any* vector  $\mathbf{y} \neq \mathbf{0}$ ,

$$\lambda_n \geq \frac{\mathbf{y}^\top \mathbf{L} \mathbf{y}}{\mathbf{y}^\top \mathbf{y}}.$$

This means we can get a test vector-based lower bound on  $\lambda_n$ . Let us apply this to the Laplacian of  $P_n$ . We'll try the vector  $\mathbf{x} \in \mathbb{R}^V$  given by  $\mathbf{x}(1) = -1$ , and  $\mathbf{x}(n) = 1$  and  $\mathbf{x}(i) = 0$  for  $i \neq 1, n$ .

Here we get

$$\lambda_n(\mathbf{L}_{P_n}) \geq \frac{\mathbf{x}^\top \mathbf{L} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{2}{2} = 1. \quad (4.3)$$

Again, it's not clear how to use the Courant-Fischer theorem to prove an upper bound on  $\lambda_n(\mathbf{L})$ . But, later we'll see how to prove an upper that shows that for  $P_n$ , the lower bound we obtained is right up to constant factors.

#### 4.4.3 Upper bounding $\lambda_n(P_n)$ via the Loewner Order

In the previous sections, we first saw a complete characterization of the eigenvalues and eigenvectors of the unit weight complete graph on  $n$  vertices,  $K_n$ . Namely,  $\mathbf{L}_{K_n} = n\mathbf{I} - \mathbf{1}\mathbf{1}^\top$ , and this means that *every* vector  $\mathbf{y} \perp \mathbf{1}$  is an eigenvector of eigenvalue  $n$ . Ideally, we would like to prove an almost matching lower bound on  $\lambda_2(P_n)$  and an almost matching upper bound on  $\lambda_n(P_n)$ , but it is not clear how to get that from the Courant-Fischer theorem.

To get there, we need to introduce some more tools. We now introduce an ordering on symmetric matrices called the *Loewner order*, which I also like to just call the positive semi-definite order. As we will see in a moment, it is a partial order on symmetric matrices, we denote it by " $\preceq$ ". For convenience, we allow ourselves to both write  $\mathbf{A} \preceq \mathbf{B}$  and equivalently  $\mathbf{B} \succeq \mathbf{A}$ .

**The Loewner Order, aka. the Positive Semi-Definite Order.** For a symmetric matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  we define that

$$\mathbf{A} \succeq \mathbf{0}$$

if and only if  $\mathbf{A}$  is positive semi-definite.

More generally, when we have two symmetric matrices  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ , we will write

$$\mathbf{A} \preceq \mathbf{B} \text{ if and only if for all } \mathbf{x} \in \mathbb{R}^n \text{ we have } \mathbf{x}^\top \mathbf{A} \mathbf{x} \leq \mathbf{x}^\top \mathbf{B} \mathbf{x} \quad (4.4)$$

This is a partial order, because it satisfies the three requirements of

1. Reflexivity:  $\mathbf{A} \preceq \mathbf{A}$ .
2. Anti-symmetry:  $\mathbf{A} \preceq \mathbf{B}$  and  $\mathbf{B} \preceq \mathbf{A}$  implies  $\mathbf{A} = \mathbf{B}$
3. Transitivity:  $\mathbf{A} \preceq \mathbf{B}$  and  $\mathbf{B} \preceq \mathbf{C}$  implies  $\mathbf{A} \preceq \mathbf{C}$

Check for yourself that these properties hold!

The PSD order has other very useful properties:  $\mathbf{A} \preceq \mathbf{B}$  implies  $\mathbf{A} + \mathbf{C} \preceq \mathbf{B} + \mathbf{C}$  for any symmetric matrix  $\mathbf{C}$ . Convince yourself of this too!

And, combining this observation with transitivity, we can see that  $\mathbf{A} \preceq \mathbf{B}$  and  $\mathbf{C} \preceq \mathbf{D}$  implies  $\mathbf{A} + \mathbf{C} \preceq \mathbf{B} + \mathbf{D}$ .

Here is another useful property: If  $\mathbf{0} \preceq \mathbf{A}$  then for all  $\alpha \geq 1$

$$\frac{1}{\alpha} \mathbf{A} \preceq \mathbf{A} \preceq \alpha \mathbf{A}.$$

Here is another one:

**Claim 4.4.1.** *If  $\mathbf{A} \preceq \mathbf{B}$ , then for all  $i$*

$$\lambda_i(\mathbf{A}) \leq \lambda_i(\mathbf{B}).$$

*Proof.* We can prove this Claim by applying the subspace version of the Courant-Fischer theorem.

$$\lambda_i(\mathbf{A}) = \min_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=i}} \max_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \leq \min_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=i}} \max_{\mathbf{x} \in W, \mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{B} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \lambda_i(\mathbf{B}).$$

□

Note that the converse of Claim 4.4.1 is very much false, for example the matrices  $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$  have equal eigenvalues, but both  $\mathbf{A} \not\preceq \mathbf{B}$  and  $\mathbf{B} \not\preceq \mathbf{A}$ .

**Remark 4.4.2.** It's useful to get used to and remember some of the properties of the Loewner order, but all the things we have established so far are almost immediate from the basic characterization in Equation (4.4). So, ideally, don't memorize all these facts, instead, try to see that they are simple consequences of the definition.

**Upper bounding  $\lambda_n(L_G)$  Using Degrees.** In an earlier chapter, we observed that for any graph  $G = (V, E, \mathbf{w})$ ,  $\mathbf{L} = \mathbf{D} - \mathbf{A} \succeq \mathbf{0}$ . We can see this from  $\mathbf{x}^\top (\mathbf{D} - \mathbf{A}) \mathbf{x} = \sum_{(u,v) \in E} \mathbf{w}(u,v) (\mathbf{x}(u) - \mathbf{x}(v))^2 \geq 0$ . Similarly  $\mathbf{D} + \mathbf{A} \succeq \mathbf{0}$  because  $\mathbf{x}^\top (\mathbf{D} + \mathbf{A}) \mathbf{x} = \sum_{(u,v) \in E} \mathbf{w}(u,v) (\mathbf{x}(u) + \mathbf{x}(v))^2 \geq 0$ . But this means that  $-\mathbf{A} \preceq \mathbf{D}$  and hence  $\mathbf{L} = \mathbf{D} - \mathbf{A} \preceq 2\mathbf{D}$ .

So, for the path graph  $P_n$ , we have  $\mathbf{L}_{P_n} \preceq \mathbf{D} - \mathbf{A} \preceq 2\mathbf{D} \preceq 4\mathbf{I}$ . So by Claim 4.4.1

$$\lambda_n(\mathbf{L}_{P_n}) \leq 4. \tag{4.5}$$

We can see that our test vector-based lower bound on  $\lambda_n(\mathbf{L}_{P_n})$  from Equation (4.3) is tight up to a factor 4.

Since this type of argument works for any unit weight graph, it proves the following claim.

**Claim 4.4.3.** *For any unit weight graph  $G$ ,  $\lambda_n(\mathbf{L}_G) \leq 2 \max_{v \in V} \mathbf{d}(v)$ .*

This is tight on a graph consisting of a single edge.

#### 4.4.4 Lower bounding $\lambda_2(P_n)$ via the Loewner Order

We will now conclude our analysis of the eigenvalues  $\lambda_2$  and  $\lambda_n$  of the path graph  $P_n$  by finally deriving a lower bound on  $\lambda_2(P_n)$ . Again, this requires us to use new tools. As in the last section, we first introduce the new tool for general graphs and then exploit the tool to derive the lower bound.

**The Loewner Order for Graphs.** It's sometimes convenient to overload the PSD order to also apply to graphs. We will write

$$G \preceq H$$

if  $L_G \preceq L_H$ .

For example, given two unit weight graphs  $G = (V, E)$  and  $H = (V, F)$ , if  $H$  is a subgraph of  $G$ , then

$$L_H \preceq L_G.$$

We can see this from the Laplacian quadratic form:

$$\mathbf{x}^\top L_G \mathbf{x} = \sum_{(u,v) \in E} w(u,v) (\mathbf{x}(u) - \mathbf{x}(v))^2.$$

Dropping edges will only decrease the value of the quadratic form. The same is for decreasing the weights of edges. The graph order notation is especially useful when we allow for scaling a graph by a constant, say  $c > 0$ ,

$$c \cdot H \preceq G$$

What is  $c \cdot H$ ? It is the same graph as  $H$ , but the weight of every edge is multiplied by  $c$ . Now we can make statements like  $\frac{1}{2}H \preceq G \preceq 2H$ , which turn out to be useful notion of the two graphs approximating each other.

**The Path Inequality.** Now, we'll see a general tool for comparing two graphs  $G$  and  $H$  to prove inequalities like  $cH \preceq G$  for some constant  $c$ . Our tools won't necessarily work well for all cases, but we'll see some examples where they do.

In the rest of the chapter, we will often need to compare two graphs defined on the same vertex set  $V = \{1, \dots, n\} = [n]$ .

We use  $G_{i,j}$  to denote the unit weight graph on vertex set  $[n]$  consisting of a single edge between vertices  $i$  and  $j$ .

**Lemma 4.4.4** (The Path Inequality).

$$(n-1) \cdot P_n \succeq G_{1,n},$$

*Proof.* We want to show that for every  $\mathbf{x} \in \mathbb{R}^n$ ,

$$(n-1) \cdot \sum_{i=1}^{n-1} (\mathbf{x}(i+1) - \mathbf{x}(i))^2 \geq (\mathbf{x}(n) - \mathbf{x}(1))^2.$$

For  $i \in [n-1]$ , set

$$\Delta(i) = \mathbf{x}(i+1) - \mathbf{x}(i).$$

The inequality we want to prove then becomes

$$(n-1) \sum_{i=1}^{n-1} (\Delta(i))^2 \geq \left( \sum_{i=1}^{n-1} \Delta(i) \right)^2.$$

But, this is immediate from the Cauchy-Schwarz inequality  $\mathbf{a}^\top \mathbf{b} \leq \|\mathbf{a}\|_2 \|\mathbf{b}\|_2$ :

$$\begin{aligned} (n-1) \sum_{i=1}^{n-1} (\Delta(i))^2 &= \|\mathbf{1}_{n-1}\|^2 \cdot \|\Delta\|^2 \\ &= (\|\mathbf{1}_{n-1}\| \cdot \|\Delta\|)^2 \\ &\geq (\mathbf{1}_{n-1}^\top \Delta)^2 \\ &= \left( \sum_{i=1}^{n-1} \Delta(i) \right)^2 \end{aligned}$$

□

**Lower Bounding  $\lambda_2$  of a Path Graph.** We will now use Lemma 4.4.4 to prove a lower bound on  $\lambda_2(\mathbf{L}_{P_n})$ . Our strategy will be to prove that the path  $P_n$  is at least some multiple of the complete graph  $K_n$ , measured by the Loewner order, i.e.  $K_n \preceq f(n)P_n$  for some function  $f : \mathbb{N} \rightarrow \mathbb{R}$ . We can combine this with our observation earlier that  $\lambda_2(\mathbf{L}_{K_n}) = n$  to show that

$$f(n)\lambda_2(\mathbf{L}_{P_n}) \geq \lambda_2(\mathbf{L}_{K_n}) = n, \tag{4.6}$$

and this will give our lower bound on  $\lambda_2(\mathbf{L}_{P_n})$ . When establishing the inequality between  $P_n$  and  $K_n$ , we can treat each edge of the complete graph separately, by first noting that

$$\mathbf{L}_{K_n} = \sum_{i < j} \mathbf{L}_{G_{i,j}}$$

For every edge  $(i, j)$  in the complete graph, we apply the Path Inequality, Lemma 4.4.4:

$$\begin{aligned} G_{i,j} &\preceq (j-i) \sum_{k=i}^{j-1} G_{k,k+1} \\ &\preceq (j-i)P_n \end{aligned}$$

This inequality says that  $G_{i,j}$  is at most  $(j - i)$  times the part of the path connecting  $i$  to  $j$ , and that this part of the path is less than the whole.

Summing inequality (4.3) over all edges  $(i, j) \in K_n$  gives

$$K_n = \sum_{i < j} G_{i,j} \preceq \sum_{i < j} (j - i) P_n.$$

To finish the proof, we compute

$$\sum_{i < j} (j - i) \leq \sum_{i < j} n \leq n^3$$

So

$$L_{K_n} \preceq n^3 \cdot L_{P_n}.$$

Plugging this into Equation (4.6) we obtain

$$\frac{1}{n^2} \leq \lambda_2(P_n).$$

This only differs from our test vector-based upper bound in Equation (4.3) by a factor 12.

We could make this considerably tighter by being more careful about the sums.

## 4.5 The Complete Binary Tree

To consolidate our new expertise, let us finally consider the complete binary tree  $T_n$  with unit weight edges on  $n$  vertices. Note that in order to have  $T_n$  be a complete binary tree, we require that  $n$  is such that there is an integer  $d$  for which  $n = 2^{d+1} - 1$ .

$T_n$  is the balanced binary tree on this many vertices, i.e. it consists of a root node, which has two children, each of those children have two children and so on until we reach a depth of  $d$  from the root, at which point the child vertices have no more children. A simple induction shows that indeed  $n = 2^{d+1} - 1$ .

We can also describe the edge set by saying that each node  $i$  has edges to its children  $2i$  and  $2i + 1$  whenever the node labels do not exceed  $n$ . We emphasize that we still think of the graph as undirected.

### 4.5.1 Deriving Bounds on $\lambda_n(T_n)$

We'll start by above bounding  $\lambda_n(\mathbf{L}_{T_n})$  using a test vector.

We let  $\mathbf{x}(i) = 0$  for all nodes that have a child node, and  $\mathbf{x}(i) = -1$  for even-numbered leaf nodes and  $\mathbf{x}(i) = +1$  for odd-numbered leaf nodes. Note that there are  $(n+1)/2$  leaf nodes, and every leaf node has a single edge, connecting it to a parent with value 0. Thus

$$\lambda_n(\mathbf{L}) = \max_{\mathbf{v} \neq \mathbf{0}} \frac{\mathbf{v}^\top \mathbf{L} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}} \geq \frac{\mathbf{x}^\top \mathbf{L} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{(n+1)/2}{(n+1)/2} = 1. \quad (4.7)$$

Meanwhile, every vertex has degree at most 3, so by Claim 4.4.3,  $\lambda_n(\mathbf{L}) \leq 6$ . So we can bound the largest eigenvalue above and below by a constant.

### 4.5.2 Upper Bounding $\lambda_2$ via Test Vectors

Let us give an upper bound on  $\lambda_2$  of the tree using a test vector. Let  $\mathbf{x} \in \mathbb{R}^v$  have  $\mathbf{x}(1) = 0$  and  $\mathbf{x}(i) = -1$  for  $i$  in the left subtree and  $\mathbf{x}(i) = +1$  in the right subtree. Then

$$\lambda_2(\mathbf{L}_{T_n}) = \min_{\substack{\mathbf{v} \neq \mathbf{0} \\ \mathbf{v}^\top \mathbf{1} = 0}} \frac{\mathbf{v}^\top \mathbf{L} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}} \leq \frac{\mathbf{x}^\top \mathbf{L} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{2}{n-1}.$$

So, we have shown  $\frac{1}{2n \log_2(n)} \leq \lambda_2(\mathbf{L}_{T_n}) \leq \frac{2}{n-1}$ , and unlike the previous examples, the gap is more than a constant.

### 4.5.3 Lower Bounding $\lambda_2$

The following lemma gives a simple lower bound on  $\lambda_2$  for any graph.

**Lemma 4.5.1.** *For any unit weight graph  $G$  with diameter  $D$ ,*

$$\lambda_2(\mathbf{L}_G) \geq \frac{1}{nD}.$$

*Proof.* We will again prove a lower bound comparing  $G$  to the complete graph. For each edge  $(i, j) \in K_n$ , let  $G^{i,j}$  denote a shortest path in  $G$  from  $i$  to  $j$ . This path will have length at most  $D$ . So, we have

$$\begin{aligned} K_n &= \sum_{i < j} G_{i,j} \\ &\preceq \sum_{i < j} D G^{i,j} \\ &\preceq \sum_{i < j} D G \\ &\preceq n^2 D G. \end{aligned}$$

So, we obtain the bound

$$n^2 D \lambda_2(G) \geq n,$$

which implies our desired statement. □



Since the complete binary tree  $T_n$  has diameter  $2d \leq 2\log_2(n)$ , by Lemma 4.5.1,  $\lambda_2(\mathbf{L}_{T_n}) \geq \frac{1}{2n\log_2(n)}$ .

In the exercises for Week 3, I will ask you to improve the lower bound to  $1/(cn)$  for some constant  $c$ .

## Chapter 5

# Conductance, Expanders and Cheeger's Inequality

A common algorithmic problem that arises is the problem of partitioning the vertex set  $V$  of a graph  $G$  into clusters  $X_1, X_2, \dots, X_k$  such that

- for each  $i$ , the *induced* graph  $G[X_i] = (X_i, E \cap (X_i \times X_i))$  is "well-connected", and
- only an  $\epsilon$ -fraction of edges  $e$  are not contained in any induced graph  $G[X_i]$  (where  $\epsilon$  is a very small constant).

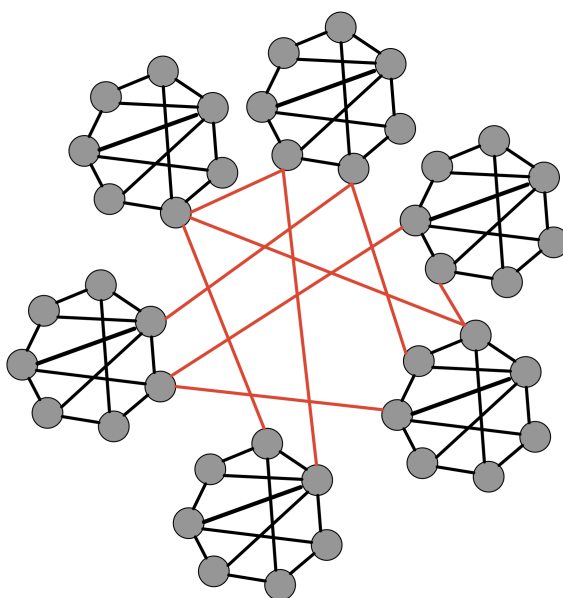


Figure 5.1: After removing the red edges (of which there are few in relation to the total number of edges), each connected component in  $G$  is "well-connected".

In this lecture, we make precise what "well-connected" means by introducing the notions of *conductance* and *expanders*.

Building on the chapter, we show that the second eigenvalue of the Laplacian  $L$  associated with graph  $G$  can be used to certify that a graph is "well-connected" (more precisely the second eigenvalue of a normalized version of the Laplacian). This result, called Cheeger's inequality, is one of the key tools in Spectral Graph Theory. Moreover, it can be turned into an algorithm that computes the partition efficiently!

## 5.1 Conductance and Expanders

**Graph Definitions.** In this lecture, we let  $G = (V, E)$  be unweighted<sup>1</sup> and always be *connected*, and let  $\mathbf{d}(v)$  be the degree of a vertex  $v$  in  $G$ . We define the *volume*  $\text{vol}(S)$  for any vertex subset  $S \subseteq V$ , to be the sum of degrees, i.e.  $\text{vol}(S) = \sum_{v \in S} \mathbf{d}(v)$ .

For any  $A, B \subseteq V$ , we define  $E(A, B)$  to be the set of edges in  $E \cap (A \times B)$ , i.e. with one endpoint in  $A$  and one endpoint in  $B$ . We let  $G[A]$  be the *induced* graph  $G$  by  $A \subseteq V$ , which is the graph  $G$  restricted to the vertices  $A$ , i.e. an edge  $e$  in  $G$  is in  $G[A]$  iff both endpoints are in  $A$ .

**Conductance.** Given set  $\emptyset \subset S \subset V$ , then we define the conductance  $\phi(S)$  of  $S$  by

$$\phi(S) = \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}}.$$

It can be seen that  $\phi(\cdot)$  is symmetric in the sense that  $\phi(S) = \phi(V \setminus S)$ . We define the conductance of the graph  $G$  denoted  $\phi(G)$  by

$$\phi(G) = \min_{\emptyset \subset S \subset V} \phi(S).$$

We note that finding the conductance of a graph  $G$  is NP-hard. However, good approximations can be found as we will see today (and in a later lecture).

**Expander and Expander Decomposition.** For any  $\phi \in (0, 1]$ , we say that a graph  $G$  is a  $\phi$ -*expander* if  $\phi(G) \geq \phi$ . We say that the partition  $X_1, X_2, \dots, X_k$  of the vertex set  $V$  is a  $\phi$ -*expander decomposition* of quality  $q$  if

- each induced graph  $G[X_i]$  is a  $\phi$ -expander, and
- the number of edges not contained in any  $G[X_i]$  is at most  $q \cdot \phi \cdot m$ .

---

<sup>1</sup>Everything we present here also works for weighted graphs, however, we focus on unweighted graphs for simplicity.

Today, we obtain a  $\phi$ -expander decomposition of quality  $q = O(\phi^{-1/2} \cdot \log n)$ . In a few lectures, we revisit the problem and obtain quality  $q = O(\log^c n)$  for some small constant  $c$ . In practice, we mostly care about values  $\phi \approx 1$ .

**An Algorithm to Compute Conductance and Expander Decomposition.** In this lecture, the main focus is *not* to obtain an algorithm to compute conductance but rather only to show that the conductance of a graph can be approximated using the eigenvalues of the "normalized" Laplacian.

However, this proof gives then rise to an algorithm  $\text{CERTIFYORCUT}(G, \phi)$  that given a graph  $G$  and a parameter  $\phi$  either:

- *Certifies* that  $G$  is a  $\phi$ -expander, or
- Presents a *cut*  $S$  such that  $\phi(S) \leq \sqrt{2\phi}$ .

Careful but rather straight-forward recursive application of this procedure then yields an algorithm to compute a  $\phi$ -expander decomposition.

To implement this algorithm above, we are required to find the second eigenvalue of the 'normalized' Laplacian  $\mathbf{N}$  that we introduce in this lecture. We can find the eigenvalues using the power method<sup>2</sup> which multiplies a random vector with the matrix multiple times to find the largest eigenvalue. In our case, we want to use the inverse of  $\mathbf{N}$  and pick a random vector orthogonal to the trivial first eigenvector which yields the 2nd eigenvalue of  $\mathbf{N}$ . The inverse-matrix vector multiplications can be done extremely fast given a solver for Laplacian systems  $\mathbf{L}\mathbf{x} = \mathbf{d}$  which we learn about in the next chapters.

## 5.2 A Lower Bound for Conductance via Eigenvalues

**An Alternative Characterization of Conductance.** Let us now take a closer look at the definition of conductance and observe that if a set  $S$  has  $\text{vol}(S) \leq \text{vol}(V)/2$  then

$$\phi(S) = \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}} = \frac{|E(S, V \setminus S)|}{\text{vol}(S)} = \frac{\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S}{\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S}.$$

To see this, observe that we can rewrite the numerator above using the Laplacian of  $G$  as

$$|E(S, V \setminus S)| = \sum_{(u,v) \in E} (\mathbf{1}_S(u) - \mathbf{1}_S(v))^2 = \mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S$$

where  $\mathbf{1}_S$  is the characteristic vector of  $S$ . Further, we can rewrite the denominator as

$$\text{vol}(S) = \mathbf{1}_S^\top \mathbf{d} = \mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S$$

---

<sup>2</sup>See [https://en.wikipedia.org/wiki/Power\\_iteration](https://en.wikipedia.org/wiki/Power_iteration).

where  $\mathbf{D} = \text{diag}(\mathbf{d})$  is the degree-matrix. We can now alternatively define the graph conductance of  $G$  by

$$\phi(G) = \min_{\substack{\emptyset \subset S \subset V, \\ \text{vol}(S) \leq \text{vol}(V)/2}} \frac{\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S}{\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S} \quad (5.1)$$

where we use that  $\phi(S) = \phi(V \setminus S)$  such that the objective value is unchanged as long as for each set  $\emptyset \subset S \subset V$  either  $S$  or  $V \setminus S$  is in the set that we minimize over.

**The Normalized Laplacian.** Let us next define the *normalized* Laplacian

$$\mathbf{N} = \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2}.$$

To learn a bit about this new matrix, let us first look at the first eigenvalue where we use the test vector  $\mathbf{y} = \mathbf{D}^{1/2} \mathbf{1}$ , to get by Courant-Fischer (see Theorem 3.1.4) that

$$\lambda_1(\mathbf{N}) = \min_{\mathbf{x} \neq \mathbf{0}} \frac{\mathbf{x}^\top \mathbf{N} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \leq \frac{\mathbf{y}^\top \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2} \mathbf{y}}{\mathbf{y}^\top \mathbf{y}} = \frac{\mathbf{1}^\top \mathbf{L} \mathbf{1}}{\mathbf{y}^\top \mathbf{y}} = 0 \quad (5.2)$$

because  $\mathbf{D}^{-1/2} \mathbf{D}^{1/2} = \mathbf{I}$  and  $\mathbf{L} \mathbf{1} = \mathbf{0}$  (for the former we use the assumption that  $G$  is connected). Since  $\mathbf{N}$  is PSD (as you will show in the exercises), we also know  $\lambda_1(\mathbf{N}) \geq 0$ , so  $\lambda_1(\mathbf{N}) = 0$ .

Let us use Courant-Fischer again to reason a bit about the second eigenvalue of  $\mathbf{N}$ :

$$\lambda_2(\mathbf{N}) = \min_{\substack{\mathbf{x} \perp \mathbf{D}^{1/2} \mathbf{1} \\ \mathbf{x} \neq \mathbf{0}}} \frac{\mathbf{x}^\top \mathbf{N} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \min_{\substack{\mathbf{z} \perp \mathbf{d} \\ \mathbf{z} \neq \mathbf{0}}} \frac{\mathbf{z}^\top \mathbf{D}^{1/2} \mathbf{N} \mathbf{D}^{1/2} \mathbf{z}}{\mathbf{z}^\top \mathbf{D}^{1/2} \mathbf{D}^{1/2} \mathbf{z}} = \min_{\substack{\mathbf{z} \perp \mathbf{d} \\ \mathbf{z} \neq \mathbf{0}}} \frac{\mathbf{z}^\top \mathbf{L} \mathbf{z}}{\mathbf{z}^\top \mathbf{D} \mathbf{z}}. \quad (5.3)$$

**Relating Conductance to the Normalized Laplacian.** At this point, it might become clearer why  $\mathbf{N}$  is a natural matrix to consider when arguing about conductance: if we could argue that for every  $\emptyset \subset S \subset V, \text{vol}(S) \leq \text{vol}(V)/2$ , we have  $\mathbf{1}_S \perp \mathbf{d}$ , then it would be easy to see that taking the second eigenvalue of  $\mathbf{N}$  in equation 5.3 is a relaxation of the minimization problem 5.1 defining  $\phi(G)$ .

While this is clearly not true, we can still argue along these lines.

**Theorem 5.2.1** (Cheeger's Inequality, Lower Bound). *We have  $\frac{\lambda_2(\mathbf{N})}{2} \leq \phi(G)$ .*

*Proof.* Instead of using  $\mathbf{1}_S$  directly, we shift  $\mathbf{1}_S$  by  $\alpha \mathbf{1}$  such that it is orthogonal to  $\mathbf{d}$ : we define  $\mathbf{z}_S = \mathbf{1}_S - \alpha \mathbf{1}$  where  $\alpha$  is the scalar that solves

$$\begin{aligned} 0 &= \mathbf{d}^\top \mathbf{z}_S \\ \iff 0 &= \mathbf{d}^\top (\mathbf{1}_S - \alpha \mathbf{1}) \\ \iff 0 &= \mathbf{d}^\top \mathbf{1}_S - \alpha \mathbf{d}^\top \mathbf{1} \\ \iff \alpha &= \frac{\mathbf{d}^\top \mathbf{1}_S}{\mathbf{d}^\top \mathbf{1}} = \frac{\text{vol}(S)}{\text{vol}(V)}. \end{aligned}$$

To conclude the proof, it remains to argue that  $\frac{\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S}{\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S} \geq \frac{1}{2} \cdot \frac{\mathbf{z}_S^\top \mathbf{L} \mathbf{z}_S}{\mathbf{z}_S^\top \mathbf{D} \mathbf{z}_S}$ :

- Numerator: since  $\mathbf{1}^\top \mathbf{L} \mathbf{1} = 0$ , we have that  $\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S = \mathbf{z}_S^\top \mathbf{L} \mathbf{z}_S$ .
- Denominator: observe by straight-forward calculations that

$$\begin{aligned} \mathbf{z}_S^\top \mathbf{D} \mathbf{z}_S &= \text{vol}(S) \cdot (1 - \alpha)^2 + \text{vol}(V \setminus S) \cdot (-\alpha)^2 \\ &= \text{vol}(S) - 2 \text{vol}(S) \cdot \alpha + \text{vol}(V) \cdot \alpha^2 \\ &= \text{vol}(S) - \frac{\text{vol}(S)^2}{\text{vol}(V)} \\ &= \text{vol}(S) - \text{vol}(S) \cdot \frac{\text{vol}(S)}{\text{vol}(V)} \\ &\geq \frac{1}{2} \text{vol}(S) = \frac{1}{2} \mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S \end{aligned}$$

where we use the assumption that  $\text{vol}(S) \leq \text{vol}(V)/2$ .

□

### 5.3 An Upper Bound for Conductance via Eigenvalues

Slightly more surprisingly, we can also show that the second eigenvalue  $\lambda_2(\mathbf{N})$  can be used to upper bound the conductance.

**Theorem 5.3.1** (Cheeger's Inequality, Upper Bound). *We have  $\phi(G) \leq \sqrt{2 \cdot \lambda_2(\mathbf{N})}$ .*

*Proof.* To prove the theorem, we want to show that for *any*  $\mathbf{z} \perp \mathbf{d}$ , we can find a set  $\emptyset \subset S \subset V$  with  $\text{vol}(S) \leq \text{vol}(V)/2$ , such that

$$\frac{\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S}{\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S} \leq \sqrt{2 \cdot \frac{\mathbf{z}^\top \mathbf{L} \mathbf{z}}{\mathbf{z}^\top \mathbf{D} \mathbf{z}}}. \quad (5.4)$$

As a first step, we would like to change  $\mathbf{z}$  slightly to make it more convenient to work with:

- we *renumber* the vertices in  $V$  such that we have

$$\mathbf{z}(1) \leq \mathbf{z}(2) \leq \dots \leq \mathbf{z}(n).$$

- we *center*  $\mathbf{z}$ , that is we let  $\mathbf{z}_c = \mathbf{z} - \alpha \mathbf{1}$  where  $\alpha$  is chosen such that

$$\sum_{\mathbf{z}_c(i) < 0} \mathbf{d}(i) < \text{vol}(V)/2 \text{ and } \sum_{\mathbf{z}_c(i) \leq 0} \mathbf{d}(i) \geq \text{vol}(V)/2$$

i.e.  $\sum_{\mathbf{z}_c(i) > 0} \mathbf{d}(i) \leq \text{vol}(V)/2$ .

- we *scale*, let  $\mathbf{z}_{sc} = \beta \mathbf{z}_c$  for some scalar  $\beta$  such that  $\mathbf{z}_{sc}(1)^2 + \mathbf{z}_{sc}(n)^2 = 1$ .

In the exercises, you will show that changing  $\mathbf{z}$  to  $\mathbf{z}_{sc}$  can only make the ratio we are interested in smaller, i.e. that  $\frac{\mathbf{z}^\top \mathbf{L} \mathbf{z}}{\mathbf{z}^\top \mathbf{D} \mathbf{z}} \geq \frac{\mathbf{z}_{sc}^\top \mathbf{L} \mathbf{z}_{sc}}{\mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc}}$ . Thus, if we can show that equation 5.4 holds for  $\mathbf{z}_{sc}$  in place of  $\mathbf{z}$ , then it also follows for  $\mathbf{z}$  itself.

We now arrive at the main idea of the proof: we define the set  $S_\tau = \{i \in V \mid \mathbf{z}_{sc}(i) < \tau\}$  for some random variable  $\tau$  with probability density function

$$p(t) = \begin{cases} 2 \cdot |t| & t \in [\mathbf{z}_{sc}(1), \mathbf{z}_{sc}(n)], \\ 0 & \text{otherwise.} \end{cases} \quad (5.5)$$

So, we have probability  $\mathbb{P}[\alpha < \tau < \beta] = \int_{t=\alpha}^{\beta} p(t) dt$ .

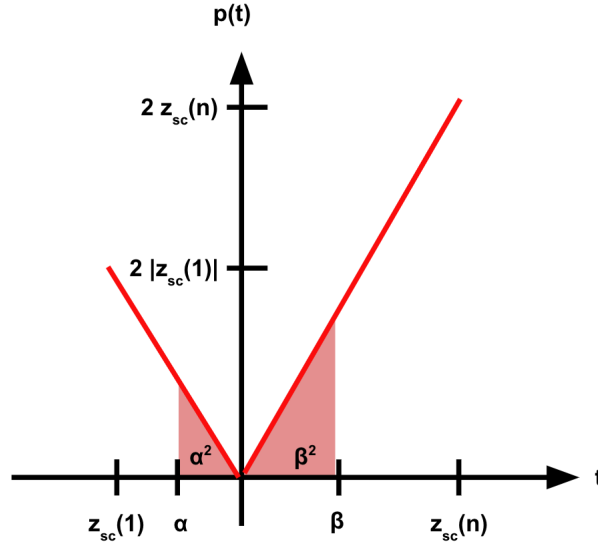


Figure 5.2: The probability density function  $p(t)$ . The area between  $\alpha$  and  $\beta$  for  $\alpha < 0 < \beta$  is  $\alpha^2 + \beta^2$ . This is the probability of the event  $\mathbb{P}[\alpha < \tau < \beta]$ . Note in particular that the total area is  $\mathbf{z}_{sc}(1)^2 + \mathbf{z}_{sc}(n)^2$  which is equal to 1 by assumption and thus  $p(t)$  is a valid pdf.

Since the volume incident to  $S_\tau$  might be quite large, let us define  $S$  for convenience by

$$S = \begin{cases} S_\tau & \text{vol}(S_\tau) < \text{vol}(V)/2, \\ V \setminus S_\tau & \text{otherwise.} \end{cases}$$

**Claim 5.3.2.** We have  $\frac{\mathbb{E}_\tau[\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S]}{\mathbb{E}_\tau[\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S]} \leq \sqrt{2 \cdot \frac{\mathbf{z}_{sc}^\top \mathbf{L} \mathbf{z}_{sc}}{\mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc}}}$ .

*Proof.* Recall  $\mathbf{1}_S^\top \mathbf{L} \mathbf{1}_S = E(S_\tau, V \setminus S_\tau)$ , and by choice of  $\tau$ , we have for any edge  $e = \{i, j\} \in E$

where  $\mathbf{z}_{sc}(i) \leq \mathbf{z}_{sc}(j)$ ,

$$\begin{aligned}\mathbb{P}_\tau[e \in E(S_\tau, V \setminus S_\tau)] &= \mathbb{P}_\tau[\mathbf{z}_{sc}(i) < \tau \leq \mathbf{z}_{sc}(j)] \\ &= \int_{t=i}^j 2|t| \, dt \\ &= \begin{cases} \mathbf{z}_{sc}(i)^2 + \mathbf{z}_{sc}(j)^2 & \mathbf{z}(i) < 0 < \mathbf{z}(j), \\ |\mathbf{z}_{sc}(i)^2 - \mathbf{z}_{sc}(j)^2| & \text{otherwise.} \end{cases}\end{aligned}$$

The last equality can be seen from Figure 5.2. We can further upper bound either case by  $|\mathbf{z}_{sc}(i) - \mathbf{z}_{sc}(j)| \cdot (|\mathbf{z}_{sc}(i)| + |\mathbf{z}_{sc}(j)|)$  (we leave this as an exercise).

Using our new upper bound, we can sum over all edges  $e \in E$  to conclude that

$$\begin{aligned}\mathbb{E}_\tau[|E(S_\tau, V \setminus S_\tau)|] &\leq \sum_{i \sim j} |\mathbf{z}_{sc}(i) - \mathbf{z}_{sc}(j)| \cdot (|\mathbf{z}_{sc}(i)| + |\mathbf{z}_{sc}(j)|) \\ &\leq \sqrt{\sum_{i \sim j} (\mathbf{z}_{sc}(i) - \mathbf{z}_{sc}(j))^2 \cdot \sum_{i \sim j} (|\mathbf{z}_{sc}(i)| + |\mathbf{z}_{sc}(j)|)^2}\end{aligned}$$

where the last line follows from  $\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle$  (i.e. Cauchy-Schwarz).

The first sum should look familiar by now: it is simply the Quadratic Laplacian Form  $\sum_{i \sim j} (\mathbf{z}_{sc}(i) - \mathbf{z}_{sc}(j))^2 = \mathbf{z}_{sc}^\top \mathbf{L} \mathbf{z}_{sc}$ .

It is not hard to reason about the second term either

$$\sum_{i \sim j} (|\mathbf{z}_{sc}(i)| + |\mathbf{z}_{sc}(j)|)^2 \leq 2 \sum_{i \sim j} \mathbf{z}_{sc}(i)^2 + \mathbf{z}_{sc}(j)^2 = 2 \sum_{i \in V} \mathbf{d}(i) \mathbf{z}_{sc}(i)^2 = 2 \mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc}.$$

Putting everything together, we obtain

$$\mathbb{E}_\tau[|E(S_\tau, V \setminus S_\tau)|] \leq \sqrt{\mathbf{z}_{sc}^\top \mathbf{L} \mathbf{z}_{sc} \cdot 2 \mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc}} = \sqrt{2 \cdot \frac{\mathbf{z}_{sc}^\top \mathbf{L} \mathbf{z}_{sc}}{\mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc}}} \cdot \mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc} \quad (5.6)$$

While this almost looks like what we want, we still have to argue that  $\mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc} = \mathbb{E}_\tau[\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S]$  to finish the proof.

To this end, when unrolling the expectation, we use a simple trick that splits by cases:

$$\begin{aligned}\mathbb{E}_\tau[\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S] &= \sum_{i \in V} \mathbf{d}(i) \cdot \mathbb{P}[i \in S] \\ &= \sum_{i \in V, \mathbf{z}_{sc}(i) < 0} \mathbf{d}(i) \cdot \mathbb{P}[i \in S \wedge S = S_\tau] + \sum_{i \in V, \mathbf{z}_{sc}(i) \geq 0} \mathbf{d}(i) \cdot \mathbb{P}[i \in S \wedge S \neq S_\tau] \\ &= \sum_{i \in V, \mathbf{z}_{sc}(i) < 0} \mathbf{d}(i) \cdot \mathbb{P}[\mathbf{z}_{sc}(i) < \tau \wedge \tau < 0] + \sum_{i \in V, \mathbf{z}_{sc}(i) \geq 0} \mathbf{d}(i) \cdot \mathbb{P}[\mathbf{z}_{sc}(i) \geq \tau \wedge \tau \geq 0]\end{aligned}$$



where we use the centering of  $\mathbf{z}_{sc}$  the definition of  $S$  and that the event  $\{i \in S \wedge S = S_\tau\}$  can be rewritten as the event  $\{i < \tau \wedge \tau < 0\}$  (the other case is analogous).

Let  $i$  be a vertex with  $\mathbf{z}_{sc}(i) < 0$ , then the probability  $\mathbb{P}[i \in S \wedge S = S_\tau]$  is exactly  $\mathbf{z}_{sc}(i)^2$  by choice of the density function of  $\tau$  (again the case for  $i$  with  $\mathbf{z}_{sc}(i)$  non-negative is analogous). Thus, summing over all vertices, we obtain

$$\begin{aligned}\mathbb{E}_\tau[\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S] &= \sum_{i \in V, \mathbf{z}_{sc}(i) < 0} \mathbf{d}(i) \cdot \mathbb{P}[\mathbf{z}_{sc}(i) < \tau \wedge \tau < 0] + \sum_{i \in V, \mathbf{z}_{sc}(i) \geq 0} \mathbb{P}[\mathbf{z}_{sc}(i) \geq \tau \wedge \tau \geq 0] \\ &= \sum_{i \in V} \mathbf{d}(i) \cdot \mathbf{z}_{sc}(i)^2 = \mathbf{z}_{sc}^\top \mathbf{D} \mathbf{z}_{sc}.\end{aligned}$$

Therefore, we can plug in our result directly into Equation 5.6 and the proof is completed by dividing both sides by  $\mathbb{E}_\tau[\mathbf{1}_S^\top \mathbf{D} \mathbf{1}_S]$ .  $\square$

While Theorem 5.3.2 only ensures our claim in expectation, this is already sufficient to conclude that there exists some set  $S$  that satisfies the same guarantees deterministically, as you will prove in Problem Set 4. This is often called the *probabilistic method of expectation* and can be seen from the definition of expectation. We have thus proven the upper bound of Cheeger's inequality.  $\square$

## 5.4 Conclusion

Today, we have introduced the concepts of conductance and formalized expanders and expander decompositions. These are crucial concepts that you will encounter often in literature and also again in this course. They are a key tool in many recent breakthroughs in Theoretical Computer Science.

In the second part of the lecture (the main part), we discussed Cheeger's inequality which allows to relate the second eigenvalue of the normalized Laplacian to a graphs conductance. We summarize the full statement here.

**Theorem 5.4.1** (Cheeger's Inequality). *We have  $\frac{\lambda_2(\mathbf{N})}{2} \leq \phi(G) \leq \sqrt{2 \cdot \lambda_2(\mathbf{N})}$ .*

We point out that this Theorem is tight as you will show in the exercises. The proof for Cheeger's inequality is probably the most advanced proof, we have seen so far in the course. The many tricks that make the proof work might sometimes seem a bit magical but it is important to remember that they are a result of many people polishing this proof over and over. The proof techniques used are extremely useful and can be re-used in various contexts. We therefore strongly encourage you to really understand the proof yourself!

# Chapter 6

## Pseudo-inverses and Effective Resistance

### 6.1 What is a (Moore-Penrose) Pseudoinverse?

Recall that for a connected graph  $G$  with Laplacian  $\mathbf{L}$ , we have  $\ker(\mathbf{L}) = \text{span}\{\mathbf{1}\}$ , which means  $\mathbf{L}$  is not invertible. However, we still want some matrix which behaves like a real inverse. To be more specific, given a Laplacian  $\mathbf{L} \in \mathbb{R}^{V \times V}$ , we want some matrix  $\mathbf{L}^+ \in \mathbb{R}^{V \times V}$  s.t.

- 1)  $(\mathbf{L}^+)^\top = \mathbf{L}^+$  (symmetric)
- 2)  $\mathbf{L}^+ \mathbf{1} = \mathbf{0}$ , or more generally,  $\mathbf{L}^+ \mathbf{v} = \mathbf{0}$  for  $\mathbf{v} \in \ker(\mathbf{L})$
- 3)  $\mathbf{L}^+ \mathbf{L} \mathbf{v} = \mathbf{L} \mathbf{L}^+ \mathbf{v} = \mathbf{v}$  for  $\mathbf{v} \perp \mathbf{1}$ , or more generally, for  $\mathbf{v} \in \ker(\mathbf{L})^\perp$

Under the above conditions,  $\mathbf{L}^+$  is uniquely defined and we call it the pseudoinverse of  $\mathbf{L}$ . Note that there are many other equivalent definitions of the pseudoinverse of some matrix  $\mathbf{A}$ , and we can also generalize the concept to matrices that aren't symmetric or even square.

Let  $\lambda_i, \mathbf{v}_i$  be the  $i$ -th pair of eigenvalue and eigenvector of  $\mathbf{L}$ , with  $\{\mathbf{v}_i\}_{i=1}^n$  forming a orthogonal basis. Then by the spectral theorem,

$$\mathbf{L} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^\top,$$

where  $\mathbf{V} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$  and  $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ . And we can show that its pseudoinverse is exactly

$$\mathbf{L}^+ = \sum_{i, \lambda_i \neq 0} \lambda_i^{-1} \mathbf{v}_i \mathbf{v}_i^\top.$$

Checking conditions 1), 2), 3) is immediate. We can also prove uniqueness, but this takes slightly more work.

## 6.2 Electrical Flows Again

Recall the incidence matrix  $\mathbf{B} \in \mathbb{R}^{V \times E}$  of a graph  $G = (V, E)$ .

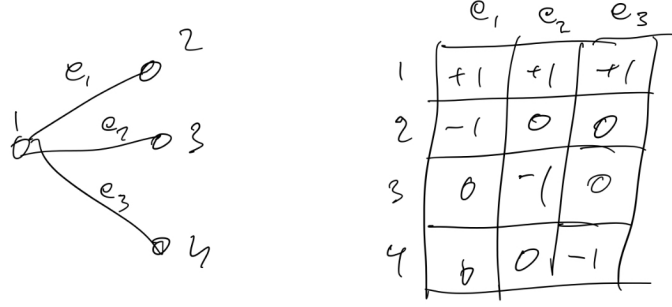


Figure 6.1: An example of a graph and its incidence matrix  $\mathbf{B}$ .

In Chapter 1, we introduced the electrical flow routing demand  $\mathbf{d} \in \mathbb{R}^V$ . Let's call the electrical flow  $\tilde{\mathbf{f}} \in \mathbb{R}^E$ . The net flow constraint requires  $\mathbf{B}\tilde{\mathbf{f}} = \mathbf{d}$ . By Ohm's Law,  $\tilde{\mathbf{f}} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{x}$  for some voltage  $\mathbf{x} \in \mathbb{R}^V$  where  $\mathbf{R} = \text{diag}(\mathbf{r})$  and  $\mathbf{r}(e)$  = resistance of edge  $e$ . We showed (in the exercises) that when  $\mathbf{d} \perp \mathbf{1}$ , there exists a voltage  $\tilde{\mathbf{x}} \perp \mathbf{1}$  s.t.  $\tilde{\mathbf{f}} = \mathbf{R}^{-1}\mathbf{B}^\top \tilde{\mathbf{x}}$  and  $\mathbf{B}\tilde{\mathbf{f}} = \mathbf{d}$ . This  $\tilde{\mathbf{x}}$  solves  $\mathbf{L}\mathbf{x} = \mathbf{d}$  where  $\mathbf{L} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top$ .

And we also made the following claim.

**Claim 6.2.1.**

$$\tilde{\mathbf{f}} = \arg \min_{\mathbf{B}\mathbf{f}=\mathbf{d}} \mathbf{f}^\top \mathbf{R}\mathbf{f} \text{ where } \mathbf{f}^\top \mathbf{R}\mathbf{f} = \sum_e \mathbf{r}(e)\mathbf{f}(e)^2, \quad (6.1)$$

You proved this in the exercises for Week 1. Let's recap the proof briefly, just to get back into thinking about electrical flows.

*Proof.* Consider any  $\mathbf{f} \in \mathbb{R}^E$  s.t.  $\mathbf{B}\mathbf{f} = \mathbf{d}$ . For any  $\mathbf{x} \in \mathbb{R}^V$ , we have

$$\begin{aligned} \frac{1}{2}\mathbf{f}^\top \mathbf{R}\mathbf{f} &= \frac{1}{2}\mathbf{f}^\top \mathbf{R}\mathbf{f} - \underbrace{\mathbf{x}^\top (\mathbf{B}\mathbf{f} - \mathbf{d})}_0 \\ &\geq \min_{\mathbf{f} \in \mathbb{R}^E} \underbrace{\frac{1}{2}\mathbf{f}^\top \mathbf{R}\mathbf{f} - \mathbf{x}^\top \mathbf{B}\mathbf{f} + \mathbf{d}^\top \mathbf{x}}_{g(\mathbf{f})} \\ &= \mathbf{d}^\top \mathbf{x} - \frac{1}{2}\mathbf{x}^\top \mathbf{L}\mathbf{x} \end{aligned}$$

since  $\nabla_{\mathbf{f}} g(\mathbf{f}) = \mathbf{0}$  gives us  $\mathbf{f} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{x}$ . Thus, for all  $\mathbf{f} \in \mathbb{R}^E$  s.t.  $\mathbf{B}\mathbf{f} = \mathbf{d}$  and all  $\mathbf{x} \in \mathbb{R}^V$ ,

$$\frac{1}{2}\mathbf{f}^\top \mathbf{R}\mathbf{f} \geq \mathbf{d}^\top \mathbf{x} - \frac{1}{2}\mathbf{x}^\top \mathbf{L}\mathbf{x}. \quad (6.2)$$

But for the electrical flow  $\tilde{\mathbf{f}}$  and electrical voltage  $\tilde{\mathbf{x}}$ , we have  $\tilde{\mathbf{f}} = \mathbf{R}^{-1}\mathbf{B}^\top \tilde{\mathbf{x}}$  and  $\mathbf{L}\tilde{\mathbf{x}} = \mathbf{d}$ . So

$$\tilde{\mathbf{f}}^\top \mathbf{R} \tilde{\mathbf{f}} = (\mathbf{R}^{-1}\mathbf{B}^\top \tilde{\mathbf{x}})^\top \mathbf{R} (\mathbf{R}^{-1}\mathbf{B}^\top \tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^\top \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top \tilde{\mathbf{x}} = \tilde{\mathbf{x}}^\top \mathbf{L} \tilde{\mathbf{x}} = \tilde{\mathbf{x}}^\top \mathbf{d}.$$

Therefore,

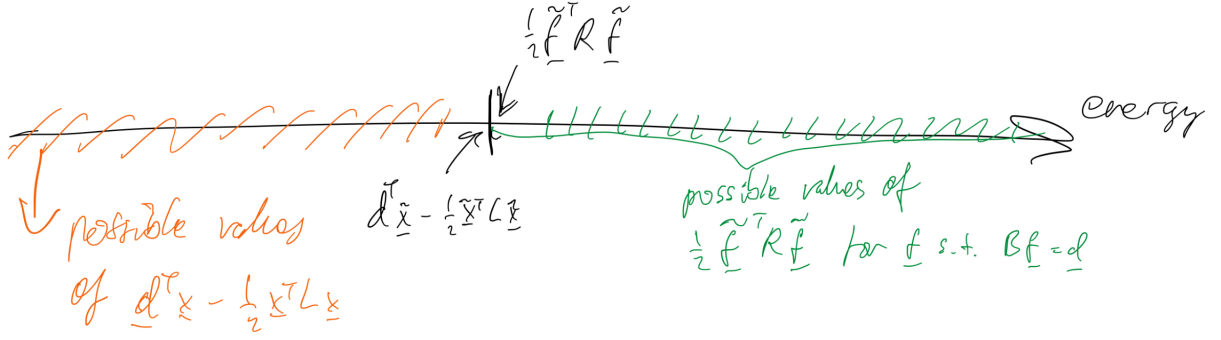
$$\frac{1}{2} \tilde{\mathbf{f}}^\top \mathbf{R} \tilde{\mathbf{f}} = \mathbf{d}^\top \tilde{\mathbf{x}} - \frac{1}{2} \tilde{\mathbf{x}}^\top \mathbf{L} \tilde{\mathbf{x}}. \quad (6.3)$$

By combining Equation (6.2) and Equation (6.3), we see that for all  $\mathbf{f}$  s.t.  $\mathbf{B}\mathbf{f} = \mathbf{d}$ ,

$$\frac{1}{2} \mathbf{f}^\top \mathbf{R} \mathbf{f} \geq \mathbf{d}^\top \tilde{\mathbf{x}} - \frac{1}{2} \tilde{\mathbf{x}}^\top \mathbf{L} \tilde{\mathbf{x}} = \frac{1}{2} \tilde{\mathbf{f}}^\top \mathbf{R} \tilde{\mathbf{f}}.$$

Thus  $\tilde{\mathbf{f}}$  is the minimum electrical energy flow among all flows that route demand  $\mathbf{d}$ , proving Equation (6.1) holds.

The drawing below shows how the quantities line up:



□

## 6.3 Effective Resistance

Given a graph  $G = (V, E)$ , for any pair of vertices  $(a, b) \in V$ , we want to compute the cost (or energy) of routing 1 unit of current from  $a$  to  $b$ . We call such cost the effective resistance between  $a$  and  $b$ , denoted by  $R_{\text{eff}}(a, b)$ . Recall for a single resistor  $r(a, b)$ ,

$$\text{energy} = r(a, b) f^2(a, b) = r(a, b).$$

So when we have a graph consisting of just one edge  $(a, b)$ , the effective resistance is just  $R_{\text{eff}}(a, b) = r(a, b)$ .

In a general graph, we can also consider the energy required to route one unit of current between two vertices. For any pair  $a, b \in V$ , we have

$$R_{\text{eff}}(a, b) = \min_{\mathbf{B}\mathbf{f} = \mathbf{e}_b - \mathbf{e}_a} \mathbf{f}^\top \mathbf{R} \mathbf{f},$$

where  $\mathbf{e}_v \in \mathbb{R}^V$  is the indicator vector of  $v$ . Note that the cost of routing  $F$  units of flow from  $a$  to  $b$  will be  $R_{\text{eff}}(a, b) \cdot F^2$ .

Since  $(\mathbf{e}_b - \mathbf{e}_a)^\top \mathbf{1} = 0$ , we know from the previous section that  $R_{\text{eff}}(a, b) = \tilde{\mathbf{f}}^\top \mathbf{R} \tilde{\mathbf{f}}$  where  $\tilde{\mathbf{f}}$  is the electrical flow. Now we can write  $\mathbf{L} \tilde{\mathbf{x}} = \mathbf{e}_b - \mathbf{e}_a$  and  $\tilde{\mathbf{x}} = \mathbf{L}^+(\mathbf{e}_b - \mathbf{e}_a)$  for the electrical voltages routing 1 unit of current from  $a$  to  $b$ . Now the energy of routing 1 unit of current from  $a$  to  $b$  is

$$R_{\text{eff}}(a, b) = \tilde{\mathbf{f}}^\top \mathbf{R} \tilde{\mathbf{f}} = \tilde{\mathbf{x}}^\top \mathbf{L} \tilde{\mathbf{x}} = (\mathbf{e}_b - \mathbf{e}_a)^\top \mathbf{L}^+ \mathbf{L} \mathbf{L}^+ (\mathbf{e}_b - \mathbf{e}_a) = (\mathbf{e}_b - \mathbf{e}_a)^\top \mathbf{L}^+ (\mathbf{e}_b - \mathbf{e}_a),$$

where the last equality is due to  $\mathbf{L}^+ \mathbf{L} \mathbf{L}^+ = \mathbf{L}^+$ .

**Remark 6.3.1.** We have now seen several different expressions that all take on the same value: the energy of the electrical flow. It's useful to remind yourself what these are. Consider an electrical flow  $\tilde{\mathbf{f}}$  routes demand  $\mathbf{d}$ , and associated electrical voltages  $\tilde{\mathbf{x}}$ . We know that  $\mathbf{B} \tilde{\mathbf{f}} = \mathbf{d}$ , and  $\tilde{\mathbf{f}} = \mathbf{R}^{-1} \mathbf{B}^\top \tilde{\mathbf{x}}$ , and  $\mathbf{L} \tilde{\mathbf{x}} = \mathbf{d}$ , where  $\mathbf{L} = \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top$ . And we have seen how to express the electrical energy using many different quantities:

$$\tilde{\mathbf{f}}^\top \mathbf{R} \tilde{\mathbf{f}} = \tilde{\mathbf{x}}^\top \mathbf{L} \tilde{\mathbf{x}} = \mathbf{d}^\top \mathbf{L}^+ \mathbf{d} = \mathbf{d}^\top \tilde{\mathbf{x}} = \tilde{\mathbf{f}}^\top \mathbf{B}^\top \tilde{\mathbf{x}}$$

**Claim 6.3.2.** Any PSD matrix  $\mathbf{A}$  has a PSD square root  $\mathbf{A}^{1/2}$  s.t.  $\mathbf{A}^{1/2} \mathbf{A}^{1/2} = \mathbf{A}$ .

*Proof.* By the spectral theorem,  $\mathbf{A} = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^\top$  where  $\{\mathbf{v}_i\}$  are orthonormal. Let  $\mathbf{A}^{1/2} = \sum_i \lambda_i^{1/2} \mathbf{v}_i \mathbf{v}_i^\top$ . Then

$$\begin{aligned} \mathbf{A}^{1/2} \mathbf{A}^{1/2} &= \left( \sum_i \lambda_i^{1/2} \mathbf{v}_i \mathbf{v}_i^\top \right)^2 \\ &= \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^\top \mathbf{v}_i \mathbf{v}_i^\top + \sum_{i \neq j} \lambda_i \mathbf{v}_i \mathbf{v}_i^\top \mathbf{v}_j \mathbf{v}_j^\top \\ &= \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^\top \end{aligned}$$

where the last equality is due to  $\mathbf{v}_i^\top \mathbf{v}_j = \delta_{ij}$ . It's easy to see that  $\mathbf{A}^{1/2}$  is also PSD.  $\square$

Let  $\mathbf{L}^{+/2}$  be the square root of  $\mathbf{L}^+$ . So

$$R_{\text{eff}}(a, b) = (\mathbf{e}_b - \mathbf{e}_a)^\top \mathbf{L}^+ (\mathbf{e}_b - \mathbf{e}_a) = \|\mathbf{L}^{+/2} (\mathbf{e}_b - \mathbf{e}_a)\|^2.$$

**Example: Effective resistance in a path.** Consider a path graph on vertices  $V = \{1, 2, 3, \dots, k+1\}$ , with resistances  $\mathbf{r}(1), \mathbf{r}(2), \dots, \mathbf{r}(k)$  on the edges of the path.



Figure 6.2: A path graph with  $k$  edges.

The effective resistance between the endpoints is

$$R_{\text{eff}}(1, k+1) = \sum_{i=1}^k r(i)$$

To see this, observe that to have 1 unit of flow going from vertex 1 to vertex  $k+1$ , we must have one unit flowing across each edge  $i$ . Let  $\Delta(i)$  be the voltage difference across edge  $i$ , and  $f(i)$  the flow on the edge. Then  $1 = f(i) = \frac{\Delta(i)}{r(i)}$ , so that  $\Delta(i) = r(i)$ . The electrical voltages are then  $\tilde{\mathbf{x}} \in \mathbb{R}^V$  where  $\tilde{\mathbf{x}}(i) = \tilde{\mathbf{x}}(1) + \sum_{j < i} \Delta(j)$ . Hence the effective resistance is

$$R_{\text{eff}}(1, k+1) = \mathbf{d}^\top \tilde{\mathbf{x}} = (\mathbf{e}_{k+1} - \mathbf{e}_1)^\top \tilde{\mathbf{x}} = \tilde{\mathbf{x}}(k+1) - \tilde{\mathbf{x}}(1) = \sum_{i=1}^k r(i).$$

This behavior is sometimes known as the fact that the resistance of resistors adds up when they are connected in series.

**Example: Effective resistance of parallel edges.** So far, we have only considered graphs with at most one edge between any two vertices. But that math also works if we allow a pair of vertices to have multiple distinct edges connecting them. We refer to this as *multi-edges*. Suppose we have a graph on just two vertices,  $V = \{1, 2\}$ , and these are connected by  $k$  parallel multi-edges with resistances  $r(1), r(2), \dots, r(k)$ .

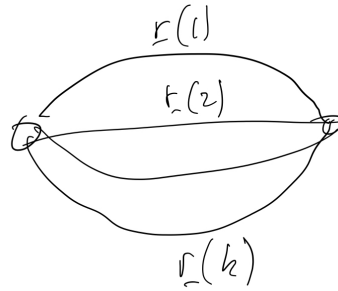


Figure 6.3: A graph on just two vertices with  $k$  parallel multiedges.

The effective resistance between the endpoints is

$$R_{\text{eff}}(1, 2) = \frac{1}{\sum_{i=1}^k 1/r(i)}.$$

Let's see why. Our electrical voltages  $\tilde{\mathbf{x}} \in \mathbb{R}^V$  can be described by just the voltage difference  $\Delta \in \mathbb{R}$  between vertex 1 and vertex 2, i.e.  $\tilde{\mathbf{x}}(2) - \tilde{\mathbf{x}}(1) = \Delta$ . which creates a flow on edge  $i$  of  $\tilde{\mathbf{f}}(i) = \Delta / \mathbf{r}(i)$ . Thus the total flow from vertex 1 to vertex 2 is  $1 = \sum_i \Delta / \mathbf{r}(i)$ , so that  $\Delta = \frac{1}{\sum_{i=1}^k 1/\mathbf{r}(i)}$ . Meanwhile, the effective resistance is also

$$R_{\text{eff}}(1, 2) = (\mathbf{e}_2 - \mathbf{e}_1)^\top \tilde{\mathbf{x}} = \Delta = \frac{1}{\sum_{i=1}^k 1/\mathbf{r}(i)}$$

### 6.3.1 Effective Resistance is a Distance

**Definition 6.3.3.** Consider a weighted undirected graph  $G$  with vertex set  $V$ . We say function  $d : V \times V \rightarrow \mathbb{R}$ , which takes a pair of vertices and returns a real number, is a *distance* if it satisfies

1.  $d(a, a) = 0$  for all  $a \in V$
2.  $d(a, b) \geq 0$  for all  $a, b \in V$ .
3.  $d(a, b) = d(b, a)$  for all  $a, b \in V$ .
4.  $d(a, b) \leq d(a, c) + d(c, b)$  for all  $a, b, c \in V$ .

**Lemma 6.3.4.**  $R_{\text{eff}}$  is a distance.

Before proving this lemma, let's see a claim that will help us finish the proof.

**Claim 6.3.5.** Let  $\mathbf{L}\tilde{\mathbf{x}} = \mathbf{e}_b - \mathbf{e}_a$ . Then for all  $c \in V$ , we have  $\tilde{\mathbf{x}}(b) \geq \tilde{\mathbf{x}}(c) \geq \tilde{\mathbf{x}}(a)$ .

We only sketch a proof of this claim:

*Proof sketch.* Consider any  $c \in V$ , where  $c \neq a, b$ . Now  $(\mathbf{L}\tilde{\mathbf{x}})(c) = 0$ , i.e.

$$\left( \sum_{(u,c)} \mathbf{w}(u, c) \right) \tilde{\mathbf{x}}(c) - \left( \sum_{(u,c)} \mathbf{w}(u, c) \tilde{\mathbf{x}}(u) \right) = 0$$

Rearranging  $\tilde{\mathbf{x}}(c) = \frac{\sum_{(u,c)} \mathbf{w}(u, c) \tilde{\mathbf{x}}(u)}{\sum_{(u,c)} \mathbf{w}(u, c)}$ . This tells us that  $\tilde{\mathbf{x}}(c)$  is a weighted average of the voltages of its neighbors. From this, we can show that  $\tilde{\mathbf{x}}(a)$  and  $\tilde{\mathbf{x}}(b)$  are the extreme values.  $\square$

*Proof.* It is easy to check that conditions 1, 2, and 3 of Definition 6.3.3 are satisfied by  $R_{\text{eff}}$ . Let us confirm condition 4.

For any  $u, v$ , let  $\tilde{\mathbf{x}}_{u,v} = \mathbf{L}^+(-\mathbf{e}_u + \mathbf{e}_v)$ . Then

$$\tilde{\mathbf{x}}_{a,b} = \mathbf{L}^+(-\mathbf{e}_a + \mathbf{e}_b) = \mathbf{L}^+(-\mathbf{e}_a + \mathbf{e}_c - \mathbf{e}_c + \mathbf{e}_b) = \tilde{\mathbf{x}}_{a,c} + \tilde{\mathbf{x}}_{c,b}.$$

Thus,

$$\begin{aligned}
R_{\text{eff}}(a, b) &= (-\mathbf{e}_a + \mathbf{e}_b)^\top \tilde{\mathbf{x}}_{a,b} = (-\mathbf{e}_a + \mathbf{e}_b)^\top (\tilde{\mathbf{x}}_{a,c} + \tilde{\mathbf{x}}_{c,b}) \\
&= -\tilde{\mathbf{x}}_{a,c}(a) + \tilde{\mathbf{x}}_{a,c}(b) - \tilde{\mathbf{x}}_{c,b}(a) + \tilde{\mathbf{x}}_{c,b}(b) \\
&\leq -\tilde{\mathbf{x}}_{a,c}(a) + \tilde{\mathbf{x}}_{a,c}(c) - \tilde{\mathbf{x}}_{c,b}(c) + \tilde{\mathbf{x}}_{c,b}(b).
\end{aligned}$$

where in the last line we applied Claim 6.3.5 to show that  $\tilde{\mathbf{x}}_{a,c}(b) \leq \tilde{\mathbf{x}}_{a,c}(c)$  and  $-\tilde{\mathbf{x}}_{c,b}(a) \leq -\tilde{\mathbf{x}}_{c,b}(c)$ .  $\square$



# Chapter 7

## Different Perspectives on Gaussian Elimination

### 7.1 An Optimization View of Gaussian Elimination for Laplacians

In this section, we will explore how to exactly minimize a Laplacian quadratic form by minimizing over one variable at a time. It turns out that this is in fact Gaussian Elimination in disguise – or, more precisely, the variant of Gaussian elimination that we tend to use on symmetric matrices, which is called Cholesky factorization.

Consider a Laplacian  $\mathbf{L}$  of a connected graph  $G = (V, E, \mathbf{w})$ , where  $\mathbf{w} \in \mathbb{R}^E$  is a vector of positive edge weights. Let  $\mathbf{W} \in \mathbb{R}^{E \times E}$  be the diagonal matrix with the edge weights on the diagonal, i.e.  $\mathbf{W} = \text{diag}(\mathbf{w})$  and  $\mathbf{L} = \mathbf{B} \mathbf{W} \mathbf{B}^\top$ . Let  $\mathbf{d} \in \mathbb{R}^V$  be a demand vector s.t.  $\mathbf{d} \perp \mathbf{1}$ .

Let us define an energy

$$\mathcal{E}(\mathbf{x}) = -\mathbf{d}^\top \mathbf{x} + \frac{1}{2} \mathbf{x}^\top \mathbf{L} \mathbf{x}$$

Note that this function is convex and is minimized at  $\mathbf{x}$  s.t.  $\mathbf{L} \mathbf{x} = \mathbf{d}$ .

We will now explore an approach to solving the minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^V} \mathcal{E}(\mathbf{x})$$

Let  $\mathbf{x} = \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix}$  where  $y \in \mathbb{R}$  and  $\mathbf{z} \in \mathbb{R}^{V \setminus \{1\}}$ .

We will now explore how to minimize over  $y$ , given any  $\mathbf{z}$ . Once we find an expression for  $y$  in terms of  $\mathbf{z}$ , we will be able to reduce it to a new quadratic minimization problem in  $\mathbf{z}$ ,

$$\mathcal{E}'(\mathbf{z}) = -\mathbf{d}'^\top \mathbf{z} + \frac{1}{2} \mathbf{z}^\top \mathbf{L}' \mathbf{z}$$

where  $\mathbf{d}'$  is a demand vector on the remaining vertices, with  $\mathbf{d} \perp \mathbf{1}$  and  $\mathbf{L}'$  is a Laplacian of a graph on the remaining vertices  $V' = V \setminus \{1\}$ . We can then repeat the procedure to eliminate another variable and so on. Eventually, we can then find all the solution to our original minimization problem.

To help us understand how to minimize over the first variable, we introduce some notation for the first row and column of the Laplacian:

$$\mathbf{L} = \begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \text{diag}(\mathbf{a}) + \mathbf{L}_{-1} \end{pmatrix}. \quad (7.1)$$

Note that  $W$  is the weighted degree of vertex 1, and that

$$\begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \text{diag}(\mathbf{a}) \end{pmatrix} \quad (7.2)$$

is the Laplacian of the subgraph of  $G$  containing only the edges incident on vertex 1, while  $\mathbf{L}_{-1}$  is the Laplacian of the subgraph of  $G$  containing all edges *not* incident on vertex 1.

Let us also write  $\mathbf{d} = \begin{pmatrix} b \\ \mathbf{c} \end{pmatrix}$  where  $b \in \mathbb{R}$  and  $\mathbf{c} \in \mathbb{R}^{V \setminus \{1\}}$ .

Now,

$$\begin{aligned} \mathcal{E}(\mathbf{x}) &= -\mathbf{d}^\top \mathbf{x} + \frac{1}{2} \mathbf{x}^\top \mathbf{L} \mathbf{x} = -\begin{pmatrix} b \\ \mathbf{c} \end{pmatrix}^\top \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix}^\top \begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \text{diag}(\mathbf{a}) + \mathbf{L}_{-1} \end{pmatrix} \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix} \\ &= -by - \mathbf{c}^\top \mathbf{z} + \frac{1}{2} (y^2 W - 2y \mathbf{a}^\top \mathbf{z} + \mathbf{z}^\top \text{diag}(\mathbf{a}) \mathbf{z} + \mathbf{z}^\top \mathbf{L}_{-1} \mathbf{z}). \end{aligned}$$

Now, to minimize over  $y$ , we set  $\frac{\partial}{\partial y} \mathcal{E}(\mathbf{x}) = 0$  and get

$$-b + yW - \mathbf{a}^\top \mathbf{z} = 0.$$

Solving for  $y$ , we get that the minimizing  $y$  is

$$y = \frac{1}{W} (b + \mathbf{a}^\top \mathbf{z}). \quad (7.3)$$

Observe that

$$\begin{aligned} \mathcal{E}(\mathbf{x}) &= -by - \mathbf{c}^\top \mathbf{z} + \frac{1}{2} (y^2 W - 2y \mathbf{a}^\top \mathbf{z} + \mathbf{z}^\top \text{diag}(\mathbf{a}) \mathbf{z} + \mathbf{z}^\top \mathbf{L}_{-1} \mathbf{z}) \\ &= -by - \mathbf{c}^\top \mathbf{z} + \frac{1}{2} \left( \frac{1}{W} (yW - \mathbf{a}^\top \mathbf{z})^2 - \underbrace{\frac{1}{W} \mathbf{z}^\top \mathbf{a} \mathbf{a}^\top \mathbf{z} + \mathbf{z}^\top \text{diag}(\mathbf{a}) \mathbf{z} + \mathbf{z}^\top \mathbf{L}_{-1} \mathbf{z}}_{\text{Let } \mathbf{S} = \text{diag}(\mathbf{a}) - \frac{1}{W} \mathbf{a} \mathbf{a}^\top + \mathbf{L}_{-1}} \right) \\ &= -by - \mathbf{c}^\top \mathbf{z} + \frac{1}{2} \left( \frac{1}{W} (yW - \mathbf{a}^\top \mathbf{z})^2 + \mathbf{z}^\top \mathbf{S} \mathbf{z} \right), \end{aligned}$$

where we simplified the expression by defining  $\mathbf{S} = \text{diag}(\mathbf{a}) - \frac{1}{W}\mathbf{a}\mathbf{a}^\top + \mathbf{L}_{-1}$ . Plugging in  $y = \frac{1}{W}(b + \mathbf{a}^\top \mathbf{z})$ , we get

$$\min_y \mathcal{E} \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix} = - \left( \mathbf{c} + b \frac{1}{W} \mathbf{a} \right)^\top \mathbf{z} - \frac{b^2}{2W} + \frac{1}{2} \mathbf{z}^\top \mathbf{S} \mathbf{z}.$$

Now, we define  $\mathbf{d}' = \mathbf{c} + b \frac{1}{W} \mathbf{a}$  and  $\mathcal{E}'(\mathbf{z}) = -\mathbf{d}'^\top \mathbf{z} + \frac{1}{2} \mathbf{z}^\top \mathbf{S} \mathbf{z}$ . And, we can see that

$$\arg \min_z \min_y \mathcal{E} \begin{pmatrix} y \\ \mathbf{z} \end{pmatrix} = \arg \min_z \mathcal{E}'(\mathbf{z}),$$

since dropping the constant term  $-\frac{b^2}{2W}$  does not change what the minimizing  $\mathbf{z}$  values are.

**Claim 7.1.1.**

1.  $\mathbf{d}' \perp \mathbf{1}$
2.  $\mathbf{S} = \text{diag}(\mathbf{a}) - \frac{1}{W}\mathbf{a}\mathbf{a}^\top + \mathbf{L}_{-1}$  is a Laplacian of a graph on the vertex set  $V \setminus \{1\}$ .

We will prove Claim 7.1.1 in a moment. From the Claim, we see that the problem of finding  $\arg \min_z \mathcal{E}'(\mathbf{z})$ , is exactly of the same form as finding  $\arg \min_x \mathcal{E}(\mathbf{x})$ , but with one fewer variables.

We can get a minimizing  $\mathbf{x}$  that solves  $\arg \min_x \mathcal{E}(\mathbf{x})$  by repeating the variable elimination procedure until we get down to a single variable and finding its value. We then have to work back up to getting a solution for  $\mathbf{z}$ , and then substitute that into Equation (7.3) to get the value for  $y$ .

**Remark 7.1.2.** In fact, this perspective on Gaussian elimination also makes sense for any positive definite matrix. In this setting, minimizing over one variable will leave us with another positive definite quadratic minimization problem.

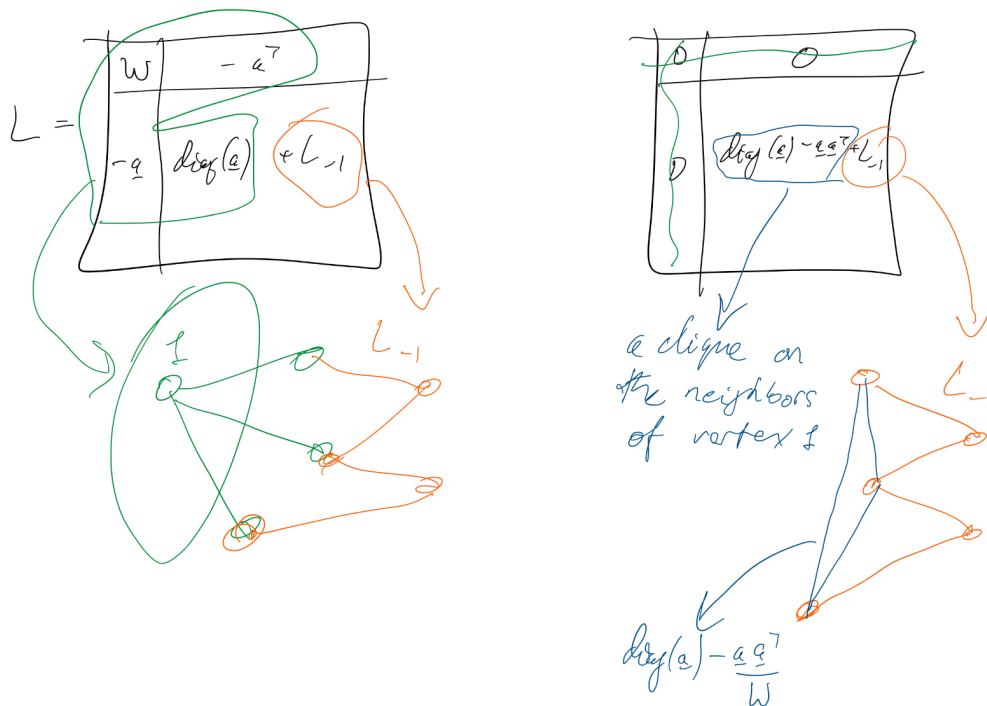
*Proof of Claim 7.1.1.* To establish the first part, we note that  $\mathbf{1}^\top \mathbf{d}' = \mathbf{1}^\top \mathbf{c} + b \frac{1}{W} \mathbf{1}^\top \mathbf{a} = \mathbf{1}^\top \mathbf{c} + b = \mathbf{1}^\top \mathbf{d} = 0$ . To establish the second part, we notice that  $\mathbf{L}_{-1}$  is a graph Laplacian by definition. Since the sum of two graph Laplacians is another graph Laplacian, it now suffices to show that  $\mathbf{S}$  is a graph Laplacian.

**Claim 7.1.3.** A matrix  $\mathbf{M}$  is a graph Laplacian if and only if it satisfies the following conditions:

- $\mathbf{M}^\top = \mathbf{M}$ .
- The diagonal entries of  $\mathbf{M}$  are non-negative, and the off-diagonal entries of  $\mathbf{M}$  are non-positive.
- $\mathbf{M}\mathbf{1} = \mathbf{0}$ .

Let's see that Claim 7.1.3 is true. Firstly, when the conditions hold we can write  $\mathbf{M} = \mathbf{D} - \mathbf{A}$  where  $\mathbf{D}$  is diagonal and non-negative, and  $\mathbf{A}$  is non-negative, symmetric, and zero on the diagonal, and from the last condition  $\mathbf{D}(i, i) = \sum_{j \neq i} \mathbf{A}(i, j)$ . Thus we can view  $\mathbf{A}$  as a graph adjacency matrix and  $\mathbf{D}$  as the corresponding diagonal matrix of weighted degrees. Secondly, it is easy to check that the conditions hold for any graph Laplacian, so the conditions indeed hold if and only if. Now we have to check that the claim applies to  $\mathbf{S}$ . We leave this as an exercise for the reader.

Finally, we want to argue that the graph corresponding to  $\mathbf{S}$  is connected. Consider any  $i, j \in V \setminus \{1\}$ . Since  $G$ , the graph of  $\mathbf{L}$ , is connected, there exists a simple path in  $G$  connecting  $i$  and  $j$ . If this path does not use vertex 1, it is a path in the graph of  $\mathbf{L}_{-1}$  and hence in the graph of  $\mathbf{S}$ . If the path does use vertex 1, it must do so by reaching the vertex on some edge  $(v, 1)$  and leaving on a different edge  $(1, u)$ . Replace this pair of edges with edge  $(u, v)$ , which appears in the graph of  $\mathbf{S}$  because  $\mathbf{S}(u, v) < 0$ . Now we have a path in the graph of  $\mathbf{S}$ .  $\square$



## 7.2 An Additive View of Gaussian Elimination

**Cholesky decomposition basics.** Again we consider a graph Laplacian  $\mathbf{L} \in \mathbb{R}^{n \times n}$  of a connected graph  $G = (V, E, \mathbf{w})$ , where as usual  $|V| = n$  and  $|E| = m$ .

In this Section, we'll study how to decompose a graph Laplacian as  $\mathbf{L} = \mathbf{L}\mathbf{L}^\top$ , where  $\mathbf{L} \in \mathbb{R}^{n \times n}$  is a lower triangular matrix, i.e.  $\mathbf{L}(i, j) = 0$  for  $i < j$ . Such a factorization is called a Cholesky decomposition. It is essentially the result of Gaussian elimination with a slight twist to ensure the matrices maintained at intermediate steps of the algorithm remain symmetric.

We use  $\text{nnz}(\mathbf{A})$  to denote the number of non-zero entries of matrix  $\mathbf{A}$ .

**Lemma 7.2.1.** *Given an invertible square lower triangular matrix  $\mathbf{L}$ , we can solve the linear equation  $\mathbf{L}\mathbf{y} = \mathbf{b}$  in time  $O(\text{nnz}(\mathbf{L}))$ . Similarly, given an upper triangular matrix  $\mathbf{U}$ , we can solve linear equations  $\mathbf{U}\mathbf{z} = \mathbf{c}$  in time  $O(\text{nnz}(\mathbf{U}))$ .*

We omit the proof, which is a straight-forward exercise. The algorithms for solving linear equations in upper and lower triangular matrices are known as forward and back substitution respectively.

**Remark 7.2.2.** Strictly speaking, the lemma requires us to have access an adjacency list representation of  $\mathbf{L}$  so that we can quickly tell where the non-zero entries are.

Using forward and back substitution, if we have a decomposition of an invertible matrix  $\mathbf{M}$  into  $\mathbf{M} = \mathbf{L}\mathbf{L}^\top$ , we can now solve linear equations in  $\mathbf{M}$  in time  $O(\text{nnz}(\mathbf{L}))$ .

**Remark 7.2.3.** We have learned about decompositions using a lower triangular matrix, and later we will see an algorithm for computing these. In fact, we can have more flexibility than that. From an algorithmic perspective, it is sufficient that there exists a permutation matrix  $\mathbf{P}$  s.t.  $\mathbf{P}\mathbf{L}\mathbf{P}^\top$  is lower triangular. If we know the ordering under which the matrix becomes lower triangular, we can perform substitution according to that order to solve linear equations in the matrix without having to explicitly apply a permutation to the matrix.

**Dealing with pseudoinverses.** But how can we solve a linear equation in  $\mathbf{L} = \mathbf{L}\mathbf{L}^\top$ , where  $\mathbf{L}$  is not invertible? For graph Laplacians we have a simple characterization of the kernel, and because of this, dealing with the lack of invertibility turns out to be fairly easy.

We can use the following lemma which you will prove in an exercise next week.

**Lemma 7.2.4.** *Consider a real symmetric matrix  $\mathbf{M} = \mathbf{X}\mathbf{Y}\mathbf{X}^\top$ , where  $\mathbf{X}$  is real and invertible and  $\mathbf{Y}$  is real symmetric. Let  $\Pi_{\mathbf{M}}$  denote the orthogonal projection to the image of  $\mathbf{M}$ . Then  $\mathbf{M}^+ = \Pi_{\mathbf{M}}(\mathbf{X}^\top)^{-1}\mathbf{Y}^+\mathbf{X}^{-1}\Pi_{\mathbf{M}}$ .*

The factorizations  $\mathbf{L} = \mathbf{L}\mathbf{L}^\top$  that we produce will have the property that all diagonal entries of  $\mathbf{L}$  are strictly non-zero, except that  $\mathbf{L}(n, n) = 0$ . Let  $\hat{\mathbf{L}}$  be the matrix whose entries agree with  $\mathbf{L}$ , except that  $\hat{\mathbf{L}}(n, n) = 1$ . Let  $\mathbf{D}$  be the diagonal matrix with  $\mathbf{D}(i, i) = 1$  for  $i < n$  and  $\mathbf{D}(n, n) = 0$ . Then  $\mathbf{L}\mathbf{L}^\top = \hat{\mathbf{L}}\mathbf{D}\hat{\mathbf{L}}^\top$ , and  $\hat{\mathbf{L}}$  is invertible, and  $\mathbf{D}^+ = \mathbf{D}$ . Finally,  $\Pi_{\mathbf{L}} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$ , because this matrix acts like identity on vectors orthogonal to  $\mathbf{1}$  and ensures  $\Pi_{\mathbf{L}}\mathbf{1} = \mathbf{0}$ , and this matrix can be applied to a vector in  $O(n)$  time. Thus  $\mathbf{L}^+ = \Pi_{\mathbf{L}}(\hat{\mathbf{L}}^\top)^{-1}\mathbf{D}\hat{\mathbf{L}}^{-1}\Pi_{\mathbf{L}}$ , and this matrix can be applied in time  $O(\text{nnz}(\mathbf{L}))$ .

**An additive view of Gaussian Elimination.** The following theorem describes Gaussian Elimination / Cholesky decomposition of a graph Laplacian.

**Theorem 7.2.5** (Cholesky Decomposition on graph Laplacians). *Let  $\mathbf{L} \in \mathbb{R}^{n \times n}$  be a graph Laplacian of a connected graph  $G = (V, E, \mathbf{w})$ , where  $|V| = n$ . Using Gaussian Elimination, we can compute in  $O(n^3)$  time a factorization  $\mathbf{L} = \mathbf{L}\mathbf{L}^\top$  where  $\mathbf{L}$  is lower triangular, and has positive diagonal entries except  $\mathbf{L}(n, n) = 0$ .*

*Proof.* Let  $\mathbf{L}^{(0)} = \mathbf{L}$ . We will use  $\mathbf{A}(:, i)$  to denote the  $i$ th column of a matrix  $\mathbf{A}$ . Now, for  $i = 1$  to  $i = n - 1$  we define

$$\mathbf{l}_i = \frac{1}{\sqrt{\mathbf{L}^{(i-1)}(i, i)}} \mathbf{L}^{(i-1)}(:, i) \text{ and } \mathbf{L}^{(i)} = \mathbf{L}^{(i-1)} - \mathbf{l}_i \mathbf{l}_i^\top$$

Finally, we let  $\mathbf{l}_n = \mathbf{0}_{n \times 1}$ . We will show later that

$$\mathbf{L}^{(n-1)} = \mathbf{0}_{n \times n}. \quad (7.4)$$

It follows that  $\mathbf{L} = \sum_i \mathbf{l}_i \mathbf{l}_i^\top$ , provided this procedure is well-defined, i.e.  $\mathbf{L}^{(i-1)}(i, i) \neq 0$  for all  $i < n$ . We will sketch a proof of this later, while also establishing several other properties of the procedure.

Given a matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $U \subseteq [n]$ , we will use  $\mathbf{A}(U, U)$  to denote the principal submatrix of  $\mathbf{A}$  obtained by restricting to the rows and columns with index in  $U$ , i.e. all entries  $\mathbf{A}(i, j)$  where  $i, j \in U$ .

**Claim 7.2.6.** *Fix some  $i < n$ . Let  $U = \{i + 1, \dots, n\}$ . Then  $\mathbf{L}^{(i)}(i, j) = 0$  if  $i \notin U$  or  $j \notin U$ . And  $\mathbf{L}^{(i)}(U, U)$  is a graph Laplacian of a connected graph on the vertex set  $U$ .*

From this claim, it follows that  $\mathbf{L}^{(i-1)}(i, i) \neq 0$  for  $i < n$ , since a connected graph Laplacian on a graph with  $|U| > 1$  vertices cannot have a zero on the diagonal, and it follows that  $\mathbf{L}^{(n-1)}(i, i) = 0$ , because the only graph we allow on one vertex is the empty graph. This shows Equation (7.4) holds.  $\square$

*Sketch of proof of Claim 7.2.6.* We will focus on the first elimination, as the remaining are similar. Adopting the same notation as in Equation (7.1), we write

$$\mathbf{L}^{(0)} = \mathbf{L} = \begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \text{diag}(\mathbf{a}) + \mathbf{L}_{-1} \end{pmatrix}$$

and, noting that

$$\mathbf{l}_1 \mathbf{l}_1^\top = \begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \frac{1}{W} \mathbf{a} \mathbf{a}^\top \end{pmatrix}$$

we see that

$$\mathbf{L}^{(1)} = \mathbf{L}^{(0)} - \mathbf{l}_1 \mathbf{l}_1^\top = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \text{diag}(\mathbf{a}) - \frac{1}{W} \mathbf{a} \mathbf{a}^\top + \mathbf{L}_{-1} \end{pmatrix}.$$

Thus the first row and column of  $\mathbf{L}^{(1)}$  are zero claimed. It also follows by Claim 7.1.1 that  $\mathbf{L}^{(1)}(\{2, \dots, n\}, \{2, \dots, n\})$  is the Laplacian of a connected graph. This proves Claim 7.2.6 for the case  $i = 1$ . An induction following the same pattern can be used to prove the claim for all  $i < n$ .  $\square$

# Chapter 8

## Random Matrix Concentration and Spectral Graph Sparsification

### 8.1 Matrix Sampling and Approximation

We want to begin understanding how sums of random matrices behave, in particular, whether they exhibit a tendency to concentrate in the same way that sum of scalar random variables do under various conditions.

First, let's recall a scalar Chernoff bound, which shows that a sum of bounded, non-negative random variables tend to concentrate around their mean.

**Theorem 8.1.1** (A Chernoff Concentration Bound). *Suppose  $X_1, \dots, X_k \in \mathbb{R}$  are independent, non-negative, random variables with  $X_i \leq R$  always. Let  $X = \sum_i X_i$ , and  $\mu = \mathbb{E}[X]$ , then for  $0 < \epsilon \leq 1$*

$$\Pr[X \geq (1 + \epsilon)\mu] \leq \exp\left(\frac{-\epsilon^2\mu}{4R}\right) \text{ and } \Pr[X \leq (1 - \epsilon)\mu] \leq \exp\left(\frac{-\epsilon^2\mu}{4R}\right).$$

The Chernoff bound should be familiar to most of you, but you may not have seen the following very similar bound. The Bernstein bound, which we will state in terms of zero-mean variables, is much like the Chernoff bound. It also requires bounded variables. But, when the variables have small variance, the Bernstein bound is sometimes stronger.

**Theorem 8.1.2** (A Bernstein Concentration Bound). *Suppose  $X_1, \dots, X_k \in \mathbb{R}$  are independent, zero-mean, random variables with  $|X_i| \leq R$  always. Let  $X = \sum_i X_i$ , and  $\sigma^2 = \text{Var}[X] = \sum_i \mathbb{E}[X_i^2]$ , then for  $t > 0$*

$$\Pr[|X| \geq t] \leq 2 \exp\left(\frac{-t^2}{2Rt + 4\sigma^2}\right).$$

We will now prove the Bernstein concentration bound for scalar random variables, as a warm-up to the next section, where we will prove a version of it for matrix-valued random variables.



To help us prove Bernstein's bound, first let's recall Markov's inequality. This is a very weak concentration inequality, but also very versatile, because it requires few assumptions.

**Lemma 8.1.3** (Markov's Inequality). *Suppose  $\mathbf{X} \in \mathbb{R}$  is a non-negative random variable, with a finite expectation. Then for any  $t > 0$ ,*

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

*Proof.*

$$\begin{aligned} \mathbb{E}[X] &= \Pr[X \geq t] \mathbb{E}[X \mid X \geq t] + \Pr[X < t] \mathbb{E}[X \mid X < t] \\ &\geq \Pr[X \geq t] \mathbb{E}[X \mid X \geq t] \\ &\geq \Pr[X \geq t] \cdot t. \end{aligned}$$

We can rearrange this to get the desired statement. □

Now, we are ready to prove Bernstein's bound.

*Proof of Theorem 8.1.2.* We will focus on bounding the probability that  $\Pr[X \geq t]$ . The proof that  $\Pr[-X \geq t]$  is small proceeds in the same way.

First we observe that

$$\begin{aligned} \Pr[X \geq t] &= \Pr[\exp(\theta X) \geq \exp(\theta t)] \\ &\quad \text{for any } \theta > 0, \text{ because } x \mapsto \exp(\theta x) \text{ is strictly increasing.} \\ &\leq \exp(-\theta t) \mathbb{E}[\exp(\theta X)] \quad \text{by Lemma 8.1.3 (Markov's Inequality).} \end{aligned}$$

Now, let's require that  $\theta \leq 1/R$ . This will allow us to use the following bound: For all  $|z| \leq 1$ ,

$$\exp(z) \leq 1 + z + z^2. \tag{8.1}$$

We omit a proof of this, but the plots in Figure 8.1 suggest that this upper bound holds. The reader should consider how to prove this. With this in mind, we see that

$$\begin{aligned} \mathbb{E}[\exp(\theta X)] &= \mathbb{E}\left[\exp\left(\theta \sum_i X_i\right)\right] \\ &= \mathbb{E}[\Pi_i \exp(\theta X_i)] \\ &= \Pi_i \mathbb{E}[\exp(\theta X_i)] \quad \text{because } \mathbb{E}[YZ] = \mathbb{E}[Y] \mathbb{E}[Z] \text{ for independent } Y \text{ and } Z. \\ &\leq \Pi_i \mathbb{E}[1 + \theta X_i + (\theta X_i)^2] \\ &= \Pi_i (1 + \theta^2 \mathbb{E}[X_i^2]) \quad \text{because } X_i \text{ are zero-mean.} \\ &\leq \Pi_i \exp(\theta^2 \mathbb{E}[X_i^2]) \quad \text{because } 1 + z \leq \exp(z) \text{ for all } z \in \mathbb{R}. \\ &= \exp\left(\sum_i \theta^2 \mathbb{E}[X_i^2]\right) = \exp(\theta^2 \sigma^2). \end{aligned}$$

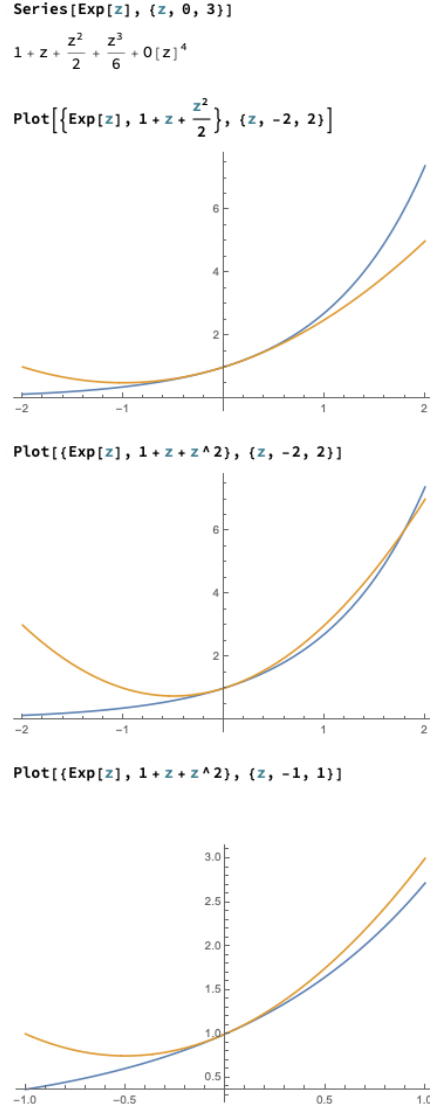


Figure 8.1: Plotting  $\exp(z)$  compared to  $1 + z + z^2$ .

Thus  $\Pr[X \geq t] \leq \exp(-\theta t) \mathbb{E}[\exp(\theta X)] \leq \exp(-\theta t + \theta^2 \sigma^2)$ . Now, to get the best possible bound, we'd like to minimize  $-\theta t + \theta^2 \sigma^2$  subject to the constraint  $0 < \theta \leq 1/R$ . Setting

$$\frac{\partial}{\partial \theta} (-\theta t + \theta^2 \sigma^2) = -t + 2\theta \sigma^2.$$

Setting this derivative to zero gives  $\theta = \frac{t}{2\sigma^2}$ , and plugging that in gives

$$-\theta t + \theta^2 \sigma^2 = -\frac{t^2}{4\sigma^2}$$

This choice only satisfies our constraints on  $\theta$  if  $\frac{t}{2\sigma^2} \leq 1/R$ . Otherwise, we let  $\theta = 1/R$  and note that in this case

$$-\theta t + \theta^2 \sigma^2 = -\frac{t}{R} + \frac{\sigma^2}{R^2} \leq -\frac{t}{R} + \frac{t}{2R} = -\frac{t}{2R}$$

where we got the inequality from  $t > 2\sigma^2/R$ . Altogether, we can conclude that there always is a choice of  $\theta$  s.t.

$$-\theta t + \theta^2 \sigma^2 \leq -\min\left(\frac{t}{2R}, \frac{t^2}{4\sigma^2}\right) \leq -\frac{t^2}{2Rt + 4\sigma^2}.$$

In fact, with the benefit of hindsight, and a little algebra, we arrive at the same conclusion in another way: One can check that the following choice of  $\theta$  is always valid and achieves the same bound:  $\theta = \frac{1}{2\sigma^2} \left( t - \frac{\sqrt{R} \cdot t^{3/2}}{\sqrt{2\sigma^2 + Rt}} \right)$ .  $\square$

We use  $\|\cdot\|$  to denote the spectral norm on matrices. Let's take a look at a version of Bernstein's bound that applies to sums of random matrices.

**Theorem 8.1.4** (A Bernstein Matrix Concentration Bound (Tropp 2011)). *Suppose  $\mathbf{X}_1, \dots, \mathbf{X}_k \in \mathbb{R}^{n \times n}$  are independent, symmetric matrix-valued random variables. Assume each  $\mathbf{X}_i$  is zero-mean, i.e.  $\mathbb{E}[\mathbf{X}_i] = \mathbf{0}_{n \times n}$ , and that  $\|\mathbf{X}_i\| \leq R$  always. Let  $\mathbf{X} = \sum_i \mathbf{X}_i$ , and  $\sigma^2 = \text{Var}[\mathbf{X}] = \sum_i \mathbb{E}[\mathbf{X}_i^2]$ , then for  $t > 0$*

$$\Pr[\|\mathbf{X}\| \geq t] \leq 2n \exp\left(\frac{-t^2}{2Rt + 4\sigma^2}\right).$$

This basically says that probability of  $\mathbf{X}$  being large in spectral norm behaves like the scalar case, except the bound is larger by a factor  $n$ , where the matrices are  $n \times n$ . We can get a feeling for why this might be a reasonable bound by considering the case of random diagonal matrices. Now  $\|\mathbf{X}\| = \max_j |\mathbf{X}(j, j)| = \max_j |\sum_i \mathbf{X}_i(j, j)|$ . In this case, we need to bound the largest of the  $n$  diagonal entries: We can do this by a union bound over  $n$  instances of the scalar problem – and this also turns out to be essentially tight in some cases, meaning we can't expect a better bound in general.

## 8.2 Matrix Concentration

In this section we will prove the Bernstein matrix concentration bound (Tropp 2011) that we saw in the previous section (Theorem 8.1.4).

**Theorem 8.2.1.** *Suppose  $\mathbf{X}_1, \dots, \mathbf{X}_k \in \mathbb{R}^{n \times n}$  are independent, symmetric matrix-valued random variables. Assume each  $\mathbf{X}_i$  is zero-mean, i.e.  $\mathbb{E}[\mathbf{X}_i] = \mathbf{0}_{n \times n}$ , and that  $\|\mathbf{X}_i\| \leq R$  always. Let  $\mathbf{X} = \sum_i \mathbf{X}_i$ , and  $\sigma^2 = \|\text{Var}[\mathbf{X}]\| = \|\sum_i \mathbb{E}[\mathbf{X}_i^2]\|$ , then for  $t > 0$*

$$\Pr[\|\mathbf{X}\| \geq t] \leq 2n \exp\left(\frac{-t^2}{2Rt + 4\sigma^2}\right).$$

But let's collect some useful tools for the proof first.

**Definition 8.2.2** (trace). The trace of a square matrix  $\mathbf{A}$  is defined as

$$\text{Tr}(\mathbf{A}) := \sum_i \mathbf{A}(i, i)$$

**Claim 8.2.3** (cyclic property of trace).  $\text{Tr}(\mathbf{AB}) = \text{Tr}(\mathbf{BA})$

Let  $S^n$  denote the set of all  $n \times n$  real symmetric matrices,  $S_+^n$  the set of all  $n \times n$  positive semidefinite matrices, and  $S_{++}^n$  the set of all  $n \times n$  positive definite matrices. Their relation is clear,  $S_{++}^n \subset S_+^n \subset S^n$ . For any  $\mathbf{A} \in S^n$  with eigenvalues  $\lambda_1(\mathbf{A}) \leq \dots \leq \lambda_n(\mathbf{A})$ , by spectral decomposition theorem,  $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top$  where  $\mathbf{\Lambda} = \text{diag}_i\{\lambda_i(\mathbf{A})\}$  and  $\mathbf{V}^\top \mathbf{V} = \mathbf{V} \mathbf{V}^\top = \mathbf{I}$ , we'll use this property without specifying in the sequel.

**Claim 8.2.4.** Given a symmetric and real matrix  $\mathbf{A}$ ,  $\text{Tr}(\mathbf{A}) = \sum_i \lambda_i$ , where  $\{\lambda_i\}$  are eigenvalues of  $\mathbf{A}$ .

*Proof.*

$$\text{Tr}(\mathbf{A}) = \text{Tr}(\mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top) = \text{Tr}\left(\mathbf{\Lambda} \underbrace{\mathbf{V}^\top \mathbf{V}}_{\mathbf{I}}\right) = \text{Tr}(\mathbf{\Lambda}) = \sum_i \lambda_i.$$

□

## 8.2.1 Matrix Functions

**Definition 8.2.5** (Matrix function). Given a real-valued function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we extend it to a matrix function  $f : S^n \rightarrow S^n$ . For  $\mathbf{A} \in S^n$  with spectral decomposition  $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top$ , let

$$f(\mathbf{A}) = \mathbf{V} \text{diag}_i\{f(\lambda_i)\} \mathbf{V}^\top.$$

**Example.** Recall that every PSD matrix  $\mathbf{A}$  has a square root  $\mathbf{A}^{1/2}$ . If  $f(x) = x^{1/2}$  for  $x \in \mathbb{R}_+$ , then  $f(\mathbf{A}) = \mathbf{A}^{1/2}$  for  $\mathbf{A} \in S_+^n$ .

**Example.** If  $f(x) = \exp(x)$  for  $x \in \mathbb{R}$ , then  $f(\mathbf{A}) = \exp(\mathbf{A}) = \mathbf{V} \exp(\mathbf{\Lambda}) \mathbf{V}^\top$  for  $\mathbf{A} \in S^n$ . Note that  $\exp(\mathbf{A})$  is positive definite for any  $\mathbf{A} \in S^n$ .

## 8.2.2 Monotonicity and Operator Monotonicity

Consider a function  $f : \mathcal{D} \rightarrow \mathcal{C}$ . If we have a partial order  $\leq_{\mathcal{D}}$  defined on  $\mathcal{D}$  and a partial order  $\leq_{\mathcal{C}}$  defined on  $\mathcal{C}$ , then we say that the function is monotone increasing (resp. decreasing) w.r.t. this pair of orderings if for all  $d_1, d_2 \in \mathcal{D}$  s.t.  $d_1 \leq_{\mathcal{D}} d_2$  we have  $f(d_1) \leq_{\mathcal{C}} f(d_2)$  (resp. decreasing if  $f(d_2) \leq_{\mathcal{C}} f(d_1)$ ).

Let's introduce some terminology for important special cases of this idea. We say that a function  $f : \mathcal{S} \rightarrow \mathbb{R}$ , where  $\mathcal{S} \subseteq S^n$ , is monotone increasing if  $\mathbf{A} \preceq \mathbf{B}$  implies  $f(\mathbf{A}) \leq f(\mathbf{B})$ .

Meanwhile, a function  $f : \mathcal{S} \rightarrow \mathcal{T}$  where  $\mathcal{S}, \mathcal{T} \subseteq S^n$  is said to be operator monotone increasing if  $\mathbf{A} \preceq \mathbf{B}$  implies  $f(\mathbf{A}) \preceq f(\mathbf{B})$ .

**Lemma 8.2.6.** *Let  $T \subseteq \mathbb{R}$ . If the scalar function  $f : T \rightarrow \mathbb{R}$  is monotone increasing, the matrix function  $\mathbf{X} \mapsto \text{Tr}(f(\mathbf{X}))$  is monotone increasing.*

*Proof.* From previous chapters, we know if  $\mathbf{A} \preceq \mathbf{B}$  then  $\lambda_i(\mathbf{A}) \leq \lambda_i(\mathbf{B})$  for all  $i$ . As  $x \mapsto f(x)$  is monotone, then  $\lambda_i(f(\mathbf{A})) \leq \lambda_i(f(\mathbf{B}))$  for all  $i$ . By Claim 8.2.4,  $\text{Tr}(f(\mathbf{A})) \leq \text{Tr}(f(\mathbf{B}))$ .  $\square$

From this, and the fact that  $x \mapsto \exp(x)$  is a monotone function on the reals, we get the following corollary.

**Corollary 8.2.7.** *If  $\mathbf{A} \preceq \mathbf{B}$ , then  $\text{Tr}(\exp(\mathbf{A})) \leq \text{Tr}(\exp(\mathbf{B}))$ , i.e.  $\mathbf{X} \mapsto \text{Tr}(\exp(\mathbf{X}))$  is monotone increasing.*

**Lemma 8.2.8.** *If  $\mathbf{0} \prec \mathbf{A} \preceq \mathbf{B}$ , then  $\mathbf{B}^{-1} \preceq \mathbf{A}^{-1}$ , i.e.  $\mathbf{X} \mapsto \mathbf{X}^{-1}$  is operator monotone decreasing on  $S_{++}^n$ .*

You will prove the above lemma in this week's exercises.

**Lemma 8.2.9.** *If  $\mathbf{0} \prec \mathbf{A} \preceq \mathbf{B}$ , then  $\log(\mathbf{A}) \preceq \log(\mathbf{B})$ .*

To prove this lemma, we first recall an integral representation of the logarithm.

**Lemma 8.2.10.**

$$\log a = \int_0^\infty \left( \frac{1}{1+t} - \frac{1}{a+t} \right) dt$$

*Proof.*

$$\begin{aligned} \int_0^\infty \left( \frac{1}{1+t} - \frac{1}{a+t} \right) dt &= \lim_{T \rightarrow \infty} \int_0^T \left( \frac{1}{1+t} - \frac{1}{a+t} \right) dt \\ &= \lim_{T \rightarrow \infty} [\log(1+t) - \log(a+t)]_0^T \\ &= \log(a) + \lim_{T \rightarrow \infty} \log \left( \frac{1+T}{a+T} \right) \\ &= \log(a) \end{aligned}$$

$\square$

*Proof sketch of Lemma 8.2.9.* Because all the matrices involved are diagonalized by the same orthogonal transformation, we can conclude from Lemma 8.2.10 that for a matrix  $\mathbf{A} \succ \mathbf{0}$ ,

$$\log(\mathbf{A}) = \int_0^\infty \left( \frac{1}{1+t} \mathbf{I} - (t\mathbf{I} + \mathbf{A})^{-1} \right) dt$$

This integration can be expressed as the limit of a sum with positive coefficients, and from this we can show that is the integrand (the term inside the integration symbol) is operator monotone increasing in  $\mathbf{A}$  by Lemma 8.2.8, the result of the integral, i.e.  $\log(\mathbf{A})$  must also be operator monotone increasing.  $\square$

The following is a more general version of Lemma 1.6.

**Lemma 8.2.11.** *Let  $T \subset \mathbb{R}$ . If the scalar function  $f : T \rightarrow \mathbb{R}$  is monotone, the matrix function  $\mathbf{X} \mapsto \text{Tr}(f(\mathbf{X}))$  is monotone.*

**Remark 8.2.12.** It is not always true that when  $f : \mathbb{R} \rightarrow \mathbb{R}$  is monotone,  $f : S^n \rightarrow S^n$  is operator monotone. For example,  $\mathbf{X} \mapsto \mathbf{X}^2$  and  $\mathbf{X} \mapsto \exp(\mathbf{X})$  are *not* operator monotone.

### 8.2.3 Some Useful Facts

**Lemma 8.2.13.**  $\exp(\mathbf{A}) \preceq \mathbf{I} + \mathbf{A} + \mathbf{A}^2$  for  $\|\mathbf{A}\| \leq 1$ .

*Proof.*

$$\begin{aligned} \mathbf{I} + \mathbf{A} + \mathbf{A}^2 - \exp(\mathbf{A}) &= \mathbf{V} \mathbf{I} \mathbf{V}^\top + \mathbf{V} \mathbf{A} \mathbf{V}^\top + \mathbf{V} \mathbf{A}^2 \mathbf{V}^\top - \mathbf{V} \exp(\mathbf{A}) \mathbf{V}^\top \\ &= \mathbf{V} (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 - \exp(\mathbf{A})) \mathbf{V}^\top \\ &= \mathbf{V} \text{diag}_i \{1 + \lambda_i + \lambda_i^2 - \exp(\lambda_i)\} \mathbf{V}^\top \end{aligned}$$

Recall  $\exp(x) \leq 1 + x + x^2$  for all  $|x| \leq 1$ . Since  $\|\mathbf{A}\| \leq 1$  i.e.  $|\lambda_i| \leq 1$  for all  $i$ , thus  $1 + \lambda_i + \lambda_i^2 - \exp(\lambda_i) \geq 0$  for all  $i$ , meaning  $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 - \exp(\mathbf{A}) \succeq 0$ .  $\square$

**Lemma 8.2.14.**  $\log(\mathbf{I} + \mathbf{A}) \preceq \mathbf{A}$  for  $\mathbf{A} \succ -\mathbf{I}$ .

*Proof.*

$$\begin{aligned} \mathbf{A} - \log(\mathbf{I} + \mathbf{A}) &= \mathbf{V} \mathbf{A} \mathbf{V}^\top - \mathbf{V} \log(\mathbf{A} + \mathbf{I}) \mathbf{V}^\top \\ &= \mathbf{V} (\mathbf{A} - \log(\mathbf{A} + \mathbf{I})) \mathbf{V}^\top \\ &= \mathbf{V} \text{diag}_i \{\lambda_i - \log(1 + \lambda_i)\} \mathbf{V}^\top \end{aligned}$$

Recall  $x \geq \log(1 + x)$  for all  $x > -1$ . Since  $\|\mathbf{A}\| \succ -\mathbf{I}$  i.e.  $\lambda_i > -1$  for all  $i$ , thus  $\lambda_i - \log(1 + \lambda_i) \geq 0$  for all  $i$ , meaning  $\mathbf{A} - \log(\mathbf{I} + \mathbf{A}) \succeq 0$ .  $\square$

**Theorem 8.2.15 (Lieb).** *Let  $f : S_{++}^n \rightarrow \mathbb{R}$  be a matrix function given by*

$$f(\mathbf{A}) = \text{Tr}(\exp(\mathbf{H} + \log(\mathbf{A})))$$

*for some  $\mathbf{H} \in S^n$ . Then  $-f$  is convex (i.e.  $f$  is concave).*

The Lieb's theorem will be crucial in our proof of Theorem 8.1.4, but it is also highly non-trivial and we will omit its proof here. The interested reader can find a proof in Chapter 8 of [T<sup>+</sup>15].

**Lemma 8.2.16 (Jensen's inequality).**  $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$  when  $f$  is convex;  $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$  when  $f$  is concave.

### 8.2.4 Proof of Matrix Bernstein Concentration Bound

Now, we are ready to prove the Bernstein matrix concentration bound.

*Proof of Theorem 8.1.4.* For any  $\mathbf{A} \in S^n$ , its spectral norm  $\|\mathbf{A}\| = \max\{|\lambda_n(\mathbf{A})|, |\lambda_1(\mathbf{A})|\} = \max\{\lambda_n(\mathbf{A}), -\lambda_1(\mathbf{A})\}$ . Let  $\lambda_1 \leq \dots \leq \lambda_n$  be the eigenvalues of  $\mathbf{X}$ . Then,

$$\Pr[\|\mathbf{X}\| \geq t] = \Pr[(\lambda_n \geq t) \vee (-\lambda_1 \geq t)] \leq \Pr[\lambda_n \geq t] + \Pr[-\lambda_1 \geq t].$$

Let  $\mathbf{Y} := \sum_i -\mathbf{X}_i$ . It's easy to see that  $-\lambda_n \leq \dots \leq -\lambda_1$  are eigenvalues of  $\mathbf{Y}$ , implying  $\lambda_n(\mathbf{Y}) = -\lambda_1(\mathbf{X})$ . Since  $\mathbb{E}[-\mathbf{X}_i] = \mathbb{E}[\mathbf{X}_i] = 0$  and  $\|-\mathbf{X}_i\| = \|\mathbf{X}_i\| \leq R$  for all  $i$ , if we can bound  $\Pr[\lambda_n(\mathbf{X}) \geq t]$ , then applying to  $\mathbf{Y}$ , we can bound  $\Pr[\lambda_n(\mathbf{Y}) \geq t]$ . As

$$\Pr[-\lambda_1(\mathbf{X}) \geq t] = \Pr[\lambda_n(\mathbf{Y}) \geq t],$$

it suffices to bound  $\Pr[\lambda_n \geq t]$ .

For any  $\theta > 0$ ,  $\lambda_n \geq t \iff \exp(\theta \lambda_n) \geq \exp(\theta t)$  and  $\text{Tr}(\exp(\theta \mathbf{X})) = \sum_i \exp(\theta \lambda_i)$  by Claim 8.2.4, thus  $\lambda_n \geq t \Rightarrow \text{Tr}(\exp(\theta \mathbf{X})) \geq \exp(\theta t)$ . Then, using Markov's inequality,

$$\begin{aligned} \Pr[\lambda_n \geq t] &\leq \Pr[\text{Tr}(\exp(\theta \mathbf{X})) \geq \exp(\theta t)] \\ &\leq \exp(-\theta t) \mathbb{E}[\text{Tr}(\exp(\theta \mathbf{X}))] \end{aligned}$$

For two independent random variables  $\mathbf{U}$  and  $\mathbf{V}$ , we have

$$\mathbb{E}_{\mathbf{U}, \mathbf{V}} f(\mathbf{U}, \mathbf{V}) = \mathbb{E}_{\mathbf{U}} \mathbb{E}_{\mathbf{V}} [f(\mathbf{U}, \mathbf{V}) | \mathbf{U}] = \mathbb{E}_{\mathbf{U}} \mathbb{E}_{\mathbf{V}} [f(\mathbf{U}, \mathbf{V})].$$

Define  $\mathbf{X}_{<i} = \sum_{j<i} \mathbf{X}_j$ . Let  $0 < \theta \leq 1/R$ ,

$$\begin{aligned}
\mathbb{E} \operatorname{Tr} (\exp(\theta \mathbf{X})) &= \mathbb{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{k-1}} \mathbb{E}_{\mathbf{X}_k} \operatorname{Tr} \exp \left( \underbrace{\theta \mathbf{X}_{<k}}_{\mathbf{H}} + \underbrace{\theta \mathbf{X}_k}_{=\log \exp(\theta \mathbf{X}_k)} \right), \quad \{\mathbf{X}_i\} \text{ are independent} \\
&\leq \mathbb{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{k-1}} \operatorname{Tr} \exp \left( \theta \mathbf{X}_{<k} + \log \mathbb{E} \exp(\theta \mathbf{X}_k) \right), \quad \text{by 8.2.15 and 8.2.16} \\
&\leq \mathbb{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{k-1}} \operatorname{Tr} \exp \left( \theta \mathbf{X}_{<k} + \log \mathbb{E} [\mathbf{I} + \theta \mathbf{X}_k + \theta^2 \mathbf{X}_k^2] \right), \quad \text{by 8.2.13, 8.2.7, and 8.2.9} \\
&\leq \mathbb{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{k-1}} \operatorname{Tr} \exp \left( \theta \mathbf{X}_{<k} + \theta^2 \mathbb{E} \mathbf{X}_k^2 \right), \quad \text{by 8.2.14 and 8.2.7} \\
&= \mathbb{E}_{\mathbf{X}_1, \dots, \mathbf{X}_{k-2}} \mathbb{E}_{\mathbf{X}_{k-1}} \operatorname{Tr} \exp \left( \underbrace{\theta^2 \mathbb{E} \mathbf{X}_k^2 + \theta \mathbf{X}_{<k-1} + \theta \mathbf{X}_{k-1}}_{\mathbf{H}} \right), \\
&\quad \vdots \\
&\leq \operatorname{Tr} \exp \left( \theta^2 \sum_i \mathbb{E} [\mathbf{X}_i^2] \right), \\
&\leq \operatorname{Tr} \exp (\theta^2 \sigma^2 \mathbf{I}), \quad \text{by 8.2.7 and } \sum_i \mathbb{E} [\mathbf{X}_i^2] \preceq \sigma^2 \mathbf{I} \\
&= n \cdot \exp(\theta^2 \sigma^2).
\end{aligned}$$

Then,

$$\Pr[\lambda_n \geq t] \leq n \cdot \exp(-\theta t + \theta^2 \sigma^2),$$

and

$$\Pr[\|\mathbf{X}\| \geq t] \leq 2n \cdot \exp(-\theta t + \theta^2 \sigma^2).$$

Similar to the proof of Bernstein concentration bound for one-dimension random variable, minimize the RHS over  $0 < \theta \leq 1/R$  yields

$$\Pr[\|\mathbf{X}\| \geq t] \leq 2n \cdot \exp \left( \frac{-t^2}{2Rt + 4\sigma^2} \right).$$

□

### 8.3 Spectral Graph Sparsification

In this section, we will see that for any dense graph, we can find another sparser graph whose graph Laplacian is approximately the same as measured by their quadratic forms. This turns out to be a very useful tool for designing algorithms.



**Definition 8.3.1.** Given  $\mathbf{A}, \mathbf{B} \in S_+^n$  and  $\epsilon > 0$ , we say

$$\mathbf{A} \approx_\epsilon \mathbf{B} \text{ if and only if } \frac{1}{1+\epsilon} \mathbf{A} \preceq \mathbf{B} \preceq (1+\epsilon) \mathbf{A}.$$

Suppose we start with a connected graph  $G = (V, E, \mathbf{w})$ , where as usual we say that  $|V| = n$  and  $|E| = m$ . We want to produce another graph  $\tilde{G} = (V, \tilde{E}, \tilde{\mathbf{w}})$  s.t.  $|\tilde{E}| \ll |E|$  and at the same time  $\mathbf{L}_G \approx_\epsilon \mathbf{L}_{\tilde{G}}$ . We call  $\tilde{G}$  a *spectral sparsifier* of  $G$ . Our construction will also ensure that  $\tilde{E} \subseteq E$ , although this is not important in most applications. Figure 8.2 shows an example of a graph  $G$  and spectral sparsifier  $\tilde{G}$ .

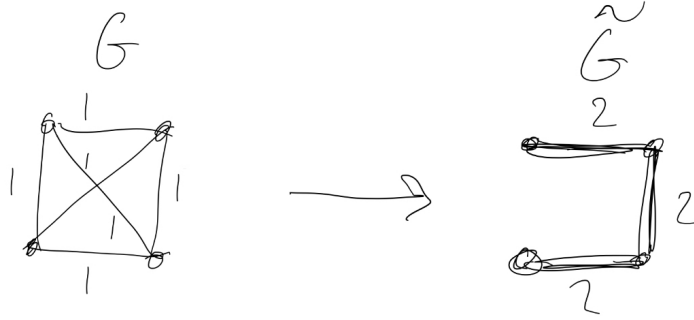


Figure 8.2: A graph  $G$  and a spectral sparsifier  $\tilde{G}$ , satisfying  $\mathbf{L}_G \approx_\epsilon \mathbf{L}_{\tilde{G}}$  for  $\epsilon = 2.42$ .

We are going to construct  $\tilde{G}$  by sampling some of the edges of  $G$  according to a suitable probability distribution and scaling up their weight to make up for the fact that we pick fewer of them.

To get a better understanding for the notion of approximation given in 8.3.1 means, let's observe a simple consequence of it.

Given a vertex subset  $T \subseteq V$ , we say that  $(T, V \setminus T)$  is a *cut* in  $G$  and that the value of the cut is

$$c_G(T) = \sum_{e \in E \cap (T \times V \setminus T)} \mathbf{w}(e).$$

Figure 8.3 shows the  $c_G(T)$  in a graph  $G$ .

**Theorem 8.3.2.** If  $\mathbf{L}_G \approx_\epsilon \mathbf{L}_{\tilde{G}}$ , then for all  $T \subseteq V$ ,

$$\frac{1}{1+\epsilon} c_G(T) \leq c_{\tilde{G}}(T) \leq (1+\epsilon) c_G(T).$$

*Proof.* Let  $\mathbf{1}_T \in \mathbb{R}^V$  be the indicator of the set  $T$ , i.e.  $\mathbf{1}_T(u) = 1$  for  $u \in T$  and  $\mathbf{1}_T(u) = 0$  otherwise. We can see that  $\mathbf{1}_T^\top \mathbf{L}_G \mathbf{1}_T = c_G(T)$ , and hence the theorem follows by comparing the quadratic forms.  $\square$

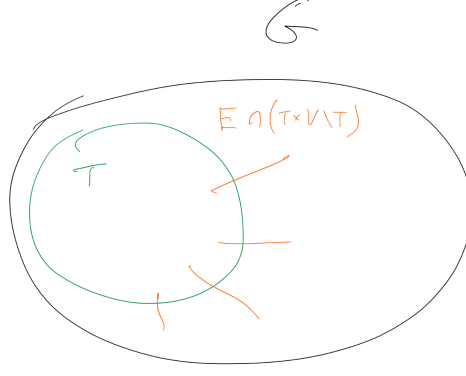


Figure 8.3: The cut  $c_G(T)$  in  $G$ .

But how well can we spectrally approximate a graph with a sparse graph? The next theorem gives us a nearly optimal answer to this question.

**Theorem 8.3.3** (Spectral Graph Approximation by Sampling, (Spielman-Srivastava 2008)). *Consider a connected graph  $G = (V, E, \mathbf{w})$ , with  $n = |V|$ . For any  $0 < \epsilon < 1$  and  $0 < \delta < 1$ , there exist sampling probabilities  $p_e$  for each edge  $e \in E$  s.t. if we include each edge  $e$  in  $\tilde{E}$  independently with probability  $p_e$  and set its weight  $\tilde{\mathbf{w}}(e) = \frac{1}{p_e} \mathbf{w}(e)$ , then with probability at least  $1 - \delta$  the graph  $\tilde{G} = (V, \tilde{E}, \tilde{\mathbf{w}})$  satisfies*

$$\mathbf{L}_G \approx_\epsilon \mathbf{L}_{\tilde{G}} \text{ and } |\tilde{E}| \leq O(n\epsilon^{-2} \log(n/\delta)).$$

The original proof can be found in [SS11].

**Remark 8.3.4.** For convenience, we will abbreviate  $\mathbf{L}_G$  as  $\mathbf{L}$  and  $\mathbf{L}_{\tilde{G}}$  as  $\tilde{\mathbf{L}}$  in the rest of this section.

We are going to analyze a sampling procedure by turning our goal into a problem of matrix concentration. Recall that

**Fact 8.3.5.**  $\mathbf{A} \preceq \mathbf{B}$  implies  $\mathbf{C}\mathbf{A}\mathbf{C}^\top \preceq \mathbf{C}\mathbf{B}\mathbf{C}^\top$  for any  $\mathbf{C} \in \mathbb{R}^{n \times n}$ .

By letting  $\mathbf{C} = \mathbf{L}^{+/2}$ , we can see that

$$\mathbf{L} \approx_\epsilon \tilde{\mathbf{L}} \text{ implies } \mathbf{\Pi}_L \approx_\epsilon \mathbf{L}^{+/2} \tilde{\mathbf{L}} \mathbf{L}^{+/2}, \quad (8.2)$$

where  $\mathbf{\Pi}_L = \mathbf{L}^{+/2} \mathbf{L} \mathbf{L}^{+/2}$  is the orthogonal projection to the complement of the kernel of  $\mathbf{L}$ .

**Definition 8.3.6.** Given a matrix  $\mathbf{A}$ , we define  $\mathbf{\Pi}_A$  to be the orthogonal projection to the complement of the kernel of  $\mathbf{A}$ , i.e.  $\mathbf{\Pi}_A \mathbf{v} = \mathbf{0}$  for  $\mathbf{v} \in \ker(\mathbf{A})$  and  $\mathbf{\Pi}_A \mathbf{v} = \mathbf{v}$  for  $\mathbf{v} \in \ker(\mathbf{A})^\perp$ . Recall that  $\ker(\mathbf{A})^\perp = \text{im}(\mathbf{A}^\top)$ .

**Claim 8.3.7.** For a matrix  $\mathbf{A} \in S^n$  with spectral decomposition  $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\top = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^\top$  s.t.  $\mathbf{V}^\top \mathbf{V} = \mathbf{I}$ , we have  $\mathbf{\Pi}_A = \sum_{i: \lambda_i \neq 0} \mathbf{v}_i \mathbf{v}_i^\top$ , and  $\mathbf{\Pi}_A = \mathbf{A}^{+/2} \mathbf{A} \mathbf{A}^{+/2} = \mathbf{A} \mathbf{A}^+ = \mathbf{A}^+ \mathbf{A}$ .

From the definition, we can see that  $\mathbf{\Pi}_L = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$ .

Now that we understand the projection  $\mathbf{\Pi}_L$ , it is not hard to show the following claim.

**Claim 8.3.8.**

1.  $\mathbf{\Pi}_L \approx_\epsilon \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2}$  implies  $\mathbf{L} \approx_\epsilon \tilde{\mathbf{L}}$ .
2. For  $\epsilon \leq 1$ , we have that  $\left\| \mathbf{\Pi}_L - \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \right\| \leq \epsilon/2$  implies  $\mathbf{\Pi}_L \approx_\epsilon \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2}$ .

Really, the only idea needed here is that when comparing quadratic forms in matrices with the same kernel, we necessarily can't have the quadratic forms disagree on vectors in the kernel. Simple! But we are going to write it out carefully, since we're still getting used to these types of calculations.

*Proof of Claim 8.3.8.* To prove Part 1, we assume  $\mathbf{\Pi}_L \approx_\epsilon \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2}$ . Recall that  $G$  is a connected graph, so  $\ker(\mathbf{L}) = \text{span}\{\mathbf{1}\}$ , while  $\tilde{\mathbf{L}}$  is the Laplacian of a graph which may or may not be connected, so  $\ker(\tilde{\mathbf{L}}) \supseteq \ker(\mathbf{L})$ , and equivalently  $\text{im}(\tilde{\mathbf{L}}) \subseteq \text{im}(\mathbf{L})$ . Now, for any  $\mathbf{v} \in \ker(\mathbf{L})$  we have  $\mathbf{v}^\top \tilde{\mathbf{L}} \mathbf{v} = 0 = \mathbf{v}^\top \mathbf{L} \mathbf{v}$ . For any  $\mathbf{v} \in \ker(\mathbf{L})^\perp$  we have  $\mathbf{v} = \mathbf{L}^{+1/2} \mathbf{z}$  for some  $\mathbf{z}$ , as  $\ker(\mathbf{L})^\perp = \text{im}(\mathbf{L}) = \text{im}(\mathbf{L}^{+1/2})$ . Hence

$$\mathbf{v}^\top \tilde{\mathbf{L}} \mathbf{v} = \mathbf{z}^\top \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \mathbf{z} \geq \frac{1}{1+\epsilon} \mathbf{z}^\top \mathbf{L}^{+1/2} \mathbf{L} \mathbf{L}^{+1/2} \mathbf{z} = \frac{1}{1+\epsilon} \mathbf{v}^\top \mathbf{L} \mathbf{v}$$

and similarly

$$\mathbf{v}^\top \tilde{\mathbf{L}} \mathbf{v} = \mathbf{z}^\top \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \mathbf{z} \leq (1+\epsilon) \mathbf{z}^\top \mathbf{L}^{+1/2} \mathbf{L} \mathbf{L}^{+1/2} \mathbf{z} = (1+\epsilon) \mathbf{v}^\top \mathbf{L} \mathbf{v}.$$

Thus we have established  $\mathbf{L} \approx_\epsilon \tilde{\mathbf{L}}$ .

To prove Part 2, we assume  $\left\| \mathbf{\Pi}_L - \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \right\| \leq \epsilon/2$ . This is equivalent to

$$-\frac{\epsilon}{2} \mathbf{I} \preceq \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} - \mathbf{\Pi}_L \preceq \frac{\epsilon}{2} \mathbf{I}$$

But since

$$\mathbf{1}^\top (\mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} - \mathbf{\Pi}_L) \mathbf{1} = 0,$$

we can in fact sharpen this to

$$-\frac{\epsilon}{2} \mathbf{\Pi}_L \preceq \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} - \mathbf{\Pi}_L \preceq \frac{\epsilon}{2} \mathbf{\Pi}_L.$$

Rearranging, we then conclude

$$(1 - \frac{\epsilon}{2}) \mathbf{\Pi}_L \preceq \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \preceq (1 + \frac{\epsilon}{2}) \mathbf{\Pi}_L.$$

Finally, we note that  $1/(1+\epsilon) \leq (1 - \frac{\epsilon}{2})$  to reach our conclusion,  $\mathbf{\Pi}_L \approx_\epsilon \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2}$ .  $\square$

We now have most of the tools to prove Theorem 8.3.3, but to help us, we are going to establish one small piece of helpful notation: We define a matrix function  $\Phi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  by

$$\Phi(\mathbf{A}) = \mathbf{L}^{+/2} \mathbf{A} \mathbf{L}^{+/2}.$$

We sometimes call this a “normalizing map”, because it transforms a matrix to the space where spectral norm bounds can be translated into relative error guarantees compare to the  $\mathbf{L}$  quadratic form.

*Proof of Theorem 8.3.3.* By Claim 8.3.8, it suffices to show

$$\left\| \mathbf{\Pi}_L - \mathbf{L}^{+/2} \tilde{\mathbf{L}} \mathbf{L}^{+/2} \right\| \leq \epsilon/2. \quad (8.3)$$

We introduce a set of independent random variables, one for each edge  $e$ , with a probability  $p_e$  associated with the edge which we will fix later. We then let

$$\mathbf{Y}_e = \begin{cases} \frac{\mathbf{w}(e)}{p_e} \mathbf{b}_e \mathbf{b}_e^\top & \text{with probability } p_e \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

This way,  $\tilde{\mathbf{L}} = \sum_e \mathbf{Y}_e$ . Note that  $\mathbb{E}[\mathbf{Y}_e] = p_e \frac{\mathbf{w}(e)}{p_e} \mathbf{b}_e \mathbf{b}_e^\top = \mathbf{w}(e) \mathbf{b}_e \mathbf{b}_e^\top$ , and so

$$\mathbb{E}[\tilde{\mathbf{L}}] = \sum_e \mathbb{E}[\mathbf{Y}_e] = \mathbf{L}.$$

By linearity of  $\Phi$ ,

$$\mathbb{E}[\Phi(\tilde{\mathbf{L}})] = \Phi(\mathbb{E}[\tilde{\mathbf{L}}]) = \mathbf{\Pi}_L.$$

Let us also define

$$\mathbf{X}_e = \Phi(\mathbf{Y}_e) - \mathbb{E}[\Phi(\mathbf{Y}_e)] \text{ and } \mathbf{X} = \sum_e \mathbf{X}_e$$

Note that this ensures  $\mathbb{E}[\mathbf{X}_e] = \mathbf{0}$ . We are now going to fix the edge sampling probabilities, in a way that depends on some overall scaling parameter  $\alpha > 0$ . We let

$$p_e = \min(\alpha \|\Phi(\mathbf{w}(e) \mathbf{b}_e \mathbf{b}_e^\top)\|, 1)$$

then we see from the definition of  $\mathbf{Y}_e$  that whenever  $p_e < 1$

$$\|\Phi(\mathbf{Y}_e)\| \leq \frac{1}{\alpha}$$

from this, we can conclude, with a bit of work, that for all  $e$

$$\|\mathbf{X}_e\| \leq \frac{1}{\alpha}. \quad (8.4)$$

We can also show that

$$\left\| \sum_e \mathbb{E}[\mathbf{X}_e^2] \right\| \leq \frac{1}{\alpha}. \quad (8.5)$$

In the exercises for this chapter, we will ask you to show that Equations (8.4) and (8.5) hold.

This means that we can apply Theorem 8.1.4 to our  $\mathbf{X} = \sum_e \mathbf{X}_e$ , with  $R = \frac{1}{\alpha}$  and  $\sigma^2 = \frac{1}{\alpha}$ , to get

$$\Pr \left[ \left\| \mathbf{\Pi}_L - \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \right\| \geq \epsilon/2 \right] \leq 2n \exp \left( \frac{-0.25\epsilon^2}{(\epsilon + 4)/\alpha} \right)$$

Since  $0 < \epsilon < 1$ , this means that if  $\alpha = 40\epsilon^{-2} \log(n/\delta)$ , then

$$\Pr \left[ \left\| \mathbf{\Pi}_L - \mathbf{L}^{+1/2} \tilde{\mathbf{L}} \mathbf{L}^{+1/2} \right\| \geq \epsilon/2 \right] \leq \frac{2n\delta^2}{n^2} \leq \delta/2.$$

In the last step, we assumed  $n \geq 4$ .

Lastly, we'd like to know that the graph  $\tilde{G}$  is sparse. The number of edges in  $\tilde{G}$  is equal to the number of  $\mathbf{Y}_e$  that come out nonzero. Thus, the expected value of  $|\tilde{E}|$  is

$$\mathbb{E} \left[ |\tilde{E}| \right] = \sum_e p_e \leq \alpha \sum_e \mathbf{w}(e) \left\| \mathbf{L}^{+1/2} \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+1/2} \right\|$$

We can bound the sum of the norms with a neat trick relating it to the trace of  $\mathbf{\Pi}_L$ . Note that in general for a vector  $\mathbf{a} \in \mathbb{R}^n$ , we have  $\|\mathbf{a}\mathbf{a}^\top\| = \mathbf{a}^\top \mathbf{a} = \text{Tr}(\mathbf{a}\mathbf{a}^\top)$ . And hence

$$\begin{aligned} \sum_e \mathbf{w}(e) \left\| \mathbf{L}^{+1/2} \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+1/2} \right\| &= \sum_e \mathbf{w}(e) \text{Tr} \left( \mathbf{L}^{+1/2} \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+1/2} \right) \\ &= \text{Tr} \left( \mathbf{L}^{+1/2} \left( \sum_e \mathbf{w}(e) \mathbf{b}_e \mathbf{b}_e^\top \right) \mathbf{L}^{+1/2} \right) \\ &= \text{Tr}(\mathbf{\Pi}_L) = n - 1. \end{aligned}$$

Thus with our choice of  $\alpha$ ,

$$\mathbb{E} \left[ |\tilde{E}| \right] \leq 40\epsilon^{-2} \log(n/\delta)n.$$

With a scalar Chernoff bound, can show that  $|\tilde{E}| \leq O(\epsilon^{-2} \log(n/\delta)n)$  with probability at least  $1 - \delta/2$ . Thus by a union bound, the this condition and Equation (8.3) are both satisfied with probability at least  $1 - \delta$ .  $\square$

**Remark 8.3.9.** Note that

$$\|\Phi(\mathbf{w}(e) \mathbf{b}_e \mathbf{b}_e^\top)\| = \mathbf{w}(e) \left\| \mathbf{L}^{+1/2} \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+1/2} \right\| \leq \mathbf{w}(e) \left\| \mathbf{L}^{+1/2} \mathbf{b}_e \right\|_2^2.$$

Recall that in Chapter 6, we saw that the effective resistance between vertex  $v$  and vertex  $u$  is given by  $\left\| \mathbf{L}^{+1/2}(\mathbf{e}_u - \mathbf{e}_v) \right\|_2^2$ , and for an edge  $e$  connecting vertex  $u$  and  $v$ , we have  $\mathbf{b}_e = \mathbf{e}_u - \mathbf{e}_v$ . That means the norm of the “baby Laplacian”  $\mathbf{w}(e) \mathbf{b}_e \mathbf{b}_e^\top$  of a single edge with weight  $\mathbf{w}(e)$  is exactly  $\mathbf{w}(e)$  times the effective resistance between the two endpoints of the edge.

We haven't shown how to compute the sampling probabilities efficiently, so right now, it isn't clear whether we can efficiently find  $\tilde{G}$ . It turns out that if we have access to a fast algorithm for solving Laplacian linear equations, then we can find sufficiently good approximations to the effective resistances quickly, and use these to compute  $\tilde{G}$ . An algorithm for this is described in [SS11].

# Chapter 9

## Solving Laplacian Linear Equations

### 9.1 Solving Linear Equations Approximately

Given a Laplacian  $\mathbf{L}$  of a connected graph and a demand vector  $\mathbf{d} \perp \mathbf{1}$ , we want to find  $\mathbf{x}^*$  solving the linear equation  $\mathbf{L}\mathbf{x}^* = \mathbf{d}$ . We are going to focus on fast algorithms for finding approximate (but highly accurate) solutions.

This means we need a notion of an approximate solution. Since our definition is not special to Laplacians, we state it more generally for positive semi-definite matrices.

**Definition 9.1.1.** Given PSD matrix  $\mathbf{M}$  and  $\mathbf{d} \in \ker(\mathbf{M})^\perp$ , let  $\mathbf{M}\mathbf{x}^* = \mathbf{d}$ . We say that  $\tilde{\mathbf{x}}$  is an  $\epsilon$ -approximate solution to the linear equation  $\mathbf{M}\mathbf{x} = \mathbf{d}$  if

$$\|\tilde{\mathbf{x}} - \mathbf{x}^*\|_{\mathbf{M}}^2 \leq \epsilon \|\mathbf{x}^*\|_{\mathbf{M}}^2.$$

**Remark 9.1.2.** The requirement  $\mathbf{d} \in \ker(\mathbf{M})^\perp$  can be removed, but this is not important for us.

**Theorem 9.1.3** (Spielman and Teng (2004) [ST04]). *Given a Laplacian  $\mathbf{L}$  of a weighted undirected graph  $G = (V, E, \mathbf{w})$  with  $|E| = m$  and  $|V| = n$  and a demand vector  $\mathbf{d} \in \mathbb{R}^V$ , we can find  $\tilde{\mathbf{x}}$  that is an  $\epsilon$ -approximate solution to  $\mathbf{L}\mathbf{x} = \mathbf{d}$ , using an algorithm that takes time  $O(m \log^c n \log(1/\epsilon))$  for some fixed constant  $c$  and succeeds with probability  $1 - 1/n^{10}$ .*

In the original algorithm of Spielman and Teng, the exponent on the log in the running time was  $c \approx 70$ .

Today, we are going to see a simpler algorithm. But first, we'll look at one of the key tools behind all algorithms for solving Laplacian linear equations quickly.

## 9.2 Preconditioning and Approximate Gaussian Elimination

Recall our definition of two positive semi-definite matrices being approximately equal.

**Definition 9.2.1** (Spectral approximation). Given  $\mathbf{A}, \mathbf{B} \in S_+^n$ , we say that

$$\mathbf{A} \approx_K \mathbf{B} \text{ if and only if } \frac{1}{1+K} \mathbf{A} \preceq \mathbf{B} \preceq (1+K) \mathbf{A}.$$

Suppose we have a positive definite matrix  $\mathbf{M} \in S_{++}^n$  and want to solve a linear equation  $\mathbf{M}\mathbf{x} = \mathbf{d}$ . We can do this using gradient descent or accelerated gradient descent, as we covered in Graded Homework 1. But if we have access to an easy-to-invert matrix that happens to also be a good spectral approximation of  $\mathbf{M}$ , then we can use this to speed up the (accelerated) gradient descent algorithm. An example of this would be that we have a factorization  $\mathbf{L}\mathbf{L}^\top \approx_K \mathbf{M}$ , where  $\mathbf{L}$  is lower triangular and sparse, which means we can invert it quickly.

The following lemma, which you will prove in Problem Set 6, makes this preconditioning precise.

**Lemma 9.2.2.** *Given a matrix  $\mathbf{M} \in S_{++}^n$ , a vector  $\mathbf{d}$  and a decomposition  $\mathbf{M} \approx_K \mathbf{L}\mathbf{L}^\top$ , we can find  $\tilde{\mathbf{x}}$  that  $\epsilon$ -approximately solves  $\mathbf{M}\mathbf{x} = \mathbf{d}$ , using  $O((1+K) \log(K/\epsilon)(T_{matvec} + T_{sol} + n))$  time.*

- $T_{matvec}$  denotes the time required to compute  $\mathbf{M}\mathbf{z}$  given a vector  $\mathbf{z}$ , i.e. a “matrix-vector multiplication”.
- $T_{sol}$  denotes the time required to compute  $\mathbf{L}^{-1}\mathbf{z}$  or  $(\mathbf{L}^\top)^{-1}\mathbf{z}$  given a vector  $\mathbf{z}$ .

**Dealing with pseudo-inverses.** When our matrices have a null space, preconditioning becomes slightly more complicated, but as long as it is easy to project to the complement of the null space, there’s no real issue. The following describes precisely what we need (but you can ignore the null-space issue when first reading these notes without losing anything significant).

**Lemma 9.2.3.** *Given a matrix  $\mathbf{M} \in S_+^n$ , a vector  $\mathbf{d} \in \ker(\mathbf{M})^\perp$  and a decomposition  $\mathbf{M} \approx_K \mathbf{L}\mathbf{D}\mathbf{L}^\top$ , where  $\mathbf{L}$  is invertible, we can find  $\tilde{\mathbf{x}}$  that  $\epsilon$ -approximately solves  $\mathbf{M}\mathbf{x} = \mathbf{d}$ , using  $O((1+K) \log(K/\epsilon)(T_{matvec} + T_{sol} + T_{proj} + n))$  time.*

- $T_{matvec}$  denotes the time required to compute  $\mathbf{M}\mathbf{z}$  given a vector  $\mathbf{z}$ , i.e. a “matrix-vector multiplication”.
- $T_{sol}$  denotes the time required to compute  $\mathbf{L}^{-1}\mathbf{z}$  and  $(\mathbf{L}^\top)^{-1}\mathbf{z}$  and  $\mathbf{D}^+\mathbf{z}$  given a vector  $\mathbf{z}$ .



- $T_{\text{proj}}$  denotes the time required to compute  $\Pi_M \mathbf{z}$  given a vector  $\mathbf{z}$ .

**Theorem 9.2.4** (Kyng and Sachdeva (2015) [KS16]). *Given a Laplacian  $\mathbf{L}$  of a weighted undirected graph  $G = (V, E, \mathbf{w})$  with  $|E| = M$  and  $|V| = n$ , we can find a decomposition  $\mathcal{L}\mathcal{L}^\top \approx_{0.5} \mathbf{L}$ , such that  $\mathcal{L}$  has number of non-zeroes  $\text{nnz}(\mathcal{L}) = O(m \log^3 n)$ , with probability at least  $1 - 3/n^5$ . in time  $O(m \log^3 n)$ .*

We can combine Theorem 9.2.4 with Lemma 9.2.3 to get a fast algorithm for solving Laplacian linear equations.

**Corollary 9.2.5.** *Given a Laplacian  $\mathbf{L}$  of a weighted undirected graph  $G = (V, E, \mathbf{w})$  with  $|E| = m$  and  $|V| = n$  and a demand vector  $\mathbf{d} \in \mathbb{R}^V$ , we can find  $\tilde{\mathbf{x}}$  that is an  $\epsilon$ -approximate solution to  $\mathbf{L}\mathbf{x} = \mathbf{d}$ , using an algorithm that takes time  $O(m \log^3 n \log(1/\epsilon))$  and succeeds with probability  $1 - 1/n^{10}$ .*

*Proof sketch.* First we need to get a factorization that confirms to Lemma 9.2.3. The decomposition  $\mathcal{L}\mathcal{L}^\top$  provided by Theorem 9.2.4 can be rewritten as  $\mathcal{L}\mathcal{L}^\top = \tilde{\mathcal{L}}\mathcal{D}(\tilde{\mathcal{L}})^\top$  where  $\tilde{\mathcal{L}}$  is equal to  $\mathcal{L}$  except  $\mathcal{L}(n, n) = 1$  and we let  $\mathcal{D}$  be the identity matrix, except  $\mathcal{D}(n, n) = 0$ . This ensures  $\mathcal{D}^+ = \mathcal{D}$  and that  $\tilde{\mathcal{L}}$  is invertible and lower triangular with  $O(m \log^3 n)$  non-zeros. We note that the inverse of an invertible lower or upper triangular matrix with  $N$  non-zeros can be applied in time  $O(N)$  given an adjacency list representation of the matrix. Finally, as  $\ker(\mathcal{L}\mathcal{L}^\top) = \text{span}\{\mathbf{1}\}$ , we have  $\Pi_{\tilde{\mathcal{L}}\mathcal{D}(\tilde{\mathcal{L}})^\top} = \mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$ , and this projection matrix can be applied in  $O(n)$  time. Altogether, this means that  $T_{\text{matvec}} + T_{\text{sol}} + T_{\text{proj}} = O(n)$ , which suffices to complete the proof.  $\square$

### 9.3 Approximate Gaussian Elimination Algorithm

Recall *Gaussian Elimination / Cholesky decomposition* of a graph Laplacian  $\mathbf{L}$ . We will use  $\mathbf{A}(:, i)$  to denote the  $i$ th column of a matrix  $\mathbf{A}$ . We can write the algorithm as

---

**Algorithm 1:** Gaussian Elimination / Cholesky Decomposition

---

**Input:** Graph Laplacian  $\mathbf{L}$

**Output:** Lower triangular  $\mathcal{L}$  s.t.  $\mathcal{L}\mathcal{L}^\top = \mathbf{L}$

Let  $\mathbf{S}_0 = \mathbf{L}$ ;

**for**  $i = 1$  **to**  $i = n - 1$  **do**

$$\left[ \begin{array}{l} \mathbf{l}_i = \frac{1}{\sqrt{\mathbf{S}_{i-1}(i, i)}} \mathbf{S}_{i-1}(:, i); \\ \mathbf{S}_i = \mathbf{S}_{i-1} - \mathbf{l}_i \mathbf{l}_i^\top. \end{array} \right.$$

$\mathbf{l}_n = \mathbf{0}_{n \times 1}$ ;

**return**  $\mathcal{L} = [\mathbf{l}_1 \cdots \mathbf{l}_n]$ ;

---

We want to introduce some notation that will help us describe and analyze a faster version of Gaussian elimination – one that uses sampling to create a sparse approximation of the decomposition.

Consider a Laplacian  $\mathbf{S}$  of a graph  $H$  and a vertex  $v$  of  $H$ . We define  $\text{STAR}(v, \mathbf{S})$  to be the Laplacian of the subgraph of  $H$  consisting of edges incident on  $v$ . We define

$$\text{CLIQUE}(v, \mathbf{S}) = \text{STAR}(v, \mathbf{S}) - \frac{1}{\mathbf{S}(v, v)} \mathbf{S}(:, v) \mathbf{S}(:, v)^\top$$

For example, suppose

$$\mathbf{L} = \begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \text{diag}(\mathbf{a}) + \mathbf{L}_{-1} \end{pmatrix}$$

Then

$$\text{STAR}(1, \mathbf{L}) = \begin{pmatrix} W & -\mathbf{a}^\top \\ -\mathbf{a} & \text{diag}(\mathbf{a}) \end{pmatrix} \text{ and } \text{CLIQUE}(1, \mathbf{L}) = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \text{diag}(\mathbf{a}) - \frac{1}{W} \mathbf{a} \mathbf{a}^\top \end{pmatrix}$$

which is illustrated in Figure 9.1.

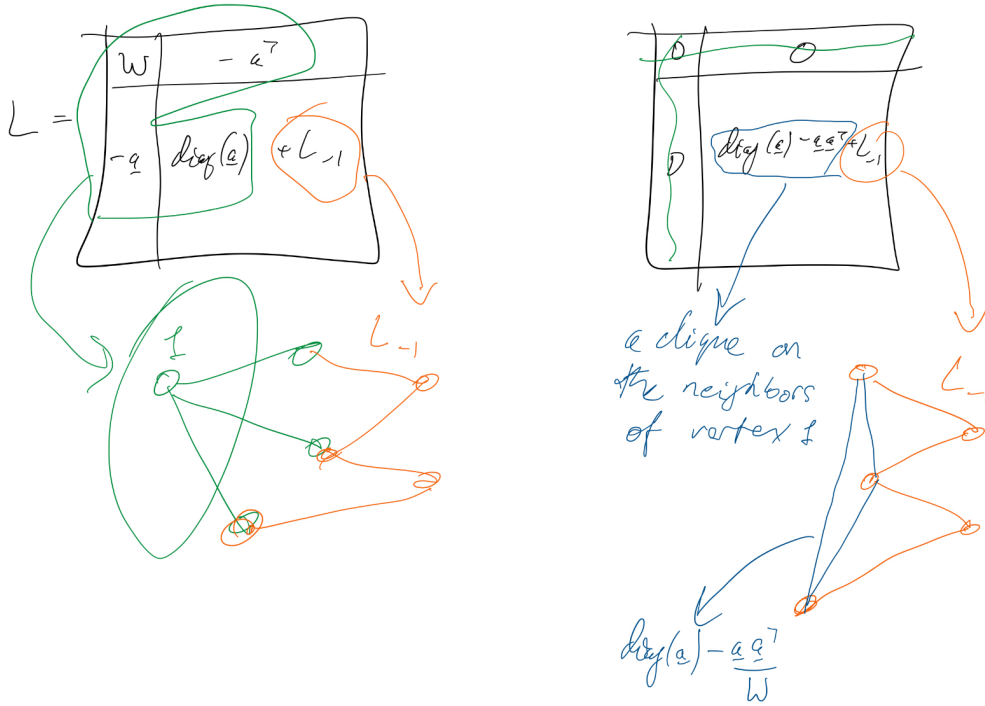


Figure 9.1: Gaussian Elimination:  $\text{CLIQUE}(1, \mathbf{L}) = \text{STAR}(1, \mathbf{L}) - \frac{1}{L(1,1)} \mathbf{L}(:, 1) \mathbf{L}(:, 1)^\top$ .

In Chapter 7, we proved that  $\text{CLIQUE}(v, \mathbf{S})$  is a graph Laplacian – it follows from the proof of Claim 7.1.1 in that chapter. Thus we have that following.

**Claim 9.3.1.** *If  $\mathbf{S}$  is the Laplacian of a connected graph, then  $\text{CLIQUE}(v, \mathbf{S})$  is a graph Laplacian.*

Note that in Algorithm 1, we have  $\mathbf{l}_i \mathbf{l}_i^\top = \text{STAR}(v_i, \mathbf{S}_{i-1}) - \text{CLIQUE}(v_i, \mathbf{S}_{i-1})$ . The update rule can be rewritten as

$$\mathbf{S}_i = \mathbf{S}_{i-1} - \text{STAR}(v_i, \mathbf{S}_{i-1}) + \text{CLIQUE}(v_i, \mathbf{S}_{i-1}),$$

This also provides way to understand why Gaussian Elimination is slow in some cases. At each step, one vertex is eliminated, but a clique is added to the subgraph on the remaining vertices, making the graph denser. And at the  $i$ th step, computing  $\text{STAR}(v_i, \mathbf{S}_{i-1})$  takes around  $\deg(v_i)$  time, but computing  $\text{CLIQUE}(v_i, \mathbf{S}_{i-1})$  requires around  $\deg(v_i)^2$  time. In order to speed up Gaussian Elimination, the algorithmic idea of [KS16] is to plug in a sparser approximate of the intended clique instead of the entire one.

The following procedure  $\text{CLIQUESAMPLE}(v, \mathbf{S})$  produces a sparse approximation of  $\text{CLIQUE}(v, \mathbf{S})$ . Let  $V$  be the vertex set of the graph associated with  $\mathbf{S}$  and  $E$  the edge set. We define  $\mathbf{b}_{i,j} \in \mathbb{R}^V$  to be the vector with

$$\mathbf{b}_{i,j}(i) = 1 \text{ and } \mathbf{b}_{i,j}(j) = -1 \text{ and } \mathbf{b}_{i,j}(k) = 0 \text{ for } k \neq i, j.$$

Given weights  $\mathbf{w} \in \mathbb{R}^E$  and a vertex  $v \in V$ , we let

$$\mathbf{w}_v = \sum_{(u,v) \in E} \mathbf{w}(u, v).$$

---

**Algorithm 2:**  $\text{CLIQUESAMPLE}(v, \mathbf{S})$

---

**Input:** Graph Laplacian  $\mathbf{S} \in \mathbb{R}^{V \times V}$ , of a graph with edge weights  $\mathbf{w}$ , and vertex  $v \in V$

**Output:**  $\mathbf{Y}_v \in \mathbb{R}^{V \times V}$  sparse approximation of  $\text{CLIQUE}(v, \mathbf{S})$

$\mathbf{Y}_v \leftarrow \mathbf{0}_{n \times n}$ ;

**foreach** *Multiedge*  $e = (v, i)$  *from*  $v$  *to* *a neighbor*  $i$  **do**

Randomly pick a neighbor  $j$  of  $v$  with probability  $\frac{\mathbf{w}(j,v)}{\mathbf{w}_v}$ ;  
 If  $i \neq j$ , let  $\mathbf{Y}_v \leftarrow \mathbf{Y}_v + \frac{\mathbf{w}(i,v)\mathbf{w}(j,v)}{\mathbf{w}(i,v) + \mathbf{w}(j,v)} \mathbf{b}_{i,j} \mathbf{b}_{i,j}^\top$ ;

**return**  $\mathbf{Y}_v$ ;

---

**Remark 9.3.2.** We can implement each sampling of a neighbor  $j$  in  $O(1)$  time using a classical algorithm known as Walker's method (also known as the Alias method or Vose's method). This algorithm requires an additional  $O(\deg_{\mathbf{S}}(v))$  time to initialize a data structure used for sampling. Overall, this means the total time for  $O(\deg_{\mathbf{S}}(v))$  samples is still  $O(\deg_{\mathbf{S}}(v))$ .

**Lemma 9.3.3.**  $\mathbb{E}[\mathbf{Y}_v] = \text{CLIQUE}(v, \mathbf{S})$ .

*Proof.* Let  $\mathbf{C} = \text{CLIQUE}(v, \mathbf{S})$ . Observe that both  $\mathbb{E}[\mathbf{Y}_v]$  and  $\mathbf{C}$  are Laplacians. Thus it suffices to verify  $\mathbb{E}_{\mathbf{Y}_v(i,j)} = \mathbf{C}(i, j)$  for  $i \neq j$ .

$$\begin{aligned} \mathbf{C}(i, j) &= -\frac{\mathbf{w}(i, v)\mathbf{w}(j, v)}{\mathbf{w}_v}, \\ \mathbb{E}_{\mathbf{Y}_v(i,j)} &= -\frac{\mathbf{w}(i, v)\mathbf{w}(j, v)}{\mathbf{w}(i, v) + \mathbf{w}(j, v)} \left( \frac{\mathbf{w}(j, v)}{\mathbf{w}_v} + \frac{\mathbf{w}(i, v)}{\mathbf{w}_v} \right) = -\frac{\mathbf{w}(i, v)\mathbf{w}(j, v)}{\mathbf{w}_v} = \mathbf{C}(i, j). \end{aligned}$$

□

**Remark 9.3.4.** Lemma 9.3.3 shows that  $\text{CLIQUESAMPLE}(v, \mathbf{L})$  produces the original  $\text{CLIQUE}(v, \mathbf{L})$  in expectation.

Now, we define *Approximate Gaussian Elimination*.

---

**Algorithm 3:** Approximate Gaussian Elimination / Cholesky Decomposition

---

**Input:** Graph Laplacian  $\mathbf{L}$

**Output:** Lower triangular<sup>a</sup>  $\mathcal{L}$  as given in Theorem 9.2.4

Let  $\mathbf{S}_0 = \mathbf{L}$ ;

Generate a random permutation  $\pi$  on  $[n]$ ;

**for**  $i = 1$  **to**  $i = n - 1$  **do**

$$\left[ \begin{array}{l} \mathbf{l}_i = \frac{1}{\sqrt{\mathbf{S}_{i-1}(\pi(i), \pi(i))}} \mathbf{S}_{i-1}(:, \pi(i)); \\ \mathbf{S}_i = \mathbf{S}_{i-1} - \text{STAR}(\pi(i), \mathbf{S}_{i-1}) + \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}) \end{array} \right.$$

$\mathbf{l}_n = \mathbf{0}_{n \times 1}$ ;

**return**  $\mathcal{L} = [\mathbf{l}_1 \cdots \mathbf{l}_n]$  and  $\pi$ ;

---

<sup>a</sup> $\mathcal{L}$  is not actually lower triangular. However, if we let  $\mathbf{P}_\pi$  be the permutation matrix corresponding to  $\pi$ , then  $\mathbf{P}_\pi \mathcal{L}$  is lower triangular. Knowing the ordering that achieves this is enough to let us implement forward and backward substitution for solving linear equations in  $\mathcal{L}$  and  $\mathcal{L}^\top$ .

Note that if we replace  $\text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1})$  by  $\text{CLIQUE}(\pi(i), \mathbf{S}_{i-1})$  at each step, then we can recover Gaussian Elimination, but with a random elimination order.

## 9.4 Analyzing Approximate Gaussian Elimination

In this Section, we're going to analyze Approximate Gaussian Elimination, and see why it works.

Ultimately, the main challenge in proving Theorem 9.2.4 will be to prove for the output  $\mathcal{L}$  of Algorithm 3 that with high probability

$$0.5\mathbf{L} \preceq \mathcal{L}\mathcal{L}^\top \preceq 1.5\mathbf{L}. \quad (9.1)$$

We can reduce this to proving that with high probability

$$\left\| \mathbf{L}^{+/2} (\mathcal{L}\mathcal{L}^\top - \mathbf{L}) \mathbf{L}^{+/2} \right\| \leq 0.5 \quad (9.2)$$

Ultimately, the proof is going to have a lot in common with our proof of Matrix Bernstein in Chapter 8. Overall, the lesson there was that when we have a sum of independent, zero-mean random matrices, we can show that the sum is likely to have small spectral norm if the spectral norm of each random matrix is small, and the matrix-valued variance is also small.

Thus, to replicate the proof, we need control over

1. The *sample norms*.

## 2. The *sample variance*.

But, there is seemingly another major obstacle: We are trying to analyze a process where the samples are far from independent. Each time we sample edges, we add new edges to the remaining graph, which we will later sample again. This creates a lot of dependencies between the samples, which we have to handle.

However, it turns out that independence is more than what is needed to prove concentration. Instead, it suffices to have a sequence of random variables such that each is mean-zero in expectation, conditional on the previous ones. This is called a martingale difference sequence. We'll now learn about those.

### 9.4.1 Normalization, a.k.a. Isotropic Position

Since our analysis requires frequently measuring matrices after right and left-multiplication by  $\mathbf{L}^{+/2}$ , we reintroduce the “normalizing map”  $\Phi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  defined by

$$\Phi(\mathbf{A}) = \mathbf{L}^{+/2} \mathbf{A} \mathbf{L}^{+/2}.$$

We previously saw this in Chapter 8.

### 9.4.2 Martingales

A scalar martingale is a sequence of random variables  $Z_0, \dots, Z_k$ , such that

$$\mathbb{E}[Z_i \mid Z_0, \dots, Z_{i-1}] = Z_{i-1}. \quad (9.3)$$

That is, conditional on the outcome of all the previous random variables, the expectation of  $Z_i$  equals  $Z_{i-1}$ . If we unravel the sequence of conditional expectations, we get that *without conditioning*  $\mathbb{E}[Z_k] = \mathbb{E}[Z_0]$ .

Typically, we use martingales to show a statement along like “ $Z_k$  is concentrated around  $\mathbb{E}[Z_k]$ ”.

We can also think of a martingale in terms of the sequence of changes in the  $Z_i$  variables. Let  $X_i = Z_i - Z_{i-1}$ . The sequence of  $X_i$ s is called a martingale difference sequence. We can now state the martingale condition as

$$\mathbb{E}[X_i \mid Z_0, \dots, Z_{i-1}] = 0.$$

And because  $Z_0$  and  $X_1, \dots, X_{i-1}$  completely determine  $Z_1, \dots, Z_{i-1}$ , we could also write the martingale condition equivalently as

$$\mathbb{E}[X_i \mid Z_0, X_1, \dots, X_{i-1}] = 0.$$

Crucially, we can write

$$Z_k = Z_0 + \sum_{i=1}^k Z_i - Z_{i-1} = Z_0 + \sum_{i=1}^k X_i$$

and when we are trying to prove concentration, the martingale difference property of the  $X_i$ 's is often “as good as” independence, meaning that  $\sum_{i=1}^k X_i$  concentrates similarly to a sum of independent random variables.

**Matrix-valued martingales.** We can also define matrix-valued martingales. In this case, we replace the martingale condition of Equation (9.3), with the condition that the whole matrix stays the same in expectation. For example, we could have a sequence of random matrices  $\mathbf{Z}_0, \dots, \mathbf{Z}_k \in \mathbb{R}^{n \times n}$ , such that

$$\mathbb{E}[\mathbf{Z}_i \mid \mathbf{Z}_0, \dots, \mathbf{Z}_{i-1}] = \mathbf{Z}_{i-1}. \quad (9.4)$$

**Lemma 9.4.1.** *Let  $\mathbf{L}_i = \mathbf{S}_i + \sum_{j=1}^i \mathbf{l}_j \mathbf{l}_j^\top$  for  $i = 1, \dots, n$  and  $\mathbf{L}_0 = \mathbf{S}_0 = \mathbf{L}$ . Then*

$$\mathbb{E}[\mathbf{L}_i \mid \text{all random variables before } \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1})] = \mathbf{L}_{i-1}.$$

*Proof.* Let's only consider  $i = 1$  here as other cases are similar.

$$\mathbf{L}_0 = \mathbf{L} = \mathbf{l}_1 \mathbf{l}_1^\top + \text{CLIQUE}(v, \mathbf{L}) + \mathbf{L}_{-1}$$

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{l}_1 \mathbf{l}_1^\top + \text{CLIQUESAMPLE}(v, \mathbf{L}) + \mathbf{L}_{-1} \\ \mathbb{E}[\mathbf{L}_1 \mid \pi(1)] &= \mathbf{l}_1 \mathbf{l}_1^\top + \mathbb{E}[\text{CLIQUESAMPLE}(v, \mathbf{L}) \mid \pi(1)] + \mathbf{L}_{-1} \\ &= \mathbf{l}_1 \mathbf{l}_1^\top + \text{CLIQUE}(v, \mathbf{L}) + \mathbf{L}_{-1} \\ &= \mathbf{L}_0 \end{aligned}$$

where we used Lemma 9.3.3 to get  $\mathbb{E}[\text{CLIQUESAMPLE}(v, \mathbf{L}) \mid \pi(1)] = \text{CLIQUE}(v, \mathbf{L})$ .  $\square$

**Remark 9.4.2.**  $\sum_{j=1}^i \mathbf{l}_j \mathbf{l}_j^\top$  can be treated as what has already been eliminated by (Approximate) Gaussian Elimination, while  $\mathbf{S}_i$  is what still left or going to be eliminated. In Approximate Gaussian Elimination,  $\mathbf{L}_n = \sum_{i=1}^n \mathbf{l}_i \mathbf{l}_i^\top$  and our goal is to show that  $\mathbf{L}_n \approx_K \mathbf{L}$ . Note that  $\mathbf{L}_i$  is always equal to the original Laplacian  $\mathbf{L}$  for all  $i$  in Gaussian Elimination. Lemma 9.4.1 demonstrates that  $\mathbf{L}_0, \mathbf{L}_1, \dots, \mathbf{L}_n$  forms a matrix martingale.

Ultimately, our plan is to use this matrix martingale structure to show that “ $\mathbf{L}_n$  is concentrated around  $\mathbf{L}$ ” in some appropriate sense. More precisely, the spectral approximation we would like to show can be established by showing that “ $\Phi(\mathbf{L}_n)$  is concentrated around  $\Phi(\mathbf{L})$ ”

### 9.4.3 Martingale Difference Sequence as Edge-Samples

We start by taking a slightly different view of the observations we used to prove Lemma 9.4.1. Recall that  $\mathbf{L}_i = \mathbf{S}_i + \sum_{j=1}^i \mathbf{l}_j \mathbf{l}_j^\top$ , and  $\mathbf{L}_{i-1} = \mathbf{S}_{i-1} + \sum_{j=1}^{i-1} \mathbf{l}_j \mathbf{l}_j^\top$  and

$$\mathbf{S}_i = \mathbf{S}_{i-1} - \text{STAR}(\pi(i), \mathbf{S}_{i-1}) + \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}).$$

Putting these together, we get

$$\begin{aligned} \mathbf{L}_i - \mathbf{L}_{i-1} &= \mathbf{l}_i \mathbf{l}_i^\top + \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}) - \text{STAR}(\pi(i), \mathbf{S}_{i-1}) \\ &= \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}) - \text{CLIQUE}(\pi(i), \mathbf{S}_{i-1}) \\ &= \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}) - \mathbb{E}[\text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}) \mid \text{preceding samples}] \\ &\quad \text{by Lemma 9.3.3.} \end{aligned} \tag{9.5}$$

In particular, recall that by Lemma 9.3.3, conditional on the randomness before the call to  $\text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1})$ , we have

$$\mathbb{E}[\text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1}) \mid \text{preceding samples}] = \text{CLIQUE}(\pi(i), \mathbf{S}_{i-1})$$

Adopting the notation of Lemma 9.3.3 we write

$$\mathbf{Y}_{\pi(i)} = \text{CLIQUESAMPLE}(\pi(i), \mathbf{S}_{i-1})$$

and we further introduce notation each multi-edge sample for  $e \in \text{STAR}(\pi(i), \mathbf{S}_{i-1})$ , as  $\mathbf{Y}_{\pi(i),e}$ , denoting the random edge Laplacian sampled when the algorithm is processing multi-edge  $e$ . Thus, conditional on preceding samples, we have

$$\mathbf{Y}_{\pi(i)} = \sum_{e \in \text{STAR}(\pi(i), \mathbf{S}_{i-1})} \mathbf{Y}_{\pi(i),e} \tag{9.6}$$

Note that even the number of multi-edges in  $\text{STAR}(\pi(i), \mathbf{S}_{i-1})$  depends on the preceding samples. We also want to associate zero-mean variables with each edge. Conditional on preceding samples, we also define

$$\mathbf{X}_{i,e} = \Phi(\mathbf{Y}_{\pi(i),e} - \mathbb{E}[\mathbf{Y}_{\pi(i),e}]) \text{ and } \mathbf{X}_i = \sum_{e \in \text{STAR}(\pi(i), \mathbf{S}_{i-1})} \mathbf{X}_{i,e}$$

and combining this with Equations (9.5) and (9.6)

$$\mathbf{X}_i = \Phi(\mathbf{Y}_{\pi(i)} - \mathbb{E}[\mathbf{Y}_{\pi(i)}]) = \Phi(\mathbf{L}_i - \mathbf{L}_{i-1})$$

Altogether, we can write

$$\Phi(\mathbf{L}_n - \mathbf{L}) = \sum_{i=1}^n \Phi(\mathbf{L}_i - \mathbf{L}_{i-1}) = \sum_{i=1}^n \mathbf{X}_i = \sum_{i=1}^n \sum_{e \in \text{STAR}(\pi(i), \mathbf{S}_{i-1})} \mathbf{X}_{i,e}$$

Note that the  $\mathbf{X}_{i,e}$  variables form a martingale difference sequence, because the linearity of  $\Phi$  ensures they are zero-mean conditional on preceding randomness.

### 9.4.4 Stopped Martingales

Unfortunately, directly analyzing the concentration properties of the  $\mathbf{L}_i$  martingale that we just introduced turns out to be difficult. The reason is that we're trying to prove some very delicate multiplicative error guarantees. And, if we analyze  $\mathbf{L}_i$ , we find that the multiplicative error is not easy to control, *after it's already gotten big*. But that's not really what we care about anyway: We want to say it never gets big in the first place, with high probability. So we need to introduce another martingale, that lets us ignore the bad case when the error has already gotten too big. At the same time, we also need to make sure that statements about our new martingale can help us prove guarantees about  $\mathbf{L}_i$ . Fortunately, we can achieve both at once. The technique we use is related to the much broader topic of martingale *stopping times*, which we only scratch the surface of here. We're also going to be quite informal about it, in the interest of brevity. Lecture notes by Tropp [Tro19] give a more formal introduction for those who are interested.

We define the stopped martingale sequence  $\tilde{\mathbf{L}}_i$  by

$$\tilde{\mathbf{L}}_i = \begin{cases} \mathbf{L}_i & \text{if for all } j < i \text{ we have } \mathbf{L}_j \preceq 1.5\mathbf{L} \\ \mathbf{L}_{j^*} & \text{for } j^* \text{ being the least } j \text{ such that } \mathbf{L}_j \not\preceq 1.5\mathbf{L} \end{cases} \quad (9.7)$$

Figure 9.2 shows the  $\tilde{\mathbf{L}}_i$  martingale getting stuck at the first time  $\mathbf{L}_{j^*} \not\preceq 1.5\mathbf{L}$ .

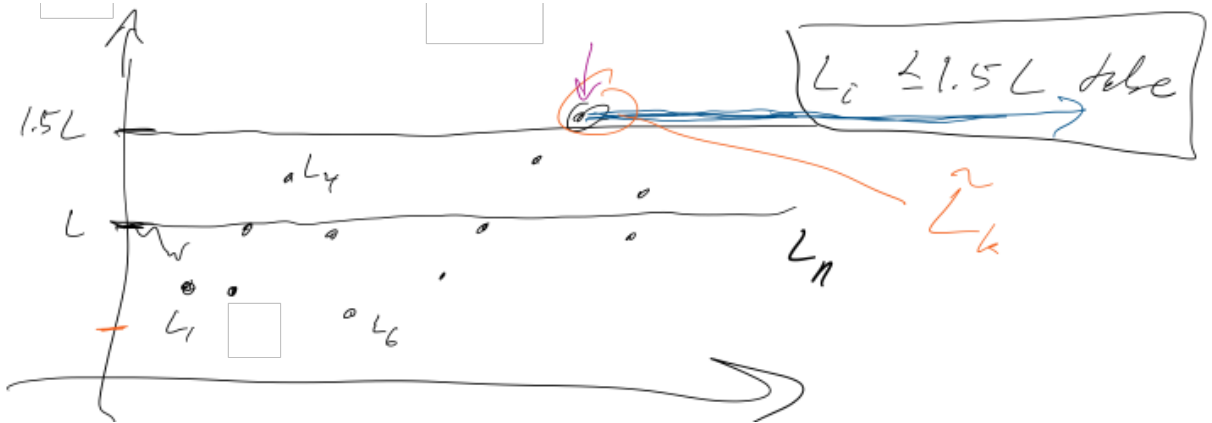


Figure 9.2: Gaussian Elimination :  $\text{CLIQUE}(1, \mathbf{L}) = \text{STAR}(1, \mathbf{L}) - \frac{1}{L(1,1)} \mathbf{L}(:, 1) \mathbf{L}(:, 1)^\top$ .

We state the following without proof:

**Claim 9.4.3.**

1. The sequence  $\{\tilde{\mathbf{L}}_i\}$  for  $i = 0, \dots, n$  is a martingale.
2.  $\|\mathbf{L}^{+/2}(\tilde{\mathbf{L}}_i - \mathbf{L})\mathbf{L}^{+/2}\| \leq 0.5$  implies  $\|\mathbf{L}^{+/2}(\mathbf{L}_i - \mathbf{L})\mathbf{L}^{+/2}\| \leq 0.5$



The martingale property also implies that the unconditional expectation satisfies  $\mathbb{E}[\tilde{\mathbf{L}}_n] = \mathbf{L}$ . The proof of the claim is easy to sketch: For Part 1, each difference is zero-mean if the condition has not been violated, and is identically zero (and hence zero-mean) if it has been violated. For Part 2, if the martingale  $\{\tilde{\mathbf{L}}_i\}$  has stopped, then  $\|\mathbf{L}^{+1/2}(\tilde{\mathbf{L}}_i - \mathbf{L})\mathbf{L}^{+1/2}\| \leq 0.5$  is false, and the implication is vacuously true. If, on the other hand, the martingale has not stopped, the quantities are equal, because  $\tilde{\mathbf{L}}_i = \mathbf{L}_i$ , and again it's easy to see the implication holds.

Thus, ultimately, our strategy is going to be to show that  $\|\mathbf{L}^{+1/2}(\tilde{\mathbf{L}}_i - \mathbf{L})\mathbf{L}^{+1/2}\| \leq 0.5$  with high probability. Expressed using the normalizing map  $\Phi(\cdot)$ , our goal is to show that with high probability

$$\|\Phi(\tilde{\mathbf{L}}_n - \mathbf{L})\| \leq 0.5.$$

**Stopped martingale difference sequence.** In order to prove the spectral norm bound, we want to express the  $\{\tilde{\mathbf{L}}_i\}$  martingale in terms of a sequence of martingale differences. To this end, we define  $\tilde{\mathbf{X}}_i = \Phi(\tilde{\mathbf{L}}_i - \tilde{\mathbf{L}}_{i-1})$ . This ensures that

$$\tilde{\mathbf{X}}_i = \begin{cases} \mathbf{X}_i & \text{if for all } j < i \text{ we have } \mathbf{L}_j \preceq 1.5\mathbf{L} \\ \mathbf{0} & \text{otherwise} \end{cases} \quad (9.8)$$

Whenever the modified martingale  $\tilde{\mathbf{X}}_i$  has not yet stopped, we also introduce individual modified edge samples  $\tilde{\mathbf{X}}_{i,e} = \mathbf{X}_{i,e}$ . If the martingale *has* stopped, i.e.  $\tilde{\mathbf{X}}_i = \mathbf{0}$ , then we can take these edge samples  $\tilde{\mathbf{X}}_{i,e}$  to be zero. We can now write

$$\Phi(\tilde{\mathbf{L}}_n - \mathbf{L}) = \sum_{i=1}^n \Phi(\tilde{\mathbf{L}}_i - \tilde{\mathbf{L}}_{i-1}) = \sum_{i=1}^n \tilde{\mathbf{X}}_i = \sum_{i=1}^n \sum_{e \in \text{STAR}(\pi(i), \mathbf{S}_{i-1})} \tilde{\mathbf{X}}_{i,e}.$$

Thus, we can see that Equation (9.2) is implied by

$$\left\| \sum_{i=1}^n \tilde{\mathbf{X}}_i \right\| \leq 0.5. \quad (9.9)$$

### 9.4.5 Sample Norm Control

In this Subsection, we're going to see that the norms of each multi-edge sample is controlled throughout the algorithm.

**Lemma 9.4.4.** *Given two Laplacians  $\mathbf{L}$  and  $\mathbf{S}$  on the same vertex set.<sup>1</sup> If each multiedge  $e$  of  $\text{STAR}(v, \mathbf{S})$  has bounded norm in the following sense,*

$$\left\| \mathbf{L}^{+1/2} \mathbf{w}_{\mathbf{S}}(e) \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+1/2} \right\| \leq R,$$

---

<sup>1</sup> $\mathbf{L}$  can be regarded as the original Laplacian we care about, while  $\mathbf{S}$  can be regarded as some intermediate Laplacian appearing during Approximate Gaussian Elimination.

then each possible sampled multiedge  $e'$  of  $\text{CLIQUE\_SAMPLE}(v, \mathbf{S})$  also satisfies

$$\left\| \mathbf{L}^{+1/2} \mathbf{w}_{\text{new}}(e') \mathbf{b}_{e'} \mathbf{b}_{e'}^\top \mathbf{L}^{+1/2} \right\| \leq R.$$

*Proof.* Let  $\mathbf{w} = \mathbf{w}_{\mathbf{S}}$  for simplicity. Consider a sampled edge between  $i$  and  $j$  with weight  $\mathbf{w}_{\text{new}}(i, j) = \mathbf{w}(i, v) \mathbf{w}(j, v) / (\mathbf{w}(i, v) + \mathbf{w}(j, v))$ .

$$\begin{aligned} \left\| \mathbf{L}^{+1/2} \mathbf{w}_{\text{new}}(i, j) \mathbf{b}_{ij} \mathbf{b}_{ij}^\top \mathbf{L}^{+1/2} \right\| &= \mathbf{w}_{\text{new}}(i, j) \left\| \mathbf{L}^{+1/2} \mathbf{b}_{ij} \mathbf{b}_{ij}^\top \mathbf{L}^{+1/2} \right\| \\ &= \mathbf{w}_{\text{new}}(i, j) \left\| \mathbf{L}^{+1/2} \mathbf{b}_{ij} \right\|^2 \\ &\leq \mathbf{w}_{\text{new}}(i, j) \left( \left\| \mathbf{L}^{+1/2} \mathbf{b}_{iv} \right\|^2 + \left\| \mathbf{L}^{+1/2} \mathbf{b}_{jv} \right\|^2 \right) \\ &= \frac{\mathbf{w}(j, v)}{\mathbf{w}(i, v) + \mathbf{w}(j, v)} \left\| \mathbf{L}^{+1/2} \mathbf{w}(i, v) \mathbf{b}_{iv} \mathbf{b}_{iv}^\top \mathbf{L}^{+1/2} \right\| + \\ &\quad \frac{\mathbf{w}(i, v)}{\mathbf{w}(i, v) + \mathbf{w}(j, v)} \left\| \mathbf{L}^{+1/2} \mathbf{w}(j, v) \mathbf{b}_{jv} \mathbf{b}_{jv}^\top \mathbf{L}^{+1/2} \right\| \\ &\leq \frac{\mathbf{w}(j, v)}{\mathbf{w}(i, v) + \mathbf{w}(j, v)} R + \frac{\mathbf{w}(i, v)}{\mathbf{w}(i, v) + \mathbf{w}(j, v)} R \\ &= R \end{aligned}$$

The first inequality uses the triangle inequality of effective resistance in  $\mathbf{L}$ , in that effective resistance is a distance as we proved in Chapter 6. The second inequality just uses the conditions of this lemma.  $\square$

**Remark 9.4.5.** Lemma 9.4.4 only requires that each single multiedge has small norm instead of that the sum of all edges between a pair of vertices have small norm. And this lemma tells us, after sampling, each multiedge in the new graph still satisfies the bounded norm condition.

From the Lemma, we can conclude that each edge sample  $\mathbf{Y}_{\pi(i), e}$  satisfies  $\|\Phi(\mathbf{Y}_{\pi(i), e})\| \leq R$  provided the assumptions of the Lemma hold. Let's record this observation as a Lemma.

**Lemma 9.4.6.** *If for all  $e \in \text{STAR}(v, \mathbf{S}_i)$ ,*

$$\left\| \Phi(\mathbf{w}_{\mathbf{S}_i}(e) \mathbf{b}_e \mathbf{b}_e^\top) \right\| \leq R.$$

*then all  $e \in \text{STAR}(\pi(i), \mathbf{S}_i)$ ,*

$$\left\| \Phi(\mathbf{Y}_{\pi(i), e}) \right\| \leq R.$$

**Preprocessing by multi-edge splitting.** In the original graph of Laplacian  $\mathbf{L}$  of graph  $G = (V, E, \mathbf{w})$ , we have for each edge  $\hat{e}$  that

$$\mathbf{w}(\hat{e}) \mathbf{b}_{\hat{e}} \mathbf{b}_{\hat{e}}^\top \preceq \sum_e \mathbf{w}(e) \mathbf{b}_e \mathbf{b}_e^\top = \mathbf{L}$$

This also implies that

$$\left\| \mathbf{L}^{+/2} \mathbf{w}(\hat{e}) \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+/2} \right\| \leq 1.$$

Now, that means that if we split every original edge  $e$  of the graph into  $K$  multi-edges  $e_1, \dots, e_K$ , with a fraction  $1/K$  of the weight, we get a new graph  $G' = (V, E', \mathbf{w}')$  such that

**Claim 9.4.7.**

1.  $G'$  and  $G$  have the same graph Laplacian.
2.  $|E'| = K |E|$
3. For every multi-edge in  $G'$

$$\left\| \mathbf{L}^{+/2} \mathbf{w}'(e) \mathbf{b}_e \mathbf{b}_e^\top \mathbf{L}^{+/2} \right\| \leq 1/K.$$

Before we run Approximate Gaussian Elimination, we are going to do this multi-edge splitting to ensure we have control over multi-edge sample norms. Combined with Lemma 9.4.4 immediately establishes the next lemma, because we start off with all multi-edges having bounded norm and only produce multi-edges with bounded norm.

**Lemma 9.4.8.** *When Algorithm 3 is run on the (multi-edge) Laplacian of  $G'$ , arising from splitting edges of  $G$  into  $K$  multi-edges, the every edge sample  $\mathbf{Y}_{\pi(i),e}$  satisfies*

$$\left\| \Phi(\mathbf{Y}_{\pi(i),e}) \right\| \leq 1/K.$$

As we will see later  $K = 200 \log^2 n$  suffices.

## 9.4.6 Random Matrix Concentration from Trace Exponentials

Let us recall how matrix-valued variances come into the picture when proving concentration following the strategy from Matrix Bernstein in Chapter 8.

For some matrix-valued random variable  $\mathbf{X} \in S^n$ , we'd like to show  $\Pr[\|\mathbf{X}\| \leq 0.5]$ . Using Markov's inequality, and some observations about matrix exponentials and traces, we saw that for all  $\theta > 0$ ,

$$\Pr[\|\mathbf{X}\| \geq 0.5] \leq \exp(-0.5\theta) (\mathbb{E}[\text{Tr}(\exp(\theta \mathbf{X}))] + \mathbb{E}[\text{Tr}(\exp(-\theta \mathbf{X}))]). \quad (9.10)$$

We then want to bound  $\mathbb{E}[\text{Tr}(\exp(\theta \mathbf{X}))]$  using Lieb's theorem. We can handle  $\mathbb{E}[\text{Tr}(\exp(-\theta \mathbf{X}))]$  similarly.

**Theorem 9.4.9 (Lieb).** *Let  $f : S_{++}^n \rightarrow \mathbb{R}$  be a matrix function given by*

$$f(\mathbf{A}) = \text{Tr}(\exp(\mathbf{H} + \log(\mathbf{A})))$$

*for some  $\mathbf{H} \in S^n$ . Then  $-f$  is convex (i.e.  $f$  is concave).*

As observed by Tropp, this is useful for proving matrix concentration statements. Combined with Jensen's inequality, it gives that for a random matrix  $\mathbf{X} \in S^n$  and a fixed  $\mathbf{H} \in S^n$

$$\mathbb{E} [\text{Tr} (\exp (\mathbf{H} + \mathbf{X}))] \leq \text{Tr} (\exp (\mathbf{H} + \log(\mathbb{E} [\exp(\mathbf{X})]))).$$

The next crucial step was to show that it suffices to obtain an upper bound on the matrix  $\mathbb{E} [\exp(\mathbf{X})]$  w.r.t the Loewner order. Using the following three lemmas, this conclusion is an immediate corollary.

**Lemma 9.4.10.** *If  $\mathbf{A} \preceq \mathbf{B}$ , then  $\text{Tr} (\exp(\mathbf{A})) \leq \text{Tr} (\exp(\mathbf{B}))$ .*

**Lemma 9.4.11.** *If  $0 \prec \mathbf{A} \preceq \mathbf{B}$ , then  $\log(\mathbf{A}) \preceq \log(\mathbf{B})$ .*

**Lemma 9.4.12.**  *$\log(\mathbf{I} + \mathbf{A}) \preceq \mathbf{A}$  for  $\mathbf{A} \succ -\mathbf{I}$ .*

**Corollary 9.4.13.** *For a random matrix  $\mathbf{X} \in S^n$  and a fixed  $\mathbf{H} \in S^n$ , if  $\mathbb{E} [\exp(\mathbf{X})] \preceq \mathbf{I} + \mathbf{U}$  where  $\mathbf{U} \succ -\mathbf{I}$ , then*

$$\mathbb{E} [\text{Tr} (\exp (\mathbf{H} + \mathbf{X}))] \leq \text{Tr} (\exp (\mathbf{H} + \mathbf{U})).$$

## 9.4.7 Mean-Exponential Bounds from Variance Bounds

To use Corollary 9.4.13, we need to construct useful upper bounds on  $\mathbb{E} [\exp(\mathbf{X})]$ . This can be done, starting from the following lemma.

**Lemma 9.4.14.**  *$\exp(\mathbf{A}) \preceq \mathbf{I} + \mathbf{A} + \mathbf{A}^2$  for  $\|\mathbf{A}\| \leq 1$ .*

If  $\mathbf{X}$  is zero-mean and  $\|\mathbf{X}\| \leq 1$ , this means that  $\mathbb{E} [\exp(\mathbf{X})] \preceq \mathbf{I} + \mathbb{E} [\mathbf{X}^2]$ , which is how we end up wanting to bound the matrix-valued variance  $\mathbb{E} [\mathbf{X}^2]$ . In the rest of this Subsection, we're going to see the matrix-valued variance of the stopped martingale is bounded throughout the algorithm.

Firstly, we note that for a single edge sample  $\tilde{\mathbf{X}}_{i,e}$ , by Lemma 9.4.8, we have that

$$\left\| \tilde{\mathbf{X}}_{i,e} \right\| \leq \left\| \Phi (\mathbf{Y}_{\pi(i),e} - \mathbb{E} [\mathbf{Y}_{\pi(i),e}]) \right\| \leq 1/K,$$

using that  $\|\mathbf{A} - \mathbf{B}\| \leq \max(\|\mathbf{A}\|, \|\mathbf{B}\|)$ , for  $\mathbf{A}, \mathbf{B} \succeq \mathbf{0}$ , and  $\|\mathbb{E}[\mathbf{A}]\| \leq \mathbb{E}[\|\mathbf{A}\|]$  by Jensen's inequality.

Thus, if  $0 < \theta \leq K$ , we have that

$$\begin{aligned} \mathbb{E} \left[ \exp(\theta \tilde{\mathbf{X}}_{i,e}) \mid \text{preceding samples} \right] &\preceq \mathbf{I} + \mathbb{E} \left[ (\theta \tilde{\mathbf{X}}_{i,e})^2 \mid \text{preceding samples} \right] \\ &\preceq \mathbf{I} + \frac{1}{K} \theta^2 \cdot \mathbb{E} \left[ \Phi(\mathbf{Y}_{\pi(i),e}) \mid \text{preceding samples} \right] \end{aligned} \quad (9.11)$$

### 9.4.8 The Overall Mean-Trace-Exponential Bound

We will use  $\mathbb{E}_{(<i)}$  to denote expectation over variables preceding the  $i$ th elimination step. We are going to refrain from explicitly writing out conditioning in our expectations, but any *inner* expectation that appears inside another *outer* expectation should be taken as conditional on the outer expectation. We are going to use  $d_i$  to denote the multi-edge degree of vertex  $\pi(i)$  in  $\mathbf{S}_{i-1}$ . This is exactly the number of edge samples in the  $i$ th elimination. Note that there is no elimination at step  $n$  (the algorithm is already finished). As a notational convenience, let's write  $\hat{n} = n - 1$ . With all that in mind, we bound the mean-trace-exponential for some parameter  $0 < \theta \leq 0.5/\sqrt{K}$

$$\begin{aligned}
& \mathbb{E} \operatorname{Tr} \left( \exp \left( \theta \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i \right) \right) \tag{9.12} \\
&= \mathbb{E}_{(<\hat{n})} \mathbb{E}_{\pi(\hat{n})} \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},1}} \cdots \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}-1}} \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}}} \operatorname{Tr} \exp \left( \underbrace{\sum_{i=1}^{\hat{n}-1} \theta \tilde{\mathbf{X}}_i + \sum_{e=1}^{d_{\hat{n}}-1} \theta \tilde{\mathbf{X}}_{\hat{n},e}}_{\mathbf{H}} + \theta \tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}} \right) \\
&\quad \tilde{\mathbf{X}}_{\hat{n},1}, \dots, \tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}} \text{ are independent conditional on } (<\hat{n}), \pi(\hat{n}) \\
&\leq \mathbb{E}_{(<\hat{n})} \mathbb{E}_{\pi(\hat{n})} \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},1}} \cdots \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}-1}} \operatorname{Tr} \exp \left( \sum_{i=1}^{\hat{n}-1} \theta \tilde{\mathbf{X}}_i + \sum_{e=1}^{d_{\hat{n}}-1} \theta \tilde{\mathbf{X}}_{\hat{n},e} + \frac{1}{K} \theta^2 \cdot \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}}} \Phi(\mathbf{Y}_{\pi(\hat{n}),d_{\hat{n}}}) \right) \\
&\quad \text{By Equation (9.11) and Corollary 9.4.13 .} \\
&\vdots \quad \text{Repeat for each multi-edge sample } \tilde{\mathbf{X}}_{\hat{n},1}, \dots, \tilde{\mathbf{X}}_{\hat{n},d_{\hat{n}}-1} \\
&\leq \mathbb{E}_{(<\hat{n})} \mathbb{E}_{\pi(\hat{n})} \operatorname{Tr} \exp \left( \sum_{i=1}^{\hat{n}-1} \theta \tilde{\mathbf{X}}_i + \sum_{e=1}^{d_{\hat{n}}} \frac{1}{K} \theta^2 \cdot \mathbb{E}_{\tilde{\mathbf{X}}_{\hat{n},e}} \Phi(\mathbf{Y}_{\pi(\hat{n}),e}) \right) \\
&= \mathbb{E}_{(<\hat{n})} \mathbb{E}_{\pi(\hat{n})} \operatorname{Tr} \exp \left( \sum_{i=1}^{\hat{n}-1} \theta \tilde{\mathbf{X}}_i + \frac{1}{K} \theta^2 \Phi(\operatorname{CLIQUE}(\pi(\hat{n}), \mathbf{S}_{\hat{n}-1})) \right)
\end{aligned}$$

To further bound this quantity, we now need to deal with the random choice of  $\pi(\hat{n})$ . We'll be able to use this to bound the trace-exponential in a very strong way. From a random matrix perspective, it's the following few steps that give the analysis its surprising strength.

We can treat  $\frac{1}{K} \theta^2 \Phi(\operatorname{CLIQUE}(\pi(\hat{n}), \mathbf{S}_{\hat{n}-1}))$  as a random matrix. It is not zero-mean, but we can still bound the trace-exponential using Corollary 9.4.13.

We can also bound the expected matrix exponential in that case, using a simple corollary of Lemma 9.4.14.

**Corollary 9.4.15.**  $\exp(\mathbf{A}) \preceq \mathbf{I} + (1 + R)\mathbf{A}$  for  $\mathbf{0} \preceq \mathbf{A}$  with  $\|\mathbf{A}\| \leq R \leq 1$ .

*Proof.* The conclusion follows after observing that for  $\mathbf{0} \preceq \mathbf{A}$  with  $\|\mathbf{A}\| \leq R$ , we have  $\mathbf{A}^2 \preceq R\mathbf{A}$ . We can see this by considering the spectral decomposition of  $\mathbf{A}$  and dealing with each eigenvalue separately.  $\square$

Next, we need a simple structural observation about the cliques created by elimination:

**Claim 9.4.16.**

$$\text{CLIQUE}(\pi(i), \mathbf{S}_i) \preceq \text{STAR}(\pi(i), \mathbf{S}_i) \preceq \mathbf{S}_i$$

*Proof.* The first inequality is immediate from  $\text{CLIQUE}(\pi(i), \mathbf{S}_i) \preceq \text{CLIQUE}(\pi(i), \mathbf{S}_i) + \mathbf{l}_i \mathbf{l}_i^\top = \text{STAR}(\pi(i), \mathbf{S}_i)$ . The latter inequality  $\text{STAR}(\pi(i), \mathbf{S}_i) \preceq \mathbf{S}_i$  follows from the star being a subgraph of the whole Laplacian  $\mathbf{S}_i$ .  $\square$

Next we make use of the fact that  $\tilde{\mathbf{X}}_i$  is from the difference sequence of the *stopped* martingale. This means we can assume

$$\mathbf{S}_i \preceq 1.5\mathbf{L},$$

since otherwise  $\tilde{\mathbf{X}}_i = \mathbf{0}$  and we get an even better bound on the trace-exponential. To make this formal, in Equation (9.12), we ought to do a case analysis that also includes the case  $\tilde{\mathbf{X}}_i = \mathbf{0}$  when the martingale has stopped, but we omit this.

Thus we can conclude by Claim 9.4.16 that

$$\|\Phi(\text{CLIQUE}(\pi(i), \mathbf{S}_i))\| \leq 1.5.$$

By our assumption  $0 < \theta \leq 0.5/\sqrt{K}$ , we have  $\|\frac{1}{K}\theta^2\Phi(\text{CLIQUE}(\pi(i), \mathbf{S}_{i-1}))\| \leq 1$ , so that by Corollary 9.4.15,

$$\begin{aligned} \mathbb{E}_{\pi(i)} \exp\left(\frac{1}{K}\theta^2\Phi(\text{CLIQUE}(\pi(i), \mathbf{S}_{i-1}))\right) &\preceq \mathbf{I} + \frac{2}{K}\theta^2 \mathbb{E}_{\pi(i)} \Phi(\text{CLIQUE}(\pi(i), \mathbf{S}_{i-1})) \\ &\preceq \mathbf{I} + \frac{2}{K}\theta^2 \mathbb{E}_{\pi(i)} \Phi(\text{STAR}(\pi(i), \mathbf{S}_{i-1})) \end{aligned} \quad (9.13)$$

by Claim 9.4.16.

Next we observe that, because every multi-edge appears in exactly two stars, and  $\pi(i)$  is chosen uniformly at random among the  $n+1-i$  vertices that  $\mathbf{S}_{i-1}$  is supported on, we have

$$\mathbb{E}_{\pi(i)} \text{STAR}(\pi(i), \mathbf{S}_{i-1}) = 2 \frac{1}{n+1-i} \mathbf{S}_{i-1}.$$

And, since we assume  $\mathbf{S}_i \preceq 1.5\mathbf{L}$ , we further get

$$\mathbb{E}_{\pi(i)} \exp\left(\frac{1}{K}\theta^2\Phi(\text{CLIQUE}(\pi(i), \mathbf{S}_{i-1}))\right) \preceq \mathbf{I} + \frac{6\theta^2}{K(n+1-i)} \mathbf{I}.$$

We can combine this with Equation (9.12) and Corollary 9.4.13 to get

$$\begin{aligned} &\mathbb{E} \text{Tr} \left( \exp\left(\theta \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i\right) \right) \\ &\leq \mathbb{E}_{(<\hat{n})} \mathbb{E}_{\pi(\hat{n})} \text{Tr} \exp \left( \sum_{i=1}^{\hat{n}-1} \theta \tilde{\mathbf{X}}_i + \frac{1}{K}\theta^2\Phi(\text{CLIQUE}(\pi(\hat{n}), \mathbf{S}_{\hat{n}-1})) \right) \\ &\leq \mathbb{E}_{(<\hat{n})} \text{Tr} \exp \left( \sum_{i=1}^{\hat{n}-1} \theta \tilde{\mathbf{X}}_i + \frac{6\theta^2}{K(n+1-i)} \mathbf{I} \right) \end{aligned}$$

And by repeating this analysis for each term  $\tilde{\mathbf{X}}_i$ , we get

$$\begin{aligned}\mathbb{E} \operatorname{Tr} \left( \exp(\theta \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i) \right) &\leq \operatorname{Tr} \exp \left( \sum_{i=1}^{\hat{n}} \frac{6\theta^2}{K(n+1-i)} \mathbf{I} \right) \\ &\leq \operatorname{Tr} \exp \left( \frac{7\theta^2 \log(n)}{K} \mathbf{I} \right) \\ &= n \exp \left( \frac{7\theta^2 \log(n)}{K} \right)\end{aligned}$$

Then, by choosing  $K = 200 \log^2 n$  and  $\theta = 0.5\sqrt{K}$ , we get

$$\exp(-0.5\theta) \mathbb{E} \operatorname{Tr} \left( \exp(\theta \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i) \right) \leq \exp(-0.5\theta)n \exp \left( \frac{7\theta^2 \log(n)}{K} \right) \leq 1/n^5.$$

$\mathbb{E} \operatorname{Tr} \left( \exp(-\theta \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i) \right)$  can be bounded by an identical argument, so that Equation (9.10) gives

$$\Pr \left[ \left\| \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i \right\| \geq 0.5 \right] \leq 2/n^5.$$

Thus we have established  $\left\| \sum_{i=1}^{\hat{n}} \tilde{\mathbf{X}}_i \right\| \leq 0.5$  with high probability (Equation (9.9)), and this in turn implies Equation (9.2), and finally Equation (9.1):

$$0.5\mathbf{L} \preceq \mathcal{L}\mathcal{L}^\top \preceq 1.5\mathbf{L}.$$

Now, all that's left to note is that the running time is linear in the multi-edge degree of the vertex being eliminated in each iteration (and this also bounds the number of non-zero entries being created in  $\mathcal{L}$ ). The total number of multi-edges left in the remaining graph stays constant at  $Km = O(m \log^2 n)$ . Thus the expected degree in the  $i$ th elimination is  $Km/(n+i-1)$ , because the remaining number of vertices is  $n+i-1$ . Hence the total running time and total number of non-zero entries created can both be bounded as

$$Km \sum_i 1/(n+i-1) = O(m \log^3 n).$$

We can further prove that the bound  $O(m \log^3 n)$  on running time and number of non-zeros in  $\mathbf{L}$  holds with high probability (e.g.  $1 - 1/n^5$ ). To show this, we essentially need a scalar Chernoff bound, in except the degrees are in fact not independent, and so we need a scalar martingale concentration result, e.g. Azuma's Inequality. This way, we complete the proof of Theorem 9.2.4.

## Part III

# Combinatorial Graph Algorithms



# Chapter 10

## Classical Algorithms for Maximum Flow I

### 10.1 Maximum Flow

In this chapter, we will study the *Maximum Flow Problem* and discuss some classical algorithms for finding solutions to it.

**Setup.** Consider a directed graph  $G = (V, E, \mathbf{c})$ , where  $V$  denotes vertices,  $E$  denotes edges and  $\mathbf{c} \in \mathbb{R}^E$ ,  $\mathbf{c} \geq 0$  denotes edge capacities. In contrast to earlier chapters, the direction of each edge will be important, and not just as a book keeping tool (which is how we previously used it in undirected graphs). We consider an edge  $(u, v) \in E$  to be *from*  $u$  and *to*  $v$ . Edge capacities are associated with directed edges, and we allow both edge  $(u, v)$  and  $(v, u)$  to exist, and they may have different capacities.

A *flow* is any vector  $\mathbf{f} \in \mathbb{R}^E$ .

We say that a flow is feasible when  $\mathbf{0} \leq \mathbf{f} \leq \mathbf{c}$ . The constraint  $\mathbf{0} \leq \mathbf{f}$  ensures that the flow respects edge directions, while the constraint  $\mathbf{f} \leq \mathbf{c}$  ensures that the flow does not exceed capacity on any edge.

We still define the edge-vertex incidence matrix  $\mathbf{B} \in \mathbb{R}^{V \times E}$  of  $G$  by

$$\mathbf{B}(v, e) = \begin{cases} 1 & \text{if } e = (u, v) \\ -1 & \text{if } e = (v, u) \\ 0 & \text{o.w.} \end{cases}$$

As in the undirected case, a *demand vector* is a vector  $\mathbf{d} \in \mathbb{R}^V$ . And as in the undirected case, we can express the net flow constraint that  $\mathbf{f}$  routes the demand  $\mathbf{d}$  by

$$\mathbf{B}\mathbf{f} = \mathbf{d}.$$

We will focus on the case:

$$\mathbf{B}\mathbf{f} = F(-\mathbf{e}_s + \mathbf{e}_t) = F\mathbf{b}_{st}.$$

Flows that satisfy the above equation for some scalar  $F$  are called  $s$ - $t$  flows where  $s, t \in V$ . The vertex  $s$  is called the source,  $t$  is called the sink and  $\mathbf{e}_s, \mathbf{e}_t$  are indicator vectors for source and sink nodes respectively. The vector  $\mathbf{b}_{st}$  has -1 at source and 1 at sink. The maximum flow can be expressed as a linear program as follows

$$\begin{aligned} \max_{\mathbf{f} \in \mathbb{R}^E, F} \quad & F \\ \text{s.t.} \quad & \mathbf{B}\mathbf{f} = F\mathbf{b}_{st} \\ & 0 \leq \mathbf{f} \leq \mathbf{c} \end{aligned} \tag{10.1}$$

We use  $\text{val}(\mathbf{f})$  to denote  $F$  when  $\mathbf{B}\mathbf{f} = F\mathbf{b}_{st}$ .

## 10.2 Flow Decomposition

We now look at a way of simplifying flows

**Definition 10.2.1.** A  $s$ - $t$  path flow is a flow  $\mathbf{f} \geq \mathbf{0}$  that can be written as

$$\mathbf{f} = \alpha \sum_{e \in p} \mathbf{e}_e$$

where  $p$  is a simple path from  $s$  to  $t$ .

**Definition 10.2.2.** A cycle flow is a flow  $\mathbf{f} \geq \mathbf{0}$  that can be written as a cycle i.e.

$$\mathbf{f} = \alpha \sum_{e \in c} \mathbf{e}_e$$

where  $c$  is a simple cycle.

**Lemma 10.2.3** (The path-cycle decomposition lemma). *Any  $s$ - $t$  flow  $\mathbf{f}$  can be decomposed into a sum of  $s$ - $t$  path flows and cycle flows such that the sum contains at most  $\text{nnz}(\mathbf{f})$  terms. Note  $\text{nnz}(\mathbf{f}) \leq |E|$ .*

*Proof.* We perform induction on the number of edges with non-zero flow, which we denote by  $\text{nnz}(\mathbf{f})$ . Note that by “the support of  $\mathbf{f}$ ”, we mean the set  $\{e \in E : \mathbf{f}(e) \neq 0\}$ .

**Base case:**  $\mathbf{f} = \mathbf{0}$ : nothing to do.

**Inductive step:** Try to find a path from  $s$  to  $t$  OR a cycle in the support of  $\mathbf{f}$ .

“*Path*” case. If there exists such a  $s$ - $t$  path, let  $\alpha$  be the minimum flow value along the edges of the path, i.e.

$$\alpha = \min_{(a,b) \in p} \mathbf{f}(a,b)$$

$$\mathbf{f}' = \alpha \sum_{e \in p} \mathbf{e}_e$$

Update the flow  $\mathbf{f}$  by

$$\mathbf{f} \leftarrow \mathbf{f} - \mathbf{f}'$$

The value of the flow will still be non-negative after this update as we subtracted the minimum entry along any positive edge on the path. The number of non-zeros,  $\text{nnz}(\mathbf{f})$ , went down by at least one. Note that the updated  $\mathbf{f}$  must again be an  $s$ - $t$  flow, as it is the difference of two  $s$ - $t$  flows.

“*Cycle*” case. Suppose we find a cycle  $c$  in the support of  $\mathbf{f}$ . Let  $\alpha$  be the minimum flow value along the edges of the cycle, i.e.

$$\alpha = \min_{(a,b) \in c} \mathbf{f}(a,b)$$

$$\mathbf{f}' = \alpha \sum_{e \in c} \mathbf{e}_e$$

Update the flow  $\mathbf{f}$  by

$$\mathbf{f} \leftarrow \mathbf{f} - \mathbf{f}'$$

As in the path case,  $\mathbf{f}$  stays non-negative, and number of non-zeros,  $\text{nnz}(\mathbf{f})$ , goes down by at least one. Note that the updated  $\mathbf{f}$  must again be an  $s$ - $t$  flow, as it is the difference of two  $s$ - $t$  flows.

“*No path or cycle*” case. Suppose we can find neither a path nor a cycle, and  $\mathbf{f} \neq \mathbf{0}$ . Then there must be an edge  $(u,v)$  with non-zero flow leading into a vertex  $v \neq s,t$  and with no outgoing edge from  $v$  in the support of  $\mathbf{f}$ . In that case, we must have  $(\mathbf{B}\mathbf{f})(v) > 0$ . But since  $v \neq t$ , this contradicts  $\mathbf{B}\mathbf{f} = F\mathbf{b}_{st}$ . So this case cannot occur.  $\square$

**Lemma 10.2.4.** *In any  $s$ - $t$  max flow problem instance, there is an optimal flow  $\mathbf{f}^*$  with a path-cycle decomposition that has only paths and no cycles.*

*Proof.* Let  $\tilde{\mathbf{f}}$  be an optimal flow. Let  $\mathbf{f}^*$  be the sum of path flows in the path cycle decomposition of  $\tilde{\mathbf{f}}$ . They route the same flow (as cycles contribute to no net flow from  $s$  to  $t$ ). Thus

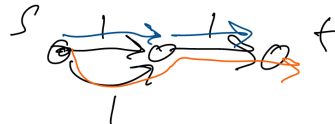
$$\mathbf{B}\mathbf{f}^* = \mathbf{B}\tilde{\mathbf{f}}$$

and hence  $\text{val}(\mathbf{f}^*) = \text{val}(\tilde{\mathbf{f}})$ . Furthermore

$$\mathbf{0} \leq \mathbf{f}^* \leq \tilde{\mathbf{f}} \leq \mathbf{c}$$

The first inequality follows from  $\mathbf{f}^*$  being a sum of positive path flow. The second inequality holds as  $\mathbf{f}^*$  is upper bounded in every single entry by  $\tilde{\mathbf{f}}$ , because we reduced it by positive entry cycles. The third inequality holds because  $\mathbf{f}$  is a feasible flow, so it is upper bounded by the capacities.  $\square$

An optimal flow solving the maximum flow problem may not be unique. For example, consider the graph below with source  $s$  and sink  $t$ :



There are two optimal paths in this example. Maximum flow is a convex optimization problem but not a strongly convex problem as the solutions are not unique, and this is part of what makes it hard to solve.

## 10.3 Cuts and Minimum Cuts

The decomposition shown earlier provides a way to show that the maximum flow in a graph is upper bounded by constructing graph cuts.

Given a vertex subset  $S \subseteq V$ , we say that  $(S, V \setminus S)$  is a *cut* in  $G$  and that the value of the cut is

$$c_G(S) = \sum_{e \in E \cap (S \times V \setminus S)} w(e).$$

Note that in a directed graph, only edges crossing the cut going *from*  $S$  and *to*  $V \setminus S$  count toward the cut.



Figure 10.1: Example of a cut: No edges go from  $S$  to  $(S, V \setminus S)$ , and so the value of this cut is zero.

**Definition.** ( $s$ - $t$  cuts). We define an  $s$ - $t$  cut to be a subset  $S \subset V$ , where  $s \in S$  and  $t \in V \setminus S$ .

**A decision problem:** “Given an instance of the Maximum Flow problem, is there a flow from  $s$  to  $t$  such that  $\mathbf{f} \neq \mathbf{0}$ ?”

**If YES:** We can decompose this flow into  $s$ - $t$  path flows and cycle flows.

**If NO:** There is no flow path from  $s$  to  $t$ . Let  $S$  be the set of vertices reachable from source  $s$ . Then  $(S, V \setminus S)$  is a cut in the graph, with no edges crossing from  $S$  to  $V \setminus S$ . Figure 10.1 gives an example.

**Upper bounding the maximum possible flow value.** How can we recognize a maximum flow? Is there a way to confirm that a flow is of maximum value?

We can now introduce the *Minimum Cut* problem.

$$\begin{aligned} \min_{S \subseteq V} c_G(S) \\ \text{s.t. } s \in V \text{ and } t \notin V \end{aligned} \tag{10.2}$$

The Minimum Cut problem can also be phrased as a linear program, although we won't see that today.

We'd like to obtain a tight connection between flows and cuts in the graph. As a first step, we won't get that, but we can at least observe that the value of any  $s$ - $t$  cut provides an upper bound to the maximum possible  $s$ - $t$  flow value.

**Theorem 10.3.1** (Max Flow  $\leq$  Min Cut). *The maximum  $s$ - $t$  flow value in a directed graph  $G$  (Program (10.1)) is upper bounded by the minimum value of any  $s$ - $t$  cut (Program (10.2)). I.e. if  $S$  is an  $s$ - $t$  cut, and  $\mathbf{f}$  a feasible  $s$ - $t$  flow then*

$$\text{val}(\mathbf{f}) \leq c_G(S)$$

*And in particular this holds for any minimum cut  $S^*$  and maximum flow  $\mathbf{f}^*$ , i.e.  $\text{val}(\mathbf{f}^*) \leq c_G(S^*)$ .*

*Proof.* Consider any feasible flow  $0 \leq \mathbf{f} \leq \mathbf{c}$  and a cut  $S, T = V \setminus S$ . Consider a path-cycle decomposition of  $\mathbf{f}$ , where each  $s$ - $t$  path must cross the cut going forward from  $S$  to  $T$  at least once. We pick a cut  $S, T$  with source on one side and sink on the other side as shown in the Figure (10.2). Every time the path flow passes through the cut, it has to use one of the edges that connect  $S$  and  $T$ . Total amount of flow crossing the cut is bounded above by total amount of capacity of the cut, otherwise the capacities would be violated, thus  $\text{val}(\mathbf{f}) \leq c_G(S, T) = \sum_{e \in E \cap S \times T} \mathbf{c}(e)$ .

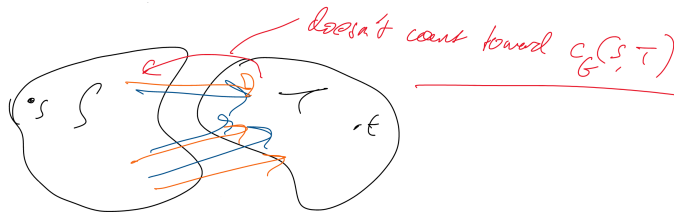


Figure 10.2:  $s$ - $t$  Cut

Note that the edges from  $T$  to  $S$  do not count towards the cut. The above equation holds for all flows with all  $s$ - $t$  cuts. This implies that **max flow**  $\leq$  **min cut**.  $\square$

This theorem is an instance of a general pattern, known as *weak duality*. Weak duality is a relationship between two optimization programs, a maximization problem and a minimization problem, where any solution to the former has its value upper bounded by the value of any solution to the latter.

## 10.4 Algorithms for Max flow

How can we find a good flow?

---

**Algorithm 4:** A first attempt – bad idea?

---

```

f  $\leftarrow$  0;
repeat
    Find an  $s$ - $t$  path flow  $\tilde{f}$  that is feasible with respect to  $c - f$ .
    f  $\leftarrow f + \tilde{f}$ 

```

---

**Does Algorithm 1 work?** Consider the graph below with directed edges with capacities 1 at every edge. If we make a single path update as shown by the orange lines in Figure 10.3, then afterwards, using the remaining capacity, there's no path flow we can route, as shown in Figure 10.3. But the max flow is 2, as shown by the flow on orange edges in Figure 10.5. So, the above algorithm does not always find a maximum flow: It can get stuck at a point where it cannot make progress despite not having reached the maximum value.

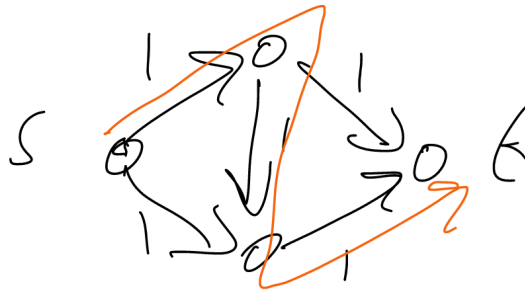


Figure 10.3: Sending a unit  $s$ - $t$  path flow through the graph.

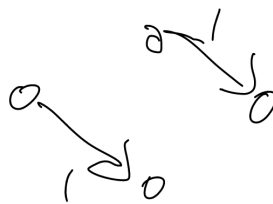


Figure 10.4: Remaining edge capacities after sending a path flow through the graph as depicted in Figure 10.3.

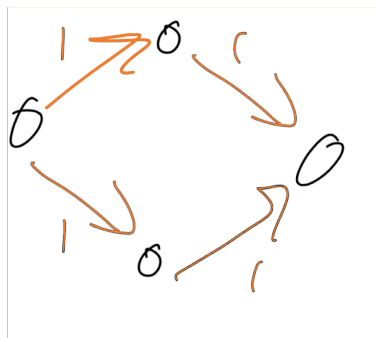


Figure 10.5: The flow depicted routes two units from  $s$  to  $t$ .

**A better approach.** It turns out we can fix the problem with the previous algorithm using a simple fix. This idea is known as *residual graphs*.

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**Algorithm 5:** Better Idea (Residual Graph)

---

$f \leftarrow 0$ ;

**repeat**

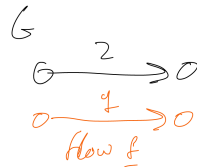
    Find an  $s$ - $t$  path flow  $\tilde{f}$  that is feasible with respect to  $-f \leq \tilde{f} \leq c - f$ .

$f \leftarrow f + \tilde{f}$

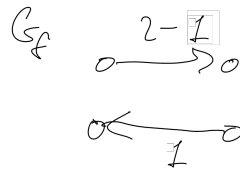
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The  $-f(e)$  can be treated as an edge going in the other direction with capacity  $f(e)$ . By convention, an edge in  $G$  with  $f(e) = c(e)$  is called *saturated*, and we do not include the edge in  $G_f$ . The graph defined above with such capacities is called **the residual graph of  $f$** ,  $G_f$ .  $G_f$  is only defined for feasible  $f$ , since otherwise the constraint  $\tilde{f} \leq c - f$  gives trouble.

Suppose we start with a graph having a single edge with capacity 2 and we introduce a flow of 1 unit.



The residual graph  $G_f$  has an edge with capacity 1 going forward and  $-1$  capacity going forward, but we can treat the latter as  $+1$  capacity going backwards. So it is an edge that allows you to undo the choice made to send flow along that direction.



Let us consider the same example with its residual graph. The original graph is shown in Figure 10.6.

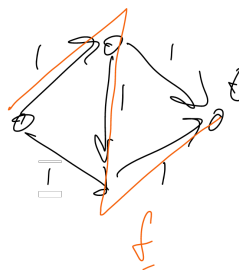


Figure 10.6: Original graph  $G$  and an  $s$ - $t$  path flow in  $G$  shown in orange.

The residual graph for the same is shown in Figure 10.7.



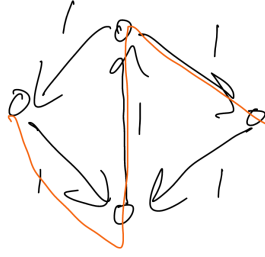


Figure 10.7: The residual graph w.r.t. the flow from Figure 10.6, and new  $s$ - $t$  path flow which is feasible in the residual graph.

Adding both flows together, we get the paths as shown in Figure 10.8 with value 2, which is the optimum.

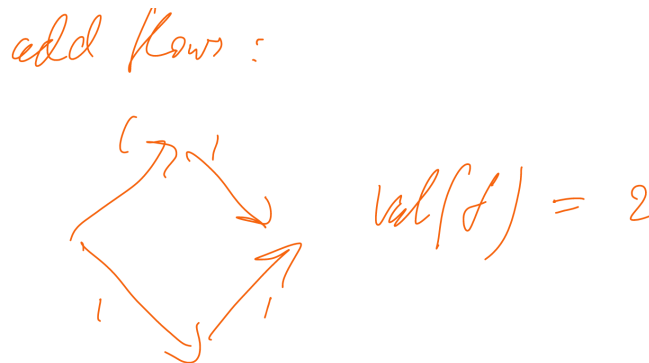


Figure 10.8: Maximum flow in the graph.

Let's prove some important properties of residual graphs.

**Lemma 10.4.1.** Suppose  $\mathbf{f}, \hat{\mathbf{f}}$  are feasible in  $G$ . Then this implies that  $\hat{\mathbf{f}} - \mathbf{f}$  (where negative entries count as flow in opposite direction) is feasible in  $G_{\mathbf{f}}$ .

*Proof.*  $\mathbf{0} \leq \mathbf{f} \leq \mathbf{c}$  and  $\mathbf{0} \leq \tilde{\mathbf{f}} \leq \mathbf{c}$ , this implies  $-\mathbf{f} \leq \tilde{\mathbf{f}} - \mathbf{f} \leq \mathbf{c} - \mathbf{f}$ . Hence, proved.  $\square$

**Lemma 10.4.2.** Suppose that  $\mathbf{f}$  is feasible in  $G$  and  $\hat{\mathbf{f}}$  is feasible in  $G_{\mathbf{f}}$ . Then,  $\mathbf{f} + \hat{\mathbf{f}}$  is feasible in  $G$ .

*Proof.*  $\mathbf{0} \leq \mathbf{f} \leq \mathbf{c}$  and  $-\mathbf{f} \leq \tilde{\mathbf{f}} \leq \mathbf{c} - \mathbf{f}$ , this implies  $\mathbf{0} \leq \tilde{\mathbf{f}} + \mathbf{f} \leq \mathbf{c}$ .  $\square$

**Lemma 10.4.3.** A feasible  $\mathbf{f}$  is optimal if and only if  $t$  is not reachable from  $s$  in  $G_{\mathbf{f}}$ .

*Proof.* Let  $\mathbf{f}$  be optimal, and suppose  $t$  is reachable from  $s$  in  $G_{\mathbf{f}}$  then, we can find a  $s$ - $t$  path flow  $\tilde{\mathbf{f}}$  that is feasible in  $G_{\mathbf{f}}$ , and  $\text{val}(\mathbf{f} + \tilde{\mathbf{f}}) > \text{val}(\mathbf{f})$ .  $\mathbf{f} + \tilde{\mathbf{f}}$  is feasible in  $G$  by Lemma 10.4.2. This is a contradiction, as we assumed  $\mathbf{f}$  was supposed to be optimal.

Suppose  $t$  is not reachable from  $s$  in  $G_f$ , and  $\mathbf{f}$  is feasible, but not optimal. Let  $\mathbf{f}^*$  be optimal, then by Lemma 10.4.1, the flow  $\mathbf{f}^* - \mathbf{f}$  is feasible in  $G_f$  and  $\text{val}(\mathbf{f}^* - \mathbf{f}) > 0$ . So there exists a  $s$ - $t$  path flow from  $s$  to  $t$  in  $G_f$  (as we can do a path decomposition of  $\mathbf{f}^* - \mathbf{f}$ ). But, this is a contradiction as  $t$  is not reachable from  $s$  in  $G_f$ .  $\square$

**Theorem 10.4.4** (Max Flow = Min Cut theorem). *The maximum flow in a directed graph  $G$  equals the minimum cut.*

*Proof.* Consider the set  $S = \{\text{vertices reachable from } s \text{ in } G_f\}$ . Note that  $\mathbf{f}$  saturates the edge capacities in cut  $S, V \setminus S$  in  $G$ : Consider any edge from  $S$  to  $T := V \setminus S$  in  $G$ . Since this edge does not exist in the residual graph, we must have  $\mathbf{f}(e) = c(e)$ .

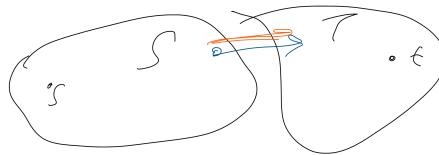


Figure 10.9: The cut between vertices reachable from  $S$  and everything else in  $G_f$  must have all outgoing edges saturated by  $\mathbf{f}$ .

This means that

$$\text{val}(\mathbf{f}) \geq c_G(S, V \setminus S).$$

Since we already know the opposite inequality by weak duality, we have shown that

$$\text{val}(\mathbf{f}) = c_G(S, V \setminus S)$$

which proves the **Max Flow = Min Cut** theorem; also called strong duality.  $\square$

**Remark 10.4.5.** Note that this proof also gives an algorithm to compute an  $s, t$ -min cut given an  $s, t$  maximum flow  $\mathbf{f}^*$ : Take one side of the cut to be the set  $S$  of nodes reachable from  $s$  in the residual graph w.r.t.  $\mathbf{f}^*$ .

## Ford-Fulkerson Algorithm.

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**Algorithm 6:** Ford-Fulkerson Algorithm

---

**repeat**

| Add update by arbitrary  $s$ - $t$  path flow in  $G_f$  (augment the flow  $\mathbf{f}$  by the path flow)

---

## Convergence properties and analysis of runtime of Ford-Fulkerson algorithm

- **Does this algorithm terminate?**

The algorithm terminates if the capacities are integers. However for irrational capacities the algorithm may not terminate.

- **Does it converge towards the max flow value?**

No, it does not converge to max flow value if the updates are poor and the capacities are irrational.

**Lemma 10.4.6.** *Consider Ford-Fulkerson algorithm with integer capacities. The algorithm terminates in  $\text{val}(\mathbf{f}^*)$  augmentations i.e.  $O(m \text{val}(\mathbf{f}^*))$  time.*

*Proof.* Each iteration increases the flow by at least one unit as the capacities in  $G_f$  are integral and each iteration can be computed in  $O(m)$  time.  $\square$

**Can we do better than this?** Suppose we pick the maximum bottleneck capacity (minimum capacity along path) augmenting path. This gives an algorithm that is better in some regimes.

How to pick the maximum bottleneck capacity augmenting path? We are going to perform a binary search on the capacities in  $G_f$ , to find a path with maximum bottleneck capacity. Each time our binary search has picked a threshold capacity, we then try to find an  $s$ - $t$  path flow in  $G_f$  using only edges with capacity above that threshold. If we find a path, the binary search tries a higher capacity next. If we don't find a path, the binary search tries a lower capacity next.

Using this approach, the time to find a single path is  $O(m \log(n))$  where  $m$  is number of edges in the graph. This path must carry at least a  $\frac{1}{m}$  fraction of the total amount of flow left in  $G_f$ . For instance, if  $\hat{F}$  is the amount of flow left in  $G_f$ , then the path must carry  $\frac{\hat{F}}{m}$  flow (from the path decomposition lemma, we know that there exists a decomposition into at most  $m$  paths, and the one carrying the most flow must carry at least the average amount).

So if the flow is integral, the algorithms completes when

$$\left(1 - \frac{1}{m}\right)^T \text{val}(\mathbf{f}^*) < 1$$

where  $T$  is the number of augmentations.

This means

$$T = m \log F$$

$$\begin{aligned} \text{Total time} &= O(m \log n T) \\ &= O(m^2 \log n \log F) \\ &\leq O(m^2 \log n \log mU) \text{ as } F \leq mU \text{ where } U \text{ is the maximum capacity} \end{aligned}$$

**Current state-of-the-art approaches for Max Flow.** For the interested reader, we'll briefly mention the current state-of-the-art for Maximum Flow algorithms, which is a very active area of research.

- Strongly polynomial time algorithm:
  - has to work with real valued capacities, then the best time  $O(mn)$  by Orlin.
- General integer capacities: For a long time, the state-of-the-art was a result by Goldberg and Rao achieving a runtime of  $O(m \min(\sqrt{m}, n^{2/3}) \log(n) \log(U))$  where  $U$  is the maximum capacity. Very recently,  $m^{1+o(1)} \log(U)$  has been claimed (not yet published in a peer-reviewed venue), see <https://arxiv.org/abs/2203.00671>. There are many other recent results prior to this work that we don't have time to cover here.

# Chapter 11

## Classical Algorithms for Maximum Flow II

### 11.1 Overview

In this chapter we continue looking at classical max flow algorithms. We derive Dinic's algorithm which (unlike Ford-Fulkerson) converges in a polynomial number of iterations. We will also show faster convergence on unit capacity graphs. The setting is the same as last chapter: we have a directed graph  $G = (V, E, \mathbf{c})$  with positive capacities on the edges. Our goal is to solve the maximum flow problem on this graph for a given source  $s$  and sink  $t \neq s$ :

$$\begin{aligned} \max_{F > 0, \mathbf{f}} F \quad & \text{s.t.} \quad \mathbf{B}\mathbf{f} = F\mathbf{b}_{s,t} \\ & \mathbf{0} \leq \mathbf{f} \leq \mathbf{c} \end{aligned} \tag{11.1}$$

### 11.2 Blocking Flows

Let  $\mathbf{f}$  be any feasible flow in  $G$  and let  $G_{\mathbf{f}}$  be its residual graph. Observe that we can partition the vertices of  $G_{\mathbf{f}}$  into 'layers': first  $s$  itself, then all vertices reachable from  $s$  in one hop, then all vertices reachable in two hops, etc. For each vertex  $v \in V$  define its **level** in  $G_{\mathbf{f}}$  as the length of the shortest path in  $G_{\mathbf{f}}$  from  $s$  to  $v$ , denoted by  $\ell_{G_{\mathbf{f}}}(v)$  (or just  $\ell(v)$  if the graph is clear from context). An edge  $(u, v) \in E$  can only take you up 'one level': if  $\ell(v) \geq 2 + \ell(u)$  this would imply we can find a shorter  $s$ - $v$  path by appending  $(u, v)$  to the shortest  $s$ - $u$  path. However, edges can 'drop down' multiple levels (or be contained in the same level).

A key strategy in Dinic's algorithm will be to focus on 'short' augmenting paths. We can use the levels we defined above to isolate a subgraph of  $G_{\mathbf{f}}$  containing all information we

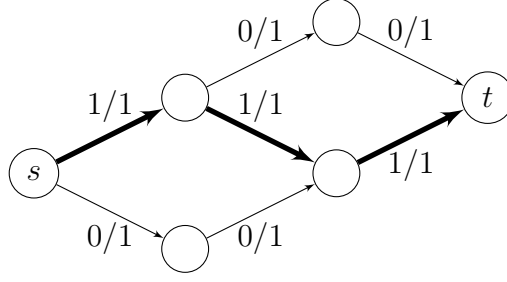


Figure 11.1: A blocking flow is not a maximum flow.

need to find shortest paths.

**Definition 11.2.1.** Call an edge  $(u, v)$  **admissible** if  $\ell(u) + 1 = \ell(v)$ . Let  $L$  be the set of admissible edges in  $G_f$  and define the **level graph** of  $G_f$  to be the subgraph induced by these edges.

**Definition 11.2.2.** Define a **blocking flow** in a residual graph  $G_f$  to be a flow  $\hat{f}$  feasible in  $G_f$ , such that  $\hat{f}$  only uses admissible edges. Furthermore we require that for any  $s$ - $t$  path in the level graph of  $G_f$ ,  $\hat{f}$  saturates at least one edge on this path.

The last condition makes a blocking flow ‘blocking’: it blocks any shortest augmenting path in  $G_f$  by saturating one of its edges. Note however that a blocking flow is not necessarily a maximum flow in the level graph.

## 11.3 Dinic’s Algorithm

We can now formulate Dinic<sup>1</sup>’s algorithm: start with  $f = 0$ , and then repeatedly add to  $f$  a blocking flow  $\hat{f}$  in  $G_f$ , until no more  $s$ - $t$  paths exist in  $G_f$ .

Note that by doing a path decomposition on  $\hat{f}$  and adding these paths to  $f$  one by one, we see that our algorithm is a ‘special case’ of Ford-Fulkerson (with a particular augmenting path strategy) and hence inherits all the behaviors/bounds we proved in the last chapter.

**Lemma 11.3.1.** Let  $f$  be a feasible flow,  $\hat{f}$  a blocking flow in  $G_f$  and define  $f' = f + \hat{f}$  (think of  $f$  and  $f'$  to be the flows at some steps  $k$  and  $k + 1$  in the algorithm). Then  $\ell_{G_{f'}}(t) \geq \ell_{G_f}(t) + 1$ .

*Proof.* Let  $L, L'$  be the edge sets of the level graphs of  $G_f, G_{f'}$  respectively. We would now like to show that  $d_{L'}(s, t) \geq 1 + d_L(s, t)$ . Let’s first assume that in fact  $d_{L'}(s, t) < d_L(s, t)$ . Now take a shortest  $s$ - $t$  path in the level graph of  $G_{f'}$ , say  $s = v_0, v_1, \dots, v_{d_{L'}(s, t)} = t$ . Let  $v_j$  be the first vertex along the path such that  $d_{L'}(s, v_j) < d_L(s, v_j)$ . As  $d_{L'}(s, t) < d_L(s, t)$ , such a  $v_j$  must exist.

<sup>1</sup>Sometimes also transliterated as Dinitz.

We'd like to understand the edges in the level graph  $L'$ . Let  $E(G_f)$  denote the edges of the  $G_f$ , and let  $\text{rev}(L)$  denote the set of reversed edges of  $L$ . The level graph edges of  $G_{f'}$  must satisfy  $L' \subseteq L \cup \text{rev}(L) \cup (E(G_f) \setminus L)$ .

In our  $s$ - $t$  path in  $L'$ , the vertex  $v_j$  is reached by an edge from  $v_{j-1}$ , and the level of  $v_{j-1}$  in  $G_f$  is at most the level it receives in  $G_{f'}$  so the edge  $(v_{j-1}, v_j) \in L$  skips at least one level forward in  $G_f$ . But, edges in  $L$  do not skip a level in  $G_f$ , and edges in  $\text{rev}(L)$  or  $E(G_f) \setminus L$  do not move from a lower level to higher level in  $G_f$ . So this edge cannot exist and we have reached a contradiction.

Now suppose that  $d_{L'}(s, t) = d_L(s, t)$ . This means there is an  $s$ - $t$  path in  $L'$  using edges in  $L$  – but such a path must contain an edge saturated by the blocking flow  $\hat{f}$ .

(Note: the reason this path must use only edges in  $L$  is similar to the previous case: the other possible types of edges do not move to higher levels in  $G_f$  at all, making the path too long if we use any of them.)

So we must have  $d_{L'}(s, t) \geq 1 + d_L(s, t)$ . □

An immediate corollary of this lemma is the convergence of Dinic's algorithm:

**Theorem 11.3.2.** *Dinic's algorithm terminates in  $O(n)$  iterations.*

*Proof.* By the previous lemma, the level of  $t$  increases by at least one each iteration. A shortest path can only contain each vertex once (otherwise it would contain a cycle) so the level of any vertex is never more than  $n$ . □

For graphs with unit capacities ( $c = 1$ ) we can prove even better bounds on the number of iterations.

**Theorem 11.3.3.** *On unit capacity graphs, Dinic's algorithm terminates in*

$$O(\min\{m^{1/2}, n^{2/3}\})$$

*iterations.*

*Proof.* We prove the two bounds separately.

1. Suppose we run Dinic's algorithm for  $k$  iterations, obtaining a flow  $\mathbf{f}$  that is feasible but not necessarily yet optimal. Let  $\tilde{\mathbf{f}}$  be an optimal flow in  $G_f$  (i.e.  $\mathbf{f} + \tilde{\mathbf{f}}$  is a maximum flow for the whole graph), and consider a path decomposition of  $\tilde{\mathbf{f}}$ . Since after  $k$  iterations any  $s$ - $t$  path has length  $k$  or more, we use up a total capacity of at least  $\text{val}(\tilde{\mathbf{f}})k$  across all edges. But the edges in  $G_f$  are either edges from the original graph  $G$  or their reversals (but never both) meaning the total capacity of  $G_f$  is at most  $m$ , hence  $\text{val}(\tilde{\mathbf{f}}) \leq m/k$ .

Recalling our earlier observation that our algorithm is a special case of Ford-Fulkerson, this implies that our algorithm will terminate after at most another  $m/k$  iterations.

Hence the number of iterations is bounded by  $k + m/k$  for any  $k > 0$ . Substituting  $k = \sqrt{m}$  gives the first desired bound.

2. Suppose again that we run Dinic's algorithm for  $k$  iterations obtaining a flow  $\mathbf{f}$ . The level graph of  $G_{\mathbf{f}}$  partitions the vertices into sets  $D_i = \{u \mid \ell(u) = i\}$  for  $i \geq 0$ . As shown before, the sink  $t$  must be at a level of at least  $k$ , meaning we have at least this many non-empty levels starting from  $D_0 = \{s\}$ . To simplify, discard all vertices and levels beyond  $D_{\ell(t)}$ .

Now, consider choosing a level  $I$  uniformly at random from  $\{1, \dots, k\}$ . Since there are at most  $n-1$  vertices in total across these levels,  $\mathbb{E}[|D_I|] \leq (n-1)/k$ , and by Markov's inequality

$$\Pr[|D_I| \geq 2n/k] \leq \frac{(n-1)/k}{2n/k} < \frac{1}{2}.$$

Thus strictly more than  $k/2$  of the levels  $i \in \{1, \dots, k\}$  have  $|D_i| < 2n/k$  and so there must be two adjacent levels  $j, j+1$  for which this upper bound on the size holds. There can be at most  $|D_j| \cdot |D_{j+1}| \leq 4n^2/k^2$  edges between these levels, and by the min-cut max-flow theorem we saw in the previous chapter, this is an upper bound on the flow in  $G_{\mathbf{f}}$ , and hence on the number of iterations still needed for our algorithm to terminate.

This means the number of iterations is bounded by  $k + 4n^2/k^2$  for any  $k > 0$ , which is  $O(n^{2/3})$  at  $k = 2n^{2/3}$ .

□

## 11.4 Finding Blocking Flows

What has been missing from our discussion so far is the process of actually finding a blocking flow. We can achieve this using repeated depth-first search. We repeatedly do a search in the level graph (so only using edges  $L$ ) for  $s$ - $t$  paths and augment these. We erase edges whose subtrees have been exhausted and do not contain any augmenting paths to  $t$ . Pseudocode for the algorithm is given below.



---

**Algorithm 7: FINDBLOCKINGFLOW**

---

```
f  $\leftarrow$  0;  
H  $\leftarrow$  L;  
repeat  
| P  $\leftarrow$  DFS(s, H, t);  
| if P  $\neq \emptyset$  then  
| | Let  $\hat{\mathbf{f}}$  be a flow that saturates path P.  
| | f  $\leftarrow$  f +  $\hat{\mathbf{f}}$ ;  
| | Remove from H all edges saturated by  $\hat{\mathbf{f}}$ .  
| else  
| | return f;  
| end
```

---

---

**Algorithm 8: DFS(**u**, **H**, **t**)**

---

```
if u = t then  
| return the path P on the dfs-stack.  
end  
for (u, v)  $\in$  H do  
| P  $\leftarrow$  DFS(v, H, t);  
| if P  $\neq \emptyset$  then  
| | return P;  
| else  
| | Erase (u, v) from H.  
| end  
end  
return  $\emptyset$ ;
```

---

**Lemma 11.4.1.** *For general graphs, FINDBLOCKINGFLOW returns a blocking flow in  $O(nm)$  time. Hence Dinic’s algorithm runs in  $O(n^2m)$  time on general capacity graphs.*

*Proof.* First, consider the amount of work spent pushing edges onto the stack which eventually results in augmentation along an  $s$ - $t$  path consisting of those edges (i.e. adding flow along that path). Since each augmenting path saturates at least one edge, we do at most  $m$  augmentations. Each path has length at most  $n$ . Thus the total amount of work pushing these edges to the stack, and removing them from the stack upon augmentation, and deleting saturated edges, can be bounded by  $O(mn)$ . Now consider the work spent pushing edges onto the stack which are later deleted because we “retreat” after not finding an  $s$ - $t$ . An edge can only be pushed onto the stack once in this way, since it is then deleted. So the total amount spent pushing edges to the stack and deleting them this way is  $O(m)$ .  $\square$

**Lemma 11.4.2.** *For unit capacity graphs, FINDBLOCKINGFLOW returns a blocking flow in  $O(m)$  time. Hence Dinic’s algorithm runs in  $O(\min\{m^{3/2}, mn^{2/3}\})$  time on unit capacity graphs.*

*Proof.* When our depth-first search traverses some edge  $(u, v)$ , one of two things will happen: either we find no augmenting path in the subtree of  $v$ , leading to the erasure of this edge, or we find an augmenting path which will necessarily saturate  $(u, v)$ , again leading to its erasure. This means each edge will be traversed at most once by the depth-first search.  $\square$

Another interesting bound on the runtime, which we will not prove here, is that Dinic's algorithm will run in  $O(|E| \sqrt{|V|})$  time on bipartite matching graphs.

## 11.5 Minimum Cut as a Linear Program

Finally we show that minimum cut may be formulated as a linear program. For a subset  $S \subseteq V$  write  $c_G(S)$  for the sum of  $\mathbf{c}(e)$  over all edges  $e = (u, v)$  that cross the cut, i.e.  $u \in S, v \notin S$ . Note that 'reverse' edges are not counted in this cut. The minimum cut problem asks us to find some  $S \subseteq V$  such that  $s \in S$  and  $t \notin S$  such that  $c_G(S)$  is minimal.

$$\begin{aligned} \min_{S \subseteq V} \quad & c_G(S) \\ \text{s.t.} \quad & s \in S \\ & t \notin S \end{aligned} \tag{11.2}$$

We claim this is equivalent to the following minimization problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^V} \quad & \sum_{e \in E} \mathbf{c}(e) \max \{ \mathbf{b}_e^\top \mathbf{x}, 0 \} \\ \text{s.t.} \quad & \mathbf{x}(s) = 0 \\ & \mathbf{x}(t) = 1 \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \end{aligned} \tag{11.3}$$

Recall  $\mathbf{b}_e^\top \mathbf{x} = \mathbf{x}(v) - \mathbf{x}(u)$  for  $e = (u, v)$ . We can rewrite this as a proper linear program by introducing extra variables for the maximum:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^V, \mathbf{u} \in \mathbb{R}^E} \quad & \mathbf{c}^\top \mathbf{u} \\ \text{s.t.} \quad & \mathbf{b}_{s,t}^\top \mathbf{x} = 1 \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \\ & \mathbf{u} \geq \mathbf{0} \\ & \mathbf{u} \geq \mathbf{B}^\top \mathbf{x} \end{aligned} \tag{11.4}$$

And, in fact, we'll also see that we don't need the constraint  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}$ , leading to the

following simpler program:

$$\begin{aligned}
\min_{\mathbf{x} \in \mathbb{R}^V, \mathbf{u} \in \mathbb{R}^E} \quad & \mathbf{c}^\top \mathbf{u} \\
s.t. \quad & \mathbf{b}_{s,t}^\top \mathbf{x} = 1 \\
& \mathbf{u} \geq \mathbf{0} \\
& \mathbf{u} \geq \mathbf{B}^\top \mathbf{x}
\end{aligned} \tag{11.5}$$

This last linear program is the dual program to the maximum flow linear program.

**Lemma 11.5.1.** *Programs (11.2), (11.4), and (11.5) have equal optimal values.*

*Proof.* We start by considering equality between optimal values of Program (11.2) and Program (11.4). Let  $S$  be an optimal solution to Program (11.2) and take  $\mathbf{x} = \mathbf{1}_{V \setminus S}$ . Then  $\mathbf{x}$  is a feasible solution to Program (11.4) with cost equal to  $c_G(S)$ .

Conversely, suppose  $\mathbf{x}$  is an optimal solution to Program (11.4). Let  $\alpha$  be uniform in  $[0, 1]$  and define  $S = \{v \in V \mid \mathbf{x}(v) \leq \alpha\}$ . This is called a threshold cut. We can verify that

$$\Pr[e \text{ is cut by } S] = \max\{\mathbf{b}_e^\top \mathbf{x}, 0\}.$$

and hence  $\mathbb{E}_t[c_G(S)]$  is exactly the optimization function of 11.3 (and hence 11.4). Since at least one outcome must do as well as the average, there is a subset  $S \subseteq V$  achieving this value (or less).

We can use the same threshold cut to show that we can round a solution to Program (11.5) to an equal or smaller value cut feasible for Program (11.2). The only difference is that in this case, we get

$$\Pr[e \text{ is cut by } S] \leq \max\{\mathbf{b}_e^\top \mathbf{x}, 0\}$$

which is still sufficient. □

# Bibliography

- [KS16] R. Kyng and S. Sachdeva. Approximate gaussian elimination for laplacians - fast, sparse, and simple. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 573–582, 2016.
- [Spi19] Daniel A Spielman. Spectral and Algebraic Graph Theory, 2019.
- [SS11] Daniel A Spielman and Nikhil Srivastava. Graph sparsification by effective resistances. *SIAM Journal on Computing*, 40(6):1913–1926, 2011.
- [ST04] Daniel A. Spielman and Shang-Hua Teng. Nearly-linear time algorithms for graph partitioning, graph sparsification, and solving linear systems. In *Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing*, STOC '04, page 81–90, New York, NY, USA, 2004. Association for Computing Machinery.
- [T<sup>+</sup>15] Joel A Tropp et al. An introduction to matrix concentration inequalities. *Foundations and Trends® in Machine Learning*, 8(1-2):1–230, 2015.
- [Tro19] Joel A Tropp. Matrix concentration & computational linear algebra. 2019.