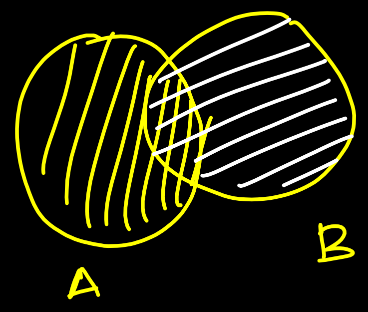


$|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion - Exclusion Principle (IEP)

A tool to find union of a finite number of sets . Union of 2 sets, 3 sets, 4 sets and so on...

- The no. of elements in the union of the two sets $A \& B$ is $|A \cup B| = |A| + |B| - |A \cap B|$
- The no.of elements in the union of three sets A, B and C is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

- The no. of elements in the union of four sets $A, B, C \& D$ is

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$$

- Let $A_1, A_2, \ldots A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \ldots A_n| = \sum_{1}^{n} |A| - \sum_{1 \le i < j}^{n} |A_i \cap A_j| + \sum_{1 \le i,j < k \le n}^{n} |A_i \cup A_j \cup A_k| \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots A_n|$$

$2^n - 1$ terms *

Example

How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion-exclusion?

$n=10 \qquad 2^n-1 = 2^{10}-1$ terms

Example

In a discrete mathematics class, every student is a major in computer science or mathematics, or both. The number of students having computer science as a major (possibly along with mathematics) is 20, the number of students having mathematics as a major (possibly along with computer science) is 15, and the number of students majoring in both computer science and mathematics is 9. How many students are in this class?

let A be the set of students who major in C.S
those
let B " mathematics.

$|A| = 20 \qquad |B| = 15$

$|A\cap B| = 9 \qquad |A\cup B| = ?$

using IEP

$|A\cup B| = |A|+|B|-|A\cap B| = 20+15-9$

$= 35-9 = 26$

Example

There are 345 students at a college who have taken a cource in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?

let A : number of students who have taken a course in calculus.

B : " DE

$|A| = 345$ $|B| = 212$

$|A \cap B| = 188$

$|A \cup B| = 345 + 212 - 188 = 369$

Example

A total of 1230 students have taken a course in Spanish, 875 have taken a course in French, and 115 have taken a course in Russian. Further, 105 have taken courses in both Spanish and French, 25 have taken courses in both Spanish and Russian, and 15 have taken courses in both French and Russian. If 2090 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

let S be the set of students who have taken a Course in Spanish

F " French

R " Russian.

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |F \cap R| - |S \cap R|$
$+ |S \cap F \cap R|$

$|S \cap F \cap R| = ?$ $|S| = 1230$ $|F| = 875$ $|R| = 115$

$|S \cap F| = 105$ $|F \cap R| = 15$ $|S \cap R| = 25$

$|S \cup F \cup R| = 2090$

$2090 = 1230 + 875 + 115 - 105 - 15 - 25 + |S \cap F \cap R|$

$|S \cap F \cap R| = 15$

5. How many positive Integers not exceeding 1000 are divisible by 7 or 11.

Ans.

A: if n is divisible by 7

B: if n is divisible by 11

$|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$

$|B| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$

$|A \cap B| = \left\lfloor \frac{1000}{7 \times 11} \right\rfloor = 12$

$|A \cup B| = 142 + 90 - 12 = 220$

6. Find the no. of positive integers not exceeding 1000 that are not divisible by 3, 17 or 35

A : If n is divisible by 3

B : " 17

C : " 35

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$|A| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$

$|B| = \left\lfloor \frac{1000}{17} \right\rfloor = 58$

$|C| = \left\lfloor \frac{1000}{35} \right\rfloor = 28$

$|A \cap B| = \left\lfloor \frac{1000}{3 \times 17} \right\rfloor = 19$

$|A \cap C| = \left\lfloor \frac{1000}{3 \times 35} \right\rfloor = 9$

$|B \cap C| = \left\lfloor \frac{1000}{17 \times 35} \right\rfloor = 1$

$|A \cap B \cap C| = \left\lfloor \frac{1000}{3 \times 17 \times 35} \right\rfloor = 0$

$|A \cup B \cup C| = 333 + 58 + 28 - 19 - 9 - 1 + 0$

$= 390$.

not d $= 1000 - 390 = 610$

7. Find the no. of positive integers not exceeding 1000 that are either the square or the cube of an integer.

$(1000)^{2}.\ (150)^{3}$

$\sqrt{1000} = 31$ squares. $\sqrt[3]{1000} = 10$ cubes.

$\sqrt[6]{1000} = 3$

$|A| = 31 \quad |B| = 10 \quad |A \cap B| = 3$

$|A \cup B| = 31 + 10 - 3 = 41 - 3 = 38$ //

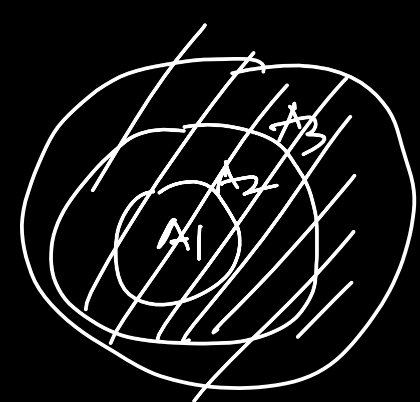
Example

Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1, 1000 in A_2, and 10,000 in A_3 if

a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$. b) the sets are pairwise disjoint

c) there are two elements common to each pair of sets and one element in all three sets.

$|A_1| = 100 \quad |A_2| = 1000 \quad |A_3| = 10,000$



c) $|A_1 \cap A_2| = 2$

$|A_1 \cap A_3| = 2 \quad |A_1 \cap A_2 \cap A_3| = 1$

$|A_2 \cap A_3| = 2$

a) $|A_1 \cup A_2 \cup A_3| = ?$

$A_1 \subseteq A_2$ and $A_2 \subseteq A_3$

$|A_1 \cup A_2 \cup A_3| = |A_3| = 10,000$

b) $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$= |A_1| + |A_2| + |A_3| = 11,100$

c) $|A_1 \cup A_2 \cup A_3| = 11100 - 2 - 2 - 2 + 1$

$= 11095$ //

$n! = n(n-1)(n-2)\ldots x)$

$3! = 3\times2\times1$

$4! = 4\times3\times2\times1$

Derangements (Nothing is in its right place)

A derangement is an arrangement of objects in which none of the objects is in its original position.

For example, the derangement of 1,2,3 is 2,3,1 and 3,1,2.

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots(-1)^n\frac{1}{n!}\right] \qquad n > 1$$

Example

List all the derangements of $\{1, 2, 3, 4\}$. $n=4$

$$D_4 = 4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$= \cancel{4!} - \cancel{4!} + \frac{4!}{2!} - \frac{4!}{3!} + \cancel{\frac{4!}{4!}}$$

$$= \frac{4\times3\times\cancel{2\times1}}{\cancel{2\times1}} - \frac{4\times\cancel{3\times2\times1}}{\cancel{3\times2\times1}} + 1$$

$$= 12 - 4 + 1 = 12 - 3 = 9$$

How many derangements of $\{1,2,3,4,5,6\}$ begin with integers 1, 2 and 3, in some order?

1 2 3

2 3 1

3 1 2

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
2	3	1	6	4	5
3	1	2	5	6	4

231564, 312645, 231645, 312564

M SIGMA GOKULAM

1	2	3	4
2	1	4	3
2	3	4	1
2	4	1	3

~~1~~	2	~~3~~	4
3	1	4	2
3	4	1	2
3	4	2	1

1	2	3	4
4	1	2	3
4	3	1	2
4	3	2	1

2143, 2341, 2413, 3142, 3412, 3421,

4123, 4312, 4321 //

M SIGMA GOKULAM
M SIGMA GOKULAM

Example

A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

$D_n = n!\,(\quad)$

The probability that no one receives the correct hat is $\frac{D_n}{n!}$

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!} \approx e^{-1}$$

$$= 0.368$$

Example

What is the probability that none of 10 people receives the correct hat if a hatcheck person hands their hats back randomly?

$n = 10$

$$\frac{D_n}{n!} = \frac{D_{10}}{10!}$$

Example

There are 8 guests at a party. Each guest brings and each receives another gift in return. No allowed to receive a gift they bought. How many ways are these to distribute gift?

$$D_8 = 8!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}\right)$$

$$= 14833$$

Example

In how many ways can a teacher distribute 10 distinct books to his 10 student (one book to each student) and then collect and redistribute the books so that each student can peruse two different books.

Number of derangements of 10 books $= D_{10}$

$$= 10!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\right]$$

Total no. of arrangements $= 10! \times D_{10}$

Onto functions

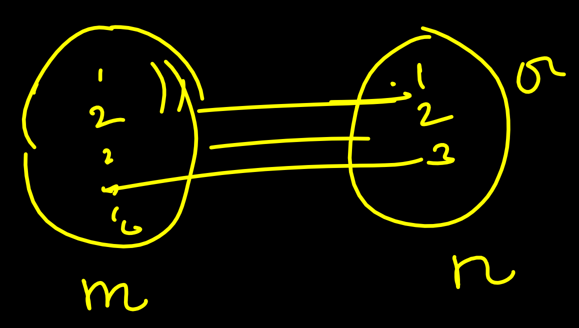
- Total number of functions from a set with m elements to a set with n elements are n^m.
- Let m and n be positive integers with $m \geq n$. Then, there are

$\therefore$ $n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - \dots + (-1)^{n-1}\,{}^nC_{n-1} \times 1^m$ onto functions from a set with n elements.

1. How many ways are there to distribute six different toys to three different children such that each child gets atleast one toy? $m = 6$ $n=3$ (540)

$m = 6$

$n = 3$



no: of onto function,

$= 3^6 - {}^3C_1\, 2^6 + {}^3C_2\, 1^6 + \cancel{{}^3C_3\, 0^6}$

$= 540$

2. How many ways are there to assign 5 different jobs to 4 different employees if every employee is assigned atleast one job. (240)

$m=5 \quad n=4$

$n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m + \ldots$

$4^5 - 4C_1 3^5 + 4C_2 2^5 - 4C_1 1^5$

$1024 - (4 \times 243) + (6 \times 32) - (4 \times 1)$

$= 240$