Maximal and Minimal element

An element of a poset is called maximal if it is not less than any element of the poset. That is, a is maximal in the poset $(S, \preceq)$ if there is no $b \in S$ such that $a \preceq b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is minimal if there is no element $b \in S$ such that $b \preceq a$.

For example, let $A = \{1, 2, 3\}$ and the relation $\preceq$ is the set inclusion.
Let $P(A)$ is the set of all subsets of A.
$P(A) = \{\phi, \{1\}, \{2, \}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. In the Poset $(P(A), \preceq)$ the set $\{1, 2, 3\}$ is the maximal elements and ϕ is the minimal element.
Let $B = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$, the collection of proper subsets of A, and the relation $\preceq$ is the set inclusion. In the poset $(B, \preceq)$ the set $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are the maximal elements and ϕ is the minimal element.

Least Element and greatest element

If $(A, \preceq)$ si a poset then an element $x \in A$ is called the greatest element of A if for all $a \in A$, $a \preceq x$.
If $(A, \preceq)$ is a poset then an element $y \in A$ is called the least element of A if for all $b \in a$, $y \preceq b$.

For example, let $A = \{1, 2, 3\}$ and the relation R is the set inclusion. Let $P(A)$ is the set of all subsets of A. In the Poset $(P(A), R)$ the set $\{1, 2, 3\}$ is the greatest element and ϕ is the least element.

Upper bound and Lower bound

Let $(A, \preceq)$ be a poset with $B \subseteq A$. An element $x \in A$ is called upper bound of B if $a \preceq x$ for all $a \in B$.
Let $(A, \preceq)$ be a poset with $B \subseteq A$. An element $y \in A$ is called lower bound of B if $y \preceq b$ for all $b \in B$.

Least upper bound (supremum)

Let $(A, \preceq)$ is a poset with $B \subseteq A$. Any element $x \in A$ is a Least upper bound (lub) of B if x is an upper bound of B nad $x \preceq x'$ for all other upper bounds x' of B.

Greatest lower bound (infimum)

Let $(A, \preceq)$ is a poset with $B \subseteq A$. Any element $y \in A$ is a Greatest lower bound (glb) of B if y is a lower bound of B and $y' \preceq y$ where y' is any lower bound of B.

lub $\{a, b\}$ $a \vee b$ | $c+b$
glb $a \wedge b$ | $c \cdot b$

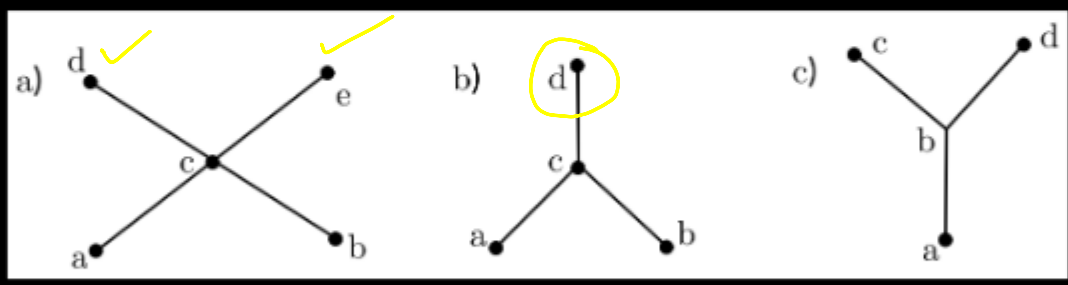
For example, let $A = \{1, 2, 3, 6, 9, 18\}$ with the relation divisibility. Then $(A, |)$ is a poset with $B = \{2, 3\} \subseteq A$. The upper bounds of $\{2, 3\}$ are 6, 18 and the least upper bound is 6.
The lower bounds of $\{2, 3\}$ is 1 and the greatest lower bound is 1.

Lattice

A lattice is a partially ordered set $(A, \preceq)$ in which every pair of elements $a, b \in A$ the least upper bound of $\{a, b\}$ and greatest lower bound of $\{a, b\}$ both exist in A^2.
~~We denote~~ $lub\{a, b\}$ by $a \bigvee b$ or $a + b$ and call it the join or sum of a and b and the $glb\{a, b\}$ by $a \bigwedge b$ or $a \cdot b$ and call it the meet or product of a and b. So, lattice $(A, \preceq)$ may be written as an algebraic system $(A, +, .)$. All posets are not lattice but all totally ordered set is a lattice.

Example 1

Find the maximal element, minimal element, greatest element and least element from the following poset with Hasse diagram



a) max elements – e, d

mini elements – a, b

no no

Example 2

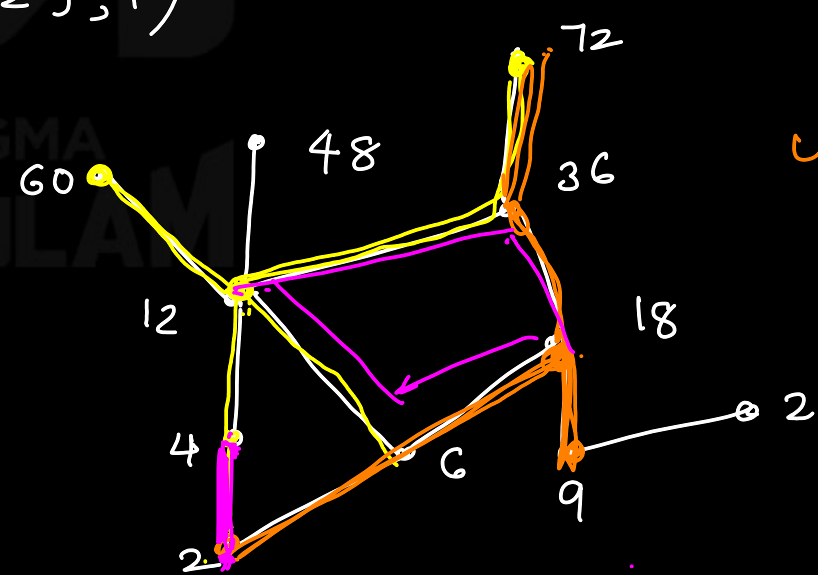
Is there a greatest element and a least element in the poset $(Z^+, |)$?

$Z^+ = \{1, 2, 3, 4, \ldots\}$

- Find the least upper bound of $\{2, 9\}$ and the greatest lower bound of $\{60, 72\}$ for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$

$A = \{60, 72\}$ lower bounds – 12, 4, 2, 6

glb – 12



$B = \{2, 9\}$

UB – 18, 36, 72

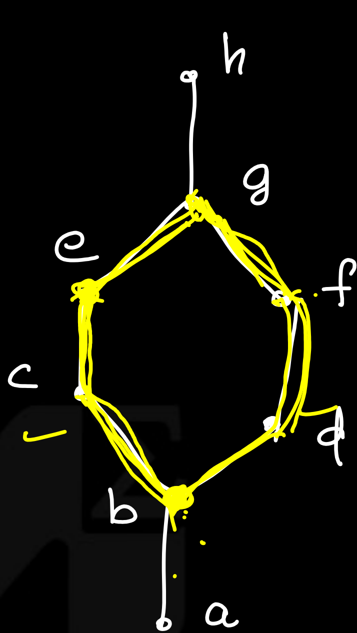
lub – 18

$C = \{2, 4\}$

UB = 4

LB – 2

- Is the given Hasse diagram a lattice?



Consider Incomparable pairs

$S = \{a, b\}$ UB – b, lub – b; LB = a, glb = a

glb(e,d) = b

lub(e,d) = g

glb(c,d) = b

lub(c,d) = g

glb(e,f) = b

lub(e,f) = g

glb(c,f) = b

lub(c,f) = g

Sublattice

Let $(A,\ R)$ be a lattice and B is a subset of A. B is called sublattice of A if for any $a,\ b \in B$, then $a \vee b$ and $a \wedge b \in B$.

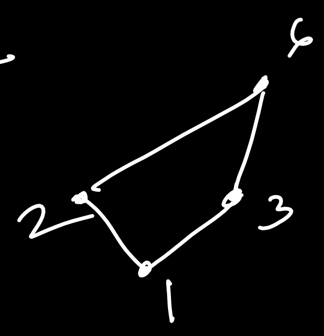
Example: The set all positive devisors of n, D_n with the relation 'divisibility' is a sublattice of $(Z^+,\ |)$.

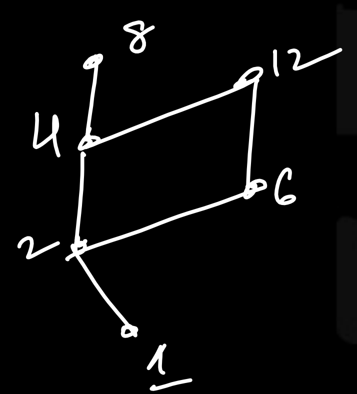
Example 9

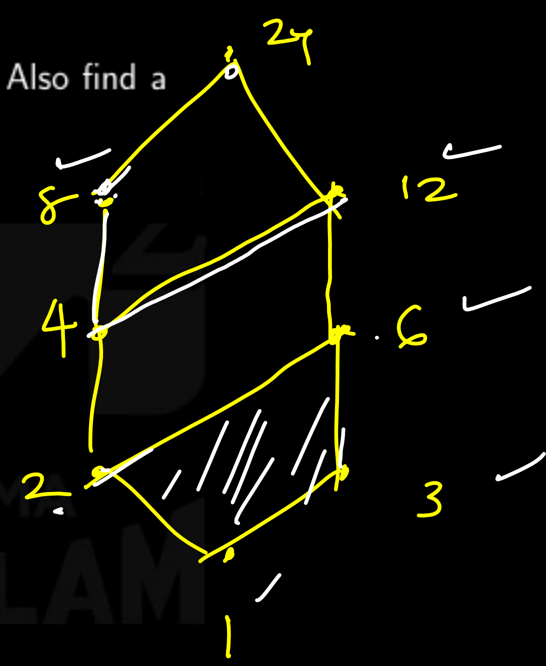
Draw a Hasse Diagram for the set of all divisors of 24. $(D_{24},\ |)$. Prove that it a lattice. Also find a sublattice of D_{24}. Also give an example of poset which is not a sublattice of D_{24}.

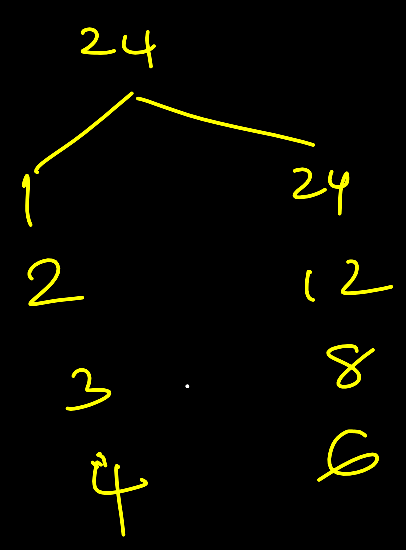
$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

sublattice









$lub(8,24) = 24 \quad lub(8,12) = 24 \quad lub(8,6) = 24$

$glb(8,24) = 8 \quad glb(8,12) = 4 \quad glb(8,6) = 2$

$lub(8,3) = 24 \quad lub(2,3) = 6 \quad lub(4,3) = 12$

$glb(8,3) = 1 \quad glb(2,3) = 1 \quad glb(4,3) = 1$

Properties of Lattice

Let (A, R) is a lattice, for any $x, y \in A$,

1) Idempotent Law
$x + x = x, \quad x \cdot x = x.$ i.e., $x \bigvee x = x, \quad x \bigwedge x = x.$

2) Commutative Law
$x + y = y + x, \quad x \cdot y = y \cdot x.$ i.e., $x \bigvee y = y \bigvee x, \quad x \bigwedge y = y \bigwedge x.$

3) Associative Law
$x + (y + z) = (x + y) + z,$ i.e., $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

i.e., $x \bigvee (y \bigvee z) = (x \bigvee y) \bigvee z,$ i.e., $x \bigwedge (y \bigwedge z) = (x \bigwedge y) \bigwedge z.$

4) Absorption Law
$x + (x \cdot y) = x \quad x \cdot (x + y) = x,$ i.e., $x \bigvee (x \bigwedge y) = x \quad x \bigwedge (x \bigvee y) = x$

Complemented Lattice

For a lattice an element b is said to be complement of a if $a \bigwedge b = 0$ and $a \bigvee b = I$. If a is complement of b then b is the complement of a. Complement of a is denoted by a'. A lattice is said to be complemented lattice if every element has at least one complement.

Distributive Lattice

Lattice $(A, +, .)$ is distributive lattice if for any $a, b, c \in A$,

$a + (b \cdot c) = (a + b) \cdot (a + c)$ — $+$ is distributive over $\cdot$.

i.e., $a \bigvee (b \bigwedge c) = (a \bigvee b) \bigwedge (a \bigvee c)$ — $\bigvee$ is distributive over $\bigwedge$.

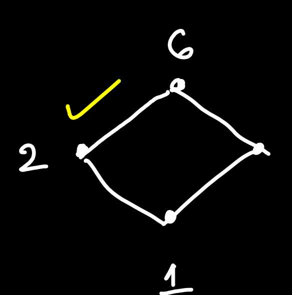
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ — $\cdot$ is distributive over $+$.

i.e., $a \bigwedge (b \bigvee c) = (a \bigwedge b) \bigvee (a \bigwedge c)$ — $\bigwedge$ is distributive over $\bigvee$.

- An element a is called the complement of b if
  (1) $a \vee b = 1$ (lub(a,b) = greatest element)
  (2) $a \wedge b = 0$ (glb(a,b) = least element)

Examples

1. Find the complements for the lattice $(A, |)$ where $A = \{1, 2, 3, 6\}$



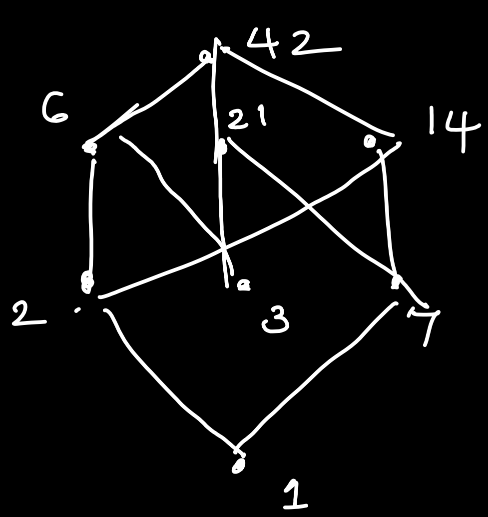
greatest element = 6
least element = 1

Trial & error method

glb(2,3) = 1 = least   glb(1,6) = 1
lub(2,3) = 6 = great   lub(1,6) = 6
$2' = 3 \quad 3' = 2$ $1' = 6 \quad 6' = 1$

2. Find the complements of each element of $(D_{42}, |)$. Is it a complemented lattice?

$D_{42} = \{\bar{1}, \bar{2}, \bar{3}, \bar{6}, \bar{7}, \bar{14}, \bar{21}, \bar{42}\}$



42

1	42
2	21
3	14
6	7

greatest element = 42

least element = 1

$lub(1, 42) = 42$ $\quad lub(2, 21) = 42$

$glb(1, 42) = 1$ $\quad glb(2, 21) = 1$

$1' = 42 \quad 42' = 1$ $\quad \bar{2} = 21 \quad \bar{21} = 2.$

$lub(3, 14) = 42$ $\quad lub(6, 7) = 42$

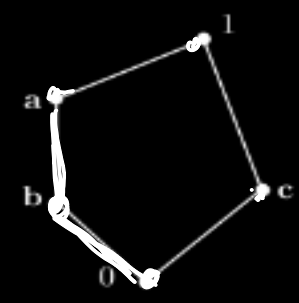
$glb(3, 14) = 1$ $\quad glb(6, 7) = 1$

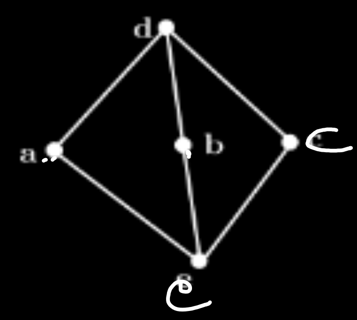
$\bar{6} = 7 \quad \bar{7} = 6$

$\bar{3} = 14 \quad \bar{14} = 3$

Show that the lattice is not distributive,

$\vee$ – lub
$\wedge$ – glb

1.


2.


1. $a \wedge (b \vee c) = a \wedge 1 = a, \quad (a \wedge b) \vee (a \wedge c) = b \vee 0 = b.$

   $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

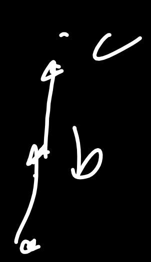
   Lattice is not distributive.

2. $a \wedge (b \vee c) = a \wedge d = a, \quad (a \wedge b) \vee (a \wedge c) = e \vee e = e$

   $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

   Lattice is not distributive.

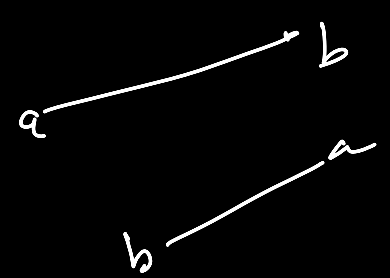
$a \leq b \leq c$



$\vee$

$a \vee c = c$

$a \wedge c = a$



Example

Show that every chain is a distributive lattice.

Solution:

Let $(L, \leq)$ be a chain. For $a, b \in L$, either $a \leq b$ or $b \leq a$

If $a \leq b$, $a \vee b = b$ and $a \wedge b = a$

If $b \leq a$, $a \vee b = a$ and $a \wedge b = b$

For $a, b, c \in L$ and $a \leq b \leq c$, $a \wedge (b \vee c) = a \wedge c = a$ and

$(a \wedge b) \vee (a \wedge c) = a \vee a = a$

Therefore, $a \wedge (b \vee c) = a \wedge b \vee a \wedge c$

Again, $a \vee (b \wedge c) = a \vee b = b$ and $(a \vee b) \wedge (a \vee c) = b$

Therefore, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Thus, every chain is a distributive lattice.