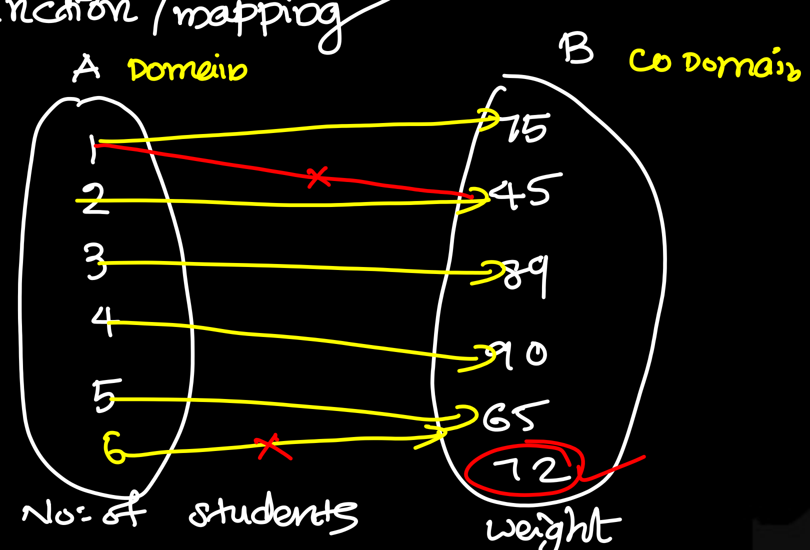
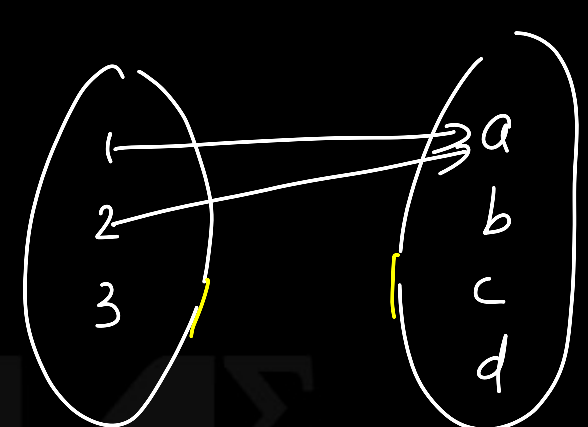
Function / mapping



non one-one

onto.



Not a function

one-one function

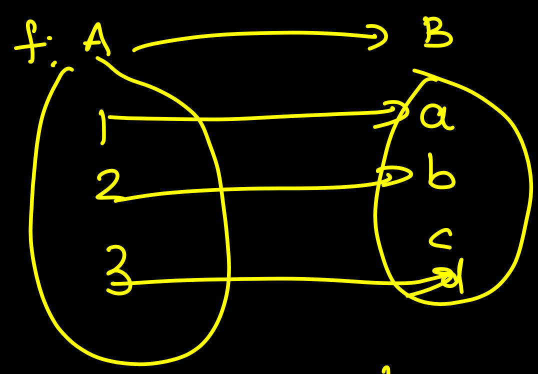
onto function.

Function

$y = f(x)$

output — function — Input

Let A and B are nonempty sets. A function (mapping) f from A to B, denoted by $f : A \to B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.
If $f : A \to B$ is a function from A to B then A is called the domain of f and B is the co-domain of f. The set of all images of A under the function f, denoted by $f(A)$, is called the range of f.



A is the domain, B is the co-domain and $\{a, b, d\}$ is the range of the function f. If $|A| = m$ and $|B| = n$ then the number of functions from A to B is n^m.

- A function is a relation that maps Inputs to outputs, where each Input has exactly one output.

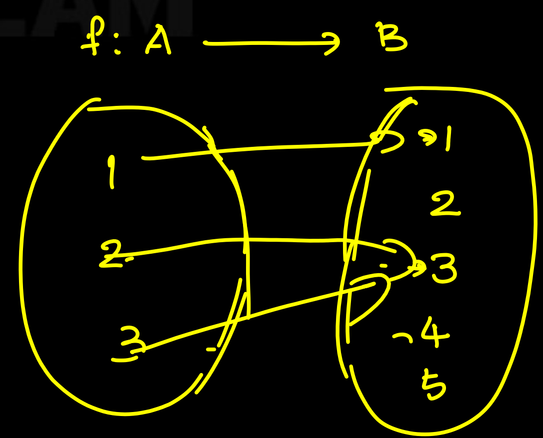
Domain = A
Codomain = B
Range = $\{a, b, d\}$

One-to-one function (Injective function)

A function $f : A \to B$ is called one-to-one or injective function if each element of B appears at most once as the image of an element of A. In other words, the function $f : A \to B$ is called one-to-one function if and only if district elements in A are mapped into distinct elements in B. i.e., the function $f : A \to B$ is one-to-one if and only if for all $a_1, a_2 \in A$,

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

$a_1, a_2 \in A$
$f(a_1) = f(a_2)$
$\Rightarrow a_1 = a_2$



Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. The function $f = \{(1, 1)(2, 3)(3, 4)\}$ is a one-to-one function from A to B. But $g = \{(1, 1)(2, 3)(3, 3)\}$ is not a one-to-one function from A to B because $g(2) = g(3)$ but $2 \neq 3$.

$2, 3 \in A$
$g(2) = g(3) = 3$
$2 \neq 3$

Note: The number of one-to-one functions from A to B is $\frac{n!}{(n-m)!}$ where $|A| = m$ and $|B| = n$.

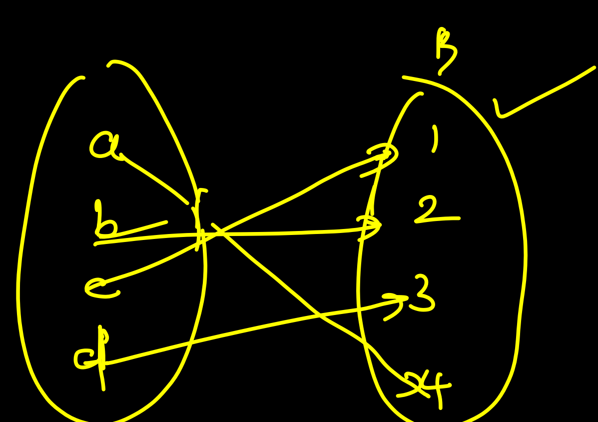
On-to-one and on-to function (Bijective function or one to one correspondence)

on-to function (surjective function)

A function $f: A \to B$ is called on-to if for all $b \in B$ there is atleast one a with $f(a) = b$

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective function.

For example, the function from $f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$ is a bijection.



Inverse Function

Let $f : A \to B$ be a function. The inverse function from B to A exists if and only if it is both one-to-one and on-to. The inverse function of f is denoted by f^{-1}.

$a, b \in A$, $(a, b) \in R \Rightarrow (b, a) \in R$ i.e., $aRb \Rightarrow bRa$.

For example, consider the relation R on the set Z where xRy if $xy \geq 0$ whenever $x, y \in Z$. The relation is symmetric since if $xy \geq 0$ then $yx \geq 0$.

Compositions of functions

If $f ; A \to B$ and $g : B \to C$ be two functions, then the composition of the functions f and g is a new function from A to C denoted by $g \circ f$ and is defined as $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

$$(g \circ f)(x) = g(f(x))$$

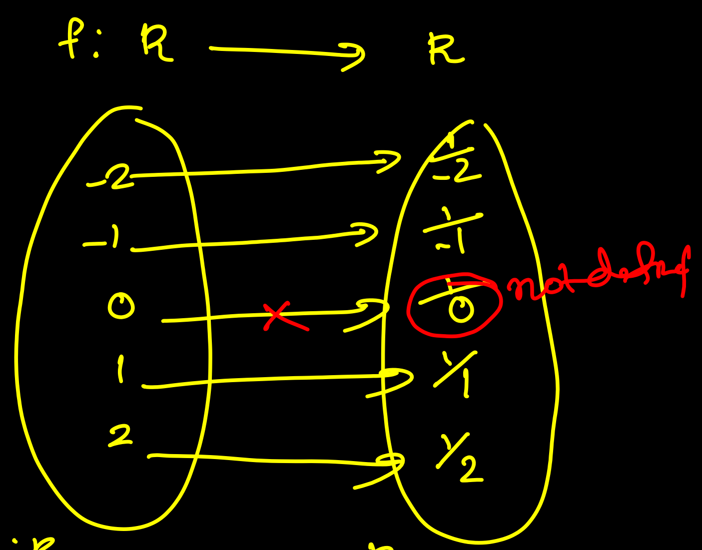
$$(f \circ g)(x) = f(g(x))$$

1. Determine whether the following relations are functions or not.

a) A relation from R to R defined by $\left\{(x, y) \mid x, y \in R, y = \frac{1}{x}\right\}$.

b) A relation from R to R defined by $\{(x, y) \mid x, y \in R, y^2 = x\}$. $x = y^2$

c) A relation from Q to Q defined by $\{(x, y) \mid x, y \in Q, x^2 + y^2 = 1\}$.

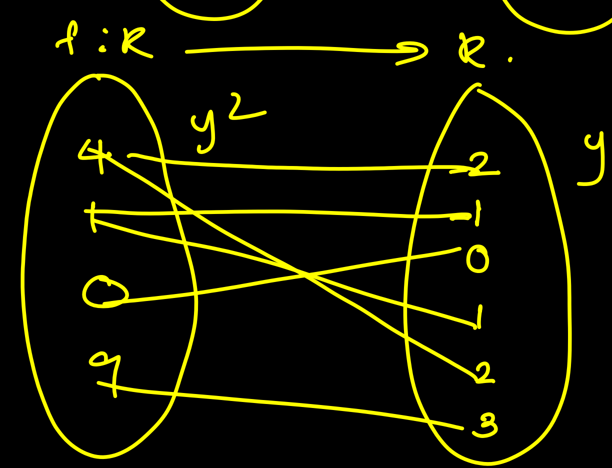


a) $(x, y) = \left(x, \frac{1}{x}\right)$

The expression $1/x$ doesnot exist for $x = 0$

$\therefore f(0)$ is not defined.

$\therefore \quad y = 1/x$ is not a function.



b) $(x, y) = (y^2, y)$

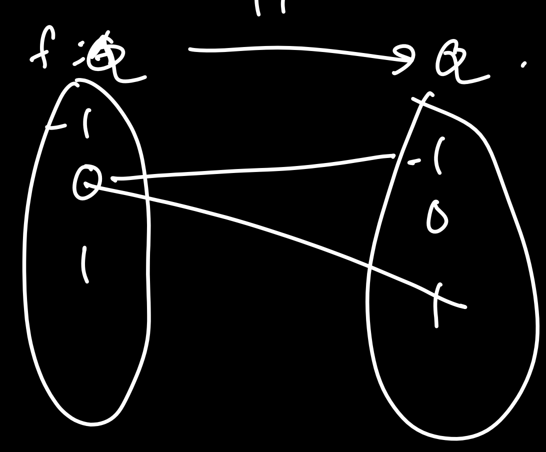
$\therefore$ The relation is not a function since 4 is mapped into 2 & -2.

c) $x^2 + y^2 = 1$

$(0, 1) \quad 0^2 + 1^2 = 1$

$(0, -1) \quad 0^2 + (-1)^2 = 1$

The relation is not a function. Since 0 is mapped into 1 & -1



2. Let $f : R \to R$ where $f(x) = x + 1$ for all $x \in R$. Prove that f is one-to-one function.

For $x_1, x_2 \in R$

Suppose $f(x_1) = f(x_2)$

$x_1 + \cancel{1} = x_2 + \cancel{1}$

$x_1 = x_2$

$\therefore$ f is 1-1 function.

3. Let $g : R \to R$ where $g(x) = x^4 - x$ for all $x \in R$. Is this one-to-one function?

For $x_1, x_2 \in R$

$g(x_1) = g(x_2)$

$x_1^4 - x_1 \neq x_2^4 - x_2$.

For $0, 1 \in R$

$g(0) = 0 = g(1)$

But $0 \neq 1$

$\therefore$ f is not 1-1

$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

4. Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one. If the domain is Z^+ is the function one to one?

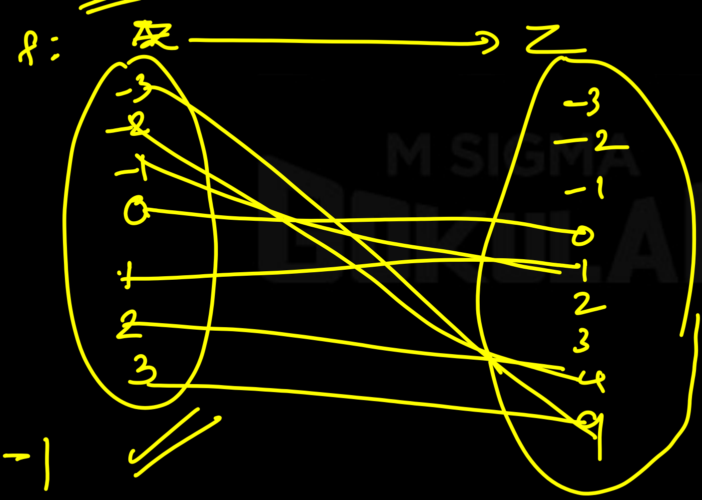
For $2, -2 \in Z$

$f(2) = 4$ $f(-2) = 4$

$f(2) = f(-2)$

But $2 \neq -2$

$\therefore$ f is not 1-1

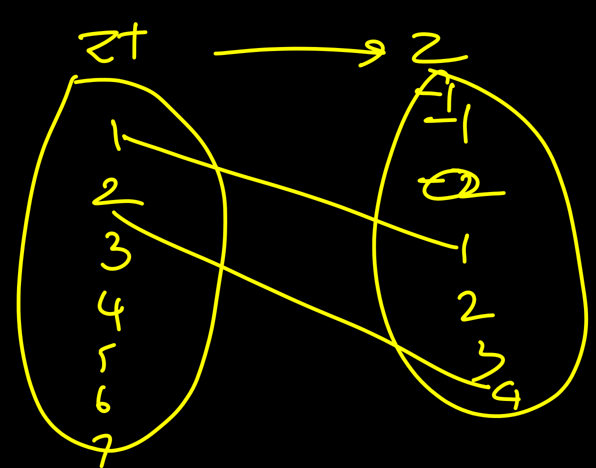


$f(x) = x^2$

(x, x^2)

$Z^+ = \{1, 2, 3, 4, \ldots\}$

If the domain is Z^+ the function is 1-1



5. Is $f(x) = \frac{1}{x^2 - 2}$ define a function from $R \to R$. Is it a function from $Z \to R$?

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$f(\sqrt{2})$ and $f(-\sqrt{2})$ is not defined in the domain of real numbers.

$\therefore$ this is not a function from $R \to R$.

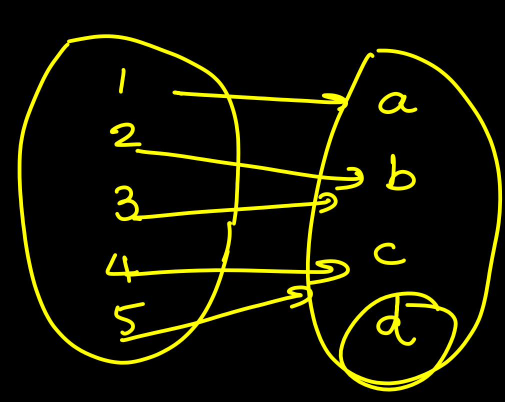
Since $\sqrt{2}, -\sqrt{2} \notin Z$

$\therefore$ $f(\sqrt{2})$ and $f(-\sqrt{2})$ is defined from $Z \to R$

$\therefore$ It is a function from $Z \to R$.

6. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$. $f : A \to B$ such that $f = \{(1, a)\ (2, b)\ (3, b)\ (4, c)\ (5, c)\}$.
Find a) Domain of f b) Range of f

$f : A \longrightarrow B$



a) Domain = A b) Range of $f = \{a, b, c\}$

7. Check whether the function $f : R \to R$ defined by $f(x) = e^{x^2}$ is one-to-one. Determine its range.

For $2, -2 \in R$

$f(2) = e^{2^2} = e^4$

$f(-2) = e^{(-2)^2} = e^4$

$f(2) = f(-2)$

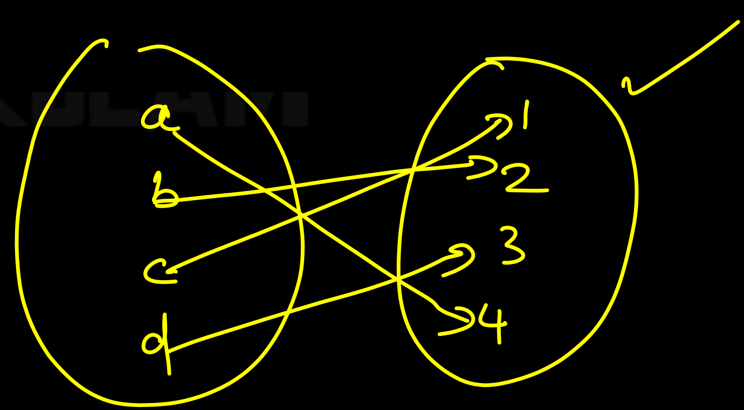
$2 \neq -2$

f is not 1-1

Range = $[0, \infty)$

8. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Yes.



9. Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

10. Let $f, g : Z \to Z$ defined by $f(x) = 2x + 1$ and $g(x) = 3x + 4$. What is the composition of f and g? What is the composition of g and f?

$f(x) = 2x+1$

$(f \circ g)(x) = f(g(x)) = f(3x+4) = 2(3x+4)+1 = 6x+8+1 = 6x+9$

$(g \circ f)(x) = g(f(x)) = g(2x+1) = 3(2x+1)+4 = 6x+3+4 = 6x+7$

11. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = x^2$, $g(x) = x + 2$. Find $(f + g)$ and (fg).

$(f+g)(x) = f(x)+g(x) = x^2+x+2$

$(fg)(x) = f(x)g(x) = x^2(x+2) = x^3+2x^2$

12. Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

13. Let $f, g : R \to R$ defined by $f(x) = 2x + 5$, $g(x) = \frac{x-5}{2}$. Find $(g \circ f)$ and $(f \circ g)$.

$$(g\circ f)(x) = g(f(x)) = g(2x+5) = \frac{2x+\cancel{5}-\cancel{5}}{\cancel{2}} = x$$

$$(f\circ g)(x) = f(g(x)) = f\left(\frac{x-5}{2}\right) = \cancel{2}\cdot\frac{x-\cancel{5}}{\cancel{2}} + \cancel{5} = x.$$

14. Let $f : R \to R^+ \cup \{0\}$ defined by $f(x) = x^2$ and $g : R^+ \cup \{0\} \to R$ defined by $g(x) = \sqrt{x}$. Find $(g \circ f)$ and $(f \circ g)$.

$$(g\circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

$$(f\circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

15. Let $f, g, h : R \to R$ defined by $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for all $x \in R$ where R is set of real numbers,
find a) $f \circ g$ b) $g \circ f$ c) $(f \circ g) \circ h$ d) $h \circ g \circ f$ e) $f \circ (g \circ h)$.

a) $(f \circ g)(x) = f(g(x)) = f(x-2) = x-2+2 = x$

b) $(g \circ f)(x) = g(f(x)) = g(x+2) = x+\cancel{2}-\cancel{2} = x$

c) $(f \circ g) \circ h = (f \circ g) h(x) = f \circ g(3x) = f(g(3x)) = f(3x-2) = 3x-2+2$
$= 3x.$

d) $h \circ g \circ f = h \circ g(f(x)) = h \circ g(x+2) = h(g(x+2)) = h(x+\cancel{2}-\cancel{2}) = h(x)$
$= 3x.$

e) $f \circ (g \circ h) = f(\underbrace{g \circ h(x)}) = f(g(h(x)))$
$= f(g(3x)) = f(3x-2) = 3x-2+2 = 3x.$

16. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$ with $f = A \to B$ and $g = B \to C$ given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, z)\}$. $g(a) = x$ $g(b) = y$ $g(c) = z$.
Find $g \circ f$.

$f(1) = a$ $f(2) = a$
$f(3) = b$ $f(4) = c$ $g \circ f(x) =$

$$g \circ f(1) = g(f(1)) = g(a) = x$$
$$g \circ f(2) = g(f(2)) = g(a) = x$$
$$g \circ f(3) = g(f(3)) = g(b) = y$$
$$g \circ f(4) = g(f(4)) = g(c) = z$$

17. If f, g and h are function of integers such that $f(n) = n^2$, $g(n) = (n+1)$, $h(n) = n - 1$, find a) $f \circ g \circ h$ b) $g \circ f \circ h$ c) $h \circ f \circ g$

a) $(f \circ g \circ h)(n) = (f \circ g)h(n) = f \circ g(h(n)) = f \circ g(n-1) = f(g(n-1))$
$= f(n-1+1)$
$= f(n) = n^2$ //

b) $(g \circ f \circ h)(n) = g \circ f(h(n))$
$= g \circ f(n-1) = g(f(n-1))$
$= g((n-1)^2)$
$= (n-1)^2 + 1 = n^2 - 2n + 1 + 1 = n^2 - 2n + 2$

c) $(h \circ f \circ g)(n) = h \circ f(g(n)) = h \circ f(n+1) = h(f(n+1))$
$= h((n+1)^2)$
$= (n+1)^2 - 1$ //
$= n^2 + 2n + 1 - 1 = n^2 + 2n.$

18. If f, g and h are function such that $f(x) = 2x$, $g(x) = x + 1$ for all $x \in R$. Find a) $f \circ g$ b) $g \circ f$ c) $f \circ f$ d) $g \circ g$

19. Let $f : Z \to Z$ be such ~~taht~~ that $f(x) = x + 5$. Is f invertible, and if it is, what is its inverse?

For $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1 + \cancel{5} = x_2 + \cancel{5}$

$x_1 = x_2$

f is 1-1

Suppose that y is the image of x.

So that $y = x + 5$

Then $x = y - 5 \in Z$

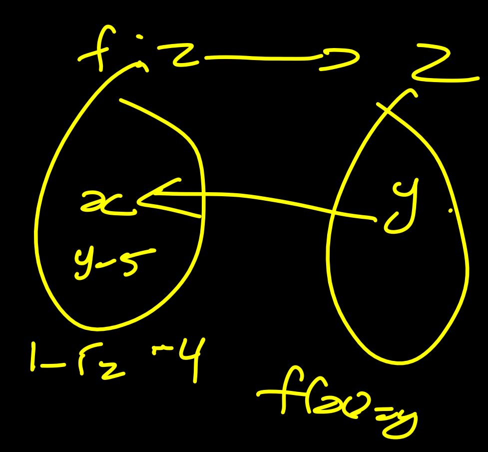
Corresponding to each element $y \in Z$

there exist a unique x s.t. $x = y - 5 \in Z$

$\therefore$ f is onto.

f is invertible

$f^{-1}(y) = x = y - 5$



20. Let f be the function from R to R with $f(x) = x^2$. Is f invertible?

$-2, 2 \in R$

$f(2) = f(-2) = 2^2 = 4$

$2 \neq -2$

f is 1-1

f is not ~~def~~ Invertible.