Discrete Mathematics

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- Set Theory
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- Functions

II Mathematical logic and Proof

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SETS, FUNCTIONS AND RELATIONS

$\{1,2,3,4,5\} = \{5,3,4,2,1\}$

belongs to

$a \in A$

Concepts of sets

A well-defined collection of objects or elements is a set. A "well-defined collection" refers to a set where its members or elements are clearly defined and can be easily determined.

For example, consider the set of "Even Natural Numbers". This set is well-defined because it includes only those natural numbers that are divisible by 2 without leaving a remainder. Every member of this set is clearly determined: $\{2, 4, 6, 8, \ldots\}$.

Roster from

Set builder from

A set is an unordered collection of distinct objects, called elements or members of the set. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

It is common for sets to be denoted using uppercase letters, while lowercase letters are typically used to denote elements of sets.

There are several ways to describe a set. One way is roster form and another way is set builder form.

In roster form, all the elements of the set are listed, separated by commas and enclosed between curly braces $\{\}$.

For example, the notation $\{a, b, c, d\}$ represents the with the four elements a, b, c, and d.

The set of positive integers less than 1000 can be denoted by $\{1, 2, 3, \ldots 999\}$.

In Set-builder form, elements are shown or represented in statements expressing relation among elements.

For example, the set A of all even positive integers less than 100 can be written as $A = \{x | x$ is an even positive integer less than $100\}$.

The usual notation of set is,

R – the set of all real numbers.

R^+ – the set of all positive real numbers.

$N = \{1, 2, 3 \ldots\}$ is the set of all natural numbers.

$Z = \{\ldots \ldots -2. -1. 0, 1, 2 \ldots\}$ is the set of all integers.

$Z^+ = \{1, 2, 3 \ldots\}$ is the set of all positive integers.

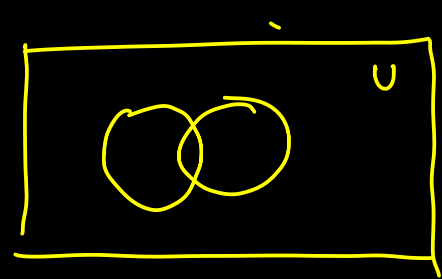
$Q = \left\{\frac{p}{q} \middle| p, q \text{ are integers, and } q \neq 0\right\}$ is the set of all rational numbers.

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{1}$ $\frac{2}{1}$

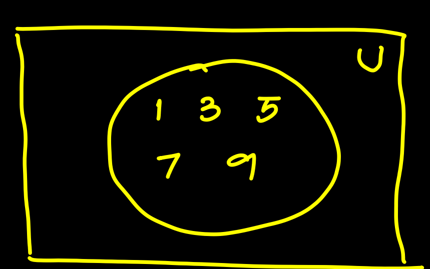
Venn Diagram

Sets can be visually represented using Venn diagrams, introduced by the English mathematician John Venn in 1881. In these diagrams, the universal set U, which includes all objects under consideration, is shown as a rectangle. Within this rectangle, circles or other shapes represent individual sets, and elements of the sets may be depicted as points within these shapes. Venn diagrams are widely used to illustrate the relationships between sets.

For example, we can draw the Venn diagram that represents, the set of all odd numbers less than 10.



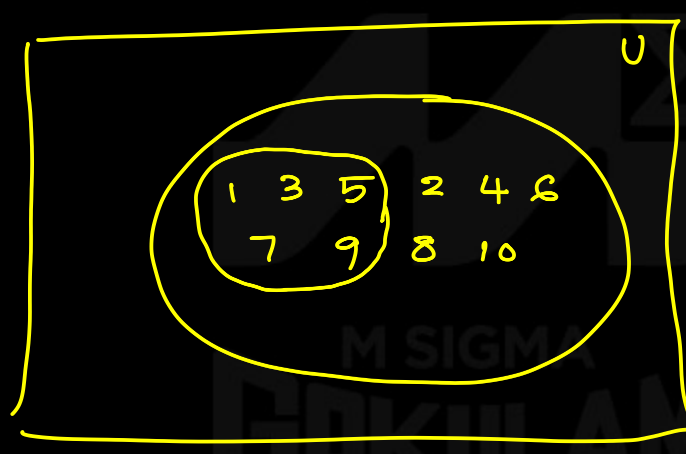
$\{1, 3, 5, 7, 9\}$



Examples

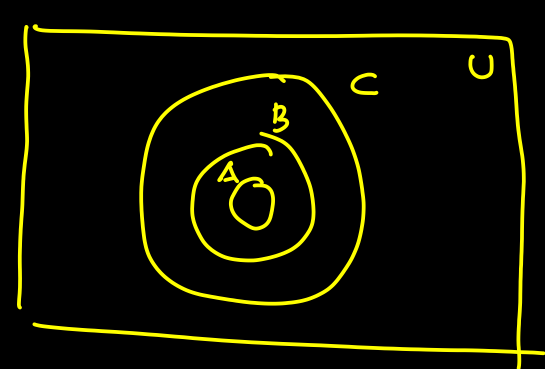
1. Use a Venn diagram to illustrate the subset of odd integers is the set of all positive integers not exceeding 10.

$\{1, \ldots 9, 10\}$ $\{1, 3, 5, 7, 9\}$



2. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$

$A \subset B \subset C$



Size of a set

If S is a set containing exactly n distinct elements, where n is a non negative integer, we describe S as a finite set, and n is called the cardinality of S. The cardinality of S is represented as $|S|$.

For example, let A be the set of odd positive integers less than 10. Then $|A| = 5$.

$|S|$

The set with no element is called empty set or null set denoted by $\emptyset$.

ϕ

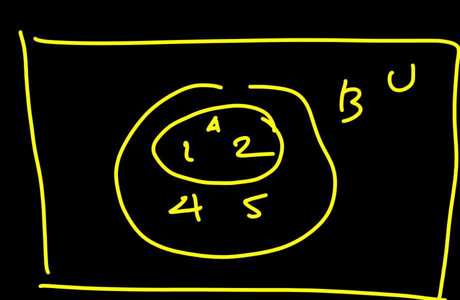
Subset

The set A is a subset of B if every element of A is an element of B. i.e., A is a subset of B if $x \in A$ then $x \in B$: B is called the superset of A. Using the set inclusion symbol $\subseteq$ (contains in) we may write $A \subseteq B$ or $B \supseteq A$. To show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$.

For example, let $A = \{1, 2\}$ is a subset of $B = \{1, 2, 4, 5\}$.

$A \subset B$

for $x \in A$ then $x \in B$



$A = \{1, 2\}$ Subsets of A

$= \{\phi, \cancel{\{1, 2\}}, \{1\}, \{2\}\}$

Proper subsets $= \{\phi, \{1\}, \{2\}\}$

Proper subset

When a set A is a subset of a set B but $A \neq B$, then we say that A is a proper subset of B and is denoted as $A \subset B$.

For example, let $A = \{1, 2\}$, the proper subset of A is $\{\emptyset, \{1\}, \{2\}\}$

Equal Set

Two sets A and B are equal when they contain identical elements.

i.e., if $A \subseteq B$ and $B \subseteq A$

For example, $A = \{1, a, d, 4\}$ and $B = \{4, a, d, 1\}$ are equal sets.

$A = B$

Power Set

The set of all subsets of a given set A is the power set of A denoted by $P(A)$.

For example, let $A = \{a, b\}$ then the power set of A,

$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. all subsets

The power set of the empty set is $\{\emptyset\}$.

The power set of $\{\emptyset\}$ (set) is $\{\emptyset \{\emptyset\}\}$. $\{\phi, \{\phi\}\}$

Number of elements in the power set of a set having n elements is 2^n.

Qn) List the members of the sets 1) $\{x \mid x$ is a real number such that $x^2 = 1\}$ (such that)

2) $\{x \mid x$ is an Integer (real no.) such that $x^2 = 2$

$x^2 = 2$, $(\sqrt{2})^2 = 2$, $(-\sqrt{2})^2 = 2$

$x^2 = 1$, $(-1)^2 = 1$, $(1)^2 = 1$

$1^2 = 1$, $2^2 = 4$, $3^2 = 9$

1) $\{-1, 1\}$ 2) $\{\}$ ϕ

Qn) Find the Powerset of the set of $\{a, b, c\}$. Also find the Proper Subsets of $\{a, b, c\}$

$A = \{a, b, c\}$

$P(A) = \{\phi, \{a, b, c\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

Proper subsets of $A = \{$

Cartesian Product

Let A and B are two sets then the cartesian product (cross product) of A and B is, denoted by $A \times B$, defined as $A \times B = \{(a, b) | a \in A, b \in B\}$. In other words, $A \times B$ is the ordered pair of elements in A and B. Cartesian product $A \times B \neq B \times A$.

For example, if $A = \{a, b\}$ and $B = \{1, 2\}$ then

$A \times B = \{(a, 1)(a, 2)(b, 1)(b, 2)\}$,
$B \times A = \{(1, a)(2, a)(1, b)(2, b)\}$.

If $(a, b), (c, d) \in A \times B$ and $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

If $|A| = m$ and $|B| = n$ then $|A \times B| = m \times n$.

(a, b) (b, c)

$A \times B = \{(a,1), (a,2), (b,1), (b,2)\}$

$B \times A = \{(1,a), (1,b), (2,a), (2,b)\}$

We extended the definition of cartesian product in more than two sets. Let $n \in Z^+$, then for any sets $A_1, A_2, \ldots A_n$ the product

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2 \ldots a_n) | a_i \in A_i\}$ is called ordered n-tuples.

If $(a_1, a_2 \ldots a_n), (b_1, b_2 \ldots b_n) \in A_1 \times A_2 \times \ldots \times A_n$ then

$(a_1, a_2 \ldots a_n) = (b_1, b_2, \ldots b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

Qn) let $A = \{2, 3, 4\}$ and $B = \{4, 5\}$. find a) $A \times B$ b) $B \times A$ c) B^2

a) $A \times B = \{(2,4), (2,5), (3,4), (3,5), (4,4), (4,5)\}$

4 5   4 5

b) $B \times A = \{(4,2), (4,3), (4,4)\ (5,2)\ (5,3), (5,4)\}$

c) $B^2 = B \times B = \{(4,4),\ (4,5),\ (5,4)\ (5,5)\}$

Qn) If $A = \{1, 2, 3, 4\}$ and $B = \{2, 5\}$ and $C = \{3, 4, 7\}$. Prove that

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

HW b) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

a) $B \cap C = \phi$

$A \times (B \cap C) = \phi$

$A \times B = \{(1,2), (1,5), (2,2), (2,5), (3,2)\ (3,5), (4,2), (4,5)\}$

$A \times C = \{(1,3), (1,4), (1,7), (2,3), (2,4), (2,7), (3,3), (3,4), (3,7), (4,3), (4,4), (4,7)\}$

$(A \times B) \cap (A \times C) = \phi$

$\therefore A \times (B \cap C) = (A \times B) \cap (A \times C)$

Q11) let A be a set. show that $\phi \times A = A \times \phi = \phi$

$\in$ belongs to

$\phi \times A = \{(x,y) \mid \underbrace{x \in \phi}, y \in A\}$

Since ϕ has no elements, there is no x s.t. $x \in \phi$.

There is no such (x,y).

$\therefore$ $\underline{\underline{\phi \times A = \phi}}$

$A \times \phi = \{(x,y) \mid x \in A, y \in \phi\} = \phi$

PYQ qn) If $A = \{1,2,3,4,5\}$ $B = \{w, x, y, z\}$. How many elements in $P(A\times B)$ where $P(A\times B)$ is the power set of $A\times B$.

→ $|A| = m$ $|B| = n$ $|A\times B| = mn$.

→ If A is set with n elements. then A has 2^n subsets.

$P(A) \Rightarrow 2^n$ elements

$P(A\times B)$ has 2^{mn} elements

$|A| = 5$ $|B| = 4$

no: of elements in $P(A\times B)$ is $2^{mn} = 2^{20}$ elements

qn) Find A^3 if a) $A = \{a\}$ b) $A = \{0, a\}$ c) $A = \{1, 2\}$

a) $A = \{a\}$ $A^2 = A\times A = \{a\}\times\{a\} = \{(a,a)\}$ $A^3 = A^2\times A = \{(a,a)\}\times\{a\} = \{(a,a,a)\}$

b) $A = \{0, a\}$ $A^2 = \{0, a\}\times\{0, a\} = \{(0,0), (0,a), (a,0), (a,a)\}$

$A^3 = A^2\times A = \{(0,0), (0,a), (a,0), (a,a)\}\times\{0, a\}$

$= \{(0,0,0), (0,a,0), (a,0,0), (a,a,0), (0,0,a), (0,a,a), (a,0,a), (a,a,a)\}$

c) $A = \{1, 2\}$ $A^2 = \{1,2\}\times\{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$ $A = \{1, 2\}$

$A^3 = A^2\times A = \{(1,1,1), (1,2,1), (2,1,1), (2,2,1), (1,1,2), (1,2,2), (2,1,2), (2,2,2)\}$

Qn) let A and B be sets with $|B|=3$. If there are 4096 elements in power set of $A \times B$. what is $|A|$?

$|A|=m \quad |B|=n \quad |A\times B| = 2^{mn}$.

$|A|=m=?\quad |B|=n=3$

$|A\times B| = 2^{mn} = 2^{3n} = 4096 = 2^{12}$

$2^{3n} = 2^{12}$

$3n=12$

$n = \frac{12}{3} = 4$ //

$|A|=4$

Qn) Prove or disprove that if A, B ~~and C~~ are sets, then

$\mathcal{P}(A\times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

No, let $A=B=\phi$ $\qquad A=\phi \quad P(A)=\{\phi\} \qquad B=\phi \quad P(B)=\{\phi\}$

$A\times B = \phi \qquad P(A)\times P(B) = \{\phi\}\times\{\phi\} = \{(\phi,\phi)\}$

$P(A\times B) = \{\phi\}$

$\therefore P(A\times B) \neq P(A)\times P(B)$

Q.11) Prove or disprove that if A, B and C are non empty sets and $A \times B = A \times C$. Then $B = C$

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

$A \times C = \{(a,c) \mid a \in A, c \in C\}$

If $A \times B = A \times C$

$(a,b) = (a,c)$

$b = c$

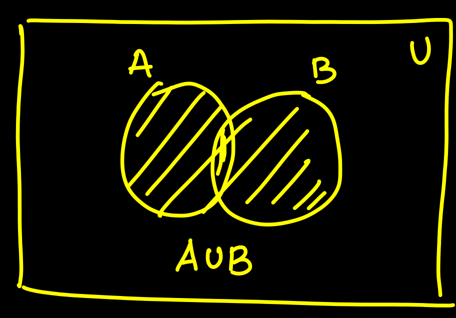
$\underline{\underline{B = C}}$

OR

Union of Sets Set Operations.

If A and B are two sets then $A \cup B$ is the set consists of all the elements that belong to either A or B or both.
An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.
i.e., $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Also, we can say
$x \in A \cup B \Rightarrow x \in A$ or $x \in B$
$x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$



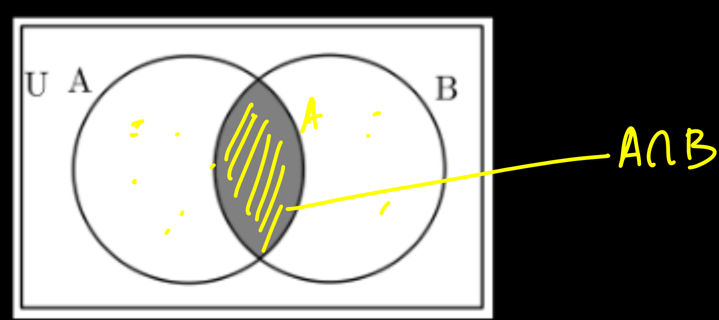
Intersection of sets

and

If A and B are two sets then $A \cap B$ is the set consists of all the elements that belongs to both A and B.
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.
i.e., $A \cap B = \{x | x \in A \text{ and } x \in B\}$

Also, we can say,
$x \in A \cap B \Rightarrow x \in A$ and $x \in B$
$x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$
For example, $A = \{a, b, c\}$ and
$B = \{a, e, f\}$ then
$A \cup B = \{a, b, c, e, f\}$ and $A \cap B = \{a\}$.

Two sets are called disjoint set if their intersection is the empty set. If A and B are two sets, then the Venn diagram of $A \cap B$ is,

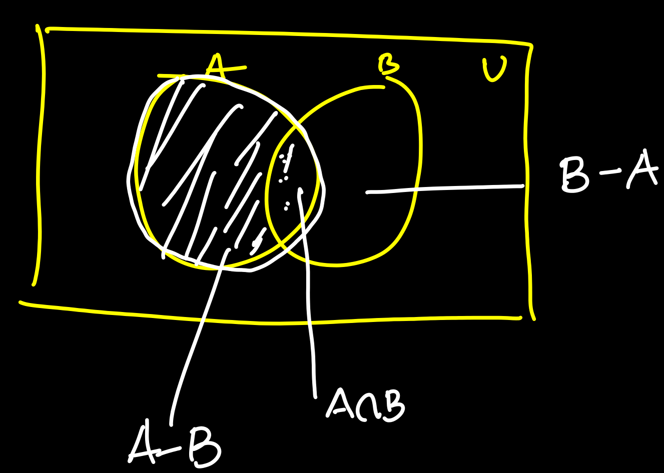


Difference of sets

If A and B are two sets then $A - B$ is the set consists of all the elements that belong to A, but are not in B. The difference of A and B is also called the complement of B with respect to A.
An element x belongs to the difference of the sets A and B if and only if x belongs to A and x does not belongs to B.
i.e., $A - B = \{x | x \in A \text{ and } x \notin B\}$

For example, $A = \{a, b, c\}$ and
$B = \{a, e, f\}$ then $A - B = \{b, c\}$

Note: If A and B are two sets, then,
$A = (A - B) \cup (A \cap B)$,
$B = (B - A) \cup (A \cap B)$
If A and B are two sets, then the Venn diagram of $A - B$ is



$A = (A-B) \cup (A \cap B)$

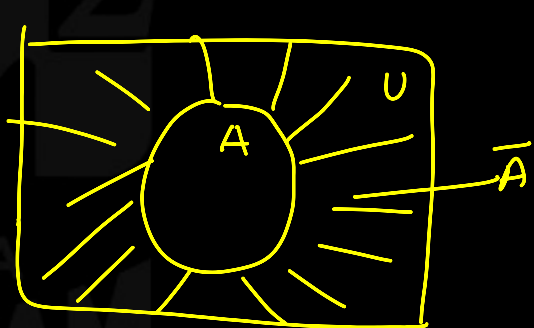
$B = (B-A) \cup (A \cap B)$

Complement of a set
The complement of a set is the set of all elements in a universal set that are not in the given set.

Let us denote the universal set as U, and the given set as A. The complement of set A, denoted as, A' or $\overline{A}$, consists of all elements that belong to U but do not belong to A.

An element x belongs to $\overline{A}$ if and only if $x \notin A$. i.e., $\overline{A} = \{x \in U | x \notin A\}$ For example, Consider the set $U = \{1, 2, 3, 4, 5, 6\}$ as universal set and $A = \{1, 2, 3\}$. Then the complement of $A = \{4, 5, 6\}$.

$\overline{A}$

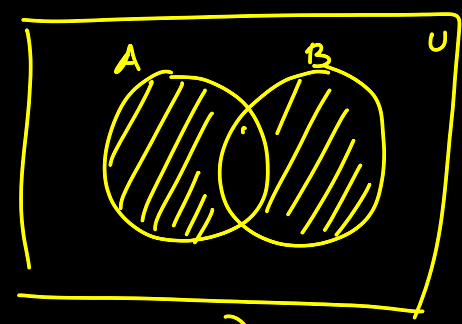


Symmetric difference of sets

The symmetric difference of sets A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. For example, the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is $\{2, 5\}$

$\{\cancel{1}, \cancel{3}, 5\}$ $\{\cancel{1}, 2, \cancel{3}\}$

$\{5, 2\}$



$(A \cup B) - (A \cap B) = A \oplus B$

Generalized Unions and Intersections

$A \cup B \cup C$ contains elements that are in at least one of the sets A, B, and C, while $A \cap B \cap C$ contains elements that are in all three sets A, B, and C.

The union of a collection of sets is the set that includes all elements that belongs to at least one of the sets in the collection.

$A_1 \cup A_2 \cup \ldots \cup A_n = \cup_{i=1}^{n} A_i$ is the union of the sets $A_1, A_2, \ldots, A_n$.

The intersection of a collection of sets is the set containing all elements that are common to every set in the collection.

$A_1 \cap A_2 \cap \ldots A_n = \cap_{i=1}^{n} A_i$ to denote the intersection of the sets $A_1, A_2, \ldots, A_n$.

Set Identities

The important set identities are

A B C

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup U = U,\ A \cap \emptyset = \emptyset$	Domination laws
$A \cup (A \cap B) = A,$ $A \cap (A \cup B) = A,$	Absorption laws
$A \cap U = A,\ A \cup \emptyset = A$	Identity laws
$A \cup A = A,\ A \cap A = A$	Idempotent law
$(\bar{A}) = A$	Complementation law
$A \cup \bar{A} = U,\ A \cap \bar{A} = \emptyset$	Complement laws

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$

$= \{x \mid x \notin A \text{ or } x \notin B\}$

$= \{x \mid x \in \bar{A} \text{ or } x \in \bar{B}\}$

$= \bar{A} \cup \bar{B}$

$A \cup U = U$

$A \cup U = \{x \mid x \in A \text{ or } x \in U\}$

$= \{x \mid x \in U\}$

$= U$

$A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$

$= A \cap (A \cup B)$

$= A$

Qn) Prove $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cup (B \cap C)} = \bar{A} \cap \overline{(B \cap C)}$ (De Morgan's law)

$= \bar{A} \cap (\bar{B} \cup \bar{C})$ "

$= (\bar{B} \cup \bar{C}) \cap \bar{A}$ Commutative Property

$= (\bar{C} \cup \bar{B}) \cap \bar{A}$ "

M SIGMA GOKULAM

Qn) let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

Find a) $A \cup B$ c) $A - B$

b) $A \cap B$ d) $B - A$

$A \cup B = \{a, b, c, d, e, f, g, h\}$

$A \cap B = \{a, b, c, d, e\}$

$A - B = \{\} \phi$

$B - A = \{f, g, h\}$

M SIGMA GOKULAM

Qn) let A, B and C be sets. show that $(A-B)-C = (A-C)-(B-C)$

LHS $= (A-B)-C = \{x \mid x \in A \text{ and } x \notin B \text{ and } x \notin C\}$

elements of A which are not in B & C.

RHS $= (A-C)-(B-C) = \{x \mid x \in A-C \text{ and } x \notin B-C\}$

$= \{x \mid x \in A \text{ and } x \notin C,\ x \notin B \text{ and } x \notin C\}$

$= \{x \mid x \in A \text{ and } x \notin B,\ x \notin C\}$

LHS = RHS

Qn) show that $A \oplus B = (A \cup B) - (A \cap B)$

$A \oplus B$ Suppose $x \in A \oplus B$

Then x must be in A or B but not in both.

ie, $x \in A \cup B$ but $x \notin A \cap B$

$x \in (A \cup B) - (A \cap B)$

$A \oplus B \subseteq (A \cup B) - (A \cap B)$

let $x \in (A \cup B) - (A \cap B)$

$x \in A \cup B$ and $x \notin A \cap B$

$x \in A$ or B, and not in both A & B

$x \in A \oplus B$

an) If $i = 1, 2, \ldots$ let $A_i = \{i, i+1, i+2, \ldots\}$. Prove that $\bigcup_{i=1}^{n} A_i = \{1, 2, 3, \ldots\}$ and $\bigcap_{i=1}^{n} A_i = A_n$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots A_n = \{1, 2, 3, \ldots n\} \cup \{2, 3, 4, \ldots n\} \cup \{3, 4, \ldots\} \cup \{n, n+1, \ldots\}$$

$$= \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \{n, n+1, n+2, \ldots\} = A_n.$$

Qn) $A_i = \{\ldots -2, -1, 0, 1, 2, 3, \ldots i\}$ Prove that $\bigcup_{i=1}^{n} A_i = A_n$ and $\bigcap_{i=1}^{n} A_i = \{\ldots -2, -1, 0, 1\}$

Qn) $A_i = \{1, 2, 3, \ldots i\}$ Prove that $\bigcup_{i=1}^{\infty} A_i = Z^+$ and $\bigcap_{i=1}^{\infty} A_i = \{1\}$

$A_1 \cup A_2 \cup \ldots A_\infty$ $=$ $A_1 = \{1\}$ $A_2 = \{1, 2\}$ $A_3 = \{1, 2, 3\}$

$\{1, 2, 3, 4 \ldots\} = Z^+$