Applications of Propositional Logic

Logic is fundamental in mathematics, computer science, and many other disciplines. Statements in mathematics, science, and everyday language are often vague, but they can be made precise by translating them into the language of logic. Logic is particularly useful in software and hardware specifications, where clarity is essential before development starts. Propositional logic and its rules are applied in designing computer circuits, writing programs, and creating expert systems. Additionally, logic helps to analyse and solve various puzzles. Software systems that use logical rules have been developed to automatically generate specific types of proofs, though they cannot construct all proofs.

1. Translating English Sentences

Translating English sentences into expressions with propositional variables and logical connectives serves several purposes. By converting sentences into logical expressions, we can analyse these expressions to evaluate their truth values.

p

For example, consider the sentence 'You can edit an article only if you are a journalist or you are not a freshman.'

q r

$p \rightarrow (q \vee \neg r)$

Let p: You can edit an article,
q: you are a journalist,
r: you are a freshman.
The logical expression for this sentence is $p \rightarrow (q \vee \neg r)$.

2. System Specifications

Converting sentences from natural language (like English) into logical expressions is crucial for specifying hardware and software systems.

For example, consider the specification "The automated reply cannot be sent when the file system is full."

$q \longrightarrow \neg p$

Let p: The automated reply can be sent.
q: file system is full.

The logical expression for this sentence is $q \rightarrow \neg p$.

3. Boolean Searches

Logical connectives are extensively used in searching large information databases, such as web page indexes. These searches, which employ techniques from propositional logic, are referred to as Boolean searches.

In Boolean searches, the AND operator is used to find records that include both search terms, the OR operator matches records containing either or both terms, and the NOT operator is used to exclude records containing a specific term.

For example, to find pages that deal with universities in New Mexico or Arizona, we can search for pages matching NEW AND MEXICO OR ARIZONA AND UNIVERSITIES.

4. Logic Puzzles

Puzzles that require logical reasoning to solve are called logic puzzles. These puzzles provide a great opportunity to practice applying the principles of logic.

5. Logic Circuits

Propositional logic is applicable in designing computer hardware. A logic circuit, or digital circuit, takes input signals $p_1, p_2, \ldots p_n$, where each input is a bit (either 0 or 1), and generates output signals $S_1, S_2, \ldots S_n$, each also represented as a bit.

Question: You can watch the movie only if you are over 18 years old or have parental permission.

p : You can watch the movie

q : You are over 18 years old

r : You have parental permission

$p \rightarrow (q \vee r)$

Question: You are eligible to be President of the United States only if you are at least 35 years old, were born in U.S. or had both parents as citizens at the time of your birth and have lived in the country for at least 14 years.

$p \rightarrow (q \wedge (r \vee s) \wedge t)$

p : You are eligible to the president of the US

q : You are atleast 35 years old

r : You were born in US

s : At the time of your birth both of your parents are US citizens.

t : You have lived in the country for atleast 14 years.

Question: You can graduate only if you have completed the requirements for your major, do not owe money to the university and have no overdue library books.

$p \rightarrow q \wedge \neg r \wedge \neg s$

$\rightarrow F \Longrightarrow$ True.

iff $\leftrightarrow$

if $\rightarrow$

either or $\vee$

and $\wedge$

$\neg r \vee s$ True.

$F \vee F \Longrightarrow F$

Question: The system is in multiuser mode if and only if it is operating normally. If the system is operating normally, the kernel is functioning. Either the kernel is not functioning, or the system is in interrupt mode. If the system is not in multiuser mode, then it is in interrupt mode. However, the system is not in interrupt mode. Are these system specifications consistent?

Solution:

$\neg s$ T $\quad s$ F

Let p, q, r and s represent propositions.

p : The system is in multiuser mode.

q : The system is operating normally.

r : The kernel is functioning.

s : The system is in interrupt mode.

$\neg p \rightarrow s$ = True

$F \rightarrow F$ = True

$\neg p$ F $\quad p$ True

The specifications can then be written as,

$p \leftrightarrow q,\ q \rightarrow r,\ \neg r \vee s,\ \neg p \rightarrow s,\ \neg s$

If $\neg s$ is true, then s is false. For $\neg p \rightarrow s$ to be true when s is false, the hypothesis $\neg p$ must be false, so p must be true. Since we want $p \leftrightarrow q$ to be true, this shows that q must also be true. Since we want $q \rightarrow r$ to be true, we must therefore have r true. But now if r is true and s is false, then the third specification, $\neg r \vee s$ is false. Therefore, we conclude that this system is not consistent.

$p \leftrightarrow q$ True. $\quad q$ True

T $\quad$ T $\quad$ $q \rightarrow r$ True.

T $\quad$ T $\quad$ r True.

Question: An island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. Two people A and B. What are A and B if

(a) A says, "B is a knight" and B says "The two of us are opposite types"?

(b) A says, "The two of us are both knights" and B says "A is a knave."

Solution:

(a) A's Statement: B is a knight.

B's Statement: The two of us are opposite types.

Case I:

Assume that A is knight.

If A is a knight, A always tells the truth. Therefore, A's statement that "B is a knight" must be true. So, B is a knight.

If B is a knight, B's statement that "The two of us are opposite types" would be false because B is supposed to be truthful, but the statement says they are opposite types, which contradicts B being a knight. Hence, B's statement is a lie, meaning B cannot be a knight.

This leads to a contradiction, so A cannot be a knight.

Case II:
Assume A is a knave. If A is a knave, A always lies.
Therefore, A's statement that 'B is a knight' must be false. Hence, B is a knave. Now, since B is a knave, B's statement that the "The two of us are opposite types" is a lie. This means the statement "The two of us are opposite types" is false, indicating that A and B are not opposite types. Given that we have already established B as a knave, A must also be a knave. We can conclude that both A and B are knaves.

(b) A's Statement: 'The two of us are both knights'
B's Statement: 'A is a knave'.
Case 1:
Assume A is a knight.
If A is a knight, A tells the truth. Therefore, A's statement that "The two of us are both knights" must be true. This means both A and B are knights.
If B is a knight, B's statement that 'A is a knave' should be true. However, this contradicts our assumption that A is a knight (since B would be lying about A being a knave). Thus, this scenario is impossible. So, A cannot be a knight.

Case 2:
Assume A is a knave:
If A is a knave, A always lies. Therefore, A's statement. The two of us are both knights are false. Hence, it is not true both A and B are knights. Since A is already a knave, B must be either a knight or a knave.
Let us know analyse B's statement under the assumption that A is a knave. B's Statement: " A is a knave."
Since B's statement is about A being a knave and A is indeed a knave, B's statement is true. Therefore, if B's statement is true, B must be a knight(because knights tell the truth).
Thus, we conclude that A is a knave and B is a knight.

$S_1 \equiv S_2$

PROPOSITIONAL EQUIVALENCE

Logical Equivalences

2-top.

Two statements S_1 and S_2 are said to be logically equivalent, we write $S_1 \equiv S_2$, when the statement $S_1 \leftrightarrow S_2$ is true. When $S_1 \equiv S_2$, the statements S_1 and S_2 provide the same truth tables because S_1 and S_2 have the same truth value for all choices for their primitive components.

Logical Implication

$\Longrightarrow$

If p and q are arbitrary statements such that $p \to q$ is a tautology, then p logically implies q, and we write $p \implies q$. When p and q are statements and $p \implies q$, then the implication $p \to q$ is a tautology.

Truth Table

P	$\neg P$
1	0
0	1

Conjunction $P \wedge q$

and

$T \wedge T = T$

P	q	$P \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction $P \vee q$

or

P	q	$P \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Exclusive or $P \oplus q$

P	q	$P \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Implication $P \rightarrow q$

P	q	$P \rightarrow q$
1	1	1
T 1	F 0	0 F
0	1	1
0	0	1
F	F	T

$T \rightarrow F$ false

Biconditional Proposition $p \leftrightarrow q$

P	q	$P \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

2. Prove that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

3. check whether the propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$ are logically equivalent.

p	q	r	$q \wedge r$	$(p \vee q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
1	1	1	1	1	1	1
1	0	0	0	1	1	0
0	1	0	0	0	1	0
0	0	1	0	0	0	0
1	1	0	0	1	1	0
1	0	1	0	1	1	1
0	1	1	1	1	1	1
0	0	0	0	0	0	0

not

4. check whether the propositions $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$ are logically equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$(p \rightarrow q)$	$(p \rightarrow q) \rightarrow r$
1	1	1	1	1	1	1
1	0	0	1	1	0	1
0	1	0	0	1	1	0
0	0	1	1	1	1	1
1	1	0	0	0	1	0
1	0	1	1	1	0	1
0	1	1	1	1	1	1
0	0	0	1	1	1	0

5. Prove that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

p	q	$p \oplus q$	$\neg(p \oplus q)$	$p \leftrightarrow q$
1	1	0	1	1
1	0	1	0	0
0	1	1	0	0
0	0	0	1	1

Laws of Logic

For any primitive statements p, q, r and any tautology T_0 and contradiction F_0, the following are the laws of logic.

p	$\neg p$	$\neg(\neg p)$
1	0	1
0	1	0

Imp

$p \vee F_0 \equiv p$

p	F_0	$p \vee F_0$
1	0	1
0	0	0

p	T_0	$p \wedge T_0$
1	1	1
0	1	0

1.	$\neg(\neg p) \equiv p$	Law of double negation
2.	$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's Laws
3.	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws
4.	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative Laws
5.	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ ☆ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ ☆	Distributive Laws
6.	$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent Laws
7.	$p \vee F_0 \equiv p, \quad p \wedge T_0 \equiv p$	Identity Laws
8.	$p \vee \neg p \equiv T_0, \quad p \wedge \neg p \equiv F_0$	Inverse Laws
9.	$p \wedge F_0 \equiv F_0, \quad p \vee T_0 \equiv T_0$	Domination Laws
10.	$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$	Absorption Laws

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Logical Equivalences Involving Conditional Statements

1.	$p \to q \equiv \neg p \vee q$ ★
2.	$p \to q \equiv \neg q \to \neg p$
3.	$p \wedge q \equiv \neg(p \to \neg q)$
4.	$p \vee q \equiv \neg p \to q$
5.	$\neg(p \to q) \equiv p \wedge \neg q$
6.	$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7.	$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8.	$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9.	$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

$p \leftrightarrow q$

$p \to q \wedge q \to p$

Logical Equivalences Involving Biconditional Statements

1.	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2.	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
3.	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
4.	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example 1

Show that $\neg(p \to q)$ and $(p \wedge \neg q)$ are logically equivalent without using truth table.

Solution:

$\neg(p \to q)$	
$\equiv \neg(\neg p \vee q)$	$p \to q \equiv \neg p \vee q$
$\equiv \neg(\neg p) \wedge \neg q$	De Morgan law
$\equiv p \wedge \neg q$	double negation

$\neg(p \to q) \equiv \neg(\neg p \vee q)$ $(p \to q \equiv \neg p \vee q)$

$\equiv \neg(\neg p) \wedge \neg(q)$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\equiv p \wedge \neg q$ double

$(p \wedge \neg p \equiv F_0)$

Example 2

Prove that $[(p \vee q) \wedge (p \vee \neg q)] \vee q \equiv p \vee q$

Solution:

$[(p \vee q) \wedge (p \vee \neg q)] \vee q$	
$\equiv [p \vee (q \wedge \neg q)] \vee q$	Distributive Law
$\equiv (p \vee F_0) \vee q$	Inverse Law
$\equiv (p \vee q)$	Identity Law

$[(p \vee q) \wedge (p \vee \neg q)] \vee q \equiv [p \vee (q \wedge \neg q)] \vee q$

$\equiv (p \vee F_0) \vee q$

$\equiv p \vee q$

Example 3

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q))$	
$\equiv \neg p \wedge \neg(\neg p \wedge q)$	De Morgan law
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	De Morgan Law
$\equiv \neg p \wedge (p \vee \neg q)$	double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	distributive law
$\equiv F_0 \vee (\neg p \wedge \neg q)$	Inverse Law
$\equiv \neg p \wedge \neg q$	identity law

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$

$\equiv \neg p \wedge \neg(\neg p) \vee \neg(q)$

$\equiv \neg p \wedge (p \vee \neg q)$

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$

$\equiv F_0 \vee (\neg p \wedge \neg q)$

$\equiv \neg p \wedge \neg q$

Example 4

Prove that $(p \wedge q) \to (p \vee q)$ is a Tautology.

$(p\wedge q)\to(p\vee q)\equiv T_0$

$\equiv \neg(p\wedge q)\vee(p\vee q)$

$\equiv (\neg p\vee\neg q)\vee(p\vee q)$

Solution:

$(p \wedge q) \to (p \vee q)$	
$\equiv \neg(p \wedge q) \vee (p \vee q)$	Since $(p \to q) \equiv (\neg p \vee q)$
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	Demorgan's Law
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	associative and commutative laws
$\equiv T_o \vee T_o$	Inverse law
$\equiv T_o$	domination law

$\equiv (\neg p\vee p)\vee(\neg q\vee q)$

$\equiv T_0 \vee T_0 \equiv T_0$ //

Example 5

Prove that $(p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$

Solution:

$(p \to q) \wedge (r \to q)$	
$\equiv (\neg p \vee q) \wedge (\neg r \vee q)$	Since $(p \to q) \equiv (\neg p \vee q)$
$\equiv (\neg p \wedge \neg r) \vee q$	Distributive Law
$\equiv \neg(p \vee r) \vee q$	Demorgan's Law
$\equiv p \vee r \to q$	Since $(p \to q) \equiv (\neg p \vee q)$

$(p\to q)\wedge(r\to q) \equiv (\neg p\vee q)\wedge(\neg r\vee q)$

$\equiv (\neg p\wedge\neg r)\vee q$

$\equiv \neg(p\vee r)\vee q$

$\equiv p\vee r\to q$

Example 6

Show that $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

Solution:

$\neg p \to (q \to r)$	
$\equiv p \vee (q \to r)$	Since $(p \to q) \equiv (\neg p \vee q)$
$\equiv p \vee (\neg q \vee r)$	Since $(p \to q) \equiv (\neg p \vee q)$
$\equiv (p \vee \neg q) \vee r$	Associative laws
$\equiv (\neg q \vee p) \vee r$	Commutative Law
$\equiv \neg q \vee (p \vee r)$	Associateive Law
$\equiv q \to (p \vee r)$	Since $(p \to q) \equiv (\neg p \vee q)$

$\neg p\to(q\to r)\equiv \neg(\neg p)\vee(q\to r)$

$\equiv p\vee(\neg q\vee r)$

$\equiv (p\vee\neg q)\vee r$

$\equiv (\neg q\vee p)\vee r$

$\equiv \neg q\vee(p\vee r)$

$\equiv q\to(p\vee r)$

Example 7

Prove that the statements are logically equivalent without using truth table
$(p \vee q) \wedge \neg(\neg p \wedge q) \equiv p$

Solution:

$(p \vee q) \wedge \neg(\neg p \wedge q)$	
$\equiv (p \vee q) \wedge (\neg\neg p \vee \neg q)$	Demorgan's Law
$\equiv (p \vee q) \wedge (p \vee \neg q)$	Law of double negation
$\equiv p \vee (q \wedge \neg q)$	Distribution Law
$\equiv p \vee F_0$	Inverse Law
$\equiv p$	Identity Law

8) Prove that $\neg[\neg((p \vee q) \wedge r) \vee \neg q] \equiv q \wedge r$

Solution:

$\neg[\neg((p \vee q) \wedge r) \vee \neg q]$	
$\equiv [\neg\neg((p \vee q) \wedge r) \wedge \neg\neg q]$	Demorgan's Law
$\equiv [(p \vee q) \wedge r] \wedge q$	Law of double negation
$\equiv (p \vee q) \wedge (r \wedge q)$	Associative Law
$\equiv (p \vee q) \wedge (q \wedge r)$	Commutative Law
$\equiv ((p \vee q) \wedge q) \wedge r$	Associative Law
$\equiv (q \wedge (p \vee q)) \wedge r$	Commutative Law
$\equiv q \wedge r$	Absorption Law

$\neg[\neg((p \vee q) \wedge r) \vee \neg q] \equiv ((p \vee q) \wedge r) \wedge q$

$\equiv (p \vee q) \wedge (r \wedge q)$

$\equiv (p \vee q) \wedge (q \wedge r)$

$\equiv ((p \vee q) \wedge q) \wedge r$

$\equiv (q \wedge (p \vee q)) \wedge r$

$\equiv q \wedge r$

Example 9

Prove that $(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \equiv (\neg p \wedge q)$

Solution:

$(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q))$	
$\equiv (p \vee q) \wedge ((\neg p \wedge \neg p) \wedge q)$	Associative Law
$\equiv (p \vee q) \wedge (\neg p \wedge q)$	Idempotent Law
$\equiv [p \wedge (\neg p \wedge q)] \vee [q \wedge (\neg p \wedge q)]$	Distributive Law
$\equiv [(p \wedge \neg p) \wedge q] \vee [(q \wedge \neg p) \wedge q]$	Associative Law
$\equiv ((p \wedge \neg p) \vee (q \wedge \neg p) \wedge q)$	Distributive Law
$\equiv (F_0 \vee (q \wedge \neg p) \wedge q)$	Inverse Law
$\equiv (q \wedge \neg p) \wedge q$	Identity Law
$\equiv (\neg p \wedge q) \wedge q$	Commutative Law
$\equiv \neg p \wedge (q \wedge q)$	Associative Law
$\equiv (\neg p \wedge q)$	Idempotent Law

Example 10

Prove that $\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

Solution:

$(\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r))$	
$\equiv [(\neg p \wedge \neg q) \wedge r] \vee (q \wedge r) \vee (p \wedge r)$	Associative Law
$\equiv [\neg(p \vee q) \wedge r] \vee (q \wedge r) \vee (p \wedge r)$	Demorgan's Law
$\equiv [\neg(p \vee q) \wedge r] \vee [(q \vee p) \wedge r]$	Distributive Law
$\equiv [\neg(p \vee q) \vee (q \vee p)] \wedge r$	Distributive Law
$\equiv [\neg(p \vee q) \vee (p \vee q)] \wedge r$	Commutative Law
$\equiv T_0 \wedge r$	~~Inverter~~ Law Inverse.
$\equiv r$	Identity Law