Reflexive
Symmetric × Antisymmetric
Transitive
If aRb and $bRa \iff a=b$

Partial Order Relation

A binary ~~relatino~~ relation R on a set A is called partial order relation if it is Reflexive, Antisymmetric and Transitive. A set A together with a partial order relation R is called partially ordered set or Poset denoted by (A, R).
In various posets symbols such as $\leq$ (less than equal to), $\subseteq$ (set inclusion), and $|$ (divisibility) are commonly used to represent partial ordering. However, when discussing the ordering relation in an arbitrary poset, a general symbol is needed. Typically, the notation $a \preceq b$ is used to indicate that $(a, b) \in R$ in an arbitrary poset (S, R).

$a \leq b$

$\leq \quad \subseteq \quad |$

$2, 3 \in \mathbb{Z}$

$2 \leq 3 \qquad 3 \not\leq 2$

$2+3=5 \qquad 3+2=5$

$a \quad c$

$2 \leq 2$

$2 \leq 2$

$Z^+, |$

$a|b \Rightarrow b=kc$

$1, 2, 3, - - - -$

$3, 5 \in Z^+$

$3 \nmid 5 \qquad 5 \nmid 3$

Comparable

The elements a and b of a ~~post~~ poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$.
When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.
For example, the poset $(Z, \preceq)$ is comparable since $a \leq b$ or $b \leq a$ when ever a and b are integers whereas the set of all positive integers with divisibility relation, $(Z^+, |)$ is not comparable since 3 and 5 are not comparable.

$Z \qquad 2, 3 \in Z \qquad 3, 4$

$2 \leq 3 \qquad 3 \leq 4 \qquad 4 \not\leq 3$

$3 \not\leq 4$

Totally Ordered Set / chain

If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order. A totally ordered set is also called a chain.

$a \longrightarrow b$

$a \longleftarrow b$

1. Determine whether the relation 'divisibility' is a partial order relation on $Z^+ \times Z^+$.

$3|3$

$2,4 \in Z^+$ $\quad 2|4 \quad 4 = 2 \times 2$

$4 \nmid 2 \quad 2 = 4(\frac{1}{2})$

$2|2$ ✓

$2|4 \quad 4|8 \Rightarrow 2|8$

① Reflexivity

For any $a \in Z^+$

a divides a itself

$\therefore a|a$

aRa

$\Rightarrow$ reflexive.

② Antisymmetric. $a, b \in Z^+$

if a divides b and b divides a only when $a=b$

$\Rightarrow$ aRb and $bRa \Leftrightarrow a=b$.

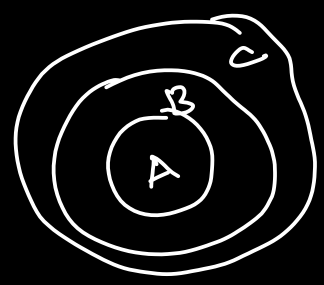
$\therefore$ R is antisymmetric.

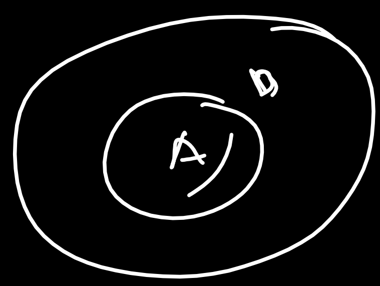
③ Transitivity.

if a divides b and b divides c then a divides c

aRb & $bRc \Rightarrow cRc$.

2. Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.





$A \subseteq B$

① For any $A \in S$, $A \subseteq A$ $\therefore$ reflexive

② For any $A, B \in S$

If $A \subseteq B$ and $B \subseteq A$ only when $A = B$

$\therefore$ antisymmetric.

③ For any $A, B, C \in S$

If $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

$\therefore$ Transitive.

3. If R is a relation on Z where xRy if $x+y$ is odd. Check whether R is a partial order relation.

$1 \in Z$

$1+1=2 \notin$ odd

$R = \{ xRy : x+y \text{ is odd} \}$

$0+1=1$ odd

$1+2=3$

$0+2=2$ eve

① for any $x \in Z$, $x+x$ is not odd.

$\therefore$ R is not reflexive.

② for any $x, y \in Z$, if $x+y$ is odd then $y+x$ is also odd. $\therefore$ The R is symmetric, not antisymmetric.

③ for any $x, y, z \in Z$

if $x+y$ is odd and $y+z$ is odd. then $x+z$ is not odd

$\therefore$ not transitive.

$\therefore$ R is not a Partial order relation

4. Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

If a person x is older than a person, y then y is older than x only when $x = y$. So, relation is antisymmetric
If a person x is older than person y and y is older than person z, then x is older than z. So, the relation is transitive.
But R is not reflexive, because no person is older than himself or herself. Hence the relation is not a partial order relation.

0   2.1   0, 2, 4, 6, 8 . . .
2.0

5. Let R be a relation on the set of all integers such that aRb if a-b is a nonnegative even integer. Verify that R is a partial order relation?

$R = \{ aRb$ if $a-b$ is a non-negative even Integer $\}$

① for any $a \in Z$

$a - a = 0$ is an non-negative even Integer

$\therefore$ re.

② for any $a, b \in Z$ $6, 2 \in Z$

If a-b is an non-negative even Integer and $b-a$ is a no — $6-2=4$ ✓ $2-6=-4$ ✗

iff $a = b$

③ for any $a, b, c \in Z$

If $a-b$ is
$a - b = 2n$, $b - c = 2m$

$a - \cancel{b} + \cancel{b} - c = 2n + 2m$

$a - c = 2(n+m)$

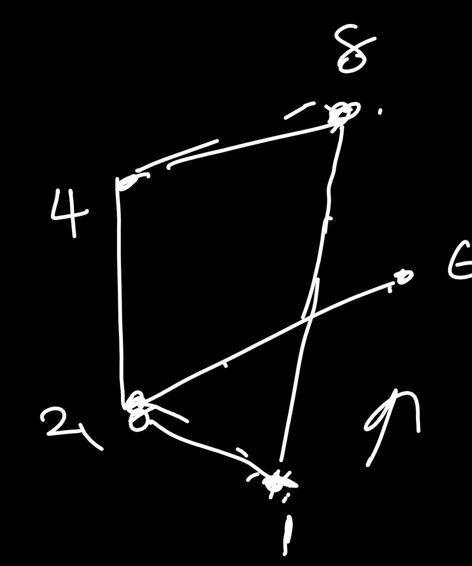
$\Rightarrow$ $\underline{aRc}$

Hasse Diagram

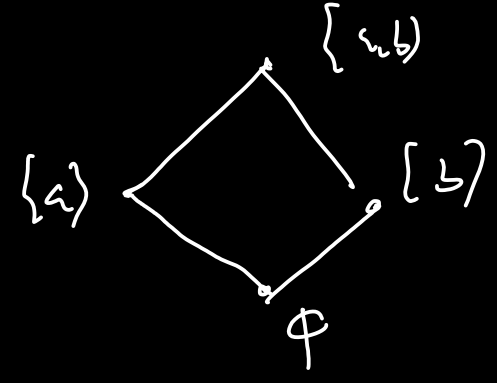
let say we have a set $A = \{1, 2, 4, 6, 8\}$ and relation R is defined on a set A

we know that (A, R) is a poset under '$|$'

$$R = \{(1,1), (2,2), (4,4), (6,6), (8,8),$$
$$(1,2), (1,4), (1,6), (1,8)$$
$$(2,4)\ (2,6), (2,8)$$
$$(4,8)\}$$



directed graph

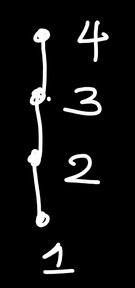


Hasse Diagram or Partially ordered set diagram

Poset can be represented by a diagram known as Hasse Diagram. In such a diagram each element is represented by . or circle

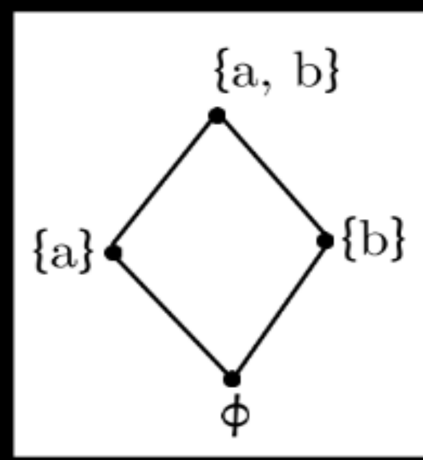
Example : 1

Let $A = \{1,\ 2,\ 3,\ 4\}$ and $\leq$ be the relation R. The Hasse diagram is

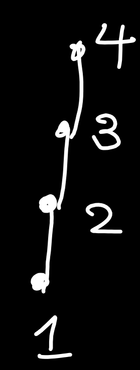


Example : 2

Let $A = \{a,\ b\}$, $P(A) = \{\phi,\ \{a\},\ \{b\},\ \{a,\ b\}\}$. The Hasse diagram of $(P(A),\ \subseteq)$ is



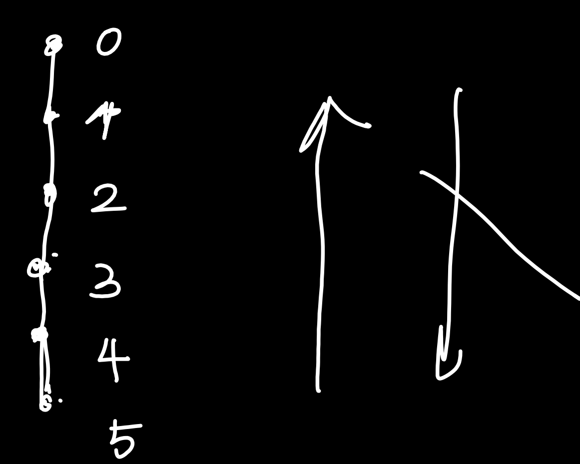
1. Draw the Hasse diagram representing the partial ordering $R = \{(a,b) | a \leq b\}$ on set $S = \{1,2,3,4\}$



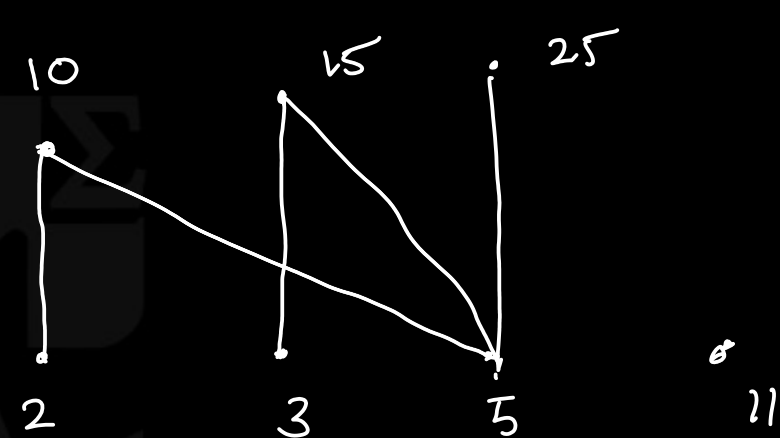
Maximal element
Minimal element
Greatest element
least element

$\geq$

2. Draw the Hasse diagram for the "greater than or equal to", relation on set $S = \{0,1,2,3,4,5\}$

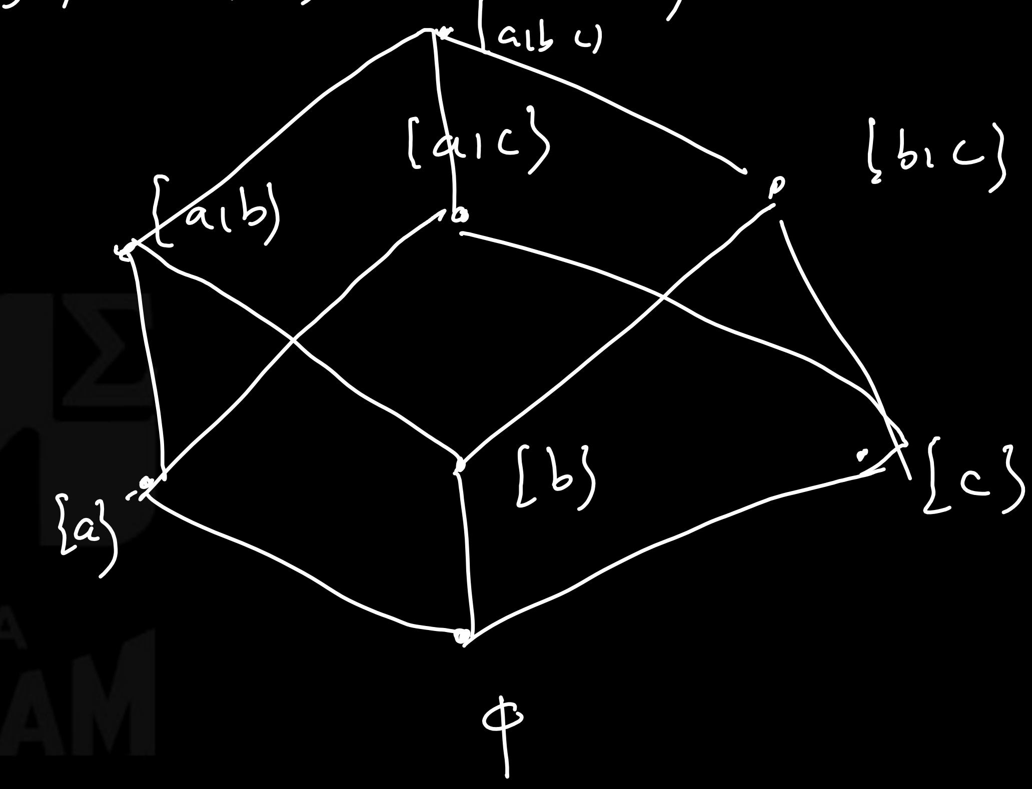


3. Draw the Hasse diagram for the divisibility on the set $A = \{2,3,5,10,11,15,25\}$



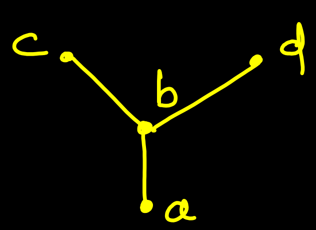
4. Draw the Hasse diagram for inclusion on the set $P(s)$ where $S = \{a, b, c\}$

$P(s) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\} \}$

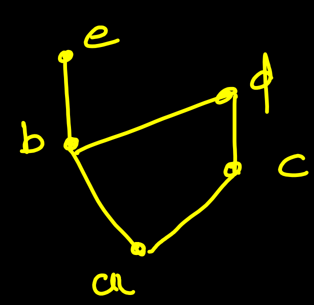


5. List all the ordered pairs in the partial ordering with the accompanying Hasse diagram.

a)



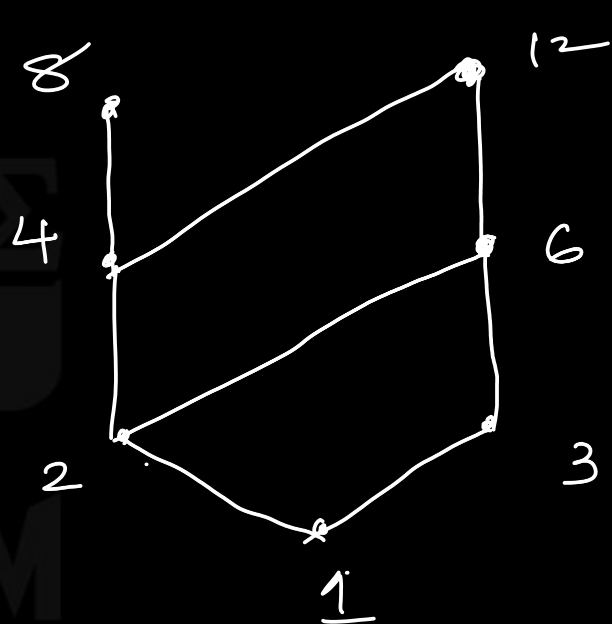
b)



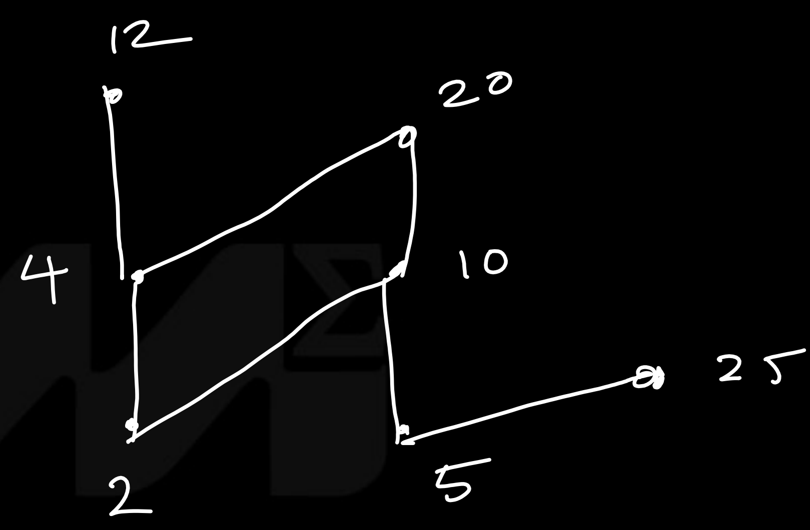
a) $R = \{ (a,a), (b,b), (c,c), (d,d), (a,b), (b,d), (b,c), (a,c)\ (a,d) \}$

b) $R = \{ (a,a), (b,b), (c,c), (d,d), (e,e), (a,c), (a,b), (c,d), (b,e), (b,d), (a,e)\ (a,d) \}$

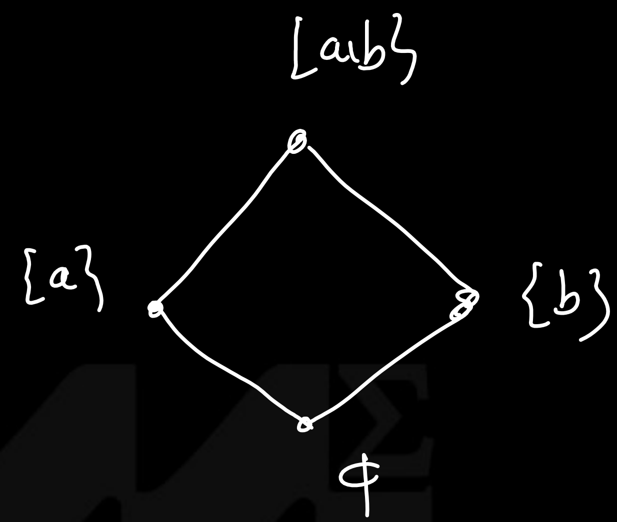
6. Constructing the Hasse diagram of $\{1, 2, 3, 4, 6, 8, 12\}$, |



7. $(\{2, 4, 5, 10, 12, 20, 25\}, |)$

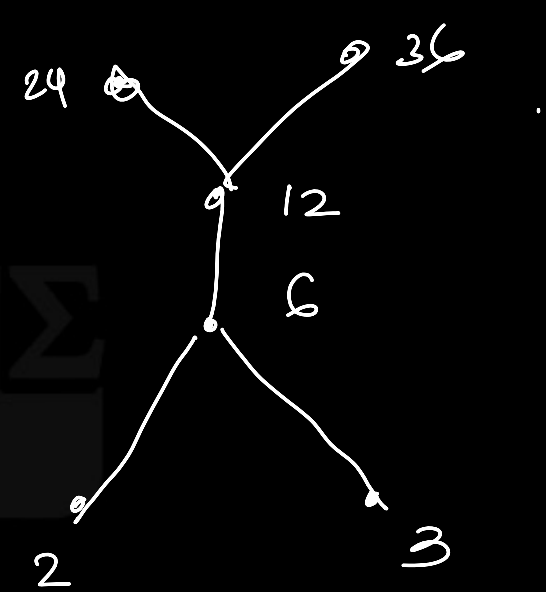


8. Let $A = \{a, b\}$. Draw the Hasse diagram of (P(A), S)



9. $A = \{2, 3, 6, 12, 24, 36\}$ Draw a Hasse diagram of (A, S)

|



10. $A = \{1, 2, 3, 6, 9, 18\}, |$

lub(2,3) =

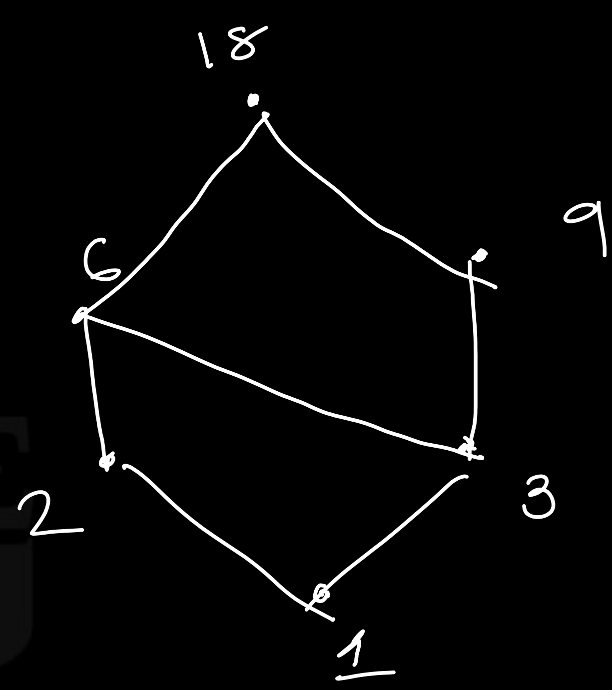
glb(2,3) =

lub(6,9) =

glb(6,9) =

lub(2,9) =

glb(2,9) =

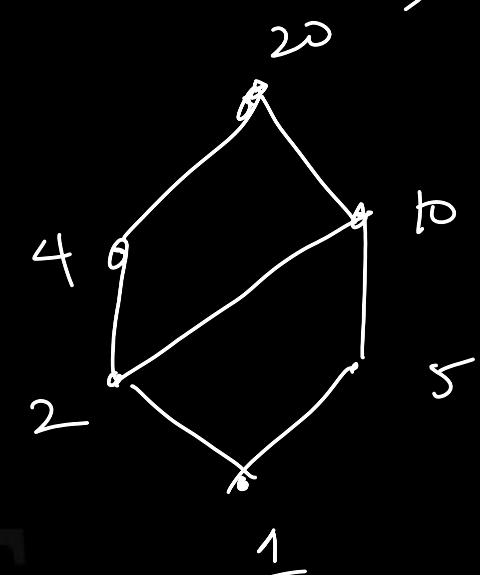


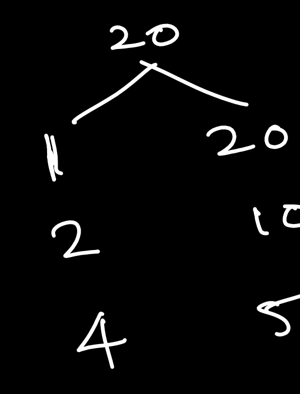
11. Draw the Hasse diagram of $(D_{20}, |)$ (| is the divides relation)

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

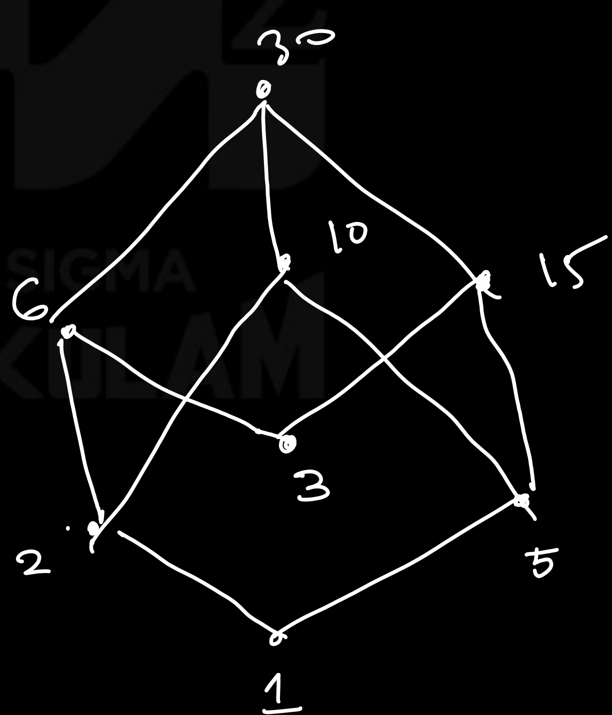
lub(4,10) =   lub(2,5) =   lub(4,5) =

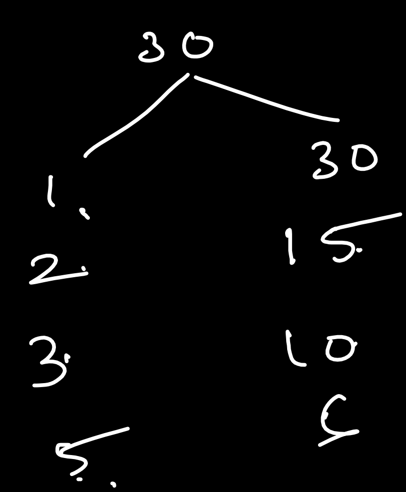
glb(4,10) =   glb(2,5) =   glb(4,5) =





12. $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}, |$





13. $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$