Predicate and Quantifiers

Predicate logic deals with predicates, which are propositions containing variables. The predicate is an expression of one or more variables defined on some domain. A predicate with variables can be made a proposition by either assigning a value to the variables or quantifying the variables.

Quantifiers Quantifiers are words that refers to quantities such as "some" or "all".
It tells for how many elements a given predicate is True.

In english, Quantifiers are used to express the quantities without giving an exact number.
Ex: all, some, many, none, few etc..

sentence like: "can I have some water?"
"Jack has many friends here"

The variables of the predicate are quantified by quantifiers. There are two types of quantifiers in predicate logic.
1) Universal Quantifier 2) Existential Quantifier

Example: Let $p(x)$ be a statement $x+1 > x$

$p(1)$: $1+1 > 1$
$2 > 1$ True.

$p(2)$: $2+1 > 2$
$3 > 2$ True.

$p(3)$: $3+1 > 3$
$4 > 3$ true.
⋮
for all +ve integer.

$p(x)$: $x+1 > x$ is true for all +ve integer x

$\forall x\, p(x)$

Example 2 : let $Q(x)$ be the statement $x<2$

$Q(x): x<2$ $\qquad$ $\forall x\, p(x)$

$Q(1): 1<2$ True.

$Q(3): 3<2$ False

$Q(4) = 4<2$ False.

there is some $x=1$ such that $x<2$.

There exist $x=1$ such that $x<2$ where x is

$\exists x\, P(x)$ $\qquad$ the set of +ve Integers

Domain or Domain of Discourse

A domain specifies the possible values of the variable under consideration

For example : Let $P(x) : x+1 > x$ and let us assume that

Domain $\rightarrow$ set of all +ve Integers

Note : It is very Important to specify the domain of discourse.

Universal Quantifier

The universal quantifier, typically denoted as $\forall$, (for all or for every) is used in logic to express that a particular property or statement is true for every element within a specified domain, known as the domain of discourse or simply the domain. When we say $\forall x p(x)$, it means that the property $p(x)$, holds for all possible values of x within the domain of discourse. The precise meaning of this universal quantification depends on clearly defining the domain, as it specifies the set of values that x can take. Without specifying the domain, the statement remains undefined, as it unclear which elements are being considered.

For example, consider the statement "Man is Mortal." It can be transformed into predicate form $\forall x p(x)$ where $p(x)$ is the predicate which denote x is mortal and the universe of discourse is the set of all men.

Existential Quantifier

The existential quantifier, commonly represented as $\exists x P(x)$, is used in logic to assert that there 1 is at least one element within a specified domain for which a particular property or statement is true. When we write $\exists x P(x)$, it means that there exists at least one value of x in the domain such that the property $P(x)$ holds. The existential quantifier indicates the existence of at least one instance where the statement is true, but it does not specify how many such instances exist.

Like the universal quantifier, the existential quantifier's meaning is contingent on the domain being clearly defined to determine the possible values for x. Here $\exists$ is called the existential quantifier.

Different ways to say $\exists x\, P(x)$:

$\exists$

Negation of Quantified statements

Let $p(x)$: x has passed the examination in the class. Then the statement $\forall x p(x)$ means that every student in the class has passed the examination. The negation of this statement is 'It is not the case that every student in the class has passed the examination.' That is there is one student in the class that has not passed the examination.

i.e,

$$\neg(\forall x p(x) \equiv \exists x \neg p(x)$$

$\neg\, \forall x\, P(x)$

$\exists x\, \neg P(x)$

Rules for negating statements

$$\neg(\forall x p(x) \equiv \exists x \neg p(x)$$
$$\neg(\exists x p(x) \equiv \forall x \neg p(x)$$
$$\neg(\forall x \neg p(x) \equiv \exists x p(x)$$
$$\neg(\exists x \neg p(x) \equiv \forall x p(x)$$

Example 1:
Let $Q(x, y)$ denote the statement $x = y + 3$. What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

$Q(x,y)$: $x = y+3$

$Q(1,2)$: $1 = 2+3$

$1 = 5$ False

$Q(3,0) =$ $3 = 0+3$

$3 = 3$ True.

a) $\exists x\,(C(x) \wedge D(x) \wedge F(x))$

b) $\forall x\,(C(x) \vee D(x) \vee F(x))$

c) $\exists x\,(C(x) \wedge F(x) \wedge \neg D(x))$

d) $\neg\exists x\,(C(x) \wedge F(x) \wedge D(x))$

Example 2:
Let $C(x)$: x has a cat. $D(x)$: x has a dog. $F(x)$: x has a ferret. Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

$\exists x$ $\wedge$

(a) A student in your class has a cat, a dog, and a ferret.
(b) All students in your class have a cat, a dog, or a ferret. $\vee$
(c) Some student in your class has a cat and a ferret, but not a dog. $\neg$
(d) No student in your class has a cat, a dog, and a ferret.
(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

e) $(\exists x\, C(x)) \wedge (\exists x\, D(x)) \wedge (\exists x\, F(x))$

$\mathbb{Z} = \{ \ldots\ -2, -1, 0, 1, 2, 3 \ldots \}$

Example 3:
Let $p(x)$: $2x + 1 = 5$, $q(x)$: $x^2 = 9$ Prove that $\exists x[p(x) \wedge q(x)]$ is not logically equivalent to $[\exists x p(x) \wedge \exists x q(x)]$ where the universe is the set of all integers. *

Solution:
If we replace x by an integer, a then $p(a)$: $2a + 1 = 5$, and $q(a)$: $a^2 = 9$ then $a = 2$ and $a \pm 3$ which shows that $p(a) \wedge q(a)$ is a false statement.
$\therefore$ $\exists x[p(x) \wedge q(x)]$ is false because there is no integer a such that $2a + 1 = 5$ and $a^2 = 9$. But there is an integer $b = 2$ such that $2b + 1 = 5$ and there is an integer $c = 3$ or -3 such that $x^2 = 9$.
$\therefore$ $[\exists x p(x) \wedge \exists x q(x)]$ is true.
i.e., $\exists x[p(x) \wedge q(x)] \not\equiv [\exists x p(x) \wedge \exists x q(x)]$.

$p(x)$: $2x+1 = 5$ $\quad$ $q(x)$: $x^2 = 9$

$p(a)$: $2a+1 = 5$ $\quad$ $q(a)$: $a^2 = 9$

$2a = 4$ $\quad$ $a^2 = 9$

$a = 2$ ✓ $\quad$ $a = \pm 3$ ✓

RHS.

$\exists x\,[p(x) \wedge q(x)] \not\equiv \exists x p(x) \wedge \exists x q(x)$

$p(a) \wedge q(a)$ False. $\quad$ False.

Example 4:
Let $Q(x)$ be the statement $x < 2$. What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

$Q(x): x<2$

$Q(1): 1<2$ True

$Q(3): 3<2$ False.

$\forall x\, Q(x)$ Truth Value is false

Example 5:
Suppose the $P(x)$ is $x^2 > 0$ Show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers.

$p(x): x^2>0$

$p(-1): (-1)^2>0$
$1>0$ True

$P(0): 0^2>0$
$0>0$ False.

$p(x): x^2>0$ will not hold for $x=0$

$p(x,y): x+y=17$

$\underset{\text{all}}{\forall x}\ \underset{a}{\exists y}\, p(x,y) \not\equiv \exists y\, \forall x\, P(x,y)$

Example 6:
Consider $p(x, y) : x+y = 17$. Show that $\forall x \exists y p(x, y)$ is not equivalent to $\exists y \forall x p(x, y)$ where the universe is the set of all integers.

Solution:
If we replace x by any integer a, then $y = 17 - a$ does exist and $a + 17 - a = 17$. Therefore $\forall x \exists y p(x, y)$ is a true statement.
Consider $\exists y \forall x p(x, y)$ the statement is false because if we select a value of y as $y = a$ then the only value of x is $17 - a$ which satisfy the statement $x + y = 17$.
Therefore $\forall x \exists y p(x, y)$ is not equivalent to $\exists y \forall x p(x, y)$.

$x=a$
$y=17-a$

$\exists y\, \forall x\, P(x,y)$
y fixed
$y=a$
$x=17-a$

Example 7:
What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

$\forall x\,(x^2 \geq x)$ Truth value : false.

If $x = \frac{1}{2} \in \mathbb{R}$

$\left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$

$\frac{1}{2} = 0.5$

$\left(\frac{1}{2}\right)^2 \not\geq \frac{1}{2}$.

$0 \geq 0$

$1^2 \geq 1$

$2^2 \geq 2$

$3^2 \geq 2$

$-1^2 \geq 1$

$-2^2 \geq -2$

Example 8:
Let $Q(x)$ denote the statement $x = x + 1$. What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

$Q(x): \ x = x+1$

$x = x+1$

$x - x = 1$

$0 \neq 1$ false.

Example 9:
Negate the statement 'If x is odd then $x^2 - 1$ is even' where the universe is the set of all integers.

$p \to q \equiv \neg p \vee q$

$p(x)$: $q(x)$: $\forall x[p(x) \longrightarrow q(x)]$

Solution:
Let $p(x)$: x is odd, $q(x)$: $x^2 - 1$ is even. Then the given statement is $\forall x[p(x) \to q(x)]$ is true.
~~The negation of this statement is determined as follows,~~

$$\begin{aligned}
\neg[\forall x(p(x) \to q(x)] &\equiv \exists x[\neg(p(x) \to q(x)]\\
&\equiv \exists x[\neg(\neg p(x) \vee q(x)] \text{ since } p \to q \equiv \neg p \vee q\\
&\equiv \exists x[(\neg\neg p(x) \wedge \neg q(x)] \quad \text{Demorgan's Law}\\
&\equiv \exists x[(p(x) \wedge \neg q(x)] \quad \text{Law of double negation}
\end{aligned}$$

Thus, the negation of the given statement is 'there exist an integer x such that x is odd and $x^2 - 1$ is odd.

$$\neg\left[\forall x(p(x) \to q(x))\right] \equiv \exists x\left[\neg\left(p(x) \to q(x)\right)\right]$$

$$\equiv \exists x\left[\neg\left[\neg p(x) \vee q(x)\right]\right]$$

$$\equiv \exists x\left[p(x) \wedge \neg q(x)\right]$$

$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$

odd   even

$\forall x\, P(x)$?

Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

(where the domain is same for all condition)

Solution:

We can prove it through a counterexample,

let $P(x)$: x is odd, $Q(x)$: x is even. Let the domain of discourse be the positive integers.

Let $D = \{3, 4\}$. Define the predicates $P(3)$ is true, $Q(3)$ is false.

$P(3) \quad Q(3)$
$T \;\vee\; F$

$P(4) \quad Q(4)$
$F \quad T$

$P(3) \vee Q(3)$ is true. $P(4)$ is false, $Q(4)$ is true, $P(4) \vee Q(4)$ is true.

$P(4) \vee Q(4)$
T

Consequently, $\forall x(P(x) \vee Q(x))$ is true, since every positive integer is either odd or even.

But $P(4)$ is false, $Q(4)$ is true $\forall x P(x) \vee \forall x Q(x)$ is false, since it is neither the case that all positive integers are odd nor the case that all of them are even.

$\forall x\, P(x) \vee \forall x\, Q(x) \not\equiv \forall x(P(x) \vee Q(x))$

Negate the statement $\exists x[p(x) \wedge q(x)]$ where $p(x) : 2x + 1 = 5,\ q(x) : x^2 = 9$ and the universe is the set of ~~all~~ integers.

$p(x): 2x+1 \neq 5 \qquad q(x) = x^2 \neq 9$

Solution:

The negation is

$$\neg\exists x[(p(x) \wedge q(x)] \equiv \forall x[\neg(p(x) \wedge q(x))]$$
$$\equiv \forall x[\neg p(x) \vee \neg q(x)] \quad \text{Demorgan's Law}$$

That for every integer x, $2x + 1 \neq 5$ or $x^2 \neq 9$ is the negation of the given statement.

$p(x): 2x+1=5 \qquad q(x): x^2=9$

$2x=4 \qquad x=\pm 3$

$x=2$

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives. The domain of discourse be all things.

a) Something is not in the correct place.
b) Everything is in the correct place and in excellent condition.
c) Nothing is in the correct place and is in excellent condition.
d) One of your tools is not in the correct place, but it is in excellent condition.

Solution:

Let $R(x)$: x is in the correct place, let $E(x)$: x is in excellent condition, let $T(x)$: x is a tool.

a) $\exists x \neg R(x)$ b) $\forall x(R(x) \wedge E(x))$

c) $\forall x \neg(R(x)E(x))$ d) $\exists x(T(x) \wedge \neg R(x) \wedge E(x))$.

$\forall x\, \neg(R(x) \wedge E(x))$

What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

$\exists x(x^2 \le x)$ $\quad$ $\forall x(x^2 \ne 2)$

Example 16:

Negate and simplify the statement

(a) $\forall x[p(x) \to q(x)]$ (b) $\exists x[(p(x) \vee q(x))]$

(c) $\forall x[p(x) \wedge \neg q(x)]$ (d) $\exists x[p(x) \vee q(x) \to r(x)]$

Solution:

a) $\neg\forall x[p(x) \to q(x)] \equiv \exists x \neg[p(x) \to q(x)]$
$\equiv \exists x\neg[\neg p(x) \vee q(x)] \equiv \exists x[p(x) \wedge \neg q(x)]$

b) $\neg\exists x[(p(x) \vee q(x)] \equiv \forall x\neg[(p(x) \vee q(x)]$
$\equiv \forall x[\neg p(x) \wedge \neg q(x)]$

c) $\neg\forall x[p(x) \wedge \neg q(x)] \equiv \exists x\neg[p(x) \wedge \neg q(x)]$
$\equiv \exists x[\neg p(x) \vee q(x)]$

d) $\neg\exists x[p(x) \vee q(x) \to r(x)]$
$\equiv \forall x\neg[p(x) \vee q(x) \to r(x)]$
$\equiv \forall x\neg[\neg(p(x) \vee q(x)) \vee r(x)]$
$\equiv \forall x\neg[(\neg p(x) \wedge \neg q(x)) \vee r(x)]$
$\equiv \forall x[(p(x) \vee q(x)) \wedge \neg r(x)]$

$\neg\exists x[p(x) \vee q(x) \longrightarrow r(x)]$

$\forall x \neg[p(x) \vee q(x) \to r(x)]$

$\forall x \neg[\neg(p(x) \vee q(x)) \vee r(x)]$

$\forall x \neg[\neg(p(x) \wedge \neg q(x)) \vee r(x)]$

$\forall x[p(x) \vee q(x)) \wedge \neg r(x)]$

Nested Quantifiers

Nested Quantifiers occur when a quantifier is used within the scope of another quantifier. For example, $\forall x\exists yP(x, y)$ is a nested quantifier. The universal quantifier $\forall$ is the outer quantifier and the existential quantifier $\exists$ is the inner quantifier.

For example : $\forall x \exists y\, Q(x,y)$

$\exists$ is within the scope of $\forall$

Note : Anything within the scope of the quantifier can be thought as a propositional function.

$\forall x \boxed{\exists y\, Q(x,y)} = \forall x\, P(x)$

$\downarrow$

$P(x)$

Different Combinations of Nested Quantifiers

order of quantifiers doesn't matter

$\forall x \forall y\ Q(x,y)$

$\forall x \exists y\ Q(x,y)$

$\exists y \forall x\ Q(x,y)$

$\exists x \exists y\ Q(x,y)$

order of quantifiers does matter

ie, $\forall x \forall y\ Q(x,y) \equiv \forall y \forall x\ Q(x,y)$

$\exists x \exists y\ Q(x,y) \equiv \exists y \exists x\ Q(x,y)$

Also, $\forall x \exists y\ Q(x,y) \not\equiv \exists y \forall x\ Q(x,y)$

proof

Example 1:
Let $Q(x,\ y)$ denote $x + y = 1$. What are the truth values of the quantifications $\exists y \forall x Q(x,\ y)$ and $\forall x \exists y Q(x,\ y)$, where the domain for all variables consists of all real numbers?

$\exists y \forall x\ Q(x,y)$

Solution:
The quantification $\exists y \forall x Q(X,\ y)$ denotes the proposition "There is a real number y such that for every real number x, $x + y = 1$ is a false statement because there is no real number y such that $x + y = 1$ for all real numbers x.
i.e., there exit a real number 5 (say), we cannot find a real number, 6, 8, 9 (say) such that $x + y = 1$.
$\forall x \exists y Q(x,\ y)$ denotes the proposition "For every real number x there is a real number y such that $x + y = 1$ is a true statement.

$\boxed{y = a_1}$ fixed

$Q(x,y): x + y = 1$

$x = 1 - a_1$

Example 2:
Let $Q(x, y, z)$ be the statement $x + y = z$. What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$, where the domain of all variables consists of all real numbers?

Solution:
The quantification $\forall x \forall y \exists z Q(X, y, z)$ is the statement for all real numbers x and for all real numbers y there is a real number z such that $x + y = z$, is a true statement.
$\exists z \forall x \forall y Q(x, y, z)$, which is the statement There is a real number z such that for all real numbers x and for all real numbers y it is true that $x + y = z$ is false.

$x + y = z$

$x + y = 1$

$x + y = z = 1$

$\frac{1}{2} + \frac{1}{2} = 1$

$\forall x (x \neq 0 \to \exists y)$

$x \times \frac{1}{x} = 1$

$x = y^2$

Example 3:
Determine the truth value of each of these statements if the domain of each variable consists of all real numbers
a) $\forall x \exists y (x = y^2)$ b) $\exists x \forall y (xy = 0)$
c) $\forall x \exists y (x + y = 1)$ d) $\forall x (x \neq 0 \to \exists y (xy = 1))$

Solution:
a) False, since no such y exists if x is negative.
b) True, since we can set $x = 0$.
c) True, since we can let $y = 1 - x$.
d) True, since we can take $y = \frac{1}{x}$.

$xy = 0$

$0 \times 1 = 0$

0×2

0×3

$x + y = 1$

$1 + 0 = 1$

$2 + -1 = 1$

$n+m=0$
$1+-1=0$
$2+-2=0$
$3+-3=$
$4+-4 \ldots$

Example 4:
Determine the truth value of each of these statements if the domain for all variables consists of all integers.
a) $\exists n \forall m (n < m^2)$ b) $\exists n \exists m (n^2 + m^2 = 5)$ c) $\forall n \exists m (n + m = 0)$

Solution:
a) The statement is there exist an n that is smaller than the square of every integer. This statement is true, since if $n = -3$, then n is less than every square, since squares are always greater than or equal to 0.
b) The statement is that the equation $n^2 + m^2 = 5$ has a solution over the integers, statement is true, since $1^2 + 2^2 = 5$
c) For all values of n we can find an m such that $n + m = 0$ is a true statement.

$n < m^2$
$0 < 1^2 = 1$
$3 < 2^2 = 4$
$-1 < 0 = 0$

$\exists n \forall m (n < m^2)$

Example 5:
Translate the statement into a logical expression
a) The sum of two positive integers is always positive.
b) Every real number except zero has a multiplicative inverse.

Solution:
a) $\forall x \forall y (x + y > 0)$, where the domain for both variables consists of all positive integers.
b) $\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$ where the domain is the set of all real numbers.

Example 6:
Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:
$p(x)$: x is female, $q(x)$: x is a parent," and
$r(x, y)$: x is the mother of y.
Therefore, the given statement is $\forall x ((p(x) \wedge q(x)) \rightarrow \exists y r(x, y))$.

Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:
Let $P(w,\ f)$: w has taken f, $Q(f,\ a)$: f is a flight an a. We can express the statement as $\exists w \forall a \exists f (P(w,\ f) \wedge Q(f,\ a))$, where the domains of discourse for w, f, and a consist of all the women in the world, all airplane f lights, and all airlines, respectively.