Module 2
Discrete Mathematics

Part - 1

1. Show that the statement $p \wedge (\neg p \wedge q)$ is a contradiction where p and q are primitive statements.

2. Construct a truth table for $(p \vee q) \rightarrow (p \oplus q)$

3. Construct a truth table for the compound statement. $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

4. Construct a truth table for the compound statement $q \leftrightarrow (\neg p \vee \neg q)$ where p and q are primitive statements. Check whether it is a tautology.

Solution: It is not a tautology.

5. Construct a truth table for the compound statement.
$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

Part - 2

1. Construct a truth table for each of the following compound statements, where p, q, r denote the primitive statements. Check whether it is a tautology.
   (a) $(p \wedge q) \rightarrow p$
   (b) $\neg(p \rightarrow q) \rightarrow \neg p$
   (c) $(\neg p \vee q) \rightarrow \neg q$
   (d) $((p \rightarrow q) \wedge [(q \wedge \neg r)) \rightarrow (p \vee r)]$
   (e) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
   (f) $q \wedge (\neg r \rightarrow p)$
   (g) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
   (h) $(p \oplus q) \wedge (p \oplus \neg q)$

2. Determine whether each of these statements are true or false.
   1) $2+2=4$ if and only if $1+1=2$
   2) If $1+1=3$, then $2+2=5$
   3) If $1+1=2$, then dogs can fly

3. Determine the truth value of the implication,
   (a) If $3+4=12$ then $3+2=6$
   (b) If $3+3=6$ then $3+4=9$

4. Write the converse, inverse and contrapositive of the following statement.
   (a) If P is a square, then P is a rectangle.
   (b) If today is Friday, then tomorrow is Saturday.

5. The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space. Are these system specifications consistent?

6. Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded. Are these system specifications consistent?

7. What Boolean search would you use to look for Web pages about beaches in New Jersy? What if you wanted to find Web pages about beaches on the isle of Jersey?

8. An island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. Two people A and B. What are A and B if

   (a) A says 'I am a knave or B is a knight' and B says nothing.

   (b) Both A and B say 'I am a knight'.

9. Check whether the propositions $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are logically equivalent.

Answers

1. (a) Tautology (b) Not a tautology (c) Not a tautology (d) Not a tautology (e) Not a tautology (f) Not a tautology (g) Tautology (h) Contradiction

2. (a) True (b) True (c) False

3. (a) True (b) False

4. (a) If P is a rectangle, then P is a square - converse
   If P is not a square, then P is not a rectangle- inverse.
   If P is not a rectangle, then P is not a square-contrapositive.
   (b) If tomorrow is Saturday, then Today is Friday-converse
   If today is not Friday, then tomorrow is not Saturday - inverse.
   If tomorrow is not Saturday, then today is not Friday-contrapositive

5. Consistent

6. Consistent

7. a) New and Jersey and Beaches b) Isle and Jersey and Beaches

8. a) A is Knight b) No conclusion

Part - 3

1. Check whether the propositions are logically equivalent or not,

   (a) $\neg p \to (q \to r)$ and $q \to (p \vee r)$

   (b) $p \to (q \wedge r)$ and $(p \to q) \wedge (p \to r)$

   (c) $[p \to (q \vee r)] \Leftrightarrow [\neg r \to (p \to q)]$

   (d) $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$

   (e) $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$

2. For primitive statements p and q verify that $p \to [q \to (p \wedge q)]$ is a tautology.

3. Assuming the truth value of p and r are false and that of q and s are true, find the truth value of the proposition
   $((p \wedge \neg q) \to (q \wedge r)) \to (s \vee \neg q)$

4. Write the negation of the following statements

   (a) Carlos will bicycle or run tomorrow

   (b) He swims if and only if the water is warm

   (c) Mei walks or takes the bus to class

   (d) Paris is in France and London is in England

   (e) If it is raining then the game is cancelled

Answers

1. a) not logically equivalent b) logically equivalent c) logically equivalent d) logically equivalent e) not logically equivalent.

3. True

4. (a) Carlos will neither bicycle nor run tomorrow
   (b) He swims if and only if the water is not warm.
   (c) Mei neither walks nor takes the bus to class.
   (d) Paris is not in France or London is not in England
   (e) It is raining and the game is not cancelled.

Part - 4

1. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language $C++$." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers and logical connectives. The domain for quantifiers consists of all students at your school.
   (a) There is a student at your school who can speak Russian and who knows $C++$.
   (b) There is a student at your school who can speak Russian but who doesn't know $C++$.
   (c) Every student at your school either can speak Russian or knows $C++$.
   (d) No student at your school can speak Russian or knows $C++$.

2. Determine the truth value of each of these statements if the domain consists of all real numbers.
   a) $\exists x(x^3 = -1)$ b) $\exists x(x^4 < x^2)$ c) $\forall x((-x^2) = x^2)$ d) $\forall x(2x > x)$

3. Let $p(x): x^2 = 2x$ where universe is the set of all integers. Determine whether the following statement is true or false.
   a) $p(0)$ b) $p(2)$ c) $\exists x p(x)$ d) $\forall x p(x)$

4. Express the negations of these propositions using quantifiers and in English
   (a) Every student in this class likes mathematics.
   (b) There is a student in this class who has never seen a computer.
   (c) There is a student in this class who has taken every mathematics course offered at this school.

5. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
   a) $\forall x \exists y(x = 1/y)$ b) $\forall x \exists y(y^2 - x < 100)$ c) $\forall x \forall y(x^2 \neq y^3)$

Answers

1. a) $\exists x(P(x) \wedge Q(x))$ b) $\exists x(P(x) \wedge \neg Q(x))$ c) $\forall x(P(x) \vee Q(x))$ d) $\forall x(\neg P(x) \wedge (\neg Q(x)))$

2. a) True b) True c) True d) False

3. a) True b) True c) True d) False

4. a) Some student in this class does not like mathematics $\exists x \neg L(x)$
   b) Every student in this class has seen a computer $\forall x S(x)$
   c) For every student in this class, there is a mathematics course that this student has not taken.
   $\forall x \exists c \neg T(x, c)$

5. (a) If $x = 2$, then there is no y among the integers such that $2 = \frac{1}{y}$, since the only solution of this equation is $y = \frac{1}{2}$.

   (b) We can rewrite $y^2 - x < 100$ as $y^2 < 100 + x$. Since squares can never be negative, no such y exists if x is, say, -200.

   (c) No true since if we take $x = 27$ and $y = 9$, since $3^6 = 9^3$

Part - 5

1. If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica will be notified. Veronica has not heard from either of these clients, that is Veronica is not notified. Consequently, Dominic did not make it to the racetrack and Ralph didn't play cards all night. Establish the validity of the statement.

2. If Raichel gets the supervisor's position and works hard, then she will get a raise. If she gets the raise, then she will buy a new car. She has not purchased a new car. Therefore, either Raichel did not get the supervisor's position or she did not work hard. Show that the conclusion follows from the premises.

3. Rita is baking a cake. If Rita is baking a cake, then she is not practicing her flute. If Rita is not practicing her flute, then her father will not buy her a car. Therefore, Rita's father will not buy her a car. Check the validity of the statement.

4. Jonathan does not have his driver's license or his new car is out of gas. Jonathan has his driver's license or he does not like to drive his new car. Jonathan's new car is not out of gas or he does not like to drive his new car. Therefore, Jonathan does not like to drive his new car.

5. If Rony is a lawyer, then he is ambitious. If Rony is an early riser, then he does not like idlies. If Rony is ambitious, then he is an early riser. Then, if Rony is a lawyer, he did not like idlies. Show that the argument is valid.

6. What rules of inference are used in this famous argument? All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

7. Show that the premises Everyone in this discrete mathematics class has taken a course in computer science and Maria is a student in this class imply the conclusion Maria has taken a course in computer science.

8. A student in this class has not read the book, and Everyone in this class passed the first exam. From these, deduce the conclusion that someone who passed the first exam has not read the book.

Answers

1. The given proposition

   p: Dominic goes to the racetrack

   q: Helen is mad

   r: Ralph plays cards

   s: Carmela is mad

   t: Veronica will be notified.

Then the hypotheses are

$$
\begin{array}{c}
p \to q \\
r \to s \\
q \vee s \to t \\
\neg t \\
\text{............} \\
\therefore \quad \neg p \wedge \neg r
\end{array}
$$

	Steps	**Reason**
1	$q \vee s \to t$	Premises
2	$\neg t$	Premises
3	$\neg(q \vee s)$	Step (1), (2) Modus Tollens
4	$\neg q \wedge \neg s$	Step 3, De Morgan's Law
5	$\neg q$	Step (4) Simplification
6	$\neg s$	Step (4) Simplification
7	$p \to q$	Premises
8	$\neg p$	Step (5), (7) Modus Tollens
9	$r \to s$	Premises
10	$\neg r$	Step (6), (9) Modus Tollens
11	$\neg p \wedge \neg r$	Step (8), (10) Conjunction

The argument is valid.

2. The given propositions are

   p: Raichel gets the supervisor's position
   q: Raichel work hard
   r: Raichel gets the raise
   s: Raichel buy a new car

   The argument is

$$
\begin{array}{c}
(p \wedge q) \to r \\
r \to s \\
\neg s \\
\text{............} \\
\therefore \quad \neg p \vee \neg q
\end{array}
$$

	Steps	**Reason**
1	$(p \wedge q) \to r$	Premises
2	$r \to s$	Premises
3	$\neg s$	Premises
4	$\neg r$	Step (3), (4) Modus Tollens
5	$\neg(p \wedge q)$	Step (1), (4) Modus Tollens
6	$\neg p \vee \neg q$	Step (5) and (3) De 'Morgan's Law

The argument is valid.

3. The given propositions are

   p: Rita is baking a cake.

   q: Rita is practicing her flute.

   r: Rita's father will buy her a car.

   The argument is

$$\begin{array}{c} p \\ p \rightarrow \neg q \\ \neg q \rightarrow \neg r \\ \hline \therefore \quad \neg r \end{array}$$

	Steps	Reason
1	p	Premises
2	$p \rightarrow \neg q$	Premises
3	$\neg q \rightarrow \neg r$	Premises
4	$p \rightarrow \neg r$	Step (2), (3) Law of Syllogism
5	$\neg r$	Step (1), (4) Modus Ponens

4. The given propositions are

   p: Jonathan has his driver's license

   q: Jonathan's new car is out of gas

   r: Jonathan likes to drive

   The argument is

$$\begin{array}{c} \neg p \vee q \\ p \vee \neg r \\ \neg q \vee \neg r \\ \hline \therefore \quad \neg r \end{array}$$

	Steps	Reason
1	$\neg p \vee q$	Premises
2	$p \vee \neg r$	Premises
3	$q \vee \neg r$	Step (1) and (2) resolution
4	$\neg q \vee \neg r$	Premises
5	$\neg r$	Step (3), (4) resolution

5. The given propositions are

   p: Rony is a lawyer

   q: Rony is ambitious

r: Rony is an early riser

s: Rony likes idles

The argument is

$$\begin{array}{rl} & p \rightarrow q \\ & r \rightarrow \neg s \\ & q \rightarrow r \\ & \ldots\ldots\ldots\ldots\ldots \\ \therefore & p \rightarrow \neg s \end{array}$$

	Steps	**Reason**
1	$p \rightarrow q$	Premises
2	$q \rightarrow r$	Premises
3	$p \rightarrow r$	Hypothetical Syllogism
4	$r \rightarrow \neg s$	Premises
5	$p \rightarrow \neg s$	Hypothetical Syllogism

6. $P(x): x$ is a man
   $Q(x): x$ is mortal
   The argument is

$$\begin{array}{c} \forall x(P(x) \rightarrow Q(x)) \\ P(\text{Socretes}) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ Q(\text{Socretes}) \end{array}$$

	Steps	**Reason**
1	$\forall x(P(x) \rightarrow Q(x))$	Premises
2	$P(\text{Socretes}) \rightarrow Q(\text{Socretes})$	Universal instantiation from(1)
3	$P(\text{Socretes})$	Premises
4	$Q(\text{Socretes})$	Modus ponens from (2) and (3)

7. The given propositions are
   $P(x): x$ is in this discrete mathematics class,
   $Q(x): x$ has taken a course in computer science
   The argument is

$$\begin{array}{rl} & \forall x(D(x) \rightarrow C(x)) \\ & D(\text{Maria}) \\ & \ldots\ldots\ldots\ldots\ldots \\ \therefore & C(\text{Maria}) \end{array}$$

	Steps	**Reason**
1	$\forall x(D(x) \rightarrow C(x))$	Premises
2	$D(\text{Maria}) \rightarrow C(\text{Maria})$	Universal instantiation from(1)
3	$D(\text{Maria})$	Premises
4	$C(\text{Maria})$	Modus ponens from (2) and (3)

8. The given propositions are

$$C(x): x \text{ is in this class,}$$
$$B(x): x \text{ has read the book,}$$
$$P(x): x \text{ passed the first exam.}$$

The argument is

$$\exists x(C(x) \wedge \neg B(x))$$
$$\forall x(C(x) \rightarrow P(x))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\therefore \quad \exists x(P(x) \wedge \neg B(x))$$

	Steps	**Reason**
1	$\exists x(C(x) \wedge \neg B(x))$	Premises
2	$C(a) \wedge \neg B(a)$	Step (1) Existential instantiation
3	$C(a)$	Step (2) Simplification
4	$\forall x(C(x) \rightarrow P(x))$	Premise
5	$C(a) \rightarrow P(a)$	Step (4) Universal instantiation
6	$P(a)$	Step (3) and (5) Modus ponens
7	$\neg B(a)$	Step (2) Simplification
8	$P(a) \wedge \neg B(a)$	Step (6) and (7) Conjunction

Part - 6

1. Use a direct proof to show that the sum of two even integers is even.
2. Use a direct proof to show that every odd integer is the difference of two squares.
3. Use a direct proof to show that the product of two rational numbers is rational.
4. Prove that if m and n are integers and mn is even, then m is even, or n is even.
5. Show that at least three of any 25 days chosen must fall in the same month of the year.
6. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.
7. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.
8. Prove that any subset of size 6 from the set $S = \{1, 2, 3, \ldots 9\}$ must contain 2 elements whose sum is 10.
9. Show that any 14 integers are selected from the set $S = \{1, 2, 3 \ldots 25\}$ there are at least 2 whose sum is 26.
10. Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.