Cardinality of Sets

The cardinality of a set refers to the number of elements within that set. It is a fundamental concept in set theory used to measure the size of the set, whether the set is finite or infinite. For finite sets, the cardinality is simply the count of distinct elements in the set. Infinite sets can have different sizes or cardinalities. Sets that can be put into a one-to-one correspondence with the natural numbers is countably infinite sets.

For example, $\{a, b, c\}$ is finite and its cardinality is 3.
The set of natural numbers $\{1, 2, 3, \ldots\}$ is a countably infinite set.
The set of integers is also a countably infinite set.
The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. Cardinality, and we write $|A| = |B|$.

Countable Sets

A countable set is a set that either has the same cardinality as the set of positive integers (countably infinite) or is finite. When an infinite set is countable, we denote the cardinality by $\aleph$ (aleph-null). In other words, a set is countable if its elements can be put into a one-to-one correspondence with the positive integers or if it contains a finite number of elements. N

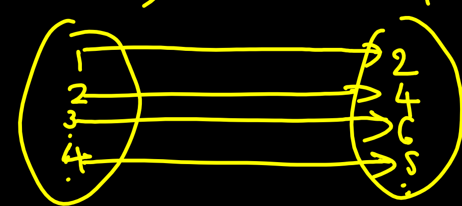
bijection

1-1, onto

A set that is not countable is called uncountable.

2, 4, 6, 8 - - - - $f(n) = 2n$

$f: N\ (Z^+) \longrightarrow 2, 4, 6, 8 - - -$



1. Show that the set of odd positive integers is a countable set.

$\{1, 3, 5, 7, - - - - -\}$

FIGURE 1 A One-to-One Correspondence Between Z^+ and the Set of Odd Positive Integers.

To prove the set of odd positive Integers is Countable.

For that, we have to define a function, $f: \cancel{Z}^+ \longrightarrow$ set of odd positive Integers

such that $f(n) = 2n - 1$.

we have to show that this function is one to one and onto.

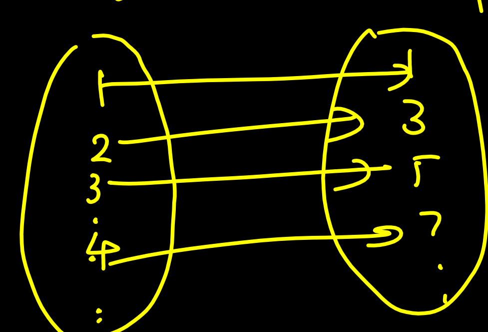
when $n = 1$, $f(1) = 1$

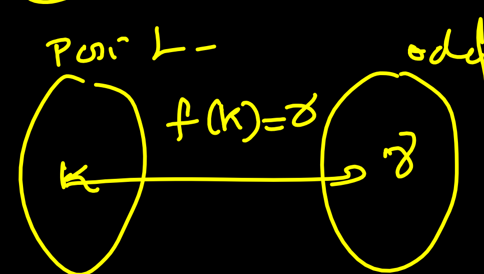
$n = 2$, $f(2) = 3$

$n = 3$, $f(3) = 5$

$n = 4$, $f(4) = 7$

so on . .





To prove f is one-to one

Suppose $f(n) = f(m)$

$2n - 1 = 2m - 1$

$2n = 2m$

$n = m$

$\therefore$ f is 1-1

To prove f is onto.

Corresponding to each odd positive Integer γ there exist k (positive Integer) such that $f(k) = 2k - 1 = \gamma$

$\therefore$ f is onto.

$\therefore$ There is a one to one correspondence between the set of positive Integers and odd positive Integers.

$\therefore$ The set of odd positive Integers is Countable.

2. Show that the set of all integers is countable.

$Z^+ = \{1, 2, 3, \ldots\} = \aleph$

$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

list : $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$

Solution: We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers: $0, 1, -1, 2, -2, \ldots$. ~~Alternatively~~, we could find a one-to-one correspondence between the set of positive integers and the set of all integers. We leave it to the reader to show that the function $f(n) = n/2$ when n is even and $f(n) = -(n-1)/2$ when n is odd is such a function. Consequently, the set of all integers is countable.

Define a function $f : Z^+ \longrightarrow Z$ such that

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

when $n=1$ $f(1) = 0$ $\quad$ $n=4$ $f(4) = 2$

$n=2$ $f(2) = 1$ $\quad$ $n=5$ $f(5) = -2$

$n=3$ $f(3) = -1$ $\quad \vdots$

To prove fn is one to one

suppose that $f(n) = f(m)$ if n & m are odd

$$\frac{n}{2} = \frac{m}{2}$$

$$n = m$$

If is n & m are even, $f(n) = f(m)$

$$\Rightarrow \frac{-(n-1)}{2} = \frac{-(m-1)}{2}$$

$$\Rightarrow n = m$$

Corresponding to each positive integer n, we can define a function $f(n) = \frac{n}{2}$ if n is even and if n is odd $f(n) = \frac{-(n-1)}{2}$.

$\therefore$ f is onto.

3. Show that the set of positive rational numbers is countable.

$q \neq 0$

Every positive rational number is of the form $\frac{p}{q}$ where p and q are positive integers. We can arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on.

First, list the positive rational numbers $\frac{p}{q}$ where $p + q = 2$, then list those where $p + q = 3$, followed by those where $p + q = 4$, and continue in this manner.

$\left[\frac{1}{1} = 1 + 1 = 2,\ \frac{1}{2} = 1 + 2 = 3,\ \frac{2}{1} = 2 + 1 = 3\right]$ and so on)

Whenever we encounter a number $\frac{p}{q}$ that is already listed, we do not list it again. [For example, when we come to $\frac{2}{2} = 1$ we do not list it because we have already listed $\frac{1}{1} = 1$.]

$p+q=2 \qquad p+q=3$

$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1},$

$\frac{5}{1} \ldots\ldots$

The initial terms in the list of positive rational numbers we have constructed are 1, $\frac{1}{2}$, 2, 3, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, 4, 5, and so on. The numbers that are circled are those we include, while the numbers that are not circled are those we omit because they have already been listed. Since each positive rational number appears exactly once, we have demonstrated that the set of positive rational numbers is countable.

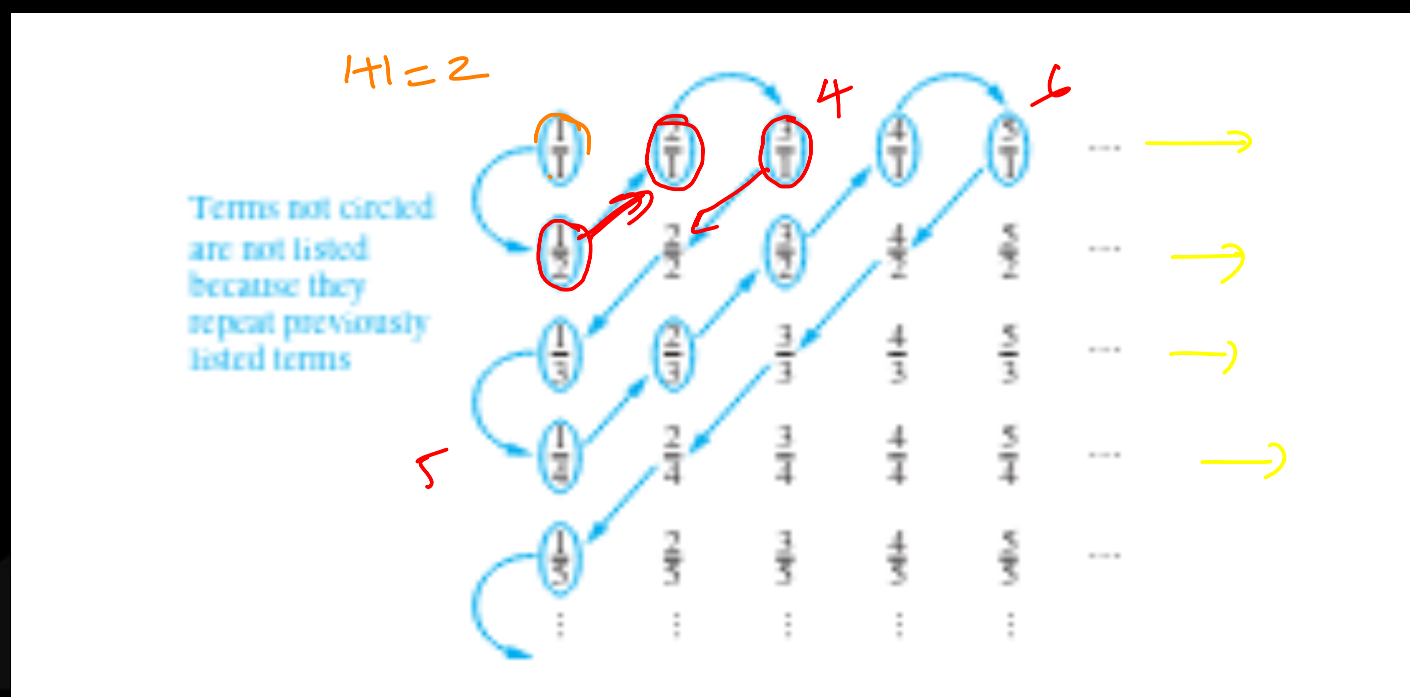


FIGURE 3 **The positive rational numbers are countable.**

EXAMPLE 4 How can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?

Solution: Because the rooms of the Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general, the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms. We illustrate this situation in Figure 2.

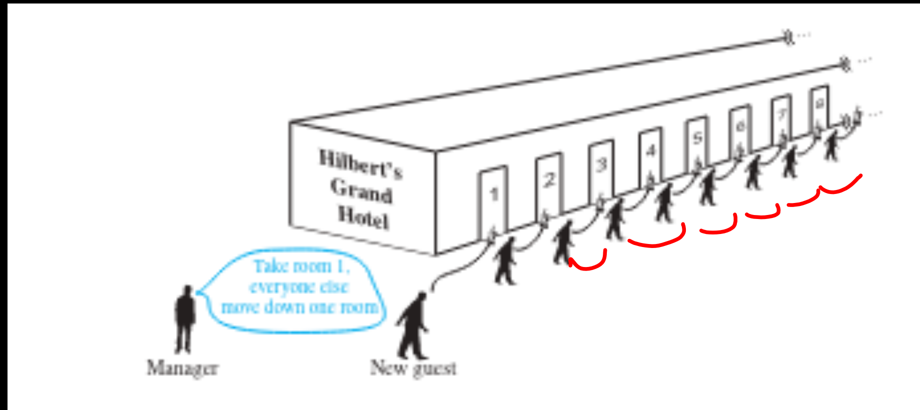


FIGURE 2 A new guest arrives at Hilbert's Grand Hotel.

5. If A and B are Countable sets, then $A \cup B$ is also Countable

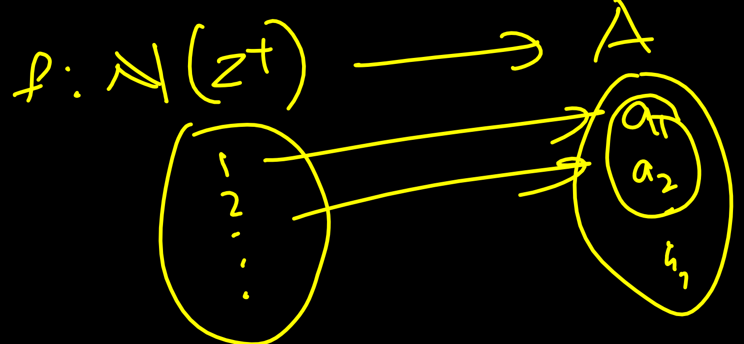
$A = \{a_1, a_2, \ldots a_n, \ldots\}$

$B = \{b_1, b_2, \ldots b_m, \ldots\}$

$A \cup B = \{a_1, b_1, a_2, b_2, \ldots a_n, b_m \ldots\}$

There are three cases to consider:
(i) A and B are both finite,
(ii) A is infinite and B is finite,
(iii) A and B are both countably infinite.
Case (i): when A and B are finite, $A \cup B$ is also finite, and therefore, countable.
Case (ii): Because A is countably infinite, its elements can be listed in an infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ and because B is finite, its terms can be listed as $b_1, b_2, \ldots, b_m$ for some positive integer m. We can list the elements of $A \cup B$ as $b_1, b_2, \ldots, b_m, a_1, a_2, a_3, \ldots, a_n, \ldots$ This means that $A \cup B$ is countably infinite.
Case (iii): Because both A and B are countably infinite, we can list their elements as $a_1, a_2, a_3, \ldots, a_n, \ldots, \ldots$ and $b_1, b_2, \ldots, b_m$, respectively. By alternating terms of these two sequences, we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$. This means $A \cup B$ must be countably infinite.

6. show that a subset of a Countable set is also Countable



Solution:
Let A be a countable set and let $B \subseteq A$. We need to prove that B is countable.
Case. 1: If A is finite, then B is also finite and therefore B is countable.
Case. 2: If A is infinite. Since A is countable and infinite, there exist a bijective function $f : N \to A$. i.e. we can list its elements as $\{a_1, a_2, a_3, \ldots a_{n\ldots}\}$ .Since $B \subseteq A$, each element of B corresponds to an infinite element in the list $\{a_1, a_2, a_3, \ldots a_{n\ldots}\}$. i.e. $B = \{b_1, b_2 \ldots \ldots\}$.
Define a function $g : N \to B$ such that $g(n) = b_n$ which is a bijective function. Which proves that B is countable.

Cantor diagonalization argument

The set of real numbers is an uncountable set.

$\begin{cases} 0.14 \\ 0.141414\ldots \end{cases}$ $\qquad 0.1314242186\ldots$

$\sqrt{2}, \sqrt{3}$

$\sqrt{2} = 1.41\ldots$

$r_1 = 0.\underline{2}3794102\ldots$ (d_{11})

$r_2 = 0.4\underline{4}590138\ldots$ (d_{22})

$r_3 = 0.09\underline{1}18764\ldots$ (d_{33})

$r_4 = 0.80553900\ldots$

$r = 0.454\ldots$

Solution: To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then, the subset of all real numbers that fall between 0 and 1 would also be countable (because any subset of a countable set is also countable; see Exercise 16). Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, $r_1, r_2, r_3, \ldots$. Let the decimal representation of these real numbers be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\cdots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\cdots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\cdots \\ r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\cdots \\ &\vdots \end{aligned}$$

$r_1 = 0.d_{11}\, d_{12}\, d_{13}\, d_{14}\ldots$

$r_2 = 0.d_{21}\, d_{22}\, d_{23}\, d_{24}\ldots$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (For example, if $r_1 = 0.23794102\ldots$, we have $d_{11} = 2$, $d_{12} = 3$, $d_{13} = 7$, and so on.) Then, form a new real number with decimal expansion $r = 0.d_1d_2d_3d_4\ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4. \end{cases}$$

$r = 0.d_1 d_2 d_3 d_4 \ldots$

(As an example, suppose that $r_1 = 0.23794102\ldots$, $r_2 = 0.44590138\ldots$, $r_3 = 0.09118764\ldots$, $r_4 = 0.80553900\ldots$, and so on. Then we have $r = 0.d_1d_2d_3d_4\ldots = 0.4544\ldots$, where $d_1 = 4$ because $d_{11} \neq 4$, $d_2 = 5$ because $d_{22} = 4$, $d_3 = 4$ because $d_{33} \neq 4$, $d_4 = 4$ because $d_{44} \neq 4$, and so on.)

Every real number has a unique decimal expansion (when the possibility that the expansion has a tail end that consists entirely of the digit 9 is excluded). Therefore, the real number r is not equal to any of $r_1, r_2, \ldots$ because the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Because there is a real number r between 0 and 1 that is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false. Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable. Any set with an uncountable subset is uncountable (see Exercise 15). Hence, the set of real numbers is uncountable. ◀