On Posterior Consistency of Bayesian Factor Model

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1. Introduction

Latent Factor models are a powerful technique for dimensionality reduction, extending Principal Component Analysis (PCA) by identifying latent factors that explain the covariance structure of the data. A Bayesian formulation of a latent factor model is as follows: given a set of N observations $Y = (y_1, \ldots, y_N)$, where $(y_i)_{i=1,\ldots,N}$ are G-dimensional vectors of variables, we assume the relationship

$$\overset{G\times 1}{y_i} = \overset{G\times K}{B}\overset{K\times 1}{\omega_i} + \overset{G\times 1}{\varepsilon_i}, \tag{1}$$

where latent factors ω_i and idiosyncratic terms ε_i are modeled by imposing specific densities. Similar to any Bayesian framework, the prior on the loadings B is crucial to the estimated model's properties. The Normal Bayesian factor model assumes the following:

$$\begin{aligned} \textbf{Likelihood:} &\quad y_i|\omega_i, B, \Sigma \overset{i.i.d.}{\sim} \mathcal{N}_G\left(B\omega_i, \Sigma\right), \\ \textbf{Latent Factors:} &\quad \omega_i \overset{i.i.d.}{\sim} \mathcal{N}_K\left(0_K, I_K\right) \end{aligned} \tag{2}$$

Similar to PCA, latent factor models are well-suited for handling large datasets with numerous observations. However, interpreting the coefficients of the loadings matrix presents two challenges. First, the loadings matrix is not identifiable because any right orthogonal transformation of the loadings matrix preserves the factorization. Specifically, for any orthogonal matrix O of dimension $K \times K$, the transformed expression $BO' \times O\omega_i = B\omega_i$ holds, and $O\omega_i$ retains the centered normal distribution with covariance $OO' = I_K$. Second, imposing sufficient restrictions on B to ensure identifiability becomes increasingly complex as the number of variables G increases, requiring many additional assumptions in both model design and inference.

To address these challenges, researchers have drawn inspiration from the LASSO technique, which suppresses the least explanatory variables by shrinking their coefficients to 0. They have applied similar methods to latent factor models, aiming to shrink to 0 the coefficients in B that capture non-significant effects of a factor on a variable. For instance, Ročková and George (2016) relies on empirical Bayes methods and a parameter-augmented version of the

Expectation-Maximization (EM) algorithm to estimate the posterior mode under a Spike-and-Slab LASSO prior imposed on the coefficients of B. Ma and Liu (2022) finds that, when factors are normally distributed the posterior distribution of the loadings exhibits "magnitude inflation" where all non-zero coefficients have absolute values larger than their true counterparts. Building on the insight that if the model is consistent under a flat prior for B, it should also be approximately consistent under a Spike-and-Slab prior, Ma and Liu (2022) propose replacing the normal factor assumption with an orthonormal factor assumption.

2. Magnitude Inflation Phenomenon

Ročková and George (2016)'s Spike-and-Slab LASSO prior is defined as:

$$\begin{aligned} B_{jk} \mid \gamma_{jk}, \lambda_0, \lambda_1 &\overset{\text{ind}}{\sim} (1-\gamma_{jk})\text{Laplace}(\lambda_0) + \gamma_{jk}\text{Laplace}(\lambda_1), \\ \gamma_{jk} &\overset{\text{ind}}{\sim} \text{Bernoulli}\left(\prod_{l=1}^{k} \nu_l\right), \\ \nu_l &\overset{\text{i.i.d.}}{\sim} \text{Beta}(\alpha, 1), \end{aligned} \tag{3}$$

where $\lambda_0 \gg \lambda_1$ are scale parameters, and γ_{jk} is a latent binary indicator stored in the $G \times K$ feature allocation matrix Γ. Because $\lambda_0 \gg \lambda_1$, the first Laplace distribution has a significantly smaller variance than the second, making it correspond to the spike prior centered around 0, while the second represents the diffuse slab prior. To estimate the mode of the posterior distribution of loadings under the Spike-and-Slab LASSO prior, one recursively maximizes an objective function resembling either a LASSO or Ridge objective, depending on λ_0 and λ_1, hence the name. Ma and Liu (2022) propose a Basic

model specification

gibbs sampler

theory + replication in the basic normal with spike and slab model

The basic gibbs sampler is very local

Inflation observable fig. 1 on both left color right curve of B_{00}, some puzzling facts on the left iter 1000 direction inconsistent, either because we explored the distribution, after converging we should draw more and more and do a mean or a mode to see the average response. or plot an empirical density function of one coeff to see

We observe some instability on fig. 2, the sixth columns which is usually zero may in certain case

Ma and Liu (2022) theoretically prove that, under the normal latent factor assumptions, as the variance of the Laplace slab priors tends to infinity (i.e., as the scale parameter approaches zero), the probability that a posterior sample B, given Y, Σ, Γ, has a matrix norm smaller than

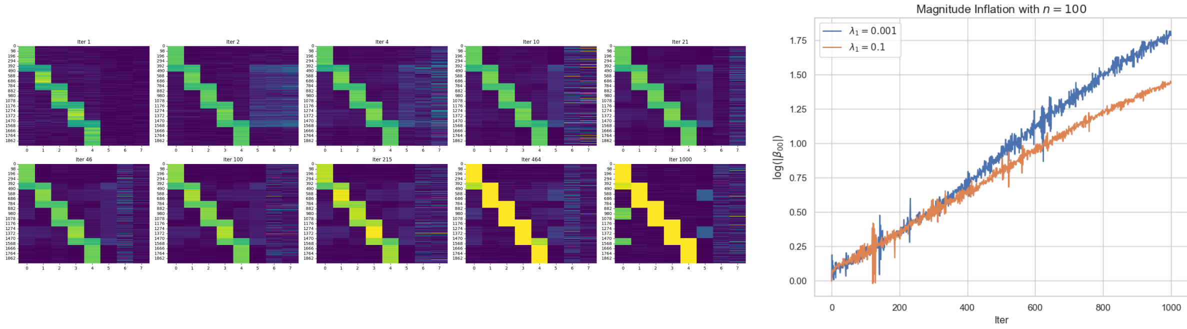

Figure 1: Factor loading matrix obtained with original Gibbs sampler (left) and the resulting inflation of the parameters (right)

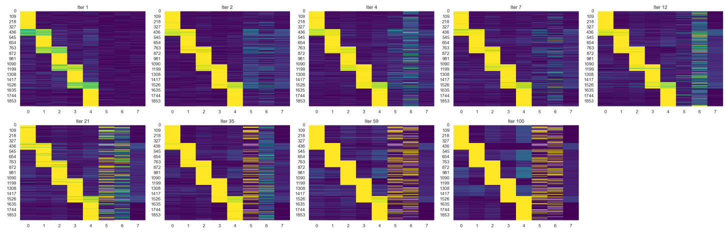

Figure 2: Sampling Instability with small scale

any constant converges to zero. Important results, for any densities on the factor, the posterior of the loadings matrix under the spike and slab prior converges in distribution as the scale parameters of the slab priors tends to 0 towards a flat slab prior. Ma and Liu (2022) remarks that the normal assumptions fails to control $\Omega\Omega'/n$ and make it converge in probability towards the identity in the small n, large s paradigm as n goes to infinity, but we have consistency when it converges towards the identity. As the first results under any distribution for the factors, Ma and Liu (2022) propose the orthonormal factor assumptions that is consistent under flat prior and better control the convergence of $\Omega\Omega'/n$.

3. Orthonormal assumptions as a solution

The posterior sample consistency (up to sign and permutation ambiguities) of the loading matrix is achieved when the posterior distribution of $\frac{\Omega\Omega^\top}{n}$, converges in probability to the identity matrix I.

This guarantees that posterior samples of the loading matrix remain robust and consistent. However, posterior samples of the unconstrained loading matrix need to be inflated correspondingly in magnitude to correct for underestimation when no control over $\frac{\Omega\Omega^\top}{n}$ is imposed. To guarantee posterior consistency, ideally, the posterior distribution of $K(\Omega)/\sqrt{n}$ converges to a point mass at the identity matrix I, where $K(\Omega)$ is L in the LQ decomposition of Ω.

Ensuring Consistency with Constraints on Ω

To impose stronger control over $\frac{\Omega\Omega^\top}{n}$, the model assumes a prior p_Ω that ensures:

1. All factors Ω are orthogonal, i.e., $\Omega^\top\Omega = I_K$, where K is the number of factors.
2. Each factor has equal norm such that:

$$\frac{\Omega}{\sqrt{n}} \in \mathrm{St}(K, n),$$

Here, $\mathrm{St}(K, n)$ denotes the Stiefel manifold, which represents the set of all ordered families of K vectors in $\mathbb{R}^n$.

Under this assumption, $\frac{\Omega\Omega^\top}{n} = I$, which guarantees that the posterior distribution of the loading matrix is consistent up to rotations. This avoids the inflation or deflation of the posterior samples.

Sampling from St(K, n)

Let $\Omega_{k\cdot}$ denote the k-th row of the factor matrix Ω, and Ω_{-k} denote the remaining rows. The conditional distribution of $\Omega_{k\cdot} \mid Y, \Omega_{-k}, B, \Sigma$ is modified from a standard multivariate normal distribution to:

$$\pi(\mathrm{d}\Omega_{k\cdot} \mid Y, \Omega_{-k}, B, \Sigma) \propto f(\Omega_{k\cdot}; \bar{\Omega}_{k\cdot}, \sigma_k^2 I_n) \cdot p_{\Omega_{-k}}(\mathrm{d}\Omega_{k\cdot}),$$

where:

- $p_{\Omega_{-k}}$ is the uniform measure on the n-radius sphere centered in the orthogonal complement of Ω_{-k},
- $f(\Omega_{k\cdot}; \bar{\Omega}_{k\cdot}, \sigma_k^2 I_n)$ is the multivariate normal density with $\bar{\Omega}_{k\cdot}$ is a centered version of column k that is not affected by the projection in the orthogonal complement of Ω_{-k} and σ_k^2 is a scaling factor.

To sample from this distribution, we look at the intersections between the $\sqrt{n}$-radius centered sphere in the orthogonal complement of Ω_{-k} and the hyperplanes orthogonal to $\bar{\Omega}_{k\cdot}$. From the resulting $(n-K)$-dimensional spheres, we choose the one at a distance d from the center along $\bar{\Omega}_{k\cdot}$. Where d is sampled according to the distribution :

$$\pi(d \mid Y, \Omega_{-k}, B, \Sigma) \propto (n - d^2)^{(n-K-2)/2} \exp\left(\frac{\|P_{\Omega_{-k}^\perp}(\bar{\Omega}_{k\cdot})\| d}{\sigma_k^2}\right),$$

where $P_{\Omega_{-k}^\perp}$ is the projection operator onto the orthogonal complement of the space spanned by Ω_{-k}. This unimodal distribution is sampled using the Metropolis algorithm.

Our results

In our implementation we managed to have $\frac{\Omega\Omega^\top}{n}$ converge to I as shown in fig. 4. However we could not show this guarantees posterior consistency of the loading matrix (up to rotations) and avoids inflation or deflation in the posterior samples as our implementation was too unstable. We think it comes from our implementation of the sampling for which you can find the pseudo-code in the appendix algorithm 2 and the code on our `github`. As you may have noticed we say the spheres from which we sample Ω_k are $(n-K)$-dimensional when the paper says $(n-k)$. We assumed it was a typo in the paper since we consider spheres in the orthogonal complement of Ω_{-k} of dimension $n-K+1$ and then take the intersection with a hyperplane so it goes down another dimension. However there may just be something we misunderstood and it makes our implementation fail.

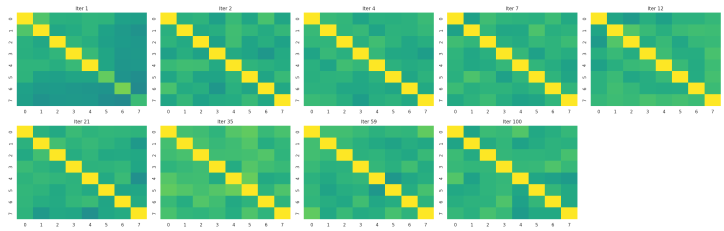

Figure 3: Orthonormal assumption of Ω

4. Conclusion

In this work, we successfully highlighted the limitations of the original Normal Sparse Factor Gibbs Sampler, demonstrating the inflation phenomenon in the posterior samples of the loading matrix. This issue, rooted in the normal latent factor assumptions, results in systematically inflated posterior coefficients, compromising the interpretability of the model.

We attempted to address this problem by replicating the orthonormal constraint on the latent factor matrix Ω, as proposed by Ma and Liu (2022). While we managed to enforce $\frac{\Omega\Omega^\top}{n}$ to converge to the identity matrix I, ensuring orthonormality, our implementation of the sampling procedure proved unstable. Consequently, we were unable to confirm whether this approach mitigates the inflation phenomenon or guarantees posterior consistency of the loading matrix.

Finally, we also explored an alternative implementation inspired by the method of Ghosh and Dunson (2009), but it failed to produce meaningful results. Due to its lack of relevance, the outcomes of this attempt are presented solely in the appendix.

Our findings underscore the challenges of implementing and evaluating advanced Bayesian factor models, particularly under orthonormal constraints, but it was a great learning opportunity.

References

Ghosh, J. and D. B. Dunson (Jan. 2009). "Default Prior Distributions and Efficient Posterior Computation in Bayesian Factor Analysis". In: *Journal of Computational and Graphical Statistics* 18.2, pp. 306–320. DOI: `10.1198/jcgs.2009.07145`.

Ma, Y. and J. S. Liu (2022). "On Posterior Consistency of Bayesian Factor Models in High Dimensions". In: *Bayesian Analysis* 17.3, pp. 901–929. DOI: `10.1214/21-BA1281`. URL: `https://doi.org/10.1214/21-BA1281`.

Ročková, V. and E. I. George (2016). "Fast Bayesian Factor Analysis via Automatic Rotations to Sparsity". In: *Journal of the American Statistical Association* 111.516, pp. 1608–1622. DOI: `10.1080/01621459.2015.1100620`. eprint: `https://doi.org/10.1080/01621459.2015.1100620`. URL: `https://doi.org/10.1080/01621459.2015.1100620`.

5. Appendix

5.1. Replicating the results

Everything can be found on our `github`. To replicate the figures in this file:

- Go on the main branch and clone it
- Install the environment with poetry (Run poetry install in the command line)
- Then open notebooks/Replicate_results.ipynb and run all cells

5.2. Sampling from the Normal Bayesian Factor under the Spike-and-slab LASSO prior

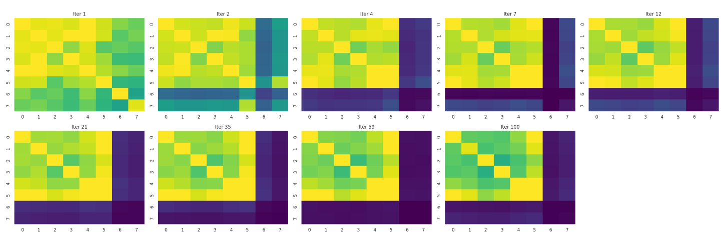

Figure 4: Orthonormal assumption of Ω

Algorithm 1 Gibbs Sampler for SpSL Factor Model

Require: Data matrix $Y \in \mathbb{R}^{G\times n}$, initial parameters $B, \Sigma, \Gamma, \Theta, \alpha, \eta, \epsilon, \lambda_0, \lambda_1$, number of iterations T

Ensure: Updated parameters $B, \Sigma, \Gamma, \Theta$

Initialize $B, \Sigma, \Gamma, \Theta$
for $t = 1$ to T **do**
  for each $i = 1, \dots, N$ **do**
    Sample ω_i from a normal distribution:

$$\omega_i | B^{(t-1)}, \Sigma^{(t-1)} \sim \mathcal{N}_K\big((I_K + B^T\Sigma^{-1}B)^{-1}B^T\Sigma^{-1}y_i,\ (I_K + B^T\Sigma^{-1}B)^{-1}\big)$$

  end for
  Update Ω with the new ω_i.
  for each $j = 1, \dots, G$ and $k = 1, \dots, K$ **do**
    Compute a_{jk}, b_{jk}, c_{jk} based on $B^{(t-1)}, \Omega^{(t-1)}, \Sigma^{(t-1)}, \Gamma^{(t-1)}$ and Y:

$$a_{jk} = \sum_{i=1}^{n} \frac{\omega_{ik}^2}{2\sigma_j^2}, \quad b_{jk} = \sum_{i=1}^{n} \frac{\omega_{ik}(y_{ij} - \sum_{l\neq k}\beta_{jl}\omega_{il})}{\sigma_j^2}, \quad c_{jk} = \lambda_1\gamma_{jk} + \lambda_0(1-\gamma_{jk})$$

    Sample β_{jk} from a mixture of truncated gaussian:

$$\beta_{jk} \overset{i.i.d.}{\sim} \mathcal{N}\left(\frac{b_{jk}-c_{jk}}{2a_{jk}}, \frac{1}{2a_{jk}}\right) \times \mathbf{1}\left[\beta_{jk} \geq 0\right] + \mathcal{N}\left(\frac{b_{jk}+c_{jk}}{2a_{jk}}, \frac{1}{2a_{jk}}\right) \times \mathbf{1}\left[\beta_{jk} < 0\right]$$

  end for
  Update B with the new β_{jk}.
  for each $j = 1, \dots, G$ and $k = 1, \dots, K$ **do**
    Sample

$$\gamma_{jk} \sim \text{Bernoulli}(p(\gamma_{jk}))$$

  end for
  for each factor $k = 1, \dots, K$ **do**
    Compute truncated Beta parameters α_k, β_k
    Sample $\theta_k \sim$ Truncated-Beta(α_k, β_k)
  end for
  for each $j = 1, \dots, G$ **do**
    Compute posterior shape and scale parameters
    Sample $\sigma_j^2 \sim \text{Inverse-Gamma}(\text{shape}, \text{scale})$
  end for
end for
Return sampled paths of $B, \Sigma, \Gamma, \Theta$ =0

5.3. Sampling of Ω_k in the orthonormal case

Algorithm 2 Sampling from the intersection of a sphere and a hyperplane

Require: d (distance of the hyperplane from the origin), $\bar{\boldsymbol{\Omega}}_{\boldsymbol{k}}$ (normal vector defining the hyperplane), k (index of the row being sampled), ϵ (small value to avoid division by zero)

Ensure: Sampled point $\boldsymbol{x}$ from the intersection of the sphere and hyperplane.

1: Compute Ω_{-k} = Remove row k from Ω, with shape $(n, K-1)$

2: Randomly sample $\boldsymbol{X} \sim \mathcal{N}(\mathbf{0}, \mathbb{I}_n)$

3: Normalize $\boldsymbol{X}$ to the unit sphere: $\boldsymbol{X}_{\text{unit}} = \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|}$

4: Project $\boldsymbol{X}_{\text{unit}}$ orthogonally to the space spanned by Ω_{-k}:

$$\boldsymbol{X}_{\text{orthogonal}} = \text{Project}(\Omega_{-k}, \boldsymbol{X}_{\text{unit}})$$

5: Further project $\boldsymbol{X}_{\text{orthogonal}}$ orthogonally to $\bar{\boldsymbol{\Omega}}_{\boldsymbol{k}}$:

$$\boldsymbol{X}_{\text{hyperplane}} = \text{Project}(\bar{\boldsymbol{\Omega}}_{\boldsymbol{k}}, \boldsymbol{X}_{\text{orthogonal}})$$

6: Compute the norm of $\boldsymbol{X}_{\text{hyperplane}}$: $\lambda = \|\boldsymbol{X}_{\text{hyperplane}}\|$

7: Rescale $\boldsymbol{X}_{\text{hyperplane}}$ to lie on the sphere of radius $\sqrt{n - d^2}$:

$$\boldsymbol{X}_{\text{sphere}} = \sqrt{n - d^2} \cdot \frac{\boldsymbol{X}_{\text{hyperplane}}}{\lambda}$$

8: Shift the point $\boldsymbol{X}_{\text{sphere}}$ by d along $\bar{\boldsymbol{\Omega}}_{\boldsymbol{k}}$:

$$\boldsymbol{x} = \boldsymbol{X}_{\text{sphere}} + \frac{d \cdot \bar{\boldsymbol{\Omega}}_{\boldsymbol{k}}}{\max(\|\bar{\boldsymbol{\Omega}}_{\boldsymbol{k}}\|, \epsilon)}$$

9: **return** $\boldsymbol{x}$ =0

5.4. Gibbs Sampler proofs

5.4.1 Motivation

During our reimplementation of Ghosh and Dunson (2009) we noticed we had the model but no new update policy. Since it was a modified version with a diffuse parameter for the loading factors, the update policy from the original paper is no longer viable. The following is our endeavour at finding the update policy for the Ghosh Dunson model

5.4.2 Base model for the base model

We first show the results on the original sampler to better understand what changes with Ghosh Dunson afterwards. **Model and Priors:**

$$y_i|\omega_i, B, \Sigma \sim \mathrm{NG}(B\omega_i, \Sigma), \omega_i \sim \mathcal{N}_K(0, I_K)$$

$$\begin{aligned}
p(\beta_{jk}|\gamma_{jk}, \lambda_0, \lambda_1) &= (1-\gamma_{jk})\psi(\beta_{jk}|\lambda_0) + \gamma_{jk}\psi(\beta_{jk}|\lambda_1), \qquad\qquad \lambda_0 \neq \lambda_1 \\
\gamma_{jk}|\theta_k &\sim \mathrm{Bernoulli}(\theta_k), \\
\theta_k &= \prod_{l=1}^{k} \nu_l, \quad \nu_l \sim \mathrm{Beta}(\alpha, 1), \\
\sigma_j^2 &\sim \text{Inverse-Gamma}\left(\frac{\eta}{2}, \frac{\eta\epsilon}{2}\right) \\
\psi(\beta_{jk}|\lambda) &\sim \mathrm{Laplace}(0, 1/\lambda)
\end{aligned}$$

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posterior for β_{jk} At fixed j and k, to compute the likelihood term, we focus on the relationship between y_{ji} and β_{jk}. Substituting the model equation for y_{ji}, we have:

$$y_{ji} = \sum_{l=1}^{K} \beta_{jl}\omega_{il} + \epsilon_{ij}$$

$$y_{ji} = \beta_{jk}\omega_{ik} + \sum_{l\neq k} \beta_{jl}\omega_{il} + \epsilon_{ij}$$

$$\text{"Residual term" without } \beta_{jk}: \quad r_{ijk} = y_{ji} - \sum_{l\neq k} \beta_{jl}\omega_{il}$$

$$f(y_{ji} \mid \beta_{jk}, \omega_{ik}) \propto \exp\left(-\frac{1}{2\sigma_j^2}(r_{ijk} - \beta_{jk}\omega_{ik})^2\right)$$

$$\text{All rows that impact } \beta_{jk}: \prod_{i=1}^{n} f(y_{ji} \mid \beta_{jk}, \omega_{ik}) \propto \exp\left(-\frac{1}{2\sigma_j^2}\sum_{i=1}^{n}(r_{ijk} - \beta_{jk}\omega_{ik})^2\right)$$

$$\sum_{i=1}^{n}(r_{ijk} - \beta_{jk}\omega_{ik})^2 = \sum_{i=1}^{n} r_{ijk}^2 - 2\beta_{jk}\sum_{i=1}^{n} r_{ijk}\omega_{ik} + \beta_{jk}^2\sum_{i=1}^{n}\omega_{ik}^2$$

$$-\frac{1}{2\sigma_j^2}\sum_{i=1}^{n}(r_{ijk} - \beta_{jk}\omega_{ik})^2 = -\frac{1}{2\sigma_j^2}\left[\beta_{jk}^2\sum_{i=1}^{n}\omega_{ik}^2 - 2\beta_{jk}\sum_{i=1}^{n} r_{ijk}\omega_{ik}\right] + \text{constant}$$

The prior for β_{jk} is a mixture of Laplace distributions:

$$\begin{aligned} p(\beta_{jk} \mid \gamma_{jk}, \lambda_0, \lambda_1) &= (1-\gamma_{jk})\psi(\beta_{jk} \mid \lambda_0) + \gamma_{jk}\psi(\beta_{jk} \mid \lambda_1) \\ \psi(\beta_{jk} \mid \lambda) &\propto \exp(-\lambda|\beta_{jk}|) \\ p(\beta_{jk} \mid \gamma_{jk}, \lambda_0, \lambda_1) &\propto \psi(\beta_{jk} \mid \lambda_1\gamma_{jk} + \lambda_0(1-\gamma_{jk})) \end{aligned}$$

Since γ_{jk} is a bernouli variable

Combining the likelihood and prior terms, the conditional posterior for β_{jk} is proportional to:

$$\begin{aligned} \pi(\beta_{jk} \mid \beta_{-jk}, \Omega, \Gamma, \Sigma) \propto \exp\left(-\frac{1}{2\sigma_j^2}\beta_{jk}^2 \sum_{i=1}^{n} \omega_{ik}^2 + \frac{1}{\sigma_j^2}\beta_{jk} \sum_{i=1}^{n} r_{ijk}\omega_{ik} - c_{jk}|\beta_{jk}|\right) \\ c_{jk} = \lambda_1\gamma_{jk} + \lambda_0(1-\gamma_{jk}) \mid a_{jk} = \frac{1}{2\sigma_j^2}\sum_{i=1}^{n} \omega_{ik}^2 \mid b_{jk} = \frac{1}{\sigma_j^2}\sum_{i=1}^{n} \omega_{ik}\left(y_{ji} - \sum_{l\neq k} \beta_{jl}\omega_{il}\right) \end{aligned}$$

This is the way to get back the results from the paper's section 2.2. Where they actually had a typo by writing y_{ij} instead of y_{ji}

5.4.3 Modified Ghosh Dunson model

Model and Priors:

$$y_i|\omega_i, B, \Sigma \sim \text{NG}(B\omega_i, \Sigma), \quad \omega_i \sim \mathcal{N}_K(0, I_K)$$

$$\begin{aligned} \beta_{jk} &= q_{jk}r_k, \\ p(r_k|\lambda) &= \psi(r_k|\lambda), \\ p(q_{jk}|\gamma_{jk}, \lambda_0, \lambda_1) &= (1-\gamma_{jk})\psi(q_{jk}|\lambda_0) + \gamma_{jk}\psi(q_{jk}|\lambda_1), \qquad \lambda_0 \neq \lambda_1 \\ \gamma_{jk}|\theta_k &\sim \text{Bernoulli}(\theta_k), \\ \theta_k &= \prod_{l=1}^{k} \nu_l, \quad \nu_l \sim \text{Beta}(\alpha, 1), \\ \sigma_j^2 &\sim \text{Inverse-Gamma}\left(\frac{\eta}{2}, \frac{\eta\epsilon}{2}\right) \\ \psi(r_k|\lambda) &\sim \text{Laplace}(0, 1/\lambda) \end{aligned}$$

posterior for q_{jk} At fixed j and k, we focus on the relationship between y_{ji} and q_{jk}. Substituting the model equation for y_{ji}, we have:

$$y_{ji} = \sum_{l=1}^{K} \beta_{jl}\omega_{il} + \epsilon_{ij}$$

$$y_{ji} = q_{jk}r_k\omega_{ik} + \sum_{l\neq k} q_{jl}r_l\omega_{il} + \epsilon_{ij}$$

$$\text{"Residual term" without } qjk: \quad res_{ijk} = y_{ji} - \sum_{l\neq k} q_{jl}r_l\omega_{il}$$

$$f(y_{ji} \mid q_{jk}, r_k, \omega_{ik}) \propto \exp\left(-\frac{1}{2\sigma_j^2}(res_{ijk} - q_{jk}r_k\omega_{ik})^2\right)$$

$$\text{All rows that impact } q_{jk}: \prod_{i=1}^{n} f(y_{ji} \mid q_{jk}, r_k, \omega_{ik}) \propto \exp\left(-\frac{1}{2\sigma_j^2}\sum_{i=1}^{n}(res_{ijk} - q_{jk}r_k\omega_{ik})^2\right)$$

$$\sum_{i=1}^{n}(res_{ijk} - q_{jk}r_k\omega_{ik})^2 = \sum_{i=1}^{n} res_{ijk}^2 - 2q_{jk}\sum_{i=1}^{n} res_{ijk}\omega_{ik}r_k + q_{jk}^2\sum_{i=1}^{n} r_k^2\omega_{ik}^2$$

$$-\frac{1}{2\sigma_j^2}\sum_{i=1}^{n}(res_{ijk} - q_{jk}r_k\omega_{ik})^2 = -\frac{1}{2\sigma_j^2}\left[q_{jk}^2\sum_{i=1}^{n}\omega_{ik}^2 r_k^2 - 2q_{jk}\sum_{i=1}^{n} res_{ijk}\omega_{ik}r_k\right] + \text{constant}$$

The prior for qjk is the same as old β_{jk}:

$$p(q_{jk} \mid \gamma_{jk}, \lambda_0, \lambda_1) = (1-\gamma_{jk})\psi(q_{jk} \mid \lambda_0) + \gamma_{jk}\psi(q_{jk} \mid \lambda_1)$$

$$\psi(q_{jk} \mid \lambda) \propto \exp(-\lambda|q_{jk}|)$$

$$p(q_{jk} \mid \gamma_{jk}, \lambda_0, \lambda_1) \propto \psi(q_{jk} \mid \lambda_1\gamma_{jk} + \lambda_0(1-\gamma_{jk}))$$

Since γ_{jk} is a bernouli variable

Combining the likelihood and prior terms, the conditional posterior for q_{jk} is proportional to:

$$\pi(q_{jk} \mid q_{-jk}, \Omega, \Gamma, \Sigma) \propto \exp\left(-\frac{1}{2\sigma_j^2}q_{jk}^2\sum_{i=1}^{n}\omega_{ik}^2 r_k^2 + \frac{1}{\sigma_j^2}q_{jk}\sum_{i=1}^{n} res_{ijk}\cdot\omega_{ik}r_k - c_{jk}|q_{jk}|\right)$$

$$c_{jk} = \lambda_1\gamma_{jk} + \lambda_0(1-\gamma_{jk}) \mid a_{jk} = \frac{1}{2\sigma_j^2}\sum_{i=1}^{n}\omega_{ik}^2 r_k^2 \mid b_{jk} = \frac{1}{\sigma_j^2}\sum_{i=1}^{n}\omega_{ik}r_k\left(y_{ji} - \sum_{l\neq k} q_{jl}r_l\omega_{il}\right)$$

Where we have the diffuse parameter intervene in the coefficients this time.

Posterior for r_k By having k fixed we go back to the likelihood. The model equation for y_{ji} is:

$$y_{ji} = q_{jk} r_k \omega_{ik} + \sum_{l \neq k} q_{jl} r_l \omega_{il} + \epsilon_{ij}.$$

Let the residual term res_{ijk} be defined as:

$$res_{ijk} = y_{ji} - \sum_{l \neq k} q_{jl} r_l \omega_{il}.$$

The likelihood for y_{ji} given r_k, q_{jk}, and ω_{ik} is:

$$f(y_{ji} \mid r_k, q_{jk}, \omega_{ik}) \propto \exp\left(-\frac{1}{2\sigma_j^2} \left(res_{ijk} - q_{jk} r_k \omega_{ik}\right)^2\right).$$

The total likelihood is the product of the likelihoods for all i and j:

$$\prod_{i=1}^{n} \prod_{j=1}^{J} f(y_{ji} \mid r_k, q_{jk}, \omega_{ik}) \propto \exp\left(-\frac{1}{2\sigma_j^2} \sum_{i=1}^{n} \sum_{j=1}^{J} \left(res_{ijk} - q_{jk} r_k \omega_{ik}\right)^2\right).$$

Expanding the quadratic term inside the sum and isolating r_k:

$$\sum_{i=1}^{n} \sum_{j=1}^{J} \left(res_{ijk} - q_{jk} r_k \omega_{ik}\right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{J} res_{ijk}^2 - 2r_k \sum_{i=1}^{n} \sum_{j=1}^{J} res_{ijk} q_{jk} \omega_{ik} q_{jk} + r_k^2 \sum_{i=1}^{n} \omega_{ik}^2 \sum_{j=1}^{J} q_{jk}^2.$$

Thus, the likelihood term becomes:

$$-\frac{1}{2\sigma_j^2} \sum_{i=1}^{n} \sum_{j=1}^{J} \left(res_{ijk} - q_{jk} r_k \omega_{ik}\right)^2 = -\frac{1}{2\sigma_j^2} \left[r_k^2 \sum_{i=1}^{n} \omega_{ik}^2 \sum_{j=1}^{J} q_{jk}^2 - 2r_k \sum_{i=1}^{n} \sum_{j=1}^{J} res_{ijk} q_{jk} \omega_{ik} \right] + \text{constant}.$$

We combine the prior for r_k and the likelihood term. The posterior for r_k is proportional

$$p(r_k \mid y_{ji}, q_{jk}, \omega_{ik}) \propto \exp\left(-\lambda |r_k|\right) \cdot \exp\left(-\frac{1}{2\sigma_j^2} \left[r_k^2 \sum_{i=1}^{n} \omega_{ik}^2 \sum_{j=1}^{J} q_{jk}^2 - 2r_k \sum_{i=1}^{n} \sum_{j=1}^{J} res_{ijk} q_{jk} \omega_{ik} \right]\right).$$

$$p(r_k \mid \dots) \propto \exp\left(a r_k^2 - 2b r_k - h|r_k|\right),$$

where:

- $a = -\frac{1}{2\sigma_j^2} \sum_{i=1}^{n} \sum_{j=1}^{J} \omega_{ik}^2 q_{jk}^2$
- $b = -\frac{1}{2\sigma_j^2} \sum_{i=1}^{n} \sum_{j=1}^{J} q_{jk}\, res_{ijk}\, \omega_{ik}$
- $h = \lambda$