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Matrix

credits: Problem	5-1	1 point
Problem	5-2	1 point

Problem 5-1 Matrix Operations

a. Consider the following matrices and vectors:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & -1 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

Calculate $\mathbf{AB}^\top$, $\mathbf{A}^\top\mathbf{B}$, $\mathbf{vw}^\top$, and $\mathbf{v}^\top\mathbf{w}$.

Note: The upper index $\top$ denotes "transposition" of the matrix, i.e. exchanging rows and columns. The 4×3-matrix $\mathbf{A}$ is transposed into the 3×4-matrix $\mathbf{A}^\top$ etc. $\mathbf{v}$ and $\mathbf{w}$ are treated here as 4×1 matrices but can also be considered as column vectors. In this case, $\mathbf{v}^\top\mathbf{w}$ and $\mathbf{vw}^\top$ are the inner and outer products of the two vectors.

b. Let

$$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Show that $\mathbf{e}_1 = (1,1)^\top$ and $\mathbf{e}_2 = (1,-1)^\top$ are eigenvectors of $\mathbf{M}$, i.e. that they satisfy the equations $\mathbf{Me}_1 = \lambda_1\mathbf{e}_1$ and $\mathbf{Me}_2 = \lambda_2\mathbf{e}_2$. Calculate the eigenvalues λ_1 and λ_2.

1 a)

$$AB^T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 3 & -1 & 0 \\ 3 & 4 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 & -1 & 4 \\ 3 & 4 & -1 & -2 \\ 6 & 9 & -2 & 4 \\ 6 & 8 & 0 & 4 \end{pmatrix}$$

$$A^TB = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & -1 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -1 \\ 3 & 5 & 8 \\ 4 & 1 & 7 \end{pmatrix}$$

$$vw^T = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 8 & 6 & 4 & 2 \\ -12 & -9 & -6 & -3 \end{pmatrix}$$

$$v^T w = \begin{pmatrix} 1 & 0 & 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = 4 + 4 - 3 = 5$$

b)

$$\underset{\underset{M}{\uparrow}}{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}} \cdot \underset{\underset{e_1}{\uparrow}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \underset{\underset{\lambda_1}{\uparrow}}{4} \cdot \underset{\underset{e_1}{\uparrow}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \rightarrow e_1 \text{ eigenvector of } M$$

$$\underset{\underset{M}{\uparrow}}{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}} \cdot \underset{\underset{e_2}{\uparrow}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \underset{\underset{\lambda_2}{\uparrow}}{2} \cdot \underset{\underset{e_2}{\uparrow}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \rightarrow e_2 \text{ eigenvector of } M$$

Problem 5-2 Principal Components

a. Consider again the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Assume now that it is the covariance matrix of the pair of data vectors $\mathbf{x} = (x_1, \ldots, x_n)^\top$ and $\mathbf{y} = (y_1, \ldots, y_n)^\top$ with zero mean ($\hat{\mathbf{x}} = \hat{\mathbf{y}} = 0$). I.e., we have $\mathrm{var}(\mathbf{x}) = \mathrm{var}(\mathbf{y}) = 3$ and $\mathrm{cov}(\mathbf{x}, \mathbf{y}) = 1$.

Calculate the variance of the combined random variables $\mathbf{z} = \mathbf{x} + \mathbf{y}$ and $\mathbf{z}' = \mathbf{x} - \mathbf{y}$ and the covariance $\mathrm{cov}(\mathbf{z}, \mathbf{z}')$

Hint: since all variables are zero mean, we have $\mathrm{var}(\mathbf{x}) = \frac{1}{n} \sum x_i^2$ etc.

b. Consider projections of the vectors $\mathbf{s}_i = (x_i, y_i)$ on a unit projection vector in direction φ, $\mathbf{p} = (\cos\varphi, \sin\varphi)^\top$:

$$z_{\varphi,i} = (\mathbf{s}_i \cdot \mathbf{p}); \qquad \mathbf{z}_\varphi = \mathbf{x}\cos\varphi + \mathbf{y}\sin\varphi.$$

Show that the variances of these projections are given as

$$\begin{aligned} \mathrm{var}(\mathbf{z}_\varphi) &= \mathbf{p}^\top \mathbf{M} \mathbf{p} \\ &= \cos^2\varphi \ \mathrm{var}(\mathbf{x}) + 2\cos\varphi\sin\varphi \ \mathrm{cov}(\mathbf{x}, \mathbf{y}) + \sin^2\varphi \ \mathrm{var}(\mathbf{y}) \\ &= 3 + 2\cos\varphi\sin\varphi \end{aligned}$$

and that they are maximal (minimal) for $\varphi = 45°$ ($\varphi = -45°$), i.e. for the projection vectors $\mathbf{p}_{\pm 45°}^\top = (\cos 45°, \pm\sin 45°) = (1, \pm 1)/\sqrt{2}$.

c. Sketch a scatter-plot of the data-points (x_i, y_i) and mark the variances $\mathrm{var}(\mathbf{x})$, $\mathrm{var}(\mathbf{y})$, $\mathrm{var}(\mathbf{z}_{45})$, $\mathrm{var}(\mathbf{z}_{-45})$ and the projection vectors $(1, 1)/\sqrt{2}$ and $(1, -1)/\sqrt{2}$. Observe that the maximal and minimal variances are obtained for projections on the eigenvectors of M.

2a) variances z and z':

$z = x + y$

$Var(z) = Var(x+y) = Var(x) + Var(y) + 2 \cdot Cov(x,y) = 3 + 3 + 2 \cdot 1 = 8$

$z' = x - y$

$Var(z') = Var(x-y) = Var(x) + Var(y) - 2 \cdot Cov(x,y) = 3 + 3 - 2 = 4$

covariance z z':

$Var(z + z') = Var(x + y + (x - y)) = Var(2x) = 2^2 \cdot Var(x) = 4\, Var(x)$

$Var(z + z') = Var(z) + Var(z') + 2 \cdot Cov(z, z')$

$4 \cdot Var(x) = Var(z) + Var(z') + 2 \cdot Cov(z, z')$

$$Cov(z, z') = \frac{4\,Var(x) - Var(z) - Var(z')}{2}$$

$$= \frac{4 \cdot 3 - 8 - 4}{2} = 0$$

b)

$$\mathrm{var}(z_\varphi) = \varphi^T M \varphi$$
$$= (\cos\varphi \quad \sin\varphi) \cdot \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$
$$= (3\cos\varphi + \sin\varphi \quad \cos\varphi + 3\sin\varphi) \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$
$$= 3\cos^2\varphi + \sin\varphi\cos\varphi + \cos\varphi \cdot \sin\varphi + 3\sin^2\varphi$$
$$= 3\sin^2\varphi + 3\cos^2\varphi + 2\sin\varphi\cos\varphi$$
$$= 3 \cdot \underbrace{(\sin^2\varphi + \cos^2\varphi)}_{1} + 2\sin\varphi\cos\varphi = 3 + 2\sin\varphi\cos\varphi$$

$$\mathrm{var}(z_\varphi) = \mathrm{var}(x\cos\varphi + y\sin\varphi)$$
$$= \mathrm{Var}(x\cos\varphi) + \mathrm{Var}(y\sin\varphi) + 2 \cdot \mathrm{Cov}(x\cos\varphi, y\sin\varphi)$$
$$= \cos^2\varphi \cdot \mathrm{Var}(x) + \sin^2\varphi \cdot \mathrm{Var}(y) + 2 \cdot \cos\varphi \cdot \sin\varphi \cdot \mathrm{Cov}(x, y)$$
$$= 3\cos^2\varphi + 3\sin^2\varphi + 2\cos\varphi\sin\varphi$$
$$= 3 + 2\sin\varphi\cos\varphi$$

$$\mathrm{Var}(z_\varphi)' = -2\sin\varphi \cdot \sin\varphi + 2\cos\varphi \cdot \cos\varphi$$
$$= -2\sin^2\varphi + 2\cos^2\varphi$$
$$= 2 \cdot (-\sin^2\varphi + \cos^2\varphi)$$

$$2 \cdot (-\sin^2(45^\circ) + \cos^2(45^\circ)) = 0 = 2 \cdot (-0{,}5 + 0{,}5) = 0$$
$$2 \cdot (-\sin^2(-45^\circ) + \cos^2(-45^\circ)) = 0 = 2 \cdot (-0{,}5 + 0{,}5) = 0$$
> extreme

$$\mathrm{Var}(z_\varphi)'' = -4\sin\varphi\cos\varphi - 4\cos\varphi\sin\varphi = -8\sin\varphi\cos\varphi$$

$-8\sin(45^\circ)\cos(45^\circ) = -4$ → max.

$-8\sin(-45^\circ)\cos(-45^\circ) = 4$ → min.