MCEN 4173/5173

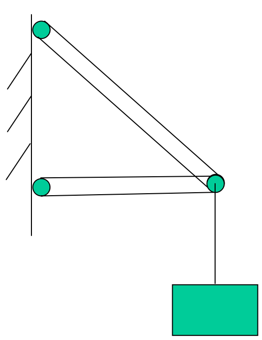
Chapter 4

2D and 3D Truss Elements

Fall, 2006

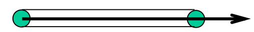
Truss

Truss is one of the simplest, yet very commonly used structural elements.



Truss

Features of a truss



- Forces can only be transmitted along the axial direction. Or a truss can only be subjected to axial load.
- The deformation is along the axial direction.
- A truss cannot sustain shear and moment.
- The load can only be applied at the two ends.

Learning points:

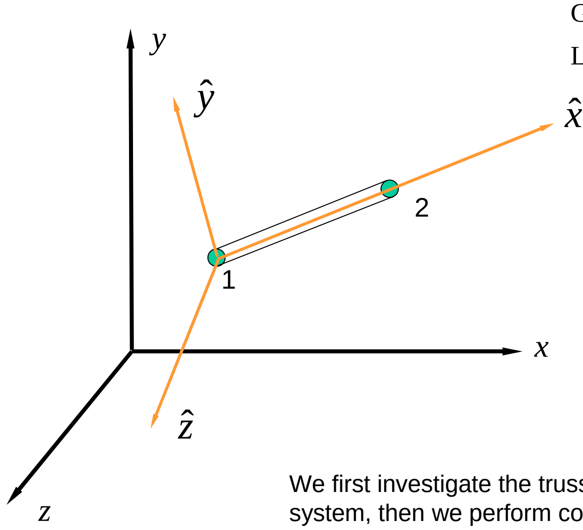
Shape Functions
Rotation of Coordinate System

Truss

Mechanics of a truss

Global Coordinate: (xyz)

Local Coordinate: $(\hat{x}\hat{y}\hat{z})$

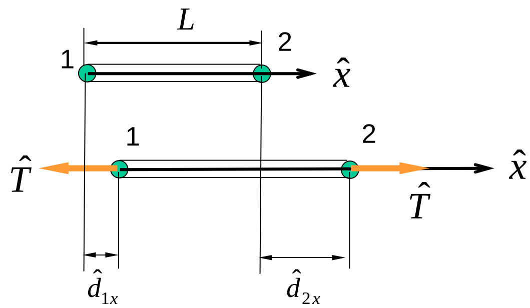


We first investigate the truss in the local coordinate system, then we perform coordinate transformation.

(Note: The procedure is different from lecture 3)

Truss

Mechanics of a truss



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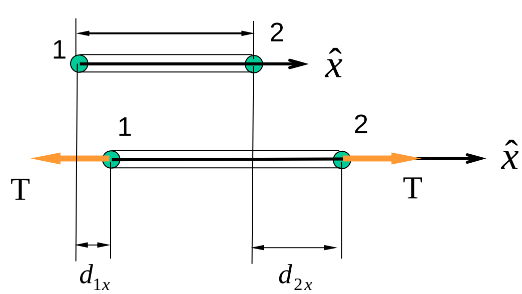
Truss

Shape Function

$$\varepsilon = \frac{du}{dx} \qquad u = ?$$



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Truss

Shape Function

$$\hat{u} = \left(1 - \frac{\hat{x}}{L}\right)\hat{d}_{1x} + \frac{\hat{x}}{L}\hat{d}_{2x}$$

$$\Rightarrow \quad \hat{u} = \begin{bmatrix} 1 - \frac{\hat{x}}{L} & \frac{\hat{x}}{L} \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \quad \Rightarrow \quad \hat{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

N_1 and N_2 are called SHAPE FUNCTIONS, or Interpolation functions.

Shape functions describe the shape of displacement field and are one of the determinant factors in governing the efficiency and accuracy of FEA.

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Truss

Shape Functions in general

In FEA, we hope

$$u(x, y, z) = a_0 + a_1 x + b_1 y + c_1 z + \ldots.$$

$f_1, f_2, f_3, \ldots.$ are shape functions

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Truss

Shape Functions in general

$$N_1 = 1 - \frac{\hat{x}}{L} \qquad N_2 = \frac{\hat{x}}{L}$$

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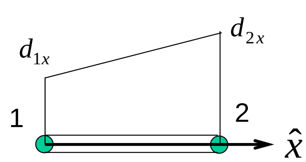
Truss

Shape Function



Here, the shape functions are linear functions

Do linear functions make sense here? Or can we use quadratic functions?

<u>Linear function:</u>

Since

<u>Quadratic function:</u>

Since

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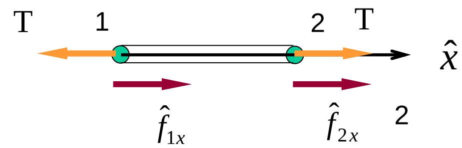
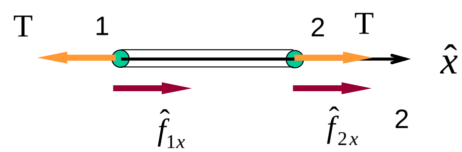
Truss Element



$$\hat{T} = EA\hat{\varepsilon} = EA\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} \Rightarrow \begin{aligned} \hat{f}_{1x} &= -\hat{T} = \frac{EA}{L}\left(\hat{d}_{1x} - \hat{d}_{2x}\right) \\ \hat{f}_{2x} &= \hat{T} = \frac{EA}{L}\left(-\hat{d}_{1x} + \hat{d}_{2x}\right) \end{aligned}$$

$$\Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Truss Element



$$\hat{T} = EA\hat{\varepsilon} = EA\frac{d\hat{u}}{d\hat{x}}$$

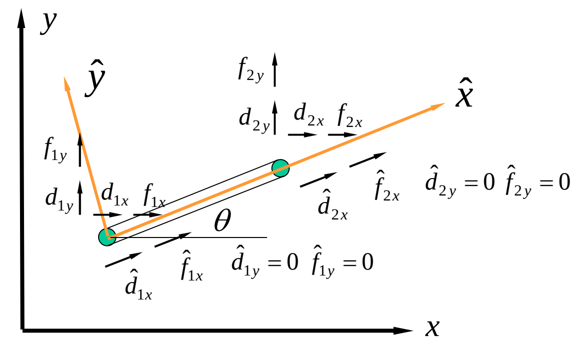
$$\hat{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\Rightarrow \begin{aligned} \hat{f}_{1x} &= -\hat{T} = EA\begin{bmatrix} -\frac{dN_1}{d\hat{x}} & -\frac{dN_2}{d\hat{x}} \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \\ \hat{f}_{2x} &= \hat{T} = EA\begin{bmatrix} \frac{dN_1}{d\hat{x}} & \frac{dN_2}{d\hat{x}} \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \end{aligned}$$

$$\Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = EA\begin{bmatrix} -\frac{dN_1}{d\hat{x}} & -\frac{dN_2}{d\hat{x}} \\ \frac{dN_1}{d\hat{x}} & \frac{dN_2}{d\hat{x}} \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\frac{dN_1}{d\hat{x}} = -\frac{1}{L} \quad \frac{dN_2}{d\hat{x}} = \frac{1}{L} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Truss Element -2D



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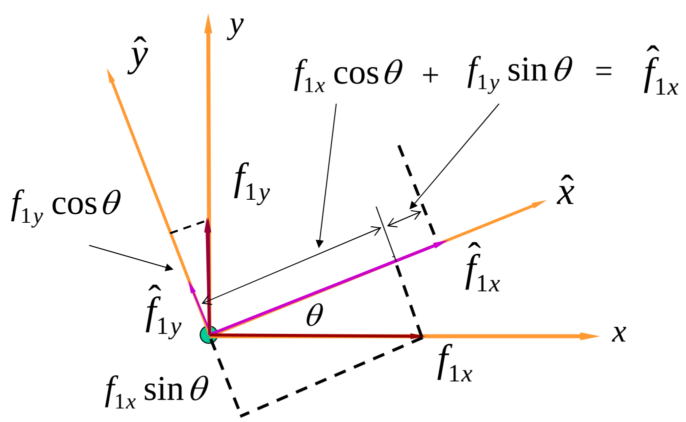
Truss Element -2D

How are quantities in different coordinate systems related?



$$-f_{1x}\sin\theta + f_{1y}\cos\theta = \hat{f}_{1y}$$

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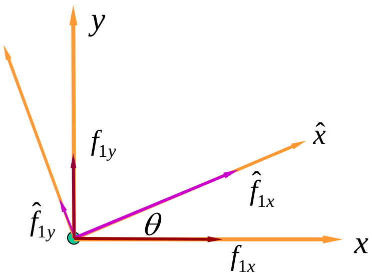
Truss Element -2D

How are quantities in different coordinate systems related?



$$\hat{f}_{1x} = f_{1x}\cos\theta + f_{1y}\sin\theta$$
$$\hat{f}_{1y} = -f_{1x}\sin\theta + f_{1y}\cos\theta$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix}$$

$$\{\hat{f}\} = [T]\{f\} \qquad \{f\} = [T]^{-1}\{\hat{f}\}$$

$$[T]^{-1} = [T]^T$$

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Truss Element -2D

How are quantities in different coordinate systems related?

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} \qquad \begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \end{Bmatrix}$$

$$C = \cos\theta \qquad S = \sin\theta$$

Now the question is:

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = [??] \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} \qquad \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} = [??] \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

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Truss Element -2D

How are quantities in different coordinate systems related?

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} \Rightarrow \begin{aligned} f_{1x} &= C\hat{f}_{1x} - S\hat{f}_{1y} \\ f_{1y} &= S\hat{f}_{1x} + C\hat{f}_{1y} \\ f_{2x} &= C\hat{f}_{2x} - S\hat{f}_{2y} \\ f_{2y} &= S\hat{f}_{2x} + C\hat{f}_{2y} \end{aligned} \Rightarrow \begin{aligned} f_{1x} &= C\hat{f}_{1x} - S\hat{f}_{1y} + 0\hat{f}_{2x} + 0\hat{f}_{2y} \\ f_{1y} &= S\hat{f}_{1x} + C\hat{f}_{1y} + 0\hat{f}_{2x} + 0\hat{f}_{2y} \\ f_{2x} &= 0\hat{f}_{1x} + 0\hat{f}_{1y} + C\hat{f}_{2x} - S\hat{f}_{2y} \\ f_{2y} &= 0\hat{f}_{1x} + 0\hat{f}_{1y} + S\hat{f}_{2x} + C\hat{f}_{2y} \end{aligned}$$

$$\Downarrow$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} \qquad \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

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Truss Element -2D

How are quantities in different coordinate systems related?

$\hat{x}\hat{y}\hat{z} \Rightarrow xyz$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} \qquad \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

$xyz \Rightarrow \hat{x}\hat{y}\hat{z}$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} \qquad \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

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Truss Element -2D

Element stiffness matrix in global coordinate

$$\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix} \Rightarrow \begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{1y}\\ \hat{f}_{2x}\\ \hat{f}_{2y}\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{1y}\\ \hat{d}_{2x}\\ \hat{d}_{2y}\end{Bmatrix}$$

$$\begin{Bmatrix}f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{Bmatrix}=\begin{bmatrix}C & -S & 0 & 0\\ S & C & 0 & 0\\ 0 & 0 & C & -S\\ 0 & 0 & S & C\end{bmatrix}\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{1y}\\ \hat{f}_{2x}\\ \hat{f}_{2y}\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}C & -S & 0 & 0\\ S & C & 0 & 0\\ 0 & 0 & C & -S\\ 0 & 0 & S & C\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{1y}\\ \hat{d}_{2x}\\ \hat{d}_{2y}\end{Bmatrix}$$

$$=\frac{EA}{L}\begin{bmatrix}C & -S & 0 & 0\\ S & C & 0 & 0\\ 0 & 0 & C & -S\\ 0 & 0 & S & C\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}C & S & 0 & 0\\ -S & C & 0 & 0\\ 0 & 0 & C & S\\ 0 & 0 & -S & C\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

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Truss Element -2D

Element stiffness matrix in global coordinate

$$\begin{Bmatrix}f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}C & -S & 0 & 0\\ S & C & 0 & 0\\ 0 & 0 & C & -S\\ 0 & 0 & S & C\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}C & S & 0 & 0\\ -S & C & 0 & 0\\ 0 & 0 & C & S\\ 0 & 0 & -S & C\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix}f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}C^2 & CS & -C^2 & -CS\\ CS & S^2 & -CS & -S^2\\ -C^2 & -CS & C^2 & CS\\ -CS & -S^2 & CS & S^2\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

$$[T]=\begin{bmatrix}C & S & 0 & 0\\ -S & C & 0 & 0\\ 0 & 0 & C & S\\ 0 & 0 & -S & C\end{bmatrix}$$

$$\{f\}=[T^T]\left[\hat{K}\right]^e[T]\{d\}$$

$$[K]=[T^T]\left[\hat{K}\right]^e[T]$$

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Truss Element -2D

Global Stiffness Matrix

Using the direct stiffness method.

Note: This time, each node has 2 degrees of freedom.

If we have N nodes in the model,

$$[K] = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}_{2N \times 2N}$$

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Truss Element -3D

Review: what did we do when we move from 1D to 2D?

Formulate 1D element stiffness matrix

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

↓

Find the transformation matrix for coordinates [T']

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix}$$

↓

Expand the transformation matrix to [T]

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix}$$

↓

$$[K]^e = [T^T]\left[\hat{K}\right]^e [T]$$

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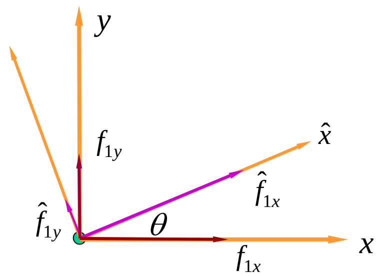
Truss Element -3D

Coordinate transformation matrix



$\cos(\langle \hat{x}, x\rangle)$ $\quad$ $\cos(\langle \hat{x}, y\rangle)$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} = \begin{array}{cc} x & y \\ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{array} \begin{array}{cc} & \\ \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} & \begin{matrix} \hat{x} \\ \hat{y} \end{matrix} \end{array}$$

$\cos(\langle \hat{y}, x\rangle)$ $\quad$ $\cos(\langle \hat{y}, y\rangle)$

$\cos(\langle x, \hat{x}\rangle)$ $\quad$ $\cos(\langle x, \hat{y}\rangle)$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{array}{cc} \hat{x} & \hat{y} \\ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{array} \begin{array}{cc} & \\ \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} & \begin{matrix} x \\ y \end{matrix} \end{array}$$

$\cos(\langle y, \hat{x}\rangle)$ $\quad$ $\cos(\langle y, \hat{y}\rangle)$

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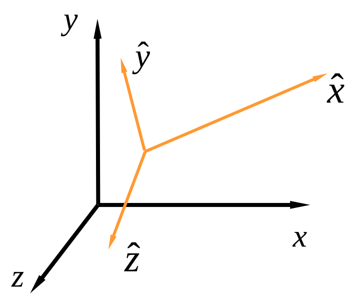
Truss Element -3D

Coordinate transformation matrix



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{1x} \end{Bmatrix} = \begin{array}{ccc} x & y & z \\ \begin{bmatrix} \cos(\langle \hat{x}, x\rangle) & \cos(\langle \hat{x}, y\rangle) & \cos(\langle \hat{x}, z\rangle) \\ \cos(\langle \hat{y}, x\rangle) & \cos(\langle \hat{y}, y\rangle) & \cos(\langle \hat{y}, z\rangle) \\ \cos(\langle \hat{z}, x\rangle) & \cos(\langle \hat{z}, y\rangle) & \cos(\langle \hat{z}, z\rangle) \end{bmatrix} \end{array} \begin{array}{cc} & \\ \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{1x} \end{Bmatrix} & \begin{matrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{matrix} \end{array}$$

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Truss Element -3D

$$[T'] = \begin{bmatrix} \cos(\langle \hat{x}, x \rangle) & \cos(\langle \hat{x}, y \rangle) & \cos(\langle \hat{x}, z \rangle) \\ \cos(\langle \hat{y}, x \rangle) & \cos(\langle \hat{y}, y \rangle) & \cos(\langle \hat{y}, z \rangle) \\ \cos(\langle \hat{z}, x \rangle) & \cos(\langle \hat{z}, y \rangle) & \cos(\langle \hat{z}, z \rangle) \end{bmatrix}$$

$$[T] = \begin{bmatrix} [T'] & 0 \\ 0 & [T'] \end{bmatrix} \qquad [\hat{K}]^e = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]^e = [T^T]\left[\hat{K}\right]^e [T]$$

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Summary of Lecture 3 and 4

So far, we learned the following important concepts:

- Element stiffness matrix
- Shape function

Element specific

- Rotation of coordinate systems

Not element specific

- Global stiffness matrix
- Features of global stiffness matrix
- Physical meaning of elements in global stiffness matrix
- Boundary conditions

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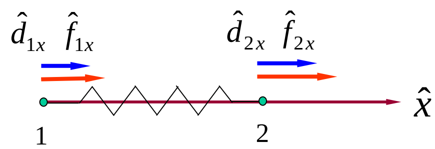
Summary of Lecture 3 and 4

Element stiffness matrix

- It represents the deformation (displacement) response of a node in an element to an nodal force.
- At element level.



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Summary of Lecture 3 and 4

Element stiffness matrix

Shape function:

- A shape function describes the shape of displacement field (not the shape of an element).
- It also describes the contribution of displacement of a node to the displacement field of an element.

$d = a_0 + a_1 x + a_2 x^2 + \ldots$ a_0=?, a_1=?, a_2=?...

Using conditions of $d=d^{(1)}$, $d=d^{(2)}$, $d=d^{(3)}$,…

$\Rightarrow$ $a_0=a_0(d^{(1)},d^{(2)}, d^{(3)},..)$ $a_1=a_1(d^{(1)},d^{(2)},d^{(3)},..)$

$$d = a_0\left(d^{(1)}, d^{(2)}, d^{(3)},\ldots\right) + a_1\left(d^{(1)}, d^{(2)}, d^{(3)},\ldots\right)x + a_2\left(d^{(1)}, d^{(2)}, d^{(3)},\ldots\right)x^2 + \ldots$$

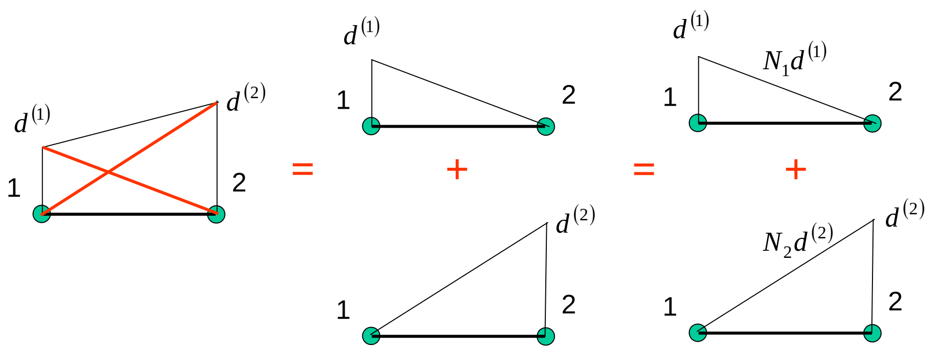
$\Rightarrow$ $d = N_1 d^{(1)} + N_2 d^{(2)} + N_3 d^{(3)} + \ldots$

Summary of Lecture 3 and 4

Element stiffness matrix

Shape function:

$$d = N_1 d^{(1)} + N_2 d^{(2)} + N_3 d^{(3)} + \ldots$$



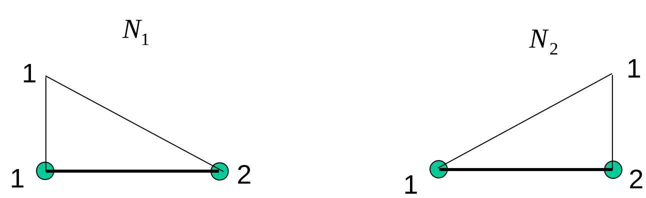
N_1=1, at the coordinate of node 1, N_1=0, at the coordinate of node other than 1.

N_2=1, at the coordinate of node 2, N_2=0, at the coordinate of node other than 2. 29

Summary of Lecture 3 and 4

Element stiffness matrix

Shape function:



Such a feature of shape function is always true. We will use this feature to build a shape function later.

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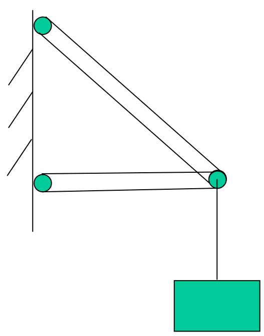
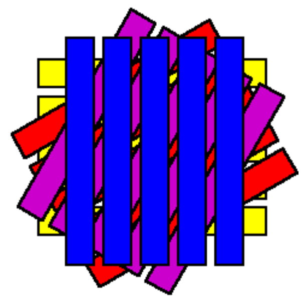
Summary of Lecture 3 and 4

Rotation of Coordinate Systems

The coordinate system used at element level might be different from the coordinate system used at global level.



Truss Structure



Top View of Rotated Fiber Layers

Composite Materials

Summary of Lecture 3 and 4

Rotation of Coordinate Systems

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

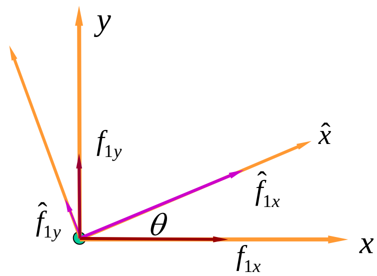
$$\Rightarrow \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

$$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$$[K] = [T^T]\left[\hat{K}\right]^e [T]$$

Summary of Lecture 3 and 4

Rotation of Coordinate Systems



$\cos(\langle \hat{x}, x\rangle)$ $\cos(\langle \hat{x}, y\rangle)$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{array} \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} \begin{array}{c} \hat{x} \\ \hat{y} \end{array}$$

$\cos(\langle \hat{y}, x\rangle)$ $\cos(\langle \hat{y}, y\rangle)$

$\cos(\langle x, \hat{x}\rangle)$ $\cos(\langle x, \hat{y}\rangle)$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{array}{c} \begin{array}{cc} \hat{x} & \hat{y} \end{array} \\ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{array} \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} \begin{array}{c} x \\ y \end{array}$$

$\cos(\langle y, \hat{x}\rangle)$ $\cos(\langle y, \hat{y}\rangle)$

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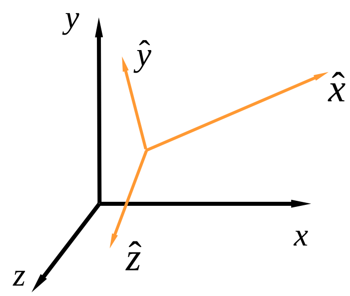
Summary of Lecture 3 and 4

Rotation of Coordinate Systems



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{1x} \end{Bmatrix} = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} \cos(\langle \hat{x}, x\rangle) & \cos(\langle \hat{x}, y\rangle) & \cos(\langle \hat{x}, z\rangle) \\ \cos(\langle \hat{y}, x\rangle) & \cos(\langle \hat{y}, y\rangle) & \cos(\langle \hat{y}, z\rangle) \\ \cos(\langle \hat{z}, x\rangle) & \cos(\langle \hat{z}, y\rangle) & \cos(\langle \hat{z}, z\rangle) \end{bmatrix} \end{array} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{1x} \end{Bmatrix} \begin{array}{c} \hat{x} \\ \hat{y} \\ \hat{z} \end{array}$$

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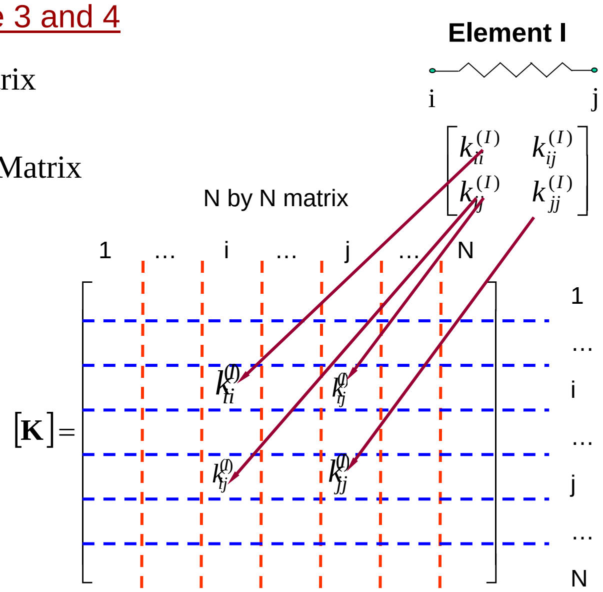
Summary of Lecture 3 and 4

Global Stiffness Matrix

Element Stiffness Matrix

Direct Stiffness Method

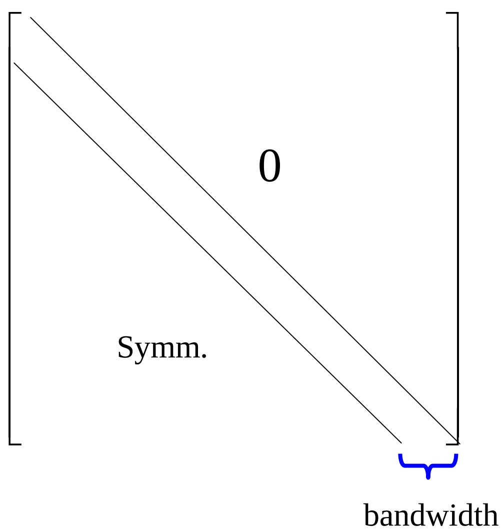
Boarding Airplane



Global Stiffness Matrix

Summary of Lecture 3 and 4

Global Stiffness Matrix



A large, symmetric, sparse matrix.

Memory requirement depends on the total DOF and bandwidth.

- Total DOF depends on mesh and type of elements
- Bandwidth depends how nodes are numbered.

Summary of Lecture 3 and 4

Global Stiffness Matrix

Physical Meaning

K_{ij} is equal to the reaction "force" on the *i*-th DOF due to a unit "displacement" on the *j*-th DOF whereas all the other DOFs are fixed.

- Here, "force" can be a force, or a moment; "displacement" can be a displacement, or a rotation.
- i-th DOF does NOT have to be in the same direction as j-th DOF.

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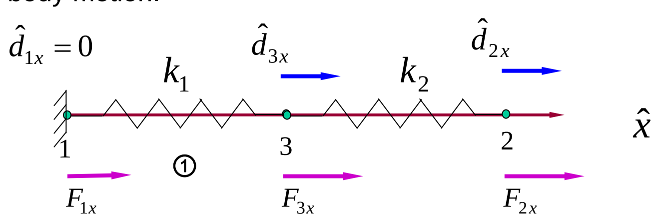
Summary of Lecture 3 and 4

Boundary Conditions

For static analysis, boundary conditions should at least get rid of rigid body motion.



$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} \cancel{k_1} & \cancel{0} & \cancel{-k_1} \\ \cancel{0} & k_2 & -k_2 \\ \cancel{-k_1} & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ \hat{d}_{2x} \\ \hat{d}_{3x} \end{Bmatrix}$$

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