3. Exercise

Foundations of Robotics

Task 1
Workspace

Consider the planar robot shown in Figure 1 with lengths l_1 and l_2. The first link is rotated by the angle q_1 via a revolute joint, the second link is adjusted to the length q_2 via a prismatic joint.

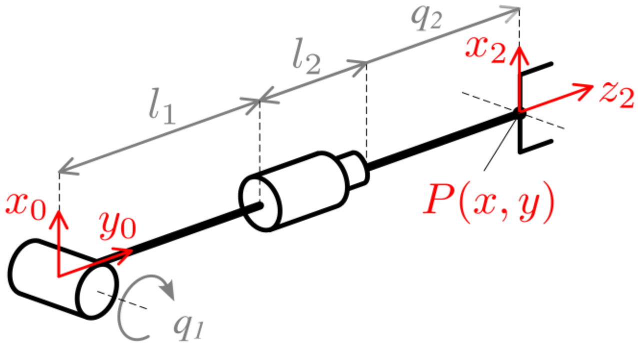
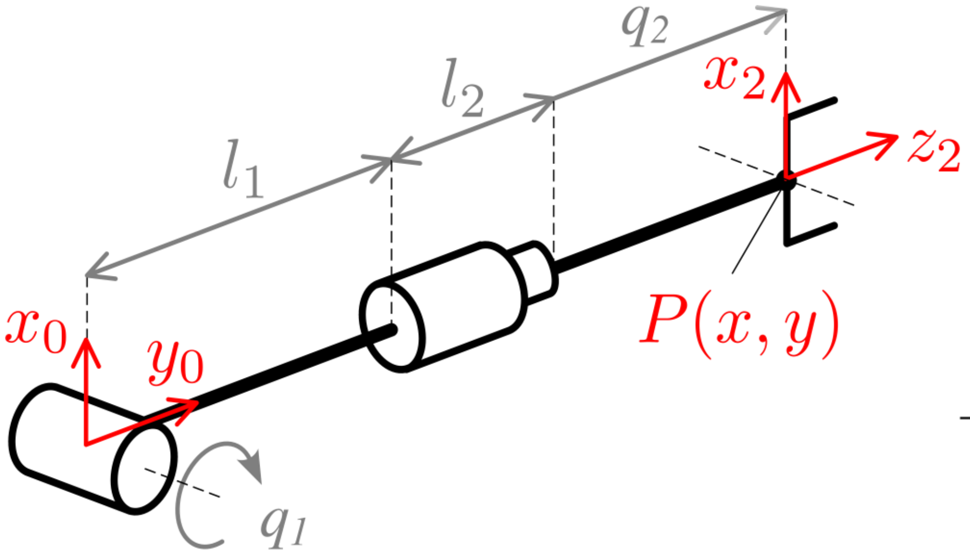


Abbildung 1: Arm with one revolute and one prismatic joint

Task 1

Workspace

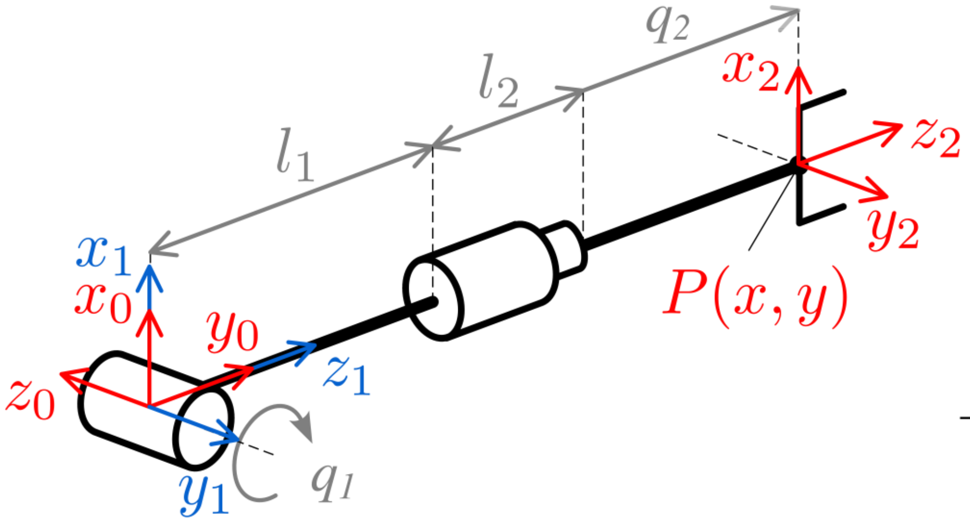
a) What are the DH parameters of the depicted robot arm? Provide them in tabular form.



i	θ_i	d_i	a_i	α_i
1	q_1	0	0	$-\frac{\pi}{2}$
2	0	$q_2 + l_1 + l_2$	0	0

Task 1
Workspace

a) What are the DH parameters of the depicted robot arm? Provide them in tabular form.



i	θ_i	d_i	a_i	α_i
1	q_1	0	0	$-\frac{\pi}{2}$
2	0	$q_2 + l_1 + l_2$	0	0

Task 1

Workspace

b) Now determine the forward kinematics model ${}^0\mathbf{T}_2$ from the DH parameters. Also provide ${}^0\mathbf{T}_1$ and ${}^1\mathbf{T}_2$.

$$ {}^0\mathbf{T}_1 = \mathrm{Rot}(z; \underbrace{\theta_1}_{q_1}) \cdot \underbrace{\mathrm{Trans}(0,0,\underbrace{d_1}_{0})}_{E_4} \cdot \underbrace{\mathrm{Trans}(\underbrace{a_1}_{0},0,0)}_{E_4} \cdot \mathrm{Rot}(x; \underbrace{\alpha_1}_{-\frac{\pi}{2}}) $$

$$ = \left(\begin{array}{ccc|c} \cos(q_1) & 0 & -\sin(q_1) & 0 \\ \sin(q_1) & 0 & \cos(q_1) & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) $$

$$ {}^1\mathbf{T}_2 = \underbrace{\mathrm{Rot}(z; \underbrace{\theta_2}_{0})}_{E_4} \cdot \mathrm{Trans}(0,0,\underbrace{d_2}_{q_2+l_1+l_2}) \cdot \underbrace{\mathrm{Trans}(\underbrace{a_1}_{0},0,0)}_{E_4} \cdot \underbrace{\mathrm{Rot}(x; \underbrace{\alpha_1}_{0})}_{E_4} $$

$$ = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_2 + l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) $$

$$ \Rightarrow {}^0\mathbf{T}_2 = {}^0\mathbf{T}_1 {}^1\mathbf{T}_2 $$

$$ = \begin{pmatrix} \cos(q_1) & 0 & -\sin(q_1) & -(q_2 + l_1 + l_2)\sin(q_1) \\ \sin(q_1) & 0 & \cos(q_1) & (q_2 + l_1 + l_2)\cos(q_1) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

i	θ_i	d_i	a_i	α_i
1	q_1	0	0	$-\frac{\pi}{2}$
2	0	$q_2 + l_1 + l_2$	0	0

Task 1

Workspace

c) Determine analytically the inverse kinematics solution for the robot using ${}^0\mathbf{T}_2$. Provide the functions $q_1(x_p, y_p)$ and $q_2(x_p, y_p)$. Note that the dependence should be exclusively on x_p and y_p. You may assume the lengths l_1 and l_2 as given.
Hint: $\sin(x)^2 + \cos(x)^2 = 1$.

$$\text{I:} \quad x_p = -(q_2 + l_1 + l_2)\sin(q_1)$$
$$\text{II:} \quad y_p = (q_2 + l_1 + l_2)\cos(q_1)$$

$$\frac{\text{I}}{\text{II}}: \quad \frac{x_p}{y_p} = -\frac{(q_2 + l_1 + l_2)\sin(q_1)}{(q_2 + l_1 + l_2)\cos(q_1)} = -\tan(q_1) \Rightarrow q_1(x_p, y_p) = -\operatorname{atan2}(x_p, y_p)$$

$$(\text{-I})^2 + \text{II}^2: \quad (-x_p)^2 + y_p^2 = (q_2 + l_1 + l_2)^2 \underbrace{(\sin^2(q_1) + \cos^2(q_1))}_{=1}$$

$$\Rightarrow q_2(x_p, y_p) = \pm\sqrt{(-x_p)^2 + y_p^2} - l_1 - l_2$$

Verification necessary to determine which solution is valid, as only one solution exists!

Task 1

Workspace

d) Based on the result of the inverse kinematics, provide criteria for point P that must be satisfied for P to lie in the workspace and for a solution of the inverse kinematics to exist. You may assume joint limits $q_1 \in [q_{1,min}, q_{1,max}]$ and $q_2 \in [0, q_{2,max}]$ in general and presuppose $l_1, l_2 > 0$.

The resulting angle for q_1 must lie within the joint angle limits:

$$q_{1,min} \leq -\text{atan2}(x_p, y_p) \leq q_{1,max}$$

Point P must lie within the annulus:

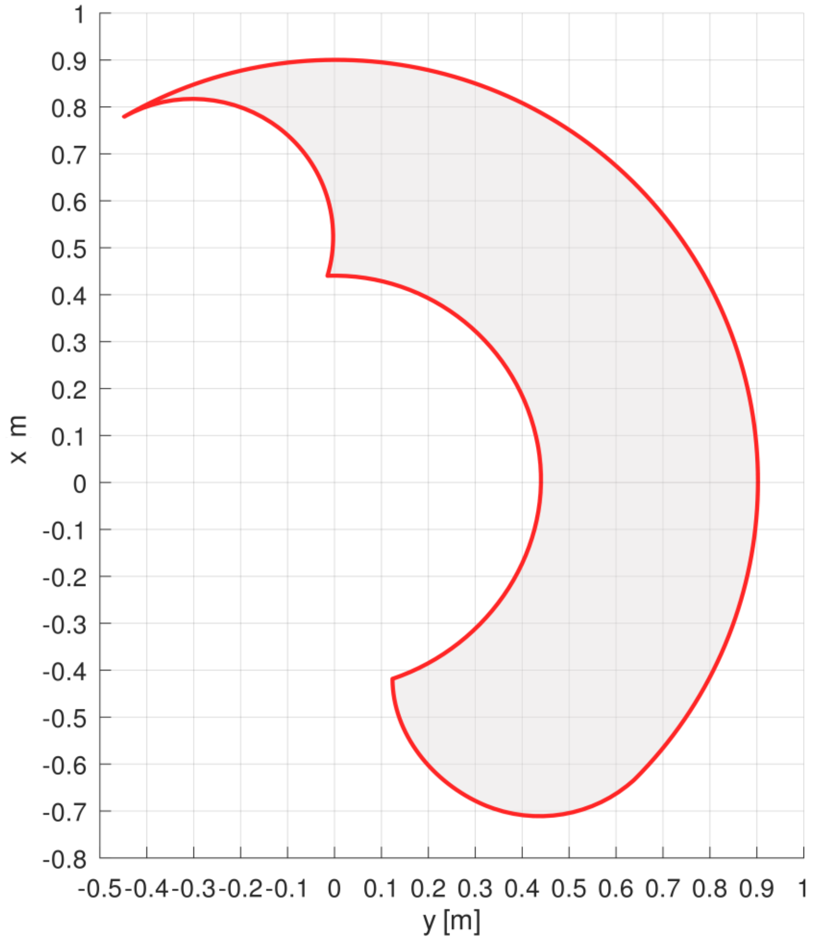
$$l_1 + l_2 \leq +\sqrt{x_p^2 + y_p^2} \leq l_1 + l_2 + q_{2,max}$$

Task 1
Workspace

e) Draw schematically a robot arm matching the workspace shown in Figure 2 with as few revolute and prismatic joints as possible. The origin of the coordinate system S_0 is located in the plot at position $(0, 0)$. Also provide the link lengths.

$$l_1 = 0.6\,\mathrm{m}$$
$$l_2 = 0.3\,\mathrm{m}$$

$$q_1 \in [-30°, 135°]$$
$$q_2 \in [0°, 135°]$$

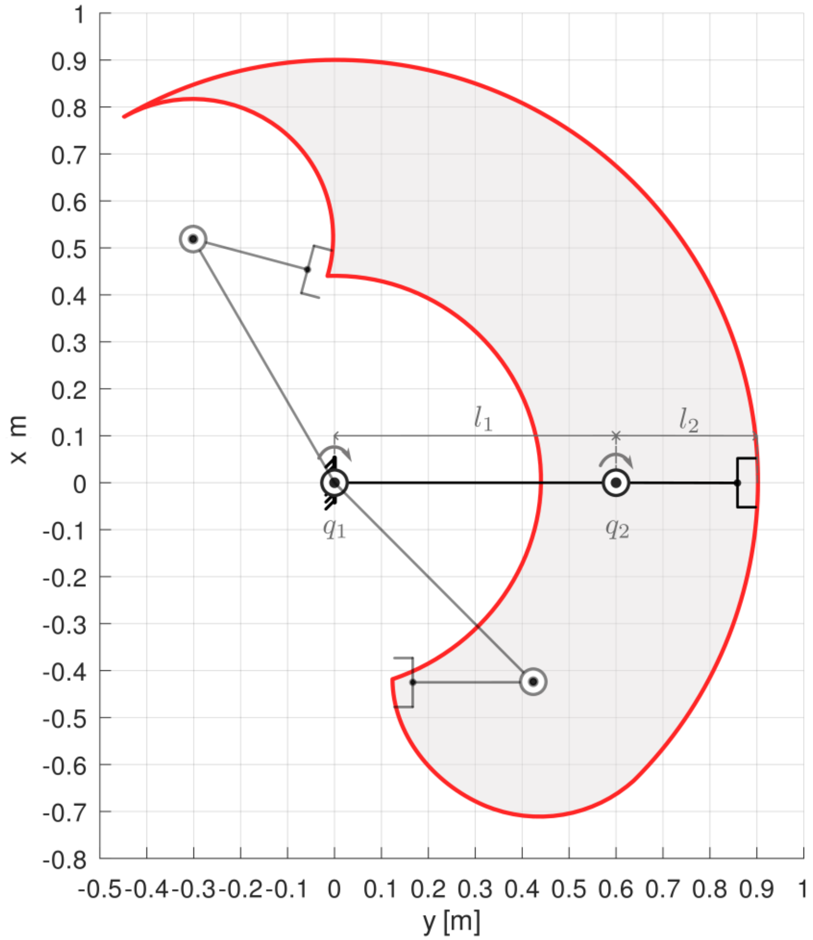


Task 1
Workspace

e) Zeichnen Sie schematisch einen zum in Abbildung 2 dargestellten Arbeitsbereich passenden Roboterarm mit so wenigen Dreh- und Schubgelenken wie möglich. Der Ursprung des Koordinatensystems S_0 befindet sich im Plot an Stelle $(0, 0)$. Geben Sie zudem die Gliedlängen an.

$$l_1 = 0.6\,\text{m}$$
$$l_2 = 0.3\,\text{m}$$

$$q_1 \in [-30°, 135°]$$
$$q_2 \in [0°, 135°]$$



Task 2

Inverse Kinematics of a 3-DOF-Manipulator

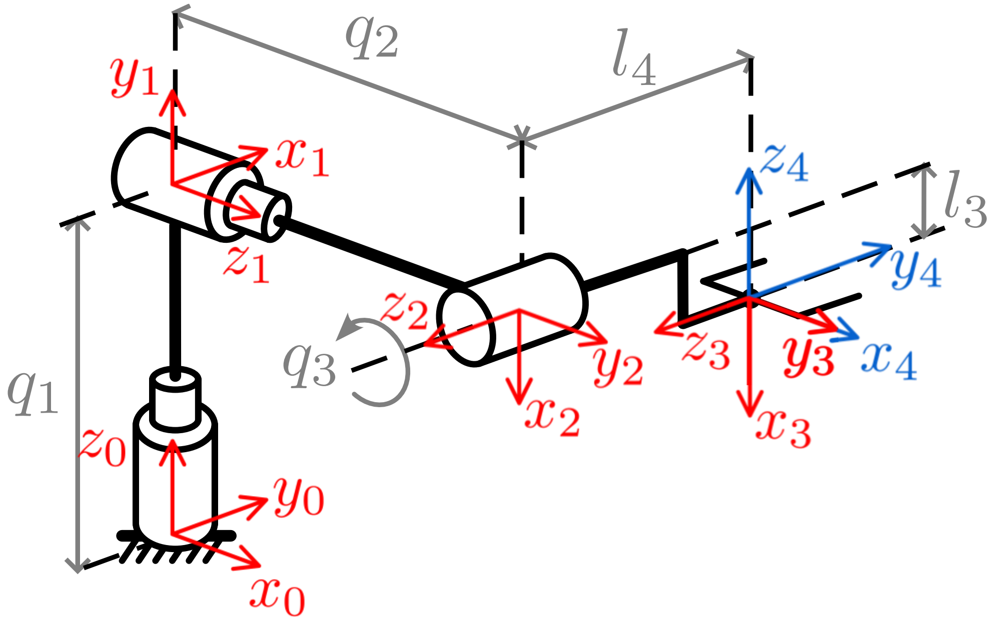
A manipulator with three joints is described by the DH parameters

i	θ_i	d_i	a_i	α_i
1	$\frac{\pi}{2}$	q_1	0	$\frac{\pi}{2}$
2	$-\frac{\pi}{2}$	q_2	0	$\frac{\pi}{2}$
3	q_3	$-l_4$	l_3	0
4	$\frac{\pi}{2}$	0	0	$-\frac{\pi}{2}$

where the first and second joints q_1, q_2 are prismatic joints and the third joint q_3 is a revolute joint. The lengths l_3 and l_4 are given by the mechanics of the manipulator and are $l_3 = 0.1\,\mathrm{m}$ and $l_4 = 0.35\,\mathrm{m}$. The fourth row of the table is only required to orient the coordinate system of the end-effector in the desired orientation and does not contain a joint variable.

Task 2

Inverse Kinematics of a 3-DOF-Manipulator



- q_1, q_2 prismatic joints
- q_3 revolute joint

i	θ_i	d_i	a_i	α_i
1	$\frac{\pi}{2}$	q_1	0	$\frac{\pi}{2}$
2	$-\frac{\pi}{2}$	q_2	0	$\frac{\pi}{2}$
3	q_3	$-l_4$	l_3	0
4	$\frac{\pi}{2}$	0	0	$-\frac{\pi}{2}$

Task 2

Inverse Kinematics of a 3-DOF-Manipulator

b) Specify the transformation ${}^0\mathbf{T}_4$ that describes the pose of the end-effector coordinate system relative to the base coordinate system. Also specify all transformations ${}^{i-1}\mathbf{T}_i$.

$$
{}^0\mathbf{T}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
{}^1\mathbf{T}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
{}^2\mathbf{T}_3 = \begin{pmatrix} \cos(q_3) & -\sin(q_3) & 0 & l_3\cos(q_3) \\ \sin(q_3) & \cos(q_3) & 0 & l_3\sin(q_3) \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\mathbf{T}_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad
\begin{aligned}
{}^0\mathbf{T}_4 &= {}^0\mathbf{T}_1\,{}^1\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4 \\
&= \begin{pmatrix} \cos(q_3) & 0 & -\sin(q_3) & l_3\sin(q_3) + q_2 \\ 0 & 1 & 0 & l_4 \\ \sin(q_3) & 0 & \cos(q_3) & -l_3\cos(q_3) + q_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Task 2

Inverse Kinematics of a 3-DOF-Manipulator

c) Specify all possible tuples of joint positions for which the end-effector coordinate system S_4 has its origin at the point $\begin{pmatrix}0.5\,\text{m} & 0.35\,\text{m} & 1.2\,\text{m}\end{pmatrix}^T$ with respect to S_0 and is rotated in orientation by $-\frac{\pi}{2}$ rad relative to S_0 about the y_0-axis, i.e., ${}^0\mathbf{R}_4 = \text{rot}(y; -\frac{\pi}{2}\text{ rad})$.

$$
{}^0\mathbf{T}_4 = \begin{pmatrix} \boxed{\begin{matrix}\cos(q_3) & 0 & -\sin(q_3) \\ 0 & 1 & 0 \\ \sin(q_3) & 0 & \cos(q_3)\end{matrix}} & \begin{matrix}\boxed{l_3 \sin(q_3) + q_2} \\ l_4 \\ \boxed{-l_3 \cos(q_3) + q_1}\end{matrix} \\ \begin{matrix}0 & 0 & 0\end{matrix} & 1 \end{pmatrix}
$$

- From ${}^0\mathbf{R}_4 = \text{rot}(y; -q_3)$ follows $q_3 = \frac{\pi}{2}$
- With the now known angle q_3, the lengths q_1 and q_2 can be determined. We obtain

$$r_x = l_3 \sin(q_3) + q_2 \qquad (1)$$
$$r_z = -l_3 \cos(q_3) + q_1 \qquad (2)$$

- By substituting $r_x = 0.5\,\text{m}$ and $r_z = 1.2\,\text{m}$, the only possible set of parameters is $\begin{pmatrix}q_1 & q_2 & q_3\end{pmatrix}^\text{T} = \begin{pmatrix}1.2\,\text{m} & 0.4\,\text{m} & \frac{\pi}{2}\,\text{rad}\end{pmatrix}^\text{T}$

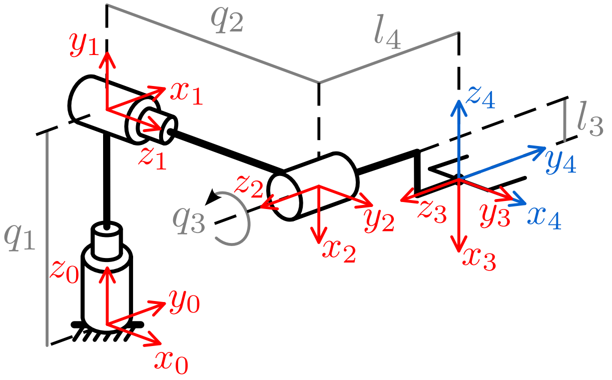
Task 2

Inverse Kinematics of a 3-DOF-Manipulator

d) What are the possible tuples of joint positions if only the origin of the end-effector coordinate system is fixed at $\begin{pmatrix} 0.5\,\mathrm{m} & 0.35\,\mathrm{m} & 1.2\,\mathrm{m} \end{pmatrix}^T$ with respect to S_0, but the orientation is arbitrary? No joint angle limits need to be considered.

- Orientation arbitrary $\Rightarrow$ q_3 can be chosen freely
- For a given angle q_3, the lengths q_1 and q_2 can be calculated as in the previous task with

$$q_1 = r_z + l_3 \cos(q_3)$$
$$q_2 = r_x - l_3 \sin(q_3)$$



- By substituting $r_x = 0.5\,\mathrm{m}$ and $r_z = 1.2\,\mathrm{m}$, all possible parameter tuples are thus $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^\mathrm{T} = \begin{pmatrix} 1.2 + 0.1\cos(q_3)\,\mathrm{m} & 0.5 - 0.1\sin(q_3)\,\mathrm{m} & q_3 \end{pmatrix}^\mathrm{T}$

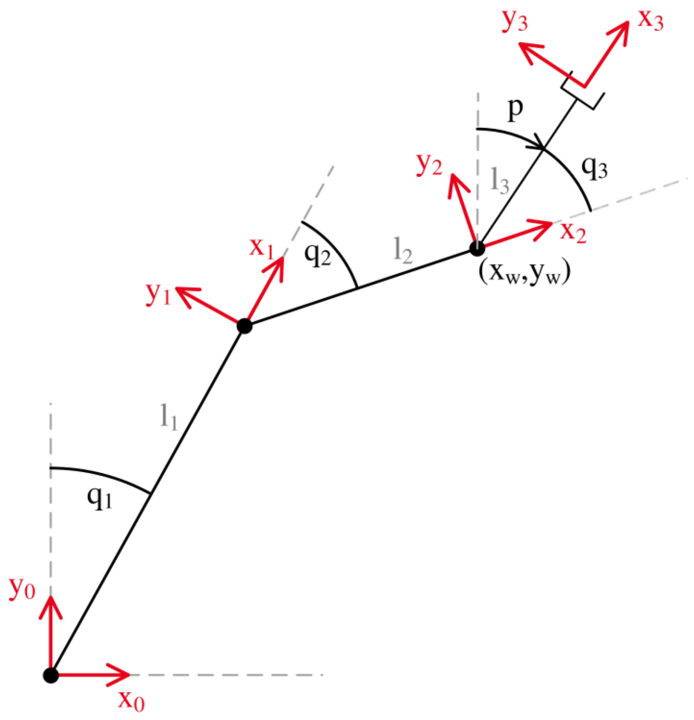
Task 3

Inverse Kinematics of a 3-DOF-SCARA Manipulator

- Element lengths l_1, l_2, l_3
- Joint angles q_1, q_2, q_3
- Positive rotation direction according to z_i-axis
- $p = \angle(y_0, 3.\text{ element})$

1. Determine the joint parameters q_1 and q_2 as a function of the position of the third revolute joint (x_w, y_w). Provide the solution for both possible elbow positions.

2. The joint angle q_3 can be calculated using the computed joint angles q_1 and q_2 as well as the given pitch p.



Task 3

Inverse Kinematics of a 3-DOF-SCARA Manipulator

- From $q_2 = \pi - \gamma$ and the properties of the cosine function it follows

$$\cos(q_2) = \cos(\pi - \gamma) = -\cos(-\gamma) = -\cos(\gamma) \qquad (1)$$

- Using $c^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos(\gamma)$ and (1) one obtains

$$-\cos(\gamma) = \left(\frac{c^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right) \Rightarrow \cos(q_2) = \left(\frac{c^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

- From $\sin(x)^2 + \cos(x)^2 = 1$ follows

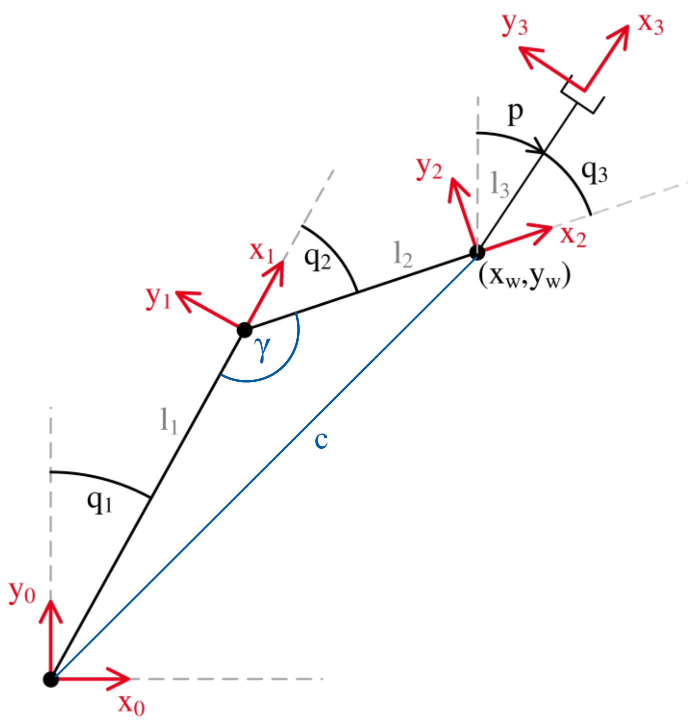
elbow ”down”

$$\sin(q_2) = \pm\sqrt{1 - \cos(q_2)^2}$$

elbow ”up”

- Finally

$$\Rightarrow q_2 = \mathrm{atan2}(\sin(q_2), \cos(q_2))$$



Task 3

Inverse Kinematics of a 3-DOF-SCARA Manipulator

- By constructing a right-angled triangle, β can be determined

$$\beta = \operatorname{atan2}(l_2 \sin(q_2), l_1 + l_2 \cos(q_2))$$

- With $\phi = \operatorname{atan2}(y_w, x_w)$ one obtains

$$\Rightarrow q_1 = \phi - \beta - \frac{\pi}{2}$$

- By using the relation (note positive rotation direction)

$$-p = q_1 + q_2 + q_3 \; ,$$

  the angle of joint 3 can be determined

$$\Rightarrow q_3 = -p - q_1 - q_2$$