

The Gas Station Problem

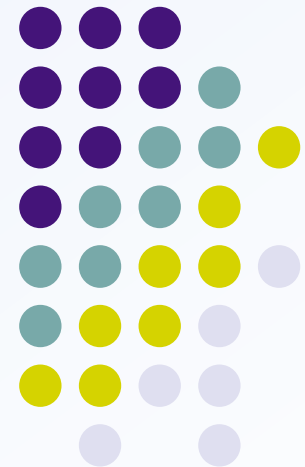
Samir Khuller

Azarakhsh Malekian

Julian Mestre

UMD

UMD \rightarrow MPI





General Description

- Suppose you want to go on a road trip across the US. You start from New York City and would like to drive to San Francisco.
- You have :
 - roadmap
 - gas station locations and their gas prices
- Want to:
 - minimize travel cost (gas expenses)

Finding gas prices online



AAA Fuel Price Finder

http://aaa.opisnet.com/index.aspx?212_D0190D88A0I

FUEL PRICE FINDER

Latest Fuel Prices At More Than 85,000 Stations Nationwide

Home Methodology Feedback Fuel Cost For A Trip AAA Maps AAA

Search Criteria

Search Location: State: Or Zip Code: 20740

Search Area: ☐ 3 mile ☐ 5 mile ☐ 10 mile

Prices in a 10 Square Mile Area Around Zip Code 20740

Self-Serve Regular Price*:

Low: \$2.759
Average: \$2.908
High: \$3.169

Average Cost To Fuel A Vehicle With a 15 Gallon Tank:

Current: \$43.83
Month Ago: \$46.96
Year Ago: \$40.38

For more detailed fuel cost by auto for trips visit the [Fuel Cost Calculator](#) website.

For information on using public transit to reduce fuel use, click [here](#).

AAA Fuel Price Finder

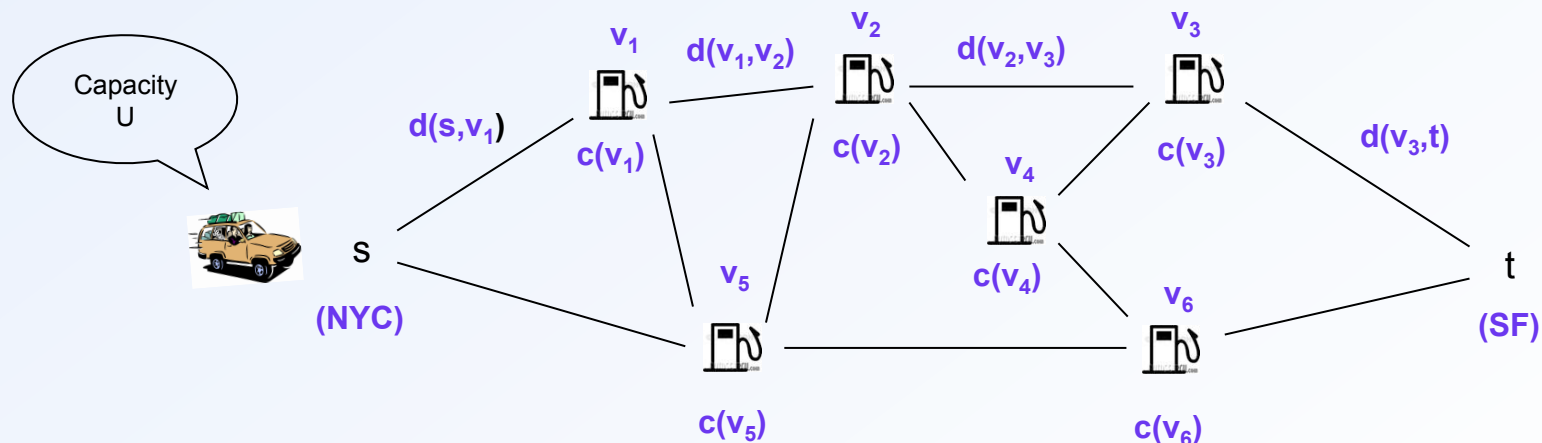
opisnet.com/index.aspx?212_D0190D88A0I

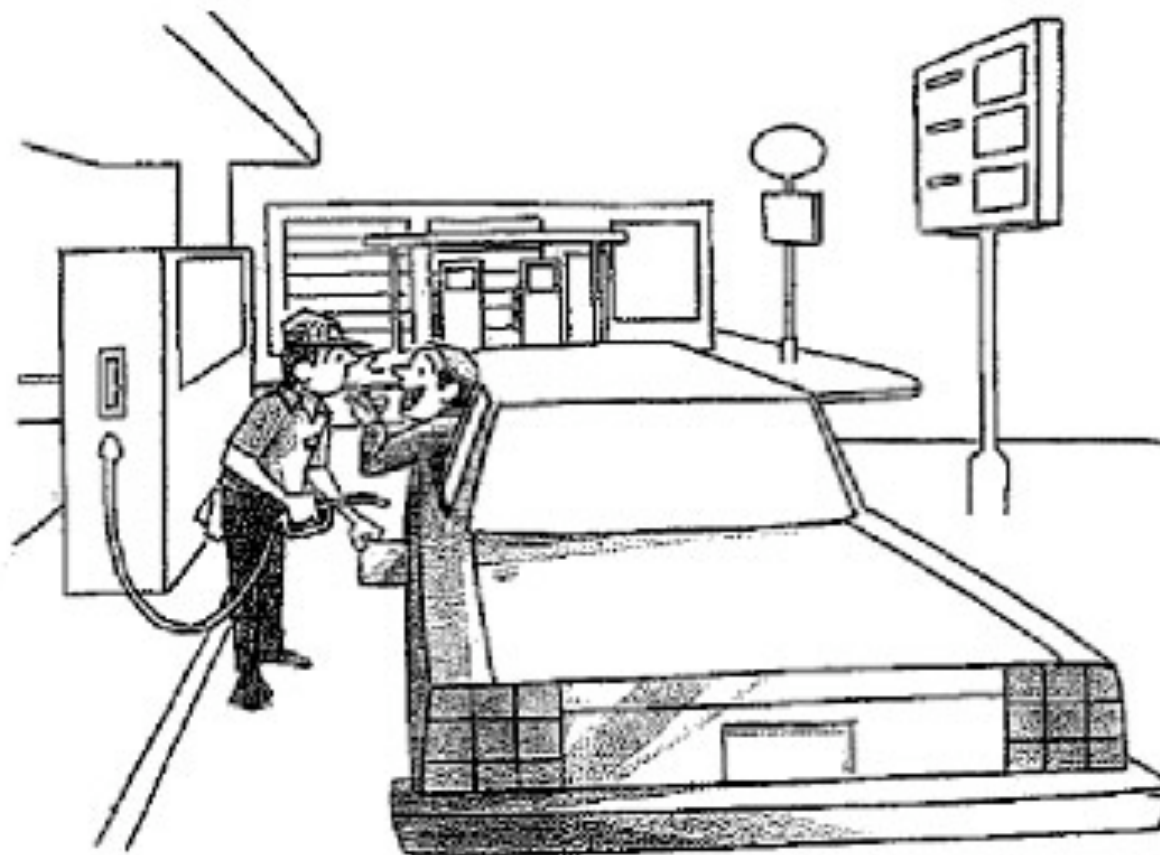
	Zip	Reg	Mid	Pre	Dsl
HAMPSHIRE AVE	20912	\$2.76	\$2.96		
rk, MD	20912	8/26/2006	8/25/2006		
HAMPSHIRE AVE	20912	\$2.76	\$2.96		\$3.06
rk, MD	20912	8/28/2006	8/27/2006		8/25/2006
DALE RD	20737	\$2.80	\$3.00		
MD	20737	8/28/2006	8/28/2006		
ERSITY BLVD E	20783	\$2.80			
MD	20783	8/26/2006			
POLIS RD	20784	\$2.82	\$2.98		
MD	20784	8/27/2006	8/25/2006		
S RD	20783	\$2.82	\$2.96		
MD	20783	8/27/2006	8/25/2006		
HAMPSHIRE AVE	20912	\$2.82		\$3.04	
rk, MD	20912	8/26/2006		8/25/2006	
POLIS RD	20706	\$2.83			\$3.10
D	20706	8/28/2006			8/28/2006
S RD	20783	\$2.83			
MD	20783	8/27/2006			
POLIS RD	20706	\$2.84	\$3.00		\$3.07
D	20706	8/26/2006	8/25/2006		8/26/2006
POLIS RD	20706	\$2.84			
D	20706	8/28/2006			
BRANCH RD	20903	\$2.84	\$3.18	\$3.28	
g, MD	20903	8/28/2006	8/28/2006	8/28/2006	
BRANCH RD	20910	\$2.84			
g, MD	20910	8/27/2006			
POLIS RD	20706	\$2.85			
D	20706	8/27/2006			
POLIS RD	20710	\$2.85			
g, MD	20710	8/28/2006			
S RD	20783	\$2.85			
D	20783	8/25/2006			
CHILLIUM SHELL	20901	\$2.86	\$3.04	\$3.16	
7-Eleven	20901	8/28/2006	8/28/2006	8/28/2006	
7-ELEVEN (STORE 23696)	20781	\$2.86	\$2.97		\$2.93
Citgo	20781	8/26/2006	8/25/2006		8/26/2006
KENILWORTH CAR WASH	20784	\$2.86	\$3.00		
Sunoco	20784	8/26/2006	8/25/2006		
HMA INC	20783	\$2.86			
Bp	20783	8/26/2006			
RIGGS RD BP	20783	\$2.86			
Hyattsville, MD	20783	8/26/2006			



Information and Constraints

- Information:
 - map of the road: $G=(V, E)$
 - length of the roads: $d: E \rightarrow \mathbb{R}^+$
 - gas prices: $c: V \rightarrow \mathbb{R}^+$
- Constraint:
 - tank capacity: U
- Goal:
 - find a min. cost solution to go from s to t



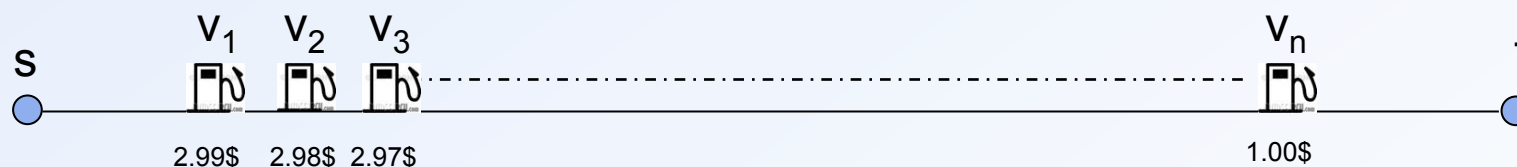


"Just enough to get me across the street
to the cheaper station."



Structure of the Optimal Solution

- Two vertices s & t
- A fixed path

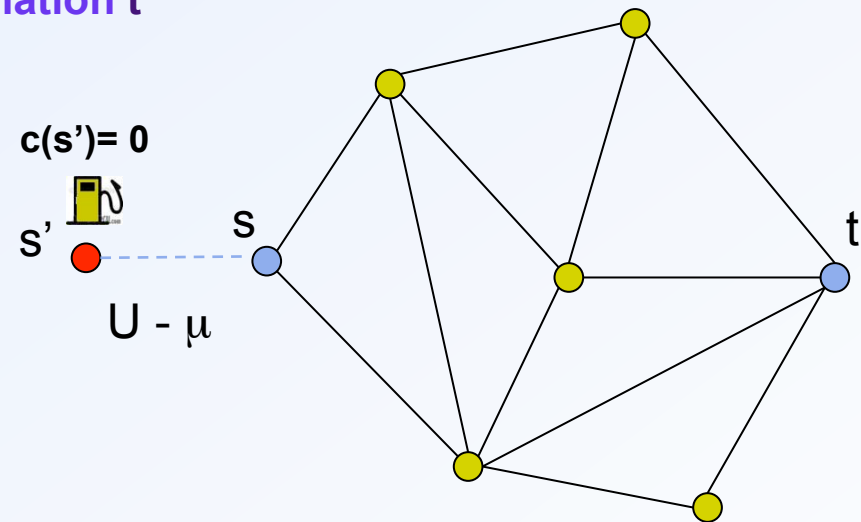


- Optimal solution involves stops at every station!
- Thus we permit at most Δ stops.

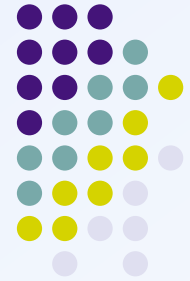
The Problem we want to solve



- Input:
 - Road map $G=(V,E)$, source s , destination t
 - U : tank capacity
 - $d: E \rightarrow \mathbb{R}^+$
 - $c: V \rightarrow \mathbb{R}^+$
 - Δ : No. of stops allowed
 - μ : The initial amount of gas at s
- Goal:
 - Minimize the cost to go from s to t .
- Output:
 - The chosen path
 - The stops
 - The amount of gas filled at each stop
- We can assume we start with 0 gas at s .



Dynamic Programming



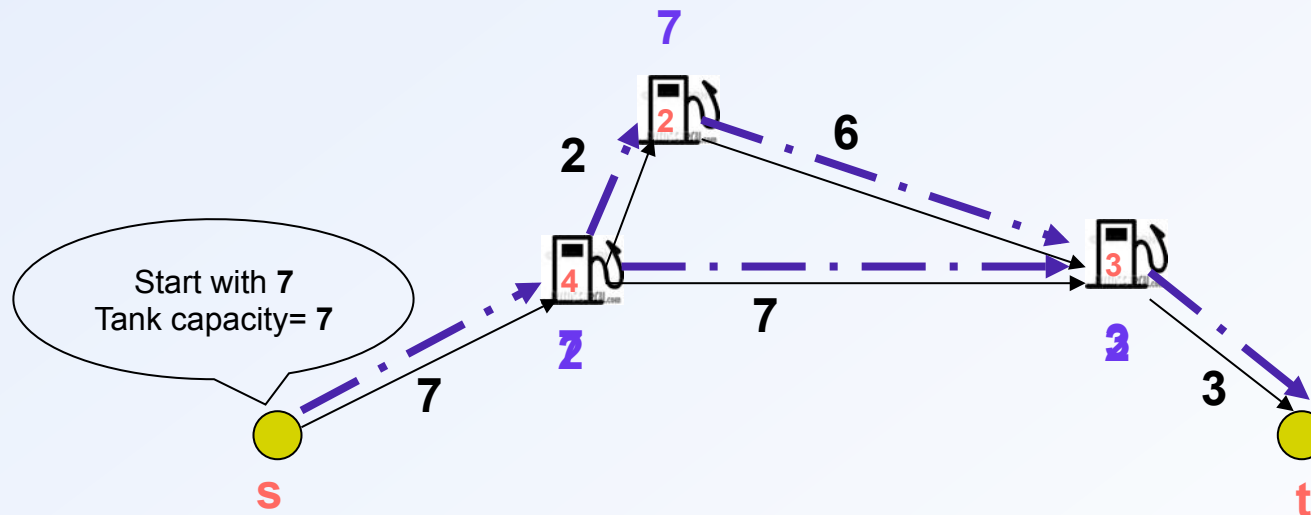
$\text{OPT}[\mathbf{x}, \mathbf{q}, \mathbf{g}] =$ Minimum cost to go from \mathbf{x} to \mathbf{t} in \mathbf{q} stops, starting with \mathbf{g} units of gas.

- Assuming all values are integral, we can find an optimal solution in $O(n^2 \Delta U^2)$ time.
- Not a strongly polynomial time algorithm.



Why not Shortest Path?

- **Shortest** path is of length **17**. $\text{Cost} = 37 = 4 \times 7 + 3 \times 3$
- **Cheapest** path is of length **18**. $\text{Cost} = 28 = 4 \times 2 + 2 \times 7 + 2 \times 3$

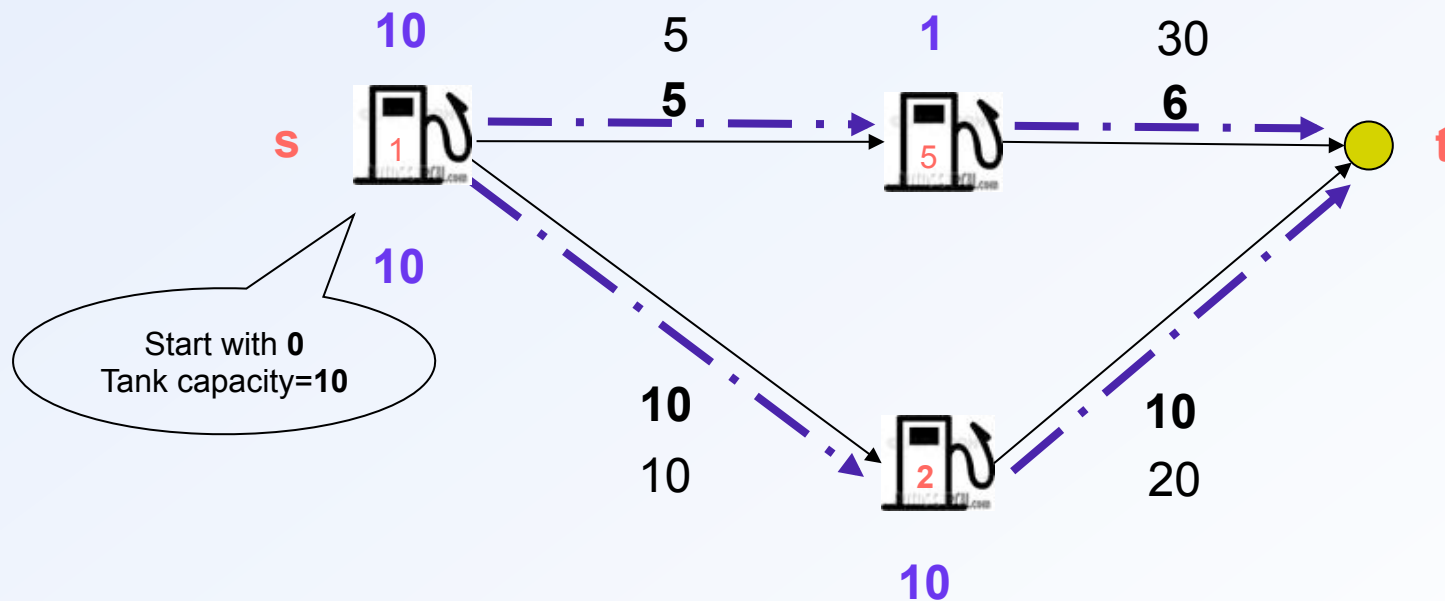




One more try at Shortest Path

Let the length of (u,v) be $d(u,v) \times c(u)$, if $d(u,v) \leq U$

- **Shortest** path has length 30. Cost = $1 \times 10 + 2 \times 10 = 30$
- **Cheapest** path has length 35. Cost = $1 \times 10 + 5 \times 1 = 15$

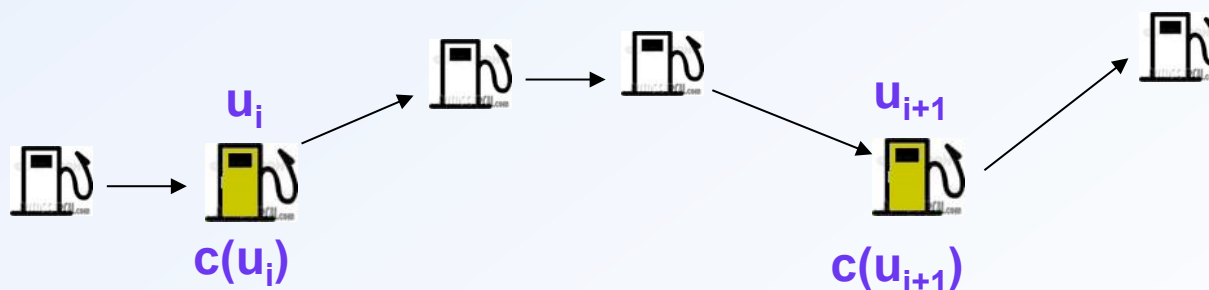




Key Property

Suppose the optimal sequence of stops is $u_1, u_2, \dots, u_{\Delta}$.

- If $c(u_i) < c(u_{i+1}) \Rightarrow$ Fill up the whole tank
- If $c(u_i) > c(u_{i+1}) \Rightarrow$ Just fill enough to reach u_{i+1} .

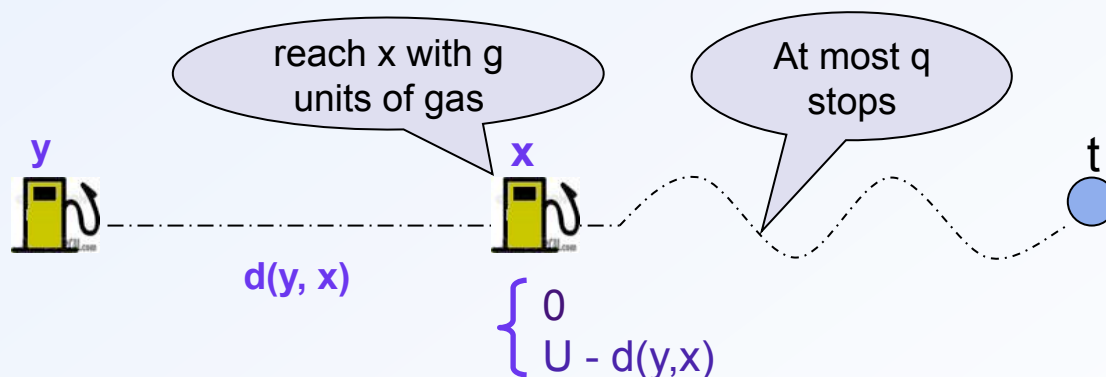




Solution

$\text{OPT}[\mathbf{x}, \mathbf{q}, \mathbf{g}] =$ Minimum cost to go from \mathbf{x} to \mathbf{t} in \mathbf{q} stops, starting with \mathbf{g} units of gas.

- Suppose the stop before x was y .
The amount of gas we reach x with is
$$\begin{cases} 0 & \text{if } c(x) < c(y) \\ U - d(y, x) & \text{if } c(x) \geq c(y) \end{cases}$$
- For each x we need to keep track of at most n different values of g .

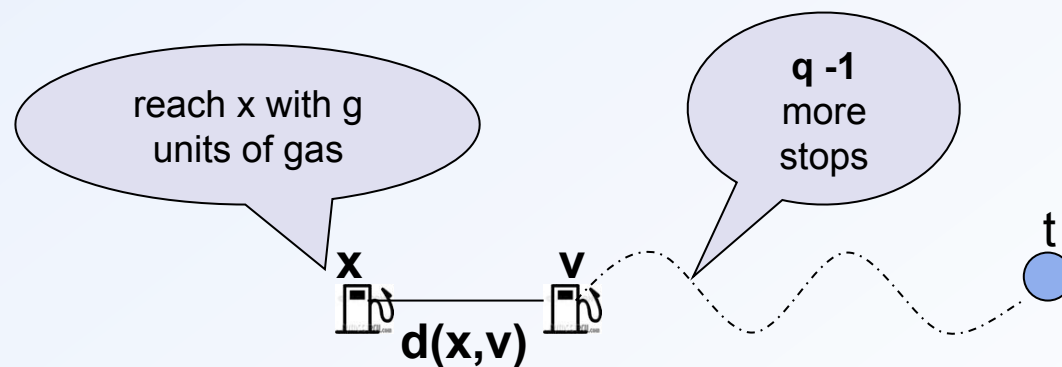




Dynamic Program

$\text{OPT}[\mathbf{x}, \mathbf{q}, \mathbf{g}] =$ Minimum cost to go from \mathbf{x} to \mathbf{t} in \mathbf{q} stops, starting with \mathbf{g} units of gas.

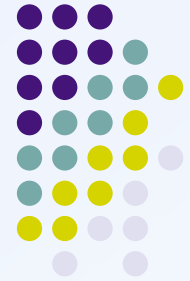
$$\min_v \begin{cases} \text{OPT}[\mathbf{v}, \mathbf{q}-1, 0] + (\mathbf{d}(\mathbf{x}, \mathbf{v}) - \mathbf{g}) \mathbf{c}(\mathbf{x}) & \text{if } \mathbf{c}(\mathbf{v}) \leq \mathbf{c}(\mathbf{x}) \text{ \& } \mathbf{d}(\mathbf{x}, \mathbf{v}) > \mathbf{g} \\ \text{OPT}[\mathbf{v}, \mathbf{q}-1, \mathbf{U} - \mathbf{d}(\mathbf{x}, \mathbf{v})] + (\mathbf{U} - \mathbf{g}) \mathbf{c}(\mathbf{x}) & \text{if } \mathbf{c}(\mathbf{v}) > \mathbf{c}(\mathbf{x}) \end{cases}$$





Problem Wrap Up

- The given DP table can be filled in:
 - $O(\Delta n^3)$ time by the direct and naïve solution
 - $O(\Delta n^2 \log n)$ time by a more intricate solution
- Faster algorithm using a different approach for the “all-pairs” version
- Faster algorithm for the fixed path with $\Delta=n$ with running time $O(n \log n)$



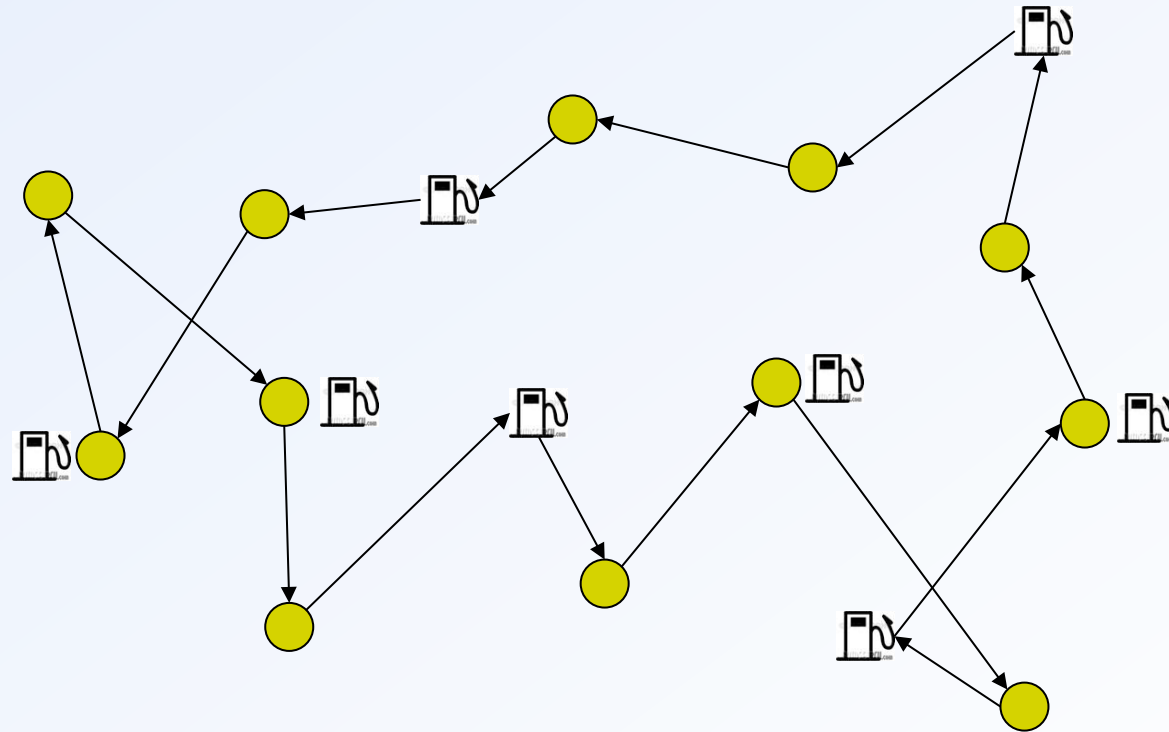
Tour Gas Station problem

- Would like to visit a set of cities **T**.
- We have access to set of gas stations **S**.
- Assume gas prices are uniform.
 - The problem is NP-hard even with this restriction.
 - Guess the range of prices the optimal solution uses, pay extra factor in approximation ratio.
 - Deals with gas companies.



Uniform Cost Tour Gas Station

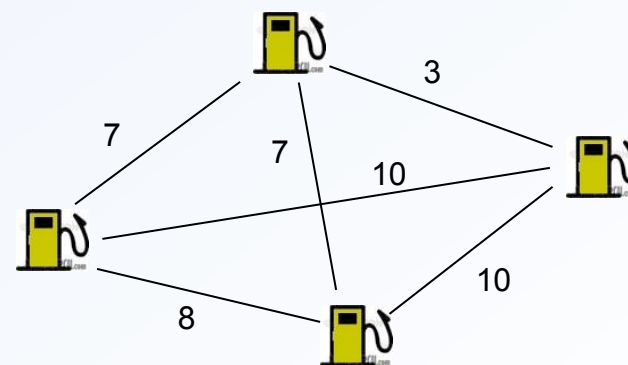
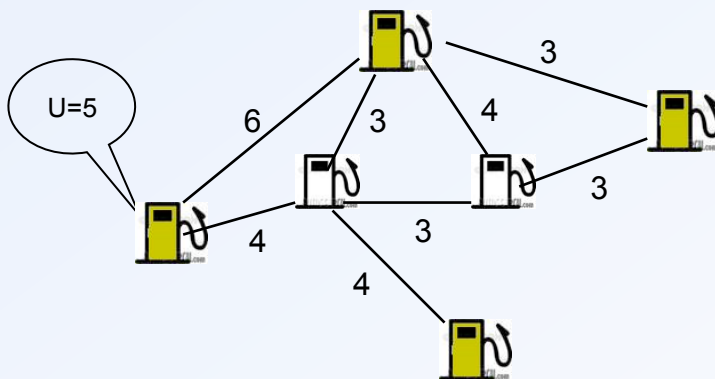
- There is a set **S** of **gas stations** and a set **T** of **cities**.
- Want to visit the cities with **min cost**.
- Gas prices are the **same**.





Uniform Cost Tour Gas Station

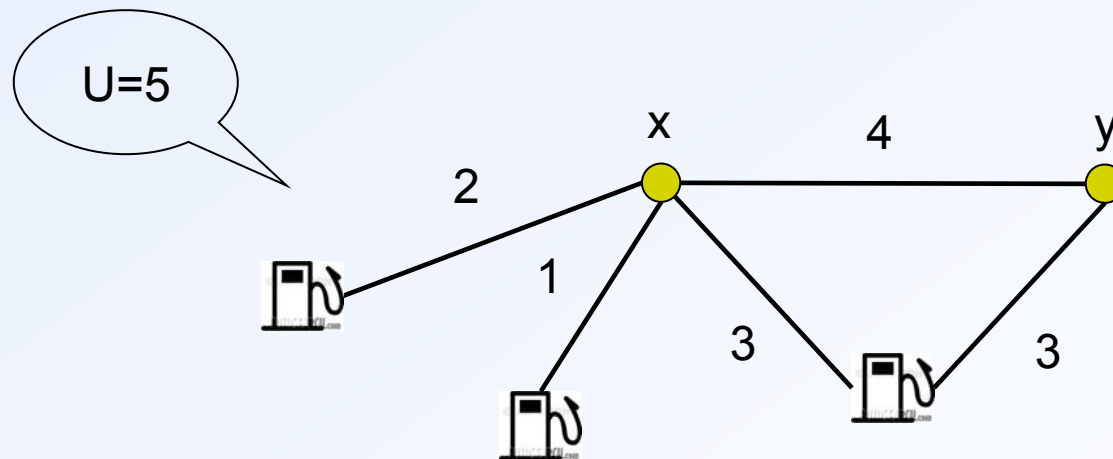
- Problem is APX-hard since it generalizes TSP.
- If each city has a gas station ($T \subseteq S$) the two problems are equivalent:
 - Let $c(x,y)$ be **shortest feasible path** from x to y .
 - Triangle inequality holds in c





Uniform Cost Tour Gas Station

- The method works only when $T \subseteq S$
 - $c(x, y)$ depends on the **last stop before x**.
- **Assumption:**
 - Each city has a gas station within distance at most $\alpha U/2$.



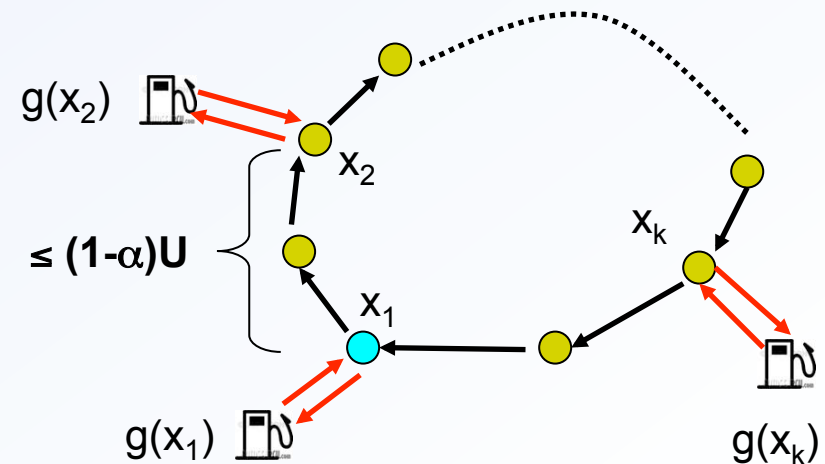
A simple case



1. Find the TSP on the cities.
2. Start from $g(x_1)$, go to x_1
3. Continue along the tour until x_2 ,
farthest city at distance at most $(1-\alpha)U$
4. Go to $g(x_2)$, repeat the procedure from $g(x_2)$
5. Continue until you reach x_1 .

For all edges (u,v) in the tour
 $d(u,v) \leq (1-\alpha)U$

Let $g(v)$ be nearest
gas station to v

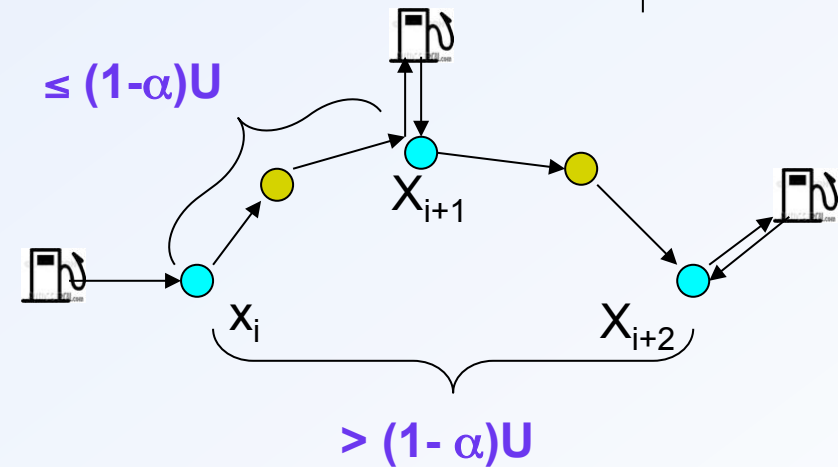


Uniform Cost Tour Gas Station



- In this solution

- $|T(x_i, x_{i+1})| \leq (1 - \alpha)U$
- $|T(x_i, x_{i+2})| > (1 - \alpha)U$



- Charge cost of trips to Gas Stations to the tour:

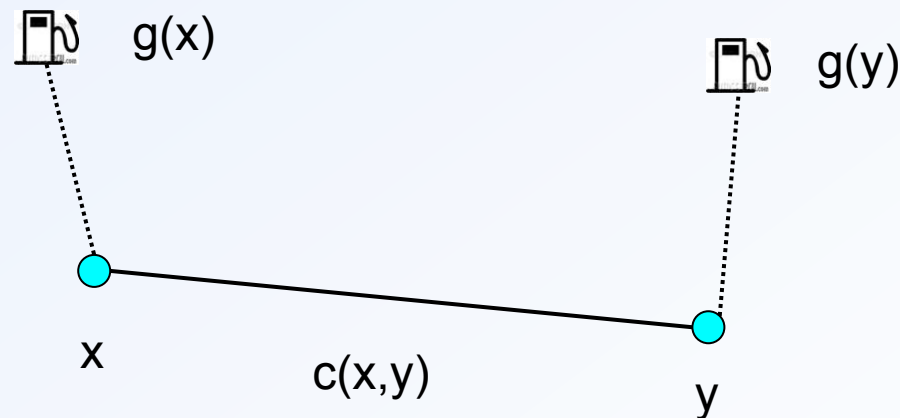
$$|T| + \alpha U k \leq (1 + 2\alpha/(1-\alpha)) |T|$$





Uniform Cost Tour Gas Station

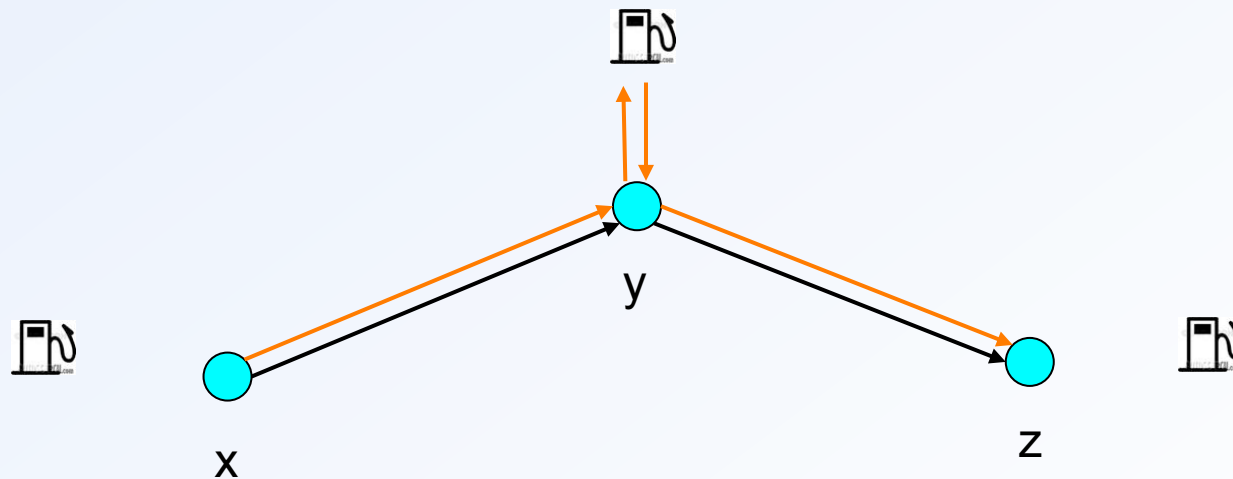
- Obtain a bound of $(1+\alpha)/(1-\alpha) 1.5 c(\text{OPT})$.
- Note that when $\alpha=0$, then we get $1.5 c(\text{OPT})$.
- Let $c(x,y)$ be cheapest traversal to go from x to y , such that we start at $g(x)$ and end at $g(y)$.





Main problem

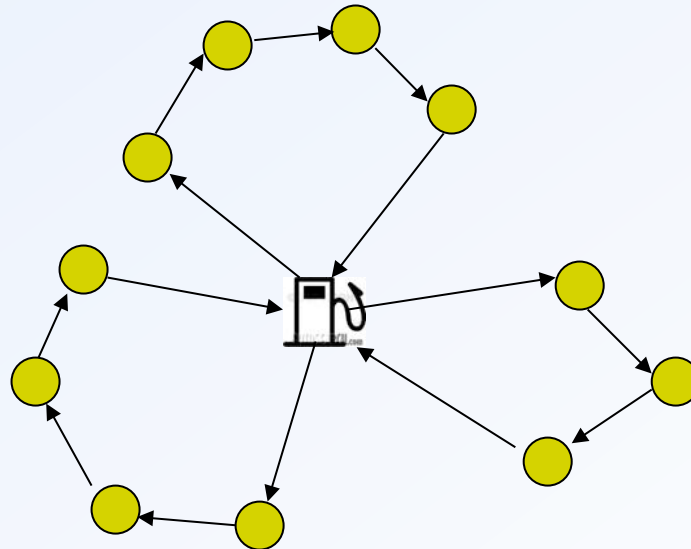
- Lack of triangle inequality.
- To get around it, use Christofides's method combine with previous approach to get a feasible solution.





Single Gas Station Problem

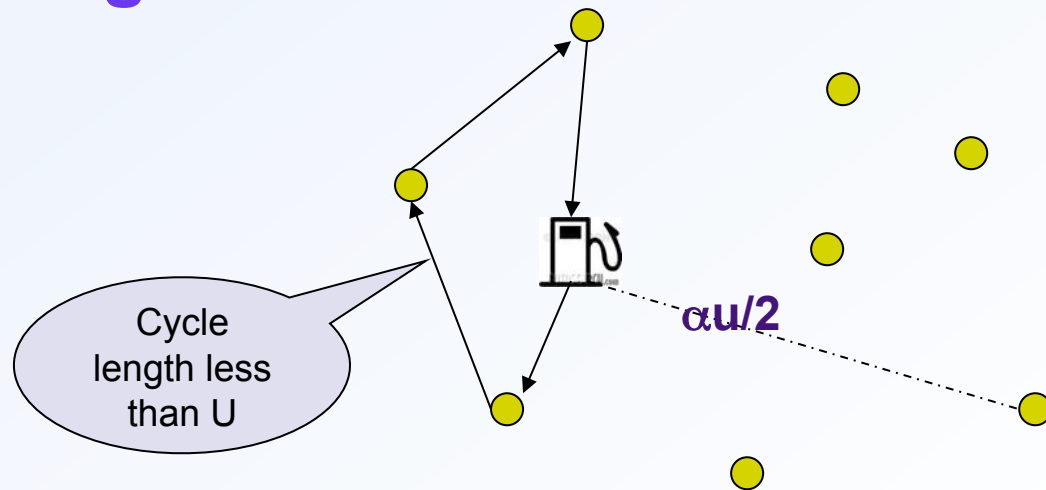
- Suppose the salesman lives near a gas station.
- He wants to go to a **set of cities**.
- In each trip we can travel a distance of **U**.





Single Gas Station

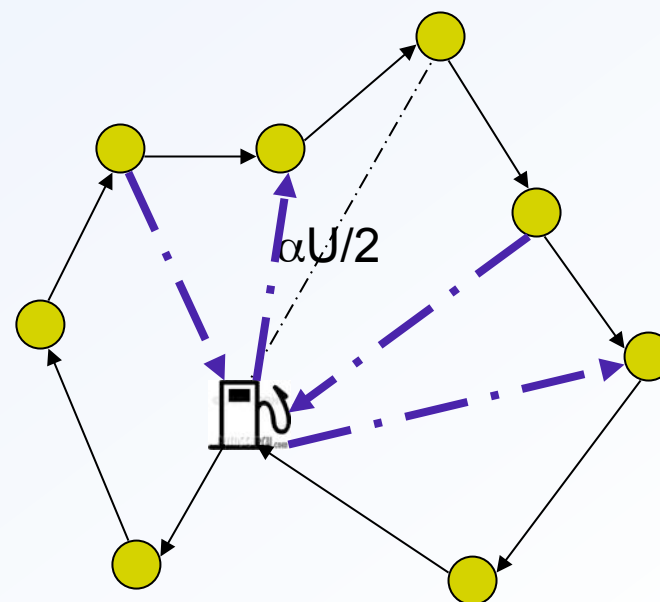
- We are given $G=(V,E)$
- We want to cover the vertices in V
- We have only **one gas station**
- Dist. of the **farthest city** to the gas station is $\alpha U/2$.
- Each cycle has **length $\leq U$**





Naïve Solution

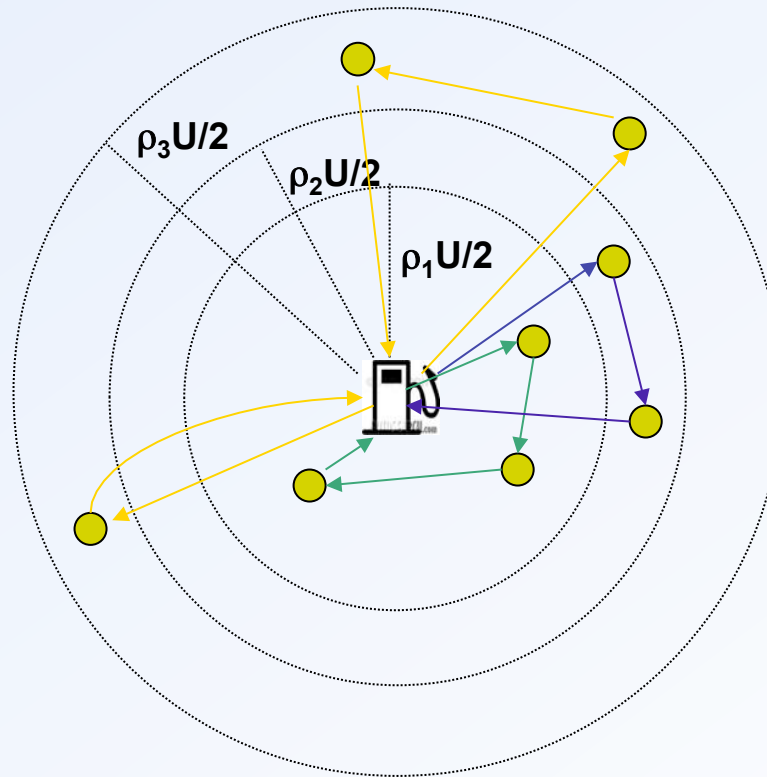
- Find the **TSP** on **cities & gas station**
- **Chop** the tour into parts of **length $(1-\alpha)U$**
- **Connect** each segment **to the root**
- This is a **$1.5/(1-\alpha)$** approximation
- Given by Li et al. [1991]





Improved Method

- **Group cities** based on their **distance** to the gas station
- Solve the problem for **each group separately**



$O(\log 1/(1-\alpha))$



Summary of The Results

Problem	Complexity	Approx. Ratio
2 Cities Graph Case	Single sink: $O(\Delta n^2 \log n)$ All pairs: $O(n^3 \Delta^2)$	
Fixed Path ($\Delta=n$)	$O(n \log n)$	
Single gas station	APX-hard	$O(\log 1/(1-\alpha))$
Uniform Tour	APX-hard	$O(1/(1-\alpha))$



Conclusion

- Incorporate the algorithms as part of a “tool” for path planning.
- Solve the tour gas station problem with arbitrary gas prices.
- Remove the assumption that every city has a gas station at distance $\alpha U/2$.
- Planar instances



Thanks for your attention